Peano arithmetic

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1 Peano arithmetic

This Logiweb page [1] defines Peano arithmetic.

1.1 The constructs of Peano arithmetic

Terms of Peano arithmetic are constructed from zero $[\dot{0}]$[1], successor $[x']$[2], plus $[x \dotplus y]$[3], and times $[x \dot{\cdot} y]$[4].

Formulas of Peano arithmeric are constructed from equality $[x \stackrel{p}{=} y]$[5], negation $[\dot{\neg} x]$[6], implication $[x \dot{\Rightarrow} y]$[7], and universal quantification $[\dot{\forall} x \colon y]$[8].

From these constructs we macro define one $[\dot{1}]$[9], two $[\dot{2}]$[10], conjunction $[x \dot{\wedge} y]$[11], disjunction $[x \dot{\vee} y]$[12], biimplication $[x \dot{\Leftrightarrow} y]$[13], and existential quantification $[\dot{\exists} x \colon y]$[14]:

$$[\dot{1} \ddot{=} \dot{0}']$$

$$[\dot{2} \ddot{=} \dot{1}']$$

$$[x \dot{\wedge} y \ddot{=} \dot{\neg}(x \dot{\Rightarrow} \dot{\neg} y)]$$

$$[x \dot{\vee} y \ddot{=} \dot{\neg} x \dot{\Rightarrow} y]$$

$$[x \dot{\Leftrightarrow} y \ddot{=} (x \dot{\Rightarrow} y) \dot{\wedge} (y \dot{\Rightarrow} x)]$$

$$[\dot{\exists} x \colon y \ddot{=} \dot{\neg} \dot{\forall} x \colon \dot{\neg} y]$$

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[1] $[\dot{0} \stackrel{\text{pyk}}{=}$ "peano zero"$]$

[2] $[x' \stackrel{\text{pyk}}{=}$ "∗ peano succ"$]$

[3] $[x \dotplus y \stackrel{\text{pyk}}{=}$ "∗ peano plus ∗"$]$

[4] $[x \dot{\cdot} y \stackrel{\text{pyk}}{=}$ "∗ peano times ∗"$]$

[5] $[x \stackrel{p}{=} y \stackrel{\text{pyk}}{=}$ "∗ peano is ∗"$]$

[6] $[\dot{\neg} x \stackrel{\text{pyk}}{=}$ "peano not ∗"$]$

[7] $[x \dot{\Rightarrow} y \stackrel{\text{pyk}}{=}$ "∗ peano imply ∗"$]$

[8] $[\dot{\forall} x \colon y \stackrel{\text{pyk}}{=}$ "peano all ∗ indeed ∗"$]$

[9] $[\dot{1} \stackrel{\text{pyk}}{=}$ "peano one"$]$

[10] $[\dot{2} \stackrel{\text{pyk}}{=}$ "peano two"$]$

[11] $[x \dot{\wedge} y \stackrel{\text{pyk}}{=}$ "∗ peano and ∗"$]$

[12] $[x \dot{\vee} y \stackrel{\text{pyk}}{=}$ "∗ peano or ∗"$]$

[13] $[x \dot{\Leftrightarrow} y \stackrel{\text{pyk}}{=}$ "∗ peano iff ∗"$]$

[14] $[\dot{\exists} x \colon y \stackrel{\text{pyk}}{=}$ "peano exist ∗ indeed ∗"$]$

1.2 Variables

We now introduce the unary operator $[\dot{\mathsf{x}}]$[15] and define that a term is a *Peano variable* (i.e. a variable of Peano arithmetic) if it has the $[\dot{\mathsf{x}}]$ operator in its root. $[\mathsf{x}^{\mathcal{P}}]$[16] is true if $[\mathsf{x}]$ is a Peano variable:

$$[\mathsf{x}^{\mathcal{P}} \doteq \mathsf{x} \stackrel{\mathrm{r}}{=} \lceil \dot{\mathsf{x}} \rceil]$$

We macro define $[\dot{a}]$[17] to be a Peano variable:

$$[\dot{a} \stackrel{..}{=} \dot{\mathsf{a}}]$$

Appendix A.2 defines Peano variables for the other letters of the English alphabet.

$[\dot{\mathrm{nonfree}}(\mathsf{x},\mathsf{y})]$[18] is true if the Peano variable $[\mathsf{x}]$ does not occur free in the Peano term/formula $[\mathsf{y}]$. $[\dot{\mathrm{nonfree}}^{*}(\mathsf{x},\mathsf{y})]$[19] is true if the Peano variable $[\mathsf{x}]$ does not occur free in the list $[\mathsf{y}]$ of Peano terms/formulas.

$$\begin{aligned}
&[\dot{\mathrm{nonfree}}(\mathsf{x},\mathsf{y}) \doteq \\
&\mathbf{if}\ \mathsf{y}^{\mathcal{P}}\ \mathbf{then}\ \neg \mathsf{x} \stackrel{\mathrm{t}}{=} \mathsf{y}\ \mathbf{else} \\
&\mathbf{if}\ \neg \mathsf{y} \stackrel{\mathrm{r}}{=} \lceil \dot{\forall} \mathsf{x}\colon \mathsf{y} \rceil\ \mathbf{then}\ \dot{\mathrm{nonfree}}^{*}(\mathsf{x},\mathsf{y}^{\mathrm{t}})\ \mathbf{else} \\
&\mathbf{if}\ \mathsf{x} \stackrel{\mathrm{t}}{=} \mathsf{y}^{1}\ \mathbf{then}\ \mathsf{T}\ \mathbf{else}\ \dot{\mathrm{nonfree}}(\mathsf{x},\mathsf{y}^{2})]
\end{aligned}$$

$$[\dot{\mathrm{nonfree}}^{*}(\mathsf{x},\mathsf{y}) \doteq \mathsf{x}!\mathrm{If}(\mathsf{y},\mathsf{T},\dot{\mathrm{nonfree}}(\mathsf{x},\mathsf{y}^{\mathrm{h}}) \ddot{\wedge} \dot{\mathrm{nonfree}}^{*}(\mathsf{x},\mathsf{y}^{\mathrm{t}}))]$$

$[\dot{\mathrm{free}}\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle]$[20] is true if the substitution $[\langle \mathsf{a}\,|\mathsf{x}{:=}\,\mathsf{b}\rangle]$ is free. $[\dot{\mathrm{free}}^{*}\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle]$[21] is the version where $[\mathsf{a}]$ is a list of terms.

$$\begin{aligned}
&[\dot{\mathrm{free}}\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \doteq \mathsf{x}!\mathsf{b}! \\
&\mathbf{if}\ \mathsf{a}^{\mathcal{P}}\ \mathbf{then}\ \mathsf{T}\ \mathbf{else} \\
&\mathbf{if}\ \neg \mathsf{a} \stackrel{\mathrm{r}}{=} \lceil \dot{\forall} \mathsf{u}\colon \mathsf{v} \rceil\ \mathbf{then}\ \dot{\mathrm{free}}^{*}\langle \mathsf{a}^{\mathrm{t}}|\mathsf{x} := \mathsf{b}\rangle\ \mathbf{else} \\
&\mathbf{if}\ \mathsf{a}^{1} \stackrel{\mathrm{t}}{=} \mathsf{x}\ \mathbf{then}\ \mathsf{T}\ \mathbf{else} \\
&\mathbf{if}\ \dot{\mathrm{nonfree}}(\mathsf{x},\mathsf{a}^{2})\ \mathbf{then}\ \mathsf{T}\ \mathbf{else} \\
&\mathbf{if}\ \neg\dot{\mathrm{nonfree}}(\mathsf{a}^{1},\mathsf{b})\ \mathbf{then}\ \mathsf{F}\ \mathbf{else} \\
&\dot{\mathrm{free}}\langle \mathsf{a}^{2}|\mathsf{x} := \mathsf{b}\rangle]
\end{aligned}$$

$$[\dot{\mathrm{free}}^{*}\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \doteq \mathsf{x}!\mathsf{b}!\mathrm{If}(\mathsf{a},\mathsf{T},\dot{\mathrm{free}}\langle \mathsf{a}^{\mathrm{h}}|\mathsf{x} := \mathsf{b}\rangle \ddot{\wedge} \dot{\mathrm{free}}^{*}\langle \mathsf{a}^{\mathrm{t}}|\mathsf{x} := \mathsf{b}\rangle)]$$

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15 $[\dot{\mathsf{x}} \stackrel{\mathrm{pyk}}{=}$ "∗ peano var"]

16 $[\mathsf{x}^{\mathcal{P}} \stackrel{\mathrm{pyk}}{=}$ "∗ is peano var"]

17 $[\dot{a} \stackrel{\mathrm{pyk}}{=}$ "peano a"]

18 $[\dot{\mathrm{nonfree}}(\mathsf{x},\mathsf{y}) \stackrel{\mathrm{pyk}}{=}$ "peano nonfree ∗ in ∗ end nonfree"]

19 $[\dot{\mathrm{nonfree}}^{*}(\mathsf{x},\mathsf{y}) \stackrel{\mathrm{pyk}}{=}$ "peano nonfree star ∗ in ∗ end nonfree"]

20 $[\dot{\mathrm{free}}\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \stackrel{\mathrm{pyk}}{=}$ "peano free ∗ set ∗ to ∗ end free"]

21 $[\dot{\mathrm{free}}^{*}\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \stackrel{\mathrm{pyk}}{=}$ "peano free star ∗ set ∗ to ∗ end free"]

$[\mathsf{a}\equiv\langle\mathsf{b}|\mathsf{x} := \mathsf{c}\rangle]$[22] is true if $[\mathsf{a}]$ equals $[\langle\mathsf{b}\,|\mathsf{x}{:=}\,\mathsf{c}\rangle]$. $[\mathsf{a}\equiv\langle^{*}\mathsf{b}|\mathsf{x} := \mathsf{c}\rangle]$[23] is the version where $[\mathsf{a}]$ and $[\mathsf{b}]$ are lists.

$$
\begin{array}{l}
[\mathsf{a}\equiv\langle\mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \doteq \mathsf{a}!\mathsf{x}!\mathsf{c}! \\
\textbf{if } \mathsf{b} \stackrel{\mathrm{r}}{=} \lceil\dot{\forall}\mathsf{u}{:}\,\mathsf{v}\rceil \ddot{\wedge} \mathsf{b}^{1} \stackrel{\mathrm{t}}{=} \mathsf{x} \textbf{ then } \mathsf{a} \stackrel{\mathrm{t}}{=} \mathsf{b} \textbf{ else} \\
\textbf{if } \mathsf{b}^{\mathcal{P}} \wedge \mathsf{b} \stackrel{\mathrm{t}}{=} \mathsf{x} \textbf{ then } \mathsf{a} \stackrel{\mathrm{t}}{=} \mathsf{c} \textbf{ else} \\
\mathsf{a} \stackrel{\mathrm{r}}{=} \mathsf{b} \ddot{\wedge} \mathsf{a}^{\mathrm{t}}\equiv\langle^{*}\mathsf{b}^{\mathrm{t}}|\mathsf{x} := \mathsf{c}\rangle]
\end{array}
$$

$$[\mathsf{a}\equiv\langle^{*}\mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \doteq \mathsf{b}!\mathsf{x}!\mathsf{c}!\mathrm{If}(\mathsf{a}, \mathsf{T}, \mathsf{a}^{\mathrm{h}}\equiv\langle\mathsf{b}^{\mathrm{h}}|\mathsf{x} := \mathsf{c}\rangle \ddot{\wedge} \mathsf{a}^{\mathrm{t}}\equiv\langle^{*}\mathsf{b}^{\mathrm{t}}|\mathsf{x} := \mathsf{c}\rangle)]$$

1.3 Mendelsons system S

System [S][24] of Mendelson [2] expresses Peano arithmetic. It comprises the axioms [A1][25], [A2][26], [A3][27], [A4][28], and [A5][29] and inference rules [MP][30] and [Gen][31] of first order predicate calculus. Furthermore, it comprises the proper axioms [S1][32], [S2][33], [S3][34], [S4][35], [S5][36], [S6][37], [S7][38], [S8][39], and [S9][40]. System [S] is defined thus:

