Peano arithmetic

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Contents

1 Peano arithmetic

This Logiweb page [1] defines Peano arithmetic.

1.1 The constructs of Peano arithmetic

Terms of Peano arithmetic are constructed from zero [$\dot{0} \stackrel{\text{pyk}}{\rightarrow}$ “peano zero”][$\dot{0} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{0}”], successor [$\mathsf{x}' \stackrel{\text{pyk}}{\rightarrow}$ “$*$ peano succ”][$\mathsf{x}' \stackrel{\text{tex}}{\rightarrow}$ “#1.
’”], plus [$\mathsf{x} \dot{+} \mathsf{y} \stackrel{\text{pyk}}{\rightarrow}$ “$*$ peano plus $*$”][$\mathsf{x} \dot{+} \mathsf{y} \stackrel{\text{tex}}{\rightarrow}$ “#1.
\mathop{\dot{+}} #2.”], and times [$\mathsf{x} \dot{\cdot} \mathsf{y} \stackrel{\text{pyk}}{\rightarrow}$ “$*$ peano times $*$”][$\mathsf{x} \dot{\cdot} \mathsf{y} \stackrel{\text{tex}}{\rightarrow}$ “#1.
\mathop{\dot{\cdot}} #2.”].

Formulas of Peano arithmeric are constructed from equality [$\mathsf{x} \stackrel{\text{p}}{=} \mathsf{y} \stackrel{\text{pyk}}{\rightarrow}$ “$*$ peano is $*$”][$\mathsf{x} \stackrel{\text{p}}{=} \mathsf{y} \stackrel{\text{tex}}{\rightarrow}$ “#1.
\stackrel{p}{=} #2.”], negation [$\dot{\neg}\, \mathsf{x} \stackrel{\text{pyk}}{\rightarrow}$ “peano not $*$”][$\dot{\neg}\, \mathsf{x} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\neg}\, #1.”], implication [$\mathsf{x} \dot{\Rightarrow} \mathsf{y} \stackrel{\text{pyk}}{\rightarrow}$ “$*$ peano imply $*$”][$\mathsf{x} \dot{\Rightarrow} \mathsf{y} \stackrel{\text{tex}}{\rightarrow}$ “#1.
\mathrel{\dot{\Rightarrow}} #2.”], and universal quantification
[$\dot{\forall} \mathsf{x} \colon \mathsf{y} \stackrel{\text{pyk}}{\rightarrow}$ “peano all $*$ indeed $*$”][$\dot{\forall} \mathsf{x} \colon \mathsf{y} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\forall} #1.
\colon #2.”].

From these constructs we macro define one [$\dot{1} \stackrel{\text{pyk}}{\rightarrow}$ “peano one”][$\dot{1} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{1}”], two [$\dot{2} \stackrel{\text{pyk}}{\rightarrow}$ “peano two”][$\dot{2} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{2}”], conjunction [$\mathsf{x} \dot{\wedge} \mathsf{y} \stackrel{\text{pyk}}{\rightarrow}$ “$*$ peano and $*$”][$\mathsf{x} \dot{\wedge} \mathsf{y} \stackrel{\text{tex}}{\rightarrow}$ “#1.
\mathrel{\dot{\wedge}} #2.”], disjunction [$\mathsf{x} \dot{\vee} \mathsf{y} \stackrel{\text{pyk}}{\rightarrow}$ “$*$ peano or $*$”][$\mathsf{x} \dot{\vee} \mathsf{y} \stackrel{\text{tex}}{\rightarrow}$ “#1.
\mathrel{\dot{\vee}} #2.”], biimplication [$\mathsf{x} \dot{\Leftrightarrow} \mathsf{y} \stackrel{\text{pyk}}{\rightarrow}$ “$*$ peano iff $*$”][$\mathsf{x} \dot{\Leftrightarrow} \mathsf{y} \stackrel{\text{tex}}{\rightarrow}$ “#1.
\mathrel{\dot{\Leftrightarrow}} #2.”], and existential quantification
[$\dot{\exists} \mathsf{x} \colon \mathsf{y} \stackrel{\text{pyk}}{\rightarrow}$ “peano exist $*$ indeed $*$”][$\dot{\exists} \mathsf{x} \colon \mathsf{y} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\exists} #1.
\colon #2.”]:

$$[\dot{1} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil [\dot{1} \doteq \dot{0}'] \rceil)]$$

$$[\dot{2} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil [\dot{2} \doteq \dot{1}'] \rceil)]$$

$$[\mathsf{x} \dot{\wedge} \mathsf{y} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil [\mathsf{x} \dot{\wedge} \mathsf{y} \doteq \dot{\neg}\, (\mathsf{x} \dot{\Rightarrow} \dot{\neg}\, \mathsf{y})] \rceil)]$$

$$[\mathsf{x} \dot{\vee} \mathsf{y} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \dot{\vee} \mathsf{y} \ddot{=} \dot{\neg} \mathsf{x} \dot{\Rightarrow} \mathsf{y}]\rceil)]$$

$$[\mathsf{x} \dot{\Leftrightarrow} \mathsf{y} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \dot{\Leftrightarrow} \mathsf{y} \ddot{=} (\mathsf{x} \dot{\Rightarrow} \mathsf{y}) \dot{\wedge} (\mathsf{y} \dot{\Rightarrow} \mathsf{x})]\rceil)]$$

$$[\dot{\exists}\mathsf{x}\colon \mathsf{y} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{\exists}\mathsf{x}\colon \mathsf{y} \ddot{=} \dot{\neg}\dot{\forall}\mathsf{x}\colon \dot{\neg}\mathsf{y}]\rceil)]$$

1.2 Variables

We now introduce the unary operator $[\dot{\mathsf{x}} \stackrel{\text{pyk}}{\to}$ "$*$ peano var"$][\dot{\mathsf{x}} \stackrel{\text{tex}}{\to}$ "
\dot{#1.
}"] and define that a term is a *Peano variable* (i.e. a variable of Peano arithmetic) if it has the $[\dot{\mathsf{x}}]$ operator in its root. $[\mathsf{x}^{\mathcal{P}} \stackrel{\text{pyk}}{\to}$ "$*$ is peano var"$][\mathsf{x}^{\mathcal{P}} \stackrel{\text{tex}}{\to}$ "#1.
{} ^ {\cal P}"] is true if $[\mathsf{x}]$ is a Peano variable:

$$[\mathsf{x}^{\mathcal{P}} \stackrel{\text{val}}{\to} \mathsf{x} \stackrel{\text{r}}{=} \lceil\dot{\mathsf{x}}\rceil]$$

We macro define $[\dot{a} \stackrel{\text{pyk}}{\to}$ "peano a"$][\dot{a} \stackrel{\text{tex}}{\to}$ "
\dot{\mathit{a}}"] to be a Peano variable:

$$[\dot{a} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{a} \ddot{=} \dot{\mathsf{a}}]\rceil)]$$

Appendix A.2 defines Peano variables for the other letters of the English alphabet.
$[\dot{\text{nonfree}}(\mathsf{x},\mathsf{y}) \stackrel{\text{pyk}}{\to}$ "peano nonfree $*$ in $*$ end nonfree"$][\dot{\text{nonfree}}(\mathsf{x},\mathsf{y}) \stackrel{\text{tex}}{\to}$ "
\dot{nonfree}(#1.
, #2.
)"] is true if the Peano variable $[\mathsf{x}]$ does not occur free in the Peano term/formula $[\mathsf{y}]$. $[\dot{\text{nonfree}}^*(\mathsf{x},\mathsf{y}) \stackrel{\text{pyk}}{\to}$ "peano nonfree star $*$ in $*$ end nonfree"$][\dot{\text{nonfree}}^*(\mathsf{x},\mathsf{y}) \stackrel{\text{tex}}{\to}$ "
\dot{nonfree}^*(#1.
, #2.
)"] is true if the Peano variable $[\mathsf{x}]$ does not occur free in the list $[\mathsf{y}]$ of Peano terms/formulas.

$$\begin{aligned}&[\dot{\text{nonfree}}(\mathsf{x},\mathsf{y}) \stackrel{\text{val}}{\to}\\ &\text{If}(\mathsf{y}^{\mathcal{P}}, \neg\mathsf{x} \stackrel{\text{t}}{=} \mathsf{y},\\ &\text{If}(\neg\mathsf{y} \stackrel{\text{r}}{=} \lceil\dot{\forall}\mathsf{x}\colon \mathsf{y}\rceil, \dot{\text{nonfree}}^*(\mathsf{x},\mathsf{y}^{\mathsf{t}}),\\ &\text{If}(\mathsf{x} \stackrel{\text{t}}{=} \mathsf{y}^1, \mathsf{T}, \dot{\text{nonfree}}(\mathsf{x},\mathsf{y}^2)))))]\end{aligned}$$

$$[\dot{\text{nonfree}}^*(\mathsf{x},\mathsf{y}) \stackrel{\text{val}}{\to} \mathsf{x}!\text{If}(\mathsf{y},\mathsf{T},\text{If}(\dot{\text{nonfree}}(\mathsf{x},\mathsf{y}^{\mathsf{h}}), \dot{\text{nonfree}}^*(\mathsf{x},\mathsf{y}^{\mathsf{t}}),\mathsf{F}))]$$

$[\dot{\text{free}}\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \stackrel{\text{pyk}}{\to}$ "peano free $*$ set $*$ to $*$ end free"$][\dot{\text{free}}\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \stackrel{\text{tex}}{\to}$ "
\dot{free}\langle #1.
| #2.

:= #3.
\rangle"] is true if the substitution $[\langle \mathsf{a} \,|\mathsf{x}{:=}\,\mathsf{b}\rangle]$ is free.
$[\dot{\text{free}}^*\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \overset{\text{pyk}}{\rightarrow}$ "peano free star $*$ set $*$ to $*$ end free"]$[\dot{\text{free}}^*\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \overset{\text{tex}}{\rightarrow}$ "
\dot{free}{}^*\langle #1.
| #2.
:= #3.
\rangle"] is the version where [a] is a list of terms.

$$[\dot{\text{free}}\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \overset{\text{val}}{\rightarrow} \mathsf{x!b!}$$
$$\text{If}(\mathsf{a}^{\mathcal{P}}, \mathsf{T},$$
$$\text{If}(\neg\mathsf{a} \overset{\text{r}}{=} \lceil\dot{\forall}\mathsf{u}\colon \mathsf{v}\rceil, \dot{\text{free}}^*\langle \mathsf{a}^{\text{t}}|\mathsf{x} := \mathsf{b}\rangle,$$
$$\text{If}(\mathsf{a}^1 \overset{\text{t}}{=} \mathsf{x}, \mathsf{T},$$
$$\text{If}(\dot{\text{nonfree}}(\mathsf{x}, \mathsf{a}^2), \mathsf{T},$$
$$\text{If}(\neg\dot{\text{nonfree}}(\mathsf{a}^1, \mathsf{b}), \mathsf{F},$$
$$\dot{\text{free}}\langle \mathsf{a}^2|\mathsf{x} := \mathsf{b}\rangle))))))]$$

$$[\dot{\text{free}}^*\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \overset{\text{val}}{\rightarrow} \mathsf{x!b!}\text{If}(\mathsf{a}, \mathsf{T}, \text{If}(\dot{\text{free}}\langle \mathsf{a}^{\text{h}}|\mathsf{x} := \mathsf{b}\rangle, \dot{\text{free}}^*\langle \mathsf{a}^{\text{t}}|\mathsf{x} := \mathsf{b}\rangle, \mathsf{F}))]$$

$[\mathsf{a}{\equiv}\langle \mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \overset{\text{pyk}}{\rightarrow}$ "peano sub $*$ is $*$ where $*$ is $*$ end sub"]$[\mathsf{a}{\equiv}\langle \mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \overset{\text{tex}}{\rightarrow}$ "#1.
{\equiv}\langle #2.
|#3.
:=#4.
\rangle"] is true if [a] equals $[\langle \mathsf{b} \,|\mathsf{x}{:=}\,\mathsf{c}\rangle]$. $[\mathsf{a}{\equiv}\langle^* \mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \overset{\text{pyk}}{\rightarrow}$ "peano sub star $*$ is $*$ where $*$ is $*$ end sub"]$[\mathsf{a}{\equiv}\langle^* \mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \overset{\text{tex}}{\rightarrow}$ "#1.
{\equiv}\langle^* #2.
|#3.
:=#4.
\rangle"] is the version where [a] and [b] are lists.

