Peano arithmetic

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1 Peano arithmetic

This Logiweb page [1] defines Peano arithmetic.

1.1 The constructs of Peano arithmetic

Terms of Peano arithmetic are constructed from zero $[\dot{0}]$[1], successor $[\mathsf{x}']$[2], plus $[\mathsf{x} \dot{+} \mathsf{y}]$[3], and times $[\mathsf{x} \dot{\cdot} \mathsf{y}]$[4].

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[1] $[\dot{0} \stackrel{\text{pyk}}{=}$ "peano zero"$]$

[2] $[\mathsf{x}' \stackrel{\text{pyk}}{=}$ "$*$ peano succ"$]$

[3] $[\mathsf{x} \dot{+} \mathsf{y} \stackrel{\text{pyk}}{=}$ "$*$ peano plus $*$"$]$

[4] $[\mathsf{x} \dot{\cdot} \mathsf{y} \stackrel{\text{pyk}}{=}$ "$*$ peano times $*$"$]$

Formulas of Peano arithmeric are constructed from equality $[\mathsf{x} \stackrel{\mathrm{p}}{=} \mathsf{y}]$[5], negation $[\dot{\neg}\, \mathsf{x}]$[6], implication $[\mathsf{x} \dot{\Rightarrow} \mathsf{y}]$[7], and universal quantification $[\dot{\forall} \mathsf{x} \colon \mathsf{y}]$[8].

From these constructs we macro define one $[\dot{1}]$[9], two $[\dot{2}]$[10], conjunction $[\mathsf{x} \dot{\wedge} \mathsf{y}]$[11], disjunction $[\mathsf{x} \dot{\vee} \mathsf{y}]$[12], biimplication $[\mathsf{x} \dot{\Leftrightarrow} \mathsf{y}]$[13], and existential quantification $[\dot{\exists} \mathsf{x} \colon \mathsf{y}]$[14]:

$$[\dot{1} \ddot{=} \dot{0}']$$

$$[\dot{2} \ddot{=} \dot{1}']$$

$$[\mathsf{x} \dot{\wedge} \mathsf{y} \ddot{=} \dot{\neg}\, (\mathsf{x} \dot{\Rightarrow} \dot{\neg}\, \mathsf{y})]$$

$$[\mathsf{x} \dot{\vee} \mathsf{y} \ddot{=} \dot{\neg}\, \mathsf{x} \dot{\Rightarrow} \mathsf{y}]$$

$$[\mathsf{x} \dot{\Leftrightarrow} \mathsf{y} \ddot{=} (\mathsf{x} \dot{\Rightarrow} \mathsf{y}) \dot{\wedge} (\mathsf{y} \dot{\Rightarrow} \mathsf{x})]$$

$$[\dot{\exists} \mathsf{x} \colon \mathsf{y} \ddot{=} \dot{\neg}\, \dot{\forall} \mathsf{x} \colon \dot{\neg}\, \mathsf{y}]$$

1.2 Variables

We now introduce the unary operator $[\dot{\mathsf{x}}]$[15] and define that a term is a *Peano variable* (i.e. a variable of Peano arithmetic) if it has the $[\dot{\mathsf{x}}]$ operator in its root. $[\mathsf{x}^{\mathcal{P}}]$[16] is true if $[\mathsf{x}]$ is a Peano variable:

$$[\mathsf{x}^{\mathcal{P}} \doteq \mathsf{x} \stackrel{\mathrm{r}}{=} \lceil \dot{\mathsf{x}} \rceil]$$

We macro define $[\dot{a}]$[17] to be a Peano variable:

$$[\dot{a} \ddot{=} \dot{\mathsf{a}}]$$

Appendix A.2 defines Peano variables for the other letters of the English alphabet.

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5 $[\mathsf{x} \stackrel{\mathrm{p}}{=} \mathsf{y} \stackrel{\mathrm{pyk}}{=}$ "$*$ peano is $*$"$]$

6 $[\dot{\neg}\, \mathsf{x} \stackrel{\mathrm{pyk}}{=}$ "peano not $*$"$]$

7 $[\mathsf{x} \dot{\Rightarrow} \mathsf{y} \stackrel{\mathrm{pyk}}{=}$ "$*$ peano imply $*$"$]$

8 $[\dot{\forall} \mathsf{x} \colon \mathsf{y} \stackrel{\mathrm{pyk}}{=}$ "peano all $*$ indeed $*$"$]$