[**Theory** S]

$$[\mathrm{S}\ \textbf{rule}\ \mathrm{A1}{:}\,\forall\mathcal{A}{:}\,\forall\mathcal{B}{:}\,\mathcal{A} \dot{\Rightarrow} \mathcal{B} \dot{\Rightarrow} \mathcal{A}]$$

$$[\mathrm{S}\ \textbf{rule}\ \mathrm{A2}{:}\,\forall\mathcal{A}{:}\,\forall\mathcal{B}{:}\,\forall\mathcal{C}{:}\,(\mathcal{A} \dot{\Rightarrow} \mathcal{B} \dot{\Rightarrow} \mathcal{C}) \dot{\Rightarrow} (\mathcal{A} \dot{\Rightarrow} \mathcal{B}) \dot{\Rightarrow} \mathcal{A} \dot{\Rightarrow} \mathcal{C}]$$

$$[\mathrm{S}\ \textbf{rule}\ \mathrm{A3}{:}\,\forall\mathcal{A}{:}\,\forall\mathcal{B}{:}\,(\dot{\neg}\mathcal{B} \dot{\Rightarrow} \dot{\neg}\mathcal{A}) \dot{\Rightarrow} (\dot{\neg}\mathcal{B} \dot{\Rightarrow} \mathcal{A}) \dot{\Rightarrow} \mathcal{B}]$$

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22 $[\mathsf{a}\equiv\langle\mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \stackrel{\mathrm{pyk}}{=}$ "peano sub ∗ is ∗ where ∗ is ∗ end sub"]

23 $[\mathsf{a}\equiv\langle^{*}\mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \stackrel{\mathrm{pyk}}{=}$ "peano sub star ∗ is ∗ where ∗ is ∗ end sub"]

24 [S $\stackrel{\mathrm{pyk}}{=}$ "system s"]

25 [A1 $\stackrel{\mathrm{pyk}}{=}$ "axiom a one"]

26 [A2 $\stackrel{\mathrm{pyk}}{=}$ "axiom a two"]

27 [A3 $\stackrel{\mathrm{pyk}}{=}$ "axiom a three"]

28 [A4 $\stackrel{\mathrm{pyk}}{=}$ "axiom a four"]

29 [A5 $\stackrel{\mathrm{pyk}}{=}$ "axiom a five"]

30 [MP $\stackrel{\mathrm{pyk}}{=}$ "rule mp"]

31 [Gen $\stackrel{\mathrm{pyk}}{=}$ "rule gen"]

32 [S1 $\stackrel{\mathrm{pyk}}{=}$ "axiom s one"]

33 [S2 $\stackrel{\mathrm{pyk}}{=}$ "axiom s two"]

34 [S3 $\stackrel{\mathrm{pyk}}{=}$ "axiom s three"]

35 [S4 $\stackrel{\mathrm{pyk}}{=}$ "axiom s four"]

36 [S5 $\stackrel{\mathrm{pyk}}{=}$ "axiom s five"]

37 [S6 $\stackrel{\mathrm{pyk}}{=}$ "axiom s six"]

38 [S7 $\stackrel{\mathrm{pyk}}{=}$ "axiom s seven"]

39 [S8 $\stackrel{\mathrm{pyk}}{=}$ "axiom s eight"]

40 [S9 $\stackrel{\mathrm{pyk}}{=}$ "axiom s nine"]

The order of quantifiers in the following axiom is such that [$\mathcal{C}$] which the current conclusion tactic cannot guess comes first. This allows to supply a value for [$\mathcal{C}$] without having to supply values for the other meta-variables.

[S **rule** A4: $\forall\mathcal{C}\colon\forall\mathcal{A}\colon\forall\mathcal{X}\colon\forall\mathcal{B}\colon \lceil\mathcal{A}\rceil\equiv\langle\lceil\mathcal{B}\rceil|\lceil\mathcal{X}\rceil := \lceil\mathcal{C}\rceil\rangle \Vdash \dot{\forall}\mathcal{X}\colon\mathcal{B} \dot{\Rightarrow} \mathcal{A}$]

[S **rule** A5: $\forall\mathcal{X}\colon\forall\mathcal{A}\colon\forall\mathcal{B}\colon \dot{\mathrm{nonfree}}(\mathcal{X},\mathcal{A}) \Vdash \dot{\forall}\mathcal{X}\colon(\mathcal{A}\dot{\Rightarrow}\mathcal{B}) \dot{\Rightarrow} \mathcal{A} \dot{\Rightarrow} \dot{\forall}\mathcal{X}\colon\mathcal{B}$]

[S **rule** MP: $\forall\mathcal{A}\colon\forall\mathcal{B}\colon\mathcal{A}\dot{\Rightarrow}\mathcal{B} \vdash \mathcal{A} \vdash \mathcal{B}$]

[S **rule** Gen: $\forall\mathcal{X}\colon\forall\mathcal{A}\colon\mathcal{A} \vdash \dot{\forall}\mathcal{X}\colon\mathcal{A}$]

Axiom [S1] to [S8] are stated as Mendelson [2] does. This is done here to test certain parts of the Logiweb system. Serious users of Peano arithmetic are advised to take Mendelson's Lemma 3.1 as axioms instead.

[S **rule** S1: $\dot{a} \stackrel{\mathrm{p}}{=} \dot{b} \dot{\Rightarrow} \dot{a} \stackrel{\mathrm{p}}{=} \dot{c} \dot{\Rightarrow} \dot{b} \stackrel{\mathrm{p}}{=} \dot{c}$]

[S **rule** S2: $\dot{a} \stackrel{\mathrm{p}}{=} \dot{b} \dot{\Rightarrow} \dot{a}' \stackrel{\mathrm{p}}{=} \dot{b}'$]

[S **rule** S3: $\dot{\neg}\dot{0} \stackrel{\mathrm{p}}{=} \dot{a}'$]

[S **rule** S4: $\dot{a}' \stackrel{\mathrm{p}}{=} \dot{b}' \dot{\Rightarrow} \dot{a} \stackrel{\mathrm{p}}{=} \dot{b}$]

[S **rule** S5: $\dot{a} \dot{+} \dot{0} \stackrel{\mathrm{p}}{=} \dot{a}$]

[S **rule** S6: $\dot{a} \dot{+} \dot{b}' \stackrel{\mathrm{p}}{=} (\dot{a} \dot{+} \dot{b})'$]

[S **rule** S7: $\dot{a} \dot{\cdot} \dot{0} \stackrel{\mathrm{p}}{=} \dot{0}$]

[S **rule** S8: $\dot{a} \dot{\cdot} (\dot{b}') \stackrel{\mathrm{p}}{=} (\dot{a} \dot{\cdot} \dot{b}) \dot{+} \dot{a}$]

[S **rule** S9: $\forall\mathcal{A}\colon\forall\mathcal{B}\colon\forall\mathcal{C}\colon\forall\mathcal{X}\colon$
$\mathcal{B}\equiv\langle\mathcal{A}|\mathcal{X} := \dot{0}\rangle \Vdash \mathcal{C}\equiv\langle\mathcal{A}|\mathcal{X} := \mathcal{X}'\rangle \Vdash$
$\mathcal{B} \dot{\Rightarrow} \dot{\forall}\mathcal{X}\colon(\mathcal{A}\dot{\Rightarrow}\mathcal{C}) \dot{\Rightarrow} \dot{\forall}\mathcal{X}\colon\mathcal{A}$]

1.4 A lemma and a proof

We now prove Lemma [L3.2(a)][41] which is an instance of the corresponding proposition in Mendelson [2]:

[S **lemma** L3.2(a): $\dot{x} \stackrel{\mathrm{p}}{=} \dot{x}$]

[41] [L3.2(a) $\stackrel{\mathrm{pyk}}{=}$ "lemma l three two a"]

S **proof of** L3.2(a):

L01:	S5 $\gg$	$\dot{a} \mathbin{\dot{+}} \dot{0} \stackrel{p}{=} \dot{a}$	;
L02:	Gen $\rhd$ L01 $\gg$	$\dot{\forall}\dot{a}\colon \dot{a} \mathbin{\dot{+}} \dot{0} \stackrel{p}{=} \dot{a}$	;
L03:	A4 @ $\dot{x}$ $\gg$	$\dot{\forall}\dot{a}\colon \dot{a} \mathbin{\dot{+}} \dot{0} \stackrel{p}{=} \dot{a} \mathrel{\dot{\Rightarrow}} \dot{x} \mathbin{\dot{+}} \dot{0} \stackrel{p}{=} \dot{x}$	;
L04:	MP $\rhd$ L03 $\rhd$ L02 $\gg$	$\dot{x} \mathbin{\dot{+}} \dot{0} \stackrel{p}{=} \dot{x}$	;
L05:	S1 $\gg$	$\dot{a} \stackrel{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{c} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{p}{=} \dot{c}$	;
L06:	Gen $\rhd$ L05 $\gg$	$\dot{\forall}\dot{c}\colon (\dot{a} \stackrel{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{c} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{p}{=} \dot{c})$	;
L07:	A4 @ $\dot{x}$ $\gg$	$\dot{\forall}\dot{c}\colon (\dot{a} \stackrel{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{c} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{p}{=} \dot{c}) \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{p}{=} \dot{x}$	;
L08:	MP $\rhd$ L07 $\rhd$ L06 $\gg$	$\dot{a} \stackrel{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{p}{=} \dot{x}$	;
L09:	Gen $\rhd$ L08 $\gg$	$\dot{\forall}\dot{b}\colon (\dot{a} \stackrel{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{p}{=} \dot{x})$	;
L10:	A4 @ $\dot{x}$ $\gg$	$\dot{\forall}\dot{b}\colon (\dot{a} \stackrel{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{p}{=} \dot{x}) \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \stackrel{p}{=} \dot{x}$	;
L11:	MP $\rhd$ L10 $\rhd$ L09 $\gg$	$\dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \stackrel{p}{=} \dot{x}$	;
L12:	Gen $\rhd$ L11 $\gg$	$\dot{\forall}\dot{a}\colon (\dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \stackrel{p}{=} \dot{x})$	;
L13:	A4 @ $\dot{x} \mathbin{\dot{+}} \dot{0}$ $\gg$	$\dot{\forall}\dot{a}\colon (\dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \stackrel{p}{=} \dot{x}) \mathrel{\dot{\Rightarrow}} \dot{x} \mathbin{\dot{+}} \dot{0} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \mathbin{\dot{+}} \dot{0} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \stackrel{p}{=} \dot{x}$	;
L14:	MP $\rhd$ L13 $\rhd$ L12 $\gg$	$\dot{x} \mathbin{\dot{+}} \dot{0} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \mathbin{\dot{+}} \dot{0} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \stackrel{p}{=} \dot{x}$	;
L15:	MP $\rhd$ L14 $\rhd$ L04 $\gg$	$\dot{x} \mathbin{\dot{+}} \dot{0} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \stackrel{p}{=} \dot{x}$	;
L16:	MP $\rhd$ L15 $\rhd$ L04 $\gg$	$\dot{x} \stackrel{p}{=} \dot{x}$	$\square$

1.5 An alternative axiomatic system

System [S′][42] is system [S] in which the proper axioms are taken from Lemma 3.1 in Mendelson [2]. It comprises the axioms [A1′][43], [A2′][44], [A3′][45], [A4′][46], and [A5′][47] and inference rules [MP′][48] and [Gen′][49] of first order predicate calculus. Furthermore, it comprises the proper axioms [S1′][50], [S2′][51], [S3′][52],

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42 [S′ $\stackrel{\text{pyk}}{=}$ “system prime s”]

43 [A1′ $\stackrel{\text{pyk}}{=}$ “axiom prime a one”]

44 [A2′ $\stackrel{\text{pyk}}{=}$ “axiom prime a two”]

45 [A3′ $\stackrel{\text{pyk}}{=}$ “axiom prime a three”]

46 [A4′ $\stackrel{\text{pyk}}{=}$ “axiom prime a four”]

47 [A5′ $\stackrel{\text{pyk}}{=}$ “axiom prime a five”]

48 [MP′ $\stackrel{\text{pyk}}{=}$ “rule prime mp”]

49 [Gen′ $\stackrel{\text{pyk}}{=}$ “rule prime gen”]

50 [S1′ $\stackrel{\text{pyk}}{=}$ “axiom prime s one”]

51 [S2′ $\stackrel{\text{pyk}}{=}$ “axiom prime s two”]

52 [S3′ $\stackrel{\text{pyk}}{=}$ “axiom prime s three”]

[S4′][53], [S5′][54], [S6′][55], [S7′][56], [S8′][57], and [S9′][58].