$$[\mathsf{a}{\equiv}\langle \mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \overset{\text{val}}{\rightarrow} \mathsf{a!x!c!}$$
$$\text{If}(\text{If}(\mathsf{b} \overset{\text{r}}{=} \lceil\dot{\forall}\mathsf{u}\colon \mathsf{v}\rceil, \mathsf{b}^1 \overset{\text{t}}{=} \mathsf{x}, \mathsf{F}), \mathsf{a} \overset{\text{t}}{=} \mathsf{b},$$
$$\text{If}(\mathsf{b}^{\mathcal{P}} \wedge \mathsf{b} \overset{\text{t}}{=} \mathsf{x}, \mathsf{a} \overset{\text{t}}{=} \mathsf{c}, \text{If}($$
$$\mathsf{a} \overset{\text{r}}{=} \mathsf{b}, \mathsf{a}^{\text{t}}{\equiv}\langle^* \mathsf{b}^{\text{t}}|\mathsf{x} := \mathsf{c}\rangle, \mathsf{F})))]$$

$$[\mathsf{a}{\equiv}\langle^* \mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \overset{\text{val}}{\rightarrow} \mathsf{b!x!c!}\text{If}(\mathsf{a}, \mathsf{T}, \text{If}(\mathsf{a}^{\text{h}}{\equiv}\langle \mathsf{b}^{\text{h}}|\mathsf{x} := \mathsf{c}\rangle, \mathsf{a}^{\text{t}}{\equiv}\langle^* \mathsf{b}^{\text{t}}|\mathsf{x} := \mathsf{c}\rangle, \mathsf{F}))]$$

1.3 Mendelsons system S

System $[\text{S} \overset{\text{pyk}}{\rightarrow}$ "system s"]$[\text{S} \overset{\text{tex}}{\rightarrow}$ "
S"] of Mendelson [2] expresses Peano arithmetic. It comprises the axioms
$[\text{A1} \overset{\text{pyk}}{\rightarrow}$ "axiom a one"]$[\text{A1} \overset{\text{tex}}{\rightarrow}$ "
A1"], $[\text{A2} \overset{\text{pyk}}{\rightarrow}$ "axiom a two"]$[\text{A2} \overset{\text{tex}}{\rightarrow}$ "
A2"], $[\text{A3} \overset{\text{pyk}}{\rightarrow}$ "axiom a three"]$[\text{A3} \overset{\text{tex}}{\rightarrow}$ "

A3"], [A4 $\stackrel{\text{pyk}}{\to}$ "axiom a four"][A4 $\stackrel{\text{tex}}{\to}$ "
A4"], and [A5 $\stackrel{\text{pyk}}{\to}$ "axiom a five"][A5 $\stackrel{\text{tex}}{\to}$ "
A5"] and inference rules [MP $\stackrel{\text{pyk}}{\to}$ "rule mp"][MP $\stackrel{\text{tex}}{\to}$ "
MP"] and [Gen $\stackrel{\text{pyk}}{\to}$ "rule gen"][Gen $\stackrel{\text{tex}}{\to}$ "
Gen"] of first order predicate calculus. Furthermore, it comprises the proper axioms [S1 $\stackrel{\text{pyk}}{\to}$ "axiom s one"][S1 $\stackrel{\text{tex}}{\to}$ "
S1"], [S2 $\stackrel{\text{pyk}}{\to}$ "axiom s two"][S2 $\stackrel{\text{tex}}{\to}$ "
S2"], [S3 $\stackrel{\text{pyk}}{\to}$ "axiom s three"][S3 $\stackrel{\text{tex}}{\to}$ "
S3"], [S4 $\stackrel{\text{pyk}}{\to}$ "axiom s four"][S4 $\stackrel{\text{tex}}{\to}$ "
S4"], [S5 $\stackrel{\text{pyk}}{\to}$ "axiom s five"][S5 $\stackrel{\text{tex}}{\to}$ "
S5"], [S6 $\stackrel{\text{pyk}}{\to}$ "axiom s six"][S6 $\stackrel{\text{tex}}{\to}$ "
S6"], [S7 $\stackrel{\text{pyk}}{\to}$ "axiom s seven"][S7 $\stackrel{\text{tex}}{\to}$ "
S7"], [S8 $\stackrel{\text{pyk}}{\to}$ "axiom s eight"][S8 $\stackrel{\text{tex}}{\to}$ "
S8"], and [S9 $\stackrel{\text{pyk}}{\to}$ "axiom s nine"][S9 $\stackrel{\text{tex}}{\to}$ "
S9"]. System [S] is defined thus:

[S $\stackrel{\text{stmt}}{\to} \dot{a} \dot{+} \dot{b}' \stackrel{p}{=} \dot{a} \dot{+} \dot{b}' \oplus \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg}\, \underline{b} \dot{\Rightarrow} \dot{\neg}\, \underline{a} \dot{\Rightarrow} \dot{\neg}\, \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{b} \oplus \dot{a} \stackrel{p}{=} \dot{b} \dot{\Rightarrow} \dot{a}' \stackrel{p}{=} \dot{b}' \oplus \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \dot{\Rightarrow} \underline{b} \vdash \underline{a} \vdash \underline{b} \oplus \dot{a}' \stackrel{p}{=} \dot{b}' \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{b} \oplus \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \oplus \forall \underline{x}\colon \forall \underline{a}\colon \forall \underline{b}\colon \mathrm{no\dot{n}free}(\underline{x}, \underline{a}) \Vdash \dot{\forall} \underline{x}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \dot{\forall} \underline{x}\colon \underline{b} \oplus \dot{a} \dot{\cdot} \dot{b}' \stackrel{p}{=} \dot{a} \dot{\cdot} \dot{b} \dot{+} \dot{a} \oplus \dot{a} \dot{+} \dot{0} \stackrel{p}{=} \dot{a} \oplus \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{c} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{c} \oplus \dot{a} \stackrel{p}{=} \dot{b} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{c} \dot{\Rightarrow} \dot{b} \stackrel{p}{=} \dot{c} \oplus \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{x}\colon \underline{b} \equiv \langle \underline{a} | \underline{x} := \dot{0} \rangle \Vdash \underline{c} \equiv \langle \underline{a} | \underline{x} := \underline{x}' \rangle \Vdash \underline{b} \dot{\Rightarrow} \dot{\forall} \underline{x}\colon \underline{a} \dot{\Rightarrow} \underline{c} \dot{\Rightarrow} \dot{\forall} \underline{x}\colon \underline{a} \oplus \neg \dot{0} \stackrel{p}{=} \dot{a}' \oplus \forall \underline{x}\colon \forall \underline{a}\colon \underline{a} \vdash \dot{\forall} \underline{x}\colon \underline{a} \oplus \forall \underline{c}\colon \forall \underline{a}\colon \forall \underline{x}\colon \forall \underline{b}\colon \lceil \underline{a} \rceil \equiv \langle \lceil \underline{b} \rceil | \lceil \underline{x} \rceil := \lceil \underline{c} \rceil \rangle \Vdash \dot{\forall} \underline{x}\colon \underline{b} \dot{\Rightarrow} \underline{a} \oplus \dot{a} \dot{\cdot} \dot{0} \stackrel{p}{=} \dot{0}$]

[A1 $\stackrel{\text{stmt}}{\to} S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a}$][A1 $\stackrel{\text{proof}}{\to}$ Rule tactic]

[A2 $\stackrel{\text{stmt}}{\to} S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{c} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{c}$][A2 $\stackrel{\text{proof}}{\to}$ Rule tactic]

[A3 $\stackrel{\text{stmt}}{\to} S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg}\, \underline{b} \dot{\Rightarrow} \dot{\neg}\, \underline{a} \dot{\Rightarrow} \dot{\neg}\, \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{b}$][A3 $\stackrel{\text{proof}}{\to}$ Rule tactic]

The order of quantifiers in the following axiom is such that [$\underline{c}$] which the current conclusion tactic cannot guess comes first. This allows to supply a value for [$\underline{c}$] without having to supply values for the other meta-variables.

[A4 $\stackrel{\text{stmt}}{\to} S \vdash \forall \underline{c}\colon \forall \underline{a}\colon \forall \underline{x}\colon \forall \underline{b}\colon \lceil \underline{a} \rceil \equiv \langle \lceil \underline{b} \rceil | \lceil \underline{x} \rceil := \lceil \underline{c} \rceil \rangle \Vdash \dot{\forall} \underline{x}\colon \underline{b} \dot{\Rightarrow} \underline{a}$][A4 $\stackrel{\text{proof}}{\to}$ Rule tactic]

[A5 $\stackrel{\text{stmt}}{\to} S \vdash \forall \underline{x}\colon \forall \underline{a}\colon \forall \underline{b}\colon \mathrm{no\dot{n}free}(\underline{x}, \underline{a}) \Vdash \dot{\forall} \underline{x}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \dot{\forall} \underline{x}\colon \underline{b}$][A5 $\stackrel{\text{proof}}{\to}$ Rule tactic]

[MP $\stackrel{\text{stmt}}{\to} S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \dot{\Rightarrow} \underline{b} \vdash \underline{a} \vdash \underline{b}$][MP $\stackrel{\text{proof}}{\to}$ Rule tactic]

[Gen $\stackrel{\text{stmt}}{\rightarrow}$ S $\vdash \forall \underline{x}\colon \forall \underline{a}\colon \underline{a} \vdash \dot{\forall}\underline{x}\colon \underline{a}$][Gen $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

Axiom [S1] to [S8] are stated as Mendelson [2] does. This is done here to test certain parts of the Logiweb system. Serious users of Peano arithmetic are advised to take Mendelson's Lemma 3.1 as axioms instead.

[S1 $\stackrel{\text{stmt}}{\rightarrow}$ S $\vdash \dot{a} \stackrel{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{c} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{p}{=} \dot{c}$][S1 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[S2 $\stackrel{\text{stmt}}{\rightarrow}$ S $\vdash \dot{a} \stackrel{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a}' \stackrel{p}{=} \dot{b}'$][S2 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[S3 $\stackrel{\text{stmt}}{\rightarrow}$ S $\vdash \neg \dot{0} \stackrel{p}{=} \dot{a}'$][S3 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[S4 $\stackrel{\text{stmt}}{\rightarrow}$ S $\vdash \dot{a}' \stackrel{p}{=} \dot{b}' \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{b}$][S4 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[S5 $\stackrel{\text{stmt}}{\rightarrow}$ S $\vdash \dot{a} \dot{+} \dot{0} \stackrel{p}{=} \dot{a}$][S5 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[S6 $\stackrel{\text{stmt}}{\rightarrow}$ S $\vdash \dot{a} \dot{+} \dot{b}' \stackrel{p}{=} \dot{a} \dot{+} \dot{b}'$][S6 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[S7 $\stackrel{\text{stmt}}{\rightarrow}$ S $\vdash \dot{a} \dot{\cdot} \dot{0} \stackrel{p}{=} \dot{0}$][S7 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[S8 $\stackrel{\text{stmt}}{\rightarrow}$ S $\vdash \dot{a} \dot{\cdot} \dot{b}' \stackrel{p}{=} \dot{a} \dot{\cdot} \dot{b} \dot{+} \dot{a}$][S8 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[S9 $\stackrel{\text{stmt}}{\rightarrow}$ S $\vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{x}\colon \underline{b} \equiv \langle \underline{a} | \underline{x} := \dot{0} \rangle \Vdash \underline{c} \equiv \langle \underline{a} | \underline{x} := \underline{x}' \rangle \Vdash \underline{b} \mathrel{\dot{\Rightarrow}} \dot{\forall}\underline{x}\colon \underline{a} \mathrel{\dot{\Rightarrow}} \underline{c} \mathrel{\dot{\Rightarrow}} \dot{\forall}\underline{x}\colon \underline{a}$][S9 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

1.4 A lemma and a proof

We now prove Lemma [L3.2(a) $\stackrel{\text{pyk}}{\rightarrow}$ "lemma l three two a"][L3.2(a) $\stackrel{\text{tex}}{\rightarrow}$ " L3.2(a)"] which is an instance of the corresponding proposition in Mendelson [2]:

[L3.2(a) $\stackrel{\text{stmt}}{\rightarrow}$ S $\vdash \dot{x} \stackrel{p}{=} \dot{x}$]

