

Peano arithmetic

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1 Peano arithmetic

This Logiweb page [1] defines Peano arithmetic.

1.1 The constructs of Peano arithmetic

Terms of Peano arithmetic are constructed from zero $[0 \xrightarrow{\text{pyk}} \text{“peano zero”}][0 \xrightarrow{\text{tex}} \text{“}$

$\backslash\text{dot}\{0\}”]$, successor $[x' \xrightarrow{\text{pyk}} \text{“* peano succ”}][x' \xrightarrow{\text{tex}} \text{“\#1.”}]]$, plus $[x \dot{+} y \xrightarrow{\text{pyk}} \text{“* peano plus *”}][x \dot{+} y \xrightarrow{\text{tex}} \text{“\#1.”}]]$.

$\backslash\text{mathop}\{\backslash\text{dot}\{+\}\} \#2.”]$, and times $[x : y \xrightarrow{\text{pyk}} \text{“* peano times *”}][x : y \xrightarrow{\text{tex}} \text{“\#1.”}][\backslash\text{mathop}\{\backslash\text{dot}\{\cdot\}\} \#2.”]$.

Formulas of Peano arithmetic are constructed from equality $[x \stackrel{\text{p}}{=} y \xrightarrow{\text{pyk}} \text{“* peano is *”}][x \stackrel{\text{p}}{=} y \xrightarrow{\text{tex}} \text{“\#1.”}]]$.

$\backslash\text{stackrel}\{p\}\{=\} \#2.”]$, negation $[\neg x \xrightarrow{\text{pyk}} \text{“peano not *”}][\neg x \xrightarrow{\text{tex}} \text{“}$

$\backslash\text{dot}\{\text{neg}\}\backslash, \#1.”]$, implication $[x \Rightarrow y \xrightarrow{\text{pyk}} \text{“* peano imply *”}][x \Rightarrow y \xrightarrow{\text{tex}} \text{“\#1.”}]]$.

$\backslash\text{mathrel}\{\backslash\text{dot}\{\rightarrow\}\} \#2.”]$, and universal quantification

$[\forall x : y \xrightarrow{\text{pyk}} \text{“peano all * indeed *”}][\forall x : y \xrightarrow{\text{tex}} \text{“}$

$\backslash\text{dot}\{\text{forall}\} \#1.$

$\backslash\text{colon} \#2.”]$.

From these constructs we macro define one $[1 \xrightarrow{\text{pyk}} \text{“peano one”}][1 \xrightarrow{\text{tex}} \text{“}$

$\backslash\text{dot}\{1\}”]$, two $[2 \xrightarrow{\text{pyk}} \text{“peano two”}][2 \xrightarrow{\text{tex}} \text{“}$

$\backslash\text{dot}\{2\}”]$, conjunction $[x \wedge y \xrightarrow{\text{pyk}} \text{“* peano and *”}][x \wedge y \xrightarrow{\text{tex}} \text{“\#1.”}]]$.

$\backslash\text{mathrel}\{\backslash\text{dot}\{\wedge\}\} \#2.”]$, disjunction $[x \vee y \xrightarrow{\text{pyk}} \text{“* peano or *”}][x \vee y \xrightarrow{\text{tex}} \text{“\#1.”}]]$.

$\backslash\text{mathrel}\{\backslash\text{dot}\{\vee\}\} \#2.”]$, biimplication $[x \Leftrightarrow y \xrightarrow{\text{pyk}} \text{“* peano iff *”}][x \Leftrightarrow y \xrightarrow{\text{tex}} \text{“\#1.”}]]$.

$\backslash\text{mathrel}\{\backslash\text{dot}\{\rightarrow\}\} \#2.”]$, and existential quantification

$[\exists x : y \xrightarrow{\text{pyk}} \text{“peano exist * indeed *”}][\exists x : y \xrightarrow{\text{tex}} \text{“}$

$\backslash\text{dot}\{\text{exists}\} \#1.$

$\backslash\text{colon} \#2.”]$:

$$[1 \xrightarrow{\text{macro}} \lambda t. \lambda s. \lambda c. \tilde{\mathcal{M}}_4(t, s, c, [[1 \doteq 0]])]$$

$$[2 \xrightarrow{\text{macro}} \lambda t. \lambda s. \lambda c. \tilde{\mathcal{M}}_4(t, s, c, [[2 \doteq 1]])]$$

$$[x \wedge y \xrightarrow{\text{macro}} \lambda t. \lambda s. \lambda c. \tilde{\mathcal{M}}_4(t, s, c, [[x \wedge y \doteq \neg (x \Rightarrow \neg y)])]]]$$

$$[x \dot{\vee} y \xrightarrow{\text{macro}} \lambda t. \lambda s. \lambda c. \tilde{\mathcal{M}}_4(t, s, c, [[x \dot{\vee} y \dot{=} \dot{\neg} x \dot{\Rightarrow} y]])]$$

$$[x \dot{\Leftrightarrow} y \xrightarrow{\text{macro}} \lambda t. \lambda s. \lambda c. \tilde{\mathcal{M}}_4(t, s, c, [[x \dot{\Leftrightarrow} y \dot{=} (x \dot{\Rightarrow} y) \dot{\wedge} (y \dot{\Rightarrow} x)])])]$$

$$[\dot{\exists} x: y \xrightarrow{\text{macro}} \lambda t. \lambda s. \lambda c. \tilde{\mathcal{M}}_4(t, s, c, [[\dot{\exists} x: y \dot{=} \dot{\neg} \dot{\forall} x: \dot{\neg} y]])]$$

1.2 Variables

We now introduce the unary operator $[\dot{x} \xrightarrow{\text{pyk}} \text{“* peano var”}][\dot{x} \xrightarrow{\text{tex}} \text{“}\backslash\text{dot}\{\#1.\}$
 $\}”]$ and define that a term is a *Peano variable* (i.e. a variable of Peano arithmetic) if it has the $[\dot{x}]$ operator in its root. $[x^{\mathcal{P}} \xrightarrow{\text{pyk}} \text{“* is peano var”}][x^{\mathcal{P}} \xrightarrow{\text{tex}} \text{“}\backslash\text{dot}\{\#1.\}$
 $\} \wedge \{\backslash\text{cal P}\}”]$ is true if $[x]$ is a Peano variable:

$$[x^{\mathcal{P}} \xrightarrow{\text{val}} x \stackrel{r}{=} [\dot{x}]]$$

We macro define $[\dot{a} \xrightarrow{\text{pyk}} \text{“peano a”}][\dot{a} \xrightarrow{\text{tex}} \text{“}\backslash\text{dot}\{\text{mathit}\{a\}\}”]$ to be a Peano variable:

$$[\dot{a} \xrightarrow{\text{macro}} \lambda t. \lambda s. \lambda c. \tilde{\mathcal{M}}_4(t, s, c, [[\dot{a} \dot{=} \dot{a}]])]$$

Appendix A.2 defines Peano variables for the other letters of the English alphabet.

$[\text{nonfree}(x, y) \xrightarrow{\text{pyk}} \text{“peano nonfree * in * end nonfree”}][\text{nonfree}(x, y) \xrightarrow{\text{tex}} \text{“}\backslash\text{dot}\{\text{nonfree}\}(\#1.\)$
 $, \#2.$
 $\}”]$ is true if the Peano variable $[x]$ does not occur free in the Peano

term/formula $[y]$. $[\text{nonfree}^*(x, y) \xrightarrow{\text{pyk}} \text{“peano nonfree star * in * end nonfree”}][\text{nonfree}^*(x, y) \xrightarrow{\text{tex}} \text{“}\backslash\text{dot}\{\text{nonfree}\}^*(\#1.\)$
 $, \#2.$
 $\}”]$ is true if the Peano variable $[x]$ does not occur free in the list $[y]$ of Peano

terms/formulas.

$$\begin{aligned} &[\text{nonfree}(x, y) \xrightarrow{\text{val}} \\ &\text{If}(y^{\mathcal{P}}, \neg x \stackrel{t}{=} y, \\ &\text{If}(\neg y \stackrel{r}{=} [\dot{\forall} x: y], \text{nonfree}^*(x, y^t), \\ &\text{If}(x \stackrel{t}{=} y^1, \text{T}, \text{nonfree}(x, y^2)))] \end{aligned}$$

$$[\text{nonfree}^*(x, y) \xrightarrow{\text{val}} x! \text{If}(y, \text{T}, \text{If}(\text{nonfree}(x, y^h), \text{nonfree}^*(x, y^t), \text{F}))]$$

$[\text{free}(a|x := b) \xrightarrow{\text{pyk}} \text{“peano free * set * to * end free”}][\text{free}(a|x := b) \xrightarrow{\text{tex}} \text{“}\backslash\text{dot}\{\text{free}\}\backslash\text{langl} \#1.$
 $| \#2.$

:= #3.

$\backslash\text{rangle}$] is true if the substitution $[\langle a \mid x := b \rangle]$ is free.

$[\text{free}^* \langle a \mid x := b \rangle \xrightarrow{\text{pyk}} \text{“peano free star * set * to * end free”}][\text{free}^* \langle a \mid x := b \rangle \xrightarrow{\text{tex}} \text{“} \backslash\text{dot}\{\text{free}\}\{\}^* \backslash\text{rangle} \#1.$

| #2.

:= #3.

$\backslash\text{rangle}$] is the version where $[a]$ is a list of terms.

$[\text{free} \langle a \mid x := b \rangle \xrightarrow{\text{val}} x!b!$
 $\text{If}(a^{\mathcal{P}}, T,$
 $\text{If}(\neg a \stackrel{r}{=} [\forall u: v], \text{free}^* \langle a^t \mid x := b \rangle,$
 $\text{If}(a^1 \stackrel{t}{=} x, T,$
 $\text{If}(\text{nonfree}(x, a^2), T,$
 $\text{If}(\neg \text{nonfree}(a^1, b), F,$
 $\text{free} \langle a^2 \mid x := b \rangle)))]$

$[\text{free}^* \langle a \mid x := b \rangle \xrightarrow{\text{val}} x!b! \text{If}(a, T, \text{If}(\text{free} \langle a^h \mid x := b \rangle, \text{free}^* \langle a^t \mid x := b \rangle, F))]$

$[a \equiv \langle b \mid x := c \rangle \xrightarrow{\text{pyk}} \text{“peano sub * is * where * is * end sub”}][a \equiv \langle b \mid x := c \rangle \xrightarrow{\text{tex}} \text{“}\#1.$
 $\{\backslash\text{equiv}\} \backslash\text{rangle} \#2.$

| #3.

:= #4.

$\backslash\text{rangle}$] is true if $[a]$ equals $[\langle b \mid x := c \rangle]$. $[a \equiv \langle *b \mid x := c \rangle \xrightarrow{\text{pyk}} \text{“peano sub star * is$
 $* \text{ where * is * end sub”}][a \equiv \langle *b \mid x := c \rangle \xrightarrow{\text{tex}} \text{“}\#1.$
 $\{\backslash\text{equiv}\} \backslash\text{rangle}^* \#2.$

| #3.

:= #4.

$\backslash\text{rangle}$] is the version where $[a]$ and $[b]$ are lists.