System [S′] is defined thus:

[**Theory** S′]

[S′ **rule** A1′: $\forall \mathcal{A} \colon \forall \mathcal{B} \colon \mathcal{A} \mathrel{\dot{\Rightarrow}} \mathcal{B} \mathrel{\dot{\Rightarrow}} \mathcal{A}$]

[S′ **rule** A2′: $\forall \mathcal{A} \colon \forall \mathcal{B} \colon \forall \mathcal{C} \colon (\mathcal{A} \mathrel{\dot{\Rightarrow}} \mathcal{B} \mathrel{\dot{\Rightarrow}} \mathcal{C}) \mathrel{\dot{\Rightarrow}} (\mathcal{A} \mathrel{\dot{\Rightarrow}} \mathcal{B}) \mathrel{\dot{\Rightarrow}} \mathcal{A} \mathrel{\dot{\Rightarrow}} \mathcal{C}$]

[S′ **rule** A3′: $\forall \mathcal{A} \colon \forall \mathcal{B} \colon (\dot{\neg} \mathcal{B} \mathrel{\dot{\Rightarrow}} \dot{\neg} \mathcal{A}) \mathrel{\dot{\Rightarrow}} (\dot{\neg} \mathcal{B} \mathrel{\dot{\Rightarrow}} \mathcal{A}) \mathrel{\dot{\Rightarrow}} \mathcal{B}$]

[S′ **rule** A4′: $\forall \mathcal{C} \colon \forall \mathcal{A} \colon \forall \mathcal{X} \colon \forall \mathcal{B} \colon \lceil \mathcal{A} \rceil \equiv \langle \lceil \mathcal{B} \rceil | \lceil \mathcal{X} \rceil := \lceil \mathcal{C} \rceil \rangle \Vdash \dot{\forall} \mathcal{X} \colon \mathcal{B} \mathrel{\dot{\Rightarrow}} \mathcal{A}$]

[S′ **rule** A5′: $\forall \mathcal{X} \colon \forall \mathcal{A} \colon \forall \mathcal{B} \colon \mathrm{nonfree}(\lceil \mathcal{X} \rceil, \lceil \mathcal{A} \rceil) \Vdash \dot{\forall} \mathcal{X} \colon (\mathcal{A} \mathrel{\dot{\Rightarrow}} \mathcal{B}) \mathrel{\dot{\Rightarrow}} \mathcal{A} \mathrel{\dot{\Rightarrow}} \dot{\forall} \mathcal{X} \colon \mathcal{B}$]

[S′ **rule** MP′: $\forall \mathcal{A} \colon \forall \mathcal{B} \colon \mathcal{A} \mathrel{\dot{\Rightarrow}} \mathcal{B} \vdash \mathcal{A} \vdash \mathcal{B}$]

[S′ **rule** Gen′: $\forall \mathcal{X} \colon \forall \mathcal{A} \colon \mathcal{A} \vdash \dot{\forall} \mathcal{X} \colon \mathcal{A}$]

[S′ **rule** S1′: $\forall \mathcal{A} \colon \forall \mathcal{B} \colon \forall \mathcal{C} \colon \mathcal{A} \stackrel{\mathrm{p}}{=} \mathcal{B} \mathrel{\dot{\Rightarrow}} \mathcal{A} \stackrel{\mathrm{p}}{=} \mathcal{C} \mathrel{\dot{\Rightarrow}} \mathcal{B} \stackrel{\mathrm{p}}{=} \mathcal{C}$]

[S′ **rule** S2′: $\forall \mathcal{A} \colon \forall \mathcal{B} \colon \mathcal{A} \stackrel{\mathrm{p}}{=} \mathcal{B} \mathrel{\dot{\Rightarrow}} \mathcal{A}' \stackrel{\mathrm{p}}{=} \mathcal{B}'$]

[S′ **rule** S3′: $\forall \mathcal{A} \colon \neg \dot{0} \stackrel{\mathrm{p}}{=} \mathcal{A}'$]

[S′ **rule** S4′: $\forall \mathcal{A} \colon \forall \mathcal{B} \colon \mathcal{A}' \stackrel{\mathrm{p}}{=} \mathcal{B}' \mathrel{\dot{\Rightarrow}} \mathcal{A} \stackrel{\mathrm{p}}{=} \mathcal{B}$]

[S′ **rule** S5′: $\forall \mathcal{A} \colon \mathcal{A} \mathbin{\dot{+}} \dot{0} \stackrel{\mathrm{p}}{=} \mathcal{A}$]

[S′ **rule** S6′: $\forall \mathcal{A} \colon \forall \mathcal{B} \colon \mathcal{A} \mathbin{\dot{+}} \mathcal{B}' \stackrel{\mathrm{p}}{=} (\mathcal{A} \mathbin{\dot{+}} \mathcal{B})'$]

[S′ **rule** S7′: $\forall \mathcal{A} \colon \mathcal{A} \mathbin{\dot{\cdot}} \dot{0} \stackrel{\mathrm{p}}{=} \dot{0}$]

[S′ **rule** S8′: $\forall \mathcal{A} \colon \forall \mathcal{B} \colon \mathcal{A} \mathbin{\dot{\cdot}} (\mathcal{B}') \stackrel{\mathrm{p}}{=} (\mathcal{A} \mathbin{\dot{\cdot}} \mathcal{B}) \mathbin{\dot{+}} \mathcal{A}$]

[S′ **rule** S9′: $\forall \mathcal{A} \colon \forall \mathcal{B} \colon \forall \mathcal{C} \colon \forall \mathcal{X} \colon$
$\mathcal{B} \equiv \langle \mathcal{A} | \mathcal{X} := \dot{0} \rangle \Vdash \mathcal{C} \equiv \langle \mathcal{A} | \mathcal{X} := \mathcal{X}' \rangle \Vdash$
$\mathcal{B} \mathrel{\dot{\Rightarrow}} \dot{\forall} \mathcal{X} \colon (\mathcal{A} \mathrel{\dot{\Rightarrow}} \mathcal{C}) \mathrel{\dot{\Rightarrow}} \dot{\forall} \mathcal{X} \colon \mathcal{A}$]

Note that [A1] and [A1′] are distinct. The former says [$\mathrm{S} \vdash \forall \mathcal{A} \colon \forall \mathcal{B} \colon \mathcal{A} \mathrel{\dot{\Rightarrow}} \mathcal{B} \mathrel{\dot{\Rightarrow}} \mathcal{A}$] and the latter says [$\mathrm{S}' \vdash \forall \mathcal{A} \colon \forall \mathcal{B} \colon \mathcal{A} \mathrel{\dot{\Rightarrow}} \mathcal{B} \mathrel{\dot{\Rightarrow}} \mathcal{A}$].

---

[53] [S4′ $\stackrel{\mathrm{pyk}}{=}$ “axiom prime s four”]

[54] [S5′ $\stackrel{\mathrm{pyk}}{=}$ “axiom prime s five”]

[55] [S6′ $\stackrel{\mathrm{pyk}}{=}$ “axiom prime s six”]

[56] [S7′ $\stackrel{\mathrm{pyk}}{=}$ “axiom prime s seven”]

[57] [S8′ $\stackrel{\mathrm{pyk}}{=}$ “axiom prime s eight”]

[58] [S9′ $\stackrel{\mathrm{pyk}}{=}$ “axiom prime s nine”]

1.6 Restatement of lemma and proof

We now prove Lemma [L3.2(a)] once again under the name of [L3.2(a)′][59]:

[S′ **lemma** L3.2(a)′: $\forall \mathcal{A}\colon \mathcal{A} \stackrel{p}{=} \mathcal{A}$]

S′ **proof of** L3.2(a)′:

L01:	Arbitrary $\gg$	$\mathcal{A}$	;
L02:	S5′ $\gg$	$\mathcal{A} \dot{+} \dot{0} \stackrel{p}{=} \mathcal{A}$	;
L03:	S1′ $\gg$	$\mathcal{A} \dot{+} \dot{0} \stackrel{p}{=} \mathcal{A} \dot{\Rightarrow}$ $\mathcal{A} \dot{+} \dot{0} \stackrel{p}{=} \mathcal{A} \dot{\Rightarrow} \mathcal{A} \stackrel{p}{=} \mathcal{A}$	;
L04:	MP′ $\rhd$ L03 $\rhd$ L02 $\gg$	$\mathcal{A} \dot{+} \dot{0} \stackrel{p}{=} \mathcal{A} \dot{\Rightarrow} \mathcal{A} \stackrel{p}{=} \mathcal{A}$	;
L05:	MP′ $\rhd$ L04 $\rhd$ L02 $\gg$	$\mathcal{A} \stackrel{p}{=} \mathcal{A}$	$\square$

2 Formal development

2.1 Propositional calculus

2.1.1 Modus ponens

We use [$\mathsf{x} \unrhd \mathsf{y}$][60] as shorthand for modus ponens:

$$[\mathsf{x} \unrhd \mathsf{y} \ddot{=} \mathrm{MP}' \rhd \mathsf{x} \rhd \mathsf{y}]$$

2.1.2 Lemma M1.7

Lemma [M1.7][61] (i.e. Lemma 1.7 in Mendelson [2]) reads:

[S′ **lemma** M1.7: $\forall \mathcal{B}\colon \mathcal{B} \dot{\Rightarrow} \mathcal{B}$]

S′ **proof of** M1.7:

L01:	Arbitrary $\gg$	$\mathcal{B}$	;
L02:	A1′ $\gg$	$\mathcal{B} \dot{\Rightarrow} (\mathcal{B} \dot{\Rightarrow} \mathcal{B}) \dot{\Rightarrow} \mathcal{B}$	;
L03:	A2′ $\gg$	$(\mathcal{B} \dot{\Rightarrow} (\mathcal{B} \dot{\Rightarrow} \mathcal{B}) \dot{\Rightarrow} \mathcal{B}) \dot{\Rightarrow}$ $(\mathcal{B} \dot{\Rightarrow} \mathcal{B} \dot{\Rightarrow} \mathcal{B}) \dot{\Rightarrow} \mathcal{B} \dot{\Rightarrow} \mathcal{B}$	;
L04:	L03 $\unrhd$ L02 $\gg$	$(\mathcal{B} \dot{\Rightarrow} \mathcal{B} \dot{\Rightarrow} \mathcal{B}) \dot{\Rightarrow} \mathcal{B} \dot{\Rightarrow} \mathcal{B}$	;
L05:	A1′ $\gg$	$\mathcal{B} \dot{\Rightarrow} \mathcal{B} \dot{\Rightarrow} \mathcal{B}$	;
L06:	L04 $\unrhd$ L05 $\gg$	$\mathcal{B} \dot{\Rightarrow} \mathcal{B}$	$\square$

2.1.3 Hypothetical modus ponens

The hypothetical version [$\mathrm{MP}'_{\mathrm{h}}$][62] of modus ponens MP′ has a hypothesis $\mathcal{H}$ on each premise and on the conclusion:

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[59] [L3.2(a)′ $\stackrel{\text{pyk}}{=}$ "lemma prime l three two a"]

[60] [$\mathsf{x} \unrhd \mathsf{y}$ $\stackrel{\text{pyk}}{=}$ "* macro modus ponens *"]