[L3.2(a) $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{S} \vdash \mathrm{S5} \gg \dot{a} \dot{+} \dot{0} \stackrel{p}{=} \dot{a}; \mathrm{Gen} \rhd \dot{a} \dot{+} \dot{0} \stackrel{p}{=} \dot{a} \gg \dot{\forall}\dot{a}\colon \dot{a} \dot{+} \dot{0} \stackrel{p}{=} \dot{a}; \mathrm{A4} @ \dot{x} \gg \dot{\forall}\dot{a}\colon \dot{a} \dot{+} \dot{0} \stackrel{p}{=} \dot{a} \mathrel{\dot{\Rightarrow}} \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x}; \mathrm{MP} \rhd \dot{\forall}\dot{a}\colon \dot{a} \dot{+} \dot{0} \stackrel{p}{=} \dot{a} \mathrel{\dot{\Rightarrow}} \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \rhd \dot{\forall}\dot{a}\colon \dot{a} \dot{+} \dot{0} \stackrel{p}{=} \dot{a} \gg \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x}; \mathrm{S1} \gg \dot{a} \stackrel{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{c} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{p}{=} \dot{c}; \mathrm{Gen} \rhd \dot{a} \stackrel{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{c} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{p}{=} \dot{c} \gg \dot{\forall}\dot{c}\colon \dot{a} \stackrel{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{c} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{p}{=} \dot{c}; \mathrm{A4} @ \dot{x} \gg \dot{\forall}\dot{c}\colon \dot{a} \stackrel{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{c} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{p}{=} \dot{c} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{p}{=} \dot{x}; \mathrm{MP} \rhd \dot{\forall}\dot{c}\colon \dot{a} \stackrel{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{c} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{p}{=} \dot{c} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{p}{=} \dot{x} \rhd \dot{\forall}\dot{c}\colon \dot{a} \stackrel{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{c} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{p}{=} \dot{c} \gg \dot{a} \stackrel{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{p}{=} \dot{x}; \mathrm{Gen} \rhd \dot{a} \stackrel{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{p}{=} \dot{x} \gg \dot{\forall}\dot{b}\colon \dot{a} \stackrel{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{p}{=} \dot{x}; \mathrm{A4} @ \dot{x} \gg \dot{\forall}\dot{b}\colon \dot{a} \stackrel{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \stackrel{p}{=} \dot{x}; \mathrm{MP} \rhd \dot{\forall}\dot{b}\colon \dot{a} \stackrel{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \stackrel{p}{=} \dot{x} \rhd \dot{\forall}\dot{b}\colon \dot{a} \stackrel{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{p}{=} \dot{x} \gg \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \stackrel{p}{=} \dot{x}; \mathrm{Gen} \rhd \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \stackrel{p}{=} \dot{x} \gg \dot{\forall}\dot{a}\colon \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \stackrel{p}{=} \dot{x}; \mathrm{A4} @ \dot{x} \dot{+} \dot{0} \gg \dot{\forall}\dot{a}\colon \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \stackrel{p}{=} \dot{x}; \mathrm{MP} \rhd \dot{\forall}\dot{a}\colon \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \stackrel{p}{=} \dot{x} \rhd \dot{\forall}\dot{a}\colon \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \stackrel{p}{=} \dot{x} \gg \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \stackrel{p}{=} \dot{x}; \mathrm{MP} \rhd \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \stackrel{p}{=} \dot{x} \rhd \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \gg \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \stackrel{p}{=} \dot{x}; \mathrm{MP} \rhd \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \stackrel{p}{=} \dot{x} \rhd \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \gg \dot{x} \stackrel{p}{=} \dot{x} \rceil, p_0, c)$]

1.5 An alternative axiomatic system

System [$S' \overset{pyk}{\rightarrow}$ “system prime s”][$S' \overset{tex}{\rightarrow}$ “ S′”] is system [S] in which the proper axioms are taken from Lemma 3.1 in Mendelson [2]. It comprises the axioms [$A1' \overset{pyk}{\rightarrow}$ “axiom prime a one”][$A1' \overset{tex}{\rightarrow}$ “ A1′”], [$A2' \overset{pyk}{\rightarrow}$ “axiom prime a two”][$A2' \overset{tex}{\rightarrow}$ “ A2′”], [$A3' \overset{pyk}{\rightarrow}$ “axiom prime a three”][$A3' \overset{tex}{\rightarrow}$ “ A3′”], [$A4' \overset{pyk}{\rightarrow}$ “axiom prime a four”][$A4' \overset{tex}{\rightarrow}$ “ A4′”], and [$A5' \overset{pyk}{\rightarrow}$ “axiom prime a five”][$A5' \overset{tex}{\rightarrow}$ “ A5′”] and inference rules [$MP' \overset{pyk}{\rightarrow}$ “rule prime mp”][$MP' \overset{tex}{\rightarrow}$ “ MP′”] and [$Gen' \overset{pyk}{\rightarrow}$ “rule prime gen”][$Gen' \overset{tex}{\rightarrow}$ “ Gen′”] of first order predicate calculus. Furthermore, it comprises the proper axioms [$S1' \overset{pyk}{\rightarrow}$ “axiom prime s one”][$S1' \overset{tex}{\rightarrow}$ “ S1′”], [$S2' \overset{pyk}{\rightarrow}$ “axiom prime s two”][$S2' \overset{tex}{\rightarrow}$ “ S2′”], [$S3' \overset{pyk}{\rightarrow}$ “axiom prime s three”][$S3' \overset{tex}{\rightarrow}$ “ S3′”], [$S4' \overset{pyk}{\rightarrow}$ “axiom prime s four”][$S4' \overset{tex}{\rightarrow}$ “ S4′”], [$S5' \overset{pyk}{\rightarrow}$ “axiom prime s five”][$S5' \overset{tex}{\rightarrow}$ “ S5′”], [$S6' \overset{pyk}{\rightarrow}$ “axiom prime s six”][$S6' \overset{tex}{\rightarrow}$ “ S6′”], [$S7' \overset{pyk}{\rightarrow}$ “axiom prime s seven”][$S7' \overset{tex}{\rightarrow}$ “ S7′”], [$S8' \overset{pyk}{\rightarrow}$ “axiom prime s eight”][$S8' \overset{tex}{\rightarrow}$ “ S8′”], and [$S9' \overset{pyk}{\rightarrow}$ “axiom prime s nine”][$S9' \overset{tex}{\rightarrow}$ “ S9′”].
System [S′] is defined thus:

$$[S' \overset{stmt}{\rightarrow} \forall \underline{a}: \forall \underline{b}: \underline{a}' \overset{p}{=} \underline{b}' \dot{\Rightarrow} \underline{a} \overset{p}{=} \underline{b} \oplus \forall \underline{h}: \forall \underline{t}: \forall \underline{r}: \forall \underline{s}: \underline{h} \dot{\Rightarrow} \underline{t} \overset{p}{=} \underline{r} \vdash \underline{h} \dot{\Rightarrow} \underline{t} \overset{p}{=} \underline{s} \vdash \underline{h} \dot{\Rightarrow} \underline{r} \overset{p}{=} \underline{s} \oplus \forall \underline{a}: \forall \underline{b}: \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \oplus \forall \underline{x}: \forall \underline{a}: \forall \underline{b}: \mathrm{nonfree}(\lceil \underline{x} \rceil, \lceil \underline{a} \rceil) \Vdash \dot{\forall} \underline{x}: \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \dot{\forall} \underline{x}: \underline{b} \oplus \forall \underline{h}: \forall \underline{t}: \forall \underline{r}: \underline{h} \dot{\Rightarrow} \underline{t} \dot{+} \underline{r}' \overset{p}{=} \underline{t} \dot{+} \underline{r}' \oplus \forall \underline{a}: \forall \underline{b}: \underline{a} \dot{\cdot} \underline{b}' \overset{p}{=} \underline{a} \dot{\cdot} \underline{b} \dot{+} \underline{a} \oplus \forall \underline{a}: \forall \underline{b}: \underline{a} \overset{p}{=} \underline{b} \dot{\Rightarrow} \underline{a}' \overset{p}{=} \underline{b}' \oplus \forall \underline{a}: \forall \underline{b}: \underline{a} \dot{\Rightarrow} \underline{b} \vdash \underline{a} \vdash \underline{b} \oplus \forall \underline{a}: \forall \underline{b}: \dot{\neg} \underline{b} \dot{\Rightarrow} \dot{\neg} \underline{a} \dot{\Rightarrow} \dot{\neg} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{b} \oplus \forall \underline{h}: \forall \underline{t}: \forall \underline{r}: \underline{h} \dot{\Rightarrow} \underline{t} \overset{p}{=} \underline{r} \vdash \underline{h} \dot{\Rightarrow} \underline{t}' \overset{p}{=} \underline{r}' \oplus \forall \underline{a}: \forall \underline{b}: \underline{a} \dot{+} \underline{b}' \overset{p}{=} \underline{a} \dot{+} \underline{b}' \oplus \forall \underline{a}: \neg \dot{0} \overset{p}{=} \underline{a}' \oplus \forall \underline{x}: \forall \underline{a}: \underline{a} \vdash \dot{\forall} \underline{x}: \underline{a} \oplus \forall \underline{c}: \forall \underline{a}: \forall \underline{x}: \forall \underline{b}: \lceil \underline{a} \rceil \equiv \langle \lceil \underline{b} \rceil | \lceil \underline{x} \rceil := \lceil \underline{c} \rceil \rangle \Vdash \dot{\forall} \underline{x}: \underline{b} \dot{\Rightarrow} \underline{a} \oplus \forall \underline{h}: \forall \underline{t}: \underline{h} \dot{\Rightarrow} \underline{t} \dot{+} \dot{0} \overset{p}{=} \underline{t} \oplus \forall \underline{a}: \underline{a} \dot{\cdot} \dot{0} \overset{p}{=} \dot{0} \oplus \forall \underline{a}: \forall \underline{b}: \forall \underline{c}: \underline{a} \overset{p}{=} \underline{b} \dot{\Rightarrow} \underline{a} \overset{p}{=} \underline{c} \dot{\Rightarrow} \underline{b} \overset{p}{=} \underline{c} \oplus \forall \underline{t}: \underline{t} \overset{p}{=} \underline{t} \oplus \forall \underline{a}: \forall \underline{b}: \forall \underline{c}: \forall \underline{x}: \underline{b} \equiv \langle \underline{a} | \underline{x} := \dot{0} \rangle \Vdash \underline{c} \equiv \langle \underline{a} | \underline{x} := \underline{x}' \rangle \Vdash \underline{b} \dot{\Rightarrow} \dot{\forall} \underline{x}: \underline{a} \dot{\Rightarrow} \underline{c} \dot{\Rightarrow} \dot{\forall} \underline{x}: \underline{a} \oplus \forall \underline{a}: \forall \underline{b}: \forall \underline{c}: \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{c} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{c} \oplus \forall \underline{h}: \forall \underline{t}: \forall \underline{r}: \forall \underline{s}: \underline{h} \dot{\Rightarrow} \underline{t} \overset{p}{=} \underline{r} \vdash \underline{h} \dot{\Rightarrow} \underline{r} \overset{p}{=} \underline{s} \vdash \underline{h} \dot{\Rightarrow} \underline{t} \overset{p}{=} \underline{s} \oplus \forall \underline{a}: \underline{a} \dot{+} \dot{0} \overset{p}{=} \underline{a}]$$

[$A1' \overset{stmt}{\rightarrow} S' \vdash \forall \underline{a}: \forall \underline{b}: \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a}$][$A1' \overset{proof}{\rightarrow}$ Rule tactic]

[$A2' \overset{stmt}{\rightarrow} S' \vdash \forall \underline{a}: \forall \underline{b}: \forall \underline{c}: \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{c} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{c}$][$A2' \overset{proof}{\rightarrow}$ Rule tactic]

[$A3' \overset{stmt}{\rightarrow} S' \vdash \forall \underline{a}: \forall \underline{b}: \dot{\neg} \underline{b} \dot{\Rightarrow} \dot{\neg} \underline{a} \dot{\Rightarrow} \dot{\neg} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{b}$][$A3' \overset{proof}{\rightarrow}$ Rule tactic]

$[\mathrm{A4'} \overset{\text{stmt}}{\rightarrow} \mathrm{S'} \vdash \forall \underline{\mathsf{c}} \colon \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{b}} \colon \lceil \underline{\mathsf{a}} \rceil \equiv \langle \lceil \underline{\mathsf{b}} \rceil | \lceil \underline{\mathsf{x}} \rceil := \lceil \underline{\mathsf{c}} \rceil \rangle \Vdash \dot{\forall} \underline{\mathsf{x}} \colon \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}}][\mathrm{A4'} \overset{\text{proof}}{\rightarrow}$ Rule tactic$]$

$[\mathrm{A5'} \overset{\text{stmt}}{\rightarrow} \mathrm{S'} \vdash \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \dot{\mathrm{nonfree}}(\lceil \underline{\mathsf{x}} \rceil, \lceil \underline{\mathsf{a}} \rceil) \Vdash \dot{\forall} \underline{\mathsf{x}} \colon \underline{\mathsf{a}} \dot{\Rightarrow} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{a}} \dot{\Rightarrow} \dot{\forall} \underline{\mathsf{x}} \colon \underline{\mathsf{b}}][\mathrm{A5'} \overset{\text{proof}}{\rightarrow}$ Rule tactic$]$

$[\mathrm{MP'} \overset{\text{stmt}}{\rightarrow} \mathrm{S'} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \dot{\Rightarrow} \underline{\mathsf{b}} \vdash \underline{\mathsf{a}} \vdash \underline{\mathsf{b}}][\mathrm{MP'} \overset{\text{proof}}{\rightarrow}$ Rule tactic$]$

$[\mathrm{Gen'} \overset{\text{stmt}}{\rightarrow} \mathrm{S'} \vdash \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{a}} \colon \underline{\mathsf{a}} \vdash \dot{\forall} \underline{\mathsf{x}} \colon \underline{\mathsf{a}}][\mathrm{Gen'} \overset{\text{proof}}{\rightarrow}$ Rule tactic$]$

$[\mathrm{S1'} \overset{\text{stmt}}{\rightarrow} \mathrm{S'} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \forall \underline{\mathsf{c}} \colon \underline{\mathsf{a}} \overset{\mathrm{p}}{=} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{a}} \overset{\mathrm{p}}{=} \underline{\mathsf{c}} \dot{\Rightarrow} \underline{\mathsf{b}} \overset{\mathrm{p}}{=} \underline{\mathsf{c}}][\mathrm{S1'} \overset{\text{proof}}{\rightarrow}$ Rule tactic$]$