$[a \equiv \langle b \mid x := c \rangle \xrightarrow{\text{val}} a!x!c!$
 $\text{If}(\text{If}(b \stackrel{r}{=} [\forall u: v], b^1 \stackrel{t}{=} x, F), a \stackrel{t}{=} b,$
 $\text{If}(b^{\mathcal{P}} \wedge b \stackrel{t}{=} x, a \stackrel{t}{=} c, \text{If}(\$
 $a \stackrel{r}{=} b, a^t \equiv \langle *b^t \mid x := c \rangle, F)))]$

$[a \equiv \langle *b \mid x := c \rangle \xrightarrow{\text{val}} b!x!c! \text{If}(a, T, \text{If}(a^h \equiv \langle b^h \mid x := c \rangle, a^t \equiv \langle *b^t \mid x := c \rangle, F))]$

1.3 Mendelsons system S

System $[S \xrightarrow{\text{pyk}} \text{“system s”}][S \xrightarrow{\text{tex}} \text{“}$
 $S”]$ of Mendelson [2] expresses Peano arithmetic. It comprises the axioms
 $[A1 \xrightarrow{\text{pyk}} \text{“axiom a one”}][A1 \xrightarrow{\text{tex}} \text{“}$
 $A1”]$, $[A2 \xrightarrow{\text{pyk}} \text{“axiom a two”}][A2 \xrightarrow{\text{tex}} \text{“}$
 $A2”]$, $[A3 \xrightarrow{\text{pyk}} \text{“axiom a three”}][A3 \xrightarrow{\text{tex}} \text{“}$

A3”], [A4 $\xrightarrow{\text{pyk}}$ “axiom a four”][A4 $\xrightarrow{\text{tex}}$ “
A4”], and [A5 $\xrightarrow{\text{pyk}}$ “axiom a five”][A5 $\xrightarrow{\text{tex}}$ “
A5”] and inference rules [MP $\xrightarrow{\text{pyk}}$ “rule mp”][MP $\xrightarrow{\text{tex}}$ “
MP”] and [Gen $\xrightarrow{\text{pyk}}$ “rule gen”][Gen $\xrightarrow{\text{tex}}$ “
Gen”] of first order predicate calculus. Furthermore, it comprises the proper
axioms [S1 $\xrightarrow{\text{pyk}}$ “axiom s one”][S1 $\xrightarrow{\text{tex}}$ “
S1”], [S2 $\xrightarrow{\text{pyk}}$ “axiom s two”][S2 $\xrightarrow{\text{tex}}$ “
S2”], [S3 $\xrightarrow{\text{pyk}}$ “axiom s three”][S3 $\xrightarrow{\text{tex}}$ “
S3”], [S4 $\xrightarrow{\text{pyk}}$ “axiom s four”][S4 $\xrightarrow{\text{tex}}$ “
S4”], [S5 $\xrightarrow{\text{pyk}}$ “axiom s five”][S5 $\xrightarrow{\text{tex}}$ “
S5”], [S6 $\xrightarrow{\text{pyk}}$ “axiom s six”][S6 $\xrightarrow{\text{tex}}$ “
S6”], [S7 $\xrightarrow{\text{pyk}}$ “axiom s seven”][S7 $\xrightarrow{\text{tex}}$ “
S7”], [S8 $\xrightarrow{\text{pyk}}$ “axiom s eight”][S8 $\xrightarrow{\text{tex}}$ “
S8”], and [S9 $\xrightarrow{\text{pyk}}$ “axiom s nine”][S9 $\xrightarrow{\text{tex}}$ “
S9”]. System [S] is defined thus:

$$[S \xrightarrow{\text{stmt}} \dot{a} \dot{+} \dot{b}' \stackrel{P}{=} \dot{a} \dot{+} \dot{b}' \oplus \forall \underline{a}: \forall \underline{b}: \dot{+} \underline{b} \Rightarrow \dot{+} \underline{a} \Rightarrow \dot{+} \underline{b} \Rightarrow \underline{a} \Rightarrow \underline{b} \oplus \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a}' \stackrel{P}{=} \dot{b}' \oplus \forall \underline{a}: \forall \underline{b}: \underline{a} \Rightarrow \underline{b} \vdash \underline{a} \vdash \underline{b} \oplus \dot{a}' \stackrel{P}{=} \dot{b}' \Rightarrow \dot{a} \stackrel{P}{=} \dot{b} \oplus \forall \underline{a}: \forall \underline{b}: \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a} \oplus \forall \underline{x}: \forall \underline{a}: \forall \underline{b}: \text{nonfree}(\underline{x}, \underline{a}) \Vdash \dot{\forall} \underline{x}: \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a} \Rightarrow \dot{\forall} \underline{x}: \underline{b} \oplus \dot{a}: \dot{b}' \stackrel{P}{=} \dot{a}: \dot{b} \dot{+} \dot{a} \oplus \dot{a} \dot{+} \dot{0} \stackrel{P}{=} \dot{a} \oplus \forall \underline{a}: \forall \underline{b}: \forall \underline{c}: \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a} \Rightarrow \underline{c} \oplus \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{c} \Rightarrow \dot{b} \stackrel{P}{=} \dot{c} \oplus \forall \underline{a}: \forall \underline{b}: \forall \underline{c}: \forall \underline{x}: \underline{b} \equiv \langle \underline{a} | \underline{x} := \dot{0} \rangle \Vdash \underline{c} \equiv \langle \underline{a} | \underline{x} := \underline{x}' \rangle \Vdash \underline{b} \Rightarrow \dot{\forall} \underline{x}: \underline{a} \Rightarrow \underline{c} \Rightarrow \dot{\forall} \underline{x}: \underline{a} \oplus \neg \dot{0} \stackrel{P}{=} \dot{a}' \oplus \forall \underline{x}: \forall \underline{a}: \underline{a} \vdash \dot{\forall} \underline{x}: \underline{a} \oplus \forall \underline{c}: \forall \underline{a}: \forall \underline{x}: \forall \underline{b}: [\underline{a}] \equiv \langle [\underline{b}] | [\underline{x}] := [\underline{c}] \rangle \Vdash \dot{\forall} \underline{x}: \underline{b} \Rightarrow \underline{a} \oplus \dot{a}: \dot{0} \stackrel{P}{=} \dot{0}]$$

$$[A1 \xrightarrow{\text{stmt}} S \vdash \forall \underline{a}: \forall \underline{b}: \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a}][A1 \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[A2 \xrightarrow{\text{stmt}} S \vdash \forall \underline{a}: \forall \underline{b}: \forall \underline{c}: \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a} \Rightarrow \underline{c}][A2 \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[A3 \xrightarrow{\text{stmt}} S \vdash \forall \underline{a}: \forall \underline{b}: \dot{+} \underline{b} \Rightarrow \dot{+} \underline{a} \Rightarrow \dot{+} \underline{b} \Rightarrow \underline{a} \Rightarrow \underline{b}][A3 \xrightarrow{\text{proof}} \text{Rule tactic}]$$

The order of quantifiers in the following axiom is such that $[\underline{c}]$ which the current conclusion tactic cannot guess comes first. This allows to supply a value for $[\underline{c}]$ without having to supply values for the other meta-variables.

$$[A4 \xrightarrow{\text{stmt}} S \vdash \forall \underline{c}: \forall \underline{a}: \forall \underline{x}: \forall \underline{b}: [\underline{a}] \equiv \langle [\underline{b}] | [\underline{x}] := [\underline{c}] \rangle \Vdash \dot{\forall} \underline{x}: \underline{b} \Rightarrow \underline{a}][A4 \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[A5 \xrightarrow{\text{stmt}} S \vdash \forall \underline{x}: \forall \underline{a}: \forall \underline{b}: \text{nonfree}(\underline{x}, \underline{a}) \Vdash \dot{\forall} \underline{x}: \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a} \Rightarrow \dot{\forall} \underline{x}: \underline{b}][A5 \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[MP \xrightarrow{\text{stmt}} S \vdash \forall \underline{a}: \forall \underline{b}: \underline{a} \Rightarrow \underline{b} \vdash \underline{a} \vdash \underline{b}][MP \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[\text{Gen} \xrightarrow{\text{stmt}} S \vdash \forall \dot{x}: \forall \dot{a}: \dot{a} \vdash \dot{x}: \dot{a}] [\text{Gen} \xrightarrow{\text{proof}} \text{Rule tactic}]$$

Axiom [S1] to [S8] are stated as Mendelson [2] does. This is done here to test certain parts of the Logiweb system. Serious users of Peano arithmetic are advised to take Mendelson's Lemma 3.1 as axioms instead.

$$[S1 \xrightarrow{\text{stmt}} S \vdash \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{c} \Rightarrow \dot{b} \stackrel{P}{=} \dot{c}] [S1 \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[S2 \xrightarrow{\text{stmt}} S \vdash \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a}' \stackrel{P}{=} \dot{b}'] [S2 \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[S3 \xrightarrow{\text{stmt}} S \vdash \neg \dot{0} \stackrel{P}{=} \dot{a}'] [S3 \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[S4 \xrightarrow{\text{stmt}} S \vdash \dot{a}' \stackrel{P}{=} \dot{b}' \Rightarrow \dot{a} \stackrel{P}{=} \dot{b}] [S4 \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[S5 \xrightarrow{\text{stmt}} S \vdash \dot{a} + \dot{0} \stackrel{P}{=} \dot{a}] [S5 \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[S6 \xrightarrow{\text{stmt}} S \vdash \dot{a} + \dot{b}' \stackrel{P}{=} \dot{a} + \dot{b}'] [S6 \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[S7 \xrightarrow{\text{stmt}} S \vdash \dot{a} : \dot{0} \stackrel{P}{=} \dot{0}] [S7 \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[S8 \xrightarrow{\text{stmt}} S \vdash \dot{a} : \dot{b}' \stackrel{P}{=} \dot{a} : \dot{b} + \dot{a}] [S8 \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[S9 \xrightarrow{\text{stmt}} S \vdash \forall \dot{a}: \forall \dot{b}: \forall \dot{c}: \forall \dot{x}: \dot{b} \equiv \langle \dot{a} | \dot{x} := \dot{0} \rangle \Vdash \dot{c} \equiv \langle \dot{a} | \dot{x} := \dot{x}' \rangle \Vdash \dot{b} \Rightarrow \dot{x}: \dot{a} \Rightarrow \dot{c} \Rightarrow \dot{x}: \dot{a}] [S9 \xrightarrow{\text{proof}} \text{Rule tactic}]$$

1.4 A lemma and a proof

We now prove Lemma [L3.2(a) $\xrightarrow{\text{pyk}}$ “lemma 1 three two a”] [L3.2(a) $\xrightarrow{\text{tex}}$ “L3.2(a)”] which is an instance of the corresponding proposition in Mendelson [2]:

$$[\text{L3.2(a)} \xrightarrow{\text{stmt}} S \vdash \dot{x} \stackrel{P}{=} \dot{x}]$$

