Peano arithmetic

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Contents

1 Peano arithmetic

This Logiweb page [1] defines Peano arithmetic.

1.1 The constructs of Peano arithmetic

Terms of Peano arithmetic are constructed from zero $[\dot{0}]$[1], successor $[x']$[2], plus $[x \dot{+} y]$[3], and times $[x \dot{\cdot} y]$[4].

Formulas of Peano arithmeric are constructed from equality $[x \stackrel{p}{=} y]$[5], negation $[\dot{\neg} x]$[6], implication $[x \dot{\Rightarrow} y]$[7], and universal quantification $[\dot{\forall} x\colon y]$[8].

From these constructs we macro define one $[\dot{1}]$[9], two $[\dot{2}]$[10], conjunction $[x \dot{\wedge} y]$[11], disjunction $[x \dot{\vee} y]$[12], biimplication $[x \dot{\Leftrightarrow} y]$[13], and existential quantification $[\dot{\exists} x\colon y]$[14]:

$$[\dot{1} \ddot{=} \dot{0}']$$

$$[\dot{2} \ddot{=} \dot{1}']$$

$$[x \dot{\wedge} y \ddot{=} \dot{\neg}(x \dot{\Rightarrow} \dot{\neg} y)]$$

$$[x \dot{\vee} y \ddot{=} \dot{\neg} x \dot{\Rightarrow} y]$$

$$[x \dot{\Leftrightarrow} y \ddot{=} (x \dot{\Rightarrow} y) \dot{\wedge} (y \dot{\Rightarrow} x)]$$

$$[\dot{\exists} x\colon y \ddot{=} \dot{\neg} \dot{\forall} x\colon \dot{\neg} y]$$

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[1] $[\dot{0} \stackrel{\text{pyk}}{=}$ "peano zero"$]$

[2] $[x' \stackrel{\text{pyk}}{=}$ "∗ peano succ"$]$

[3] $[x \dot{+} y \stackrel{\text{pyk}}{=}$ "∗ peano plus ∗"$]$

[4] $[x \dot{\cdot} y \stackrel{\text{pyk}}{=}$ "∗ peano times ∗"$]$

[5] $[x \stackrel{p}{=} y \stackrel{\text{pyk}}{=}$ "∗ peano is ∗"$]$

[6] $[\dot{\neg} x \stackrel{\text{pyk}}{=}$ "peano not ∗"$]$

[7] $[x \dot{\Rightarrow} y \stackrel{\text{pyk}}{=}$ "∗ peano imply ∗"$]$

[8] $[\dot{\forall} x\colon y \stackrel{\text{pyk}}{=}$ "peano all ∗ indeed ∗"$]$

[9] $[\dot{1} \stackrel{\text{pyk}}{=}$ "peano one"$]$

[10] $[\dot{2} \stackrel{\text{pyk}}{=}$ "peano two"$]$

[11] $[x \dot{\wedge} y \stackrel{\text{pyk}}{=}$ "∗ peano and ∗"$]$

[12] $[x \dot{\vee} y \stackrel{\text{pyk}}{=}$ "∗ peano or ∗"$]$

[13] $[x \dot{\Leftrightarrow} y \stackrel{\text{pyk}}{=}$ "∗ peano iff ∗"$]$

[14] $[\dot{\exists} x\colon y \stackrel{\text{pyk}}{=}$ "peano exist ∗ indeed ∗"$]$

1.2 Variables

We now introduce the unary operator $[\dot{\mathsf{x}}]$[15] and define that a term is a *Peano variable* (i.e. a variable of Peano arithmetic) if it has the $[\dot{\mathsf{x}}]$ operator in its root. $[\mathsf{x}^{\mathcal{P}}]$[16] is true if $[\mathsf{x}]$ is a Peano variable:

$$[\mathsf{x}^{\mathcal{P}} \doteq \mathsf{x} \stackrel{\mathrm{r}}{=} \lceil \dot{\mathsf{x}} \rceil]$$

We macro define $[\dot{a}]$[17] to be a Peano variable:

$$[\dot{a} \ddot{=} \dot{\mathsf{a}}]$$

Appendix A.2 defines Peano variables for the other letters of the English alphabet.

$[\dot{\mathrm{nonfree}}(\mathsf{x},\mathsf{y})]$[18] is true if the Peano variable $[\mathsf{x}]$ does not occur free in the Peano term/formula $[\mathsf{y}]$. $[\dot{\mathrm{nonfree}}^*(\mathsf{x},\mathsf{y})]$[19] is true if the Peano variable $[\mathsf{x}]$ does not occur free in the list $[\mathsf{y}]$ of Peano terms/formulas.

$$\begin{aligned}&[\dot{\mathrm{nonfree}}(\mathsf{x},\mathsf{y}) \doteq \\ &\textbf{if } \mathsf{y}^{\mathcal{P}} \textbf{ then } \neg \mathsf{x} \stackrel{\mathrm{t}}{=} \mathsf{y} \textbf{ else} \\ &\textbf{if } \neg \mathsf{y} \stackrel{\mathrm{r}}{=} \lceil \dot{\forall} \mathsf{x}\colon \mathsf{y} \rceil \textbf{ then } \dot{\mathrm{nonfree}}^*(\mathsf{x},\mathsf{y}^{\mathrm{t}}) \textbf{ else} \\ &\textbf{if } \mathsf{x} \stackrel{\mathrm{t}}{=} \mathsf{y}^1 \textbf{ then } \mathsf{T} \textbf{ else } \dot{\mathrm{nonfree}}(\mathsf{x},\mathsf{y}^2)]\end{aligned}$$

$$[\dot{\mathrm{nonfree}}^*(\mathsf{x},\mathsf{y}) \doteq \mathsf{x}!\mathrm{If}(\mathsf{y},\mathsf{T},\dot{\mathrm{nonfree}}(\mathsf{x},\mathsf{y}^{\mathrm{h}}) \ddot{\wedge} \dot{\mathrm{nonfree}}^*(\mathsf{x},\mathsf{y}^{\mathrm{t}}))]$$

$[\dot{\mathrm{free}}\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle]$[20] is true if the substitution $[\langle \mathsf{a}\,|\mathsf{x}{:=}\,\mathsf{b}\rangle]$ is free. $[\dot{\mathrm{free}}^*\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle]$[21] is the version where $[\mathsf{a}]$ is a list of terms.

$$\begin{aligned}&[\dot{\mathrm{free}}\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \doteq \mathsf{x}!\mathsf{b}! \\ &\textbf{if } \mathsf{a}^{\mathcal{P}} \textbf{ then } \mathsf{T} \textbf{ else} \\ &\textbf{if } \neg \mathsf{a} \stackrel{\mathrm{r}}{=} \lceil \dot{\forall} \mathsf{u}\colon \mathsf{v} \rceil \textbf{ then } \dot{\mathrm{free}}^*\langle \mathsf{a}^{\mathrm{t}}|\mathsf{x} := \mathsf{b}\rangle \textbf{ else} \\ &\textbf{if } \mathsf{a}^1 \stackrel{\mathrm{t}}{=} \mathsf{x} \textbf{ then } \mathsf{T} \textbf{ else} \\ &\textbf{if } \dot{\mathrm{nonfree}}(\mathsf{x},\mathsf{a}^2) \textbf{ then } \mathsf{T} \textbf{ else} \\ &\textbf{if } \neg\dot{\mathrm{nonfree}}(\mathsf{a}^1,\mathsf{b}) \textbf{ then } \mathsf{F} \textbf{ else} \\ &\dot{\mathrm{free}}\langle \mathsf{a}^2|\mathsf{x} := \mathsf{b}\rangle]\end{aligned}$$

$$[\dot{\mathrm{free}}^*\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \doteq \mathsf{x}!\mathsf{b}!\mathrm{If}(\mathsf{a},\mathsf{T},\dot{\mathrm{free}}\langle \mathsf{a}^{\mathrm{h}}|\mathsf{x} := \mathsf{b}\rangle \ddot{\wedge} \dot{\mathrm{free}}^*\langle \mathsf{a}^{\mathrm{t}}|\mathsf{x} := \mathsf{b}\rangle)]$$

[15] $[\dot{\mathsf{x}} \stackrel{\mathrm{pyk}}{=}$ "* peano var"$]$

[16] $[\mathsf{x}^{\mathcal{P}} \stackrel{\mathrm{pyk}}{=}$ "* is peano var"$]$

[17] $[\dot{a} \stackrel{\mathrm{pyk}}{=}$ "peano a"$]$

[18] $[\dot{\mathrm{nonfree}}(\mathsf{x},\mathsf{y}) \stackrel{\mathrm{pyk}}{=}$ "peano nonfree * in * end nonfree"$]$

[19] $[\dot{\mathrm{nonfree}}^*(\mathsf{x},\mathsf{y}) \stackrel{\mathrm{pyk}}{=}$ "peano nonfree star * in * end nonfree"$]$

[20] $[\dot{\mathrm{free}}\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \stackrel{\mathrm{pyk}}{=}$ "peano free * set * to * end free"$]$

[21] $[\dot{\mathrm{free}}^*\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \stackrel{\mathrm{pyk}}{=}$ "peano free star * set * to * end free"$]$

$[a \equiv \langle b|x := c\rangle]$[22] is true if $[a]$ equals $[\langle b\,|x{:=}c\rangle]$. $[a \equiv \langle^* b|x := c\rangle]$[23] is the version where $[a]$ and $[b]$ are lists.

$$[a \equiv \langle b|x := c\rangle \doteq a!x!c!$$
$$\textbf{if } b \stackrel{r}{=} \lceil \dot{\forall} u\colon v\rceil \ddot{\wedge} b^1 \stackrel{t}{=} x \textbf{ then } a \stackrel{t}{=} b \textbf{ else}$$
$$\textbf{if } b^{\mathcal{P}} \wedge b \stackrel{t}{=} x \textbf{ then } a \stackrel{t}{=} c \textbf{ else}$$
$$a \stackrel{r}{=} b \ddot{\wedge} a^t \equiv \langle^* b^t|x := c\rangle]$$

$$[a \equiv \langle^* b|x := c\rangle \doteq b!x!c!\mathrm{If}(a, T, a^h \equiv \langle b^h|x := c\rangle \ddot{\wedge} a^t \equiv \langle^* b^t|x := c\rangle)]$$

1.3 Mendelsons system S

System [S][24] of Mendelson [2] expresses Peano arithmetic. It comprises the axioms [A1][25], [A2][26], [A3][27], [A4][28], and [A5][29] and inference rules [MP][30] and [Gen][31] of first order predicate calculus. Furthermore, it comprises the proper axioms [S1][32], [S2][33], [S3][34], [S4][35], [S5][36], [S6][37], [S7][38], [S8][39], and [S9][40]. System [S] is defined thus:

[**Theory** S]

$[\mathrm{S}\ \textbf{rule}\ \mathrm{A1}\colon \forall \mathcal{A}\colon \forall \mathcal{B}\colon \mathcal{A} \dot{\Rightarrow} \mathcal{B} \dot{\Rightarrow} \mathcal{A}]$

$[\mathrm{S}\ \textbf{rule}\ \mathrm{A2}\colon \forall \mathcal{A}\colon \forall \mathcal{B}\colon \forall \mathcal{C}\colon (\mathcal{A} \dot{\Rightarrow} \mathcal{B} \dot{\Rightarrow} \mathcal{C}) \dot{\Rightarrow} (\mathcal{A} \dot{\Rightarrow} \mathcal{B}) \dot{\Rightarrow} \mathcal{A} \dot{\Rightarrow} \mathcal{C}]$

$[\mathrm{S}\ \textbf{rule}\ \mathrm{A3}\colon \forall \mathcal{A}\colon \forall \mathcal{B}\colon (\dot{\neg} \mathcal{B} \dot{\Rightarrow} \dot{\neg} \mathcal{A}) \dot{\Rightarrow} (\dot{\neg} \mathcal{B} \dot{\Rightarrow} \mathcal{A}) \dot{\Rightarrow} \mathcal{B}]$