[61] [M1.7 $\stackrel{\text{pyk}}{=}$ "mendelson one seven"]

[62] [$\mathrm{MP}'_{\mathrm{h}}$ $\stackrel{\text{pyk}}{=}$ "hypothetical rule prime mp"]

[S′ **lemma** $\mathrm{MP}'_{\mathrm{h}}\colon \forall\mathcal{H}\colon\forall\mathcal{A}\colon\forall\mathcal{B}\colon \mathcal{H} \dot{\Rightarrow} \mathcal{A} \dot{\Rightarrow} \mathcal{B} \vdash \mathcal{H} \dot{\Rightarrow} \mathcal{A} \vdash \mathcal{H} \dot{\Rightarrow} \mathcal{B}$]

S′ **proof of** $\mathrm{MP}'_{\mathrm{h}}$:

L01:	Arbitrary ≫	$\mathcal{H}$	;
L02:	Arbitrary ≫	$\mathcal{A}$	;
L03:	Arbitrary ≫	$\mathcal{B}$	;
L04:	Premise ≫	$\mathcal{H} \dot{\Rightarrow} \mathcal{A} \dot{\Rightarrow} \mathcal{B}$	;
L05:	Premise ≫	$\mathcal{H} \dot{\Rightarrow} \mathcal{A}$	;
L06:	A2′ ≫	$(\mathcal{H} \dot{\Rightarrow} \mathcal{A} \dot{\Rightarrow} \mathcal{B}) \dot{\Rightarrow} (\mathcal{H} \dot{\Rightarrow} \mathcal{A}) \dot{\Rightarrow} \mathcal{H} \dot{\Rightarrow} \mathcal{B}$	;
L07:	L06 $\unrhd$ L04 ≫	$(\mathcal{H} \dot{\Rightarrow} \mathcal{A}) \dot{\Rightarrow} \mathcal{H} \dot{\Rightarrow} \mathcal{B}$	;
L08:	L07 $\unrhd$ L05 ≫	$\mathcal{H} \dot{\Rightarrow} \mathcal{B}$	□

We use [$\mathsf{x} \unrhd_{\mathrm{h}} \mathsf{y}$][63] as shorthand for hypothetical modus ponens:

$$[\mathsf{x} \unrhd_{\mathrm{h}} \mathsf{y} \doteq \mathrm{MP}'_{\mathrm{h}} \rhd \mathsf{x} \rhd \mathsf{y}]$$

2.1.4 Turning lemmas to hypothetical lemmas

Lemma [Hypothesize][64] turns a lemma with no premises into one that assumes the hypothesis [$\mathcal{H}$] to hold:

[S′ **lemma** Hypothesize: $\forall\mathcal{H}\colon\forall\mathcal{A}\colon \mathcal{A} \vdash \mathcal{H} \dot{\Rightarrow} \mathcal{A}$]

S′ **proof of** Hypothesize:

L01:	Arbitrary ≫	$\mathcal{H}$	;
L02:	Arbitrary ≫	$\mathcal{A}$	;
L03:	Premise ≫	$\mathcal{A}$	;
L04:	A1′ ≫	$\mathcal{A} \dot{\Rightarrow} \mathcal{H} \dot{\Rightarrow} \mathcal{A}$	;
L05:	L04 $\unrhd$ L03 ≫	$\mathcal{H} \dot{\Rightarrow} \mathcal{A}$	□

2.2 First order predicate calculus

2.2.1 Hypothetical generalization

The hypothetical version [$\mathrm{Gen}'_{\mathrm{h}}$][65] of generalisation Gen′ has a hypothesis $\mathcal{H}$ on premise and conclusion:

[S′ **lemma** $\mathrm{Gen}'_{\mathrm{h}}\colon \forall\mathcal{H}\colon\forall\mathcal{X}\colon\forall\mathcal{A}\colon \mathrm{non\dot{f}ree}(\lceil\mathcal{X}\rceil, \lceil\mathcal{H}\rceil) \Vdash \mathcal{H} \dot{\Rightarrow} \mathcal{A} \vdash \mathcal{H} \dot{\Rightarrow} \dot{\forall}\mathcal{X}\colon \mathcal{A}$]

S′ **proof of** $\mathrm{Gen}'_{\mathrm{h}}$:

L01:	Arbitrary ≫	$\mathcal{H}$	;
L02:	Arbitrary ≫	$\mathcal{X}$	;
L03:	Arbitrary ≫	$\mathcal{A}$	;
L04:	Side-condition ≫	$\mathrm{non\dot{f}ree}(\lceil\mathcal{X}\rceil, \lceil\mathcal{H}\rceil)$	;

[63] [$\mathsf{x} \unrhd_{\mathrm{h}} \mathsf{y} \overset{\mathrm{pyk}}{=}$ "∗ hypothetical modus ponens ∗"]

[64] [Hypothesize $\overset{\mathrm{pyk}}{=}$ "hypothesize"]

[65] [$\mathrm{Gen}'_{\mathrm{h}} \overset{\mathrm{pyk}}{=}$ "hypothetical rule prime gen"]

L05:	Premise $\gg$	$\mathcal{H} \dot{\Rightarrow} \mathcal{A}$	;
L06:	A5′ $\trianglerighteq\!\!\!\triangleright$ L04 $\gg$	$\dot{\forall}\mathcal{X}\colon(\mathcal{H} \dot{\Rightarrow} \mathcal{A}) \dot{\Rightarrow} \mathcal{H} \dot{\Rightarrow} \dot{\forall}\mathcal{X}\colon\mathcal{A}$	;
L07:	Gen′ $\triangleright$ L05 $\gg$	$\dot{\forall}\mathcal{X}\colon(\mathcal{H} \dot{\Rightarrow} \mathcal{A})$	;
L08:	L06 $\trianglerighteq$ L07 $\gg$	$\mathcal{H} \dot{\Rightarrow} \dot{\forall}\mathcal{X}\colon\mathcal{A}$	$\square$

2.3 Peano arithmetic

2.3.1 Lemma M3.2(a)

Lemma [M3.2(a)][66] and the associated hypothetical lemma [M3.2(a)$_\mathrm{h}$][67] read:

[S′ **rule** M3.2(a)$\colon \forall\mathcal{T}\colon\mathcal{T} \stackrel{\mathrm{p}}{=} \mathcal{T}$]

[S′ **lemma** M3.2(a)$_\mathrm{h}\colon \forall\mathcal{H}\colon\forall\mathcal{T}\colon\mathcal{H} \dot{\Rightarrow} \mathcal{T} \stackrel{\mathrm{p}}{=} \mathcal{T}$]

Above we cheat in stating M3.2(a) as a rule and not as a lemma. A reasonable way to construct a large proof is to start stating everything as rules and then changing the rules to lemmas one at a time until only the rules of the theory are left.

S′ **proof of** M3.2(a)$_\mathrm{h}$:

L01:	Arbitrary $\gg$	$\mathcal{H}$	;
L02:	Arbitrary $\gg$	$\mathcal{T}$	;
L03:	M3.2(a) $\gg$	$\mathcal{T} \stackrel{\mathrm{p}}{=} \mathcal{T}$	;
L04:	Hypothesize $\triangleright$ L03 $\gg$	$\mathcal{H} \dot{\Rightarrow} \mathcal{T} \stackrel{\mathrm{p}}{=} \mathcal{T}$	$\square$

2.3.2 Lemma M3.2(b)

Lemma [M3.2(b)$_\mathrm{h}$][68] reads:

[S′ **rule** M3.2(b)$_\mathrm{h}\colon \forall\mathcal{H}\colon\forall\mathcal{T}\colon\forall\mathcal{R}\colon\mathcal{H} \dot{\Rightarrow} \mathcal{T} \stackrel{\mathrm{p}}{=} \mathcal{R} \vdash \mathcal{H} \dot{\Rightarrow} \mathcal{R} \stackrel{\mathrm{p}}{=} \mathcal{T}$]

S′ **proof of** M3.2(b)$_\mathrm{h}$:

L01:	Arbitrary $\gg$	$\mathcal{H}$	;
L02:	Arbitrary $\gg$	$\mathcal{T}$	;
L03:	Arbitrary $\gg$	$\mathcal{R}$	;
L04:	Premise $\gg$	$\mathcal{H} \dot{\Rightarrow} \mathcal{T} \stackrel{\mathrm{p}}{=} \mathcal{R}$	;
L05:	S1′ $\gg$	$\mathcal{T} \stackrel{\mathrm{p}}{=} \mathcal{R} \dot{\Rightarrow} \mathcal{T} \stackrel{\mathrm{p}}{=} \mathcal{T} \dot{\Rightarrow} \mathcal{R} \stackrel{\mathrm{p}}{=} \mathcal{T}$	;
L06:	Hypothesize $\triangleright$ L05 $\gg$	$\mathcal{H} \dot{\Rightarrow} \mathcal{T} \stackrel{\mathrm{p}}{=} \mathcal{R} \dot{\Rightarrow} \mathcal{T} \stackrel{\mathrm{p}}{=} \mathcal{T} \dot{\Rightarrow} \mathcal{R} \stackrel{\mathrm{p}}{=} \mathcal{T}$	;
L07:	L06 $\trianglerighteq_\mathrm{h}$ L04 $\gg$	$\mathcal{H} \dot{\Rightarrow} \mathcal{T} \stackrel{\mathrm{p}}{=} \mathcal{T} \dot{\Rightarrow} \mathcal{R} \stackrel{\mathrm{p}}{=} \mathcal{T}$	;
L08:	M3.2(a)$_\mathrm{h}$ $\gg$	$\mathcal{H} \dot{\Rightarrow} \mathcal{T} \stackrel{\mathrm{p}}{=} \mathcal{T}$	;
L09:	L07 $\trianglerighteq_\mathrm{h}$ L08 $\gg$	$\mathcal{H} \dot{\Rightarrow} \mathcal{R} \stackrel{\mathrm{p}}{=} \mathcal{T}$	$\square$

[66] [M3.2(a) $\stackrel{\mathrm{pyk}}{=}$ "mendelson three two a"]

[67] [M3.2(a)$_\mathrm{h}$ $\stackrel{\mathrm{pyk}}{=}$ "hypothetical three two a"]

[68] [M3.2(b)$_\mathrm{h}$ $\stackrel{\mathrm{pyk}}{=}$ "hypothetical three two b"]

2.3.3 Lemma M3.1(S1)

Lemma [M3.1(S1′)$_\mathrm{h}$][69] is the hypothetical version of Mendelson Lemma 3.1(S1′):

[S′ **rule** M3.1(S1′)$_\mathrm{h}$: $\forall\mathcal{H}: \forall\mathcal{T}: \forall\mathcal{R}: \forall\mathcal{S}: \mathcal{H} \dot{\Rightarrow} \mathcal{T} \stackrel{\mathrm{p}}{=} \mathcal{R} \vdash \mathcal{H} \dot{\Rightarrow} \mathcal{T} \stackrel{\mathrm{p}}{=} \mathcal{S} \vdash \mathcal{H} \dot{\Rightarrow} \mathcal{R} \stackrel{\mathrm{p}}{=} \mathcal{S}$]

2.3.4 Lemma M3.2(c)

Lemma [M3.2(c)$_\mathrm{h}$][70] is the hypothetical version of Mendelson Lemma 3.2(c) which expresses ordinary transitivity:

[S′ **rule** M3.2(c)$_\mathrm{h}$: $\forall\mathcal{H}: \forall\mathcal{T}: \forall\mathcal{R}: \forall\mathcal{S}: \mathcal{H} \dot{\Rightarrow} \mathcal{T} \stackrel{\mathrm{p}}{=} \mathcal{R} \vdash \mathcal{H} \dot{\Rightarrow} \mathcal{R} \stackrel{\mathrm{p}}{=} \mathcal{S} \vdash \mathcal{H} \dot{\Rightarrow} \mathcal{T} \stackrel{\mathrm{p}}{=} \mathcal{S}$]