$$\begin{aligned} & [\text{L3.2(a)} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([S \vdash S5 \gg \dot{a} + \dot{0} \stackrel{P}{=} \dot{a}; \text{Gen} \triangleright \dot{a} + \dot{0} \stackrel{P}{=} \dot{a} \gg \dot{V}\dot{a}: \dot{a} + \dot{0} \stackrel{P}{=} \\ & \dot{a}; A4 @ \dot{x} \gg \dot{V}\dot{a}: \dot{a} + \dot{0} \stackrel{P}{=} \dot{a} \Rightarrow \dot{x} + \dot{0} \stackrel{P}{=} \dot{x}; \text{MP} \triangleright \dot{V}\dot{a}: \dot{a} + \dot{0} \stackrel{P}{=} \dot{a} \Rightarrow \dot{x} + \dot{0} \stackrel{P}{=} \\ & \dot{x} \triangleright \dot{V}\dot{a}: \dot{a} + \dot{0} \stackrel{P}{=} \dot{a} \gg \dot{x} + \dot{0} \stackrel{P}{=} \dot{x}; S1 \gg \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{c} \Rightarrow \dot{b} \stackrel{P}{=} \dot{c}; \text{Gen} \triangleright \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \\ & \dot{c} \Rightarrow \dot{b} \stackrel{P}{=} \dot{c} \gg \dot{V}\dot{c}: \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{c} \Rightarrow \dot{b} \stackrel{P}{=} \dot{c}; A4 @ \dot{x} \gg \dot{V}\dot{c}: \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{c} \Rightarrow \dot{b} \stackrel{P}{=} \\ & \dot{c} \Rightarrow \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{b} \stackrel{P}{=} \dot{x}; \text{MP} \triangleright \dot{V}\dot{c}: \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{c} \Rightarrow \dot{b} \stackrel{P}{=} \dot{c} \Rightarrow \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \\ & \dot{x} \Rightarrow \dot{b} \stackrel{P}{=} \dot{x} \triangleright \dot{V}\dot{c}: \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{c} \Rightarrow \dot{b} \stackrel{P}{=} \dot{c} \gg \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{b} \stackrel{P}{=} \dot{x}; \text{Gen} \triangleright \dot{a} \stackrel{P}{=} \\ & \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{b} \stackrel{P}{=} \dot{x} \gg \dot{V}\dot{b}: \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{b} \stackrel{P}{=} \dot{x}; A4 @ \dot{x} \gg \dot{V}\dot{b}: \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \\ & \dot{x} \Rightarrow \dot{b} \stackrel{P}{=} \dot{x} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x}; \text{MP} \triangleright \dot{V}\dot{b}: \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{b} \stackrel{P}{=} \dot{x} \Rightarrow \dot{a} \stackrel{P}{=} \\ & \dot{x} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x} \triangleright \dot{V}\dot{b}: \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{b} \stackrel{P}{=} \dot{x} \gg \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \\ & \dot{x}; \text{Gen} \triangleright \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x} \gg \dot{V}\dot{a}: \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x}; A4 @ \dot{x} + \dot{0} \gg \\ & \dot{V}\dot{a}: \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} + \dot{0} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} + \dot{0} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x} \triangleright \dot{V}\dot{a}: \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x} \gg \\ & \dot{x} + \dot{0} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} + \dot{0} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x}; \text{MP} \triangleright \dot{x} + \dot{0} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} + \dot{0} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x} \triangleright \dot{x} + \dot{0} \stackrel{P}{=} \\ & \dot{x} \gg \dot{x} + \dot{0} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x}; \text{MP} \triangleright \dot{x} + \dot{0} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x} \triangleright \dot{x} + \dot{0} \stackrel{P}{=} \dot{x} \gg \dot{x} \stackrel{P}{=} \dot{x}], p_0, c)] \end{aligned}$$

1.5 An alternative axiomatic system

System $[S' \xrightarrow{\text{pyk}} \text{“system prime s”}][S' \xrightarrow{\text{tex}} \text{“S”}]$ is system $[S]$ in which the proper axioms are taken from Lemma 3.1 in Mendelson [2]. It comprises the axioms $[A1' \xrightarrow{\text{pyk}} \text{“axiom prime a one”}][A1' \xrightarrow{\text{tex}} \text{“A1”}]$, $[A2' \xrightarrow{\text{pyk}} \text{“axiom prime a two”}][A2' \xrightarrow{\text{tex}} \text{“A2”}]$, $[A3' \xrightarrow{\text{pyk}} \text{“axiom prime a three”}][A3' \xrightarrow{\text{tex}} \text{“A3”}]$, $[A4' \xrightarrow{\text{pyk}} \text{“axiom prime a four”}][A4' \xrightarrow{\text{tex}} \text{“A4”}]$, $[A5' \xrightarrow{\text{pyk}} \text{“axiom prime a five”}][A5' \xrightarrow{\text{tex}} \text{“A5”}]$ and inference rules $[MP' \xrightarrow{\text{pyk}} \text{“rule prime mp”}][MP' \xrightarrow{\text{tex}} \text{“MP”}]$ and $[Gen' \xrightarrow{\text{pyk}} \text{“rule prime gen”}][Gen' \xrightarrow{\text{tex}} \text{“Gen”}]$ of first order predicate calculus. Furthermore, it comprises the proper axioms $[S1' \xrightarrow{\text{pyk}} \text{“axiom prime s one”}][S1' \xrightarrow{\text{tex}} \text{“S1”}]$, $[S2' \xrightarrow{\text{pyk}} \text{“axiom prime s two”}][S2' \xrightarrow{\text{tex}} \text{“S2”}]$, $[S3' \xrightarrow{\text{pyk}} \text{“axiom prime s three”}][S3' \xrightarrow{\text{tex}} \text{“S3”}]$, $[S4' \xrightarrow{\text{pyk}} \text{“axiom prime s four”}][S4' \xrightarrow{\text{tex}} \text{“S4”}]$, $[S5' \xrightarrow{\text{pyk}} \text{“axiom prime s five”}][S5' \xrightarrow{\text{tex}} \text{“S5”}]$, $[S6' \xrightarrow{\text{pyk}} \text{“axiom prime s six”}][S6' \xrightarrow{\text{tex}} \text{“S6”}]$, $[S7' \xrightarrow{\text{pyk}} \text{“axiom prime s seven”}][S7' \xrightarrow{\text{tex}} \text{“S7”}]$, $[S8' \xrightarrow{\text{pyk}} \text{“axiom prime s eight”}][S8' \xrightarrow{\text{tex}} \text{“S8”}]$, and $[S9' \xrightarrow{\text{pyk}} \text{“axiom prime s nine”}][S9' \xrightarrow{\text{tex}} \text{“S9”}]$. System $[S']$ is defined thus:

$$[S' \xrightarrow{\text{stmt}} \forall a: \forall b: \underline{a} \stackrel{P}{=} \underline{b'} \Rightarrow \underline{a} \stackrel{P}{=} \underline{b} \oplus \forall h: \forall t: \forall r: \forall s: \underline{h} \Rightarrow \underline{t} \stackrel{P}{=} \underline{r} \vdash \underline{h} \Rightarrow \underline{t} \stackrel{P}{=} \underline{s} \vdash \underline{h} \Rightarrow \underline{r} \stackrel{P}{=} \underline{s} \oplus \forall a: \forall b: \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a} \oplus \forall x: \forall a: \forall b: \text{nonfree}(\langle \underline{x} \rangle, \langle \underline{a} \rangle) \Vdash \forall x: \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a} \Rightarrow \dot{\forall} x: \underline{b} \oplus \forall h: \forall t: \forall r: \underline{h} \Rightarrow \underline{t} \dot{+} \underline{r'} \stackrel{P}{=} \underline{t} \dot{+} \underline{r'} \oplus \forall a: \forall b: \underline{a} \cdot \underline{b'} \stackrel{P}{=} \underline{a} \cdot \underline{b} \dot{+} \underline{a} \oplus \forall a: \forall b: \underline{a} \stackrel{P}{=} \underline{b} \Rightarrow \underline{a'} \stackrel{P}{=} \underline{b'} \oplus \forall a: \forall b: \underline{a} \Rightarrow \underline{b} \vdash \underline{a} \vdash \underline{b} \oplus \forall a: \forall b: \neg \underline{b} \Rightarrow \neg \underline{a} \Rightarrow \neg \underline{b} \Rightarrow \underline{a} \Rightarrow \underline{b} \oplus \forall h: \forall t: \forall r: \underline{h} \Rightarrow \underline{t} \stackrel{P}{=} \underline{r} \vdash \underline{h} \Rightarrow \underline{t'} \stackrel{P}{=} \underline{r'} \oplus \forall a: \forall b: \underline{a} \dot{+} \underline{b'} \stackrel{P}{=} \underline{a} \dot{+} \underline{b} \oplus \forall a: \neg \dot{0} \stackrel{P}{=} \underline{a} \oplus \forall x: \forall a: \underline{a} \vdash \dot{\forall} x: \underline{a} \oplus \forall c: \forall a: \forall x: \forall b: \langle \underline{a} \rangle \equiv \langle \underline{b} \rangle \parallel \langle \underline{x} \rangle := \langle \underline{c} \rangle \Vdash \forall x: \underline{b} \Rightarrow \underline{a} \oplus \forall h: \forall t: \underline{h} \Rightarrow \underline{t} \dot{+} \dot{0} \stackrel{P}{=} \underline{t} \oplus \forall a: \underline{a} \cdot \dot{0} \stackrel{P}{=} \dot{0} \oplus \forall a: \forall b: \forall c: \underline{a} \stackrel{P}{=} \underline{b} \Rightarrow \underline{a} \stackrel{P}{=} \underline{c} \Rightarrow \underline{b} \stackrel{P}{=} \underline{c} \oplus \forall t: \underline{t} \stackrel{P}{=} \underline{t} \oplus \forall a: \forall b: \forall c: \forall x: \underline{b} \equiv \langle \underline{a} \rangle \underline{x} := \dot{0} \Vdash \underline{c} \equiv \langle \underline{a} \rangle \underline{x} := \underline{x'} \Vdash \underline{b} \Rightarrow \forall x: \underline{a} \Rightarrow \underline{c} \Rightarrow \forall x: \underline{a} \oplus \forall a: \forall b: \forall c: \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a} \Rightarrow \underline{c} \oplus \forall h: \forall t: \forall r: \forall s: \underline{h} \Rightarrow \underline{t} \stackrel{P}{=} \underline{r} \vdash \underline{h} \Rightarrow \underline{r} \stackrel{P}{=} \underline{s} \vdash \underline{h} \Rightarrow \underline{t} \stackrel{P}{=} \underline{s} \oplus \forall a: \underline{a} \dot{+} \dot{0} \stackrel{P}{=} \underline{a}]$$

$$[A1' \xrightarrow{\text{stmt}} S' \vdash \forall a: \forall b: \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a}][A1' \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[A2' \xrightarrow{\text{stmt}} S' \vdash \forall a: \forall b: \forall c: \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a} \Rightarrow \underline{c}][A2' \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[A3' \xrightarrow{\text{stmt}} S' \vdash \forall a: \forall b: \neg \underline{b} \Rightarrow \neg \underline{a} \Rightarrow \neg \underline{b} \Rightarrow \underline{a} \Rightarrow \underline{b}][A3' \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$[A4' \xrightarrow{\text{stmt}} S' \vdash \forall \underline{c}: \forall \underline{a}: \forall \underline{x}: \forall \underline{b}: [\underline{a}] \equiv \langle [\underline{b}] \mid [\underline{x}] := [\underline{c}] \rangle \Vdash \dot{\forall} \underline{x}: \underline{b} \Rightarrow \underline{a}] [A4' \xrightarrow{\text{proof}} \text{Rule tactic}]$

$[A5' \xrightarrow{\text{stmt}} S' \vdash \forall \underline{x}: \forall \underline{a}: \forall \underline{b}: \text{nonfree}([\underline{x}], [\underline{a}]) \Vdash \dot{\forall} \underline{x}: \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a} \Rightarrow \dot{\forall} \underline{x}: \underline{b}] [A5' \xrightarrow{\text{proof}} \text{Rule tactic}]$

$[MP' \xrightarrow{\text{stmt}} S' \vdash \forall \underline{a}: \forall \underline{b}: \underline{a} \Rightarrow \underline{b} \vdash \underline{a} \vdash \underline{b}] [MP' \xrightarrow{\text{proof}} \text{Rule tactic}]$

$[\text{Gen}' \xrightarrow{\text{stmt}} S' \vdash \forall \underline{x}: \forall \underline{a}: \underline{a} \vdash \dot{\forall} \underline{x}: \underline{a}] [\text{Gen}' \xrightarrow{\text{proof}} \text{Rule tactic}]$

$[S1' \xrightarrow{\text{stmt}} S' \vdash \forall \underline{a}: \forall \underline{b}: \forall \underline{c}: \underline{a} \stackrel{P}{=} \underline{b} \Rightarrow \underline{a} \stackrel{P}{=} \underline{c} \Rightarrow \underline{b} \stackrel{P}{=} \underline{c}] [S1' \xrightarrow{\text{proof}} \text{Rule tactic}]$

$[S2' \xrightarrow{\text{stmt}} S' \vdash \forall \underline{a}: \forall \underline{b}: \underline{a} \stackrel{P}{=} \underline{b} \Rightarrow \underline{a}' \stackrel{P}{=} \underline{b}'] [S2' \xrightarrow{\text{proof}} \text{Rule tactic}]$

$[S3' \xrightarrow{\text{stmt}} S' \vdash \forall \underline{a}: \neg \dot{0} \stackrel{P}{=} \underline{a}'] [S3' \xrightarrow{\text{proof}} \text{Rule tactic}]$

