Peano arithmetic

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GRD-2005-06-29.UTC:12:32:14.418881

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1 Peano arithmetic

This Logiweb page [1] defines Peano arithmetic.

1.1 The constructs of Peano arithmetic

Terms of Peano arithmetic are constructed from zero $[\dot{0} \stackrel{\text{pyk}}{\to}$ “peano zero”$][\dot{0} \stackrel{\text{tex}}{\to}$ “
\dot{0}”], successor $[x' \stackrel{\text{pyk}}{\to}$ “∗ peano succ”$][x' \stackrel{\text{tex}}{\to}$ “#1.’”], plus $[x \dot{+} y \stackrel{\text{pyk}}{\to}$ “∗ peano plus ∗”$][x \dot{+} y \stackrel{\text{tex}}{\to}$ “#1.
\mathop{\dot{+}} #2.”], and times $[x \dot{\cdot} y \stackrel{\text{pyk}}{\to}$ “∗ peano times ∗”$][x \dot{\cdot} y \stackrel{\text{tex}}{\to}$ “#1.
\mathop{\dot{\cdot}} #2.”].

Formulas of Peano arithmeric are constructed from equality $[x \stackrel{p}{=} y \stackrel{\text{pyk}}{\to}$ “∗ peano is ∗”$][x \stackrel{p}{=} y \stackrel{\text{tex}}{\to}$ “#1.
\stackrel{p}{=} #2.”], negation $[\dot{\neg}\, x \stackrel{\text{pyk}}{\to}$ “peano not ∗”$][\dot{\neg}\, x \stackrel{\text{tex}}{\to}$ “
\dot{\neg}\, #1.”], implication $[x \dot{\Rightarrow} y \stackrel{\text{pyk}}{\to}$ “∗ peano imply ∗”$][x \dot{\Rightarrow} y \stackrel{\text{tex}}{\to}$ “#1.
\mathrel{\dot{\Rightarrow}} #2.”], and universal quantification
$[\dot{\forall} x \colon y \stackrel{\text{pyk}}{\to}$ “peano all ∗ indeed ∗”$][\dot{\forall} x \colon y \stackrel{\text{tex}}{\to}$ “
\dot{\forall} #1.
\colon #2.”].

From these constructs we macro define one $[\dot{1} \stackrel{\text{pyk}}{\to}$ “peano one”$][\dot{1} \stackrel{\text{tex}}{\to}$ “
\dot{1}”], two $[\dot{2} \stackrel{\text{pyk}}{\to}$ “peano two”$][\dot{2} \stackrel{\text{tex}}{\to}$ “
\dot{2}”], conjunction $[x \dot{\wedge} y \stackrel{\text{pyk}}{\to}$ “∗ peano and ∗”$][x \dot{\wedge} y \stackrel{\text{tex}}{\to}$ “#1.
\mathrel{\dot{\wedge}} #2.”], disjunction $[x \dot{\vee} y \stackrel{\text{pyk}}{\to}$ “∗ peano or ∗”$][x \dot{\vee} y \stackrel{\text{tex}}{\to}$ “#1.
\mathrel{\dot{\vee}} #2.”], biimplication $[x \dot{\Leftrightarrow} y \stackrel{\text{pyk}}{\to}$ “∗ peano iff ∗”$][x \dot{\Leftrightarrow} y \stackrel{\text{tex}}{\to}$ “#1.
\mathrel{\dot{\Leftrightarrow}} #2.”], and existential quantification
$[\dot{\exists} x \colon y \stackrel{\text{pyk}}{\to}$ “peano exist ∗ indeed ∗”$][\dot{\exists} x \colon y \stackrel{\text{tex}}{\to}$ “
\dot{\exists} #1.
\colon #2.”]:

$$[\dot{1} \stackrel{\text{macro}}{\to} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, \lceil [\dot{1} \doteq \dot{0}'] \rceil)]$$

$$[\dot{2} \stackrel{\text{macro}}{\to} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, \lceil [\dot{2} \doteq \dot{1}'] \rceil)]$$

$$[x \dot{\wedge} y \stackrel{\text{macro}}{\to} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, \lceil [x \dot{\wedge} y \doteq \dot{\neg}\,(x \dot{\Rightarrow} \dot{\neg}\, y)] \rceil)]$$

$$[\mathsf{x} \mathbin{\dot\vee} \mathsf{y} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \mathbin{\dot\vee} \mathsf{y} \ddot{=} \dot\neg \mathsf{x} \mathbin{\dot\Rightarrow} \mathsf{y}]\rceil)]$$

$$[\mathsf{x} \mathbin{\dot\Leftrightarrow} \mathsf{y} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \mathbin{\dot\Leftrightarrow} \mathsf{y} \ddot{=} (\mathsf{x} \mathbin{\dot\Rightarrow} \mathsf{y}) \mathbin{\dot\wedge} (\mathsf{y} \mathbin{\dot\Rightarrow} \mathsf{x})]\rceil)]$$

$$[\dot\exists \mathsf{x} \colon \mathsf{y} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot\exists \mathsf{x} \colon \mathsf{y} \ddot{=} \dot\neg \dot\forall \mathsf{x} \colon \dot\neg \mathsf{y}]\rceil)]$$

1.2 Variables

We now introduce the unary operator $[\dot{\mathsf{x}} \stackrel{\text{pyk}}{\to}$ "∗ peano var"$][\dot{\mathsf{x}} \stackrel{\text{tex}}{\to}$ "
\dot{#1.
}"] and define that a term is a *Peano variable* (i.e. a variable of Peano arithmetic) if it has the $[\dot{\mathsf{x}}]$ operator in its root. $[\mathsf{x}^{\mathcal{P}} \stackrel{\text{pyk}}{\to}$ "∗ is peano var"$][\mathsf{x}^{\mathcal{P}} \stackrel{\text{tex}}{\to}$ "#1.
{} ^ {\cal P}"] is true if $[\mathsf{x}]$ is a Peano variable:

$$[\mathsf{x}^{\mathcal{P}} \stackrel{\text{val}}{\to} \mathsf{x} \stackrel{\text{r}}{=} \lceil \dot{\mathsf{x}} \rceil]$$

We macro define $[\dot{a} \stackrel{\text{pyk}}{\to}$ "peano a"$][\dot{a} \stackrel{\text{tex}}{\to}$ "
\dot{\mathit{a}}"] to be a Peano variable:

$$[\dot{a} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{a} \ddot{=} \dot{\mathsf{a}}]\rceil)]$$

Appendix A.2 defines Peano variables for the other letters of the English alphabet.
$[\dot{\text{nonfree}}(\mathsf{x},\mathsf{y}) \stackrel{\text{pyk}}{\to}$ "peano nonfree ∗ in ∗ end nonfree"$][\dot{\text{nonfree}}(\mathsf{x},\mathsf{y}) \stackrel{\text{tex}}{\to}$ "
\dot{nonfree}(#1.
, #2.
)"] is true if the Peano variable $[\mathsf{x}]$ does not occur free in the Peano term/formula $[\mathsf{y}]$. $[\dot{\text{nonfree}}^*(\mathsf{x},\mathsf{y}) \stackrel{\text{pyk}}{\to}$ "peano nonfree star ∗ in ∗ end nonfree"$][\dot{\text{nonfree}}^*(\mathsf{x},\mathsf{y}) \stackrel{\text{tex}}{\to}$ "
\dot{nonfree}^*(#1.
, #2.
)"] is true if the Peano variable $[\mathsf{x}]$ does not occur free in the list $[\mathsf{y}]$ of Peano terms/formulas.

$$\begin{aligned}&[\dot{\text{nonfree}}(\mathsf{x},\mathsf{y}) \stackrel{\text{val}}{\to}\\ &\text{If}(\mathsf{y}^{\mathcal{P}}, \neg \mathsf{x} \stackrel{\text{t}}{=} \mathsf{y},\\ &\text{If}(\neg \mathsf{y} \stackrel{\text{r}}{=} \lceil \dot\forall \mathsf{x} \colon \mathsf{y} \rceil, \dot{\text{nonfree}}^*(\mathsf{x},\mathsf{y}^{\mathsf{t}}),\\ &\text{If}(\mathsf{x} \stackrel{\text{t}}{=} \mathsf{y}^1, \mathsf{T}, \dot{\text{nonfree}}(\mathsf{x},\mathsf{y}^2))))]\end{aligned}$$

$$[\dot{\text{nonfree}}^*(\mathsf{x},\mathsf{y}) \stackrel{\text{val}}{\to} \mathsf{x}!\text{If}(\mathsf{y}, \mathsf{T}, \text{If}(\dot{\text{nonfree}}(\mathsf{x},\mathsf{y}^{\mathsf{h}}), \dot{\text{nonfree}}^*(\mathsf{x},\mathsf{y}^{\mathsf{t}}), \mathsf{F}))]$$

$[\dot{\text{free}}\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \stackrel{\text{pyk}}{\to}$ "peano free ∗ set ∗ to ∗ end free"$][\dot{\text{free}}\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \stackrel{\text{tex}}{\to}$ "
\dot{free}\langle #1.
| #2.

:= #3.
\rangle”] is true if the substitution [$\langle \mathsf{a} \,|\mathsf{x}{:=}\,\mathsf{b}\rangle$] is free.
[$\dot{\mathrm{free}}^*\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \stackrel{\mathrm{pyk}}{\rightarrow}$ “peano free star * set * to * end free”][$\dot{\mathrm{free}}^*\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \stackrel{\mathrm{tex}}{\rightarrow}$ “
\dot{free}{}^*\langle #1.
| #2.
:= #3.
\rangle”] is the version where [a] is a list of terms.

$$[\dot{\mathrm{free}}\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \stackrel{\mathrm{val}}{\rightarrow} \mathsf{x!b!}$$
$$\mathrm{If}(\mathsf{a}^{\mathcal{P}}, \mathsf{T},$$
$$\mathrm{If}(\neg \mathsf{a} \stackrel{\mathrm{r}}{=} \lceil \dot{\forall} \mathsf{u}\colon \mathsf{v}\rceil, \dot{\mathrm{free}}^*\langle \mathsf{a}^{\mathrm{t}}|\mathsf{x} := \mathsf{b}\rangle,$$
$$\mathrm{If}(\mathsf{a}^1 \stackrel{\mathrm{t}}{=} \mathsf{x}, \mathsf{T},$$
$$\mathrm{If}(\mathrm{non}\dot{\mathrm{free}}(\mathsf{x}, \mathsf{a}^2), \mathsf{T},$$
$$\mathrm{If}(\neg \mathrm{non}\dot{\mathrm{free}}(\mathsf{a}^1, \mathsf{b}), \mathsf{F},$$
$$\dot{\mathrm{free}}\langle \mathsf{a}^2|\mathsf{x} := \mathsf{b}\rangle)))))]$$

$$[\dot{\mathrm{free}}^*\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \stackrel{\mathrm{val}}{\rightarrow} \mathsf{x!b!}\mathrm{If}(\mathsf{a}, \mathsf{T}, \mathrm{If}(\dot{\mathrm{free}}\langle \mathsf{a}^{\mathrm{h}}|\mathsf{x} := \mathsf{b}\rangle, \dot{\mathrm{free}}^*\langle \mathsf{a}^{\mathrm{t}}|\mathsf{x} := \mathsf{b}\rangle, \mathsf{F}))]$$

[$\mathsf{a}\equiv\langle \mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \stackrel{\mathrm{pyk}}{\rightarrow}$ “peano sub * is * where * is * end sub”][$\mathsf{a}\equiv\langle \mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \stackrel{\mathrm{tex}}{\rightarrow}$ “#1.
{\equiv}\langle #2.
|#3.
:=#4.
\rangle”] is true if [a] equals [$\langle \mathsf{b}\,|\mathsf{x}{:=}\,\mathsf{c}\rangle$]. [$\mathsf{a}\equiv\langle^* \mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \stackrel{\mathrm{pyk}}{\rightarrow}$ “peano sub star * is * where * is * end sub”][$\mathsf{a}\equiv\langle^* \mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \stackrel{\mathrm{tex}}{\rightarrow}$ “#1.
{\equiv}\langle^* #2.
|#3.
:=#4.
\rangle”] is the version where [a] and [b] are lists.