2.3.5 Lemma M3.1(S2)

Lemma [M3.1(S2′)$_\mathrm{h}$][71] is the hypothetical version of Mendelson Lemma 3.1(S2′):

[S′ **rule** M3.1(S2′)$_\mathrm{h}$: $\forall\mathcal{H}: \forall\mathcal{T}: \forall\mathcal{R}: \mathcal{H} \dot{\Rightarrow} \mathcal{T} \stackrel{\mathrm{p}}{=} \mathcal{R} \vdash \mathcal{H} \dot{\Rightarrow} \mathcal{T}' \stackrel{\mathrm{p}}{=} \mathcal{R}'$]

2.3.6 Lemma M3.1(S5)

Lemma [M3.1(S5′)$_\mathrm{h}$][72] is the hypothetical version of Mendelson Lemma 3.1(S5′):

[S′ **rule** M3.1(S5′)$_\mathrm{h}$: $\forall\mathcal{H}: \forall\mathcal{T}: \mathcal{H} \dot{\Rightarrow} \mathcal{T} \dot{+} \dot{0} \stackrel{\mathrm{p}}{=} \mathcal{T}$]

2.3.7 Lemma M3.1(S6)

Lemma [M3.1(S6′)$_\mathrm{h}$][73] is the hypothetical version of Mendelson Lemma 3.1(S6′):

[S′ **rule** M3.1(S6′)$_\mathrm{h}$: $\forall\mathcal{H}: \forall\mathcal{T}: \forall\mathcal{R}: \mathcal{H} \dot{\Rightarrow} \mathcal{T} \dot{+} \mathcal{R}' \stackrel{\mathrm{p}}{=} (\mathcal{T} \dot{+} \mathcal{R})'$]

2.3.8 Lemma M3.2(f)

Lemma [M3.2(f)][74] is the closure of Mendelson Lemma 3.2(f) for the concrete variable [$\dot{t}$]:

[S′ **lemma** M3.2(f): $\dot{\forall}\dot{t}: \dot{t} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{t}$]

The proof below uses local macro definitions.

[69] [M3.1(S1′)$_\mathrm{h}$ $\stackrel{\mathrm{pyk}}{=}$ "hypothetical three one s one"]

[70] [M3.2(c)$_\mathrm{h}$ $\stackrel{\mathrm{pyk}}{=}$ "hypothetical three two c"]

[71] [M3.1(S2′)$_\mathrm{h}$ $\stackrel{\mathrm{pyk}}{=}$ "hypothetical three one s two"]

[72] [M3.1(S5′)$_\mathrm{h}$ $\stackrel{\mathrm{pyk}}{=}$ "hypothetical three one s five"]

[73] [M3.1(S6′)$_\mathrm{h}$ $\stackrel{\mathrm{pyk}}{=}$ "hypothetical three one s six"]

[74] [M3.2(f) $\stackrel{\mathrm{pyk}}{=}$ "mendelson three two f"]

S′ **proof of** M3.2(f):

L01:	Local $\gg$	$\mathcal{Z} = \mathsf{x} \dot{\Rightarrow} \mathsf{x} \dot{\Rightarrow} \mathsf{x}$	;
L02:	A1′ $\gg$	$\mathcal{Z}$	;
L03:	M3.1(S5′)$_{\mathrm{h}}$ $\gg$	$\mathcal{Z} \dot{\Rightarrow} \dot{0} \dot{+} \dot{0} \stackrel{\mathrm{p}}{=} \dot{0}$	;
L04:	M3.2(b)$_{\mathrm{h}}$ $\rhd$ L03 $\gg$	$\mathcal{Z} \dot{\Rightarrow} \dot{0} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{0}$	;
L05:	L04 $\unrhd$ L02 $\gg$	$\dot{0} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{0}$	;
L06:	Local $\gg$	$\mathcal{H} = \dot{t} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{t}$	;
L07:	M1.7 $\gg$	$\mathcal{H} \dot{\Rightarrow} \dot{t} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{t}$	;
L08:	M3.1(S2′)$_{\mathrm{h}}$ $\rhd$ L07 $\gg$	$\mathcal{H} \dot{\Rightarrow} \dot{t}' \stackrel{\mathrm{p}}{=} (\dot{0} \dot{+} \dot{t})'$	;
L09:	M3.1(S6′)$_{\mathrm{h}}$ $\gg$	$\mathcal{H} \dot{\Rightarrow} \dot{0} \dot{+} \dot{t}' \stackrel{\mathrm{p}}{=} (\dot{0} \dot{+} \dot{t})'$	;
L10:	M3.2(b)$_{\mathrm{h}}$ $\rhd$ L09 $\gg$	$\mathcal{H} \dot{\Rightarrow} (\dot{0} \dot{+} \dot{t})' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{t}'$	;
L11:	M3.2(c)$_{\mathrm{h}}$ $\rhd$ L08 $\rhd$ L10 $\gg$	$\mathcal{H} \dot{\Rightarrow} \dot{t}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{t}'$	;
L12:	Gen′ $\rhd$ L11 $\gg$	$\dot{\forall} \dot{t} \colon (\mathcal{H} \dot{\Rightarrow} \dot{t}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{t}')$	;
L13:	S9′ $\gg$	$\dot{0} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{0} \dot{\Rightarrow} \dot{\forall} \dot{t} \colon (\dot{t} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{t} \dot{\Rightarrow} \dot{t}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{t}') \dot{\Rightarrow} \dot{\forall} \dot{t} \colon \dot{t} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{t}$	;
L14:	L13 $\unrhd$ L05 $\gg$	$\dot{\forall} \dot{t} \colon (\mathcal{H} \dot{\Rightarrow} \dot{t}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{t}') \dot{\Rightarrow} \dot{\forall} \dot{t} \colon \dot{t} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{t}$	;
L15:	L14 $\unrhd$ L12 $\gg$	$\dot{\forall} \dot{t} \colon \dot{t} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{t}$	$\square$

A Chores

A.1 The name of the page

This defines the name of the page:

$$[\text{peano} \stackrel{\text{pyk}}{=} \text{“peano”}]$$

A.2 Variables of Peano arithmetic

We use $[\dot{b}]$[75], $[\dot{c}]$[76], $[\dot{d}]$[77], $[\dot{e}]$[78], $[\dot{f}]$[79], $[\dot{g}]$[80], $[\dot{h}]$[81], $[\dot{i}]$[82], $[\dot{j}]$[83], $[\dot{k}]$[84], $[\dot{l}]$[85], $[\dot{m}]$[86], $[\dot{n}]$[87], $[\dot{o}]$[88], $[\dot{p}]$[89], $[\dot{q}]$[90], $[\dot{r}]$[91], $[\dot{s}]$[92], $[\dot{t}]$[93], $[\dot{u}]$[94], $[\dot{v}]$[95], $[\dot{w}]$[96], $[\dot{x}]$[97], $[\dot{y}]$[98], and $[\dot{z}]$[99] to denote variables of Peano arithmetic:

$[\dot{b} \ddot{=} \dot{\mathsf{b}}]$, $[\dot{c} \ddot{=} \dot{\mathsf{c}}]$, $[\dot{d} \ddot{=} \dot{\mathsf{d}}]$, $[\dot{e} \ddot{=} \dot{\mathsf{e}}]$, $[\dot{f} \ddot{=} \dot{\mathsf{f}}]$, $[\dot{g} \ddot{=} \dot{\mathsf{g}}]$, $[\dot{h} \ddot{=} \dot{\mathsf{h}}]$, $[\dot{i} \ddot{=} \dot{\mathsf{i}}]$, $[\dot{j} \ddot{=} \dot{\mathsf{j}}]$, $[\dot{k} \ddot{=} \dot{\mathsf{k}}]$, $[\dot{l} \ddot{=} \dot{\mathsf{l}}]$, $[\dot{m} \ddot{=} \dot{\mathsf{m}}]$, $[\dot{n} \ddot{=} \dot{\mathsf{n}}]$, $[\dot{o} \ddot{=} \dot{\mathsf{o}}]$, $[\dot{p} \ddot{=} \dot{\mathsf{p}}]$, $[\dot{q} \ddot{=} \dot{\mathsf{q}}]$, $[\dot{r} \ddot{=} \dot{\mathsf{r}}]$, $[\dot{s} \ddot{=} \dot{\mathsf{s}}]$, $[\dot{t} \ddot{=} \dot{\mathsf{t}}]$, $[\dot{u} \ddot{=} \dot{\mathsf{u}}]$, $[\dot{v} \ddot{=} \dot{\mathsf{v}}]$, $[\dot{w} \ddot{=} \dot{\mathsf{w}}]$, $[\dot{x} \ddot{=} \dot{\mathsf{x}}]$, $[\dot{y} \ddot{=} \dot{\mathsf{y}}]$, and $[\dot{z} \ddot{=} \dot{\mathsf{z}}]$.

A.3 TEX definitions

$[\dot{0} \overset{\text{tex}}{=}$ "
\dot{0}"]

$[\mathsf{x}' \overset{\text{tex}}{=}$ "#1.
'"]

---

[75] $[\dot{b} \overset{\text{pyk}}{=}$ "peano b"]

[76] $[\dot{c} \overset{\text{pyk}}{=}$ "peano c"]

[77] $[\dot{d} \overset{\text{pyk}}{=}$ "peano d"]

[78] $[\dot{e} \overset{\text{pyk}}{=}$ "peano e"]

[79] $[\dot{f} \overset{\text{pyk}}{=}$ "peano f"]

[80] $[\dot{g} \overset{\text{pyk}}{=}$ "peano g"]

[81] $[\dot{h} \overset{\text{pyk}}{=}$ "peano h"]

[82] $[\dot{i} \overset{\text{pyk}}{=}$ "peano i"]

[83] $[\dot{j} \overset{\text{pyk}}{=}$ "peano j"]

[84] $[\dot{k} \overset{\text{pyk}}{=}$ "peano k"]

[85] $[\dot{l} \overset{\text{pyk}}{=}$ "peano l"]

[86] $[\dot{m} \overset{\text{pyk}}{=}$ "peano m"]

[87] $[\dot{n} \overset{\text{pyk}}{=}$ "peano n"]

[88] $[\dot{o} \overset{\text{pyk}}{=}$ "peano o"]

[89] $[\dot{p} \overset{\text{pyk}}{=}$ "peano p"]

[90] $[\dot{q} \overset{\text{pyk}}{=}$ "peano q"]

[91] $[\dot{r} \overset{\text{pyk}}{=}$ "peano r"]

[92] $[\dot{s} \overset{\text{pyk}}{=}$ "peano s"]

[93] $[\dot{t} \overset{\text{pyk}}{=}$ "peano t"]

[94] $[\dot{u} \overset{\text{pyk}}{=}$ "peano u"]

[95] $[\dot{v} \overset{\text{pyk}}{=}$ "peano v"]

[96] $[\dot{w} \overset{\text{pyk}}{=}$ "peano w"]

[97] $[\dot{x} \overset{\text{pyk}}{=}$ "peano x"]

[98] $[\dot{y} \overset{\text{pyk}}{=}$ "peano y"]

[99] $[\dot{z} \overset{\text{pyk}}{=}$ "peano z"]

[$\mathsf{x} \mathop{\dot{+}} \mathsf{y}$ $\stackrel{\text{tex}}{=}$ "#1.
\mathop{\dot{+}} #2."]

[$\mathsf{x} \mathop{\dot{\cdot}} \mathsf{y}$ $\stackrel{\text{tex}}{=}$ "#1.
\mathop{\dot{\cdot}} #2."]

[$\mathsf{x} \stackrel{p}{=} \mathsf{y}$ $\stackrel{\text{tex}}{=}$ "#1.
\stackrel{p}{=} #2."]

[$\dot{\neg}\, \mathsf{x}$ $\stackrel{\text{tex}}{=}$ "
\dot{\neg}\, #1."]

[$\mathsf{x} \mathrel{\dot{\Rightarrow}} \mathsf{y}$ $\stackrel{\text{tex}}{=}$ "#1.
\mathrel{\dot{\Rightarrow}} #2."]