$[\mathrm{S2'} \overset{\text{stmt}}{\rightarrow} \mathrm{S'} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \overset{\mathrm{p}}{=} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{a}}' \overset{\mathrm{p}}{=} \underline{\mathsf{b}}'][\mathrm{S2'} \overset{\text{proof}}{\rightarrow}$ Rule tactic$]$

$[\mathrm{S3'} \overset{\text{stmt}}{\rightarrow} \mathrm{S'} \vdash \forall \underline{\mathsf{a}} \colon \dot{\neg} \dot{0} \overset{\mathrm{p}}{=} \underline{\mathsf{a}}'][\mathrm{S3'} \overset{\text{proof}}{\rightarrow}$ Rule tactic$]$

$[\mathrm{S4'} \overset{\text{stmt}}{\rightarrow} \mathrm{S'} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}}' \overset{\mathrm{p}}{=} \underline{\mathsf{b}}' \dot{\Rightarrow} \underline{\mathsf{a}} \overset{\mathrm{p}}{=} \underline{\mathsf{b}}][\mathrm{S4'} \overset{\text{proof}}{\rightarrow}$ Rule tactic$]$

$[\mathrm{S5'} \overset{\text{stmt}}{\rightarrow} \mathrm{S'} \vdash \forall \underline{\mathsf{a}} \colon \underline{\mathsf{a}} \dot{+} \dot{0} \overset{\mathrm{p}}{=} \underline{\mathsf{a}}][\mathrm{S5'} \overset{\text{proof}}{\rightarrow}$ Rule tactic$]$

$[\mathrm{S6'} \overset{\text{stmt}}{\rightarrow} \mathrm{S'} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \dot{+} \underline{\mathsf{b}}' \overset{\mathrm{p}}{=} \underline{\mathsf{a}} \dot{+} \underline{\mathsf{b}}'][\mathrm{S6'} \overset{\text{proof}}{\rightarrow}$ Rule tactic$]$

$[\mathrm{S7'} \overset{\text{stmt}}{\rightarrow} \mathrm{S'} \vdash \forall \underline{\mathsf{a}} \colon \underline{\mathsf{a}} \dot{\cdot} \dot{0} \overset{\mathrm{p}}{=} \dot{0}][\mathrm{S7'} \overset{\text{proof}}{\rightarrow}$ Rule tactic$]$

$[\mathrm{S8'} \overset{\text{stmt}}{\rightarrow} \mathrm{S'} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \dot{\cdot} \underline{\mathsf{b}}' \overset{\mathrm{p}}{=} \underline{\mathsf{a}} \dot{\cdot} \underline{\mathsf{b}} \dot{+} \underline{\mathsf{a}}][\mathrm{S8'} \overset{\text{proof}}{\rightarrow}$ Rule tactic$]$

$[\mathrm{S9'} \overset{\text{stmt}}{\rightarrow} \mathrm{S'} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \forall \underline{\mathsf{c}} \colon \forall \underline{\mathsf{x}} \colon \underline{\mathsf{b}} \equiv \langle \underline{\mathsf{a}} | \underline{\mathsf{x}} := \dot{0} \rangle \Vdash \underline{\mathsf{c}} \equiv \langle \underline{\mathsf{a}} | \underline{\mathsf{x}} := \underline{\mathsf{x}}' \rangle \Vdash \underline{\mathsf{b}} \dot{\Rightarrow} \dot{\forall} \underline{\mathsf{x}} \colon \underline{\mathsf{a}} \dot{\Rightarrow} \underline{\mathsf{c}} \dot{\Rightarrow} \dot{\forall} \underline{\mathsf{x}} \colon \underline{\mathsf{a}}][\mathrm{S9'} \overset{\text{proof}}{\rightarrow}$ Rule tactic$]$

Note that [A1] and [A1′] are distinct. The former says $[\mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \dot{\Rightarrow} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{a}}]$ and the latter says $[\mathrm{S'} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \dot{\Rightarrow} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{a}}]$.

1.6 Restatement of lemma and proof

We now prove Lemma [L3.2(a)] once again under the name of $[\mathrm{L3.2(a)'} \overset{\text{pyk}}{\rightarrow}$ "lemma prime l three two a"$][\mathrm{L3.2(a)'} \overset{\text{tex}}{\rightarrow}$ " L3.2(a)'"$]$:

$[\mathrm{L3.2(a)'} \overset{\text{stmt}}{\rightarrow} \mathrm{S'} \vdash \forall \underline{\mathsf{a}} \colon \underline{\mathsf{a}} \overset{\mathrm{p}}{=} \underline{\mathsf{a}}]$

$[\mathrm{L3.2(a)'} \overset{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}([\mathrm{S'} \vdash \forall \underline{\mathsf{a}} \colon \mathrm{S5'} \gg \underline{\mathsf{a}} \dot{+} \dot{0} \overset{\mathrm{p}}{=} \underline{\mathsf{a}}; \mathrm{S1'} \gg \underline{\mathsf{a}} \dot{+} \dot{0} \overset{\mathrm{p}}{=} \underline{\mathsf{a}} \dot{\Rightarrow} \underline{\mathsf{a}} \dot{+} \dot{0} \overset{\mathrm{p}}{=} \underline{\mathsf{a}} \dot{\Rightarrow} \underline{\mathsf{a}} \overset{\mathrm{p}}{=} \underline{\mathsf{a}}; \mathrm{MP'} \rhd \underline{\mathsf{a}} \dot{+} \dot{0} \overset{\mathrm{p}}{=} \underline{\mathsf{a}} \dot{\Rightarrow} \underline{\mathsf{a}} \dot{+} \dot{0} \overset{\mathrm{p}}{=} \underline{\mathsf{a}} \dot{\Rightarrow} \underline{\mathsf{a}} \overset{\mathrm{p}}{=} \underline{\mathsf{a}} \rhd \underline{\mathsf{a}} \dot{+} \dot{0} \overset{\mathrm{p}}{=} \underline{\mathsf{a}} \gg \underline{\mathsf{a}} \dot{+} \dot{0} \overset{\mathrm{p}}{=} \underline{\mathsf{a}} \dot{\Rightarrow} \underline{\mathsf{a}} \overset{\mathrm{p}}{=} \underline{\mathsf{a}}; \mathrm{MP'} \rhd \underline{\mathsf{a}} \dot{+} \dot{0} \overset{\mathrm{p}}{=} \underline{\mathsf{a}} \dot{\Rightarrow} \underline{\mathsf{a}} \overset{\mathrm{p}}{=} \underline{\mathsf{a}} \rhd \underline{\mathsf{a}} \dot{+} \dot{0} \overset{\mathrm{p}}{=} \underline{\mathsf{a}} \gg \underline{\mathsf{a}} \overset{\mathrm{p}}{=} \underline{\mathsf{a}}], \mathrm{p}_0, \mathsf{c})]$

2 Formal development

2.1 Propositional calculus

2.1.1 Modus ponens

We use $[\mathsf{x} \trianglerighteq \mathsf{y} \overset{\text{pyk}}{\to}$ "$*$ macro modus ponens $*$"$][\mathsf{x} \trianglerighteq \mathsf{y} \overset{\text{tex}}{\to}$ "#1. \unrhd #2."$]$ as shorthand for modus ponens:

$$[\mathsf{x} \trianglerighteq \mathsf{y} \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil [\mathsf{x} \trianglerighteq \mathsf{y} \doteqdot \mathrm{MP}' \rhd \mathsf{x} \rhd \mathsf{y}] \rceil)]$$

2.1.2 Lemma M1.7

Lemma $[\mathrm{M1.7} \overset{\text{pyk}}{\to}$ "mendelson one seven"$][\mathrm{M1.7} \overset{\text{tex}}{\to}$ " M1.7"$]$ (i.e. Lemma 1.7 in Mendelson [2]) reads:

$$[\mathrm{M1.7} \overset{\text{stmt}}{\to} \mathrm{S}' \vdash \forall \underline{\mathsf{b}} \colon \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}}]$$

$$[\mathrm{M1.7} \overset{\text{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{S}' \vdash \forall \underline{\mathsf{b}} \colon \mathrm{A1}' \gg \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}}; \mathrm{A2}' \gg \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}}; \mathrm{MP}' \rhd \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}} \rhd \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}} \gg \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}}; \mathrm{A1}' \gg \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}}; \mathrm{MP}' \rhd \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}} \rhd \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}} \gg \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{b}} \rceil, \mathrm{p}_0, \mathsf{c})]$$

2.1.3 Hypothetical modus ponens

The hypothetical version $[\mathrm{MP}'_{\mathrm{h}} \overset{\text{pyk}}{\to}$ "hypothetical rule prime mp"$][\mathrm{MP}'_{\mathrm{h}} \overset{\text{tex}}{\to}$ " MP'_h"$]$ of modus ponens MP' has a hypothesis $\underline{\mathsf{h}}$ on each premise and on the conclusion:

$$[\mathrm{MP}'_{\mathrm{h}} \overset{\text{stmt}}{\to} \mathrm{S}' \vdash \forall \underline{\mathsf{h}} \colon \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{a}} \dot{\Rightarrow} \underline{\mathsf{b}} \vdash \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{a}} \vdash \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{b}}]$$

$$[\mathrm{MP}'_{\mathrm{h}} \overset{\text{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{S}' \vdash \forall \underline{\mathsf{h}} \colon \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{a}} \dot{\Rightarrow} \underline{\mathsf{b}} \vdash \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{a}} \vdash \mathrm{A2}' \gg \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{a}} \dot{\Rightarrow} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{a}} \dot{\Rightarrow} \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{b}}; \mathrm{MP}' \rhd \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{a}} \dot{\Rightarrow} \underline{\mathsf{b}} \dot{\Rightarrow} \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{a}} \dot{\Rightarrow} \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{b}} \rhd \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{a}} \dot{\Rightarrow} \underline{\mathsf{b}} \gg \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{a}} \dot{\Rightarrow} \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{b}}; \mathrm{MP}' \rhd \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{a}} \dot{\Rightarrow} \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{b}} \rhd \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{a}} \gg \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{b}} \rceil, \mathrm{p}_0, \mathsf{c})]$$

We use $[\mathsf{x} \trianglerighteq_{\mathrm{h}} \mathsf{y} \overset{\text{pyk}}{\to}$ "$*$ hypothetical modus ponens $*$"$][\mathsf{x} \trianglerighteq_{\mathrm{h}} \mathsf{y} \overset{\text{tex}}{\to}$ "#1. \unrhd_h #2."$]$ as shorthand for hypothetical modus ponens:

$$[\mathsf{x} \trianglerighteq_{\mathrm{h}} \mathsf{y} \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil [\mathsf{x} \trianglerighteq_{\mathrm{h}} \mathsf{y} \doteqdot \mathrm{MP}'_{\mathrm{h}} \rhd \mathsf{x} \rhd \mathsf{y}] \rceil)]$$

2.1.4 Turning lemmas to hypothetical lemmas

Lemma $[\mathrm{Hypothesize} \overset{\text{pyk}}{\to}$ "hypothesize"$][\mathrm{Hypothesize} \overset{\text{tex}}{\to}$ " Hypothesize"$]$ turns a lemma with no premises into one that assumes the hypothesis $[\underline{\mathsf{h}}]$ to hold:

$$[\mathrm{Hypothesize} \overset{\text{stmt}}{\to} \mathrm{S}' \vdash \forall \underline{\mathsf{h}} \colon \forall \underline{\mathsf{a}} \colon \underline{\mathsf{a}} \vdash \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{a}}]$$

$[\text{Hypothesize} \stackrel{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil S' \vdash \forall \underline{h}\colon \forall \underline{a}\colon \underline{a} \vdash A1' \gg \underline{a} \Rightarrow \underline{h} \Rightarrow \underline{a}; MP' \rhd \underline{a} \Rightarrow \underline{h} \Rightarrow \underline{a} \rhd \underline{a} \gg \underline{h} \dot{\Rightarrow} \underline{a}\rceil, p_0, c)]$

2.2 First order predicate calculus

2.2.1 Hypothetical generalization

The hypothetical version $[\text{Gen}'_h \stackrel{\text{pyk}}{\rightarrow}$ "hypothetical rule prime gen"$][\text{Gen}'_h \stackrel{\text{tex}}{\rightarrow}$ " Gen'_h"] of generalisation Gen′ has a hypothesis $\underline{h}$ on premise and conclusion:

$[\text{Gen}'_h \stackrel{\text{stmt}}{\rightarrow} S' \vdash \forall \underline{h}\colon \forall \underline{x}\colon \forall \underline{a}\colon \dot{\text{nonfree}}(\lceil \underline{x}\rceil, \lceil \underline{h}\rceil) \Vdash \underline{h} \dot{\Rightarrow} \underline{a} \vdash \underline{h} \dot{\Rightarrow} \dot{\forall}\underline{x}\colon \underline{a}]$