$$[\mathsf{a}\equiv\langle \mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \stackrel{\mathrm{val}}{\rightarrow} \mathsf{a!x!c!}$$
$$\mathrm{If}(\mathrm{If}(\mathsf{b} \stackrel{\mathrm{r}}{=} \lceil \dot{\forall} \mathsf{u}\colon \mathsf{v}\rceil, \mathsf{b}^1 \stackrel{\mathrm{t}}{=} \mathsf{x}, \mathsf{F}), \mathsf{a} \stackrel{\mathrm{t}}{=} \mathsf{b},$$
$$\mathrm{If}(\mathsf{b}^{\mathcal{P}} \wedge \mathsf{b} \stackrel{\mathrm{t}}{=} \mathsf{x}, \mathsf{a} \stackrel{\mathrm{t}}{=} \mathsf{c}, \mathrm{If}($$
$$\mathsf{a} \stackrel{\mathrm{r}}{=} \mathsf{b}, \mathsf{a}^{\mathrm{t}}\equiv\langle^* \mathsf{b}^{\mathrm{t}}|\mathsf{x} := \mathsf{c}\rangle, \mathsf{F})))]$$

$$[\mathsf{a}\equiv\langle^* \mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \stackrel{\mathrm{val}}{\rightarrow} \mathsf{b!x!c!}\mathrm{If}(\mathsf{a}, \mathsf{T}, \mathrm{If}(\mathsf{a}^{\mathrm{h}}\equiv\langle \mathsf{b}^{\mathrm{h}}|\mathsf{x} := \mathsf{c}\rangle, \mathsf{a}^{\mathrm{t}}\equiv\langle^* \mathsf{b}^{\mathrm{t}}|\mathsf{x} := \mathsf{c}\rangle, \mathsf{F}))]$$

1.3 Mendelsons system S

System [$\mathrm{S} \stackrel{\mathrm{pyk}}{\rightarrow}$ “system s”][$\mathrm{S} \stackrel{\mathrm{tex}}{\rightarrow}$ “
S”] of Mendelson [2] expresses Peano arithmetic. It comprises the axioms
[$\mathrm{A1} \stackrel{\mathrm{pyk}}{\rightarrow}$ “axiom a one”][$\mathrm{A1} \stackrel{\mathrm{tex}}{\rightarrow}$ “
A1”], [$\mathrm{A2} \stackrel{\mathrm{pyk}}{\rightarrow}$ “axiom a two”][$\mathrm{A2} \stackrel{\mathrm{tex}}{\rightarrow}$ “
A2”], [$\mathrm{A3} \stackrel{\mathrm{pyk}}{\rightarrow}$ “axiom a three”][$\mathrm{A3} \stackrel{\mathrm{tex}}{\rightarrow}$ “

A3”], [A4 $\overset{\text{pyk}}{\to}$ “axiom a four”][A4 $\overset{\text{tex}}{\to}$ “
A4”], and [A5 $\overset{\text{pyk}}{\to}$ “axiom a five”][A5 $\overset{\text{tex}}{\to}$ “
A5”] and inference rules [MP $\overset{\text{pyk}}{\to}$ “rule mp”][MP $\overset{\text{tex}}{\to}$ “
MP”] and [Gen $\overset{\text{pyk}}{\to}$ “rule gen”][Gen $\overset{\text{tex}}{\to}$ “
Gen”] of first order predicate calculus. Furthermore, it comprises the proper axioms [S1 $\overset{\text{pyk}}{\to}$ “axiom s one”][S1 $\overset{\text{tex}}{\to}$ “
S1”], [S2 $\overset{\text{pyk}}{\to}$ “axiom s two”][S2 $\overset{\text{tex}}{\to}$ “
S2”], [S3 $\overset{\text{pyk}}{\to}$ “axiom s three”][S3 $\overset{\text{tex}}{\to}$ “
S3”], [S4 $\overset{\text{pyk}}{\to}$ “axiom s four”][S4 $\overset{\text{tex}}{\to}$ “
S4”], [S5 $\overset{\text{pyk}}{\to}$ “axiom s five”][S5 $\overset{\text{tex}}{\to}$ “
S5”], [S6 $\overset{\text{pyk}}{\to}$ “axiom s six”][S6 $\overset{\text{tex}}{\to}$ “
S6”], [S7 $\overset{\text{pyk}}{\to}$ “axiom s seven”][S7 $\overset{\text{tex}}{\to}$ “
S7”], [S8 $\overset{\text{pyk}}{\to}$ “axiom s eight”][S8 $\overset{\text{tex}}{\to}$ “
S8”], and [S9 $\overset{\text{pyk}}{\to}$ “axiom s nine”][S9 $\overset{\text{tex}}{\to}$ “
S9”]. System [S] is defined thus:

[S $\overset{\text{stmt}}{\to}$ $\dot{a} \dot{+} \dot{b}' \overset{p}{=} \dot{a} \dot{+} \dot{b}' \oplus \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg}\underline{b} \dot{\Rightarrow} \dot{\neg}\underline{a} \dot{\Rightarrow} \dot{\neg}\underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{b} \oplus \dot{a} \overset{p}{=} \dot{b} \dot{\Rightarrow} \dot{a}' \overset{p}{=} \dot{b}' \oplus \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \dot{\Rightarrow} \underline{b} \vdash \underline{a} \vdash \underline{b} \oplus \dot{a}' \overset{p}{=} \dot{b}' \dot{\Rightarrow} \dot{a} \overset{p}{=} \dot{b} \oplus \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \oplus \forall \underline{x}\colon \forall \underline{a}\colon \forall \underline{b}\colon \mathrm{nonfree}(\underline{x}, \underline{a}) \Vdash \dot{\forall} \underline{x}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \dot{\forall} \underline{x}\colon \underline{b} \oplus \dot{a} \dot{\cdot} \dot{b}' \overset{p}{=} \dot{a} \dot{\cdot} \dot{b} \dot{+} \dot{a} \oplus \dot{a} \dot{+} \dot{0} \overset{p}{=} \dot{a} \oplus \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{c} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{c} \oplus \dot{a} \overset{p}{=} \dot{b} \dot{\Rightarrow} \dot{a} \overset{p}{=} \dot{c} \dot{\Rightarrow} \dot{b} \overset{p}{=} \dot{c} \oplus \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{x}\colon \underline{b} \equiv \langle \underline{a} | \underline{x} := \dot{0} \rangle \Vdash \underline{c} \equiv \langle \underline{a} | \underline{x} := \underline{x}' \rangle \Vdash \underline{b} \dot{\Rightarrow} \dot{\forall} \underline{x}\colon \underline{a} \dot{\Rightarrow} \underline{c} \dot{\Rightarrow} \dot{\forall} \underline{x}\colon \underline{a} \oplus \dot{\neg} \dot{0} \overset{p}{=} \dot{a}' \oplus \forall \underline{x}\colon \forall \underline{a}\colon \underline{a} \vdash \dot{\forall} \underline{x}\colon \underline{a} \oplus \forall \underline{c}\colon \forall \underline{a}\colon \forall \underline{x}\colon \forall \underline{b}\colon \lceil \underline{a} \rceil \equiv \langle \lceil \underline{b} \rceil | \lceil \underline{x} \rceil := \lceil \underline{c} \rceil \rangle \Vdash \dot{\forall} \underline{x}\colon \underline{b} \dot{\Rightarrow} \underline{a} \oplus \dot{a} \dot{\cdot} \dot{0} \overset{p}{=} \dot{0}$]

[A1 $\overset{\text{stmt}}{\to}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a}$][A1 $\overset{\text{proof}}{\to}$ Rule tactic]

[A2 $\overset{\text{stmt}}{\to}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{c} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{c}$][A2 $\overset{\text{proof}}{\to}$ Rule tactic]

[A3 $\overset{\text{stmt}}{\to}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg}\underline{b} \dot{\Rightarrow} \dot{\neg}\underline{a} \dot{\Rightarrow} \dot{\neg}\underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{b}$][A3 $\overset{\text{proof}}{\to}$ Rule tactic]

The order of quantifiers in the following axiom is such that [$\underline{c}$] which the current conclusion tactic cannot guess comes first. This allows to supply a value for [$\underline{c}$] without having to supply values for the other meta-variables.

[A4 $\overset{\text{stmt}}{\to}$ $S \vdash \forall \underline{c}\colon \forall \underline{a}\colon \forall \underline{x}\colon \forall \underline{b}\colon \lceil \underline{a} \rceil \equiv \langle \lceil \underline{b} \rceil | \lceil \underline{x} \rceil := \lceil \underline{c} \rceil \rangle \Vdash \dot{\forall} \underline{x}\colon \underline{b} \dot{\Rightarrow} \underline{a}$][A4 $\overset{\text{proof}}{\to}$ Rule tactic]

[A5 $\overset{\text{stmt}}{\to}$ $S \vdash \forall \underline{x}\colon \forall \underline{a}\colon \forall \underline{b}\colon \mathrm{nonfree}(\underline{x}, \underline{a}) \Vdash \dot{\forall} \underline{x}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \dot{\forall} \underline{x}\colon \underline{b}$][A5 $\overset{\text{proof}}{\to}$ Rule tactic]

[MP $\overset{\text{stmt}}{\to}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \dot{\Rightarrow} \underline{b} \vdash \underline{a} \vdash \underline{b}$][MP $\overset{\text{proof}}{\to}$ Rule tactic]

$[\mathrm{Gen} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \forall \underline{x}\colon \forall \underline{a}\colon \underline{a} \vdash \dot{\forall} \underline{x}\colon \underline{a}][\mathrm{Gen} \stackrel{\mathrm{proof}}{\rightarrow} \text{Rule tactic}]$

Axiom [S1] to [S8] are stated as Mendelson [2] does. This is done here to test certain parts of the Logiweb system. Serious users of Peano arithmetic are advised to take Mendelson's Lemma 3.1 as axioms instead.

$[\mathrm{S1} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \dot{a} \stackrel{p}{=} \dot{b} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{c} \dot{\Rightarrow} \dot{b} \stackrel{p}{=} \dot{c}][\mathrm{S1} \stackrel{\mathrm{proof}}{\rightarrow} \text{Rule tactic}]$

$[\mathrm{S2} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \dot{a} \stackrel{p}{=} \dot{b} \dot{\Rightarrow} \dot{a}' \stackrel{p}{=} \dot{b}'][\mathrm{S2} \stackrel{\mathrm{proof}}{\rightarrow} \text{Rule tactic}]$

$[\mathrm{S3} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \dot{\neg} \dot{0} \stackrel{p}{=} \dot{a}'][\mathrm{S3} \stackrel{\mathrm{proof}}{\rightarrow} \text{Rule tactic}]$

$[\mathrm{S4} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \dot{a}' \stackrel{p}{=} \dot{b}' \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{b}][\mathrm{S4} \stackrel{\mathrm{proof}}{\rightarrow} \text{Rule tactic}]$

$[\mathrm{S5} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \dot{a} \dot{+} \dot{0} \stackrel{p}{=} \dot{a}][\mathrm{S5} \stackrel{\mathrm{proof}}{\rightarrow} \text{Rule tactic}]$

$[\mathrm{S6} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \dot{a} \dot{+} \dot{b}' \stackrel{p}{=} \dot{a} \dot{+} \dot{b}'][\mathrm{S6} \stackrel{\mathrm{proof}}{\rightarrow} \text{Rule tactic}]$

$[\mathrm{S7} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \dot{a} \dot{\cdot} \dot{0} \stackrel{p}{=} \dot{0}][\mathrm{S7} \stackrel{\mathrm{proof}}{\rightarrow} \text{Rule tactic}]$

$[\mathrm{S8} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \dot{a} \dot{\cdot} \dot{b}' \stackrel{p}{=} \dot{a} \dot{\cdot} \dot{b} \dot{+} \dot{a}][\mathrm{S8} \stackrel{\mathrm{proof}}{\rightarrow} \text{Rule tactic}]$

$[\mathrm{S9} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{x}\colon \underline{b} \equiv \langle \underline{a} | \underline{x} := \dot{0} \rangle \Vdash \underline{c} \equiv \langle \underline{a} | \underline{x} := \underline{x}' \rangle \Vdash \underline{b} \dot{\Rightarrow} \dot{\forall} \underline{x}\colon \underline{a} \dot{\Rightarrow} \underline{c} \dot{\Rightarrow} \dot{\forall} \underline{x}\colon \underline{a}][\mathrm{S9} \stackrel{\mathrm{proof}}{\rightarrow} \text{Rule tactic}]$

1.4 A lemma and a proof

We now prove Lemma [L3.2(a) $\stackrel{\mathrm{pyk}}{\rightarrow}$ "lemma l three two a"][L3.2(a) $\stackrel{\mathrm{tex}}{\rightarrow}$ " L3.2(a)"] which is an instance of the corresponding proposition in Mendelson [2]:

$[\mathrm{L3.2(a)} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \dot{x} \stackrel{p}{=} \dot{x}]$