$[S4' \xrightarrow{\text{stmt}} S' \vdash \forall \underline{a}: \forall \underline{b}: \underline{a}' \stackrel{P}{=} \underline{b}' \Rightarrow \underline{a} \stackrel{P}{=} \underline{b}] [S4' \xrightarrow{\text{proof}} \text{Rule tactic}]$

$[S5' \xrightarrow{\text{stmt}} S' \vdash \forall \underline{a}: \underline{a} \dot{+} \dot{0} \stackrel{P}{=} \underline{a}] [S5' \xrightarrow{\text{proof}} \text{Rule tactic}]$

$[S6' \xrightarrow{\text{stmt}} S' \vdash \forall \underline{a}: \forall \underline{b}: \underline{a} \dot{+} \underline{b}' \stackrel{P}{=} \underline{a} \dot{+} \underline{b}'] [S6' \xrightarrow{\text{proof}} \text{Rule tactic}]$

$[S7' \xrightarrow{\text{stmt}} S' \vdash \forall \underline{a}: \underline{a} \dot{:} \dot{0} \stackrel{P}{=} \dot{0}] [S7' \xrightarrow{\text{proof}} \text{Rule tactic}]$

$[S8' \xrightarrow{\text{stmt}} S' \vdash \forall \underline{a}: \forall \underline{b}: \underline{a} \dot{:} \underline{b}' \stackrel{P}{=} \underline{a} \dot{:} \underline{b} \dot{+} \underline{a}] [S8' \xrightarrow{\text{proof}} \text{Rule tactic}]$

$[S9' \xrightarrow{\text{stmt}} S' \vdash \forall \underline{a}: \forall \underline{b}: \forall \underline{c}: \forall \underline{x}: \underline{b} \equiv \langle \underline{a} \mid \underline{x} := \dot{0} \rangle \Vdash \underline{c} \equiv \langle \underline{a} \mid \underline{x} := \underline{x}' \rangle \Vdash \underline{b} \Rightarrow \dot{\forall} \underline{x}: \underline{a} \Rightarrow \underline{c} \Rightarrow \dot{\forall} \underline{x}: \underline{a}] [S9' \xrightarrow{\text{proof}} \text{Rule tactic}]$

Note that $[A1]$ and $[A1']$ are distinct. The former says $[S \vdash \forall \underline{a}: \forall \underline{b}: \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a}]$ and the latter says $[S' \vdash \forall \underline{a}: \forall \underline{b}: \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a}]$.

1.6 Restatement of lemma and proof

We now prove Lemma [L3.2(a)] once again under the name of

$[L3.2(a)' \xrightarrow{\text{pyk}} \text{“lemma prime l three two a”}] [L3.2(a)' \xrightarrow{\text{tex}} \text{“L3.2(a)”}]$:

$[L3.2(a)' \xrightarrow{\text{stmt}} S' \vdash \forall \underline{a}: \underline{a} \stackrel{P}{=} \underline{a}]$

$[L3.2(a)' \xrightarrow{\text{proof}} \lambda c. \lambda x. \lambda \mathcal{P}. ([S' \vdash \forall \underline{a}: S5' \gg \underline{a} \dot{+} \dot{0} \stackrel{P}{=} \underline{a}; S1' \gg \underline{a} \dot{+} \dot{0} \stackrel{P}{=} \underline{a} \Rightarrow \underline{a} \dot{+} \dot{0} \stackrel{P}{=} \underline{a} \Rightarrow \underline{a} \stackrel{P}{=} \underline{a}; MP' \triangleright \underline{a} \dot{+} \dot{0} \stackrel{P}{=} \underline{a} \Rightarrow \underline{a} \dot{+} \dot{0} \stackrel{P}{=} \underline{a} \Rightarrow \underline{a} \stackrel{P}{=} \underline{a} \triangleright \underline{a} \dot{+} \dot{0} \stackrel{P}{=} \underline{a} \gg \underline{a} \dot{+} \dot{0} \stackrel{P}{=} \underline{a} \Rightarrow \underline{a} \stackrel{P}{=} \underline{a}; MP' \triangleright \underline{a} \dot{+} \dot{0} \stackrel{P}{=} \underline{a} \Rightarrow \underline{a} \stackrel{P}{=} \underline{a} \triangleright \underline{a} \dot{+} \dot{0} \stackrel{P}{=} \underline{a} \gg \underline{a} \stackrel{P}{=} \underline{a}], p_0, c)]$

[Hypothesize $\xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([S' \vdash \forall \underline{h}. \forall \underline{a}. \underline{a} \vdash A1' \gg \underline{a} \Rightarrow \underline{h} \Rightarrow \underline{a}; MP' \triangleright \underline{a} \Rightarrow \underline{h} \Rightarrow \underline{a} \triangleright \underline{a} \gg \underline{h} \Rightarrow \underline{a}])$, p_0, c)]

2.2 First order predicate calculus

2.2.1 Hypothetical generalization

The hypothetical version $[\text{Gen}'_h \xrightarrow{\text{pyk}} \text{“hypothetical rule prime gen”}][\text{Gen}'_h \xrightarrow{\text{tex}} \text{“Gen}'_h$ ”] of generalisation Gen' has a hypothesis $\underline{h}$ on premise and conclusion:

$[\text{Gen}'_h \xrightarrow{\text{stmt}} S' \vdash \forall \underline{h}. \forall \underline{x}. \forall \underline{a}. \text{nonfree}([\underline{x}], [\underline{h}]) \Vdash \underline{h} \Rightarrow \underline{a} \vdash \underline{h} \Rightarrow \underline{a}]$

$[\text{Gen}'_h \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([S' \vdash \forall \underline{h}. \forall \underline{x}. \forall \underline{a}. \text{nonfree}([\underline{x}], [\underline{h}]) \Vdash \underline{h} \Rightarrow \underline{a} \vdash A5' \triangleright \text{nonfree}([\underline{x}], [\underline{h}]) \gg \underline{a} \Rightarrow \underline{h} \Rightarrow \underline{a}; \text{Gen}' \triangleright \underline{h} \Rightarrow \underline{a} \gg \underline{a} \Rightarrow \underline{a}; MP' \triangleright \underline{a} \Rightarrow \underline{h} \Rightarrow \underline{a} \triangleright \underline{a} \gg \underline{h} \Rightarrow \underline{a}])$, p_0, c)]

2.3 Peano arithmetic

2.3.1 Lemma M3.2(a)

Lemma $[\text{M3.2(a)} \xrightarrow{\text{pyk}} \text{“mendelson three two a”}][\text{M3.2(a)} \xrightarrow{\text{tex}} \text{“M3.2(a)”}]$ and the associated hypothetical lemma $[\text{M3.2(a)}_h \xrightarrow{\text{pyk}} \text{“hypothetical three two a”}][\text{M3.2(a)}_h \xrightarrow{\text{tex}} \text{“M3.2(a)}_h$ ”] read:

$[\text{M3.2(a)} \xrightarrow{\text{stmt}} S' \vdash \forall \underline{t}. \underline{t} \stackrel{P}{=} \underline{t}][\text{M3.2(a)} \xrightarrow{\text{proof}} \text{Rule tactic}]$

$[\text{M3.2(a)}_h \xrightarrow{\text{stmt}} S' \vdash \forall \underline{h}. \forall \underline{t}. \underline{h} \Rightarrow \underline{t} \stackrel{P}{=} \underline{t}]$

Above we cheat in stating M3.2(a) as a rule and not as a lemma. A reasonable way to construct a large proof is to start stating everything as rules and then changing the rules to lemmas one at a time until only the rules of the theory are left.

$[\text{M3.2(a)}_h \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([S' \vdash \forall \underline{h}. \forall \underline{t}. \text{M3.2(a)} \gg \underline{t} \stackrel{P}{=} \underline{t}; \text{Hypothesize} \triangleright \underline{t} \stackrel{P}{=} \underline{t} \gg \underline{h} \Rightarrow \underline{t} \stackrel{P}{=} \underline{t}])$, p_0, c)]

2.3.2 Lemma M3.2(b)

Lemma $[\text{M3.2(b)}_h \xrightarrow{\text{pyk}} \text{“hypothetical three two b”}][\text{M3.2(b)}_h \xrightarrow{\text{tex}} \text{“M3.2(b)}_h$ ”] reads:

$[\text{M3.2(b)}_h \xrightarrow{\text{stmt}} S' \vdash \forall \underline{h}. \forall \underline{t}. \forall \underline{r}. \underline{h} \Rightarrow \underline{t} \stackrel{P}{=} \underline{r} \vdash \underline{h} \Rightarrow \underline{r} \stackrel{P}{=} \underline{t}][\text{M3.2(b)}_h \xrightarrow{\text{proof}} \text{Rule tactic}]$

$[\text{M3.2(b)}_h \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([S' \vdash \forall \underline{h}. \forall \underline{t}. \forall \underline{r}. \underline{h} \Rightarrow \underline{t} \stackrel{P}{=} \underline{r} \vdash S1' \gg \underline{t} \stackrel{P}{=} \underline{r} \Rightarrow \underline{t} \stackrel{P}{=} \underline{t} \Rightarrow \underline{r} \stackrel{P}{=} \underline{t}; \text{Hypothesize} \triangleright \underline{t} \stackrel{P}{=} \underline{r} \Rightarrow \underline{t} \stackrel{P}{=} \underline{t} \Rightarrow \underline{r} \stackrel{P}{=} \underline{t} \gg \underline{h} \Rightarrow \underline{t} \stackrel{P}{=} \underline{r} \Rightarrow \underline{t} \stackrel{P}{=} \underline{t} \Rightarrow \underline{r} \stackrel{P}{=} \underline{t}; MP'_h \triangleright \underline{h} \Rightarrow \underline{t} \stackrel{P}{=} \underline{r} \Rightarrow \underline{t} \stackrel{P}{=} \underline{t} \triangleright \underline{h} \Rightarrow \underline{t} \stackrel{P}{=} \underline{r} \gg \underline{h} \Rightarrow \underline{t} \stackrel{P}{=} \underline{t} \Rightarrow \underline{r} \stackrel{P}{=} \underline{t}; \text{M3.2(a)}_h \gg \underline{h} \Rightarrow \underline{t} \stackrel{P}{=} \underline{t}; MP'_h \triangleright \underline{h} \Rightarrow \underline{t} \stackrel{P}{=} \underline{t} \Rightarrow \underline{r} \stackrel{P}{=} \underline{t} \triangleright \underline{h} \Rightarrow \underline{t} \stackrel{P}{=} \underline{t} \gg \underline{h} \Rightarrow \underline{t} \stackrel{P}{=} \underline{t}])$, p_0, c)]

2.3.3 Lemma M3.1(S1)

Lemma $[M3.1(S1')_h \xrightarrow{\text{pyk}} \text{“hypothetical three one s one”}][M3.1(S1')_h \xrightarrow{\text{tex}} \text{“} M3.1(S1')_h \text{”}]$ is the hypothetical version of Mendelson Lemma 3.1(S1'):

$$[M3.1(S1')_h \xrightarrow{\text{stmt}} S' \vdash \forall \underline{h}: \forall \underline{t}: \forall \underline{r}: \forall \underline{s}: \underline{h} \Rightarrow \underline{t} \stackrel{P}{=} \underline{r} \vdash \underline{h} \Rightarrow \underline{t} \stackrel{P}{=} \underline{s} \vdash \underline{h} \Rightarrow \underline{r} \stackrel{P}{=} \underline{s}] [M3.1(S1')_h \xrightarrow{\text{proof}} \text{Rule tactic}]$$