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[22] $[a \equiv \langle b|x := c\rangle \stackrel{\text{pyk}}{=}$ "peano sub ∗ is ∗ where ∗ is ∗ end sub"]

[23] $[a \equiv \langle^* b|x := c\rangle \stackrel{\text{pyk}}{=}$ "peano sub star ∗ is ∗ where ∗ is ∗ end sub"]

[24] [S $\stackrel{\text{pyk}}{=}$ "system s"]

[25] [A1 $\stackrel{\text{pyk}}{=}$ "axiom a one"]

[26] [A2 $\stackrel{\text{pyk}}{=}$ "axiom a two"]

[27] [A3 $\stackrel{\text{pyk}}{=}$ "axiom a three"]

[28] [A4 $\stackrel{\text{pyk}}{=}$ "axiom a four"]

[29] [A5 $\stackrel{\text{pyk}}{=}$ "axiom a five"]

[30] [MP $\stackrel{\text{pyk}}{=}$ "rule mp"]

[31] [Gen $\stackrel{\text{pyk}}{=}$ "rule gen"]

[32] [S1 $\stackrel{\text{pyk}}{=}$ "axiom s one"]

[33] [S2 $\stackrel{\text{pyk}}{=}$ "axiom s two"]

[34] [S3 $\stackrel{\text{pyk}}{=}$ "axiom s three"]

[35] [S4 $\stackrel{\text{pyk}}{=}$ "axiom s four"]

[36] [S5 $\stackrel{\text{pyk}}{=}$ "axiom s five"]

[37] [S6 $\stackrel{\text{pyk}}{=}$ "axiom s six"]

[38] [S7 $\stackrel{\text{pyk}}{=}$ "axiom s seven"]

[39] [S8 $\stackrel{\text{pyk}}{=}$ "axiom s eight"]

[40] [S9 $\stackrel{\text{pyk}}{=}$ "axiom s nine"]

The order of quantifiers in the following axiom is such that $[\mathcal{C}]$ which the current conclusion tactic cannot guess comes first. This allows to supply a value for $[\mathcal{C}]$ without having to supply values for the other meta-variables.

[S **rule** A4: $\forall\mathcal{C}\colon\forall\mathcal{A}\colon\forall\mathcal{X}\colon\forall\mathcal{B}\colon \lceil\mathcal{A}\rceil\equiv\langle\lceil\mathcal{B}\rceil|\lceil\mathcal{X}\rceil := \lceil\mathcal{C}\rceil\rangle \Vdash \dot{\forall}\mathcal{X}\colon\mathcal{B}\mathrel{\dot{\Rightarrow}}\mathcal{A}$]

[S **rule** A5: $\forall\mathcal{X}\colon\forall\mathcal{A}\colon\forall\mathcal{B}\colon \dot{\text{nonfree}}(\mathcal{X},\mathcal{A}) \Vdash \dot{\forall}\mathcal{X}\colon(\mathcal{A}\mathrel{\dot{\Rightarrow}}\mathcal{B})\mathrel{\dot{\Rightarrow}}\mathcal{A}\mathrel{\dot{\Rightarrow}}\dot{\forall}\mathcal{X}\colon\mathcal{B}$]

[S **rule** MP: $\forall\mathcal{A}\colon\forall\mathcal{B}\colon\mathcal{A}\mathrel{\dot{\Rightarrow}}\mathcal{B}\vdash\mathcal{A}\vdash\mathcal{B}$]

[S **rule** Gen: $\forall\mathcal{X}\colon\forall\mathcal{A}\colon\mathcal{A}\vdash\dot{\forall}\mathcal{X}\colon\mathcal{A}$]

Axiom [S1] to [S8] are stated as Mendelson [2] does. This is done here to test certain parts of the Logiweb system. Serious users of Peano arithmetic are advised to take Mendelson's Lemma 3.1 as axioms instead.

[S **rule** S1: $\dot{a}\stackrel{p}{=}\dot{b}\mathrel{\dot{\Rightarrow}}\dot{a}\stackrel{p}{=}\dot{c}\mathrel{\dot{\Rightarrow}}\dot{b}\stackrel{p}{=}\dot{c}$]

[S **rule** S2: $\dot{a}\stackrel{p}{=}\dot{b}\mathrel{\dot{\Rightarrow}}\dot{a}'\stackrel{p}{=}\dot{b}'$]

[S **rule** S3: $\dot{\neg}\dot{0}\stackrel{p}{=}\dot{a}'$]

[S **rule** S4: $\dot{a}'\stackrel{p}{=}\dot{b}'\mathrel{\dot{\Rightarrow}}\dot{a}\stackrel{p}{=}\dot{b}$]

[S **rule** S5: $\dot{a}\mathbin{\dot{+}}\dot{0}\stackrel{p}{=}\dot{a}$]

[S **rule** S6: $\dot{a}\mathbin{\dot{+}}\dot{b}'\stackrel{p}{=}(\dot{a}\mathbin{\dot{+}}\dot{b})'$]

[S **rule** S7: $\dot{a}\mathbin{\dot{\cdot}}\dot{0}\stackrel{p}{=}\dot{0}$]

[S **rule** S8: $\dot{a}\mathbin{\dot{\cdot}}(\dot{b}')\stackrel{p}{=}(\dot{a}\mathbin{\dot{\cdot}}\dot{b})\mathbin{\dot{+}}\dot{a}$]

[S **rule** S9: $\forall\mathcal{A}\colon\forall\mathcal{B}\colon\forall\mathcal{C}\colon\forall\mathcal{X}\colon$
$\mathcal{B}\equiv\langle\mathcal{A}|\mathcal{X} := \dot{0}\rangle \Vdash \mathcal{C}\equiv\langle\mathcal{A}|\mathcal{X} := \mathcal{X}'\rangle \Vdash$
$\mathcal{B}\mathrel{\dot{\Rightarrow}}\dot{\forall}\mathcal{X}\colon(\mathcal{A}\mathrel{\dot{\Rightarrow}}\mathcal{C})\mathrel{\dot{\Rightarrow}}\dot{\forall}\mathcal{X}\colon\mathcal{A}$]

1.4 A lemma and a proof

We now prove Lemma [L3.2(a)][41] which is an instance of the corresponding proposition in Mendelson [2]:

[S **lemma** L3.2(a): $\dot{x}\stackrel{p}{=}\dot{x}$]

[41] [L3.2(a) $\stackrel{\text{pyk}}{=}$ "lemma 1 three two a"]

S **proof of** L3.2(a):

L01:	S5 $\gg$	$\dot{a} \dot{+} \dot{0} \stackrel{p}{=} \dot{a}$	;
L02:	Gen $\rhd$ L01 $\gg$	$\dot{\forall} \dot{a} \colon \dot{a} \dot{+} \dot{0} \stackrel{p}{=} \dot{a}$	;
L03:	A4 @ $\dot{x}$ $\gg$	$\dot{\forall} \dot{a} \colon \dot{a} \dot{+} \dot{0} \stackrel{p}{=} \dot{a} \dot{\Rightarrow} \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x}$	;
L04:	MP $\rhd$ L03 $\rhd$ L02 $\gg$	$\dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x}$	;
L05:	S1 $\gg$	$\dot{a} \stackrel{p}{=} \dot{b} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{c} \dot{\Rightarrow} \dot{b} \stackrel{p}{=} \dot{c}$	;
L06:	Gen $\rhd$ L05 $\gg$	$\dot{\forall} \dot{c} \colon (\dot{a} \stackrel{p}{=} \dot{b} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{c} \dot{\Rightarrow} \dot{b} \stackrel{p}{=} \dot{c})$	;
L07:	A4 @ $\dot{x}$ $\gg$	$\dot{\forall} \dot{c} \colon (\dot{a} \stackrel{p}{=} \dot{b} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{c} \dot{\Rightarrow} \dot{b} \stackrel{p}{=} \dot{c}) \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{b} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{b} \stackrel{p}{=} \dot{x}$	;
L08:	MP $\rhd$ L07 $\rhd$ L06 $\gg$	$\dot{a} \stackrel{p}{=} \dot{b} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{b} \stackrel{p}{=} \dot{x}$	;
L09:	Gen $\rhd$ L08 $\gg$	$\dot{\forall} \dot{b} \colon (\dot{a} \stackrel{p}{=} \dot{b} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{b} \stackrel{p}{=} \dot{x})$	;
L10:	A4 @ $\dot{x}$ $\gg$	$\dot{\forall} \dot{b} \colon (\dot{a} \stackrel{p}{=} \dot{b} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{b} \stackrel{p}{=} \dot{x}) \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{p}{=} \dot{x}$	;
L11:	MP $\rhd$ L10 $\rhd$ L09 $\gg$	$\dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{p}{=} \dot{x}$	;
L12:	Gen $\rhd$ L11 $\gg$	$\dot{\forall} \dot{a} \colon (\dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{p}{=} \dot{x})$	;
L13:	A4 @ $\dot{x} \dot{+} \dot{0}$ $\gg$	$\dot{\forall} \dot{a} \colon (\dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{p}{=} \dot{x}) \dot{\Rightarrow} \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{p}{=} \dot{x}$	;
L14:	MP $\rhd$ L13 $\rhd$ L12 $\gg$	$\dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{p}{=} \dot{x}$	;
L15:	MP $\rhd$ L14 $\rhd$ L04 $\gg$	$\dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{p}{=} \dot{x}$	;
L16:	MP $\rhd$ L15 $\rhd$ L04 $\gg$	$\dot{x} \stackrel{p}{=} \dot{x}$	$\square$

1.5 An alternative axiomatic system

System [S′][42] is system [S] in which the proper axioms are taken from Lemma 3.1 in Mendelson [2]. It comprises the axioms [A1′][43], [A2′][44], [A3′][45], [A4′][46], and [A5′][47] and inference rules [MP′][48] and [Gen′][49] of first order predicate calculus. Furthermore, it comprises the proper axioms [S1′][50], [S2′][51], [S3′][52],

[42] [S′ $\stackrel{pyk}{=}$ "system prime s"]

[43] [A1′ $\stackrel{pyk}{=}$ "axiom prime a one"]

[44] [A2′ $\stackrel{pyk}{=}$ "axiom prime a two"]

[45] [A3′ $\stackrel{pyk}{=}$ "axiom prime a three"]

[46] [A4′ $\stackrel{pyk}{=}$ "axiom prime a four"]

[47] [A5′ $\stackrel{pyk}{=}$ "axiom prime a five"]

[48] [MP′ $\stackrel{pyk}{=}$ "rule prime mp"]

[49] [Gen′ $\stackrel{pyk}{=}$ "rule prime gen"]

[50] [S1′ $\stackrel{pyk}{=}$ "axiom prime s one"]

[51] [S2′ $\stackrel{pyk}{=}$ "axiom prime s two"]

[52] [S3′ $\stackrel{pyk}{=}$ "axiom prime s three"]

[S4′][53], [S5′][54], [S6′][55], [S7′][56], [S8′][57], and [S9′][58].