$[\mathrm{L3.2(a)} \stackrel{\mathrm{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{S} \vdash \mathrm{S5} \gg \dot{a} \dot{+} \dot{0} \stackrel{p}{=} \dot{a}; \mathrm{Gen} \rhd \dot{a} \dot{+} \dot{0} \stackrel{p}{=} \dot{a} \gg \dot{\forall} \dot{a}\colon \dot{a} \dot{+} \dot{0} \stackrel{p}{=} \dot{a}; \mathrm{A4} @ \dot{x} \gg \dot{\forall} \dot{a}\colon \dot{a} \dot{+} \dot{0} \stackrel{p}{=} \dot{a} \dot{\Rightarrow} \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x}; \mathrm{MP} \rhd \dot{\forall} \dot{a}\colon \dot{a} \dot{+} \dot{0} \stackrel{p}{=} \dot{a} \dot{\Rightarrow} \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \rhd \dot{\forall} \dot{a}\colon \dot{a} \dot{+} \dot{0} \stackrel{p}{=} \dot{a} \gg \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x}; \mathrm{S1} \gg \dot{a} \stackrel{p}{=} \dot{b} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{c} \dot{\Rightarrow} \dot{b} \stackrel{p}{=} \dot{c}; \mathrm{Gen} \rhd \dot{a} \stackrel{p}{=} \dot{b} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{c} \dot{\Rightarrow} \dot{b} \stackrel{p}{=} \dot{c} \gg \dot{\forall} \dot{c}\colon \dot{a} \stackrel{p}{=} \dot{b} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{c} \dot{\Rightarrow} \dot{b} \stackrel{p}{=} \dot{c}; \mathrm{A4} @ \dot{x} \gg \dot{\forall} \dot{c}\colon \dot{a} \stackrel{p}{=} \dot{b} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{c} \dot{\Rightarrow} \dot{b} \stackrel{p}{=} \dot{c} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{b} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{b} \stackrel{p}{=} \dot{x}; \mathrm{MP} \rhd \dot{\forall} \dot{c}\colon \dot{a} \stackrel{p}{=} \dot{b} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{c} \dot{\Rightarrow} \dot{b} \stackrel{p}{=} \dot{c} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{b} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{b} \stackrel{p}{=} \dot{x} \rhd \dot{\forall} \dot{c}\colon \dot{a} \stackrel{p}{=} \dot{b} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{c} \dot{\Rightarrow} \dot{b} \stackrel{p}{=} \dot{c} \gg \dot{a} \stackrel{p}{=} \dot{b} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{b} \stackrel{p}{=} \dot{x}; \mathrm{Gen} \rhd \dot{a} \stackrel{p}{=} \dot{b} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{b} \stackrel{p}{=} \dot{x} \gg \dot{\forall} \dot{b}\colon \dot{a} \stackrel{p}{=} \dot{b} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{b} \stackrel{p}{=} \dot{x}; \mathrm{A4} @ \dot{x} \gg \dot{\forall} \dot{b}\colon \dot{a} \stackrel{p}{=} \dot{b} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{b} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{p}{=} \dot{x}; \mathrm{MP} \rhd \dot{\forall} \dot{b}\colon \dot{a} \stackrel{p}{=} \dot{b} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{b} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{p}{=} \dot{x} \rhd \dot{\forall} \dot{b}\colon \dot{a} \stackrel{p}{=} \dot{b} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{b} \stackrel{p}{=} \dot{x} \gg \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{p}{=} \dot{x}; \mathrm{Gen} \rhd \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{p}{=} \dot{x} \gg \dot{\forall} \dot{a}\colon \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{p}{=} \dot{x}; \mathrm{A4} @ \dot{x} \dot{+} \dot{0} \gg \dot{\forall} \dot{a}\colon \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{p}{=} \dot{x}; \mathrm{MP} \rhd \dot{\forall} \dot{a}\colon \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{p}{=} \dot{x} \rhd \dot{\forall} \dot{a}\colon \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{p}{=} \dot{x} \gg \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{p}{=} \dot{x}; \mathrm{MP} \rhd \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{p}{=} \dot{x} \rhd \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \gg \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{p}{=} \dot{x}; \mathrm{MP} \rhd \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{p}{=} \dot{x} \rhd \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \gg \dot{x} \stackrel{p}{=} \dot{x} \rceil, p_0, c)]$

1.5 An alternative axiomatic system

System [S′ $\stackrel{\text{pyk}}{\rightarrow}$ “system prime s”][S′ $\stackrel{\text{tex}}{\rightarrow}$ “
S’”] is system [S] in which the proper axioms are taken from Lemma 3.1 in Mendelson [2]. It comprises the axioms [A1′ $\stackrel{\text{pyk}}{\rightarrow}$ “axiom prime a one”][A1′ $\stackrel{\text{tex}}{\rightarrow}$ “
A1’”], [A2′ $\stackrel{\text{pyk}}{\rightarrow}$ “axiom prime a two”][A2′ $\stackrel{\text{tex}}{\rightarrow}$ “
A2’”], [A3′ $\stackrel{\text{pyk}}{\rightarrow}$ “axiom prime a three”][A3′ $\stackrel{\text{tex}}{\rightarrow}$ “
A3’”], [A4′ $\stackrel{\text{pyk}}{\rightarrow}$ “axiom prime a four”][A4′ $\stackrel{\text{tex}}{\rightarrow}$ “
A4’”], and [A5′ $\stackrel{\text{pyk}}{\rightarrow}$ “axiom prime a five”][A5′ $\stackrel{\text{tex}}{\rightarrow}$ “
A5’”] and inference rules [MP′ $\stackrel{\text{pyk}}{\rightarrow}$ “rule prime mp”][MP′ $\stackrel{\text{tex}}{\rightarrow}$ “
MP’”] and [Gen′ $\stackrel{\text{pyk}}{\rightarrow}$ “rule prime gen”][Gen′ $\stackrel{\text{tex}}{\rightarrow}$ “
Gen’”] of first order predicate calculus. Furthermore, it comprises the proper axioms [S1′ $\stackrel{\text{pyk}}{\rightarrow}$ “axiom prime s one”][S1′ $\stackrel{\text{tex}}{\rightarrow}$ “
S1’”], [S2′ $\stackrel{\text{pyk}}{\rightarrow}$ “axiom prime s two”][S2′ $\stackrel{\text{tex}}{\rightarrow}$ “
S2’”], [S3′ $\stackrel{\text{pyk}}{\rightarrow}$ “axiom prime s three”][S3′ $\stackrel{\text{tex}}{\rightarrow}$ “
S3’”], [S4′ $\stackrel{\text{pyk}}{\rightarrow}$ “axiom prime s four”][S4′ $\stackrel{\text{tex}}{\rightarrow}$ “
S4’”], [S5′ $\stackrel{\text{pyk}}{\rightarrow}$ “axiom prime s five”][S5′ $\stackrel{\text{tex}}{\rightarrow}$ “
S5’”], [S6′ $\stackrel{\text{pyk}}{\rightarrow}$ “axiom prime s six”][S6′ $\stackrel{\text{tex}}{\rightarrow}$ “
S6’”], [S7′ $\stackrel{\text{pyk}}{\rightarrow}$ “axiom prime s seven”][S7′ $\stackrel{\text{tex}}{\rightarrow}$ “
S7’”], [S8′ $\stackrel{\text{pyk}}{\rightarrow}$ “axiom prime s eight”][S8′ $\stackrel{\text{tex}}{\rightarrow}$ “
S8’”], and [S9′ $\stackrel{\text{pyk}}{\rightarrow}$ “axiom prime s nine”][S9′ $\stackrel{\text{tex}}{\rightarrow}$ “
S9’”].
System [S′] is defined thus:

$$[S' \stackrel{\text{stmt}}{\rightarrow} \forall \underline{a}\colon \forall \underline{b}\colon \underline{a}' \stackrel{p}{=} \underline{b}' \dot{\Rightarrow} \underline{a} \stackrel{p}{=} \underline{b} \oplus \forall \underline{h}\colon \forall \underline{t}\colon \forall \underline{r}\colon \forall \underline{s}\colon \underline{h} \dot{\Rightarrow} \underline{t} \stackrel{p}{=} \underline{r} \vdash \underline{h} \dot{\Rightarrow} \underline{t} \stackrel{p}{=} \underline{s} \vdash \underline{h} \dot{\Rightarrow} \underline{r} \stackrel{p}{=} \underline{s} \oplus \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \oplus \forall \underline{x}\colon \forall \underline{a}\colon \forall \underline{b}\colon \mathrm{nonfree}(\lceil \underline{x} \rceil, \lceil \underline{a} \rceil) \Vdash \dot{\forall} \underline{x}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \dot{\forall} \underline{x}\colon \underline{b} \oplus \forall \underline{h}\colon \forall \underline{t}\colon \forall \underline{r}\colon \underline{h} \dot{\Rightarrow} \underline{t} \dot{+} \underline{r}' \stackrel{p}{=} \underline{t} \dot{+} \underline{r}' \oplus \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \mathbin{\dot{\cdot}} \underline{b}' \stackrel{p}{=} \underline{a} \mathbin{\dot{\cdot}} \underline{b} \dot{+} \underline{a} \oplus \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \stackrel{p}{=} \underline{b} \dot{\Rightarrow} \underline{a}' \stackrel{p}{=} \underline{b}' \oplus \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \dot{\Rightarrow} \underline{b} \vdash \underline{a} \vdash \underline{b} \oplus \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg} \underline{b} \dot{\Rightarrow} \dot{\neg} \underline{a} \dot{\Rightarrow} \dot{\neg} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{b} \oplus \forall \underline{h}\colon \forall \underline{t}\colon \forall \underline{r}\colon \underline{h} \dot{\Rightarrow} \underline{t} \stackrel{p}{=} \underline{r} \vdash \underline{h} \dot{\Rightarrow} \underline{t}' \stackrel{p}{=} \underline{r}' \oplus \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \dot{+} \underline{b}' \stackrel{p}{=} \underline{a} \dot{+} \underline{b}' \oplus \forall \underline{a}\colon \neg \dot{0} \stackrel{p}{=} \underline{a}' \oplus \forall \underline{x}\colon \forall \underline{a}\colon \underline{a} \vdash \dot{\forall} \underline{x}\colon \underline{a} \oplus \forall \underline{c}\colon \forall \underline{a}\colon \forall \underline{x}\colon \forall \underline{b}\colon \lceil \underline{a} \rceil \equiv \langle \lceil \underline{b} \rceil | \lceil \underline{x} \rceil := \lceil \underline{c} \rceil \rangle \Vdash \dot{\forall} \underline{x}\colon \underline{b} \dot{\Rightarrow} \underline{a} \oplus \forall \underline{h}\colon \forall \underline{t}\colon \underline{h} \dot{\Rightarrow} \underline{t} \dot{+} \dot{0} \stackrel{p}{=} \underline{t} \oplus \forall \underline{a}\colon \underline{a} \mathbin{\dot{\cdot}} \dot{0} \stackrel{p}{=} \dot{0} \oplus \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} \stackrel{p}{=} \underline{b} \dot{\Rightarrow} \underline{a} \stackrel{p}{=} \underline{c} \dot{\Rightarrow} \underline{b} \stackrel{p}{=} \underline{c} \oplus \forall \underline{t}\colon \underline{t} \stackrel{p}{=} \underline{t} \oplus \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{x}\colon \underline{b} \equiv \langle \underline{a} | \underline{x} := \dot{0} \rangle \Vdash \underline{c} \equiv \langle \underline{a} | \underline{x} := \underline{x}' \rangle \Vdash \underline{b} \dot{\Rightarrow} \dot{\forall} \underline{x}\colon \underline{a} \dot{\Rightarrow} \underline{c} \dot{\Rightarrow} \dot{\forall} \underline{x}\colon \underline{a} \oplus \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{c} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{c} \oplus \forall \underline{h}\colon \forall \underline{t}\colon \forall \underline{r}\colon \forall \underline{s}\colon \underline{h} \dot{\Rightarrow} \underline{t} \stackrel{p}{=} \underline{r} \vdash \underline{h} \dot{\Rightarrow} \underline{r} \stackrel{p}{=} \underline{s} \vdash \underline{h} \dot{\Rightarrow} \underline{t} \stackrel{p}{=} \underline{s} \oplus \forall \underline{a}\colon \underline{a} \dot{+} \dot{0} \stackrel{p}{=} \underline{a}]$$

$$[A1' \stackrel{\text{stmt}}{\rightarrow} S' \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a}][A1' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$$

$$[A2' \stackrel{\text{stmt}}{\rightarrow} S' \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{c} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{c}][A2' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$$

$$[A3' \stackrel{\text{stmt}}{\rightarrow} S' \vdash \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg} \underline{b} \dot{\Rightarrow} \dot{\neg} \underline{a} \dot{\Rightarrow} \dot{\neg} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{b}][A3' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$$