$[\text{Gen}'_h \stackrel{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil S' \vdash \forall \underline{h}\colon \forall \underline{x}\colon \forall \underline{a}\colon \dot{\text{nonfree}}(\lceil \underline{x}\rceil, \lceil \underline{h}\rceil) \Vdash \underline{h} \dot{\Rightarrow} \underline{a} \vdash A5' \unrhd \dot{\text{nonfree}}(\lceil \underline{x}\rceil, \lceil \underline{h}\rceil) \gg \dot{\forall}\underline{x}\colon \underline{h} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{h} \dot{\Rightarrow} \dot{\forall}\underline{x}\colon \underline{a}; \text{Gen}' \rhd \underline{h} \dot{\Rightarrow} \underline{a} \gg \dot{\forall}\underline{x}\colon \underline{h} \dot{\Rightarrow} \underline{a}; MP' \rhd \dot{\forall}\underline{x}\colon \underline{h} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{h} \dot{\Rightarrow} \dot{\forall}\underline{x}\colon \underline{a} \rhd \dot{\forall}\underline{x}\colon \underline{h} \dot{\Rightarrow} \underline{a} \gg \underline{h} \dot{\Rightarrow} \dot{\forall}\underline{x}\colon \underline{a}\rceil, p_0, c)]$

2.3 Peano arithmetic

2.3.1 Lemma M3.2(a)

Lemma $[\text{M3.2(a)} \stackrel{\text{pyk}}{\rightarrow}$ "mendelson three two a"$][\text{M3.2(a)} \stackrel{\text{tex}}{\rightarrow}$ " M3.2(a)"] and the associated hypothetical lemma $[\text{M3.2(a)}_h \stackrel{\text{pyk}}{\rightarrow}$ "hypothetical three two a"$][\text{M3.2(a)}_h \stackrel{\text{tex}}{\rightarrow}$ " M3.2(a)_h"] read:

$[\text{M3.2(a)} \stackrel{\text{stmt}}{\rightarrow} S' \vdash \forall \underline{t}\colon \underline{t} \stackrel{p}{=} \underline{t}][\text{M3.2(a)} \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[\text{M3.2(a)}_h \stackrel{\text{stmt}}{\rightarrow} S' \vdash \forall \underline{h}\colon \forall \underline{t}\colon \underline{h} \dot{\Rightarrow} \underline{t} \stackrel{p}{=} \underline{t}]$

Above we cheat in stating M3.2(a) as a rule and not as a lemma. A reasonable way to construct a large proof is to start stating everything as rules and then changing the rules to lemmas one at a time until only the rules of the theory are left.

$[\text{M3.2(a)}_h \stackrel{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil S' \vdash \forall \underline{h}\colon \forall \underline{t}\colon \text{M3.2(a)} \gg \underline{t} \stackrel{p}{=} \underline{t}; \text{Hypothesize} \rhd \underline{t} \stackrel{p}{=} \underline{t} \gg \underline{h} \dot{\Rightarrow} \underline{t} \stackrel{p}{=} \underline{t}\rceil, p_0, c)]$

2.3.2 Lemma M3.2(b)

Lemma $[\text{M3.2(b)}_h \stackrel{\text{pyk}}{\rightarrow}$ "hypothetical three two b"$][\text{M3.2(b)}_h \stackrel{\text{tex}}{\rightarrow}$ " M3.2(b)_h"] reads:

$[\text{M3.2(b)}_h \stackrel{\text{stmt}}{\rightarrow} S' \vdash \forall \underline{h}\colon \forall \underline{t}\colon \forall \underline{r}\colon \underline{h} \dot{\Rightarrow} \underline{t} \stackrel{p}{=} \underline{r} \vdash \underline{h} \dot{\Rightarrow} \underline{r} \stackrel{p}{=} \underline{t}][\text{M3.2(b)}_h \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[\text{M3.2(b)}_h \stackrel{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil S' \vdash \forall \underline{h}\colon \forall \underline{t}\colon \forall \underline{r}\colon \underline{h} \dot{\Rightarrow} \underline{t} \stackrel{p}{=} \underline{r} \vdash S1' \gg \underline{t} \stackrel{p}{=} \underline{r} \dot{\Rightarrow} \underline{t} \stackrel{p}{=} \underline{t} \dot{\Rightarrow} \underline{r} \stackrel{p}{=} \underline{t}; \text{Hypothesize} \rhd \underline{t} \stackrel{p}{=} \underline{r} \dot{\Rightarrow} \underline{t} \stackrel{p}{=} \underline{t} \dot{\Rightarrow} \underline{r} \stackrel{p}{=} \underline{t} \gg \underline{h} \dot{\Rightarrow} \underline{t} \stackrel{p}{=} \underline{r} \dot{\Rightarrow} \underline{t} \stackrel{p}{=} \underline{t} \dot{\Rightarrow} \underline{r} \stackrel{p}{=} \underline{t}; MP'_h \rhd \underline{h} \dot{\Rightarrow} \underline{t} \stackrel{p}{=} \underline{r} \dot{\Rightarrow} \underline{t} \stackrel{p}{=} \underline{t} \dot{\Rightarrow} \underline{r} \stackrel{p}{=} \underline{t} \rhd \underline{h} \dot{\Rightarrow} \underline{t} \stackrel{p}{=} \underline{r} \gg \underline{h} \dot{\Rightarrow} \underline{t} \stackrel{p}{=} \underline{t} \dot{\Rightarrow} \underline{r} \stackrel{p}{=} \underline{t}; \text{M3.2(a)}_h \gg \underline{h} \dot{\Rightarrow} \underline{t} \stackrel{p}{=} \underline{t}; MP'_h \rhd \underline{h} \dot{\Rightarrow} \underline{t} \stackrel{p}{=} \underline{t} \dot{\Rightarrow} \underline{r} \stackrel{p}{=} \underline{t} \rhd \underline{h} \dot{\Rightarrow} \underline{t} \stackrel{p}{=} \underline{t} \gg \underline{h} \dot{\Rightarrow} \underline{r} \stackrel{p}{=} \underline{t}\rceil, p_0, c)]$

2.3.3 Lemma M3.1(S1)

Lemma [M3.1(S1′)$_{\mathrm{h}}$ $\overset{\text{pyk}}{\rightarrow}$ "hypothetical three one s one"][M3.1(S1′)$_{\mathrm{h}}$ $\overset{\text{tex}}{\rightarrow}$ " M3.1(S1')_h"] is the hypothetical version of Mendelson Lemma 3.1(S1′):

[M3.1(S1′)$_{\mathrm{h}}$ $\overset{\text{stmt}}{\rightarrow}$ $\mathsf{S}' \vdash \forall \underline{\mathsf{h}}\colon \forall \underline{\mathsf{t}}\colon \forall \underline{\mathsf{r}}\colon \forall \underline{\mathsf{s}}\colon \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}} \overset{\mathrm{p}}{=} \underline{\mathsf{r}} \vdash \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}} \overset{\mathrm{p}}{=} \underline{\mathsf{s}} \vdash \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{r}} \overset{\mathrm{p}}{=} \underline{\mathsf{s}}$][M3.1(S1′)$_{\mathrm{h}}$ $\overset{\text{proof}}{\rightarrow}$ Rule tactic]

2.3.4 Lemma M3.2(c)

Lemma [M3.2(c)$_{\mathrm{h}}$ $\overset{\text{pyk}}{\rightarrow}$ "hypothetical three two c"][M3.2(c)$_{\mathrm{h}}$ $\overset{\text{tex}}{\rightarrow}$ " M3.2(c)_h"] is the hypothetical version of Mendelson Lemma 3.2(c) which expresses ordinary transitivity:

[M3.2(c)$_{\mathrm{h}}$ $\overset{\text{stmt}}{\rightarrow}$ $\mathsf{S}' \vdash \forall \underline{\mathsf{h}}\colon \forall \underline{\mathsf{t}}\colon \forall \underline{\mathsf{r}}\colon \forall \underline{\mathsf{s}}\colon \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}} \overset{\mathrm{p}}{=} \underline{\mathsf{r}} \vdash \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{r}} \overset{\mathrm{p}}{=} \underline{\mathsf{s}} \vdash \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}} \overset{\mathrm{p}}{=} \underline{\mathsf{s}}$][M3.2(c)$_{\mathrm{h}}$ $\overset{\text{proof}}{\rightarrow}$ Rule tactic]

2.3.5 Lemma M3.1(S2)

Lemma [M3.1(S2′)$_{\mathrm{h}}$ $\overset{\text{pyk}}{\rightarrow}$ "hypothetical three one s two"][M3.1(S2′)$_{\mathrm{h}}$ $\overset{\text{tex}}{\rightarrow}$ " M3.1(S2')_h"] is the hypothetical version of Mendelson Lemma 3.1(S2′):

[M3.1(S2′)$_{\mathrm{h}}$ $\overset{\text{stmt}}{\rightarrow}$ $\mathsf{S}' \vdash \forall \underline{\mathsf{h}}\colon \forall \underline{\mathsf{t}}\colon \forall \underline{\mathsf{r}}\colon \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}} \overset{\mathrm{p}}{=} \underline{\mathsf{r}} \vdash \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}}' \overset{\mathrm{p}}{=} \underline{\mathsf{r}}'$][M3.1(S2′)$_{\mathrm{h}}$ $\overset{\text{proof}}{\rightarrow}$ Rule tactic]

2.3.6 Lemma M3.1(S5)

Lemma [M3.1(S5′)$_{\mathrm{h}}$ $\overset{\text{pyk}}{\rightarrow}$ "hypothetical three one s five"][M3.1(S5′)$_{\mathrm{h}}$ $\overset{\text{tex}}{\rightarrow}$ " M3.1(S5')_h"] is the hypothetical version of Mendelson Lemma 3.1(S5′):

[M3.1(S5′)$_{\mathrm{h}}$ $\overset{\text{stmt}}{\rightarrow}$ $\mathsf{S}' \vdash \forall \underline{\mathsf{h}}\colon \forall \underline{\mathsf{t}}\colon \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}} \dot{+} \dot{0} \overset{\mathrm{p}}{=} \underline{\mathsf{t}}$][M3.1(S5′)$_{\mathrm{h}}$ $\overset{\text{proof}}{\rightarrow}$ Rule tactic]

2.3.7 Lemma M3.1(S6)

Lemma [M3.1(S6′)$_{\mathrm{h}}$ $\overset{\text{pyk}}{\rightarrow}$ "hypothetical three one s six"][M3.1(S6′)$_{\mathrm{h}}$ $\overset{\text{tex}}{\rightarrow}$ " M3.1(S6')_h"] is the hypothetical version of Mendelson Lemma 3.1(S6′):

[M3.1(S6′)$_{\mathrm{h}}$ $\overset{\text{stmt}}{\rightarrow}$ $\mathsf{S}' \vdash \forall \underline{\mathsf{h}}\colon \forall \underline{\mathsf{t}}\colon \forall \underline{\mathsf{r}}\colon \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}} \dot{+} \underline{\mathsf{r}}' \overset{\mathrm{p}}{=} \underline{\mathsf{t}} \dot{+} \underline{\mathsf{r}}'$][M3.1(S6′)$_{\mathrm{h}}$ $\overset{\text{proof}}{\rightarrow}$ Rule tactic]

2.3.8 Lemma M3.2(f)

Lemma [M3.2(f) $\overset{\text{pyk}}{\rightarrow}$ "mendelson three two f"][M3.2(f) $\overset{\text{tex}}{\rightarrow}$ " M3.2(f)"] is the closure of Mendelson Lemma 3.2(f) for the concrete variable [$\dot{\mathsf{t}}$]:

[M3.2(f) $\overset{\text{stmt}}{\rightarrow}$ $\mathsf{S}' \vdash \dot{\forall} \dot{\mathsf{t}}\colon \dot{\mathsf{t}} \overset{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}$]

The proof below uses local macro definitions.

