Chapter 9, Operations Research (OR)

Kent Andersen

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1 The transportation problem

We continue considering the transportation problem (P).

$$\begin{aligned} \text{Minimize } Z = \sum_{i\in S}\sum_{j\in D} c_{i,j}x_{i,j} & \\ \text{s.t.} & \\ \sum_{j\in D} x_{i,j} = s_i, \quad & \text{for all } i \in S \qquad (u_i) \\ \sum_{i\in S} x_{i,j} = d_j, \quad & \text{for all } j \in D \qquad (v_j) \\ x_{i,j} \geq 0. \text{ for all } (i,j) \in S \times D & \end{aligned}$$

and the dual (D) of (P):

$$\begin{aligned} \text{Maximize } Z = \sum_{i\in S} u_i s_i + \sum_{j\in D} v_j d_j & \\ \text{s.t.} & \\ u_i + v_j \leq c_{i,j}, \quad & \text{for all } (i,j) \in S \times D \quad (x_{i,j}) \\ u_i, v_j \in \mathbb{R}. & \end{aligned}$$

In the last chapter, we introduced a graph that represented the transportation problem. This graph has a node for every source $i \in S$, and a node for every destination $j \in D$. Furthermore, for every pair $(i,j) \in S \times D$ of source and destination, there is an edge joining the pair.

We first use a result from the last chapter to obtain the following.

Lemma 1 *Let "$Ax = b$" denote the equality system involved in the definition of the transportation problem, and suppose $\sum_{i\in S} s_i = \sum_{j\in D} d_j$. The equality system $Ax = b$ has one redundant equality, and the rank of A is $n + m - 1$.*

Proof: We showed last time that deleting any of the $n+m$ rows defining the transportation problem does not change the rank of the matrix A. Without loss of generality, suppose we delete the last row corresponding to the destination n. let $A'x = b'$ denote the system obtained from $Ax = b$ by deleting the last equality. Also, let $a_{i.}$ and $a_{j.}$ denote the rows of A, where $i \in S$ and $j \in D$. The rank of A is $(n + m - 1)$ if and only if the system $\sum_{i\in S} a_{i.}\alpha_i + \sum_{j\in D\setminus\{n\}} a_{j.}\beta_j = 0_{|S\times D|}$ implies that $\alpha_i = 0$ for all $i \in S$, and $\beta_j = 0$ for all $j \in D$.

Observe that the matrix A has exactly two non-zero values for every column of A. Therefore, in the system $\sum_{i\in S} \alpha_i a_{i.} + \sum_{j\in D\setminus\{n\}} \beta_j a_{j.} = 0_{|S\times D|}$ includes the equalities $\alpha_i = 0$ for $i \in S$ corresponding to the variables $x_{i,n}$ for $i \in S$. Finally, for any (i,j) such that $i \in S$ and $j \in D\setminus\{n\}$, the system $\sum_{i\in S} a_{i.}\alpha_i + \sum_{j\in D\setminus\{n\}} a_{j.}\beta_j = 0_{|S\times D|}$ includes the equality $\alpha_i + \beta_j = 0$, which implies $\beta_j = 0$ for all $j \in D \setminus \{n\}$. Hence the rank of A is $(m + n - 1)$. ∎

For a general LP with constraints $Ax = b$ and $x \geq 0_n$, where the coefficient A is an $m \times n$ matrix, a basis is defined as a maximal subset of the columns of A such that the corresponding columns of A are linearly independent. Earlier, this meant that a basis would have size m, since we assumed that the rank of A is m. In the case of the transportation problem, the rank of the equality matrix is $(m + n - 1)$, and therefore a basis has size $(m + n - 1)$.

Definition 1 *Let "$Ax = b$" denote the equality system involved in the definition of the transportation problem. A* basis *for the transportation problem is a subset $B \subseteq S \times D$, such that:*

(i) The columns of A corresponding to B are linearly independent, and

(ii) $|B| = m + n - 1$.

We note that, if $B \subseteq S \times D$ is a basis for the transportation problem, then the *basic solution* x^B to the following system is unique.

$$
\begin{aligned}
\sum_{j \in D} x_{i,j} &= s_i, && \text{for all } i \in S \\
\sum_{i \in S} x_{i,j} &= d_j, && \text{for all } j \in D \\
x_{i,j} &= 0, && \text{for all } (i,j) \in (S \times D) \setminus B.
\end{aligned}
$$

The edge-induced subgraph of the transportation graph that corresponds to the edges in a basis B was shown in the last chapter to have the following property.

Lemma 2 *For a basis B, the graph obtained from the transportation graph by only drawing the edges corresponding to edges $(i,j) \in B$ has no cycles.*

We also introduced the concept of a tree in a general graph $G = (V, E)$, and we also discussed the concepts of forests and paths.