[A4′ $\stackrel{\text{stmt}}{\rightarrow}$ $\mathrm{S}' \vdash \forall \underline{\mathsf{c}}\colon \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{b}}\colon \lceil \underline{\mathsf{a}} \rceil \equiv \langle \lceil \underline{\mathsf{b}} \rceil | \lceil \underline{\mathsf{x}} \rceil := \lceil \underline{\mathsf{c}} \rceil \rangle \Vdash \dot{\forall} \underline{\mathsf{x}}\colon \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{a}}$][A4′ $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[A5′ $\stackrel{\text{stmt}}{\rightarrow}$ $\mathrm{S}' \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \mathrm{non\dot{f}ree}(\lceil \underline{\mathsf{x}} \rceil, \lceil \underline{\mathsf{a}} \rceil) \Vdash \dot{\forall} \underline{\mathsf{x}}\colon \underline{\mathsf{a}} \dot{\Rightarrow} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{a}} \dot{\Rightarrow} \dot{\forall} \underline{\mathsf{x}}\colon \underline{\mathsf{b}}$][A5′ $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[MP′ $\stackrel{\text{stmt}}{\rightarrow}$ $\mathrm{S}' \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{a}} \dot{\Rightarrow} \underline{\mathsf{b}} \vdash \underline{\mathsf{a}} \vdash \underline{\mathsf{b}}$][MP′ $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[Gen′ $\stackrel{\text{stmt}}{\rightarrow}$ $\mathrm{S}' \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{a}}\colon \underline{\mathsf{a}} \vdash \dot{\forall} \underline{\mathsf{x}}\colon \underline{\mathsf{a}}$][Gen′ $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[S1′ $\stackrel{\text{stmt}}{\rightarrow}$ $\mathrm{S}' \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \forall \underline{\mathsf{c}}\colon \underline{\mathsf{a}} \stackrel{p}{=} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{a}} \stackrel{p}{=} \underline{\mathsf{c}} \Rightarrow \underline{\mathsf{b}} \stackrel{p}{=} \underline{\mathsf{c}}$][S1′ $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[S2′ $\stackrel{\text{stmt}}{\rightarrow}$ $\mathrm{S}' \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{a}} \stackrel{p}{=} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{a}}' \stackrel{p}{=} \underline{\mathsf{b}}'$][S2′ $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[S3′ $\stackrel{\text{stmt}}{\rightarrow}$ $\mathrm{S}' \vdash \forall \underline{\mathsf{a}}\colon \neg \dot{0} \stackrel{p}{=} \underline{\mathsf{a}}'$][S3′ $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[S4′ $\stackrel{\text{stmt}}{\rightarrow}$ $\mathrm{S}' \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{a}}' \stackrel{p}{=} \underline{\mathsf{b}}' \dot{\Rightarrow} \underline{\mathsf{a}} \stackrel{p}{=} \underline{\mathsf{b}}$][S4′ $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[S5′ $\stackrel{\text{stmt}}{\rightarrow}$ $\mathrm{S}' \vdash \forall \underline{\mathsf{a}}\colon \underline{\mathsf{a}} \dot{+} \dot{0} \stackrel{p}{=} \underline{\mathsf{a}}$][S5′ $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[S6′ $\stackrel{\text{stmt}}{\rightarrow}$ $\mathrm{S}' \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{a}} \dot{+} \underline{\mathsf{b}}' \stackrel{p}{=} \underline{\mathsf{a}} \dot{+} \underline{\mathsf{b}}'$][S6′ $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[S7′ $\stackrel{\text{stmt}}{\rightarrow}$ $\mathrm{S}' \vdash \forall \underline{\mathsf{a}}\colon \underline{\mathsf{a}} \dot{\cdot} \dot{0} \stackrel{p}{=} \dot{0}$][S7′ $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[S8′ $\stackrel{\text{stmt}}{\rightarrow}$ $\mathrm{S}' \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{a}} \dot{\cdot} \underline{\mathsf{b}}' \stackrel{p}{=} \underline{\mathsf{a}} \dot{\cdot} \underline{\mathsf{b}} \dot{+} \underline{\mathsf{a}}$][S8′ $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[S9′ $\stackrel{\text{stmt}}{\rightarrow}$ $\mathrm{S}' \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \forall \underline{\mathsf{c}}\colon \forall \underline{\mathsf{x}}\colon \underline{\mathsf{b}} \equiv \langle \underline{\mathsf{a}} | \underline{\mathsf{x}} := \dot{0} \rangle \Vdash \underline{\mathsf{c}} \equiv \langle \underline{\mathsf{a}} | \underline{\mathsf{x}} := \underline{\mathsf{x}}' \rangle \Vdash \underline{\mathsf{b}} \dot{\Rightarrow} \dot{\forall} \underline{\mathsf{x}}\colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{c}} \dot{\Rightarrow} \dot{\forall} \underline{\mathsf{x}}\colon \underline{\mathsf{a}}$][S9′ $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

Note that [A1] and [A1′] are distinct. The former says [$\mathrm{S} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{a}} \dot{\Rightarrow} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{a}}$] and the latter says [$\mathrm{S}' \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{a}} \dot{\Rightarrow} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{a}}$].

1.6 Restatement of lemma and proof

We now prove Lemma [L3.2(a)] once again under the name of [L3.2(a)′ $\stackrel{\text{pyk}}{\rightarrow}$ "lemma prime l three two a"][L3.2(a)′ $\stackrel{\text{tex}}{\rightarrow}$ " L3.2(a)′"]:

[L3.2(a)′ $\stackrel{\text{stmt}}{\rightarrow}$ $\mathrm{S}' \vdash \forall \underline{\mathsf{a}}\colon \underline{\mathsf{a}} \stackrel{p}{=} \underline{\mathsf{a}}$]

[L3.2(a)′ $\stackrel{\text{proof}}{\rightarrow}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{S}' \vdash \forall \underline{\mathsf{a}}\colon \mathrm{S5}' \gg \underline{\mathsf{a}} \dot{+} \dot{0} \stackrel{p}{=} \underline{\mathsf{a}}; \mathrm{S1}' \gg \underline{\mathsf{a}} \dot{+} \dot{0} \stackrel{p}{=} \underline{\mathsf{a}} \dot{\Rightarrow} \underline{\mathsf{a}} \dot{+} \dot{0} \stackrel{p}{=} \underline{\mathsf{a}} \dot{\Rightarrow} \underline{\mathsf{a}} \stackrel{p}{=} \underline{\mathsf{a}}; \mathrm{MP}' \rhd \underline{\mathsf{a}} \dot{+} \dot{0} \stackrel{p}{=} \underline{\mathsf{a}} \dot{\Rightarrow} \underline{\mathsf{a}} \dot{+} \dot{0} \stackrel{p}{=} \underline{\mathsf{a}} \dot{\Rightarrow} \underline{\mathsf{a}} \stackrel{p}{=} \underline{\mathsf{a}} \rhd \underline{\mathsf{a}} \dot{+} \dot{0} \stackrel{p}{=} \underline{\mathsf{a}} \gg \underline{\mathsf{a}} \dot{+} \dot{0} \stackrel{p}{=} \underline{\mathsf{a}} \dot{\Rightarrow} \underline{\mathsf{a}} \stackrel{p}{=} \underline{\mathsf{a}}; \mathrm{MP}' \rhd \underline{\mathsf{a}} \dot{+} \dot{0} \stackrel{p}{=} \underline{\mathsf{a}} \dot{\Rightarrow} \underline{\mathsf{a}} \stackrel{p}{=} \underline{\mathsf{a}} \rhd \underline{\mathsf{a}} \dot{+} \dot{0} \stackrel{p}{=} \underline{\mathsf{a}} \gg \underline{\mathsf{a}} \stackrel{p}{=} \underline{\mathsf{a}} \rceil, \mathrm{p}_0, \mathsf{c})$]

2 Formal development

2.1 Propositional calculus

2.1.1 Modus ponens

We use $[\mathsf{x} \unrhd \mathsf{y} \stackrel{\text{pyk}}{\rightarrow}$ "$*$ macro modus ponens $*$"$][\mathsf{x} \unrhd \mathsf{y} \stackrel{\text{tex}}{\rightarrow}$ "#1. \unrhd #2."] as shorthand for modus ponens:

$$[\mathsf{x} \unrhd \mathsf{y} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil [\mathsf{x} \unrhd \mathsf{y} \ddot{=} \mathrm{MP}' \rhd \mathsf{x} \rhd \mathsf{y}] \rceil)]$$

2.1.2 Lemma M1.7

Lemma $[\mathrm{M1.7} \stackrel{\text{pyk}}{\rightarrow}$ "mendelson one seven"$][\mathrm{M1.7} \stackrel{\text{tex}}{\rightarrow}$ " M1.7"] (i.e. Lemma 1.7 in Mendelson [2]) reads:

$$[\mathrm{M1.7} \stackrel{\text{stmt}}{\rightarrow} \mathrm{S}' \vdash \forall \underline{b} \colon \underline{b} \dot{\Rightarrow} \underline{b}]$$

$$[\mathrm{M1.7} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{S}' \vdash \forall \underline{b} \colon \mathrm{A1}' \gg \underline{b} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{b}; \mathrm{A2}' \gg \underline{b} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{b}; \mathrm{MP}' \rhd \underline{b} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{b} \rhd \underline{b} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{b} \gg \underline{b} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{b}; \mathrm{A1}' \gg \underline{b} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{b}; \mathrm{MP}' \rhd \underline{b} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{b} \rhd \underline{b} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{b} \gg \underline{b} \dot{\Rightarrow} \underline{b} \rceil, \mathrm{p}_0, \mathsf{c})]$$

2.1.3 Hypothetical modus ponens

The hypothetical version $[\mathrm{MP}'_{\mathrm{h}} \stackrel{\text{pyk}}{\rightarrow}$ "hypothetical rule prime mp"$][\mathrm{MP}'_{\mathrm{h}} \stackrel{\text{tex}}{\rightarrow}$ " MP'_h"] of modus ponens MP' has a hypothesis $\underline{h}$ on each premise and on the conclusion:

$$[\mathrm{MP}'_{\mathrm{h}} \stackrel{\text{stmt}}{\rightarrow} \mathrm{S}' \vdash \forall \underline{h} \colon \forall \underline{a} \colon \forall \underline{b} \colon \underline{h} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{b} \vdash \underline{h} \dot{\Rightarrow} \underline{a} \vdash \underline{h} \dot{\Rightarrow} \underline{b}]$$

$$[\mathrm{MP}'_{\mathrm{h}} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{S}' \vdash \forall \underline{h} \colon \forall \underline{a} \colon \forall \underline{b} \colon \underline{h} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{b} \vdash \underline{h} \dot{\Rightarrow} \underline{a} \vdash \mathrm{A2}' \gg \underline{h} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{h} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{h} \dot{\Rightarrow} \underline{b}; \mathrm{MP}' \rhd \underline{h} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{h} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{h} \dot{\Rightarrow} \underline{b} \rhd \underline{h} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{b} \gg \underline{h} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{h} \dot{\Rightarrow} \underline{b}; \mathrm{MP}' \rhd \underline{h} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{h} \dot{\Rightarrow} \underline{b} \rhd \underline{h} \dot{\Rightarrow} \underline{a} \gg \underline{h} \dot{\Rightarrow} \underline{b} \rceil, \mathrm{p}_0, \mathsf{c})]$$

We use $[\mathsf{x} \unrhd_{\mathrm{h}} \mathsf{y} \stackrel{\text{pyk}}{\rightarrow}$ "$*$ hypothetical modus ponens $*$"$][\mathsf{x} \unrhd_{\mathrm{h}} \mathsf{y} \stackrel{\text{tex}}{\rightarrow}$ "#1. \unrhd_h #2."] as shorthand for hypothetical modus ponens:

$$[\mathsf{x} \unrhd_{\mathrm{h}} \mathsf{y} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil [\mathsf{x} \unrhd_{\mathrm{h}} \mathsf{y} \ddot{=} \mathrm{MP}'_{\mathrm{h}} \rhd \mathsf{x} \rhd \mathsf{y}] \rceil)]$$