[$\dot{\forall} \mathsf{x} \colon \mathsf{y}$ $\stackrel{\text{tex}}{=}$ "
\dot{\forall} #1.
\colon #2."]

[$\dot{1}$ $\stackrel{\text{tex}}{=}$ "
\dot{1}"]

[$\dot{2}$ $\stackrel{\text{tex}}{=}$ "
\dot{2}"]

[$\mathsf{x} \mathrel{\dot{\wedge}} \mathsf{y}$ $\stackrel{\text{tex}}{=}$ "#1.
\mathrel{\dot{\wedge}} #2."]

[$\mathsf{x} \mathrel{\dot{\vee}} \mathsf{y}$ $\stackrel{\text{tex}}{=}$ "#1.
\mathrel{\dot{\vee}} #2."]

[$\mathsf{x} \mathrel{\dot{\Leftrightarrow}} \mathsf{y}$ $\stackrel{\text{tex}}{=}$ "#1.
\mathrel{\dot{\Leftrightarrow}} #2."]

[$\dot{\exists} \mathsf{x} \colon \mathsf{y}$ $\stackrel{\text{tex}}{=}$ "
\dot{\exists} #1.
\colon #2."]

[$\dot{\mathsf{x}}$ $\stackrel{\text{tex}}{=}$ "
\dot{#1.
}"]

[$\mathsf{x}^{\mathcal{P}}$ $\stackrel{\text{tex}}{=}$ "#1.
{} ^ {\cal P}"]

[$\dot{a}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{a}}"]

[$\dot{\text{nonfree}}(\mathsf{x},\mathsf{y}) \overset{\text{tex}}{=}$ "
\dot{nonfree}(#1.
, #2.
)"]

[$\dot{\text{nonfree}}^*(\mathsf{x},\mathsf{y}) \overset{\text{tex}}{=}$ "
\dot{nonfree}^*(#1.
, #2.
)"]

[$\dot{\text{free}}\langle\mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \overset{\text{tex}}{=}$ "
\dot{free}\langle #1.
| #2.
:= #3.
\rangle"]

[$\dot{\text{free}}^*\langle\mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \overset{\text{tex}}{=}$ "
\dot{free}{}^*\langle #1.
| #2.
:= #3.
\rangle"]

[$\mathsf{a}{\equiv}\langle\mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \overset{\text{tex}}{=}$ "#1.
{\equiv}\langle #2.
|#3.
:=#4.
\rangle"]

[$\mathsf{a}{\equiv}\langle^*\mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \overset{\text{tex}}{=}$ "#1.
{\equiv}\langle^* #2.
|#3.
:=#4.
\rangle"]

[S $\overset{\text{tex}}{=}$ "
S"]

[A1 $\overset{\text{tex}}{=}$ "
A1"]

[A2 $\overset{\text{tex}}{=}$ "
A2"]

[A3 $\overset{\text{tex}}{=}$ "
A3"]

[A4 $\overset{\text{tex}}{=}$ "
A4"]

[A5 $\stackrel{\text{tex}}{=}$ "
A5"]

[MP $\stackrel{\text{tex}}{=}$ "
MP"]

[Gen $\stackrel{\text{tex}}{=}$ "
Gen"]

[S1 $\stackrel{\text{tex}}{=}$ "
S1"]

[S2 $\stackrel{\text{tex}}{=}$ "
S2"]

[S3 $\stackrel{\text{tex}}{=}$ "
S3"]

[S4 $\stackrel{\text{tex}}{=}$ "
S4"]

[S5 $\stackrel{\text{tex}}{=}$ "
S5"]

[S6 $\stackrel{\text{tex}}{=}$ "
S6"]

[S7 $\stackrel{\text{tex}}{=}$ "
S7"]

[S8 $\stackrel{\text{tex}}{=}$ "
S8"]

[S9 $\stackrel{\text{tex}}{=}$ "
S9"]

[L3.2(a) $\stackrel{\text{tex}}{=}$ "
L3.2(a)"]

[S′ $\stackrel{\text{tex}}{=}$ "
S'"]

[A1′ $\stackrel{\text{tex}}{=}$ "
A1'"]

[A2′ $\stackrel{\text{tex}}{=}$ "
A2'"]

[A3′ $\stackrel{\text{tex}}{=}$ "
A3"']

[A4′ $\stackrel{\text{tex}}{=}$ "
A4"']

[A5′ $\stackrel{\text{tex}}{=}$ "
A5"']

[MP′ $\stackrel{\text{tex}}{=}$ "
MP"']

[Gen′ $\stackrel{\text{tex}}{=}$ "
Gen"']

[S1′ $\stackrel{\text{tex}}{=}$ "
S1"']

[S2′ $\stackrel{\text{tex}}{=}$ "
S2"']

[S3′ $\stackrel{\text{tex}}{=}$ "
S3"']

[S4′ $\stackrel{\text{tex}}{=}$ "
S4"']

[S5′ $\stackrel{\text{tex}}{=}$ "
S5"']

[S6′ $\stackrel{\text{tex}}{=}$ "
S6"']

[S7′ $\stackrel{\text{tex}}{=}$ "
S7"']

[S8′ $\stackrel{\text{tex}}{=}$ "
S8"']

[S9′ $\stackrel{\text{tex}}{=}$ "
S9"']

[L3.2(a)′ $\stackrel{\text{tex}}{=}$ "
L3.2(a)"']

[$\mathsf{x} \unrhd \mathsf{y}$ $\stackrel{\text{tex}}{=}$ "#1.
\unrhd #2."]

[M1.7 $\stackrel{\text{tex}}{=}$ "
M1.7"]

[$\mathrm{MP}'_{\mathrm{h}}$ $\stackrel{\text{tex}}{=}$ "
MP'_h"]

[$\mathsf{x} \unrhd_{\mathrm{h}} \mathsf{y}$ $\stackrel{\text{tex}}{=}$ "#1.
\unrhd_h #2."]

[Hypothesize $\stackrel{\text{tex}}{=}$ "
Hypothesize"]

[$\mathrm{Gen}'_{\mathrm{h}}$ $\stackrel{\text{tex}}{=}$ "
Gen'_h"]

[M3.2(a) $\stackrel{\text{tex}}{=}$ "
M3.2(a)"]

[$\mathrm{M3.2(a)}_{\mathrm{h}}$ $\stackrel{\text{tex}}{=}$ "
M3.2(a)_h"]

[$\mathrm{M3.2(b)}_{\mathrm{h}}$ $\stackrel{\text{tex}}{=}$ "
M3.2(b)_h"]

[$\mathrm{M3.1(S1')}_{\mathrm{h}}$ $\stackrel{\text{tex}}{=}$ "
M3.1(S1')_h"]

[$\mathrm{M3.2(c)}_{\mathrm{h}}$ $\stackrel{\text{tex}}{=}$ "
M3.2(c)_h"]

[$\mathrm{M3.1(S2')}_{\mathrm{h}}$ $\stackrel{\text{tex}}{=}$ "
M3.1(S2')_h"]

[$\mathrm{M3.1(S5')}_{\mathrm{h}}$ $\stackrel{\text{tex}}{=}$ "
M3.1(S5')_h"]

[$\mathrm{M3.1(S6')}_{\mathrm{h}}$ $\stackrel{\text{tex}}{=}$ "
M3.1(S6')_h"]

[M3.2(f) $\stackrel{\text{tex}}{=}$ "
M3.2(f)"]

[$\dot{b}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{b}}"]

[$\dot{c}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{c}}"]

[$\dot{d} \stackrel{\text{tex}}{=}$ "
\dot{\mathit{d}}"]

[$\dot{e} \stackrel{\text{tex}}{=}$ "
\dot{\mathit{e}}"]

[$\dot{f} \stackrel{\text{tex}}{=}$ "
\dot{\mathit{f}}"]

[$\dot{g} \stackrel{\text{tex}}{=}$ "
\dot{\mathit{g}}"]

[$\dot{h} \stackrel{\text{tex}}{=}$ "
\dot{\mathit{h}}"]

[$\dot{i} \stackrel{\text{tex}}{=}$ "
\dot{\mathit{i}}"]

[$\dot{j} \stackrel{\text{tex}}{=}$ "
\dot{\mathit{j}}"]

[$\dot{k} \stackrel{\text{tex}}{=}$ "
\dot{\mathit{k}}"]

[$\dot{l} \stackrel{\text{tex}}{=}$ "
\dot{\mathit{l}}"]

[$\dot{m} \stackrel{\text{tex}}{=}$ "
\dot{\mathit{m}}"]

[$\dot{n} \stackrel{\text{tex}}{=}$ "
\dot{\mathit{n}}"]

[$\dot{o} \stackrel{\text{tex}}{=}$ "
\dot{\mathit{o}}"]

[$\dot{p} \stackrel{\text{tex}}{=}$ "
\dot{\mathit{p}}"]

[$\dot{q} \stackrel{\text{tex}}{=}$ "
\dot{\mathit{q}}"]

[$\dot{r} \stackrel{\text{tex}}{=}$ "
\dot{\mathit{r}}"]

[$\dot{s} \stackrel{\text{tex}}{=}$ "
\dot{\mathit{s}}"]

$[\dot{t} \stackrel{\text{tex}}{=}$ "

\dot{\mathit{t}}"]

$[\dot{u} \stackrel{\text{tex}}{=}$ "

\dot{\mathit{u}}"]

$[\dot{v} \stackrel{\text{tex}}{=}$ "

\dot{\mathit{v}}"]

$[\dot{w} \stackrel{\text{tex}}{=}$ "

\dot{\mathit{w}}"]

$[\dot{x} \stackrel{\text{tex}}{=}$ "

\dot{\mathit{x}}"]

$[\dot{y} \stackrel{\text{tex}}{=}$ "

\dot{\mathit{y}}"]

$[\dot{z} \stackrel{\text{tex}}{=}$ "

\dot{\mathit{z}}"]

A.4 Test

$[\lceil \dot{a} \rceil^{\mathcal{P}}]^{\cdot}$

$[\lceil \mathsf{a} \rceil^{\mathcal{P}}]^{-}$

$[\dot{\mathrm{nonfree}}(\lceil \dot{x} \rceil, \lceil \dot{y} \stackrel{\mathrm{p}}{=} \dot{z} \dot{\Rightarrow} \dot{\forall} \dot{x} \colon \dot{x} \stackrel{\mathrm{p}}{=} \dot{y} \rceil)]^{\cdot}$

$[\dot{\mathrm{nonfree}}(\lceil \dot{x} \rceil, \lceil \dot{x} \stackrel{\mathrm{p}}{=} \dot{z} \dot{\Rightarrow} \dot{\forall} \dot{x} \colon \dot{x} \stackrel{\mathrm{p}}{=} \dot{y} \rceil)]^{-}$

$[\dot{\mathrm{nonfree}}(\lceil \dot{x} \rceil, \lceil \dot{y} \stackrel{\mathrm{p}}{=} \dot{x} \dot{\Rightarrow} \dot{\forall} \dot{x} \colon \dot{x} \stackrel{\mathrm{p}}{=} \dot{y} \rceil)]^{-}$

$[\dot{\mathrm{nonfree}}(\lceil \dot{x} \rceil, \lceil \dot{y} \stackrel{\mathrm{p}}{=} \dot{z} \dot{\Rightarrow} \dot{\forall} \dot{y} \colon \dot{x} \stackrel{\mathrm{p}}{=} \dot{y} \rceil)]^{-}$

$[\dot{\mathrm{free}}\langle \lceil \dot{\forall} \dot{x} \colon \mathsf{b} :: \dot{x} :: \mathsf{c} \rceil | \lceil \dot{x} \rceil := \lceil \mathsf{x} :: \dot{y} :: \mathsf{z} \rceil \rangle]^{\cdot}$