Definition 2 *Let $G = (V, E)$ be a graph.*

(i) If the graph G contains no cycles, then G is called a forest*.*

(ii) If G is a forest, and there is a path between every pair of vertices of G, then G is called a spanning tree*.*

Two vertices $u, v \in V$ of a general graph $G = (V, E)$ are said to be *connected*, if there is a path from u to v in G. A node $v \in V$ is always, by definition, said to be connected to itself. We note that the concept of connectivity defines an *equivalence relation* on the vertices of G. An equivalence relation on a set is a relation that satisfy three properties called reflectivity, symmetry and transitivity.

(a) A node $v \in V$ is connected to itself (reflectivity).

(b) If $u \in V$ is connected to $v \in V$, then v is also connected to u (symmetry).

(c) If $u \in V$ is connected to $v \in V$, and v is connected to $w \in V$, then u is connected to w (transitivity).

An equivalence relation divides its members into *equivalence classes*, or in other words, into classes of members that are all pairwise related to each other (another example of an equivalence relation is "="). In our example of connectivity in a graph, connectivity divides the vertices into *connected components* of vertices that are all pairwise connected.

Since a tree is a forest in which all nodes are pairwise connected, the above discussion demonstrates that a forest is a collection of trees, where each tree connects the vertices of each connected component.

The *degree* $d_G(v)$ of a node $v \in V$ in a graph G is defined to be the number of edges in G that are connected to v. We showed the following important properties of a forest in the last chapter.

Lemma 3 *Let $G = (V, E)$ be a forest.*

(i) If $E \neq \emptyset$, then G has a node of degree one.

(ii) If $v \in V$ is a node of degree one, and $e \in E$ is the edge connected to v, then the graph $G' = (V \setminus \{v\}, E \setminus \{e\})$ is also a forest.

From the above properties, it is easy to prove the following.

Lemma 4 *Let $G = (V, E)$ be a graph that is a spanning tree.*

(i) If $v \in V$ is a node of degree one, and $e \in E$ is the edge connected to v, then the graph $G' = (V \setminus \{v\}, E \setminus \{e\})$ is also spanning tree.

(ii) For every two vertices $u, v \in V$, there is a unique path in G from u to v.

(iii) $|E| = |V| - 1$.

Proof: (i): it is clear that, if there is a path between two nodes $u, w \in V \setminus \{v\}$ in the graph G, then this path still remains in the graph G', and therefore G' is also a spanning tree.(ii) If there are two (different) paths between two vertices $u, v \in V$, then a cycle can be constructed, which contradicts the fact that spanning trees do not have cycles.(iii): This follows from repeated application of (i), or in other words, deleting one vertex and one edge at a time. ■

We now finish the proof that a basis corresponds to a spanning tree.

Lemma 5 *Let $B \subseteq S \times D$ be a basis for the transportation problem. Then the subgraph of the transportation graph induced by the edges in B forms a spanning tree.*

Proof: Let $G = (V, E)$ be the subgraph of the transportation graph with nodes index by $S \cup D$ and edges $B \subseteq S \times D$. Since $|B| = n + m - 1$, the graph G has $(n + m - 1)$ edges and $|S \cup D| = m + n$ vertices. Furthermore, since B is a basis, G does not have any cycles, and therefore G is a forest. If G is not a spanning tree, G consists of several spanning trees, say $G^i = (V^i, E^i)$ for $i = 1, 2, \ldots, k$, where k denotes the number of connected components, $V = V^1 \cup V^2 \ldots \cup V^k$ and $E = B = E^1 \cup E^2 \ldots \cup E^k$. Since G^i is a spanning tree, we have $|E^i| = |V^i| - 1$. However, we now have $n + m - 1 = |B| = |E| = \sum_{i=1}^{k} |E^i| = \sum_{i=1}^{k} (|V^i| - 1) = |V| - k = n + m - k$. This implies $k = 1$, and therefore G is connected. ■

The *transportation simplex method* can now be stated as follows.

Step 1: Find an initial basic feasible solution x^B for the transportation problem (see the next section), and a corresponding basis $B \subseteq S \times D$.

Step 2: Compute the corresponding dual solution $\{u_i\}_{i \in S}$ and $\{v_i\}_{j \in D}$. If the reduced costs $r_{i,j} := c_{i,j} - u_i - v_j$ are all non-negative - STOP. B is an optimal basis.

Step 3: Consider the spanning tree for the transportation graph defined by the edges indexed by B. by the edges Let $(\bar{i}, \bar{j}) \in S \times D$ be such that $r_{\bar{i},\bar{j}} < 0$. Find the unique cycle obtained by adding the edges $(\bar{i}, \bar{j})$ to the spanning tree, and use this cycle to identify the first basic variable (i', j') that becomes zero when $x_{\bar{i},\bar{j}}$ is increased from its current value of zero. Update the basis to $B' := (B \cup \{(\bar{i}, \bar{j})\}) \setminus \{(i', j')\}$, and return to step 2.