$$[\mathrm{M3.2(f)} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{S}' \vdash \mathrm{A1}' \gg \mathsf{x} \mathrel{\dot{\Rightarrow}} \mathsf{x} \mathrel{\dot{\Rightarrow}} \mathsf{x}; \mathrm{M3.1(S5')_h} \gg \mathsf{x} \mathrel{\dot{\Rightarrow}} \mathsf{x} \mathrel{\dot{\Rightarrow}} \mathsf{x} \mathrel{\dot{\Rightarrow}} \dot{0} \mathbin{\dot{+}} \dot{0} \stackrel{p}{=} \dot{0}; \mathrm{M3.2(b)_h} \rhd \mathsf{x} \mathrel{\dot{\Rightarrow}} \mathsf{x} \mathrel{\dot{\Rightarrow}} \mathsf{x} \mathrel{\dot{\Rightarrow}} \dot{0} \mathbin{\dot{+}} \dot{0} \stackrel{p}{=} \dot{0} \gg \mathsf{x} \mathrel{\dot{\Rightarrow}} \mathsf{x} \mathrel{\dot{\Rightarrow}} \mathsf{x} \mathrel{\dot{\Rightarrow}} \dot{0} \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{0}; \mathrm{MP}' \rhd \mathsf{x} \mathrel{\dot{\Rightarrow}} \mathsf{x} \mathrel{\dot{\Rightarrow}} \mathsf{x} \mathrel{\dot{\Rightarrow}} \dot{0} \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{0} \rhd \mathsf{x} \mathrel{\dot{\Rightarrow}} \mathsf{x} \mathrel{\dot{\Rightarrow}} \mathsf{x} \gg \dot{0} \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{0}; \mathrm{M1.7} \gg \dot{\mathsf{t}} \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}} \mathrel{\dot{\Rightarrow}} \dot{\mathsf{t}} \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}}; \mathrm{M3.1(S2')_h} \rhd \dot{\mathsf{t}} \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}} \mathrel{\dot{\Rightarrow}} \dot{\mathsf{t}} \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}} \gg \dot{\mathsf{t}} \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}} \mathrel{\dot{\Rightarrow}} \dot{\mathsf{t}}' \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}}'; \mathrm{M3.1(S6')_h} \gg \dot{\mathsf{t}} \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}} \mathrel{\dot{\Rightarrow}} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}}' \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}}'; \mathrm{M3.2(b)_h} \rhd \dot{\mathsf{t}} \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}} \mathrel{\dot{\Rightarrow}} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}}' \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}}' \gg \dot{\mathsf{t}} \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}} \mathrel{\dot{\Rightarrow}} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}}' \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}}'; \mathrm{M3.2(c)_h} \rhd \dot{\mathsf{t}} \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}} \mathrel{\dot{\Rightarrow}} \dot{\mathsf{t}}' \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}}' \rhd \dot{\mathsf{t}} \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}} \mathrel{\dot{\Rightarrow}} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}}' \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}}' \gg \dot{\mathsf{t}} \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}} \mathrel{\dot{\Rightarrow}} \dot{\mathsf{t}}' \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}}'; \mathrm{Gen}' \rhd \dot{\mathsf{t}} \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}} \mathrel{\dot{\Rightarrow}} \dot{\mathsf{t}}' \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}}' \gg \dot{\forall}\dot{\mathsf{t}}\colon \dot{\mathsf{t}} \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}} \mathrel{\dot{\Rightarrow}} \dot{\mathsf{t}}' \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}}'; \mathrm{S9}' \gg \dot{0} \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{0} \mathrel{\dot{\Rightarrow}} \dot{\forall}\dot{\mathsf{t}}\colon \dot{\mathsf{t}} \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}} \mathrel{\dot{\Rightarrow}} \dot{\mathsf{t}}' \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}}' \mathrel{\dot{\Rightarrow}} \dot{\forall}\dot{\mathsf{t}}\colon \dot{\mathsf{t}} \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}}; \mathrm{MP}' \rhd \dot{0} \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{0} \mathrel{\dot{\Rightarrow}} \dot{\forall}\dot{\mathsf{t}}\colon \dot{\mathsf{t}} \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}} \mathrel{\dot{\Rightarrow}} \dot{\mathsf{t}}' \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}}' \mathrel{\dot{\Rightarrow}} \dot{\forall}\dot{\mathsf{t}}\colon \dot{\mathsf{t}} \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}} \rhd \dot{0} \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{0} \gg \dot{\forall}\dot{\mathsf{t}}\colon \dot{\mathsf{t}} \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}} \mathrel{\dot{\Rightarrow}} \dot{\mathsf{t}}' \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}}' \mathrel{\dot{\Rightarrow}} \dot{\forall}\dot{\mathsf{t}}\colon \dot{\mathsf{t}} \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}}; \mathrm{MP}' \rhd \dot{\forall}\dot{\mathsf{t}}\colon \dot{\mathsf{t}} \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}} \mathrel{\dot{\Rightarrow}} \dot{\mathsf{t}}' \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}}' \mathrel{\dot{\Rightarrow}} \dot{\forall}\dot{\mathsf{t}}\colon \dot{\mathsf{t}} \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}} \rhd \dot{\forall}\dot{\mathsf{t}}\colon \dot{\mathsf{t}} \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}} \mathrel{\dot{\Rightarrow}} \dot{\mathsf{t}}' \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}}' \gg \dot{\forall}\dot{\mathsf{t}}\colon \dot{\mathsf{t}} \stackrel{p}{=} \dot{0} \mathbin{\dot{+}} \dot{\mathsf{t}} \rceil, \mathrm{p}_0, \mathsf{c})]$$

A Chores

A.1 The name of the page

This defines the name of the page:

$[\mathrm{peano} \stackrel{\mathrm{pyk}}{\rightarrow}$ "peano"$]$

A.2 Variables of Peano arithmetic

We use $[\dot{b} \stackrel{\mathrm{pyk}}{\rightarrow}$ "peano b"$][\dot{b} \stackrel{\mathrm{tex}}{\rightarrow}$ "
\dot{\mathit{b}}"], $[\dot{c} \stackrel{\mathrm{pyk}}{\rightarrow}$ "peano c"$][\dot{c} \stackrel{\mathrm{tex}}{\rightarrow}$ "
\dot{\mathit{c}}"], $[\dot{d} \stackrel{\mathrm{pyk}}{\rightarrow}$ "peano d"$][\dot{d} \stackrel{\mathrm{tex}}{\rightarrow}$ "
\dot{\mathit{d}}"], $[\dot{e} \stackrel{\mathrm{pyk}}{\rightarrow}$ "peano e"$][\dot{e} \stackrel{\mathrm{tex}}{\rightarrow}$ "
\dot{\mathit{e}}"], $[\dot{f} \stackrel{\mathrm{pyk}}{\rightarrow}$ "peano f"$][\dot{f} \stackrel{\mathrm{tex}}{\rightarrow}$ "
\dot{\mathit{f}}"], $[\dot{g} \stackrel{\mathrm{pyk}}{\rightarrow}$ "peano g"$][\dot{g} \stackrel{\mathrm{tex}}{\rightarrow}$ "
\dot{\mathit{g}}"], $[\dot{h} \stackrel{\mathrm{pyk}}{\rightarrow}$ "peano h"$][\dot{h} \stackrel{\mathrm{tex}}{\rightarrow}$ "
\dot{\mathit{h}}"], $[\dot{i} \stackrel{\mathrm{pyk}}{\rightarrow}$ "peano i"$][\dot{i} \stackrel{\mathrm{tex}}{\rightarrow}$ "
\dot{\mathit{i}}"], $[\dot{j} \stackrel{\mathrm{pyk}}{\rightarrow}$ "peano j"$][\dot{j} \stackrel{\mathrm{tex}}{\rightarrow}$ "
\dot{\mathit{j}}"], $[\dot{k} \stackrel{\mathrm{pyk}}{\rightarrow}$ "peano k"$][\dot{k} \stackrel{\mathrm{tex}}{\rightarrow}$ "
\dot{\mathit{k}}"], $[\dot{l} \stackrel{\mathrm{pyk}}{\rightarrow}$ "peano l"$][\dot{l} \stackrel{\mathrm{tex}}{\rightarrow}$ "
\dot{\mathit{l}}"], $[\dot{m} \stackrel{\mathrm{pyk}}{\rightarrow}$ "peano m"$][\dot{m} \stackrel{\mathrm{tex}}{\rightarrow}$ "
\dot{\mathit{m}}"], $[\dot{n} \stackrel{\mathrm{pyk}}{\rightarrow}$ "peano n"$][\dot{n} \stackrel{\mathrm{tex}}{\rightarrow}$ "
\dot{\mathit{n}}"], $[\dot{o} \stackrel{\mathrm{pyk}}{\rightarrow}$ "peano o"$][\dot{o} \stackrel{\mathrm{tex}}{\rightarrow}$ "
\dot{\mathit{o}}"], $[\dot{p} \stackrel{\mathrm{pyk}}{\rightarrow}$ "peano p"$][\dot{p} \stackrel{\mathrm{tex}}{\rightarrow}$ "
\dot{\mathit{p}}"], $[\dot{q} \stackrel{\mathrm{pyk}}{\rightarrow}$ "peano q"$][\dot{q} \stackrel{\mathrm{tex}}{\rightarrow}$ "
\dot{\mathit{q}}"], $[\dot{r} \stackrel{\mathrm{pyk}}{\rightarrow}$ "peano r"$][\dot{r} \stackrel{\mathrm{tex}}{\rightarrow}$ "
\dot{\mathit{r}}"], $[\dot{s} \stackrel{\mathrm{pyk}}{\rightarrow}$ "peano s"$][\dot{s} \stackrel{\mathrm{tex}}{\rightarrow}$ "

\dot{\mathit{s}}"], [$\dot{t} \overset{\text{pyk}}{\to}$ "peano t"][$\dot{t} \overset{\text{tex}}{\to}$ "
\dot{\mathit{t}}"], [$\dot{u} \overset{\text{pyk}}{\to}$ "peano u"][$\dot{u} \overset{\text{tex}}{\to}$ "
\dot{\mathit{u}}"], [$\dot{v} \overset{\text{pyk}}{\to}$ "peano v"][$\dot{v} \overset{\text{tex}}{\to}$ "
\dot{\mathit{v}}"], [$\dot{w} \overset{\text{pyk}}{\to}$ "peano w"][$\dot{w} \overset{\text{tex}}{\to}$ "
\dot{\mathit{w}}"], [$\dot{x} \overset{\text{pyk}}{\to}$ "peano x"][$\dot{x} \overset{\text{tex}}{\to}$ "
\dot{\mathit{x}}"], [$\dot{y} \overset{\text{pyk}}{\to}$ "peano y"][$\dot{y} \overset{\text{tex}}{\to}$ "
\dot{\mathit{y}}"], and [$\dot{z} \overset{\text{pyk}}{\to}$ "peano z"][$\dot{z} \overset{\text{tex}}{\to}$ "
\dot{\mathit{z}}"] to denote variables of Peano arithmetic:
$[\dot{b} \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{b} \doteqdot \dot{\mathsf{b}}]\rceil)]$, $[\dot{c} \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{c} \doteqdot \dot{\mathsf{c}}]\rceil)]$,
$[\dot{d} \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{d} \doteqdot \dot{\mathsf{d}}]\rceil)]$, $[\dot{e} \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{e} \doteqdot \dot{\mathsf{e}}]\rceil)]$,
$[\dot{f} \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{f} \doteqdot \dot{\mathsf{f}}]\rceil)]$, $[\dot{g} \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{g} \doteqdot \dot{\mathsf{g}}]\rceil)]$,
$[\dot{h} \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{h} \doteqdot \dot{\mathsf{h}}]\rceil)]$, $[\dot{i} \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{i} \doteqdot \dot{\mathsf{i}}]\rceil)]$,
$[\dot{j} \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{j} \doteqdot \dot{\mathsf{j}}]\rceil)]$, $[\dot{k} \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{k} \doteqdot \dot{\mathsf{k}}]\rceil)]$,
$[\dot{l} \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{l} \doteqdot \dot{\mathsf{l}}]\rceil)]$, $[\dot{m} \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{m} \doteqdot \dot{\mathsf{m}}]\rceil)]$,
$[\dot{n} \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{n} \doteqdot \dot{\mathsf{n}}]\rceil)]$, $[\dot{o} \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{o} \doteqdot \dot{\mathsf{o}}]\rceil)]$,
$[\dot{p} \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{p} \doteqdot \dot{\mathsf{p}}]\rceil)]$, $[\dot{q} \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{q} \doteqdot \dot{\mathsf{q}}]\rceil)]$,
$[\dot{r} \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{r} \doteqdot \dot{\mathsf{r}}]\rceil)]$, $[\dot{s} \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{s} \doteqdot \dot{\mathsf{s}}]\rceil)]$,
$[\dot{t} \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{t} \doteqdot \dot{\mathsf{t}}]\rceil)]$, $[\dot{u} \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{u} \doteqdot \dot{\mathsf{u}}]\rceil)]$,
$[\dot{v} \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{v} \doteqdot \dot{\mathsf{v}}]\rceil)]$, $[\dot{w} \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{w} \doteqdot \dot{\mathsf{w}}]\rceil)]$,
$[\dot{x} \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{x} \doteqdot \dot{\mathsf{x}}]\rceil)]$, $[\dot{y} \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{y} \doteqdot \dot{\mathsf{y}}]\rceil)]$,
and $[\dot{z} \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{z} \doteqdot \dot{\mathsf{z}}]\rceil)]$.