$[\dot{\mathrm{free}}\langle \lceil \dot{\forall} \dot{y} \colon \mathsf{b} :: \dot{x} :: \mathsf{c} \rceil | \lceil \dot{x} \rceil := \lceil \mathsf{x} :: \dot{y} :: \mathsf{z} \rceil \rangle]^{-}$

$[\dot{\mathrm{free}}\langle \lceil \dot{\forall} \dot{x} \colon \mathsf{b} :: \dot{x} :: \mathsf{c} \rceil | \lceil \dot{y} \rceil := \lceil \mathsf{x} :: \dot{y} :: \mathsf{z} \rceil \rangle]^{\cdot}$

$[\dot{\mathrm{free}}\langle \lceil \dot{\forall} \dot{y} \colon \mathsf{b} :: \dot{x} :: \mathsf{c} \rceil | \lceil \dot{y} \rceil := \lceil \mathsf{x} :: \dot{y} :: \mathsf{z} \rceil \rangle]^{\cdot}$

$[\dot{a} \equiv \langle \dot{a} | \dot{b} := \dot{c} \rangle]^{\cdot}$

$[\dot{c} \equiv \langle \dot{b} | \dot{b} := \dot{c} \rangle]^{\cdot}$

$[\forall \dot{a} \colon \dot{a} \stackrel{\mathrm{p}}{=} \dot{b} \equiv \langle \forall \dot{a} \colon \dot{a} \stackrel{\mathrm{p}}{=} \dot{b} | \dot{a} := \dot{c} \rangle]^{\cdot}$

$[\forall \dot{a}\colon \dot{a} \stackrel{\mathrm{p}}{=} \dot{c} \equiv \langle \forall \dot{a}\colon \dot{a} \stackrel{\mathrm{p}}{=} \dot{b} | \dot{b} := \dot{c} \rangle]^{\cdot}$

$[\dot{\forall} \dot{a}\colon \dot{a} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{a} \mathrel{\dot{\Rightarrow}} \dot{c} \mathrel{\dot{:}} \dot{d} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{c} \mathrel{\dot{:}} \dot{d} \equiv \langle \dot{\forall} \dot{a}\colon \dot{a} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{a} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{b} | \dot{b} := \dot{c} \mathrel{\dot{:}} \dot{d} \rangle]^{\cdot}$

$[\dot{\forall} \dot{a}\colon \dot{a} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{a} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{b} \equiv \langle \dot{\forall} \dot{a}\colon \dot{a} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{a} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{b} | \dot{a} := \dot{c} \rangle]^{\cdot}$

A.5 Priority table

Priority table
Preassociative
[peano], [base], [bracket $*$ end bracket], [big bracket $*$ end bracket], [math $*$ end math], [**flush left** [$*$]], [x], [y], [z], [[$* \bowtie *$]], [[$* \stackrel{*}{\rightarrow} *$]], [pyk], [tex], [name], [prio], [$*$], [T], [if($*, *, *$)], [[$* \stackrel{*}{\Rightarrow} *$]], [val], [claim], [$\perp$], [f($*$)], [$(*)^{\mathrm{I}}$], [F], [$\underline{0}$], [$\underline{1}$], [$\underline{2}$], [$\underline{3}$], [$\underline{4}$], [$\underline{5}$], [$\underline{6}$], [$\underline{7}$], [$\underline{8}$], [$\underline{9}$], [0], [1], [2], [3], [4], [5], [6], [7], [8], [9], [a], [b], [c], [d], [e], [f], [g], [h], [i], [j], [k], [l], [m], [n], [o], [p], [q], [r], [s], [t], [u], [v], [w], [$(*)^{\mathrm{M}}$], [If($*, *, *$)], [array{$*$} $*$ end array], [l], [c], [r], [empty], [$\langle * \mid * := * \rangle$], [$\mathcal{M}(*)$], [$\tilde{\mathcal{U}}(*)$], [$\mathcal{U}(*)$], [$\mathcal{U}^{\mathrm{M}}(*)$], [**apply**($*, *$)], [$\mathbf{apply}_1(*, *)$], [identifier($*$)], [$\mathrm{identifier}_1(*, *)$], [array-plus($*, *$)], [array-remove($*, *, *$)], [array-put($*, *, *, *$)], [array-add($*, *, *, *, *$)], [bit($*, *$)], [$\mathrm{bit}_1(*, *)$], [rack], ["vector"], ["bibliography"], ["dictionary"], ["body"], ["codex"], ["expansion"], ["code"], ["cache"], ["diagnose"], ["pyk"], ["tex"], ["texname"], ["value"], ["message"], ["macro"], ["definition"], ["unpack"], ["claim"], ["priority"], ["lambda"], ["apply"], ["true"], ["if"], ["quote"], ["proclaim"], ["define"], ["introduce"], ["hide"], ["pre"], ["post"], [$\mathcal{E}(*, *, *)$], [$\mathcal{E}_2(*, *, *, *, *)$], [$\mathcal{E}_3(*, *, *, *)$], [$\mathcal{E}_4(*, *, *, *)$], [**lookup**($*, *, *$)], [**abstract**($*, *, *, *$)], [[$\lceil * \rceil$]], [$\mathcal{M}(*, *, *)$], [$\mathcal{M}_2(*, *, *, *)$], [$\mathcal{M}^*(*, *, *)$], [macro], [$\mathrm{s}_0$], [**zip**($*, *$)], [$\mathbf{assoc}_1(*, *, *)$], [$(*)^{\mathbf{p}}$], [self], [[$* \stackrel{..}{=} *$]], [[$* \doteq *$]], [[$* \stackrel{\prime}{=} *$]], [[$* \stackrel{\mathrm{pyk}}{=} *$]], [[$* \stackrel{\mathrm{tex}}{=} *$]], [[$* \stackrel{\mathrm{name}}{=} *$]], [**Priority table**[$*$]], [$\tilde{\mathcal{M}}_1$], [$\tilde{\mathcal{M}}_2(*)$], [$\tilde{\mathcal{M}}_3(*)$], [$\tilde{\mathcal{M}}_4(*, *, *, *)$], [$\tilde{\mathcal{M}}(*, *, *)$], [$\tilde{\mathcal{Q}}(*, *, *)$], [$\tilde{\mathcal{Q}}_2(*, *, *)$], [$\tilde{\mathcal{Q}}_3(*, *, *, *)$], [$\tilde{\mathcal{Q}}^*(*, *, *)$], [$(*)$], [**aspect**($*, *$)], [**aspect**($*, *, *$)], [$\langle * \rangle$], [$\mathbf{tuple}_1(*)$], [$\mathbf{tuple}_2(*)$], [$\mathrm{let}_2(*, *)$], [$\mathrm{let}_1(*, *)$], [[$* \stackrel{\mathrm{claim}}{=} *$]], [checker], [**check**($*, *$)], [$\mathbf{check}_2(*, *, *)$], [$\mathbf{check}_3(*, *, *)$], [$\mathbf{check}^*(*, *)$], [$\mathbf{check}^*_2(*, *, *)$], [$[*]^{\cdot}$], [$[*]^{-}$], [$[*]^{\circ}$], [msg], [[$* \stackrel{\mathrm{msg}}{=} *$]], [<stmt>], [stmt], [[$* \stackrel{\mathrm{stmt}}{=} *$]], [HeadNil$'$], [HeadPair$'$], [Transitivity$'$], [$\perp\!\!\!\perp$], [Contra$'$], [$\mathrm{T}'_{\mathrm{E}}$], [$\mathrm{L}_1$], [$\underline{*}$], [$\mathcal{A}$], [$\mathcal{B}$], [$\mathcal{C}$], [$\mathcal{D}$], [$\mathcal{E}$], [$\mathcal{F}$], [$\mathcal{G}$], [$\mathcal{H}$], [$\mathcal{I}$], [$\mathcal{J}$], [$\mathcal{K}$], [$\mathcal{L}$], [$\mathcal{M}$], [$\mathcal{N}$], [$\mathcal{O}$], [$\mathcal{P}$], [$\mathcal{Q}$], [$\mathcal{R}$], [$\mathcal{S}$], [$\mathcal{T}$], [$\mathcal{U}$], [$\mathcal{V}$], [$\mathcal{W}$], [$\mathcal{X}$], [$\mathcal{Y}$], [$\mathcal{Z}$], [$\langle * \mid * := * \rangle$], [$\langle^* * \mid * := * \rangle$], [$\emptyset$], [Remainder], [$(*)^{\mathbf{v}}$], [intro($*, *, *, *$)], [intro($*, *, *$)], [error($*, *$)], [$\mathrm{error}_2(*, *)$], [proof($*, *, *$)], [$\mathrm{proof}_2(*, *)$], [$\mathcal{S}(*, *)$], [$\mathcal{S}^{\mathrm{I}}(*, *)$], [$\mathcal{S}^{\rhd}(*, *)$], [$\mathcal{S}^{\rhd}_1(*, *, *)$], [$\mathcal{S}^{\mathrm{E}}(*, *)$], [$\mathcal{S}^{\mathrm{E}}_1(*, *, *)$], [$\mathcal{S}^{+}(*, *)$], [$\mathcal{S}^{+}_1(*, *, *)$], [$\mathcal{S}^{-}(*, *)$], [$\mathcal{S}^{-}_1(*, *, *)$], [$\mathcal{S}^{*}(*, *)$], [$\mathcal{S}^{*}_1(*, *, *)$], [$\mathcal{S}^{*}_2(*, *, *, *)$], [$\mathcal{S}^{@}(*, *)$], [$\mathcal{S}^{@}_1(*, *, *)$], [$\mathcal{S}^{\vdash}(*, *)$], [$\mathcal{S}^{\vdash}_1(*, *, *, *)$], [$\mathcal{S}^{\Vdash}(*, *)$], [$\mathcal{S}^{\Vdash}_1(*, *, *, *)$], [$\mathcal{S}^{\mathrm{i.e.}}(*, *)$], [$\mathcal{S}^{\mathrm{i.e.}}_1(*, *, *, *)$], [$\mathcal{S}^{\mathrm{i.e.}}_2(*, *, *, *, *)$], [$\mathcal{S}^{\forall}(*, *)$], [$\mathcal{S}^{\forall}_1(*, *, *, *)$], [$\mathcal{S}^{;}(*, *)$], [$\mathcal{S}^{;}_1(*, *, *)$], [$\mathcal{S}^{;}_2(*, *, *, *)$], [$\mathcal{T}(*)$], [claims($*, *, *$)], [$\mathrm{claims}_2(*, *, *)$], [<proof>], [proof], [[**Lemma** $*\colon *$]], [[**Proof of** $*\colon *$]], [[$*$ **lemma** $*\colon *$]], [[$*$ **antilemma** $*\colon *$]], [[$*$ **rule** $*\colon *$]], [[$*$ **antirule** $*\colon *$]], [verifier], [$\mathcal{V}_1(*)$], [$\mathcal{V}_2(*, *)$], [$\mathcal{V}_3(*, *, *, *)$], [$\mathcal{V}_4(*, *)$], [$\mathcal{V}_5(*, *, *, *)$], [$\mathcal{V}_6(*, *, *, *)$],