2.1.4 Turning lemmas to hypothetical lemmas

Lemma $[\mathrm{Hypothesize} \stackrel{\text{pyk}}{\rightarrow}$ "hypothesize"$][\mathrm{Hypothesize} \stackrel{\text{tex}}{\rightarrow}$ " Hypothesize"] turns a lemma with no premises into one that assumes the hypothesis $[\underline{h}]$ to hold:

$$[\mathrm{Hypothesize} \stackrel{\text{stmt}}{\rightarrow} \mathrm{S}' \vdash \forall \underline{h} \colon \forall \underline{a} \colon \underline{a} \vdash \underline{h} \dot{\Rightarrow} \underline{a}]$$

[Hypothesize $\overset{\text{proof}}{\to}$ $\lambda c.\lambda x.\mathcal{P}(\lceil S' \vdash \forall \underline{h}\colon \forall \underline{a}\colon \underline{a} \vdash A1' \gg \underline{a} \dot{\Rightarrow} \underline{h} \dot{\Rightarrow} \underline{a}; MP' \rhd \underline{a} \dot{\Rightarrow} \underline{h} \dot{\Rightarrow} \underline{a} \rhd \underline{a} \gg \underline{h} \dot{\Rightarrow} \underline{a}\rceil, p_0, c)$]

2.2 First order predicate calculus

2.2.1 Hypothetical generalization

The hypothetical version [$Gen'_h \overset{\text{pyk}}{\to}$ "hypothetical rule prime gen"][$Gen'_h \overset{\text{tex}}{\to}$ " Gen'_h"] of generalisation Gen′ has a hypothesis $\underline{h}$ on premise and conclusion:

[$Gen'_h \overset{\text{stmt}}{\to} S' \vdash \forall \underline{h}\colon \forall \underline{x}\colon \forall \underline{a}\colon \mathrm{nonfree}(\lceil \underline{x}\rceil, \lceil \underline{h}\rceil) \Vdash \underline{h} \dot{\Rightarrow} \underline{a} \vdash \underline{h} \dot{\Rightarrow} \dot{\forall} \underline{x}\colon \underline{a}$]

[$Gen'_h \overset{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil S' \vdash \forall \underline{h}\colon \forall \underline{x}\colon \forall \underline{a}\colon \mathrm{nonfree}(\lceil \underline{x}\rceil, \lceil \underline{h}\rceil) \Vdash \underline{h} \dot{\Rightarrow} \underline{a} \vdash A5' \unrhd \mathrm{nonfree}(\lceil \underline{x}\rceil, \lceil \underline{h}\rceil) \gg \dot{\forall} \underline{x}\colon \underline{h} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{h} \dot{\Rightarrow} \dot{\forall} \underline{x}\colon \underline{a}; Gen' \rhd \underline{h} \dot{\Rightarrow} \underline{a} \gg \dot{\forall} \underline{x}\colon \underline{h} \dot{\Rightarrow} \underline{a}; MP' \rhd \dot{\forall} \underline{x}\colon \underline{h} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{h} \dot{\Rightarrow} \dot{\forall} \underline{x}\colon \underline{a} \rhd \dot{\forall} \underline{x}\colon \underline{h} \dot{\Rightarrow} \underline{a} \gg \underline{h} \dot{\Rightarrow} \dot{\forall} \underline{x}\colon \underline{a}\rceil, p_0, c)$]

2.3 Peano arithmetic

2.3.1 Lemma M3.2(a)

Lemma [M3.2(a) $\overset{\text{pyk}}{\to}$ "mendelson three two a"][M3.2(a) $\overset{\text{tex}}{\to}$ "
M3.2(a)"] and the associated hypothetical lemma [$M3.2(a)_h \overset{\text{pyk}}{\to}$ "hypothetical three two a"][$M3.2(a)_h \overset{\text{tex}}{\to}$ "
M3.2(a)_h"] read:

[M3.2(a) $\overset{\text{stmt}}{\to} S' \vdash \forall \underline{t}\colon \underline{t} \overset{p}{=} \underline{t}$][M3.2(a) $\overset{\text{proof}}{\to}$ Rule tactic]

[$M3.2(a)_h \overset{\text{stmt}}{\to} S' \vdash \forall \underline{h}\colon \forall \underline{t}\colon \underline{h} \dot{\Rightarrow} \underline{t} \overset{p}{=} \underline{t}$]

Above we cheat in stating M3.2(a) as a rule and not as a lemma. A reasonable way to construct a large proof is to start stating everything as rules and then changing the rules to lemmas one at a time until only the rules of the theory are left.

[$M3.2(a)_h \overset{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil S' \vdash \forall \underline{h}\colon \forall \underline{t}\colon M3.2(a) \gg \underline{t} \overset{p}{=} \underline{t}; Hypothesize \rhd \underline{t} \overset{p}{=} \underline{t} \gg \underline{h} \dot{\Rightarrow} \underline{t} \overset{p}{=} \underline{t}\rceil, p_0, c)$]

2.3.2 Lemma M3.2(b)

Lemma [$M3.2(b)_h \overset{\text{pyk}}{\to}$ "hypothetical three two b"][$M3.2(b)_h \overset{\text{tex}}{\to}$ "
M3.2(b)_h"] reads:

[$M3.2(b)_h \overset{\text{stmt}}{\to} S' \vdash \forall \underline{h}\colon \forall \underline{t}\colon \forall \underline{r}\colon \underline{h} \dot{\Rightarrow} \underline{t} \overset{p}{=} \underline{r} \vdash \underline{h} \dot{\Rightarrow} \underline{r} \overset{p}{=} \underline{t}$][$M3.2(b)_h \overset{\text{proof}}{\to}$ Rule tactic]

[$M3.2(b)_h \overset{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil S' \vdash \forall \underline{h}\colon \forall \underline{t}\colon \forall \underline{r}\colon \underline{h} \dot{\Rightarrow} \underline{t} \overset{p}{=} \underline{r} \vdash S1' \gg \underline{t} \overset{p}{=} \underline{r} \dot{\Rightarrow} \underline{t} \overset{p}{=} \underline{t} \dot{\Rightarrow} \underline{r} \overset{p}{=} \underline{t}; Hypothesize \rhd \underline{t} \overset{p}{=} \underline{r} \dot{\Rightarrow} \underline{t} \overset{p}{=} \underline{t} \dot{\Rightarrow} \underline{r} \overset{p}{=} \underline{t} \gg \underline{h} \dot{\Rightarrow} \underline{t} \overset{p}{=} \underline{r} \dot{\Rightarrow} \underline{t} \overset{p}{=} \underline{t} \dot{\Rightarrow} \underline{r} \overset{p}{=} \underline{t}; MP'_h \rhd \underline{h} \dot{\Rightarrow} \underline{t} \overset{p}{=} \underline{r} \dot{\Rightarrow} \underline{t} \overset{p}{=} \underline{t} \dot{\Rightarrow} \underline{r} \overset{p}{=} \underline{t} \rhd \underline{h} \dot{\Rightarrow} \underline{t} \overset{p}{=} \underline{r} \gg \underline{h} \dot{\Rightarrow} \underline{t} \overset{p}{=} \underline{t} \dot{\Rightarrow} \underline{r} \overset{p}{=} \underline{t}; M3.2(a)_h \gg \underline{h} \dot{\Rightarrow} \underline{t} \overset{p}{=} \underline{t}; MP'_h \rhd \underline{h} \dot{\Rightarrow} \underline{t} \overset{p}{=} \underline{t} \dot{\Rightarrow} \underline{r} \overset{p}{=} \underline{t} \rhd \underline{h} \dot{\Rightarrow} \underline{t} \overset{p}{=} \underline{t} \gg \underline{h} \dot{\Rightarrow} \underline{r} \overset{p}{=} \underline{t}\rceil, p_0, c)$]

2.3.3 Lemma M3.1(S1)

Lemma $[\mathrm{M3.1(S1')_h} \stackrel{\mathrm{pyk}}{\to}$ "hypothetical three one s one"$][\mathrm{M3.1(S1')_h} \stackrel{\mathrm{tex}}{\to}$ " M3.1(S1')_h"] is the hypothetical version of Mendelson Lemma 3.1(S1′):

$[\mathrm{M3.1(S1')_h} \stackrel{\mathrm{stmt}}{\to} \mathsf{S}' \vdash \forall \underline{\mathsf{h}}\colon \forall \underline{\mathsf{t}}\colon \forall \underline{\mathsf{r}}\colon \forall \underline{\mathsf{s}}\colon \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{r}} \vdash \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{s}} \vdash \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{r}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{s}}][\mathrm{M3.1(S1')_h} \stackrel{\mathrm{proof}}{\to}$ Rule tactic$]$

2.3.4 Lemma M3.2(c)

Lemma $[\mathrm{M3.2(c)_h} \stackrel{\mathrm{pyk}}{\to}$ "hypothetical three two c"$][\mathrm{M3.2(c)_h} \stackrel{\mathrm{tex}}{\to}$ " M3.2(c)_h"] is the hypothetical version of Mendelson Lemma 3.2(c) which expresses ordinary transitivity:

$[\mathrm{M3.2(c)_h} \stackrel{\mathrm{stmt}}{\to} \mathsf{S}' \vdash \forall \underline{\mathsf{h}}\colon \forall \underline{\mathsf{t}}\colon \forall \underline{\mathsf{r}}\colon \forall \underline{\mathsf{s}}\colon \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{r}} \vdash \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{r}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{s}} \vdash \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{s}}][\mathrm{M3.2(c)_h} \stackrel{\mathrm{proof}}{\to}$ Rule tactic$]$

2.3.5 Lemma M3.1(S2)

Lemma $[\mathrm{M3.1(S2')_h} \stackrel{\mathrm{pyk}}{\to}$ "hypothetical three one s two"$][\mathrm{M3.1(S2')_h} \stackrel{\mathrm{tex}}{\to}$ " M3.1(S2')_h"] is the hypothetical version of Mendelson Lemma 3.1(S2′):

$[\mathrm{M3.1(S2')_h} \stackrel{\mathrm{stmt}}{\to} \mathsf{S}' \vdash \forall \underline{\mathsf{h}}\colon \forall \underline{\mathsf{t}}\colon \forall \underline{\mathsf{r}}\colon \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{r}} \vdash \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}}' \stackrel{\mathrm{p}}{=} \underline{\mathsf{r}}'][\mathrm{M3.1(S2')_h} \stackrel{\mathrm{proof}}{\to}$ Rule tactic$]$

2.3.6 Lemma M3.1(S5)

Lemma $[\mathrm{M3.1(S5')_h} \stackrel{\mathrm{pyk}}{\to}$ "hypothetical three one s five"$][\mathrm{M3.1(S5')_h} \stackrel{\mathrm{tex}}{\to}$ " M3.1(S5')_h"] is the hypothetical version of Mendelson Lemma 3.1(S5′):

$[\mathrm{M3.1(S5')_h} \stackrel{\mathrm{stmt}}{\to} \mathsf{S}' \vdash \forall \underline{\mathsf{h}}\colon \forall \underline{\mathsf{t}}\colon \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}} \dot{+} \dot{0} \stackrel{\mathrm{p}}{=} \underline{\mathsf{t}}][\mathrm{M3.1(S5')_h} \stackrel{\mathrm{proof}}{\to}$ Rule tactic$]$

2.3.7 Lemma M3.1(S6)

Lemma $[\mathrm{M3.1(S6')_h} \stackrel{\mathrm{pyk}}{\to}$ "hypothetical three one s six"$][\mathrm{M3.1(S6')_h} \stackrel{\mathrm{tex}}{\to}$ " M3.1(S6')_h"] is the hypothetical version of Mendelson Lemma 3.1(S6′):

$[\mathrm{M3.1(S6')_h} \stackrel{\mathrm{stmt}}{\to} \mathsf{S}' \vdash \forall \underline{\mathsf{h}}\colon \forall \underline{\mathsf{t}}\colon \forall \underline{\mathsf{r}}\colon \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}} \dot{+} \underline{\mathsf{r}}' \stackrel{\mathrm{p}}{=} \underline{\mathsf{t}} \dot{+} \underline{\mathsf{r}}'][\mathrm{M3.1(S6')_h} \stackrel{\mathrm{proof}}{\to}$ Rule tactic$]$

2.3.8 Lemma M3.2(f)

Lemma $[\mathrm{M3.2(f)} \stackrel{\mathrm{pyk}}{\to}$ "mendelson three two f"$][\mathrm{M3.2(f)} \stackrel{\mathrm{tex}}{\to}$ " M3.2(f)"] is the closure of Mendelson Lemma 3.2(f) for the concrete variable $[\dot{\mathsf{t}}]$:

$[\mathrm{M3.2(f)} \stackrel{\mathrm{stmt}}{\to} \mathsf{S}' \vdash \dot{\forall} \dot{\mathsf{t}}\colon \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}]$