System [S′] is defined thus:

[**Theory** S′]

[S′ **rule** A1′: $\forall \mathcal{A}\colon \forall \mathcal{B}\colon \mathcal{A} \dot{\Rightarrow} \mathcal{B} \dot{\Rightarrow} \mathcal{A}$]

[S′ **rule** A2′: $\forall \mathcal{A}\colon \forall \mathcal{B}\colon \forall \mathcal{C}\colon (\mathcal{A} \dot{\Rightarrow} \mathcal{B} \dot{\Rightarrow} \mathcal{C}) \dot{\Rightarrow} (\mathcal{A} \dot{\Rightarrow} \mathcal{B}) \dot{\Rightarrow} \mathcal{A} \dot{\Rightarrow} \mathcal{C}$]

[S′ **rule** A3′: $\forall \mathcal{A}\colon \forall \mathcal{B}\colon (\dot{\neg} \mathcal{B} \dot{\Rightarrow} \dot{\neg} \mathcal{A}) \dot{\Rightarrow} (\dot{\neg} \mathcal{B} \dot{\Rightarrow} \mathcal{A}) \dot{\Rightarrow} \mathcal{B}$]

[S′ **rule** A4′: $\forall \mathcal{C}\colon \forall \mathcal{A}\colon \forall \mathcal{X}\colon \forall \mathcal{B}\colon \lceil \mathcal{A} \rceil \equiv \langle \lceil \mathcal{B} \rceil | \lceil \mathcal{X} \rceil := \lceil \mathcal{C} \rceil \rangle \Vdash \dot{\forall} \mathcal{X}\colon \mathcal{B} \dot{\Rightarrow} \mathcal{A}$]

[S′ **rule** A5′: $\forall \mathcal{X}\colon \forall \mathcal{A}\colon \forall \mathcal{B}\colon \mathrm{nonfree}(\lceil \mathcal{X} \rceil, \lceil \mathcal{A} \rceil) \Vdash \dot{\forall} \mathcal{X}\colon (\mathcal{A} \dot{\Rightarrow} \mathcal{B}) \dot{\Rightarrow} \mathcal{A} \dot{\Rightarrow} \dot{\forall} \mathcal{X}\colon \mathcal{B}$]

[S′ **rule** MP′: $\forall \mathcal{A}\colon \forall \mathcal{B}\colon \mathcal{A} \dot{\Rightarrow} \mathcal{B} \vdash \mathcal{A} \vdash \mathcal{B}$]

[S′ **rule** Gen′: $\forall \mathcal{X}\colon \forall \mathcal{A}\colon \mathcal{A} \vdash \dot{\forall} \mathcal{X}\colon \mathcal{A}$]

[S′ **rule** S1′: $\forall \mathcal{A}\colon \forall \mathcal{B}\colon \forall \mathcal{C}\colon \mathcal{A} \stackrel{p}{=} \mathcal{B} \dot{\Rightarrow} \mathcal{A} \stackrel{p}{=} \mathcal{C} \dot{\Rightarrow} \mathcal{B} \stackrel{p}{=} \mathcal{C}$]

[S′ **rule** S2′: $\forall \mathcal{A}\colon \forall \mathcal{B}\colon \mathcal{A} \stackrel{p}{=} \mathcal{B} \dot{\Rightarrow} \mathcal{A}' \stackrel{p}{=} \mathcal{B}'$]

[S′ **rule** S3′: $\forall \mathcal{A}\colon \neg \dot{0} \stackrel{p}{=} \mathcal{A}'$]

[S′ **rule** S4′: $\forall \mathcal{A}\colon \forall \mathcal{B}\colon \mathcal{A}' \stackrel{p}{=} \mathcal{B}' \dot{\Rightarrow} \mathcal{A} \stackrel{p}{=} \mathcal{B}$]

[S′ **rule** S5′: $\forall \mathcal{A}\colon \mathcal{A} \dot{+} \dot{0} \stackrel{p}{=} \mathcal{A}$]

[S′ **rule** S6′: $\forall \mathcal{A}\colon \forall \mathcal{B}\colon \mathcal{A} \dot{+} \mathcal{B}' \stackrel{p}{=} (\mathcal{A} \dot{+} \mathcal{B})'$]

[S′ **rule** S7′: $\forall \mathcal{A}\colon \mathcal{A} \dot{\cdot} \dot{0} \stackrel{p}{=} \dot{0}$]

[S′ **rule** S8′: $\forall \mathcal{A}\colon \forall \mathcal{B}\colon \mathcal{A} \dot{\cdot} (\mathcal{B}') \stackrel{p}{=} (\mathcal{A} \dot{\cdot} \mathcal{B}) \dot{+} \mathcal{A}$]

[S′ **rule** S9′: $\forall \mathcal{A}\colon \forall \mathcal{B}\colon \forall \mathcal{C}\colon \forall \mathcal{X}\colon$
$\mathcal{B} \equiv \langle \mathcal{A} | \mathcal{X} := \dot{0} \rangle \Vdash \mathcal{C} \equiv \langle \mathcal{A} | \mathcal{X} := \mathcal{X}' \rangle \Vdash$
$\mathcal{B} \dot{\Rightarrow} \dot{\forall} \mathcal{X}\colon (\mathcal{A} \dot{\Rightarrow} \mathcal{C}) \dot{\Rightarrow} \dot{\forall} \mathcal{X}\colon \mathcal{A}$]

Note that [A1] and [A1′] are distinct. The former says [S $\vdash \forall \mathcal{A}\colon \forall \mathcal{B}\colon \mathcal{A} \dot{\Rightarrow} \mathcal{B} \dot{\Rightarrow} \mathcal{A}$] and the latter says [S′ $\vdash \forall \mathcal{A}\colon \forall \mathcal{B}\colon \mathcal{A} \dot{\Rightarrow} \mathcal{B} \dot{\Rightarrow} \mathcal{A}$].

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[53] [S4′ $\stackrel{\text{pyk}}{=}$ "axiom prime s four"]

[54] [S5′ $\stackrel{\text{pyk}}{=}$ "axiom prime s five"]

[55] [S6′ $\stackrel{\text{pyk}}{=}$ "axiom prime s six"]

[56] [S7′ $\stackrel{\text{pyk}}{=}$ "axiom prime s seven"]

[57] [S8′ $\stackrel{\text{pyk}}{=}$ "axiom prime s eight"]

[58] [S9′ $\stackrel{\text{pyk}}{=}$ "axiom prime s nine"]

1.6 Restatement of lemma and proof

We now prove Lemma [L3.2(a)] once again under the name of [L3.2(a)′][59]:

[S′ **lemma** L3.2(a)′: $\forall \mathcal{A}\colon \mathcal{A} \stackrel{\mathrm{p}}{=} \mathcal{A}$]

S′ **proof of** L3.2(a)′:

L01:	Arbitrary ≫	$\mathcal{A}$	;
L02:	S5′ ≫	$\mathcal{A} \dot{+} \dot{0} \stackrel{\mathrm{p}}{=} \mathcal{A}$	;
L03:	S1′ ≫	$\mathcal{A} \dot{+} \dot{0} \stackrel{\mathrm{p}}{=} \mathcal{A} \dot{\Rightarrow}$ $\mathcal{A} \dot{+} \dot{0} \stackrel{\mathrm{p}}{=} \mathcal{A} \dot{\Rightarrow} \mathcal{A} \stackrel{\mathrm{p}}{=} \mathcal{A}$	;
L04:	MP′ ▷ L03 ▷ L02 ≫	$\mathcal{A} \dot{+} \dot{0} \stackrel{\mathrm{p}}{=} \mathcal{A} \dot{\Rightarrow} \mathcal{A} \stackrel{\mathrm{p}}{=} \mathcal{A}$	;
L05:	MP′ ▷ L04 ▷ L02 ≫	$\mathcal{A} \stackrel{\mathrm{p}}{=} \mathcal{A}$	□

2 Formal development

2.1 Propositional calculus

2.1.1 Modus ponens

We use [x ⊵ y][60] as shorthand for modus ponens:

[x ⊵ y $\ddot{=}$ MP′ ▷ x ▷ y]

2.1.2 Lemma M1.7

Lemma [M1.7][61] (i.e. Lemma 1.7 in Mendelson [2]) reads:

[S′ **lemma** M1.7: $\forall \mathcal{B}\colon \mathcal{B} \dot{\Rightarrow} \mathcal{B}$]

S′ **proof of** M1.7:

L01:	Arbitrary ≫	$\mathcal{B}$	;
L02:	A1′ ≫	$\mathcal{B} \dot{\Rightarrow} (\mathcal{B} \dot{\Rightarrow} \mathcal{B}) \dot{\Rightarrow} \mathcal{B}$	;
L03:	A2′ ≫	$(\mathcal{B} \dot{\Rightarrow} (\mathcal{B} \dot{\Rightarrow} \mathcal{B}) \dot{\Rightarrow} \mathcal{B}) \dot{\Rightarrow}$ $(\mathcal{B} \dot{\Rightarrow} \mathcal{B} \dot{\Rightarrow} \mathcal{B}) \dot{\Rightarrow} \mathcal{B} \dot{\Rightarrow} \mathcal{B}$	;
L04:	L03 ⊵ L02 ≫	$(\mathcal{B} \dot{\Rightarrow} \mathcal{B} \dot{\Rightarrow} \mathcal{B}) \dot{\Rightarrow} \mathcal{B} \dot{\Rightarrow} \mathcal{B}$	;
L05:	A1′ ≫	$\mathcal{B} \dot{\Rightarrow} \mathcal{B} \dot{\Rightarrow} \mathcal{B}$	;
L06:	L04 ⊵ L05 ≫	$\mathcal{B} \dot{\Rightarrow} \mathcal{B}$	□

2.1.3 Hypothetical modus ponens

The hypothetical version [$\mathrm{MP}'_{\mathrm{h}}$][62] of modus ponens MP′ has a hypothesis $\mathcal{H}$ on each premise and on the conclusion:

---

[59] [L3.2(a)′ $\stackrel{\mathrm{pyk}}{=}$ "lemma prime l three two a"]

[60] [x ⊵ y $\stackrel{\mathrm{pyk}}{=}$ "* macro modus ponens *"]

[61] [M1.7 $\stackrel{\mathrm{pyk}}{=}$ "mendelson one seven"]

[62] [$\mathrm{MP}'_{\mathrm{h}}$ $\stackrel{\mathrm{pyk}}{=}$ "hypothetical rule prime mp"]

[S′ **lemma** $\mathrm{MP}'_{\mathrm{h}}$: $\forall\mathcal{H}\colon\forall\mathcal{A}\colon\forall\mathcal{B}\colon \mathcal{H}\dot{\Rightarrow}\mathcal{A}\dot{\Rightarrow}\mathcal{B}\vdash\mathcal{H}\dot{\Rightarrow}\mathcal{A}\vdash\mathcal{H}\dot{\Rightarrow}\mathcal{B}$]

S′ **proof of** $\mathrm{MP}'_{\mathrm{h}}$:

L01:	Arbitrary ≫	$\mathcal{H}$	;
L02:	Arbitrary ≫	$\mathcal{A}$	;
L03:	Arbitrary ≫	$\mathcal{B}$	;
L04:	Premise ≫	$\mathcal{H}\dot{\Rightarrow}\mathcal{A}\dot{\Rightarrow}\mathcal{B}$	;
L05:	Premise ≫	$\mathcal{H}\dot{\Rightarrow}\mathcal{A}$	;
L06:	A2′ ≫	$(\mathcal{H}\dot{\Rightarrow}\mathcal{A}\dot{\Rightarrow}\mathcal{B})\dot{\Rightarrow}(\mathcal{H}\dot{\Rightarrow}\mathcal{A})\dot{\Rightarrow}$ $\mathcal{H}\dot{\Rightarrow}\mathcal{B}$	;
L07:	L06 $\unrhd$ L04 ≫	$(\mathcal{H}\dot{\Rightarrow}\mathcal{A})\dot{\Rightarrow}\mathcal{H}\dot{\Rightarrow}\mathcal{B}$	;
L08:	L07 $\unrhd$ L05 ≫	$\mathcal{H}\dot{\Rightarrow}\mathcal{B}$	□

We use [$\mathsf{x}\unrhd_{\mathrm{h}}\mathsf{y}$][63] as shorthand for hypothetical modus ponens:

$$[\mathsf{x}\unrhd_{\mathrm{h}}\mathsf{y}\ \ddot{=}\ \mathrm{MP}'_{\mathrm{h}}\rhd\mathsf{x}\rhd\mathsf{y}]$$

2.1.4 Turning lemmas to hypothetical lemmas

Lemma [Hypothesize][64] turns a lemma with no premises into one that assumes the hypothesis [$\mathcal{H}$] to hold:

[S′ **lemma** Hypothesize: $\forall\mathcal{H}\colon\forall\mathcal{A}\colon\mathcal{A}\vdash\mathcal{H}\dot{\Rightarrow}\mathcal{A}$]

S′ **proof of** Hypothesize:

L01:	Arbitrary ≫	$\mathcal{H}$	;
L02:	Arbitrary ≫	$\mathcal{A}$	;
L03:	Premise ≫	$\mathcal{A}$	;
L04:	A1′ ≫	$\mathcal{A}\dot{\Rightarrow}\mathcal{H}\dot{\Rightarrow}\mathcal{A}$	;
L05:	L04 $\unrhd$ L03 ≫	$\mathcal{H}\dot{\Rightarrow}\mathcal{A}$	□

2.2 First order predicate calculus

2.2.1 Hypothetical generalization

The hypothetical version [$\mathrm{Gen}'_{\mathrm{h}}$][65] of generalisation Gen′ has a hypothesis $\mathcal{H}$ on premise and conclusion:

[S′ **lemma** $\mathrm{Gen}'_{\mathrm{h}}$: $\forall\mathcal{H}\colon\forall\mathcal{X}\colon\forall\mathcal{A}\colon \mathrm{non\dot{f}ree}(\lceil\mathcal{X}\rceil,\lceil\mathcal{H}\rceil)\Vdash\mathcal{H}\dot{\Rightarrow}\mathcal{A}\vdash\mathcal{H}\dot{\Rightarrow}\dot{\forall}\mathcal{X}\colon\mathcal{A}$]

S′ **proof of** $\mathrm{Gen}'_{\mathrm{h}}$:

L01:	Arbitrary ≫	$\mathcal{H}$	;
L02:	Arbitrary ≫	$\mathcal{X}$	;
L03:	Arbitrary ≫	$\mathcal{A}$	;
L04:	Side-condition ≫	$\mathrm{non\dot{f}ree}(\lceil\mathcal{X}\rceil,\lceil\mathcal{H}\rceil)$	;

[63] [$\mathsf{x}\unrhd_{\mathrm{h}}\mathsf{y}\overset{\mathrm{pyk}}{=}$ "∗ hypothetical modus ponens ∗"]

[64] [Hypothesize $\overset{\mathrm{pyk}}{=}$ "hypothesize"]

[65] [$\mathrm{Gen}'_{\mathrm{h}}\overset{\mathrm{pyk}}{=}$ "hypothetical rule prime gen"]

L05:	Premise $\gg$	$\mathcal{H} \dot{\Rightarrow} \mathcal{A}$	;
L06:	A5′ $\rhd\!\!\rhd$ L04 $\gg$	$\dot{\forall} \mathcal{X} \colon (\mathcal{H} \dot{\Rightarrow} \mathcal{A}) \dot{\Rightarrow} \mathcal{H} \dot{\Rightarrow} \dot{\forall} \mathcal{X} \colon \mathcal{A}$	;
L07:	Gen′ $\rhd$ L05 $\gg$	$\dot{\forall} \mathcal{X} \colon (\mathcal{H} \dot{\Rightarrow} \mathcal{A})$	;
L08:	L06 $\unrhd$ L07 $\gg$	$\mathcal{H} \dot{\Rightarrow} \dot{\forall} \mathcal{X} \colon \mathcal{A}$	$\Box$

2.3 Peano arithmetic

2.3.1 Lemma M3.2(a)

Lemma [M3.2(a)][66] and the associated hypothetical lemma [M3.2(a)$_\mathrm{h}$][67] read:

[S′ **rule** M3.2(a): $\forall \mathcal{T} \colon \mathcal{T} \stackrel{\mathrm{p}}{=} \mathcal{T}$]

[S′ **lemma** M3.2(a)$_\mathrm{h}$: $\forall \mathcal{H} \colon \forall \mathcal{T} \colon \mathcal{H} \dot{\Rightarrow} \mathcal{T} \stackrel{\mathrm{p}}{=} \mathcal{T}$]

Above we cheat in stating M3.2(a) as a rule and not as a lemma. A reasonable way to construct a large proof is to start stating everything as rules and then changing the rules to lemmas one at a time until only the rules of the theory are left.

S′ **proof of** M3.2(a)$_\mathrm{h}$:

L01:	Arbitrary $\gg$	$\mathcal{H}$	;
L02:	Arbitrary $\gg$	$\mathcal{T}$	;
L03:	M3.2(a) $\gg$	$\mathcal{T} \stackrel{\mathrm{p}}{=} \mathcal{T}$	;
L04:	Hypothesize $\rhd$ L03 $\gg$	$\mathcal{H} \dot{\Rightarrow} \mathcal{T} \stackrel{\mathrm{p}}{=} \mathcal{T}$	$\Box$

2.3.2 Lemma M3.2(b)

Lemma [M3.2(b)$_\mathrm{h}$][68] reads:

[S′ **rule** M3.2(b)$_\mathrm{h}$: $\forall \mathcal{H} \colon \forall \mathcal{T} \colon \forall \mathcal{R} \colon \mathcal{H} \dot{\Rightarrow} \mathcal{T} \stackrel{\mathrm{p}}{=} \mathcal{R} \vdash \mathcal{H} \dot{\Rightarrow} \mathcal{R} \stackrel{\mathrm{p}}{=} \mathcal{T}$]

S′ **proof of** M3.2(b)$_\mathrm{h}$:

L01:	Arbitrary $\gg$	$\mathcal{H}$	;
L02:	Arbitrary $\gg$	$\mathcal{T}$	;
L03:	Arbitrary $\gg$	$\mathcal{R}$	;
L04:	Premise $\gg$	$\mathcal{H} \dot{\Rightarrow} \mathcal{T} \stackrel{\mathrm{p}}{=} \mathcal{R}$	;
L05:	S1′ $\gg$	$\mathcal{T} \stackrel{\mathrm{p}}{=} \mathcal{R} \dot{\Rightarrow} \mathcal{T} \stackrel{\mathrm{p}}{=} \mathcal{T} \dot{\Rightarrow} \mathcal{R} \stackrel{\mathrm{p}}{=} \mathcal{T}$	;
L06:	Hypothesize $\rhd$ L05 $\gg$	$\mathcal{H} \dot{\Rightarrow} \mathcal{T} \stackrel{\mathrm{p}}{=} \mathcal{R} \dot{\Rightarrow} \mathcal{T} \stackrel{\mathrm{p}}{=} \mathcal{T} \dot{\Rightarrow} \mathcal{R} \stackrel{\mathrm{p}}{=} \mathcal{T}$	;
L07:	L06 $\unrhd_\mathrm{h}$ L04 $\gg$	$\mathcal{H} \dot{\Rightarrow} \mathcal{T} \stackrel{\mathrm{p}}{=} \mathcal{T} \dot{\Rightarrow} \mathcal{R} \stackrel{\mathrm{p}}{=} \mathcal{T}$	;
L08:	M3.2(a)$_\mathrm{h}$ $\gg$	$\mathcal{H} \dot{\Rightarrow} \mathcal{T} \stackrel{\mathrm{p}}{=} \mathcal{T}$	;
L09:	L07 $\unrhd_\mathrm{h}$ L08 $\gg$	$\mathcal{H} \dot{\Rightarrow} \mathcal{R} \stackrel{\mathrm{p}}{=} \mathcal{T}$	$\Box$

[66] [M3.2(a) $\stackrel{\mathrm{pyk}}{=}$ "mendelson three two a"]

[67] [M3.2(a)$_\mathrm{h}$ $\stackrel{\mathrm{pyk}}{=}$ "hypothetical three two a"]

[68] [M3.2(b)$_\mathrm{h}$ $\stackrel{\mathrm{pyk}}{=}$ "hypothetical three two b"]

2.3.3 Lemma M3.1(S1)

Lemma [M3.1(S1′)$_h$][69] is the hypothetical version of Mendelson Lemma 3.1(S1′):

[S′ **rule** M3.1(S1′)$_h$: $\forall \mathcal{H}\colon \forall \mathcal{T}\colon \forall \mathcal{R}\colon \forall \mathcal{S}\colon \mathcal{H} \dot{\Rightarrow} \mathcal{T} \stackrel{p}{=} \mathcal{R} \vdash \mathcal{H} \dot{\Rightarrow} \mathcal{T} \stackrel{p}{=} \mathcal{S} \vdash \mathcal{H} \dot{\Rightarrow} \mathcal{R} \stackrel{p}{=} \mathcal{S}$]

2.3.4 Lemma M3.2(c)

Lemma [M3.2(c)$_h$][70] is the hypothetical version of Mendelson Lemma 3.2(c) which expresses ordinary transitivity:

[S′ **rule** M3.2(c)$_h$: $\forall \mathcal{H}\colon \forall \mathcal{T}\colon \forall \mathcal{R}\colon \forall \mathcal{S}\colon \mathcal{H} \dot{\Rightarrow} \mathcal{T} \stackrel{p}{=} \mathcal{R} \vdash \mathcal{H} \dot{\Rightarrow} \mathcal{R} \stackrel{p}{=} \mathcal{S} \vdash \mathcal{H} \dot{\Rightarrow} \mathcal{T} \stackrel{p}{=} \mathcal{S}$]

2.3.5 Lemma M3.1(S2)

Lemma [M3.1(S2′)$_h$][71] is the hypothetical version of Mendelson Lemma 3.1(S2′):

[S′ **rule** M3.1(S2′)$_h$: $\forall \mathcal{H}\colon \forall \mathcal{T}\colon \forall \mathcal{R}\colon \mathcal{H} \dot{\Rightarrow} \mathcal{T} \stackrel{p}{=} \mathcal{R} \vdash \mathcal{H} \dot{\Rightarrow} \mathcal{T}' \stackrel{p}{=} \mathcal{R}'$]

2.3.6 Lemma M3.1(S5)

Lemma [M3.1(S5′)$_h$][72] is the hypothetical version of Mendelson Lemma 3.1(S5′):

[S′ **rule** M3.1(S5′)$_h$: $\forall \mathcal{H}\colon \forall \mathcal{T}\colon \mathcal{H} \dot{\Rightarrow} \mathcal{T} \dot{+} \dot{0} \stackrel{p}{=} \mathcal{T}$]

2.3.7 Lemma M3.1(S6)

Lemma [M3.1(S6′)$_h$][73] is the hypothetical version of Mendelson Lemma 3.1(S6′):

[S′ **rule** M3.1(S6′)$_h$: $\forall \mathcal{H}\colon \forall \mathcal{T}\colon \forall \mathcal{R}\colon \mathcal{H} \dot{\Rightarrow} \mathcal{T} \dot{+} \mathcal{R}' \stackrel{p}{=} (\mathcal{T} \dot{+} \mathcal{R})'$]

2.3.8 Lemma M3.2(f)

Lemma [M3.2(f)][74] is the closure of Mendelson Lemma 3.2(f) for the concrete variable [$\dot{t}$]:

[S′ **lemma** M3.2(f): $\dot{\forall} \dot{t}\colon \dot{t} \stackrel{p}{=} \dot{0} \dot{+} \dot{t}$]

The proof below uses local macro definitions.