2.3.4 Lemma M3.2(c)

Lemma $[M3.2(c)_h \xrightarrow{\text{pyk}} \text{“hypothetical three two c”}][M3.2(c)_h \xrightarrow{\text{tex}} \text{“} M3.2(c)_h \text{”}]$ is the hypothetical version of Mendelson Lemma 3.2(c) which expresses ordinary transitivity:

$$[M3.2(c)_h \xrightarrow{\text{stmt}} S' \vdash \forall \underline{h}: \forall \underline{t}: \forall \underline{r}: \forall \underline{s}: \underline{h} \Rightarrow \underline{t} \stackrel{P}{=} \underline{r} \vdash \underline{h} \Rightarrow \underline{r} \stackrel{P}{=} \underline{s} \vdash \underline{h} \Rightarrow \underline{t} \stackrel{P}{=} \underline{s}] [M3.2(c)_h \xrightarrow{\text{proof}} \text{Rule tactic}]$$

2.3.5 Lemma M3.1(S2)

Lemma $[M3.1(S2')_h \xrightarrow{\text{pyk}} \text{“hypothetical three one s two”}][M3.1(S2')_h \xrightarrow{\text{tex}} \text{“} M3.1(S2')_h \text{”}]$ is the hypothetical version of Mendelson Lemma 3.1(S2'):

$$[M3.1(S2')_h \xrightarrow{\text{stmt}} S' \vdash \forall \underline{h}: \forall \underline{t}: \forall \underline{r}: \underline{h} \Rightarrow \underline{t} \stackrel{P}{=} \underline{r} \vdash \underline{h} \Rightarrow \underline{t}' \stackrel{P}{=} \underline{r}'] [M3.1(S2')_h \xrightarrow{\text{proof}} \text{Rule tactic}]$$

2.3.6 Lemma M3.1(S5)

Lemma $[M3.1(S5')_h \xrightarrow{\text{pyk}} \text{“hypothetical three one s five”}][M3.1(S5')_h \xrightarrow{\text{tex}} \text{“} M3.1(S5')_h \text{”}]$ is the hypothetical version of Mendelson Lemma 3.1(S5'):

$$[M3.1(S5')_h \xrightarrow{\text{stmt}} S' \vdash \forall \underline{h}: \forall \underline{t}: \underline{h} \Rightarrow \underline{t} \dot{+} \dot{0} \stackrel{P}{=} \underline{t}] [M3.1(S5')_h \xrightarrow{\text{proof}} \text{Rule tactic}]$$

2.3.7 Lemma M3.1(S6)

Lemma $[M3.1(S6')_h \xrightarrow{\text{pyk}} \text{“hypothetical three one s six”}][M3.1(S6')_h \xrightarrow{\text{tex}} \text{“} M3.1(S6')_h \text{”}]$ is the hypothetical version of Mendelson Lemma 3.1(S6'):

$$[M3.1(S6')_h \xrightarrow{\text{stmt}} S' \vdash \forall \underline{h}: \forall \underline{t}: \forall \underline{r}: \underline{h} \Rightarrow \underline{t} \dot{+} \underline{r}' \stackrel{P}{=} \underline{t} \dot{+} \underline{r}] [M3.1(S6')_h \xrightarrow{\text{proof}} \text{Rule tactic}]$$

2.3.8 Lemma M3.2(f)

Lemma $[M3.2(f) \xrightarrow{\text{pyk}} \text{“mendelson three two f”}][M3.2(f) \xrightarrow{\text{tex}} \text{“} M3.2(f) \text{”}]$ is the closure of Mendelson Lemma 3.2(f) for the concrete variable $[\dot{t}]$:

$$[M3.2(f) \xrightarrow{\text{stmt}} S' \vdash \forall \dot{t}: \dot{t} \stackrel{P}{=} \dot{0} \dot{+} \dot{t}]$$

The proof below uses local macro definitions.

$$\begin{aligned} & [\text{M3.2(f)} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\ulcorner S' \vdash A1' \gg x \Rightarrow x \Rightarrow x; \text{M3.1}(S5')_h \gg x \Rightarrow x \Rightarrow x \Rightarrow \\ & \dot{0} + \dot{0} \stackrel{P}{=} \dot{0}; \text{M3.2(b)}_h \triangleright x \Rightarrow x \Rightarrow x \Rightarrow \dot{0} + \dot{0} \stackrel{P}{=} \dot{0} \gg x \Rightarrow x \Rightarrow x \Rightarrow \dot{0} \stackrel{P}{=} \\ & \dot{0} + \dot{0}; \text{MP}' \triangleright x \Rightarrow x \Rightarrow x \Rightarrow \dot{0} \stackrel{P}{=} \dot{0} + \dot{0} \triangleright x \Rightarrow x \Rightarrow x \gg \dot{0} \stackrel{P}{=} \dot{0} + \dot{0}; \text{M1.7} \gg \mathfrak{t} \stackrel{P}{=} \\ & \dot{0} + \mathfrak{t} \Rightarrow \mathfrak{t} \stackrel{P}{=} \dot{0} + \mathfrak{t}; \text{M3.1}(S2')_h \triangleright \mathfrak{t} \stackrel{P}{=} \dot{0} + \mathfrak{t} \Rightarrow \mathfrak{t} \stackrel{P}{=} \dot{0} + \mathfrak{t} \gg \mathfrak{t} \stackrel{P}{=} \dot{0} + \mathfrak{t} \Rightarrow \mathfrak{t}' \stackrel{P}{=} \\ & \dot{0} + \mathfrak{t}'; \text{M3.1}(S6')_h \gg \mathfrak{t} \stackrel{P}{=} \dot{0} + \mathfrak{t} \Rightarrow \dot{0} + \mathfrak{t}' \stackrel{P}{=} \dot{0} + \mathfrak{t}'; \text{M3.2(b)}_h \triangleright \mathfrak{t} \stackrel{P}{=} \dot{0} + \mathfrak{t} \Rightarrow \dot{0} + \mathfrak{t}' \stackrel{P}{=} \\ & \dot{0} + \mathfrak{t}' \gg \mathfrak{t} \stackrel{P}{=} \dot{0} + \mathfrak{t} \Rightarrow \dot{0} + \mathfrak{t}' \stackrel{P}{=} \dot{0} + \mathfrak{t}'; \text{M3.2(c)}_h \triangleright \mathfrak{t} \stackrel{P}{=} \dot{0} + \mathfrak{t} \Rightarrow \mathfrak{t}' \stackrel{P}{=} \dot{0} + \mathfrak{t}' \triangleright \mathfrak{t} \stackrel{P}{=} \\ & \dot{0} + \mathfrak{t} \Rightarrow \dot{0} + \mathfrak{t}' \stackrel{P}{=} \dot{0} + \mathfrak{t}' \gg \mathfrak{t} \stackrel{P}{=} \dot{0} + \mathfrak{t} \Rightarrow \mathfrak{t}' \stackrel{P}{=} \dot{0} + \mathfrak{t}'; \text{Gen}' \triangleright \mathfrak{t} \stackrel{P}{=} \dot{0} + \mathfrak{t} \Rightarrow \mathfrak{t}' \stackrel{P}{=} \dot{0} + \mathfrak{t}' \gg \\ & \check{\mathfrak{v}}\mathfrak{t}: \mathfrak{t} \stackrel{P}{=} \dot{0} + \mathfrak{t} \Rightarrow \mathfrak{t}' \stackrel{P}{=} \dot{0} + \mathfrak{t}'; \text{S9}' \gg \dot{0} \stackrel{P}{=} \dot{0} + \dot{0} \Rightarrow \check{\mathfrak{v}}\mathfrak{t}: \mathfrak{t} \stackrel{P}{=} \dot{0} + \mathfrak{t} \Rightarrow \mathfrak{t}' \stackrel{P}{=} \dot{0} + \mathfrak{t}' \Rightarrow \\ & \check{\mathfrak{v}}\mathfrak{t}: \mathfrak{t} \stackrel{P}{=} \dot{0} + \mathfrak{t}; \text{MP}' \triangleright \dot{0} \stackrel{P}{=} \dot{0} + \dot{0} \Rightarrow \check{\mathfrak{v}}\mathfrak{t}: \mathfrak{t} \stackrel{P}{=} \dot{0} + \mathfrak{t} \Rightarrow \mathfrak{t}' \stackrel{P}{=} \dot{0} + \mathfrak{t}' \Rightarrow \check{\mathfrak{v}}\mathfrak{t}: \mathfrak{t} \stackrel{P}{=} \dot{0} + \mathfrak{t} \triangleright \dot{0} \stackrel{P}{=} \\ & \dot{0} + \dot{0} \gg \check{\mathfrak{v}}\mathfrak{t}: \mathfrak{t} \stackrel{P}{=} \dot{0} + \mathfrak{t} \Rightarrow \mathfrak{t}' \stackrel{P}{=} \dot{0} + \mathfrak{t}' \Rightarrow \check{\mathfrak{v}}\mathfrak{t}: \mathfrak{t} \stackrel{P}{=} \dot{0} + \mathfrak{t}; \text{MP}' \triangleright \check{\mathfrak{v}}\mathfrak{t}: \mathfrak{t} \stackrel{P}{=} \dot{0} + \mathfrak{t} \Rightarrow \mathfrak{t}' \stackrel{P}{=} \\ & \dot{0} + \mathfrak{t}' \Rightarrow \check{\mathfrak{v}}\mathfrak{t}: \mathfrak{t} \stackrel{P}{=} \dot{0} + \mathfrak{t} \triangleright \check{\mathfrak{v}}\mathfrak{t}: \mathfrak{t} \stackrel{P}{=} \dot{0} + \mathfrak{t} \Rightarrow \mathfrak{t}' \stackrel{P}{=} \dot{0} + \mathfrak{t}' \gg \check{\mathfrak{v}}\mathfrak{t}: \mathfrak{t} \stackrel{P}{=} \dot{0} + \mathfrak{t} \rceil, p_0, c)] \end{aligned}$$

A Chores

A.1 The name of the page

This defines the name of the page:

$$[\text{peano} \xrightarrow{\text{pyk}} \text{“peano”}]$$

A.2 Variables of Peano arithmetic

We use $[\dot{b} \xrightarrow{\text{pyk}} \text{“peano b”}][\dot{b} \xrightarrow{\text{tex}} \text{“} \backslash \dot{\text{mathit}}\{\text{b}\} \text{”}]$, $[\dot{c} \xrightarrow{\text{pyk}} \text{“peano c”}][\dot{c} \xrightarrow{\text{tex}} \text{“} \backslash \dot{\text{mathit}}\{\text{c}\} \text{”}]$, $[\dot{d} \xrightarrow{\text{pyk}} \text{“peano d”}][\dot{d} \xrightarrow{\text{tex}} \text{“} \backslash \dot{\text{mathit}}\{\text{d}\} \text{”}]$, $[\dot{e} \xrightarrow{\text{pyk}} \text{“peano e”}][\dot{e} \xrightarrow{\text{tex}} \text{“} \backslash \dot{\text{mathit}}\{\text{e}\} \text{”}]$, $[\dot{f} \xrightarrow{\text{pyk}} \text{“peano f”}][\dot{f} \xrightarrow{\text{tex}} \text{“} \backslash \dot{\text{mathit}}\{\text{f}\} \text{”}]$, $[\dot{g} \xrightarrow{\text{pyk}} \text{“peano g”}][\dot{g} \xrightarrow{\text{tex}} \text{“} \backslash \dot{\text{mathit}}\{\text{g}\} \text{”}]$, $[\dot{h} \xrightarrow{\text{pyk}} \text{“peano h”}][\dot{h} \xrightarrow{\text{tex}} \text{“} \backslash \dot{\text{mathit}}\{\text{h}\} \text{”}]$, $[\dot{i} \xrightarrow{\text{pyk}} \text{“peano i”}][\dot{i} \xrightarrow{\text{tex}} \text{“} \backslash \dot{\text{mathit}}\{\text{i}\} \text{”}]$, $[\dot{j} \xrightarrow{\text{pyk}} \text{“peano j”}][\dot{j} \xrightarrow{\text{tex}} \text{“} \backslash \dot{\text{mathit}}\{\text{j}\} \text{”}]$, $[\dot{k} \xrightarrow{\text{pyk}} \text{“peano k”}][\dot{k} \xrightarrow{\text{tex}} \text{“} \backslash \dot{\text{mathit}}\{\text{k}\} \text{”}]$, $[\dot{l} \xrightarrow{\text{pyk}} \text{“peano l”}][\dot{l} \xrightarrow{\text{tex}} \text{“} \backslash \dot{\text{mathit}}\{\text{l}\} \text{”}]$, $[\dot{m} \xrightarrow{\text{pyk}} \text{“peano m”}][\dot{m} \xrightarrow{\text{tex}} \text{“} \backslash \dot{\text{mathit}}\{\text{m}\} \text{”}]$, $[\dot{n} \xrightarrow{\text{pyk}} \text{“peano n”}][\dot{n} \xrightarrow{\text{tex}} \text{“} \backslash \dot{\text{mathit}}\{\text{n}\} \text{”}]$, $[\dot{o} \xrightarrow{\text{pyk}} \text{“peano o”}][\dot{o} \xrightarrow{\text{tex}} \text{“} \backslash \dot{\text{mathit}}\{\text{o}\} \text{”}]$, $[\dot{p} \xrightarrow{\text{pyk}} \text{“peano p”}][\dot{p} \xrightarrow{\text{tex}} \text{“} \backslash \dot{\text{mathit}}\{\text{p}\} \text{”}]$, $[\dot{q} \xrightarrow{\text{pyk}} \text{“peano q”}][\dot{q} \xrightarrow{\text{tex}} \text{“} \backslash \dot{\text{mathit}}\{\text{q}\} \text{”}]$, $[\dot{r} \xrightarrow{\text{pyk}} \text{“peano r”}][\dot{r} \xrightarrow{\text{tex}} \text{“} \backslash \dot{\text{mathit}}\{\text{r}\} \text{”}]$, $[\dot{s} \xrightarrow{\text{pyk}} \text{“peano s”}][\dot{s} \xrightarrow{\text{tex}} \text{“} \backslash \dot{\text{mathit}}\{\text{s}\} \text{”}]$