The proof below uses local macro definitions.

$$[\mathrm{M3.2(f)} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{S}' \vdash \mathrm{A1}' \gg \mathsf{x} \dot{\Rightarrow} \mathsf{x} \dot{\Rightarrow} \mathsf{x}; \mathrm{M3.1(S5')_h} \gg \mathsf{x} \dot{\Rightarrow} \mathsf{x} \dot{\Rightarrow} \mathsf{x} \dot{\Rightarrow} \dot{0} \dot{+} \dot{0} \stackrel{\mathrm{p}}{=} \dot{0}; \mathrm{M3.2(b)_h} \rhd \mathsf{x} \dot{\Rightarrow} \mathsf{x} \dot{\Rightarrow} \mathsf{x} \dot{\Rightarrow} \dot{0} \dot{+} \dot{0} \stackrel{\mathrm{p}}{=} \dot{0} \gg \mathsf{x} \dot{\Rightarrow} \mathsf{x} \dot{\Rightarrow} \mathsf{x} \dot{\Rightarrow} \dot{0} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{0}; \mathrm{MP}' \rhd \mathsf{x} \dot{\Rightarrow} \mathsf{x} \dot{\Rightarrow} \mathsf{x} \dot{\Rightarrow} \dot{0} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{0} \rhd \mathsf{x} \dot{\Rightarrow} \mathsf{x} \dot{\Rightarrow} \mathsf{x} \gg \dot{0} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{0}; \mathrm{M1.7} \gg \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \dot{\Rightarrow} \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}; \mathrm{M3.1(S2')_h} \rhd \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \dot{\Rightarrow} \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \gg \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \dot{\Rightarrow} \dot{\mathsf{t}}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}'; \mathrm{M3.1(S6')_h} \gg \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \dot{\Rightarrow} \dot{0} \dot{+} \dot{\mathsf{t}}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}'; \mathrm{M3.2(b)_h} \rhd \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \dot{\Rightarrow} \dot{0} \dot{+} \dot{\mathsf{t}}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}' \gg \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \dot{\Rightarrow} \dot{0} \dot{+} \dot{\mathsf{t}}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}'; \mathrm{M3.2(c)_h} \rhd \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \dot{\Rightarrow} \dot{\mathsf{t}}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}' \rhd \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \dot{\Rightarrow} \dot{0} \dot{+} \dot{\mathsf{t}}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}' \gg \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \dot{\Rightarrow} \dot{\mathsf{t}}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}'; \mathrm{Gen}' \rhd \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \dot{\Rightarrow} \dot{\mathsf{t}}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}' \gg \dot{\forall} \dot{\mathsf{t}} \colon \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \dot{\Rightarrow} \dot{\mathsf{t}}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}'; \mathrm{S9}' \gg \dot{0} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{0} \dot{\Rightarrow} \dot{\forall} \dot{\mathsf{t}} \colon \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \dot{\Rightarrow} \dot{\mathsf{t}}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}' \dot{\Rightarrow} \dot{\forall} \dot{\mathsf{t}} \colon \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}; \mathrm{MP}' \rhd \dot{0} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{0} \dot{\Rightarrow} \dot{\forall} \dot{\mathsf{t}} \colon \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \dot{\Rightarrow} \dot{\mathsf{t}}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}' \dot{\Rightarrow} \dot{\forall} \dot{\mathsf{t}} \colon \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \rhd \dot{0} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{0} \gg \dot{\forall} \dot{\mathsf{t}} \colon \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \dot{\Rightarrow} \dot{\mathsf{t}}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}' \dot{\Rightarrow} \dot{\forall} \dot{\mathsf{t}} \colon \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}; \mathrm{MP}' \rhd \dot{\forall} \dot{\mathsf{t}} \colon \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \dot{\Rightarrow} \dot{\mathsf{t}}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}' \dot{\Rightarrow} \dot{\forall} \dot{\mathsf{t}} \colon \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \rhd \dot{\forall} \dot{\mathsf{t}} \colon \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \dot{\Rightarrow} \dot{\mathsf{t}}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}' \gg \dot{\forall} \dot{\mathsf{t}} \colon \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \rceil, \mathrm{p}_0, \mathsf{c})]$$

A Chores

A.1 The name of the page

This defines the name of the page:

$$[\mathrm{peano} \stackrel{\mathrm{pyk}}{\rightarrow} \text{“peano”}]$$

A.2 Variables of Peano arithmetic

We use [$\dot{b} \stackrel{\mathrm{pyk}}{\rightarrow}$ “peano b”][$\dot{b} \stackrel{\mathrm{tex}}{\rightarrow}$ “
\dot{\mathit{b}}”], [$\dot{c} \stackrel{\mathrm{pyk}}{\rightarrow}$ “peano c”][$\dot{c} \stackrel{\mathrm{tex}}{\rightarrow}$ “
\dot{\mathit{c}}”], [$\dot{d} \stackrel{\mathrm{pyk}}{\rightarrow}$ “peano d”][$\dot{d} \stackrel{\mathrm{tex}}{\rightarrow}$ “
\dot{\mathit{d}}”], [$\dot{e} \stackrel{\mathrm{pyk}}{\rightarrow}$ “peano e”][$\dot{e} \stackrel{\mathrm{tex}}{\rightarrow}$ “
\dot{\mathit{e}}”], [$\dot{f} \stackrel{\mathrm{pyk}}{\rightarrow}$ “peano f”][$\dot{f} \stackrel{\mathrm{tex}}{\rightarrow}$ “
\dot{\mathit{f}}”], [$\dot{g} \stackrel{\mathrm{pyk}}{\rightarrow}$ “peano g”][$\dot{g} \stackrel{\mathrm{tex}}{\rightarrow}$ “
\dot{\mathit{g}}”], [$\dot{h} \stackrel{\mathrm{pyk}}{\rightarrow}$ “peano h”][$\dot{h} \stackrel{\mathrm{tex}}{\rightarrow}$ “
\dot{\mathit{h}}”], [$\dot{i} \stackrel{\mathrm{pyk}}{\rightarrow}$ “peano i”][$\dot{i} \stackrel{\mathrm{tex}}{\rightarrow}$ “
\dot{\mathit{i}}”], [$\dot{j} \stackrel{\mathrm{pyk}}{\rightarrow}$ “peano j”][$\dot{j} \stackrel{\mathrm{tex}}{\rightarrow}$ “
\dot{\mathit{j}}”], [$\dot{k} \stackrel{\mathrm{pyk}}{\rightarrow}$ “peano k”][$\dot{k} \stackrel{\mathrm{tex}}{\rightarrow}$ “
\dot{\mathit{k}}”], [$\dot{l} \stackrel{\mathrm{pyk}}{\rightarrow}$ “peano l”][$\dot{l} \stackrel{\mathrm{tex}}{\rightarrow}$ “
\dot{\mathit{l}}”], [$\dot{m} \stackrel{\mathrm{pyk}}{\rightarrow}$ “peano m”][$\dot{m} \stackrel{\mathrm{tex}}{\rightarrow}$ “
\dot{\mathit{m}}”], [$\dot{n} \stackrel{\mathrm{pyk}}{\rightarrow}$ “peano n”][$\dot{n} \stackrel{\mathrm{tex}}{\rightarrow}$ “
\dot{\mathit{n}}”], [$\dot{o} \stackrel{\mathrm{pyk}}{\rightarrow}$ “peano o”][$\dot{o} \stackrel{\mathrm{tex}}{\rightarrow}$ “
\dot{\mathit{o}}”], [$\dot{p} \stackrel{\mathrm{pyk}}{\rightarrow}$ “peano p”][$\dot{p} \stackrel{\mathrm{tex}}{\rightarrow}$ “
\dot{\mathit{p}}”], [$\dot{q} \stackrel{\mathrm{pyk}}{\rightarrow}$ “peano q”][$\dot{q} \stackrel{\mathrm{tex}}{\rightarrow}$ “
\dot{\mathit{q}}”], [$\dot{r} \stackrel{\mathrm{pyk}}{\rightarrow}$ “peano r”][$\dot{r} \stackrel{\mathrm{tex}}{\rightarrow}$ “
\dot{\mathit{r}}”], [$\dot{s} \stackrel{\mathrm{pyk}}{\rightarrow}$ “peano s”][$\dot{s} \stackrel{\mathrm{tex}}{\rightarrow}$ “

\dot{\mathit{s}}”], [$\dot{t} \stackrel{\text{pyk}}{\to}$ “peano t”][$\dot{t} \stackrel{\text{tex}}{\to}$ “
\dot{\mathit{t}}”], [$\dot{u} \stackrel{\text{pyk}}{\to}$ “peano u”][$\dot{u} \stackrel{\text{tex}}{\to}$ “
\dot{\mathit{u}}”], [$\dot{v} \stackrel{\text{pyk}}{\to}$ “peano v”][$\dot{v} \stackrel{\text{tex}}{\to}$ “
\dot{\mathit{v}}”], [$\dot{w} \stackrel{\text{pyk}}{\to}$ “peano w”][$\dot{w} \stackrel{\text{tex}}{\to}$ “
\dot{\mathit{w}}”], [$\dot{x} \stackrel{\text{pyk}}{\to}$ “peano x”][$\dot{x} \stackrel{\text{tex}}{\to}$ “
\dot{\mathit{x}}”], [$\dot{y} \stackrel{\text{pyk}}{\to}$ “peano y”][$\dot{y} \stackrel{\text{tex}}{\to}$ “
\dot{\mathit{y}}”], and [$\dot{z} \stackrel{\text{pyk}}{\to}$ “peano z”][$\dot{z} \stackrel{\text{tex}}{\to}$ “
\dot{\mathit{z}}”] to denote variables of Peano arithmetic:
$[\dot{b} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{b} \ddot{=} \dot{\mathsf{b}}]\rceil)]$, $[\dot{c} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{c} \ddot{=} \dot{\mathsf{c}}]\rceil)]$, $[\dot{d} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{d} \ddot{=} \dot{\mathsf{d}}]\rceil)]$, $[\dot{e} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{e} \ddot{=} \dot{\mathsf{e}}]\rceil)]$, $[\dot{f} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{f} \ddot{=} \dot{\mathsf{f}}]\rceil)]$, $[\dot{g} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{g} \ddot{=} \dot{\mathsf{g}}]\rceil)]$, $[\dot{h} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{h} \ddot{=} \dot{\mathsf{h}}]\rceil)]$, $[\dot{i} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{i} \ddot{=} \dot{\mathsf{i}}]\rceil)]$, $[\dot{j} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{j} \ddot{=} \dot{\mathsf{j}}]\rceil)]$, $[\dot{k} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{k} \ddot{=} \dot{\mathsf{k}}]\rceil)]$, $[\dot{l} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{l} \ddot{=} \dot{\mathsf{l}}]\rceil)]$, $[\dot{m} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{m} \ddot{=} \dot{\mathsf{m}}]\rceil)]$, $[\dot{n} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{n} \ddot{=} \dot{\mathsf{n}}]\rceil)]$, $[\dot{o} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{o} \ddot{=} \dot{\mathsf{o}}]\rceil)]$, $[\dot{p} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{p} \ddot{=} \dot{\mathsf{p}}]\rceil)]$, $[\dot{q} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{q} \ddot{=} \dot{\mathsf{q}}]\rceil)]$, $[\dot{r} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{r} \ddot{=} \dot{\mathsf{r}}]\rceil)]$, $[\dot{s} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{s} \ddot{=} \dot{\mathsf{s}}]\rceil)]$, $[\dot{t} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{t} \ddot{=} \dot{\mathsf{t}}]\rceil)]$, $[\dot{u} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{u} \ddot{=} \dot{\mathsf{u}}]\rceil)]$, $[\dot{v} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{v} \ddot{=} \dot{\mathsf{v}}]\rceil)]$, $[\dot{w} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{w} \ddot{=} \dot{\mathsf{w}}]\rceil)]$, $[\dot{x} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{x} \ddot{=} \dot{\mathsf{x}}]\rceil)]$, $[\dot{y} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{y} \ddot{=} \dot{\mathsf{y}}]\rceil)]$, and $[\dot{z} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{z} \ddot{=} \dot{\mathsf{z}}]\rceil)]$.