A.3 TEX definitions

A.4 Test

$[\lceil \dot{\mathsf{a}} \rceil^{\mathcal{P}}]^{\cdot}$

$[\lceil \mathsf{a} \rceil^{\mathcal{P}}]^{-}$

$[\dot{\text{nonfree}}(\lceil \dot{\mathsf{x}} \rceil, \lceil \dot{\mathsf{y}} \overset{\text{p}}{=} \dot{\mathsf{z}} \dot{\Rightarrow} \dot{\forall} \dot{\mathsf{x}}\colon \dot{\mathsf{x}} \overset{\text{p}}{=} \dot{\mathsf{y}} \rceil)]^{\cdot}$

$[\dot{\text{nonfree}}(\lceil \dot{\mathsf{x}} \rceil, \lceil \dot{\mathsf{x}} \overset{\text{p}}{=} \dot{\mathsf{z}} \dot{\Rightarrow} \dot{\forall} \dot{\mathsf{x}}\colon \dot{\mathsf{x}} \overset{\text{p}}{=} \dot{\mathsf{y}} \rceil)]^{-}$

$[\dot{\text{nonfree}}(\lceil \dot{\mathsf{x}} \rceil, \lceil \dot{\mathsf{y}} \overset{\text{p}}{=} \dot{\mathsf{x}} \dot{\Rightarrow} \dot{\forall} \dot{\mathsf{x}}\colon \dot{\mathsf{x}} \overset{\text{p}}{=} \dot{\mathsf{y}} \rceil)]^{-}$

$[\dot{\text{nonfree}}(\lceil \dot{\mathsf{x}} \rceil, \lceil \dot{\mathsf{y}} \overset{\text{p}}{=} \dot{\mathsf{z}} \dot{\Rightarrow} \dot{\forall} \dot{\mathsf{y}}\colon \dot{\mathsf{x}} \overset{\text{p}}{=} \dot{\mathsf{y}} \rceil)]^{-}$

$[\dot{\text{free}}\langle \lceil \dot{\forall} \dot{\mathsf{x}}\colon \mathsf{b} :: \dot{\mathsf{x}} :: \mathsf{c} \rceil | \lceil \dot{\mathsf{x}} \rceil := \lceil \mathsf{x} :: \dot{\mathsf{y}} :: \mathsf{z} \rceil \rangle]^{\cdot}$

$[\dot{\text{free}}\langle \lceil \dot{\forall} \dot{\mathsf{y}}\colon \mathsf{b} :: \dot{\mathsf{x}} :: \mathsf{c} \rceil | \lceil \dot{\mathsf{x}} \rceil := \lceil \mathsf{x} :: \dot{\mathsf{y}} :: \mathsf{z} \rceil \rangle]^{-}$

$[\dot{\text{free}}\langle \lceil \dot{\forall} \dot{\mathsf{x}}\colon \mathsf{b} :: \dot{\mathsf{x}} :: \mathsf{c} \rceil | \lceil \dot{\mathsf{y}} \rceil := \lceil \mathsf{x} :: \dot{\mathsf{y}} :: \mathsf{z} \rceil \rangle]^{\cdot}$

$[\mathrm{free}\langle\lceil\dot\forall\dot{\mathsf{y}}\colon \mathsf{b} :: \dot{\mathsf{x}} :: \mathsf{c}\rceil | \lceil\dot{\mathsf{y}}\rceil := \lceil \mathsf{x} :: \dot{\mathsf{y}} :: \mathsf{z}\rceil\rangle]^{\cdot}$

$[\dot{\mathsf{a}}\equiv\langle\dot{\mathsf{a}}|\dot{\mathsf{b}} := \dot{\mathsf{c}}\rangle]^{\cdot}$

$[\dot{\mathsf{c}}\equiv\langle\dot{\mathsf{b}}|\dot{\mathsf{b}} := \dot{\mathsf{c}}\rangle]^{\cdot}$

$[\forall\dot{\mathsf{a}}\colon \dot{\mathsf{a}} \stackrel{\mathrm{p}}{=} \dot{\mathsf{b}}\equiv\langle\forall\dot{\mathsf{a}}\colon \dot{\mathsf{a}} \stackrel{\mathrm{p}}{=} \dot{\mathsf{b}}|\dot{\mathsf{a}} := \dot{\mathsf{c}}\rangle]^{\cdot}$

$[\forall\dot{\mathsf{a}}\colon \dot{\mathsf{a}} \stackrel{\mathrm{p}}{=} \dot{\mathsf{c}}\equiv\langle\forall\dot{\mathsf{a}}\colon \dot{\mathsf{a}} \stackrel{\mathrm{p}}{=} \dot{\mathsf{b}}|\dot{\mathsf{b}} := \dot{\mathsf{c}}\rangle]^{\cdot}$

$[\dot\forall\dot{\mathsf{a}}\colon \dot{\mathsf{a}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{a}} \dot{\Rightarrow} \dot{\mathsf{c}} \mathbin{\dot{:}} \dot{\mathsf{d}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{c}} \mathbin{\dot{:}} \dot{\mathsf{d}}\equiv\langle\dot\forall\dot{\mathsf{a}}\colon \dot{\mathsf{a}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{a}} \dot{\Rightarrow} \dot{\mathsf{b}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{b}}|\dot{\mathsf{b}} := \dot{\mathsf{c}} \mathbin{\dot{:}} \dot{\mathsf{d}}\rangle]^{\cdot}$

$[\dot\forall\dot{\mathsf{a}}\colon \dot{\mathsf{a}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{a}} \dot{\Rightarrow} \dot{\mathsf{b}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{b}}\equiv\langle\dot\forall\dot{\mathsf{a}}\colon \dot{\mathsf{a}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{a}} \dot{\Rightarrow} \dot{\mathsf{b}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{b}}|\dot{\mathsf{a}} := \dot{\mathsf{c}}\rangle]^{\cdot}$

A.5 Priority table

[peano $\stackrel{\mathrm{prio}}{\rightarrow}$
Preassociative
[peano], [base], [bracket $*$ end bracket], [big bracket $*$ end bracket], [math $*$ end math], [**flush left** [$*$]], [x], [y], [z], [[$* \bowtie *$]], [[$* \stackrel{*}{\rightarrow} *$]], [pyk], [tex], [name], [prio], [$*$], [T], [if($*, *, *$)], [[$* \stackrel{*}{\Rightarrow} *$]], [val], [claim], [$\perp$], [f($*$)], [$(*)^{\mathrm{I}}$], [$\mathsf{F}$], [$\underline{0}$], [$\underline{1}$], [$\underline{2}$], [$\underline{3}$], [$\underline{4}$], [$\underline{5}$], [$\underline{6}$], [$\underline{7}$], [$\underline{8}$], [$\underline{9}$], [0], [1], [2], [3], [4], [5], [6], [7], [8], [9], [a], [b], [c], [d], [e], [f], [g], [h], [i], [j], [k], [l], [m], [n], [o], [p], [q], [r], [s], [t], [u], [v], [w], [$(*)^{\mathrm{M}}$], [If($*, *, *$)], [array{$*$} $*$ end array], [l], [c], [r], [empty], [$\langle * \mid * := * \rangle$], [$\mathcal{M}(*)$], [$\tilde{\mathcal{U}}(*)$], [$\mathcal{U}(*)$], [$\mathcal{U}^{\mathrm{M}}(*)$], [**apply**($*, *$)], [$\mathbf{apply}_1(*, *)$], [identifier($*$)], [$\mathrm{identifier}_1(*, *)$], [array-plus($*, *$)], [array-remove($*, *, *$)], [array-put($*, *, *, *$)], [array-add($*, *, *, *, *$)], [bit($*, *$)], [$\mathrm{bit}_1(*, *)$], [rack], ["vector"], ["bibliography"], ["dictionary"], ["body"], ["codex"], ["expansion"], ["code"], ["cache"], ["diagnose"], ["pyk"], ["tex"], ["texname"], ["value"], ["message"], ["macro"], ["definition"], ["unpack"], ["claim"], ["priority"], ["lambda"], ["apply"], ["true"], ["if"], ["quote"], ["proclaim"], ["define"], ["introduce"], ["hide"], ["pre"], ["post"], [$\mathcal{E}(*, *, *)$], [$\mathcal{E}_2(*, *, *, *, *)$], [$\mathcal{E}_3(*, *, *, *)$], [$\mathcal{E}_4(*, *, *, *)$], [**lookup**($*, *, *$)], [**abstract**($*, *, *, *$)], [[$\lceil *\rceil$]], [$\mathcal{M}(*, *, *)$], [$\mathcal{M}_2(*, *, *, *)$], [$\mathcal{M}^*(*, *, *)$], [macro], [$\mathrm{s}_0$], [**zip**($*, *$)], [$\mathbf{assoc}_1(*, *, *)$], [$(*)^{\mathbf{P}}$], [self], [[$* \stackrel{..}{=} *$]], [[$* \doteq *$]], [[$* \stackrel{\prime}{=} *$]], [[$* \stackrel{\mathrm{pyk}}{=} *$]], [[$* \stackrel{\mathrm{tex}}{=} *$]], [[$* \stackrel{\mathrm{name}}{=} *$]], [**Priority table**[$*$]], [$\tilde{\mathcal{M}}_1$], [$\tilde{\mathcal{M}}_2(*)$], [$\tilde{\mathcal{M}}_3(*)$], [$\tilde{\mathcal{M}}_4(*, *, *, *)$], [$\tilde{\mathcal{M}}(*, *, *)$], [$\tilde{\mathcal{Q}}(*, *, *)$], [$\tilde{\mathcal{Q}}_2(*, *, *)$], [$\tilde{\mathcal{Q}}_3(*, *, *, *)$], [$\tilde{\mathcal{Q}}^*(*, *, *)$], [$(*)$], [**aspect**($*, *$)], [**aspect**($*, *, *$)], [$\langle * \rangle$], [$\mathbf{tuple}_1(*)$], [$\mathbf{tuple}_2(*)$], [$\mathrm{let}_2(*, *)$], [$\mathrm{let}_1(*, *)$], [[$* \stackrel{\mathrm{claim}}{=} *$]], [checker], [**check**($*, *$)], [$\mathbf{check}_2(*, *, *)$], [$\mathbf{check}_3(*, *, *)$], [$\mathbf{check}^*(*, *)$], [$\mathbf{check}^*_2(*, *, *)$], [$[*]^{\cdot}$], [$[*]^{-}$], [$[*]^{\circ}$], [msg], [[$* \stackrel{\mathrm{msg}}{=} *$]], [<stmt>], [stmt], [[$* \stackrel{\mathrm{stmt}}{=} *$]], [HeadNil$'$], [HeadPair$'$], [Transitivity$'$], [$\perp\!\!\!\perp$], [Contra$'$], [$\mathrm{T}'_{\mathrm{E}}$], [$\mathrm{L}_1$], [$\underline{*}$], [$\mathcal{A}$], [$\mathcal{B}$], [$\mathcal{C}$], [$\mathcal{D}$], [$\mathcal{E}$], [$\mathcal{F}$], [$\mathcal{G}$], [$\mathcal{H}$], [$\mathcal{I}$], [$\mathcal{J}$], [$\mathcal{K}$], [$\mathcal{L}$], [$\mathcal{M}$], [$\mathcal{N}$], [$\mathcal{O}$], [$\mathcal{P}$], [$\mathcal{Q}$], [$\mathcal{R}$], [$\mathcal{S}$], [$\mathcal{T}$], [$\mathcal{U}$], [$\mathcal{V}$], [$\mathcal{W}$], [$\mathcal{X}$], [$\mathcal{Y}$], [$\mathcal{Z}$], [$\langle * \mid * := * \rangle$], [$\langle^* * \mid * := * \rangle$], [$\emptyset$], [Remainder], [$(*)^{\mathbf{v}}$], [intro($*, *, *, *$)], [intro($*, *, *$)], [error($*, *$)], [$\mathrm{error}_2(*, *)$], [proof($*, *, *$)], [$\mathrm{proof}_2(*, *)$], [$\mathcal{S}(*, *)$], [$\mathcal{S}^{\mathrm{I}}(*, *)$], [$\mathcal{S}^{\rhd}(*, *)$], [$\mathcal{S}^{\rhd}_1(*, *, *)$], [$\mathcal{S}^{\mathrm{E}}(*, *)$], [$\mathcal{S}^{\mathrm{E}}_1(*, *, *)$],