$\backslash \text{dot}\{\backslash \text{mathit}\{\text{s}\}\}$ ”, $[\dot{t} \xrightarrow{\text{pyk}} \text{“peano t”}][\dot{t} \xrightarrow{\text{tex}} \text{“}$
 $\backslash \text{dot}\{\backslash \text{mathit}\{\text{t}\}\}$ ”, $[\dot{u} \xrightarrow{\text{pyk}} \text{“peano u”}][\dot{u} \xrightarrow{\text{tex}} \text{“}$
 $\backslash \text{dot}\{\backslash \text{mathit}\{\text{u}\}\}$ ”, $[\dot{v} \xrightarrow{\text{pyk}} \text{“peano v”}][\dot{v} \xrightarrow{\text{tex}} \text{“}$
 $\backslash \text{dot}\{\backslash \text{mathit}\{\text{v}\}\}$ ”, $[\dot{w} \xrightarrow{\text{pyk}} \text{“peano w”}][\dot{w} \xrightarrow{\text{tex}} \text{“}$
 $\backslash \text{dot}\{\backslash \text{mathit}\{\text{w}\}\}$ ”, $[\dot{x} \xrightarrow{\text{pyk}} \text{“peano x”}][\dot{x} \xrightarrow{\text{tex}} \text{“}$
 $\backslash \text{dot}\{\backslash \text{mathit}\{\text{x}\}\}$ ”, $[\dot{y} \xrightarrow{\text{pyk}} \text{“peano y”}][\dot{y} \xrightarrow{\text{tex}} \text{“}$
 $\backslash \text{dot}\{\backslash \text{mathit}\{\text{y}\}\}$ ”, and $[\dot{z} \xrightarrow{\text{pyk}} \text{“peano z”}][\dot{z} \xrightarrow{\text{tex}} \text{“}$
 $\backslash \text{dot}\{\backslash \text{mathit}\{\text{z}\}\}$ ”] to denote variables of Peano arithmetic:
 $[\dot{b} \xrightarrow{\text{macro}} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, [[\dot{b} \doteq \dot{b}]])]$, $[\dot{c} \xrightarrow{\text{macro}} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, [[\dot{c} \doteq \dot{c}]])]$,
 $[\dot{d} \xrightarrow{\text{macro}} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, [[\dot{d} \doteq \dot{d}]])]$, $[\dot{e} \xrightarrow{\text{macro}} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, [[\dot{e} \doteq \dot{e}]])]$,
 $[\dot{f} \xrightarrow{\text{macro}} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, [[\dot{f} \doteq \dot{f}]])]$, $[\dot{g} \xrightarrow{\text{macro}} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, [[\dot{g} \doteq \dot{g}]])]$,
 $[\dot{h} \xrightarrow{\text{macro}} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, [[\dot{h} \doteq \dot{h}]])]$, $[\dot{i} \xrightarrow{\text{macro}} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, [[\dot{i} \doteq \dot{i}]])]$,
 $[\dot{j} \xrightarrow{\text{macro}} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, [[\dot{j} \doteq \dot{j}]])]$, $[\dot{k} \xrightarrow{\text{macro}} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, [[\dot{k} \doteq \dot{k}]])]$,
 $[\dot{l} \xrightarrow{\text{macro}} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, [[\dot{l} \doteq \dot{l}]])]$, $[\dot{m} \xrightarrow{\text{macro}} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, [[\dot{m} \doteq \dot{m}]])]$,
 $[\dot{n} \xrightarrow{\text{macro}} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, [[\dot{n} \doteq \dot{n}]])]$, $[\dot{o} \xrightarrow{\text{macro}} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, [[\dot{o} \doteq \dot{o}]])]$,
 $[\dot{p} \xrightarrow{\text{macro}} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, [[\dot{p} \doteq \dot{p}]])]$, $[\dot{q} \xrightarrow{\text{macro}} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, [[\dot{q} \doteq \dot{q}]])]$,
 $[\dot{r} \xrightarrow{\text{macro}} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, [[\dot{r} \doteq \dot{r}]])]$, $[\dot{s} \xrightarrow{\text{macro}} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, [[\dot{s} \doteq \dot{s}]])]$,
 $[\dot{t} \xrightarrow{\text{macro}} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, [[\dot{t} \doteq \dot{t}]])]$, $[\dot{u} \xrightarrow{\text{macro}} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, [[\dot{u} \doteq \dot{u}]])]$,
 $[\dot{v} \xrightarrow{\text{macro}} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, [[\dot{v} \doteq \dot{v}]])]$, $[\dot{w} \xrightarrow{\text{macro}} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, [[\dot{w} \doteq \dot{w}]])]$,
 $[\dot{x} \xrightarrow{\text{macro}} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, [[\dot{x} \doteq \dot{x}]])]$, $[\dot{y} \xrightarrow{\text{macro}} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, [[\dot{y} \doteq \dot{y}]])]$,
and $[\dot{z} \xrightarrow{\text{macro}} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, [[\dot{z} \doteq \dot{z}]])]$.

A.3 T_EX definitions

A.4 Test

$[[\dot{a}]^{\mathcal{P}}]$.
 $[[\dot{a}]^{\mathcal{P}}]^-$
 $[\text{nonfree}(\lceil \dot{x} \rceil, [\dot{y} \stackrel{\mathcal{P}}{=} \dot{z} \Rightarrow \dot{v}\dot{x} : \dot{x} \stackrel{\mathcal{P}}{=} \dot{y}])]$.
 $[\text{nonfree}(\lceil \dot{x} \rceil, [\dot{x} \stackrel{\mathcal{P}}{=} \dot{z} \Rightarrow \dot{v}\dot{x} : \dot{x} \stackrel{\mathcal{P}}{=} \dot{y}])^-$
 $[\text{nonfree}(\lceil \dot{x} \rceil, [\dot{y} \stackrel{\mathcal{P}}{=} \dot{x} \Rightarrow \dot{v}\dot{x} : \dot{x} \stackrel{\mathcal{P}}{=} \dot{y}])^-$
 $[\text{nonfree}(\lceil \dot{x} \rceil, [\dot{y} \stackrel{\mathcal{P}}{=} \dot{z} \Rightarrow \dot{v}\dot{y} : \dot{x} \stackrel{\mathcal{P}}{=} \dot{y}])^-$
 $[\text{free}(\lceil \dot{v}\dot{x} : \mathbf{b} :: \dot{x} :: \mathbf{c} \rceil \lceil \dot{x} \rceil := [\mathbf{x} :: \dot{y} :: \mathbf{z}])]$.
 $[\text{free}(\lceil \dot{v}\dot{y} : \mathbf{b} :: \dot{x} :: \mathbf{c} \rceil \lceil \dot{x} \rceil := [\mathbf{x} :: \dot{y} :: \mathbf{z}])^-$
 $[\text{free}(\lceil \dot{v}\dot{x} : \mathbf{b} :: \dot{x} :: \mathbf{c} \rceil \lceil \dot{y} \rceil := [\mathbf{x} :: \dot{y} :: \mathbf{z}])]$.

$\llbracket \text{free}(\llbracket \dot{\forall} \dot{y} : \dot{b} :: \dot{x} :: \dot{c} \rrbracket \mid \llbracket \dot{y} \rrbracket := \llbracket \dot{x} :: \dot{y} :: \dot{z} \rrbracket) \rrbracket$

$\llbracket \dot{a} \equiv \langle \dot{a} \mid \dot{b} := \dot{c} \rangle \rrbracket$

$\llbracket \dot{c} \equiv \langle \dot{b} \mid \dot{b} := \dot{c} \rangle \rrbracket$

$\llbracket \forall \dot{a} : \dot{a} \stackrel{P}{=} \dot{b} \equiv \langle \forall \dot{a} : \dot{a} \stackrel{P}{=} \dot{b} \mid \dot{a} := \dot{c} \rangle \rrbracket$

$\llbracket \forall \dot{a} : \dot{a} \stackrel{P}{=} \dot{c} \equiv \langle \forall \dot{a} : \dot{a} \stackrel{P}{=} \dot{b} \mid \dot{b} := \dot{c} \rangle \rrbracket$

$\llbracket \dot{\forall} \dot{a} : \dot{a} \stackrel{P}{=} \dot{0} \dot{+} \dot{a} \Rightarrow \dot{c} : \dot{d} \stackrel{P}{=} \dot{0} \dot{+} \dot{c} : \dot{d} \equiv \langle \dot{\forall} \dot{a} : \dot{a} \stackrel{P}{=} \dot{0} \dot{+} \dot{a} \Rightarrow \dot{b} \stackrel{P}{=} \dot{0} \dot{+} \dot{b} \mid \dot{b} := \dot{c} : \dot{d} \rangle \rrbracket$

$\llbracket \dot{\forall} \dot{a} : \dot{a} \stackrel{P}{=} \dot{0} \dot{+} \dot{a} \Rightarrow \dot{b} \stackrel{P}{=} \dot{0} \dot{+} \dot{b} \equiv \langle \dot{\forall} \dot{a} : \dot{a} \stackrel{P}{=} \dot{0} \dot{+} \dot{a} \Rightarrow \dot{b} \stackrel{P}{=} \dot{0} \dot{+} \dot{b} \mid \dot{a} := \dot{c} \rangle \rrbracket$