9 $[\dot{1} \stackrel{\mathrm{pyk}}{=}$ "peano one"$]$

10 $[\dot{2} \stackrel{\mathrm{pyk}}{=}$ "peano two"$]$

11 $[\mathsf{x} \dot{\wedge} \mathsf{y} \stackrel{\mathrm{pyk}}{=}$ "$*$ peano and $*$"$]$

12 $[\mathsf{x} \dot{\vee} \mathsf{y} \stackrel{\mathrm{pyk}}{=}$ "$*$ peano or $*$"$]$

13 $[\mathsf{x} \dot{\Leftrightarrow} \mathsf{y} \stackrel{\mathrm{pyk}}{=}$ "$*$ peano iff $*$"$]$

14 $[\dot{\exists} \mathsf{x} \colon \mathsf{y} \stackrel{\mathrm{pyk}}{=}$ "peano exist $*$ indeed $*$"$]$

15 $[\dot{\mathsf{x}} \stackrel{\mathrm{pyk}}{=}$ "$*$ peano var"$]$

16 $[\mathsf{x}^{\mathcal{P}} \stackrel{\mathrm{pyk}}{=}$ "$*$ is peano var"$]$

17 $[\dot{a} \stackrel{\mathrm{pyk}}{=}$ "peano a"$]$

$[\mathrm{non\dot{f}ree}(\mathsf{x},\mathsf{y})]$[18] is true if the Peano variable $[\mathsf{x}]$ does not occur free in the Peano term/formula $[\mathsf{y}]$. $[\mathrm{non\dot{f}ree}^{*}(\mathsf{x},\mathsf{y})]$[19] is true if the Peano variable $[\mathsf{x}]$ does not occur free in the list $[\mathsf{y}]$ of Peano terms/formulas.

$$
\begin{aligned}
&[\mathrm{non\dot{f}ree}(\mathsf{x},\mathsf{y}) \doteq \\
&\textbf{if } \mathsf{y}^{\mathcal{P}} \textbf{ then } \neg \mathsf{x} \stackrel{t}{=} \mathsf{y} \textbf{ else} \\
&\textbf{if } \neg \mathsf{y} \stackrel{r}{=} \lceil \dot{\forall} \mathsf{x}\colon \mathsf{y} \rceil \textbf{ then } \mathrm{non\dot{f}ree}^{*}(\mathsf{x},\mathsf{y}^{t}) \textbf{ else} \\
&\textbf{if } \mathsf{x} \stackrel{t}{=} \mathsf{y}^{1} \textbf{ then } \mathsf{T} \textbf{ else } \mathrm{non\dot{f}ree}(\mathsf{x},\mathsf{y}^{2})]
\end{aligned}
$$

$$[\mathrm{non\dot{f}ree}^{*}(\mathsf{x},\mathsf{y}) \doteq \mathsf{x}!\mathrm{If}(\mathsf{y},\mathsf{T},\mathrm{non\dot{f}ree}(\mathsf{x},\mathsf{y}^{h}) \ddot{\wedge} \mathrm{non\dot{f}ree}^{*}(\mathsf{x},\mathsf{y}^{t}))]$$

$[\mathrm{\dot{f}ree}\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle]$[20] is true if the substitution $[\langle \mathsf{a}\,|\mathsf{x}{:=}\,\mathsf{b}\rangle]$ is free. $[\mathrm{\dot{f}ree}^{*}\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle]$[21] is the version where $[\mathsf{a}]$ is a list of terms.

$$
\begin{aligned}
&[\mathrm{\dot{f}ree}\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \doteq \mathsf{x}!\mathsf{b}! \\
&\textbf{if } \mathsf{a}^{\mathcal{P}} \textbf{ then } \mathsf{T} \textbf{ else} \\
&\textbf{if } \neg \mathsf{a} \stackrel{r}{=} \lceil \dot{\forall} \mathsf{u}\colon \mathsf{v} \rceil \textbf{ then } \mathrm{\dot{f}ree}^{*}\langle \mathsf{a}^{t}|\mathsf{x} := \mathsf{b}\rangle \textbf{ else} \\
&\textbf{if } \mathsf{a}^{1} \stackrel{t}{=} \mathsf{x} \textbf{ then } \mathsf{T} \textbf{ else} \\
&\textbf{if } \mathrm{non\dot{f}ree}(\mathsf{x},\mathsf{a}^{2}) \textbf{ then } \mathsf{T} \textbf{ else} \\
&\textbf{if } \neg\mathrm{non\dot{f}ree}(\mathsf{a}^{1},\mathsf{b}) \textbf{ then } \mathsf{F} \textbf{ else} \\
&\mathrm{\dot{f}ree}\langle \mathsf{a}^{2}|\mathsf{x} := \mathsf{b}\rangle]
\end{aligned}
$$

$$[\mathrm{\dot{f}ree}^{*}\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \doteq \mathsf{x}!\mathsf{b}!\mathrm{If}(\mathsf{a},\mathsf{T},\mathrm{\dot{f}ree}\langle \mathsf{a}^{h}|\mathsf{x} := \mathsf{b}\rangle \ddot{\wedge} \mathrm{\dot{f}ree}^{*}\langle \mathsf{a}^{t}|\mathsf{x} := \mathsf{b}\rangle)]$$