69 [M3.1(S1′)$_h$ $\stackrel{pyk}{=}$ "hypothetical three one s one"]

70 [M3.2(c)$_h$ $\stackrel{pyk}{=}$ "hypothetical three two c"]

71 [M3.1(S2′)$_h$ $\stackrel{pyk}{=}$ "hypothetical three one s two"]

72 [M3.1(S5′)$_h$ $\stackrel{pyk}{=}$ "hypothetical three one s five"]

73 [M3.1(S6′)$_h$ $\stackrel{pyk}{=}$ "hypothetical three one s six"]

74 [M3.2(f) $\stackrel{pyk}{=}$ "mendelson three two f"]

S′ **proof of** M3.2(f):

L01:	Local $\gg$	$\mathcal{Z} = \mathsf{x} \dot{\Rightarrow} \mathsf{x} \dot{\Rightarrow} \mathsf{x}$	;
L02:	A1′ $\gg$	$\mathcal{Z}$	;
L03:	M3.1(S5′)$_{\mathrm{h}}$ $\gg$	$\mathcal{Z} \dot{\Rightarrow} \dot{0} \dot{+} \dot{0} \stackrel{\mathrm{p}}{=} \dot{0}$	;
L04:	M3.2(b)$_{\mathrm{h}}$ $\rhd$ L03 $\gg$	$\mathcal{Z} \dot{\Rightarrow} \dot{0} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{0}$	;
L05:	L04 $\unrhd$ L02 $\gg$	$\dot{0} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{0}$	;
L06:	Local $\gg$	$\mathcal{H} = \dot{t} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{t}$	;
L07:	M1.7 $\gg$	$\mathcal{H} \dot{\Rightarrow} \dot{t} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{t}$	;
L08:	M3.1(S2′)$_{\mathrm{h}}$ $\rhd$ L07 $\gg$	$\mathcal{H} \dot{\Rightarrow} \dot{t}' \stackrel{\mathrm{p}}{=} (\dot{0} \dot{+} \dot{t})'$	;
L09:	M3.1(S6′)$_{\mathrm{h}}$ $\gg$	$\mathcal{H} \dot{\Rightarrow} \dot{0} \dot{+} \dot{t}' \stackrel{\mathrm{p}}{=} (\dot{0} \dot{+} \dot{t})'$	;
L10:	M3.2(b)$_{\mathrm{h}}$ $\rhd$ L09 $\gg$	$\mathcal{H} \dot{\Rightarrow} (\dot{0} \dot{+} \dot{t})' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{t}'$	;
L11:	M3.2(c)$_{\mathrm{h}}$ $\rhd$ L08 $\rhd$ L10 $\gg$	$\mathcal{H} \dot{\Rightarrow} \dot{t}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{t}'$	;
L12:	Gen′ $\rhd$ L11 $\gg$	$\dot{\forall} \dot{t}\colon (\mathcal{H} \dot{\Rightarrow} \dot{t}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{t}')$	;
L13:	S9′ $\gg$	$\dot{0} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{0} \dot{\Rightarrow} \dot{\forall} \dot{t}\colon (\dot{t} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{t} \dot{\Rightarrow} \dot{t}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{t}') \dot{\Rightarrow} \dot{\forall} \dot{t}\colon \dot{t} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{t}$	;
L14:	L13 $\unrhd$ L05 $\gg$	$\dot{\forall} \dot{t}\colon (\mathcal{H} \dot{\Rightarrow} \dot{t}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{t}') \dot{\Rightarrow} \dot{\forall} \dot{t}\colon \dot{t} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{t}$	;
L15:	L14 $\unrhd$ L12 $\gg$	$\dot{\forall} \dot{t}\colon \dot{t} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{t}$	$\square$

A Chores

A.1 The name of the page

This defines the name of the page:

$$[\mathrm{peano} \stackrel{\mathrm{pyk}}{=} \text{“peano”}]$$

A.2 Variables of Peano arithmetic

We use $[\dot{b}]$[75], $[\dot{c}]$[76], $[\dot{d}]$[77], $[\dot{e}]$[78], $[\dot{f}]$[79], $[\dot{g}]$[80], $[\dot{h}]$[81], $[\dot{i}]$[82], $[\dot{j}]$[83], $[\dot{k}]$[84], $[\dot{l}]$[85], $[\dot{m}]$[86], $[\dot{n}]$[87], $[\dot{o}]$[88], $[\dot{p}]$[89], $[\dot{q}]$[90], $[\dot{r}]$[91], $[\dot{s}]$[92], $[\dot{t}]$[93], $[\dot{u}]$[94], $[\dot{v}]$[95], $[\dot{w}]$[96], $[\dot{x}]$[97], $[\dot{y}]$[98], and $[\dot{z}]$[99] to denote variables of Peano arithmetic:

$[\dot{b} \ddot{=} \dot{\mathsf{b}}]$, $[\dot{c} \ddot{=} \dot{\mathsf{c}}]$, $[\dot{d} \ddot{=} \dot{\mathsf{d}}]$, $[\dot{e} \ddot{=} \dot{\mathsf{e}}]$, $[\dot{f} \ddot{=} \dot{\mathsf{f}}]$, $[\dot{g} \ddot{=} \dot{\mathsf{g}}]$, $[\dot{h} \ddot{=} \dot{\mathsf{h}}]$, $[\dot{i} \ddot{=} \dot{\mathsf{i}}]$, $[\dot{j} \ddot{=} \dot{\mathsf{j}}]$, $[\dot{k} \ddot{=} \dot{\mathsf{k}}]$, $[\dot{l} \ddot{=} \dot{\mathsf{l}}]$, $[\dot{m} \ddot{=} \dot{\mathsf{m}}]$, $[\dot{n} \ddot{=} \dot{\mathsf{n}}]$, $[\dot{o} \ddot{=} \dot{\mathsf{o}}]$, $[\dot{p} \ddot{=} \dot{\mathsf{p}}]$, $[\dot{q} \ddot{=} \dot{\mathsf{q}}]$, $[\dot{r} \ddot{=} \dot{\mathsf{r}}]$, $[\dot{s} \ddot{=} \dot{\mathsf{s}}]$, $[\dot{t} \ddot{=} \dot{\mathsf{t}}]$, $[\dot{u} \ddot{=} \dot{\mathsf{u}}]$, $[\dot{v} \ddot{=} \dot{\mathsf{v}}]$, $[\dot{w} \ddot{=} \dot{\mathsf{w}}]$, $[\dot{x} \ddot{=} \dot{\mathsf{x}}]$, $[\dot{y} \ddot{=} \dot{\mathsf{y}}]$, and $[\dot{z} \ddot{=} \dot{\mathsf{z}}]$.

A.3 TEX definitions

$[\dot{0} \stackrel{\text{tex}}{=}$ "
\dot{0}"]

$[\mathsf{x}' \stackrel{\text{tex}}{=}$ "#1.'"]

$[\mathsf{x} \dot{+} \mathsf{y} \stackrel{\text{tex}}{=}$ "#1.
\mathop{\dot{+}} #2."]

---

[75] $[\dot{b} \stackrel{\text{pyk}}{=}$ "peano b"]

[76] $[\dot{c} \stackrel{\text{pyk}}{=}$ "peano c"]

[77] $[\dot{d} \stackrel{\text{pyk}}{=}$ "peano d"]

[78] $[\dot{e} \stackrel{\text{pyk}}{=}$ "peano e"]

[79] $[\dot{f} \stackrel{\text{pyk}}{=}$ "peano f"]

[80] $[\dot{g} \stackrel{\text{pyk}}{=}$ "peano g"]

[81] $[\dot{h} \stackrel{\text{pyk}}{=}$ "peano h"]

[82] $[\dot{i} \stackrel{\text{pyk}}{=}$ "peano i"]

[83] $[\dot{j} \stackrel{\text{pyk}}{=}$ "peano j"]

[84] $[\dot{k} \stackrel{\text{pyk}}{=}$ "peano k"]

[85] $[\dot{l} \stackrel{\text{pyk}}{=}$ "peano l"]

[86] $[\dot{m} \stackrel{\text{pyk}}{=}$ "peano m"]

[87] $[\dot{n} \stackrel{\text{pyk}}{=}$ "peano n"]

[88] $[\dot{o} \stackrel{\text{pyk}}{=}$ "peano o"]

[89] $[\dot{p} \stackrel{\text{pyk}}{=}$ "peano p"]

[90] $[\dot{q} \stackrel{\text{pyk}}{=}$ "peano q"]

[91] $[\dot{r} \stackrel{\text{pyk}}{=}$ "peano r"]

[92] $[\dot{s} \stackrel{\text{pyk}}{=}$ "peano s"]

[93] $[\dot{t} \stackrel{\text{pyk}}{=}$ "peano t"]

[94] $[\dot{u} \stackrel{\text{pyk}}{=}$ "peano u"]

[95] $[\dot{v} \stackrel{\text{pyk}}{=}$ "peano v"]

[96] $[\dot{w} \stackrel{\text{pyk}}{=}$ "peano w"]

[97] $[\dot{x} \stackrel{\text{pyk}}{=}$ "peano x"]

[98] $[\dot{y} \stackrel{\text{pyk}}{=}$ "peano y"]

[99] $[\dot{z} \stackrel{\text{pyk}}{=}$ "peano z"]

[$\mathsf{x} \mathop{\dot{\cdot}} \mathsf{y} \stackrel{\text{tex}}{=}$ “#1.
\mathop{\dot{\cdot}} #2.”]

[$\mathsf{x} \stackrel{p}{=} \mathsf{y} \stackrel{\text{tex}}{=}$ “#1.
\stackrel{p}{=} #2.”]

[$\dot{\neg}\, \mathsf{x} \stackrel{\text{tex}}{=}$ “
\dot{\neg}\, #1.”]

[$\mathsf{x} \mathrel{\dot{\Rightarrow}} \mathsf{y} \stackrel{\text{tex}}{=}$ “#1.
\mathrel{\dot{\Rightarrow}} #2.”]

[$\dot{\forall} \mathsf{x} \colon \mathsf{y} \stackrel{\text{tex}}{=}$ “
\dot{\forall} #1.
\colon #2.”]

[$\dot{1} \stackrel{\text{tex}}{=}$ “
\dot{1}”]

[$\dot{2} \stackrel{\text{tex}}{=}$ “
\dot{2}”]

[$\mathsf{x} \mathrel{\dot{\wedge}} \mathsf{y} \stackrel{\text{tex}}{=}$ “#1.
\mathrel{\dot{\wedge}} #2.”]

[$\mathsf{x} \mathrel{\dot{\vee}} \mathsf{y} \stackrel{\text{tex}}{=}$ “#1.
\mathrel{\dot{\vee}} #2.”]

[$\mathsf{x} \mathrel{\dot{\Leftrightarrow}} \mathsf{y} \stackrel{\text{tex}}{=}$ “#1.
\mathrel{\dot{\Leftrightarrow}} #2.”]

[$\dot{\exists} \mathsf{x} \colon \mathsf{y} \stackrel{\text{tex}}{=}$ “
\dot{\exists} #1.
\colon #2.”]

[$\dot{\mathsf{x}} \stackrel{\text{tex}}{=}$ “
\dot{#1.
}”]

[$\mathsf{x}^{\mathcal{P}} \stackrel{\text{tex}}{=}$ “#1.
{} ^ {\cal P}”]

[$\dot{\mathit{a}} \stackrel{\text{tex}}{=}$ “
\dot{\mathit{a}}”]

[$\dot{\mathrm{nonfree}}(\mathsf{x}, \mathsf{y}) \stackrel{\text{tex}}{=}$ “
\dot{nonfree}(#1.
, #2.
)”]

[$\dot{\text{nonfree}}^*(\mathsf{x},\mathsf{y}) \stackrel{\text{tex}}{=}$ "
\dot{nonfree}ˆ*(#1.
, #2.
)"]

[$\dot{\text{free}}\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \stackrel{\text{tex}}{=}$ "
\dot{free}\langle #1.
| #2.
:= #3.
\rangle"]

[$\dot{\text{free}}^*\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \stackrel{\text{tex}}{=}$ "
\dot{free}{}ˆ*\langle #1.
| #2.
:= #3.
\rangle"]

[$\mathsf{a}\equiv\langle \mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \stackrel{\text{tex}}{=}$ "#1.
{\equiv}\langle #2.
|#3.
:=#4.
\rangle"]

[$\mathsf{a}\equiv\langle^* \mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \stackrel{\text{tex}}{=}$ "#1.
{\equiv}\langleˆ* #2.
|#3.
:=#4.
\rangle"]

[S $\stackrel{\text{tex}}{=}$ "
S"]

[A1 $\stackrel{\text{tex}}{=}$ "
A1"]

[A2 $\stackrel{\text{tex}}{=}$ "
A2"]

[A3 $\stackrel{\text{tex}}{=}$ "
A3"]

[A4 $\stackrel{\text{tex}}{=}$ "
A4"]

[A5 $\stackrel{\text{tex}}{=}$ "
A5"]

[MP $\stackrel{\text{tex}}{=}$ "
MP"]

[Gen $\stackrel{\text{tex}}{=}$ "
Gen"]

[S1 $\stackrel{\text{tex}}{=}$ "
S1"]

[S2 $\stackrel{\text{tex}}{=}$ "
S2"]

[S3 $\stackrel{\text{tex}}{=}$ "
S3"]