A.5 Priority table

$\llbracket \text{peano} \xrightarrow{\text{prio}} \rrbracket$

Preassociative

$\llbracket \text{peano} \rrbracket, \llbracket \text{base} \rrbracket, \llbracket \text{bracket} * \text{end bracket} \rrbracket, \llbracket \text{big bracket} * \text{end bracket} \rrbracket,$
 $\llbracket \text{math} * \text{end math} \rrbracket, \llbracket \text{flush left} * \rrbracket, \llbracket x \rrbracket, \llbracket y \rrbracket, \llbracket z \rrbracket, \llbracket [* \bowtie *] \rrbracket, \llbracket [* \xrightarrow{*} *] \rrbracket, \llbracket \text{pyk} \rrbracket, \llbracket \text{tex} \rrbracket,$
 $\llbracket \text{name} \rrbracket, \llbracket \text{prio} \rrbracket, \llbracket * \rrbracket, \llbracket \text{T} \rrbracket, \llbracket \text{if}(*, *, *) \rrbracket, \llbracket [* \xrightarrow{*} *] \rrbracket, \llbracket \text{val} \rrbracket, \llbracket \text{claim} \rrbracket, \llbracket \perp \rrbracket, \llbracket \text{f}(*, *) \rrbracket, \llbracket (*, *)^I \rrbracket, \llbracket \text{F} \rrbracket, \llbracket \text{Q} \rrbracket,$
 $\llbracket \underline{1} \rrbracket, \llbracket \underline{2} \rrbracket, \llbracket \underline{3} \rrbracket, \llbracket \underline{4} \rrbracket, \llbracket \underline{5} \rrbracket, \llbracket \underline{6} \rrbracket, \llbracket \underline{7} \rrbracket, \llbracket \underline{8} \rrbracket, \llbracket \underline{9} \rrbracket, \llbracket 0 \rrbracket, \llbracket 1 \rrbracket, \llbracket 2 \rrbracket, \llbracket 3 \rrbracket, \llbracket 4 \rrbracket, \llbracket 5 \rrbracket, \llbracket 6 \rrbracket, \llbracket 7 \rrbracket, \llbracket 8 \rrbracket, \llbracket 9 \rrbracket, \llbracket a \rrbracket, \llbracket b \rrbracket, \llbracket c \rrbracket, \llbracket d \rrbracket,$
 $\llbracket e \rrbracket, \llbracket f \rrbracket, \llbracket g \rrbracket, \llbracket h \rrbracket, \llbracket i \rrbracket, \llbracket j \rrbracket, \llbracket k \rrbracket, \llbracket l \rrbracket, \llbracket m \rrbracket, \llbracket n \rrbracket, \llbracket o \rrbracket, \llbracket p \rrbracket, \llbracket q \rrbracket, \llbracket r \rrbracket, \llbracket s \rrbracket, \llbracket t \rrbracket, \llbracket u \rrbracket, \llbracket v \rrbracket, \llbracket w \rrbracket, \llbracket (*, *)^M \rrbracket, \llbracket \text{If}(*, *, *) \rrbracket,$
 $\llbracket \text{array}\{*\} * \text{end array} \rrbracket, \llbracket \text{I} \rrbracket, \llbracket \text{C} \rrbracket, \llbracket \text{R} \rrbracket, \llbracket \text{empty} \rrbracket, \llbracket \langle * \mid * := * \rangle \rrbracket, \llbracket \mathcal{M}(*, *) \rrbracket, \llbracket \mathcal{U}(*, *) \rrbracket, \llbracket \mathcal{U}^M(*, *) \rrbracket,$
 $\llbracket \text{apply}(*, *) \rrbracket, \llbracket \text{apply}_1(*, *) \rrbracket, \llbracket \text{identifier}(*, *) \rrbracket, \llbracket \text{identifier}_1(*, *) \rrbracket, \llbracket \text{array-plus}(*, *) \rrbracket, \llbracket \text{array-remove}(*, *, *) \rrbracket, \llbracket \text{array-put}(*, *, *, *) \rrbracket, \llbracket \text{array-add}(*, *, *, *, *) \rrbracket,$
 $\llbracket \text{bit}(*, *) \rrbracket, \llbracket \text{bit}_1(*, *) \rrbracket, \llbracket \text{rack} \rrbracket, \llbracket \text{"vector"} \rrbracket, \llbracket \text{"bibliography"} \rrbracket, \llbracket \text{"dictionary"} \rrbracket,$
 $\llbracket \text{"body"} \rrbracket, \llbracket \text{"codex"} \rrbracket, \llbracket \text{"expansion"} \rrbracket, \llbracket \text{"code"} \rrbracket, \llbracket \text{"cache"} \rrbracket, \llbracket \text{"diagnose"} \rrbracket, \llbracket \text{"pyk"} \rrbracket,$
 $\llbracket \text{"tex"} \rrbracket, \llbracket \text{"texname"} \rrbracket, \llbracket \text{"value"} \rrbracket, \llbracket \text{"message"} \rrbracket, \llbracket \text{"macro"} \rrbracket, \llbracket \text{"definition"} \rrbracket,$
 $\llbracket \text{"unpack"} \rrbracket, \llbracket \text{"claim"} \rrbracket, \llbracket \text{"priority"} \rrbracket, \llbracket \text{"lambda"} \rrbracket, \llbracket \text{"apply"} \rrbracket, \llbracket \text{"true"} \rrbracket, \llbracket \text{"if"} \rrbracket,$
 $\llbracket \text{"quote"} \rrbracket, \llbracket \text{"proclaim"} \rrbracket, \llbracket \text{"define"} \rrbracket, \llbracket \text{"introduce"} \rrbracket, \llbracket \text{"hide"} \rrbracket, \llbracket \text{"pre"} \rrbracket, \llbracket \text{"post"} \rrbracket,$
 $\llbracket \mathcal{E}(*, *, *) \rrbracket, \llbracket \mathcal{E}_2(*, *, *, *) \rrbracket, \llbracket \mathcal{E}_3(*, *, *, *) \rrbracket, \llbracket \mathcal{E}_4(*, *, *, *) \rrbracket, \llbracket \text{lookup}(*, *, *) \rrbracket,$
 $\llbracket \text{abstract}(*, *, *, *) \rrbracket, \llbracket [*] \rrbracket, \llbracket \mathcal{M}(*, *, *) \rrbracket, \llbracket \mathcal{M}_2(*, *, *, *) \rrbracket, \llbracket \mathcal{M}^*(*, *, *) \rrbracket, \llbracket \text{macro} \rrbracket,$
 $\llbracket s_0 \rrbracket, \llbracket \text{zip}(*, *) \rrbracket, \llbracket \text{assoc}_1(*, *, *) \rrbracket, \llbracket (*, *)^P \rrbracket, \llbracket \text{self} \rrbracket, \llbracket [* \ddot{=} *] \rrbracket, \llbracket [* \dot{=} *] \rrbracket, \llbracket [* \hat{=} *] \rrbracket,$
 $\llbracket [* \stackrel{\text{pyk}}{=} *] \rrbracket, \llbracket [* \stackrel{\text{tex}}{=} *] \rrbracket, \llbracket [* \stackrel{\text{name}}{=} *] \rrbracket, \llbracket \text{Priority table}[*] \rrbracket, \llbracket \tilde{\mathcal{M}}_1 \rrbracket, \llbracket \tilde{\mathcal{M}}_2(*, *) \rrbracket, \llbracket \tilde{\mathcal{M}}_3(*, *) \rrbracket,$
 $\llbracket \tilde{\mathcal{M}}_4(*, *, *, *) \rrbracket, \llbracket \mathcal{M}(*, *, *) \rrbracket, \llbracket \mathcal{Q}(*, *, *) \rrbracket, \llbracket \tilde{\mathcal{Q}}_2(*, *, *) \rrbracket, \llbracket \tilde{\mathcal{Q}}_3(*, *, *, *) \rrbracket, \llbracket \tilde{\mathcal{Q}}^*(*, *, *) \rrbracket,$
 $\llbracket (*, *) \rrbracket, \llbracket \text{aspect}(*, *) \rrbracket, \llbracket \text{aspect}(*, *, *) \rrbracket, \llbracket \langle * \rangle \rrbracket, \llbracket \text{tuple}_1(*, *) \rrbracket, \llbracket \text{tuple}_2(*, *) \rrbracket, \llbracket \text{let}_2(*, *) \rrbracket,$
 $\llbracket \text{let}_1(*, *) \rrbracket, \llbracket [* \stackrel{\text{claim}}{=} *] \rrbracket, \llbracket \text{checker} \rrbracket, \llbracket \text{check}(*, *) \rrbracket, \llbracket \text{check}_2(*, *, *) \rrbracket, \llbracket \text{check}_3(*, *, *) \rrbracket,$
 $\llbracket \text{check}^*(*, *) \rrbracket, \llbracket \text{check}_2^*(*, *, *) \rrbracket, \llbracket [*] \rrbracket, \llbracket [*]^- \rrbracket, \llbracket [*]^\circ \rrbracket, \llbracket \text{msg} \rrbracket, \llbracket [* \stackrel{\text{msg}}{=} *] \rrbracket, \llbracket \langle \text{stmt} \rangle \rrbracket,$
 $\llbracket \text{stmt} \rrbracket, \llbracket [* \stackrel{\text{stmt}}{=} *] \rrbracket, \llbracket \text{HeadNil}' \rrbracket, \llbracket \text{HeadPair}' \rrbracket, \llbracket \text{Transitivity}' \rrbracket, \llbracket \perp \rrbracket, \llbracket \text{Contra}' \rrbracket, \llbracket \text{T}'_{\text{E}} \rrbracket,$
 $\llbracket \text{L}_1 \rrbracket, \llbracket [*] \rrbracket, \llbracket \mathcal{A} \rrbracket, \llbracket \mathcal{B} \rrbracket, \llbracket \mathcal{C} \rrbracket, \llbracket \mathcal{D} \rrbracket, \llbracket \mathcal{E} \rrbracket, \llbracket \mathcal{F} \rrbracket, \llbracket \mathcal{G} \rrbracket, \llbracket \mathcal{H} \rrbracket, \llbracket \mathcal{I} \rrbracket, \llbracket \mathcal{J} \rrbracket, \llbracket \mathcal{K} \rrbracket, \llbracket \mathcal{L} \rrbracket, \llbracket \mathcal{M} \rrbracket, \llbracket \mathcal{N} \rrbracket, \llbracket \mathcal{O} \rrbracket, \llbracket \mathcal{P} \rrbracket, \llbracket \mathcal{Q} \rrbracket,$
 $\llbracket \mathcal{R} \rrbracket, \llbracket \mathcal{S} \rrbracket, \llbracket \mathcal{T} \rrbracket, \llbracket \mathcal{U} \rrbracket, \llbracket \mathcal{V} \rrbracket, \llbracket \mathcal{W} \rrbracket, \llbracket \mathcal{X} \rrbracket, \llbracket \mathcal{Y} \rrbracket, \llbracket \mathcal{Z} \rrbracket, \llbracket \langle * \mid * := * \rangle \rrbracket, \llbracket \langle * \mid * := * \rangle \rrbracket, \llbracket \emptyset \rrbracket, \llbracket \text{Remainder} \rrbracket,$
 $\llbracket (*, *)^\vee \rrbracket, \llbracket \text{intro}(*, *, *, *) \rrbracket, \llbracket \text{intro}(*, *, *) \rrbracket, \llbracket \text{error}(*, *) \rrbracket, \llbracket \text{error}_2(*, *) \rrbracket, \llbracket \text{proof}(*, *, *) \rrbracket,$
 $\llbracket \text{proof}_2(*, *) \rrbracket, \llbracket \mathcal{S}(*, *) \rrbracket, \llbracket \mathcal{S}^I(*, *) \rrbracket, \llbracket \mathcal{S}^\triangleright(*, *) \rrbracket, \llbracket \mathcal{S}_1^\triangleright(*, *, *) \rrbracket, \llbracket \mathcal{S}^E(*, *) \rrbracket, \llbracket \mathcal{S}_1^E(*, *, *) \rrbracket,$