$[\mathcal{V}_7(*,*,*,*)]$, $[\mathrm{Cut}(*,*)]$, $[\mathrm{Head}_\oplus(*)]$, $[\mathrm{Tail}_\oplus(*)]$, $[\mathrm{rule}_1(*,*)]$, $[\mathrm{rule}(*,*)]$, [Rule tactic], $[\mathrm{Plus}(*,*)]$, [[**Theory** $*$]], $[\mathrm{theory}_2(*,*)]$, $[\mathrm{theory}_3(*,*)]$, $[\mathrm{theory}_4(*,*,*)]$, [HeadNil″], [HeadPair″], [Transitivity″], [Contra″], [HeadNil], [HeadPair], [Transitivity], [Contra], $[\mathrm{T_E}]$, [ragged right], [ragged right expansion], $[\mathrm{parm}(*,*,*)]$, $[\mathrm{parm}^*(*,*,*)]$, $[\mathrm{inst}(*,*)]$, $[\mathrm{inst}^*(*,*)]$, $[\mathrm{occur}(*,*,*)]$, $[\mathrm{occur}^*(*,*,*)]$, $[\mathrm{unify}(*=*,*)]$, $[\mathrm{unify}^*(*=*,*)]$, $[\mathrm{unify}_2(*=*,*)]$, $[\mathrm{L_a}]$, $[\mathrm{L_b}]$, $[\mathrm{L_c}]$, $[\mathrm{L_d}]$, $[\mathrm{L_e}]$, $[\mathrm{L_f}]$, $[\mathrm{L_g}]$, $[\mathrm{L_h}]$, $[\mathrm{L_i}]$, $[\mathrm{L_j}]$, $[\mathrm{L_k}]$, $[\mathrm{L_l}]$, $[\mathrm{L_m}]$, $[\mathrm{L_n}]$, $[\mathrm{L_o}]$, $[\mathrm{L_p}]$, $[\mathrm{L_q}]$, $[\mathrm{L_r}]$, $[\mathrm{L_s}]$, $[\mathrm{L_t}]$, $[\mathrm{L_u}]$, $[\mathrm{L_v}]$, $[\mathrm{L_w}]$, $[\mathrm{L_x}]$, $[\mathrm{L_y}]$, $[\mathrm{L_z}]$, $[\mathrm{L_A}]$, $[\mathrm{L_B}]$, $[\mathrm{L_C}]$, $[\mathrm{L_D}]$, $[\mathrm{L_E}]$, $[\mathrm{L_F}]$, $[\mathrm{L_G}]$, $[\mathrm{L_H}]$, $[\mathrm{L_I}]$, $[\mathrm{L_J}]$, $[\mathrm{L_K}]$, $[\mathrm{L_L}]$, $[\mathrm{L_M}]$, $[\mathrm{L_N}]$, $[\mathrm{L_O}]$, $[\mathrm{L_P}]$, $[\mathrm{L_Q}]$, $[\mathrm{L_R}]$, $[\mathrm{L_S}]$, $[\mathrm{L_T}]$, $[\mathrm{L_U}]$, $[\mathrm{L_V}]$, $[\mathrm{L_W}]$, $[\mathrm{L_X}]$, $[\mathrm{L_Y}]$, $[\mathrm{L_Z}]$, $[\mathrm{L_?}]$, [Reflexivity], $[\mathrm{Reflexivity}_1]$, [Commutativity], $[\mathrm{Commutativity}_1]$, [<tactic>], [tactic], $[[* \overset{\mathrm{tactic}}{=} *]]$, $[\mathcal{P}(*,*,*)]$, $[\mathcal{P}^*(*,*,*)]$, $[\mathrm{p}_0]$, $[\mathrm{conclude}_1(*,*)]$, $[\mathrm{conclude}_2(*,*,*)]$, $[\mathrm{conclude}_3(*,*,*,*)]$, $[\mathrm{conclude}_4(*,*)]$, $[\dot{0}]$, $[\dot{1}]$, $[\dot{2}]$, $[\dot{a}]$, $[\dot{b}]$, $[\dot{c}]$, $[\dot{d}]$, $[\dot{e}]$, $[\dot{f}]$, $[\dot{g}]$, $[\dot{h}]$, $[\dot{i}]$, $[\dot{j}]$, $[\dot{k}]$, $[\dot{l}]$, $[\dot{m}]$, $[\dot{n}]$, $[\dot{o}]$, $[\dot{p}]$, $[\dot{q}]$, $[\dot{r}]$, $[\dot{s}]$, $[\dot{t}]$, $[\dot{u}]$, $[\dot{v}]$, $[\dot{w}]$, $[\dot{x}]$, $[\dot{y}]$, $[\dot{z}]$, $[\mathrm{nonfree}(*,*)]$, $[\mathrm{nonfree}^*(*,*)]$, $[\mathrm{free}\langle *|*:=*\rangle]$, $[\mathrm{free}^*\langle *|*:=*\rangle]$, $[*\equiv\langle *|*:=*\rangle]$, $[*\equiv\langle^* *|*:=*\rangle]$, [S], [A1], [A2], [A3], [A4], [A5], [S1], [S2], [S3], [S4], [S5], [S6], [S7], [S8], [S9], [MP], [Gen], [L3.2(a)], [S′], [A1′], [A2′], [A3′], [A4′], [A5′], [S1′], [S2′], [S3′], [S4′], [S5′], [S6′], [S7′], [S8′], [S9′], [MP′], [Gen′], [L3.2(a)′], [M1.7], $[\mathrm{MP}'_\mathrm{h}]$, [Hypothesize], $[\mathrm{Gen}'_\mathrm{h}]$, [M3.2(a)], $[\mathrm{M3.2(a)_h}]$, $[\mathrm{M3.2(b)_h}]$, $[\mathrm{M3.1(S1')_h}]$, $[\mathrm{M3.2(c)_h}]$, $[\mathrm{M3.1(S2')_h}]$, $[\mathrm{M3.1(S5')_h}]$, $[\mathrm{M3.1(S6')_h}]$, [M3.2(f)];

Preassociative

$[*\_\{*\}]$, $[*']$, $[*[\,*\,]]$, $[*[*\rightarrow*]]$, $[*[*\Rightarrow*]]$, $[\dot{*}]$;

Preassociative

[“ $*$ ”], [], $[(*)^{\mathbf{t}}]$, [string($*$) + $*$], [string($*$) ++ $*$], [
$*$], [$*$], [!$*$], [”$*$], [#$*$], [$\$*$], [%$*$], [&$*$], [’$*$], [($*$], [)$*$], [$**$], [+$*$], [, $*$], [-$*$], [.$*$], [/$*$], [0$*$], [1$*$], [2$*$], [3$*$], [4$*$], [5$*$], [6$*$], [7$*$], [8$*$], [9$*$], [:$*$], [; $*$], [$<*$], [$=*$], [$>*$], [?$*$], [@$*$], [A$*$], [B$*$], [C$*$], [D$*$], [E$*$], [F$*$], [G$*$], [H$*$], [I$*$], [J$*$], [K$*$], [L$*$], [M$*$], [N$*$], [O$*$], [P$*$], [Q$*$], [R$*$], [S$*$], [T$*$], [U$*$], [V$*$], [W$*$], [X$*$], [Y$*$], [Z$*$], [[$*$], [\$*$], []$*$], [^$*$], [_$*$], [‘$*$], [a$*$], [b$*$], [c$*$], [d$*$], [e$*$], [f$*$], [g$*$], [h$*$], [i$*$], [j$*$], [k$*$], [l$*$], [m$*$], [n$*$], [o$*$], [p$*$], [q$*$], [r$*$], [s$*$], [t$*$], [u$*$], [v$*$], [w$*$], [x$*$], [y$*$], [z$*$], [{$*$], [|$*$], [}$*$], [~$*$], [**Preassociative** $*;*$], [**Postassociative** $*;*$], [[$*$], $*$], [priority $*$ end], [newline $*$], [macro newline $*$];

Preassociative

[$*$0], [$*$1], [0b], [$*$-color($*$)], [$*$-color*($*$)];

Preassociative

[$*$ ’ $*$], [$*$ ‘ $*$];

Preassociative

$[*^{\mathrm{H}}]$, $[*^{\mathrm{T}}]$, $[*^{\mathrm{U}}]$, $[*^{\mathrm{h}}]$, $[*^{\mathrm{t}}]$, $[*^{\mathrm{s}}]$, $[*^{\mathrm{c}}]$, $[*^{\mathrm{d}}]$, $[*^{\mathrm{a}}]$, $[*^{\mathrm{C}}]$, $[*^{\mathrm{M}}]$, $[*^{\mathrm{B}}]$, $[*^{\mathrm{r}}]$, $[*^{\mathrm{i}}]$, $[*^{\mathrm{d}}]$, $[*^{\mathrm{R}}]$, $[*^{0}]$, $[*^{1}]$, $[*^{2}]$, $[*^{3}]$, $[*^{4}]$, $[*^{5}]$, $[*^{6}]$, $[*^{7}]$, $[*^{8}]$, $[*^{9}]$, $[*^{\mathrm{E}}]$, $[*^{\mathcal{V}}]$, $[*^{\mathcal{C}}]$, $[*^{\mathcal{C}^*}]$, $[*']$;

Preassociative

$[*\cdot*]$, $[*\cdot_0*]$, $[*\mathbin{\dot{:}}*]$;

Preassociative

$[*+*]$, $[*+_0*]$, $[*+_1*]$, $[*-*]$, $[*-_0*]$, $[*-_1*]$, $[*\dot{+}*]$;

Preassociative

$[*\cup\{*\}]$, $[*\cup*]$, $[*\backslash\{*\}]$;

Postassociative
$[* \therefore *], [* \underline{\therefore} *], [* \underline{::} *], [* \underline{+2*} *], [* :: *], [* +2* *]$;
Postassociative
$[*, *]$;
Preassociative
$[* \overset{\mathrm{B}}{\approx} *], [* \overset{\mathrm{D}}{\approx} *], [* \overset{\mathrm{C}}{\approx} *], [* \overset{\mathrm{P}}{\approx} *], [* \approx *], [* = *], [* \overset{+}{\rightarrow} *], [* \overset{\mathrm{t}}{=} *], [* \overset{\mathrm{t}^*}{=} *], [* \overset{\mathrm{r}}{=} *]$,
$[* \in_{\mathrm{t}} *], [* \subseteq_{\mathrm{T}} *], [* \overset{\mathrm{T}}{=} *], [* \overset{\mathrm{s}}{=} *], [* \text{ free in } *], [* \text{ free in}^* *], [* \text{ free for } * \text{ in } *]$,
$[* \text{ free for}^* * \text{ in } *], [* \in_{\mathrm{c}} *], [* < *], [* <' *], [* \leq' *], [* \overset{\mathrm{p}}{=} *], [*^{\mathcal{P}}]$;
Preassociative
$[\neg *], [\dot{\neg} *]$;
Preassociative
$[* \wedge *], [* \ddot{\wedge} *], [* \tilde{\wedge} *], [* \wedge_{\mathrm{c}} *], [* \dot{\wedge} *]$;
Preassociative
$[* \vee *], [* \parallel *], [* \ddot{\vee} *], [* \dot{\vee} *]$;
Preassociative
$[\dot{\forall} *\colon *], [\dot{\exists} *\colon *]$;
Postassociative
$[* \ddot{\Rightarrow} *], [* \dot{\Rightarrow} *], [* \dot{\Leftrightarrow} *]$;
Postassociative
$[* : *], [*!*]$;
Preassociative
$[* \begin{cases} * \\ * \end{cases}]$;
Preassociative
$[\lambda * .*], [\Lambda *]$, [**if** $*$ **then** $*$ **else** $*$], [**let** $* = *$ **in** $*$], [**let** $* \ddot{=} *$ in $*$];
Preassociative
$[*^{\mathrm{I}}], [*^{\triangleright}], [*^{\mathrm{V}}], [*^{+}], [*^{-}], [*^{*}]$;
Preassociative
$[* @ *], [* \triangleright *], [* \blacktriangleright *], [* \gg *], [* \trianglerighteq *], [* \trianglerighteq_{\mathrm{h}} *]$;
Postassociative
$[* \vdash *], [* \Vdash *]$, [$*$ i.e. $*$];
Preassociative
$[\forall *\colon *]$;
Postassociative
$[* \oplus *]$;
Postassociative
$[*; *]$;
Preassociative
[$*$ proves $*$];
Preassociative
[$*$ **proof of** $* : *$], [Line $* : * \gg *; *$], [Last line $* \gg * \square$],
[Line $*$: Premise $\gg *; *$], [Line $*$: Side-condition $\gg *; *$], [Arbitrary $\gg *; *$],
[Local $\gg * = *; *$];
Postassociative
[$*$ then $*$], $[*[*]*]$;

Preassociative
[$*\&*$];
Preassociative
[$*\backslash\backslash*$]; **End table**

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