[S4 $\stackrel{\text{tex}}{=}$ "
S4"]

[S5 $\stackrel{\text{tex}}{=}$ "
S5"]

[S6 $\stackrel{\text{tex}}{=}$ "
S6"]

[S7 $\stackrel{\text{tex}}{=}$ "
S7"]

[S8 $\stackrel{\text{tex}}{=}$ "
S8"]

[S9 $\stackrel{\text{tex}}{=}$ "
S9"]

[L3.2(a) $\stackrel{\text{tex}}{=}$ "
L3.2(a)"]

[S′ $\stackrel{\text{tex}}{=}$ "
S'"]

[A1′ $\stackrel{\text{tex}}{=}$ "
A1'"]

[A2′ $\stackrel{\text{tex}}{=}$ "
A2'"]

[A3′ $\stackrel{\text{tex}}{=}$ "
A3'"]

[A4′ $\stackrel{\text{tex}}{=}$ "
A4'"]

[A5′ $\stackrel{\text{tex}}{=}$ "
A5'"]

[MP′ $\stackrel{\text{tex}}{=}$ "
MP'"]

[Gen′ $\stackrel{\text{tex}}{=}$ "
Gen'"]

[S1′ $\stackrel{\text{tex}}{=}$ "
S1'"]

[S2′ $\stackrel{\text{tex}}{=}$ "
S2'"]

[S3′ $\stackrel{\text{tex}}{=}$ "
S3'"]

[S4′ $\stackrel{\text{tex}}{=}$ "
S4'"]

[S5′ $\stackrel{\text{tex}}{=}$ "
S5'"]

[S6′ $\stackrel{\text{tex}}{=}$ "
S6'"]

[S7′ $\stackrel{\text{tex}}{=}$ "
S7'"]

[S8′ $\stackrel{\text{tex}}{=}$ "
S8'"]

[S9′ $\stackrel{\text{tex}}{=}$ "
S9'"]

[L3.2(a)′ $\stackrel{\text{tex}}{=}$ "
L3.2(a)'"]

[$\mathsf{x} \unrhd \mathsf{y}$ $\stackrel{\text{tex}}{=}$ "#1.
\unrhd #2."]

[M1.7 $\stackrel{\text{tex}}{=}$ "
M1.7"]

[$\mathrm{MP}'_{\mathrm{h}} \stackrel{\mathrm{tex}}{=}$ “
MP’_h”]

[$\mathsf{x} \unrhd_{\mathrm{h}} \mathsf{y} \stackrel{\mathrm{tex}}{=}$ “#1.
\unrhd_h #2.”]

[Hypothesize $\stackrel{\mathrm{tex}}{=}$ “
Hypothesize”]

[$\mathrm{Gen}'_{\mathrm{h}} \stackrel{\mathrm{tex}}{=}$ “
Gen’_h”]

[M3.2(a) $\stackrel{\mathrm{tex}}{=}$ “
M3.2(a)”]

[$\mathrm{M3.2(a)}_{\mathrm{h}} \stackrel{\mathrm{tex}}{=}$ “
M3.2(a)_h”]

[$\mathrm{M3.2(b)}_{\mathrm{h}} \stackrel{\mathrm{tex}}{=}$ “
M3.2(b)_h”]

[$\mathrm{M3.1(S1')}_{\mathrm{h}} \stackrel{\mathrm{tex}}{=}$ “
M3.1(S1’)_h”]

[$\mathrm{M3.2(c)}_{\mathrm{h}} \stackrel{\mathrm{tex}}{=}$ “
M3.2(c)_h”]

[$\mathrm{M3.1(S2')}_{\mathrm{h}} \stackrel{\mathrm{tex}}{=}$ “
M3.1(S2’)_h”]

[$\mathrm{M3.1(S5')}_{\mathrm{h}} \stackrel{\mathrm{tex}}{=}$ “
M3.1(S5’)_h”]

[$\mathrm{M3.1(S6')}_{\mathrm{h}} \stackrel{\mathrm{tex}}{=}$ “
M3.1(S6’)_h”]

[M3.2(f) $\stackrel{\mathrm{tex}}{=}$ “
M3.2(f)”]

[$\dot{b} \stackrel{\mathrm{tex}}{=}$ “
\dot{\mathit{b}}”]

[$\dot{c} \stackrel{\mathrm{tex}}{=}$ “
\dot{\mathit{c}}”]

[$\dot{d} \stackrel{\mathrm{tex}}{=}$ “
\dot{\mathit{d}}”]

[$\dot{e} \overset{\text{tex}}{=}$ “
\dot{\mathit{e}}”]

[$\dot{f} \overset{\text{tex}}{=}$ “
\dot{\mathit{f}}”]

[$\dot{g} \overset{\text{tex}}{=}$ “
\dot{\mathit{g}}”]

[$\dot{h} \overset{\text{tex}}{=}$ “
\dot{\mathit{h}}”]

[$\dot{i} \overset{\text{tex}}{=}$ “
\dot{\mathit{i}}”]

[$\dot{j} \overset{\text{tex}}{=}$ “
\dot{\mathit{j}}”]

[$\dot{k} \overset{\text{tex}}{=}$ “
\dot{\mathit{k}}”]

[$\dot{l} \overset{\text{tex}}{=}$ “
\dot{\mathit{l}}”]

[$\dot{m} \overset{\text{tex}}{=}$ “
\dot{\mathit{m}}”]

[$\dot{n} \overset{\text{tex}}{=}$ “
\dot{\mathit{n}}”]

[$\dot{o} \overset{\text{tex}}{=}$ “
\dot{\mathit{o}}”]

[$\dot{p} \overset{\text{tex}}{=}$ “
\dot{\mathit{p}}”]

[$\dot{q} \overset{\text{tex}}{=}$ “
\dot{\mathit{q}}”]

[$\dot{r} \overset{\text{tex}}{=}$ “
\dot{\mathit{r}}”]

[$\dot{s} \overset{\text{tex}}{=}$ “
\dot{\mathit{s}}”]

[$\dot{t} \overset{\text{tex}}{=}$ “
\dot{\mathit{t}}”]

$[\dot{u} \stackrel{\text{tex}}{=}$ “
\dot{\mathit{u}}”]

$[\dot{v} \stackrel{\text{tex}}{=}$ “
\dot{\mathit{v}}”]

$[\dot{w} \stackrel{\text{tex}}{=}$ “
\dot{\mathit{w}}”]

$[\dot{x} \stackrel{\text{tex}}{=}$ “
\dot{\mathit{x}}”]

$[\dot{y} \stackrel{\text{tex}}{=}$ “
\dot{\mathit{y}}”]

$[\dot{z} \stackrel{\text{tex}}{=}$ “
\dot{\mathit{z}}”]

A.4 Test

$[\lceil \dot{a} \rceil^{\mathcal{P}}]^{\cdot}$

$[\lceil \mathsf{a} \rceil^{\mathcal{P}}]^{-}$

$[\dot{\text{nonfree}}(\lceil \dot{x} \rceil, \lceil \dot{y} \stackrel{\text{p}}{=} \dot{z} \dot{\Rightarrow} \dot{\forall} \dot{x} \colon \dot{x} \stackrel{\text{p}}{=} \dot{y} \rceil)]^{\cdot}$

$[\dot{\text{nonfree}}(\lceil \dot{x} \rceil, \lceil \dot{x} \stackrel{\text{p}}{=} \dot{z} \dot{\Rightarrow} \dot{\forall} \dot{x} \colon \dot{x} \stackrel{\text{p}}{=} \dot{y} \rceil)]^{-}$

$[\dot{\text{nonfree}}(\lceil \dot{x} \rceil, \lceil \dot{y} \stackrel{\text{p}}{=} \dot{x} \dot{\Rightarrow} \dot{\forall} \dot{x} \colon \dot{x} \stackrel{\text{p}}{=} \dot{y} \rceil)]^{-}$

$[\dot{\text{nonfree}}(\lceil \dot{x} \rceil, \lceil \dot{y} \stackrel{\text{p}}{=} \dot{z} \dot{\Rightarrow} \dot{\forall} \dot{y} \colon \dot{x} \stackrel{\text{p}}{=} \dot{y} \rceil)]^{-}$

$[\dot{\text{free}}\langle \lceil \dot{\forall} \dot{x} \colon \mathsf{b} :: \dot{x} :: \mathsf{c} \rceil | \lceil \dot{x} \rceil := \lceil \mathsf{x} :: \dot{y} :: \mathsf{z} \rceil \rangle]^{\cdot}$

$[\dot{\text{free}}\langle \lceil \dot{\forall} \dot{y} \colon \mathsf{b} :: \dot{x} :: \mathsf{c} \rceil | \lceil \dot{x} \rceil := \lceil \mathsf{x} :: \dot{y} :: \mathsf{z} \rceil \rangle]^{-}$

$[\dot{\text{free}}\langle \lceil \dot{\forall} \dot{x} \colon \mathsf{b} :: \dot{x} :: \mathsf{c} \rceil | \lceil \dot{y} \rceil := \lceil \mathsf{x} :: \dot{y} :: \mathsf{z} \rceil \rangle]^{\cdot}$

$[\dot{\text{free}}\langle \lceil \dot{\forall} \dot{y} \colon \mathsf{b} :: \dot{x} :: \mathsf{c} \rceil | \lceil \dot{y} \rceil := \lceil \mathsf{x} :: \dot{y} :: \mathsf{z} \rceil \rangle]^{\cdot}$

$[\dot{a} \equiv \langle \dot{a} | \dot{b} := \dot{c} \rangle]^{\cdot}$

$[\dot{c} \equiv \langle \dot{b} | \dot{b} := \dot{c} \rangle]^{\cdot}$

$[\forall \dot{a} \colon \dot{a} \stackrel{\text{p}}{=} \dot{b} \equiv \langle \forall \dot{a} \colon \dot{a} \stackrel{\text{p}}{=} \dot{b} | \dot{a} := \dot{c} \rangle]^{\cdot}$

$[\forall \dot{a} \colon \dot{a} \stackrel{\text{p}}{=} \dot{c} \equiv \langle \forall \dot{a} \colon \dot{a} \stackrel{\text{p}}{=} \dot{b} | \dot{b} := \dot{c} \rangle]^{\cdot}$

$[\dot{\forall} \dot{a} \colon \dot{a} \stackrel{\text{p}}{=} \dot{0} \dot{+} \dot{a} \dot{\Rightarrow} \dot{c} \mathrel{\dot{:}} \dot{d} \stackrel{\text{p}}{=} \dot{0} \dot{+} \dot{c} \mathrel{\dot{:}} \dot{d} \equiv \langle \dot{\forall} \dot{a} \colon \dot{a} \stackrel{\text{p}}{=} \dot{0} \dot{+} \dot{a} \dot{\Rightarrow} \dot{b} \stackrel{\text{p}}{=} \dot{0} \dot{+} \dot{b} | \dot{b} := \dot{c} \mathrel{\dot{:}} \dot{d} \rangle]^{\cdot}$

$[\dot\forall \dot a\colon \dot a \overset{\mathrm{p}}{=} \dot 0 \dot+ \dot a \Rightarrow \dot b \overset{\mathrm{p}}{=} \dot 0 \dot+ \dot b \equiv \langle \dot\forall \dot a\colon \dot a \overset{\mathrm{p}}{=} \dot 0 \dot+ \dot a \Rightarrow \dot b \overset{\mathrm{p}}{=} \dot 0 \dot+ \dot b | \dot a := \dot c\rangle]^{\cdot}$