$[S^+(*,*), [S_1^+(*,*,*), [\mathcal{S}^-(*,*), [S_1^-(*,*,*), [S^*(*,*), [S_1^*(*,*,*),$
 $[S_2^*(*,*,*,*), [S^{\textcircled{A}}(*,*), [S_1^{\textcircled{A}}(*,*,*), [S^+(*,*), [S_1^+(*,*,*,*), [S^{\text{H}}(*,*),$
 $[S_1^{\text{H}}(*,*,*,*), [S^{\text{i.e.}}(*,*), [S_1^{\text{i.e.}}(*,*,*,*), [S_2^{\text{i.e.}}(*,*,*,*,*), [S^{\text{V}}(*,*)],$
 $[S_1^{\text{V}}(*,*,*,*), [S^{\text{i}}(*,*)], [S_1^{\text{i}}(*,*,*), [S_2^{\text{i}}(*,*,*,*), [\mathcal{T}(*)], [\text{claims}(*,*,*)],$
 $[\text{claims}_2(*,*,*)], [<\text{proof}>], [\text{proof}], [[\text{Lemma } *: *]], [[\text{Proof of } *: *]],$
 $[[* \text{ lemma } *: *]], [[* \text{ antilemma } *: *]], [[* \text{ rule } *: *]], [[* \text{ antirule } *: *]],$
 $[\text{verifier}], [\mathcal{V}_1(*)], [\mathcal{V}_2(*,*)], [\mathcal{V}_3(*,*,*,*)], [\mathcal{V}_4(*,*)], [\mathcal{V}_5(*,*,*,*)], [\mathcal{V}_6(*,*,*,*)],$
 $[\mathcal{V}_7(*,*,*,*)], [\text{Cut}(*,*)], [\text{Head}_{\oplus}(*)], [\text{Tail}_{\oplus}(*)], [\text{rule}_1(*,*)], [\text{rule}(*,*)],$
 $[\text{Rule tactic}], [\text{Plus}(*,*)], [[\text{Theory } *]], [\text{theory}_2(*,*)], [\text{theory}_3(*,*)],$
 $[\text{theory}_4(*,*,*)], [\text{HeadNil}'''], [\text{HeadPair}'''], [\text{Transitivity}'''], [\text{Contra}'''], [\text{HeadNil}],$
 $[\text{HeadPair}], [\text{Transitivity}], [\text{Contra}], [\text{T}_{\text{E}}], [\text{ragged right}],$
 $[\text{ragged right expansion }], [\text{parm}(*,*,*)], [\text{parm}^*(*,*,*)], [\text{inst}(*,*)],$
 $[\text{inst}^*(*,*)], [\text{occur}(*,*,*)], [\text{occur}^*(*,*,*)], [\text{unify}(* = *,*)], [\text{unify}^*(* = *,*)],$
 $[\text{unify}_2(* = *,*)], [\text{L}_a], [\text{L}_b], [\text{L}_c], [\text{L}_d], [\text{L}_e], [\text{L}_f], [\text{L}_g], [\text{L}_h], [\text{L}_i], [\text{L}_j], [\text{L}_k], [\text{L}_l], [\text{L}_m],$
 $[\text{L}_n], [\text{L}_o], [\text{L}_p], [\text{L}_q], [\text{L}_r], [\text{L}_s], [\text{L}_t], [\text{L}_u], [\text{L}_v], [\text{L}_w], [\text{L}_x], [\text{L}_y], [\text{L}_z], [\text{L}_A], [\text{L}_B], [\text{L}_C],$
 $[\text{L}_D], [\text{L}_E], [\text{L}_F], [\text{L}_G], [\text{L}_H], [\text{L}_I], [\text{L}_J], [\text{L}_K], [\text{L}_L], [\text{L}_M], [\text{L}_N], [\text{L}_O], [\text{L}_P], [\text{L}_Q], [\text{L}_R],$
 $[\text{L}_S], [\text{L}_T], [\text{L}_U], [\text{L}_V], [\text{L}_W], [\text{L}_X], [\text{L}_Y], [\text{L}_Z], [\text{L}_?], [\text{Reflexivity}], [\text{Reflexivity}_1],$
 $[\text{Commutativity}], [\text{Commutativity}_1], [<\text{tactic}>], [\text{tactic}], [[* \stackrel{\text{tactic}}{=} *]], [\mathcal{P}(*,*,*)],$
 $[\mathcal{P}^*(*,*,*)], [\text{p}_0], [\text{conclude}_1(*,*)], [\text{conclude}_2(*,*,*)], [\text{conclude}_3(*,*,*,*)],$
 $[\text{conclude}_4(*,*)], [\hat{0}], [\hat{1}], [\hat{2}], [\hat{a}], [\hat{b}], [\hat{c}], [\hat{d}], [\hat{e}], [\hat{f}], [\hat{g}], [\hat{h}], [\hat{i}], [\hat{j}], [\hat{k}], [\hat{l}], [\hat{m}], [\hat{n}],$
 $[\hat{o}], [\hat{p}], [\hat{q}], [\hat{r}], [\hat{s}], [\hat{t}], [\hat{u}], [\hat{v}], [\hat{w}], [\hat{x}], [\hat{y}], [\hat{z}], [\text{nonfree}(*,*)], [\text{nonfree}^*(*,*)],$
 $[\text{free}(*|* := *)], [\text{free}^*(** := *)], [* \equiv (*|* := *)], [* \equiv^*(** := *)], [\text{S}], [\text{A1}], [\text{A2}],$
 $[\text{A3}], [\text{A4}], [\text{A5}], [\text{S1}], [\text{S2}], [\text{S3}], [\text{S4}], [\text{S5}], [\text{S6}], [\text{S7}], [\text{S8}], [\text{S9}], [\text{MP}], [\text{Gen}],$
 $[\text{L3.2(a)}], [\text{S}'], [\text{A1}'], [\text{A2}'], [\text{A3}'], [\text{A4}'], [\text{A5}'], [\text{S1}'], [\text{S2}'], [\text{S3}'], [\text{S4}'], [\text{S5}'], [\text{S6}'],$
 $[\text{S7}'], [\text{S8}'], [\text{S9}'], [\text{MP}'], [\text{Gen}'], [\text{L3.2(a)}'], [\text{M1.7}], [\text{MP}'_{\text{h}}], [\text{Hypothesize}], [\text{Gen}'_{\text{h}}],$
 $[\text{M3.2(a)}], [\text{M3.2(a)}_{\text{h}}], [\text{M3.2(b)}_{\text{h}}], [\text{M3.1(S1')}_{\text{h}}], [\text{M3.2(c)}_{\text{h}}], [\text{M3.1(S2')}_{\text{h}}],$
 $[\text{M3.1(S5')}_{\text{h}}], [\text{M3.1(S6')}_{\text{h}}], [\text{M3.2(f)}];$

Preassociative

$[_* \{ * \}], [*'], [*|*], [* \mapsto *], [* \Rightarrow *], [*];$

Preassociative

$["*"], [], [(*)^{\text{t}}], [\text{string}(* + *)], [\text{string}(* ++ *)], [$
 $*, [*, [!*, [!^*], [\#*], [\$*], [\%*], [\&*], [^*], [(*)], [!^*], [**], [+*], [*, [-*], [.*], [/^*],$
 $[0*], [1*], [2*], [3*], [4*], [5*], [6*], [7*], [8*], [9*], [:*], [; *], [< *], [= *], [> *], [? *],$
 $[@*], [A*], [B*], [C*], [D*], [E*], [F*], [G*], [H*], [I*], [J*], [K*], [L*], [M*], [N*],$
 $[O*], [P*], [Q*], [R*], [S*], [T*], [U*], [V*], [W*], [X*], [Y*], [Z*], [[*], [\backslash *], [\hat{*}], [\wedge *],$
 $[*], [\text{'*}], [a*], [b*], [c*], [d*], [e*], [f*], [g*], [h*], [i*], [j*], [k*], [l*], [m*], [n*], [o*],$
 $[p*], [q*], [r*], [s*], [t*], [u*], [v*], [w*], [x*], [y*], [z*], [\{ *], [| *], [} *], [\sim *],$
 $[\text{Preassociative } *; *], [\text{Postassociative } *; *], [[*], *], [\text{priority } * \text{ end}],$
 $[\text{newline } *], [\text{macro newline } *];$

Preassociative

$[*0], [*1], [0b], [* \text{-color}(*)], [* \text{-color}^*(*)];$

Preassociative

$[* ' *], [* ' *];$

Preassociative

$[*^{\text{H}}], [*^{\text{T}}], [*^{\text{U}}], [*^{\text{h}}], [*^{\text{t}}], [*^{\text{s}}], [*^{\text{c}}], [*^{\text{d}}], [*^{\text{a}}], [*^{\text{C}}], [*^{\text{M}}], [*^{\text{B}}], [*^{\text{r}}], [*^{\text{i}}], [*^{\text{d}}], [*^{\text{R}}], [*^0],$

$[*^1], [*^2], [*^3], [*^4], [*^5], [*^6], [*^7], [*^8], [*^9], [*^E], [*^V], [*^C], [*^{C*}], [*^/];$

Preassociative

$[* \cdot *], [* \cdot_0 *], [* \dot{\cdot} *];$

Preassociative

$[* + *], [* +_0 *], [* +_1 *], [* - *], [* -_0 *], [* -_1 *], [* \dot{+} *];$

Preassociative

$[* \cup \{*\}], [* \cup *], [* \setminus \{*\}];$

Postassociative

$[* \dot{:} *], [* \dot{:} *], [* \dot{:} : *], [* \underline{+2} *], [* :: *], [* +2 * *];$

Postassociative

$[*, *];$

Preassociative

$[* \overset{B}{\approx} *], [* \overset{D}{\approx} *], [* \overset{C}{\approx} *], [* \overset{P}{\approx} *], [* \approx *], [* = *], [* \overset{+}{\rightarrow} *], [* \overset{t}{=} *], [* \overset{t^*}{=} *], [* \overset{r}{=} *],$

$[* \in_t *], [* \subseteq_T *], [* \overset{T}{=} *], [* \overset{s}{=} *], [* \text{ free in } *], [* \text{ free in }^* *], [* \text{ free for } * \text{ in } *],$

$[* \text{ free for }^* * \text{ in } *], [* \in_c *], [* < *], [* <' *], [* \leq' *], [* \overset{p}{=} *], [*^P];$

Preassociative

$[* \neg *], [* \dot{\neg} *];$

Preassociative

$[* \wedge *], [* \tilde{\wedge} *], [* \tilde{\wedge} *], [* \wedge_c *], [* \dot{\wedge} *];$

Preassociative

$[* \vee *], [* \parallel *], [* \ddot{\vee} *], [* \dot{\vee} *];$

Preassociative

$[\dot{\vee} * : *], [\dot{\exists} * : *];$

Postassociative

$[* \rightrightarrows *], [* \Rightarrow *], [* \Leftrightarrow *];$

Postassociative

$[* : *], [* ! *];$

Preassociative

$[* \left\{ \begin{array}{c} * \\ * \end{array} \right.];$

Preassociative

$[\lambda * . *], [\Lambda *], [\text{if } * \text{ then } * \text{ else } *], [\text{let } * = * \text{ in } *], [\text{let } * \doteq * \text{ in } *];$

Preassociative

$[*^I], [*^\triangleright], [*^V], [*^+], [*^-], [*^*];$

Preassociative

$[* @ *], [* \triangleright *], [* \blacktriangleright *], [* \gg *], [* \trianglerighteq *], [* \trianglerighteq_h *];$

Postassociative

$[* \vdash *], [* \Vdash *], [* \text{ i.e. } *];$

Preassociative

$[\forall * : *];$

Postassociative

$[* \oplus *];$

Postassociative

$[*, *];$

Preassociative

[* proves *];

Preassociative

[* **proof of** * : *], [Line * : * $\gg$ *; *], [Last line * $\gg$ * $\square$],
[Line * : Premise $\gg$ *; *], [Line * : Side-condition $\gg$ *; *], [Arbitrary $\gg$ *; *],
[Local $\gg$ * = *; *];

Postassociative

[* then *], [* [*] *];

Preassociative

[*&*];

Preassociative

[* \ \ *];]

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C Bibliography

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- [2] E. Mendelson. *Introduction to Mathematical Logic*. Wadsworth and Brooks, 3. edition, 1987.