2 Finding an initial basic feasible solution

We now describe a method for finding an initial basic feasible solution for the transportation problem. This method is called the *Northwest corner rule*. Start by selecting the variable $x_{1,1}$ to be in the basis, and attempt to ship as much from source 1 to destination 1 as possible. If supply limits the amount that can be shipped from 1 to 1, set $x_{1,1} = s_1$, update the remaining demand at destination 1 to $d_1 := d_1 - s_1$ and eliminate source 1 for further consideration as a source for further delivery. Otherwise, if demand limits the amount that can be shipped, set $x_{1,1} = d_1$, update the remaining supply at source 1 to $s_1 := s_1 - d_1$ and eliminate destination 1 for further consideration for delivery.

Now, iteratively, suppose the variable $x_{i,j}$ was the last basic variable selected. If supply was the fact that limited the amount that could be shipped from i to j, choose the variable $x_{i+1,j}$ as the next basic variable. Otherwise, if demand was the limiting factor, choose the variable $x_{i,j+1}$. Now, set the new basic variable to as large a value as remaining demand and supply permits, and iterate the procedure.

Example 1 *Consider, again, the transportation problem with supply data* $(s_1, s_2, s_3, s_4) = (50, 50, 50, 50)$ *and demand data* $(d_1, d_2, d_3, d_4, d_5) = (30, 20, 70, 30, 50)$ *(*$m = 4$ *and* $n = 5$*).*

The variable $x_{1,1}$ *is selected initially, and since* $d_1 = 30$ *and* $s_1 = 50$*, we set* $x_{1,1} = 30$*, and satisfy all demand at destination* 1 *with supply from source* 1*. Destination* 1 *is now eliminated from further consideration, and* s_1 *is updated to* $s_1 = s_1 - 30 = 20$*.*

Since supply remains at source 1, we choose $x_{1,2}$ *as the next basic variable. We have* $s_1 = 20$ *(updated), and* $d_2 = 20$*, so we can satisfy all of the demand of destination* 2 *from source* 1 *and therefore set* $x_{1,2} = 20$*. We next say that source* 1 *is the limiting factor (destination* 2 *could also have been chosen, since* $s_1 = d_2 = 20$*). We therefore eliminate source* 1 *from further consideration, and update the demand at destination* 2 *to* $d_2 = d_2 - 20 = 0$*.*

Since we considered source 1 *as the limiting factor, the next basic variable is* $x_{2,2}$*. Since* $s_2 = 50$ *and* $d_2 = 0$ *(updated), we have that destination* 2 *is the limiting factor. We therefore set* $x_{2,2} = 0$ *(a basic variable with value zero). No update is necessary, since the variable was assigned the value zero. We now eliminate destination* 2 *from further consideration.*

Since destination 2 *was the limiting factor, we consider the variable* $x_{2,3}$ *as the next basic variable. We have* $s_2 = 50$ *and* $d_3 = 70$*. We therefore set* $x_{2,3} = 50$*, eliminate source 2 for further consideration, and update* $d_3 = d_3 - 50 = 20$*.*

Continuing in this way, we obtain the basic solution $x_{1,1} = 30$*,* $x_{1,2} = 20$*,* $x_{2,2} = 0$*,* $x_{2,3} = 50$*,* $x_{3,3} = 20$*,* $x_{3,4} = 30$*,* $x_{4,4} = 0$ *and* $x_{4,5} = 50$*.*

3 The assignment problem

Consider the special of a transportation problem, where $s_i = 1$ for all $i \in S$ and $d_j = 1$ for all $j \in D$

$$
\begin{aligned}
\text{Minimize } Z = \sum_{i \in S} \sum_{j \in D} & c_{i,j} x_{i,j} \\
\text{s.t.} & \\
\sum_{j \in D} x_{i,j} &= 1, && \text{for all } i \in S \\
\sum_{i \in S} x_{i,j} &= 1, && \text{for all } j \in D \\
x_{i,j} &\geq 0. && \text{for all } i \in S, j \in D
\end{aligned}
$$

Recall the following fact of transportation problems.

Lemma 6 *If the numbers* $\{s_i\}_{i \in S}$ *and* $\{d_j\}_{j \in D}$ *are integers, then* every *basic feasible solution* x^B *to the transportation problem is such that* x^B *is integer.*

Observe that for the assignment problem, the values of the variables are bounded to be between zero and one. Hence, from the above integrality property, the values of the variables are *either* zero *or* one.

We can therefore interpret a basic solution to the assignment problem as an assignment of a source to a destination. The problem is then to find such an assignment that has the smallest total cost. This is an important application, since many applications involve the assignment of certain things to other things, such as personnel to machines, personnel to schedules or other things.

The method for solving the assignment problem is the same as the transportation problem, and it can be shown to be very efficient for solving assignment problems (large problems can be solved).