A.5 Priority table

Priority table
Preassociative
[peano], [base], [bracket $*$ end bracket], [big bracket $*$ end bracket], [math $*$ end math], [**flush left** $[*]$], [x], [y], [z], $[[* \bowtie *]]$, $[[* \overset{*}{\rightarrow} *]]$, [pyk], [tex], [name], [prio], $[*]$, [T], $[\mathrm{if}(*, *, *)]$, $[[* \overset{*}{\Rightarrow} *]]$, [val], [claim], $[\bot]$, $[\mathrm{f}(*)]$, $[(*)^{\mathrm{I}}]$, [F], $[\underline{0}]$, $[\underline{1}]$, $[\underline{2}]$, $[\underline{3}]$, $[\underline{4}]$, $[\underline{5}]$, $[\underline{6}]$, $[\underline{7}]$, $[\underline{8}]$, $[\underline{9}]$, [0], [1], [2], [3], [4], [5], [6], [7], [8], [9], [a], [b], [c], [d], [e], [f], [g], [h], [i], [j], [k], [l], [m], [n], [o], [p], [q], [r], [s], [t], [u], [v], [w], $[(*)^{\mathrm{M}}]$, $[\mathrm{If}(*, *, *)]$, [array$\{*\}$ $*$ end array], [l], [c], [r], [empty], $[\langle * \mid * := * \rangle]$, $[\mathcal{M}(*)]$, $[\tilde{\mathcal{U}}(*)]$, $[\mathcal{U}(*)]$, $[\mathcal{U}^{\mathrm{M}}(*)]$, $[\mathbf{apply}(*, *)]$, $[\mathbf{apply}_1(*, *)]$, $[\mathrm{identifier}(*)]$, $[\mathrm{identifier}_1(*, *)]$, [array-plus$(*, *)$], [array-remove$(*, *, *)$], [array-put$(*, *, *, *)$], [array-add$(*, *, *, *, *)$], $[\mathrm{bit}(*, *)]$, $[\mathrm{bit}_1(*, *)]$, [rack], ["vector"], ["bibliography"], ["dictionary"], ["body"], ["codex"], ["expansion"], ["code"], ["cache"], ["diagnose"], ["pyk"], ["tex"], ["texname"], ["value"], ["message"], ["macro"], ["definition"], ["unpack"], ["claim"], ["priority"], ["lambda"], ["apply"], ["true"], ["if"], ["quote"], ["proclaim"], ["define"], ["introduce"], ["hide"], ["pre"], ["post"], $[\mathcal{E}(*, *, *)]$, $[\mathcal{E}_2(*, *, *, *, *)]$, $[\mathcal{E}_3(*, *, *, *)]$, $[\mathcal{E}_4(*, *, *, *)]$, $[\mathbf{lookup}(*, *, *)]$, $[\mathbf{abstract}(*, *, *, *)]$, $[[\lceil *]]$, $[\mathcal{M}(*, *, *)]$, $[\mathcal{M}_2(*, *, *, *)]$, $[\mathcal{M}^*(*, *, *)]$, [macro], $[\mathrm{s}_0]$, $[\mathbf{zip}(*, *)]$, $[\mathbf{assoc}_1(*, *, *)]$, $[(*)^{\mathsf{P}}]$, [self], $[[* \ddot{=} *]]$, $[[* \doteq *]]$, $[[* \acute{=} *]]$, $[[* \overset{\mathrm{pyk}}{=} *]]$, $[[* \overset{\mathrm{tex}}{=} *]]$, $[[* \overset{\mathrm{name}}{=} *]]$, [**Priority table**$[*]$], $[\tilde{\mathcal{M}}_1]$, $[\tilde{\mathcal{M}}_2(*)]$, $[\tilde{\mathcal{M}}_3(*)]$, $[\tilde{\mathcal{M}}_4(*, *, *, *)]$, $[\tilde{\mathcal{M}}(*, *, *)]$, $[\tilde{\mathcal{Q}}(*, *, *)]$, $[\tilde{\mathcal{Q}}_2(*, *, *)]$, $[\tilde{\mathcal{Q}}_3(*, *, *, *)]$, $[\tilde{\mathcal{Q}}^*(*, *, *)]$, $[(*)]$, $[\mathbf{aspect}(*, *)]$, $[\mathbf{aspect}(*, *, *)]$, $[\langle * \rangle]$, $[\mathbf{tuple}_1(*)]$, $[\mathbf{tuple}_2(*)]$, $[\mathrm{let}_2(*, *)]$, $[\mathrm{let}_1(*, *)]$, $[[* \overset{\mathrm{claim}}{=} *]]$, [checker], $[\mathbf{check}(*, *)]$, $[\mathbf{check}_2(*, *, *)]$, $[\mathbf{check}_3(*, *, *)]$, $[\mathbf{check}^*(*, *)]$, $[\mathbf{check}^*_2(*, *, *)]$, $[[*]^{\cdot}]$, $[[*]^{-}]$, $[[*]^{\circ}]$, [msg], $[[* \overset{\mathrm{msg}}{=} *]]$, [<stmt>], [stmt], $[[* \overset{\mathrm{stmt}}{=} *]]$, [HeadNil$'$], [HeadPair$'$], [Transitivity$'$], $[\perp\!\!\!\perp]$, [Contra$'$], $[\mathrm{T}'_{\mathrm{E}}]$, $[\mathrm{L}_1]$, $[\underline{*}]$, $[\mathcal{A}]$, $[\mathcal{B}]$, $[\mathcal{C}]$, $[\mathcal{D}]$, $[\mathcal{E}]$, $[\mathcal{F}]$, $[\mathcal{G}]$, $[\mathcal{H}]$, $[\mathcal{I}]$, $[\mathcal{J}]$, $[\mathcal{K}]$, $[\mathcal{L}]$, $[\mathcal{M}]$, $[\mathcal{N}]$, $[\mathcal{O}]$, $[\mathcal{P}]$, $[\mathcal{Q}]$, $[\mathcal{R}]$, $[\mathcal{S}]$, $[\mathcal{T}]$, $[\mathcal{U}]$, $[\mathcal{V}]$, $[\mathcal{W}]$, $[\mathcal{X}]$, $[\mathcal{Y}]$, $[\mathcal{Z}]$, $[\langle * \mid * := * \rangle]$, $[\langle^* * \mid * := * \rangle]$, $[\emptyset]$, [Remainder], $[(*)^{\mathsf{v}}]$, $[\mathrm{intro}(*, *, *, *)]$, $[\mathrm{intro}(*, *, *)]$, $[\mathrm{error}(*, *)]$, $[\mathrm{error}_2(*, *)]$, $[\mathrm{proof}(*, *, *)]$, $[\mathrm{proof}_2(*, *)]$, $[\mathcal{S}(*, *)]$, $[\mathcal{S}^{\mathrm{I}}(*, *)]$, $[\mathcal{S}^{\rhd}(*, *)]$, $[\mathcal{S}^{\rhd}_1(*, *, *)]$, $[\mathcal{S}^{\mathrm{E}}(*, *)]$, $[\mathcal{S}^{\mathrm{E}}_1(*, *, *)]$, $[\mathcal{S}^{+}(*, *)]$, $[\mathcal{S}^{+}_1(*, *, *)]$, $[\mathcal{S}^{-}(*, *)]$, $[\mathcal{S}^{-}_1(*, *, *)]$, $[\mathcal{S}^{*}(*, *)]$, $[\mathcal{S}^{*}_1(*, *, *)]$, $[\mathcal{S}^{*}_2(*, *, *, *)]$, $[\mathcal{S}^{@}(*, *)]$, $[\mathcal{S}^{@}_1(*, *, *)]$, $[\mathcal{S}^{\vdash}(*, *)]$, $[\mathcal{S}^{\vdash}_1(*, *, *, *)]$, $[\mathcal{S}^{\Vdash}(*, *)]$, $[\mathcal{S}^{\Vdash}_1(*, *, *, *)]$, $[\mathcal{S}^{\mathrm{i.e.}}(*, *)]$, $[\mathcal{S}^{\mathrm{i.e.}}_1(*, *, *, *)]$, $[\mathcal{S}^{\mathrm{i.e.}}_2(*, *, *, *, *)]$, $[\mathcal{S}^{\forall}(*, *)]$, $[\mathcal{S}^{\forall}_1(*, *, *, *)]$, $[\mathcal{S}^{;}(*, *)]$, $[\mathcal{S}^{;}_1(*, *, *)]$, $[\mathcal{S}^{;}_2(*, *, *, *)]$, $[\mathcal{T}(*)]$, $[\mathrm{claims}(*, *, *)]$, $[\mathrm{claims}_2(*, *, *)]$, [<proof>], [proof], [[**Lemma** $*\colon *$]], [[**Proof of** $*\colon *$]], [[$*$ **lemma** $*\colon *$]], [[$*$ **antilemma** $*\colon *$]], [[$*$ **rule** $*\colon *$]], [[$*$ **antirule** $*\colon *$]], [verifier], $[\mathcal{V}_1(*)]$, $[\mathcal{V}_2(*, *)]$, $[\mathcal{V}_3(*, *, *, *)]$, $[\mathcal{V}_4(*, *)]$, $[\mathcal{V}_5(*, *, *, *)]$, $[\mathcal{V}_6(*, *, *, *)]$, $[\mathcal{V}_7(*, *, *, *)]$, $[\mathrm{Cut}(*, *)]$, $[\mathrm{Head}_{\oplus}(*)]$, $[\mathrm{Tail}_{\oplus}(*)]$, $[\mathrm{rule}_1(*, *)]$, $[\mathrm{rule}(*, *)]$, [Rule tactic], $[\mathrm{Plus}(*, *)]$, [[**Theory** $*$]], $[\mathrm{theory}_2(*, *)]$, $[\mathrm{theory}_3(*, *)]$, $[\mathrm{theory}_4(*, *, *)]$, [HeadNil$''$], [HeadPair$''$], [Transitivity$''$], [Contra$''$], [HeadNil], [HeadPair], [Transitivity], [Contra], $[\mathrm{T}_{\mathrm{E}}]$, [ragged right],

[ragged right expansion], [parm$(*, *, *)$], [parm$^*(*, *, *)$], [inst$(*, *)$], [inst$^*(*, *)$], [occur$(*, *, *)$], [occur$^*(*, *, *)$], [unify$(* = *, *)$], [unify$^*(* = *, *)$], [unify$_2(* = *, *)$], [$\mathrm{L_a}$], [$\mathrm{L_b}$], [$\mathrm{L_c}$], [$\mathrm{L_d}$], [$\mathrm{L_e}$], [$\mathrm{L_f}$], [$\mathrm{L_g}$], [$\mathrm{L_h}$], [$\mathrm{L_i}$], [$\mathrm{L_j}$], [$\mathrm{L_k}$], [$\mathrm{L_l}$], [$\mathrm{L_m}$], [$\mathrm{L_n}$], [$\mathrm{L_o}$], [$\mathrm{L_p}$], [$\mathrm{L_q}$], [$\mathrm{L_r}$], [$\mathrm{L_s}$], [$\mathrm{L_t}$], [$\mathrm{L_u}$], [$\mathrm{L_v}$], [$\mathrm{L_w}$], [$\mathrm{L_x}$], [$\mathrm{L_y}$], [$\mathrm{L_z}$], [$\mathrm{L_A}$], [$\mathrm{L_B}$], [$\mathrm{L_C}$], [$\mathrm{L_D}$], [$\mathrm{L_E}$], [$\mathrm{L_F}$], [$\mathrm{L_G}$], [$\mathrm{L_H}$], [$\mathrm{L_I}$], [$\mathrm{L_J}$], [$\mathrm{L_K}$], [$\mathrm{L_L}$], [$\mathrm{L_M}$], [$\mathrm{L_N}$], [$\mathrm{L_O}$], [$\mathrm{L_P}$], [$\mathrm{L_Q}$], [$\mathrm{L_R}$], [$\mathrm{L_S}$], [$\mathrm{L_T}$], [$\mathrm{L_U}$], [$\mathrm{L_V}$], [$\mathrm{L_W}$], [$\mathrm{L_X}$], [$\mathrm{L_Y}$], [$\mathrm{L_Z}$], [$\mathrm{L_?}$], [Reflexivity], [Reflexivity$_1$], [Commutativity], [Commutativity$_1$], [<tactic>], [tactic], [[$* \stackrel{\text{tactic}}{=} *$]], [$\mathcal{P}(*, *, *)$], [$\mathcal{P}^*(*, *, *)$], [$\mathrm{p_0}$], [conclude$_1(*, *)$], [conclude$_2(*, *, *)$], [conclude$_3(*, *, *, *)$], [conclude$_4(*, *)$], [$\dot{0}$], [$\dot{1}$], [$\dot{2}$], [$\dot{a}$], [$\dot{b}$], [$\dot{c}$], [$\dot{d}$], [$\dot{e}$], [$\dot{f}$], [$\dot{g}$], [$\dot{h}$], [$\dot{i}$], [$\dot{j}$], [$\dot{k}$], [$\dot{l}$], [$\dot{m}$], [$\dot{n}$], [$\dot{o}$], [$\dot{p}$], [$\dot{q}$], [$\dot{r}$], [$\dot{s}$], [$\dot{t}$], [$\dot{u}$], [$\dot{v}$], [$\dot{w}$], [$\dot{x}$], [$\dot{y}$], [$\dot{z}$], [nonfree$(*, *)$], [nonfree$^*(*, *)$], [free$\langle *|* := *\rangle$], [free$^*\langle *|* := *\rangle$], [$*\equiv\langle *|* := *\rangle$], [$*\equiv\langle^* *|* := *\rangle$], [S], [A1], [A2], [A3], [A4], [A5], [S1], [S2], [S3], [S4], [S5], [S6], [S7], [S8], [S9], [MP], [Gen], [L3.2(a)], [S$'$], [A1$'$], [A2$'$], [A3$'$], [A4$'$], [A5$'$], [S1$'$], [S2$'$], [S3$'$], [S4$'$], [S5$'$], [S6$'$], [S7$'$], [S8$'$], [S9$'$], [MP$'$], [Gen$'$], [L3.2(a)$'$], [M1.7], [$\mathrm{MP'_h}$], [Hypothesize], [$\mathrm{Gen'_h}$], [M3.2(a)], [M3.2(a)$_\mathrm{h}$], [M3.2(b)$_\mathrm{h}$], [M3.1(S1$'$)$_\mathrm{h}$], [M3.2(c)$_\mathrm{h}$], [M3.1(S2$'$)$_\mathrm{h}$], [M3.1(S5$'$)$_\mathrm{h}$], [M3.1(S6$'$)$_\mathrm{h}$], [M3.2(f)];