A.3 TEX definitions

A.4 Test

$[\lceil \dot{\mathsf{a}} \rceil^{\mathcal{P}}]^{\cdot}$

$[\lceil \mathsf{a} \rceil^{\mathcal{P}}]^{-}$

$[\dot{\text{nonfree}}(\lceil \dot{\mathsf{x}} \rceil, \lceil \dot{\mathsf{y}} \stackrel{\text{p}}{=} \dot{\mathsf{z}} \dot{\Rightarrow} \dot{\forall} \dot{\mathsf{x}}\colon \dot{\mathsf{x}} \stackrel{\text{p}}{=} \dot{\mathsf{y}} \rceil)]^{\cdot}$

$[\dot{\text{nonfree}}(\lceil \dot{\mathsf{x}} \rceil, \lceil \dot{\mathsf{x}} \stackrel{\text{p}}{=} \dot{\mathsf{z}} \dot{\Rightarrow} \dot{\forall} \dot{\mathsf{x}}\colon \dot{\mathsf{x}} \stackrel{\text{p}}{=} \dot{\mathsf{y}} \rceil)]^{-}$

$[\dot{\text{nonfree}}(\lceil \dot{\mathsf{x}} \rceil, \lceil \dot{\mathsf{y}} \stackrel{\text{p}}{=} \dot{\mathsf{x}} \dot{\Rightarrow} \dot{\forall} \dot{\mathsf{x}}\colon \dot{\mathsf{x}} \stackrel{\text{p}}{=} \dot{\mathsf{y}} \rceil)]^{-}$

$[\dot{\text{nonfree}}(\lceil \dot{\mathsf{x}} \rceil, \lceil \dot{\mathsf{y}} \stackrel{\text{p}}{=} \dot{\mathsf{z}} \dot{\Rightarrow} \dot{\forall} \dot{\mathsf{y}}\colon \dot{\mathsf{x}} \stackrel{\text{p}}{=} \dot{\mathsf{y}} \rceil)]^{-}$

$[\dot{\text{free}}\langle \lceil \dot{\forall} \dot{\mathsf{x}}\colon \mathsf{b} :: \dot{\mathsf{x}} :: \mathsf{c} \rceil | \lceil \dot{\mathsf{x}} \rceil := \lceil \mathsf{x} :: \dot{\mathsf{y}} :: \mathsf{z} \rceil \rangle]^{\cdot}$

$[\dot{\text{free}}\langle \lceil \dot{\forall} \dot{\mathsf{y}}\colon \mathsf{b} :: \dot{\mathsf{x}} :: \mathsf{c} \rceil | \lceil \dot{\mathsf{x}} \rceil := \lceil \mathsf{x} :: \dot{\mathsf{y}} :: \mathsf{z} \rceil \rangle]^{-}$

$[\dot{\text{free}}\langle \lceil \dot{\forall} \dot{\mathsf{x}}\colon \mathsf{b} :: \dot{\mathsf{x}} :: \mathsf{c} \rceil | \lceil \dot{\mathsf{y}} \rceil := \lceil \mathsf{x} :: \dot{\mathsf{y}} :: \mathsf{z} \rceil \rangle]^{\cdot}$

$[\dot{\text{free}}\langle\lceil\dot\forall\dot{\mathsf{y}}\colon \mathsf{b} :: \dot{\mathsf{x}} :: \mathsf{c}\rceil | \lceil\dot{\mathsf{y}}\rceil := \lceil \mathsf{x} :: \dot{\mathsf{y}} :: \mathsf{z}\rceil\rangle]^{\cdot}$

$[\dot{\mathsf{a}}\equiv\langle\dot{\mathsf{a}}|\dot{\mathsf{b}} := \dot{\mathsf{c}}\rangle]^{\cdot}$

$[\dot{\mathsf{c}}\equiv\langle\dot{\mathsf{b}}|\dot{\mathsf{b}} := \dot{\mathsf{c}}\rangle]^{\cdot}$

$[\forall\dot{\mathsf{a}}\colon \dot{\mathsf{a}} \stackrel{\mathrm{p}}{=} \dot{\mathsf{b}}\equiv\langle\forall\dot{\mathsf{a}}\colon \dot{\mathsf{a}} \stackrel{\mathrm{p}}{=} \dot{\mathsf{b}}|\dot{\mathsf{a}} := \dot{\mathsf{c}}\rangle]^{\cdot}$

$[\forall\dot{\mathsf{a}}\colon \dot{\mathsf{a}} \stackrel{\mathrm{p}}{=} \dot{\mathsf{c}}\equiv\langle\forall\dot{\mathsf{a}}\colon \dot{\mathsf{a}} \stackrel{\mathrm{p}}{=} \dot{\mathsf{b}}|\dot{\mathsf{b}} := \dot{\mathsf{c}}\rangle]^{\cdot}$

$[\dot\forall\dot{\mathsf{a}}\colon \dot{\mathsf{a}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{a}} \dot{\Rightarrow} \dot{\mathsf{c}} \dot{:} \dot{\mathsf{d}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{c}} \dot{:} \dot{\mathsf{d}}\equiv\langle\dot\forall\dot{\mathsf{a}}\colon \dot{\mathsf{a}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{a}} \dot{\Rightarrow} \dot{\mathsf{b}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{b}}|\dot{\mathsf{b}} := \dot{\mathsf{c}} \dot{:} \dot{\mathsf{d}}\rangle]^{\cdot}$

$[\dot\forall\dot{\mathsf{a}}\colon \dot{\mathsf{a}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{a}} \dot{\Rightarrow} \dot{\mathsf{b}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{b}}\equiv\langle\dot\forall\dot{\mathsf{a}}\colon \dot{\mathsf{a}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{a}} \dot{\Rightarrow} \dot{\mathsf{b}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{b}}|\dot{\mathsf{a}} := \dot{\mathsf{c}}\rangle]^{\cdot}$

A.5 Priority table

[peano $\stackrel{\text{prio}}{\rightarrow}$
Preassociative
[peano], [base], [bracket $*$ end bracket], [big bracket $*$ end bracket], [math $*$ end math], [**flush left** [$*$]], [x], [y], [z], [[$* \bowtie *$]], [[$* \stackrel{*}{\rightarrow} *$]], [pyk], [tex], [name], [prio], [$*$], [T], [if($*,*,*$)], [[$* \stackrel{*}{\Rightarrow} *$]], [val], [claim], [$\bot$], [f($*$)], [$(*)^{\mathrm{I}}$], [$\mathsf{F}$], [$\underline{0}$], [$\underline{1}$], [$\underline{2}$], [$\underline{3}$], [$\underline{4}$], [$\underline{5}$], [$\underline{6}$], [$\underline{7}$], [$\underline{8}$], [$\underline{9}$], [0], [1], [2], [3], [4], [5], [6], [7], [8], [9], [a], [b], [c], [d], [e], [f], [g], [h], [i], [j], [k], [l], [m], [n], [o], [p], [q], [r], [s], [t], [u], [v], [w], [$(*)^{\mathrm{M}}$], [If($*,*,*$)], [array{$*$} $*$ end array], [l], [c], [r], [empty], [$\langle * \mid * := * \rangle$], [$\mathcal{M}(*)$], [$\tilde{\mathcal{U}}(*)$], [$\mathcal{U}(*)$], [$\mathcal{U}^{\mathrm{M}}(*)$], [**apply**($*,*$)], [$\mathbf{apply}_1(*,*)$], [identifier($*$)], [$\text{identifier}_1(*,*)$], [array-plus($*,*$)], [array-remove($*,*,*$)], [array-put($*,*,*,*$)], [array-add($*,*,*,*,*$)], [bit($*,*$)], [$\text{bit}_1(*,*)$], [rack], ["vector"], ["bibliography"], ["dictionary"], ["body"], ["codex"], ["expansion"], ["code"], ["cache"], ["diagnose"], ["pyk"], ["tex"], ["texname"], ["value"], ["message"], ["macro"], ["definition"], ["unpack"], ["claim"], ["priority"], ["lambda"], ["apply"], ["true"], ["if"], ["quote"], ["proclaim"], ["define"], ["introduce"], ["hide"], ["pre"], ["post"], [$\mathcal{E}(*,*,*)$], [$\mathcal{E}_2(*,*,*,*,*)$], [$\mathcal{E}_3(*,*,*,*)$], [$\mathcal{E}_4(*,*,*,*)$], [**lookup**($*,*,*$)], [**abstract**($*,*,*,*$)], [[$\lceil *\rceil$]], [$\mathcal{M}(*,*,*)$], [$\mathcal{M}_2(*,*,*,*)$], [$\mathcal{M}^*(*,*,*)$], [macro], [s_0], [**zip**($*,*$)], [$\mathbf{assoc}_1(*,*,*)$], [$(*)^{\mathbf{P}}$], [self], [[$* \stackrel{..}{=} *$]], [[$* \doteq *$]], [[$* \stackrel{\prime}{=} *$]], [[$* \stackrel{\text{pyk}}{=} *$]], [[$* \stackrel{\text{tex}}{=} *$]], [[$* \stackrel{\text{name}}{=} *$]], [**Priority table**[$*$]], [$\tilde{\mathcal{M}}_1$], [$\tilde{\mathcal{M}}_2(*)$], [$\tilde{\mathcal{M}}_3(*)$], [$\tilde{\mathcal{M}}_4(*,*,*,*)$], [$\tilde{\mathcal{M}}(*,*,*)$], [$\tilde{\mathcal{Q}}(*,*,*)$], [$\tilde{\mathcal{Q}}_2(*,*,*)$], [$\tilde{\mathcal{Q}}_3(*,*,*,*)$], [$\tilde{\mathcal{Q}}^*(*,*,*)$], [($*$)], [**aspect**($*,*$)], [**aspect**($*,*,*$)], [$\langle * \rangle$], [$\mathbf{tuple}_1(*)$], [$\mathbf{tuple}_2(*)$], [$\text{let}_2(*,*)$], [$\text{let}_1(*,*)$], [[$* \stackrel{\text{claim}}{=} *$]], [checker], [**check**($*,*$)], [$\mathbf{check}_2(*,*,*)$], [$\mathbf{check}_3(*,*,*)$], [$\mathbf{check}^*(*,*)$], [$\mathbf{check}^*_2(*,*,*)$], [$[*]^{\cdot}$], [$[*]^{-}$], [$[*]^{\circ}$], [msg], [[$* \stackrel{\text{msg}}{=} *$]], [<stmt>], [stmt], [[$* \stackrel{\text{stmt}}{=} *$]], [HeadNil′], [HeadPair′], [Transitivity′], [$\bot\!\!\!\bot$], [Contra′], [T'_{E}], [L_1], [$\underline{*}$], [$\mathcal{A}$], [$\mathcal{B}$], [$\mathcal{C}$], [$\mathcal{D}$], [$\mathcal{E}$], [$\mathcal{F}$], [$\mathcal{G}$], [$\mathcal{H}$], [$\mathcal{I}$], [$\mathcal{J}$], [$\mathcal{K}$], [$\mathcal{L}$], [$\mathcal{M}$], [$\mathcal{N}$], [$\mathcal{O}$], [$\mathcal{P}$], [$\mathcal{Q}$], [$\mathcal{R}$], [$\mathcal{S}$], [$\mathcal{T}$], [$\mathcal{U}$], [$\mathcal{V}$], [$\mathcal{W}$], [$\mathcal{X}$], [$\mathcal{Y}$], [$\mathcal{Z}$], [$\langle * \mid * := * \rangle$], [$\langle^* * \mid * := * \rangle$], [$\emptyset$], [Remainder], [$(*)^{\mathbf{v}}$], [intro($*,*,*,*$)], [intro($*,*,*$)], [error($*,*$)], [$\text{error}_2(*,*)$], [proof($*,*,*$)], [$\text{proof}_2(*,*)$], [$\mathcal{S}(*,*)$], [$\mathcal{S}^{\mathrm{I}}(*,*)$], [$\mathcal{S}^{\rhd}(*,*)$], [$\mathcal{S}^{\rhd}_1(*,*,*)$], [$\mathcal{S}^{\mathrm{E}}(*,*)$], [$\mathcal{S}^{\mathrm{E}}_1(*,*,*)$],