$[\mathcal{S}^{+}(\ast,\ast)], [\mathcal{S}_1^{+}(\ast,\ast,\ast)], [\mathcal{S}^{-}(\ast,\ast)], [\mathcal{S}_1^{-}(\ast,\ast,\ast)], [\mathcal{S}^{\ast}(\ast,\ast)], [\mathcal{S}_1^{\ast}(\ast,\ast,\ast)],$
$[\mathcal{S}_2^{\ast}(\ast,\ast,\ast,\ast)], [\mathcal{S}^{@}(\ast,\ast)], [\mathcal{S}_1^{@}(\ast,\ast,\ast)], [\mathcal{S}^{\vdash}(\ast,\ast)], [\mathcal{S}_1^{\vdash}(\ast,\ast,\ast,\ast)], [\mathcal{S}^{\Vdash}(\ast,\ast)],$
$[\mathcal{S}_1^{\Vdash}(\ast,\ast,\ast,\ast)], [\mathcal{S}^{\text{i.e.}}(\ast,\ast)], [\mathcal{S}_1^{\text{i.e.}}(\ast,\ast,\ast,\ast)], [\mathcal{S}_2^{\text{i.e.}}(\ast,\ast,\ast,\ast,\ast)], [\mathcal{S}^{\forall}(\ast,\ast)],$
$[\mathcal{S}_1^{\forall}(\ast,\ast,\ast,\ast)], [\mathcal{S}^{;}(\ast,\ast)], [\mathcal{S}_1^{;}(\ast,\ast,\ast)], [\mathcal{S}_2^{;}(\ast,\ast,\ast,\ast)], [\mathcal{T}(\ast)], [\text{claims}(\ast,\ast,\ast)],$
$[\text{claims}_2(\ast,\ast,\ast)]$, [<proof>], [proof], [[**Lemma** $\ast{:}\,\ast$]], [[**Proof of** $\ast{:}\,\ast$]],
[[$\ast$ **lemma** $\ast{:}\,\ast$]], [[$\ast$ **antilemma** $\ast{:}\,\ast$]], [[$\ast$ **rule** $\ast{:}\,\ast$]], [[$\ast$ **antirule** $\ast{:}\,\ast$]],
$[\text{verifier}], [\mathcal{V}_1(\ast)], [\mathcal{V}_2(\ast,\ast)], [\mathcal{V}_3(\ast,\ast,\ast,\ast)], [\mathcal{V}_4(\ast,\ast)], [\mathcal{V}_5(\ast,\ast,\ast,\ast)], [\mathcal{V}_6(\ast,\ast,\ast,\ast)],$
$[\mathcal{V}_7(\ast,\ast,\ast,\ast)], [\text{Cut}(\ast,\ast)], [\text{Head}_{\oplus}(\ast)], [\text{Tail}_{\oplus}(\ast)], [\text{rule}_1(\ast,\ast)], [\text{rule}(\ast,\ast)],$
[Rule tactic], $[\text{Plus}(\ast,\ast)]$, [[**Theory** $\ast$]], $[\text{theory}_2(\ast,\ast)], [\text{theory}_3(\ast,\ast)],$
$[\text{theory}_4(\ast,\ast,\ast)], [\text{HeadNil}''], [\text{HeadPair}''], [\text{Transitivity}''], [\text{Contra}''], [\text{HeadNil}],$
[HeadPair], [Transitivity], [Contra], $[\mathrm{T_E}]$, [ragged right],
[ragged right expansion], $[\text{parm}(\ast,\ast,\ast)], [\text{parm}^{\ast}(\ast,\ast,\ast)], [\text{inst}(\ast,\ast)],$
$[\text{inst}^{\ast}(\ast,\ast)], [\text{occur}(\ast,\ast,\ast)], [\text{occur}^{\ast}(\ast,\ast,\ast)], [\text{unify}(\ast=\ast,\ast)], [\text{unify}^{\ast}(\ast=\ast,\ast)],$
$[\text{unify}_2(\ast=\ast,\ast)], [\mathrm{L_a}], [\mathrm{L_b}], [\mathrm{L_c}], [\mathrm{L_d}], [\mathrm{L_e}], [\mathrm{L_f}], [\mathrm{L_g}], [\mathrm{L_h}], [\mathrm{L_i}], [\mathrm{L_j}], [\mathrm{L_k}], [\mathrm{L_l}], [\mathrm{L_m}],$
$[\mathrm{L_n}], [\mathrm{L_o}], [\mathrm{L_p}], [\mathrm{L_q}], [\mathrm{L_r}], [\mathrm{L_s}], [\mathrm{L_t}], [\mathrm{L_u}], [\mathrm{L_v}], [\mathrm{L_w}], [\mathrm{L_x}], [\mathrm{L_y}], [\mathrm{L_z}], [\mathrm{L_A}], [\mathrm{L_B}], [\mathrm{L_C}],$
$[\mathrm{L_D}], [\mathrm{L_E}], [\mathrm{L_F}], [\mathrm{L_G}], [\mathrm{L_H}], [\mathrm{L_I}], [\mathrm{L_J}], [\mathrm{L_K}], [\mathrm{L_L}], [\mathrm{L_M}], [\mathrm{L_N}], [\mathrm{L_O}], [\mathrm{L_P}], [\mathrm{L_Q}], [\mathrm{L_R}],$
$[\mathrm{L_S}], [\mathrm{L_T}], [\mathrm{L_U}], [\mathrm{L_V}], [\mathrm{L_W}], [\mathrm{L_X}], [\mathrm{L_Y}], [\mathrm{L_Z}], [\mathrm{L_?}]$, [Reflexivity], $[\text{Reflexivity}_1],$
[Commutativity], $[\text{Commutativity}_1]$, [<tactic>], [tactic], $[[\ast \overset{\text{tactic}}{=} \ast]], [\mathcal{P}(\ast,\ast,\ast)],$
$[\mathcal{P}^{\ast}(\ast,\ast,\ast)], [\mathrm{p}_0], [\text{conclude}_1(\ast,\ast)], [\text{conclude}_2(\ast,\ast,\ast)], [\text{conclude}_3(\ast,\ast,\ast,\ast)],$
$[\text{conclude}_4(\ast,\ast)], [\dot{0}], [\dot{1}], [\dot{2}], [\dot{a}], [\dot{b}], [\dot{c}], [\dot{d}], [\dot{e}], [\dot{f}], [\dot{g}], [\dot{h}], [\dot{i}], [\dot{j}], [\dot{k}], [\dot{l}], [\dot{m}], [\dot{n}],$
$[\dot{o}], [\dot{p}], [\dot{q}], [\dot{r}], [\dot{s}], [\dot{t}], [\dot{u}], [\dot{v}], [\dot{w}], [\dot{x}], [\dot{y}], [\dot{z}], [\text{nonfree}(\ast,\ast)], [\text{nonfree}^{\ast}(\ast,\ast)],$
$[\text{free}\langle\ast|\ast:=\ast\rangle], [\text{free}^{\ast}\langle\ast|\ast:=\ast\rangle], [\ast{\equiv}\langle\ast|\ast:=\ast\rangle], [\ast{\equiv}\langle^{\ast}\ast|\ast:=\ast\rangle]$, [S], [A1], [A2],
[A3], [A4], [A5], [S1], [S2], [S3], [S4], [S5], [S6], [S7], [S8], [S9], [MP], [Gen],
$[\text{L3.2(a)}], [\text{S}'], [\text{A1}'], [\text{A2}'], [\text{A3}'], [\text{A4}'], [\text{A5}'], [\text{S1}'], [\text{S2}'], [\text{S3}'], [\text{S4}'], [\text{S5}'], [\text{S6}'],$
$[\text{S7}'], [\text{S8}'], [\text{S9}'], [\text{MP}'], [\text{Gen}'], [\text{L3.2(a)}'], [\text{M1.7}], [\text{MP}'_{\mathrm{h}}], [\text{Hypothesize}], [\text{Gen}'_{\mathrm{h}}],$
$[\text{M3.2(a)}], [\text{M3.2(a)}_{\mathrm{h}}], [\text{M3.2(b)}_{\mathrm{h}}], [\text{M3.1(S1}')_{\mathrm{h}}], [\text{M3.2(c)}_{\mathrm{h}}], [\text{M3.1(S2}')_{\mathrm{h}}],$
$[\text{M3.1(S5}')_{\mathrm{h}}], [\text{M3.1(S6}')_{\mathrm{h}}], [\text{M3.2(f)}];$
Preassociative
$[\ast\_\{\ast\}], [\ast'], [\ast[\ast]], [\ast[\ast\to\ast]], [\ast[\ast\Rightarrow\ast]], [\dot{\ast}];$
Preassociative
$[\text{“}\ast\text{”}], [], [(\ast)^{\mathbf{t}}], [\text{string}(\ast)+\ast], [\text{string}(\ast)++\ast]$, [
*], [*], [!*], ["*], [#*], $[\$\ast]$, [%*], [&*], ['*], [(*], [)*], [**], [+*], [,*], [-*], [.*], [/*],
[0*], [1*], [2*], [3*], [4*], [5*], [6*], [7*], [8*], [9*], [:*], [;*], [<*], [=*], [>*], [?*],
[@*], [A*], [B*], [C*], [D*], [E*], [F*], [G*], [H*], [I*], [J*], [K*], [L*], [M*], [N*],
[O*], [P*], [Q*], [R*], [S*], [T*], [U*], [V*], [W*], [X*], [Y*], [Z*], [[*], [*], []*], [^*],
[_*], [`*], [a*], [b*], [c*], [d*], [e*], [f*], [g*], [h*], [i*], [j*], [k*], [l*], [m*], [n*], [o*],
[p*], [q*], [r*], [s*], [t*], [u*], [v*], [w*], [x*], [y*], [z*], [{*], [|*], [}*], [~*],
[**Preassociative** $\ast;\ast$], [**Postassociative** $\ast;\ast$], [[*], *], [priority * end],
[newline *], [macro newline *];
Preassociative
$[\ast 0], [\ast 1], [0\mathrm{b}], [\ast\text{-color}(\ast)], [\ast\text{-color}^{\ast}(\ast)];$
Preassociative
[* ’ *], [* ‘ *];
Preassociative
$[\ast^{\mathrm{H}}], [\ast^{\mathrm{T}}], [\ast^{\mathrm{U}}], [\ast^{\mathrm{h}}], [\ast^{\mathrm{t}}], [\ast^{\mathrm{s}}], [\ast^{\mathrm{c}}], [\ast^{\mathrm{d}}], [\ast^{\mathrm{a}}], [\ast^{\mathrm{C}}], [\ast^{\mathrm{M}}], [\ast^{\mathrm{B}}], [\ast^{\mathrm{r}}], [\ast^{\mathrm{i}}], [\ast^{\mathrm{d}}], [\ast^{\mathrm{R}}], [\ast^{0}],$

$[*^1], [*^2], [*^3], [*^4], [*^5], [*^6], [*^7], [*^8], [*^9], [*^{\mathrm{E}}], [*^{\mathcal{V}}], [*^{\mathcal{C}}], [*^{\mathcal{C}^*}], [*']$;
Preassociative
$[* \cdot *], [* \cdot_0 *], [* \colon *]$;
Preassociative
$[* + *], [* +_0 *], [* +_1 *], [* - *], [* -_0 *], [* -_1 *], [* \dot{+} *]$;
Preassociative
$[* \cup \{*\}], [* \cup *], [* \backslash \{*\}]$;
Postassociative
$[* \therefore *], [* \underline{\therefore} *], [* \underline{::} *], [* \underline{+2*} *], [* :: *], [* +2* *]$;
Postassociative
$[*, *]$;
Preassociative
$[* \stackrel{\mathrm{B}}{\approx} *], [* \stackrel{\mathrm{D}}{\approx} *], [* \stackrel{\mathrm{C}}{\approx} *], [* \stackrel{\mathrm{P}}{\approx} *], [* \approx *], [* = *], [* \stackrel{+}{\rightarrow} *], [* \stackrel{\mathrm{t}}{=} *], [* \stackrel{\mathrm{t}^*}{=} *], [* \stackrel{\mathrm{r}}{=} *]$,
$[* \in_{\mathrm{t}} *], [* \subseteq_{\mathrm{T}} *], [* \stackrel{\mathrm{T}}{=} *], [* \stackrel{\mathrm{s}}{=} *], [* \text{ free in } *], [* \text{ free in}^* *], [* \text{ free for } * \text{ in } *]$,
$[* \text{ free for}^* * \text{ in } *], [* \in_{\mathrm{c}} *], [* < *], [* <' *], [* \leq' *], [* \stackrel{\mathrm{p}}{=} *], [*^{\mathcal{P}}]$;
Preassociative
$[\neg *], [\dot{\neg} *]$;
Preassociative
$[* \wedge *], [* \ddot{\wedge} *], [* \tilde{\wedge} *], [* \wedge_{\mathrm{c}} *], [* \dot{\wedge} *]$;
Preassociative
$[* \vee *], [* \parallel *], [* \ddot{\vee} *], [* \dot{\vee} *]$;
Preassociative
$[\dot{\forall} *\colon *], [\dot{\exists} *\colon *]$;
Postassociative
$[* \ddot{\Rightarrow} *], [* \dot{\Rightarrow} *], [* \dot{\Leftrightarrow} *]$;
Postassociative
$[* : *], [* ! *]$;
Preassociative
$[* \left\{ \begin{array}{c} * \\ * \end{array} \right.]$;
Preassociative
$[\lambda * . *], [\Lambda *], [\textbf{if } * \textbf{ then } * \textbf{ else } *], [\textbf{let } * = * \textbf{ in } *], [\textbf{let } * \ddot{=} * \text{ in } *]$;
Preassociative
$[*^{\mathrm{I}}], [*^{\rhd}], [*^{\mathrm{V}}], [*^+], [*^-], [*^*]$;
Preassociative
$[* @ *], [* \rhd *], [* \blacktriangleright *], [* \gg *], [* \unrhd *], [* \unrhd_{\mathrm{h}} *]$;
Postassociative
$[* \vdash *], [* \Vdash *], [* \text{ i.e. } *]$;
Preassociative
$[\forall *\colon *]$;
Postassociative
$[* \oplus *]$;
Postassociative
$[*; *]$;
Preassociative

[$*$ proves $*$];
Preassociative
[$*$ **proof of** $* : *$], [Line $* : * \gg *; *$], [Last line $* \gg * \square$],
[Line $*$: Premise $\gg *; *$], [Line $*$: Side-condition $\gg *; *$], [Arbitrary $\gg *; *$],
[Local $\gg * = *; *$];
Postassociative
[$*$ then $*$], [$*[*]*$];
Preassociative
[$*\&*$];
Preassociative
[$*\backslash\backslash*$];]

B Index

C Bibliography

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