Preassociative

[$*_{\{*\}}$], [$*'$], [$*[*]$], [$*[*\rightarrow*]$], [$*[*\Rightarrow*]$], [$\dot{*}$];

Preassociative

[“ $*$ ”], [], [$(*)^{\mathbf{t}}$], [string$(*) + *$], [string$(*) ++ *$], [
$*$], [$*$], [!$*$], [”$*$], [#$*$], [\$$*$], [%$*$], [&$*$], [’$*$], [($*$], [)$*$], [$**$], [+$*$], [,$*$], [-$*$], [.$*$], [/$*$], [0$*$], [1$*$], [2$*$], [3$*$], [4$*$], [5$*$], [6$*$], [7$*$], [8$*$], [9$*$], [:$*$], [;$*$], [<$*$], [=$*$], [>$*$], [?$*$], [@$*$], [A$*$], [B$*$], [C$*$], [D$*$], [E$*$], [F$*$], [G$*$], [H$*$], [I$*$], [J$*$], [K$*$], [L$*$], [M$*$], [N$*$], [O$*$], [P$*$], [Q$*$], [R$*$], [S$*$], [T$*$], [U$*$], [V$*$], [W$*$], [X$*$], [Y$*$], [Z$*$], [[$*$], [\$*$], []$*$], [^$*$], [_$*$], [‘$*$], [a$*$], [b$*$], [c$*$], [d$*$], [e$*$], [f$*$], [g$*$], [h$*$], [i$*$], [j$*$], [k$*$], [l$*$], [m$*$], [n$*$], [o$*$], [p$*$], [q$*$], [r$*$], [s$*$], [t$*$], [u$*$], [v$*$], [w$*$], [x$*$], [y$*$], [z$*$], [{$*$], [|$*$], [}$*$], [~$*$], [**Preassociative** $*; *$], [**Postassociative** $*; *$], [[$*$], $*$], [priority $*$ end], [newline $*$], [macro newline $*$];

Preassociative

[$*0$], [$*1$], [0b], [$*$-color$(*)$], [$*$-color$^*(*)$];

Preassociative

[$*$ ’ $*$], [$*$ ‘ $*$];

Preassociative

[$*^{\mathrm{H}}$], [$*^{\mathrm{T}}$], [$*^{\mathrm{U}}$], [$*^{\mathrm{h}}$], [$*^{\mathrm{t}}$], [$*^{\mathrm{s}}$], [$*^{\mathrm{c}}$], [$*^{\mathrm{d}}$], [$*^{\mathrm{a}}$], [$*^{\mathrm{C}}$], [$*^{\mathrm{M}}$], [$*^{\mathrm{B}}$], [$*^{\mathrm{r}}$], [$*^{\mathrm{i}}$], [$*^{\mathrm{d}}$], [$*^{\mathrm{R}}$], [$*^{0}$], [$*^{1}$], [$*^{2}$], [$*^{3}$], [$*^{4}$], [$*^{5}$], [$*^{6}$], [$*^{7}$], [$*^{8}$], [$*^{9}$], [$*^{\mathrm{E}}$], [$*^{\mathcal{V}}$], [$*^{\mathcal{C}}$], [$*^{\mathcal{C}^*}$], [$*'$];

Preassociative

[$* \cdot *$], [$* \cdot_0 *$], [$* \stackrel{\cdot}{\cdot} *$];

Preassociative

[$* + *$], [$* +_0 *$], [$* +_1 *$], [$* - *$], [$* -_0 *$], [$* -_1 *$], [$* \dot{+} *$];

Preassociative

[$* \cup \{*\}$], [$* \cup *$], [$* \backslash \{*\}$];

Postassociative

[$* \therefore *$], [$* \underline{\therefore} *$], [$* \underline{::} *$], [$* \underline{+2*} *$], [$* :: *$], [$* +2* *$];

Postassociative

[$*, *$];

Preassociative
$[* \stackrel{\rm B}{\approx} *], [* \stackrel{\rm D}{\approx} *], [* \stackrel{\rm C}{\approx} *], [* \stackrel{\rm P}{\approx} *], [* \approx *], [* = *], [* \stackrel{+}{\rightarrow} *], [* \stackrel{\rm t}{=} *], [* \stackrel{\rm t^*}{=} *], [* \stackrel{\rm r}{=} *],$
$[* \in_{\rm t} *], [* \subseteq_{\rm T} *], [* \stackrel{\rm T}{=} *], [* \stackrel{\rm s}{=} *], [* \text{ free in } *], [* \text{ free in}^* *], [* \text{ free for } * \text{ in } *],$
$[* \text{ free for}^* * \text{ in } *], [* \in_{\rm c} *], [* < *], [* <' *], [* \leq' *], [* \stackrel{\rm p}{=} *], [*^{\mathcal{P}}];$
Preassociative
$[\neg *], [\dot{\neg} *];$
Preassociative
$[* \wedge *], [* \ddot{\wedge} *], [* \tilde{\wedge} *], [* \wedge_{\rm c} *], [* \dot{\wedge} *];$
Preassociative
$[* \vee *], [* \parallel *], [* \ddot{\vee} *], [* \dot{\vee} *];$
Preassociative
$[\dot{\forall} * \colon *], [\dot{\exists} * \colon *];$
Postassociative
$[* \ddot{\Rightarrow} *], [* \dot{\Rightarrow} *], [* \dot{\Leftrightarrow} *];$
Postassociative
$[* : *], [*!*];$
Preassociative
$[* \left\{ \begin{array}{c} * \\ * \end{array} \right.];$
Preassociative
$[\lambda * .*], [\Lambda *], [\textbf{if } * \textbf{ then } * \textbf{ else } *], [\textbf{let } * = * \textbf{ in } *], [\textbf{let } * \ddot{=} * \text{ in } *];$
Preassociative
$[*^{\rm I}], [*^{\triangleright}], [*^{\rm V}], [*^{+}], [*^{-}], [*^{*}];$
Preassociative
$[* @ *], [* \triangleright *], [* \blacktriangleright *], [* \gg *], [* \trianglerighteq *], [* \trianglerighteq_{\rm h} *];$
Postassociative
$[* \vdash *], [* \Vdash *], [* \text{ i.e. } *];$
Preassociative
$[\forall * \colon *];$
Postassociative
$[* \oplus *];$
Postassociative
$[*; *];$
Preassociative
$[* \text{ proves } *];$
Preassociative
$[* \textbf{ proof of } * : *], [\text{Line} * : * \gg *; *], [\text{Last line} * \gg * \square],$
$[\text{Line} * : \text{Premise} \gg *; *], [\text{Line} * : \text{Side-condition} \gg *; *], [\text{Arbitrary} \gg *; *],$
$[\text{Local} \gg * = *; *];$
Postassociative
$[* \text{ then } *], [*[*]*];$
Preassociative
$[* \& *];$
Preassociative
$[* \backslash\backslash *];$ **End table**

B Index

pyk: peano a [$\dot{a}$], 3
pyk: peano all * indeed * [$\dot{\forall}\mathsf{x}\colon \mathsf{y}$], 2
pyk: peano b [$\dot{b}$], 13
pyk: peano c [$\dot{c}$], 13
pyk: peano d [$\dot{d}$], 13
pyk: peano e [$\dot{e}$], 13
pyk: peano exist * indeed * [$\dot{\exists}\mathsf{x}\colon \mathsf{y}$], 2
pyk: peano f [$\dot{f}$], 13
pyk: peano free * set * to * end free [$\dot{\mathrm{free}}\langle \mathsf{a},\ \mathsf{x} := \mathsf{b}\rangle$]3
pyk: peano free star * set * to * end free [$\underset{\cdot}{\mathrm{free}}^{*}\langle \mathsf{a},\ \mathsf{x} := \mathsf{b}\rangle$]3
pyk: peano g [$\dot{g}$], 13
pyk: peano h [$\dot{h}$], 13
pyk: peano i [$\dot{i}$], 13
pyk: peano j [$\dot{j}$], 13
pyk: peano k [$\dot{k}$], 13
pyk: peano l [$\dot{l}$], 13
pyk: peano m [$\dot{m}$], 13
pyk: peano n [$\dot{n}$], 13
pyk: peano nonfree * in * end nonfree [$\dot{\mathrm{nonfree}}(\mathsf{x}, \mathsf{y})$], 3
pyk: peano nonfree star * in * end nonfree [$\dot{\mathrm{nonfree}}^{*}(\mathsf{x}, \mathsf{y})$], 3
pyk: peano not * [$\dot{\neg}\,\mathsf{x}$], 2
pyk: peano o [$\dot{o}$], 13
pyk: peano one [$\dot{1}$], 2
pyk: peano p [$\dot{p}$], 13
pyk: peano q [$\dot{q}$], 13
pyk: peano r [$\dot{r}$], 13
pyk: peano s [$\dot{s}$], 13
pyk: peano sub * is * where * is * end sub [$\mathsf{a} \dot{\equiv} \langle \mathsf{b},\ \mathsf{x} := \mathsf{c}\rangle$]4
pyk: peano sub star * is * where * is * end sub [$\mathsf{a} \dot{\equiv} \langle^{*} \mathsf{b},\ \mathsf{x} := \mathsf{c}\rangle$]4
pyk: peano t [$\dot{t}$], 13
pyk: peano two [$\dot{2}$], 2
pyk: peano u [$\dot{u}$], 13
pyk: peano v [$\dot{v}$], 13
pyk: peano w [$\dot{w}$], 13
pyk: peano x [$\dot{x}$], 13
pyk: peano y [$\dot{y}$], 13
pyk: peano z [$\dot{z}$], 13
pyk: peano zero [$\dot{0}$], 2
pyk: rule gen [Gen], 4
pyk: rule mp [MP], 4
pyk: rule prime gen [Gen′], 6
pyk: rule prime mp [MP′], 6
pyk: system prime s [S′], 6
pyk: system s [S], 4

q: [$\dot{q}$] peano q, 13

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