$[\mathcal{S}^+(*,*)], [\mathcal{S}^+_1(*,*,*)], [\mathcal{S}^-(*,*)], [\mathcal{S}^-_1(*,*,*)], [\mathcal{S}^*(*,*)], [\mathcal{S}^*_1(*,*,*)],$
$[\mathcal{S}^*_2(*,*,*,*)], [\mathcal{S}^@(*,*)], [\mathcal{S}^@_1(*,*,*)], [\mathcal{S}^\vdash(*,*)], [\mathcal{S}^\vdash_1(*,*,*,*)], [\mathcal{S}^\Vdash(*,*)],$
$[\mathcal{S}^\Vdash_1(*,*,*,*)], [\mathcal{S}^{\text{i.e.}}(*,*)], [\mathcal{S}^{\text{i.e.}}_1(*,*,*,*)], [\mathcal{S}^{\text{i.e.}}_2(*,*,*,*,*)], [\mathcal{S}^\forall(*,*)],$
$[\mathcal{S}^\forall_1(*,*,*,*)], [\mathcal{S}^;(*,*)], [\mathcal{S}^;_1(*,*,*)], [\mathcal{S}^;_2(*,*,*,*)], [\mathcal{T}(*)]$, [claims$(*,*,*)$],
[claims$_2(*,*,*)$], [<proof>], [proof], [[**Lemma** $*{:}\,*$]], [[**Proof of** $*{:}\,*$]],
[[$*$ **lemma** $*{:}\,*$]], [[$*$ **antilemma** $*{:}\,*$]], [[$*$ **rule** $*{:}\,*$]], [[$*$ **antirule** $*{:}\,*$]],
[verifier], $[\mathcal{V}_1(*)], [\mathcal{V}_2(*,*)], [\mathcal{V}_3(*,*,*,*)], [\mathcal{V}_4(*,*)], [\mathcal{V}_5(*,*,*,*)], [\mathcal{V}_6(*,*,*,*)],$
$[\mathcal{V}_7(*,*,*,*)]$, [Cut$(*,*)$], [Head$_\oplus(*)$], [Tail$_\oplus(*)$], [rule$_1(*,*)$], [rule$(*,*)$],
[Rule tactic], [Plus$(*,*)$], [[**Theory** $*$]], [theory$_2(*,*)$], [theory$_3(*,*)$],
[theory$_4(*,*,*)$], [HeadNil$''$], [HeadPair$''$], [Transitivity$''$], [Contra$''$], [HeadNil],
[HeadPair], [Transitivity], [Contra], [$\mathrm{T_E}$], [ragged right],
[ragged right expansion], [parm$(*,*,*)$], [parm$^*(*,*,*)$], [inst$(*,*)$],
[inst$^*(*,*)$], [occur$(*,*,*)$], [occur$^*(*,*,*)$], [unify$(*=*,*)$], [unify$^*(*=*,*)$],
[unify$_2(*=*,*)$], [$\mathrm{L_a}$], [$\mathrm{L_b}$], [$\mathrm{L_c}$], [$\mathrm{L_d}$], [$\mathrm{L_e}$], [$\mathrm{L_f}$], [$\mathrm{L_g}$], [$\mathrm{L_h}$], [$\mathrm{L_i}$], [$\mathrm{L_j}$], [$\mathrm{L_k}$], [$\mathrm{L_l}$], [$\mathrm{L_m}$],
[$\mathrm{L_n}$], [$\mathrm{L_o}$], [$\mathrm{L_p}$], [$\mathrm{L_q}$], [$\mathrm{L_r}$], [$\mathrm{L_s}$], [$\mathrm{L_t}$], [$\mathrm{L_u}$], [$\mathrm{L_v}$], [$\mathrm{L_w}$], [$\mathrm{L_x}$], [$\mathrm{L_y}$], [$\mathrm{L_z}$], [$\mathrm{L_A}$], [$\mathrm{L_B}$], [$\mathrm{L_C}$],
[$\mathrm{L_D}$], [$\mathrm{L_E}$], [$\mathrm{L_F}$], [$\mathrm{L_G}$], [$\mathrm{L_H}$], [$\mathrm{L_I}$], [$\mathrm{L_J}$], [$\mathrm{L_K}$], [$\mathrm{L_L}$], [$\mathrm{L_M}$], [$\mathrm{L_N}$], [$\mathrm{L_O}$], [$\mathrm{L_P}$], [$\mathrm{L_Q}$], [$\mathrm{L_R}$],
[$\mathrm{L_S}$], [$\mathrm{L_T}$], [$\mathrm{L_U}$], [$\mathrm{L_V}$], [$\mathrm{L_W}$], [$\mathrm{L_X}$], [$\mathrm{L_Y}$], [$\mathrm{L_Z}$], [$\mathrm{L_?}$], [Reflexivity], [Reflexivity$_1$],
[Commutativity], [Commutativity$_1$], [<tactic>], [tactic], [[$* \overset{\text{tactic}}{=} *$]], $[\mathcal{P}(*,*,*)]$,
$[\mathcal{P}^*(*,*,*)]$, [$\mathrm{p_0}$], [conclude$_1(*,*)$], [conclude$_2(*,*,*)$], [conclude$_3(*,*,*,*)$],
[conclude$_4(*,*)$], $[\dot{0}], [\dot{1}], [\dot{2}], [\dot{a}], [\dot{b}], [\dot{c}], [\dot{d}], [\dot{e}], [\dot{f}], [\dot{g}], [\dot{h}], [\dot{i}], [\dot{j}], [\dot{k}], [\dot{l}], [\dot{m}], [\dot{n}],$
$[\dot{o}], [\dot{p}], [\dot{q}], [\dot{r}], [\dot{s}], [\dot{t}], [\dot{u}], [\dot{v}], [\dot{w}], [\dot{x}], [\dot{y}], [\dot{z}]$, [nonfree$(*,*)$], [nonfree$^*(*,*)$],
[free$\langle *|*:=*\rangle$], [free$^*\langle *|*:=*\rangle$], [$*\equiv\langle *|*:=*\rangle$], [$*\equiv\langle^* *|*:=*\rangle$], [S], [A1], [A2],
[A3], [A4], [A5], [S1], [S2], [S3], [S4], [S5], [S6], [S7], [S8], [S9], [MP], [Gen],
[L3.2(a)], [S$'$], [A1$'$], [A2$'$], [A3$'$], [A4$'$], [A5$'$], [S1$'$], [S2$'$], [S3$'$], [S4$'$], [S5$'$], [S6$'$],
[S7$'$], [S8$'$], [S9$'$], [MP$'$], [Gen$'$], [L3.2(a)$'$], [M1.7], [MP$'_\mathrm{h}$], [Hypothesize], [Gen$'_\mathrm{h}$],
[M3.2(a)], [M3.2(a)$_\mathrm{h}$], [M3.2(b)$_\mathrm{h}$], [M3.1(S1$'$)$_\mathrm{h}$], [M3.2(c)$_\mathrm{h}$], [M3.1(S2$'$)$_\mathrm{h}$],
[M3.1(S5$'$)$_\mathrm{h}$], [M3.1(S6$'$)$_\mathrm{h}$], [M3.2(f)];

Preassociative

[$*_{\{*\}}$], [$*'$], [$*[*]$], [$*[*\to*]$], [$*[*\Rightarrow*]$], [$\dot{*}$];

Preassociative

[“ * ”], [], [$(*)^{\mathbf{t}}$], [string(*) + *], [string(*) ++ *], [
*], [*], [!*], [”*], [#*], [\$*], [%*], [&*], [’*], [(*], [)*], [**], [+*], [, *], [-*], [.*], [/*],
[0*], [1*], [2*], [3*], [4*], [5*], [6*], [7*], [8*], [9*], [:*], [; *], [<*], [=*], [>*], [?*],
[@*], [A*], [B*], [C*], [D*], [E*], [F*], [G*], [H*], [I*], [J*], [K*], [L*], [M*], [N*],
[O*], [P*], [Q*], [R*], [S*], [T*], [U*], [V*], [W*], [X*], [Y*], [Z*], [[*], [*], []*], [^*],
[_*], [‘*], [a*], [b*], [c*], [d*], [e*], [f*], [g*], [h*], [i*], [j*], [k*], [l*], [m*], [n*], [o*],
[p*], [q*], [r*], [s*], [t*], [u*], [v*], [w*], [x*], [y*], [z*], [{*], [|*], [}*], [~*],
[**Preassociative** *; *], [**Postassociative** *; *], [[*], *], [priority * end],
[newline *], [macro newline *];

Preassociative

[*0], [*1], [0b], [*-color(*)], [*-color*(*)];

Preassociative

[* ’ *], [* ‘ *];

Preassociative

[$*^{\mathrm{H}}$], [$*^{\mathrm{T}}$], [$*^{\mathrm{U}}$], [$*^{\mathrm{h}}$], [$*^{\mathrm{t}}$], [$*^{\mathrm{s}}$], [$*^{\mathrm{c}}$], [$*^{\mathrm{d}}$], [$*^{\mathrm{a}}$], [$*^{\mathrm{C}}$], [$*^{\mathrm{M}}$], [$*^{\mathrm{B}}$], [$*^{\mathrm{r}}$], [$*^{\mathrm{i}}$], [$*^{\mathrm{d}}$], [$*^{\mathrm{R}}$], [$*^{0}$],

$[*^1], [*^2], [*^3], [*^4], [*^5], [*^6], [*^7], [*^8], [*^9], [*^{\mathrm{E}}], [*^{\mathcal{V}}], [*^{\mathcal{C}}], [*^{\mathcal{C}^*}], [*']$;
Preassociative
$[* \cdot *], [* \cdot_0 *], [* \dot{:} *]$;
Preassociative
$[* + *], [* +_0 *], [* +_1 *], [* - *], [* -_0 *], [* -_1 *], [* \dot{+} *]$;
Preassociative
$[* \cup \{*\}], [* \cup *], [* \backslash \{*\}]$;
Postassociative
$[* \therefore *], [* \underline{\therefore} *], [* \underline{::} *], [* \underline{+2*} *], [* :: *], [* +2* *]$;
Postassociative
$[*, *]$;
Preassociative
$[* \overset{\mathrm{B}}{\approx} *], [* \overset{\mathrm{D}}{\approx} *], [* \overset{\mathrm{C}}{\approx} *], [* \overset{\mathrm{P}}{\approx} *], [* \approx *], [* = *], [* \overset{+}{\rightarrow} *], [* \overset{\mathrm{t}}{=} *], [* \overset{\mathrm{t}^*}{=} *], [* \overset{\mathrm{r}}{=} *]$,
$[* \in_{\mathrm{t}} *], [* \subseteq_{\mathrm{T}} *], [* \overset{\mathrm{T}}{=} *], [* \overset{\mathrm{s}}{=} *]$, [∗ free in ∗], [∗ free in* ∗], [∗ free for ∗ in ∗],
[∗ free for* ∗ in ∗], $[* \in_{\mathrm{c}} *], [* < *], [* <' *], [* \leq' *], [* \overset{\mathrm{p}}{=} *], [*^{\mathcal{P}}]$;
Preassociative
$[\neg *], [\dot{\neg} *]$;
Preassociative
$[* \wedge *], [* \ddot{\wedge} *], [* \tilde{\wedge} *], [* \wedge_{\mathrm{c}} *], [* \dot{\wedge} *]$;
Preassociative
$[* \vee *], [* \parallel *], [* \ddot{\vee} *], [* \dot{\vee} *]$;
Preassociative
$[\dot{\forall} * \colon *], [\dot{\exists} * \colon *]$;
Postassociative
$[* \ddot{\Rightarrow} *], [* \dot{\Rightarrow} *], [* \dot{\Leftrightarrow} *]$;
Postassociative
$[* : *], [* ! *]$;
Preassociative
$[* \left\{ \begin{array}{c} * \\ * \end{array} \right.]$;
Preassociative
$[\lambda * . *], [\Lambda *], [\mathbf{if}\ *\ \mathbf{then}\ *\ \mathbf{else}\ *], [\mathbf{let}\ * = *\ \mathbf{in}\ *], [\mathbf{let}\ * \ddot{=} *\ \mathrm{in}\ *]$;
Preassociative
$[*^{\mathrm{I}}], [*^{\rhd}], [*^{\mathrm{V}}], [*^{+}], [*^{-}], [*^{*}]$;
Preassociative
$[* @ *], [* \rhd *], [* \mathrel{|\!\!\rhd} *], [* \gg *], [* \unrhd *], [* \unrhd_{\mathrm{h}} *]$;
Postassociative
$[* \vdash *], [* \Vdash *]$, [∗ i.e. ∗];
Preassociative
$[\forall * \colon *]$;
Postassociative
$[* \oplus *]$;
Postassociative
$[*; *]$;
Preassociative

$[* \text{ proves } *]$;
Preassociative
$[* \textbf{ proof of } * : *]$, $[\text{Line} * : * \gg *; *]$, $[\text{Last line} * \gg * \Box]$,
$[\text{Line} * : \text{Premise} \gg *; *]$, $[\text{Line} * : \text{Side-condition} \gg *; *]$, $[\text{Arbitrary} \gg *; *]$,
$[\text{Local} \gg * = *; *]$;
Postassociative
$[* \text{ then } *]$, $[*[*]*]$;
Preassociative
$[*\&*]$;
Preassociative
[**];]

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C Bibliography

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