$[\mathsf{a}{\equiv}\langle \mathsf{b}|\mathsf{x} := \mathsf{c}\rangle]$[22] is true if $[\mathsf{a}]$ equals $[\langle \mathsf{b}\,|\mathsf{x}{:=}\,\mathsf{c}\rangle]$. $[\mathsf{a}{\equiv}\langle^{*} \mathsf{b}|\mathsf{x} := \mathsf{c}\rangle]$[23] is the version where $[\mathsf{a}]$ and $[\mathsf{b}]$ are lists.

$$
\begin{aligned}
&[\mathsf{a}{\equiv}\langle \mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \doteq \mathsf{a}!\mathsf{x}!\mathsf{c}! \\
&\textbf{if } \mathsf{b} \stackrel{r}{=} \lceil \dot{\forall} \mathsf{u}\colon \mathsf{v} \rceil \ddot{\wedge} \mathsf{b}^{1} \stackrel{t}{=} \mathsf{x} \textbf{ then } \mathsf{a} \stackrel{t}{=} \mathsf{b} \textbf{ else} \\
&\textbf{if } \mathsf{b}^{\mathcal{P}} \wedge \mathsf{b} \stackrel{t}{=} \mathsf{x} \textbf{ then } \mathsf{a} \stackrel{t}{=} \mathsf{c} \textbf{ else} \\
&\mathsf{a} \stackrel{r}{=} \mathsf{b} \ddot{\wedge} \mathsf{a}^{t}{\equiv}\langle^{*} \mathsf{b}^{t}|\mathsf{x} := \mathsf{c}\rangle]
\end{aligned}
$$

$$[\mathsf{a}{\equiv}\langle^{*} \mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \doteq \mathsf{b}!\mathsf{x}!\mathsf{c}!\mathrm{If}(\mathsf{a},\mathsf{T},\mathsf{a}^{h}{\equiv}\langle \mathsf{b}^{h}|\mathsf{x} := \mathsf{c}\rangle \ddot{\wedge} \mathsf{a}^{t}{\equiv}\langle^{*} \mathsf{b}^{t}|\mathsf{x} := \mathsf{c}\rangle)]$$

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[18] $[\mathrm{non\dot{f}ree}(\mathsf{x},\mathsf{y}) \stackrel{\mathrm{pyk}}{=}$ "peano nonfree $*$ in $*$ end nonfree"$]$

[19] $[\mathrm{non\dot{f}ree}^{*}(\mathsf{x},\mathsf{y}) \stackrel{\mathrm{pyk}}{=}$ "peano nonfree star $*$ in $*$ end nonfree"$]$

[20] $[\mathrm{\dot{f}ree}\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \stackrel{\mathrm{pyk}}{=}$ "peano free $*$ set $*$ to $*$ end free"$]$

[21] $[\mathrm{\dot{f}ree}^{*}\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \stackrel{\mathrm{pyk}}{=}$ "peano free star $*$ set $*$ to $*$ end free"$]$

[22] $[\mathsf{a}{\equiv}\langle \mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \stackrel{\mathrm{pyk}}{=}$ "peano sub $*$ is $*$ where $*$ is $*$ end sub"$]$

[23] $[\mathsf{a}{\equiv}\langle^{*} \mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \stackrel{\mathrm{pyk}}{=}$ "peano sub star $*$ is $*$ where $*$ is $*$ end sub"$]$

1.3 Mendelsons system S

System [S][24] of Mendelson [2] expresses Peano arithmetic. It comprises the axioms [A1][25], [A2][26], [A3][27], [A4][28], and [A5][29] and inference rules [MP][30] and [Gen][31] of first order predicate calculus. Furthermore, it comprises the proper axioms [S1][32], [S2][33], [S3][34], [S4][35], [S5][36], [S6][37], [S7][38], [S8][39], and [S9][40]. System [S] is defined thus:

[**Theory** S]

[S **rule** A1: $\forall \mathcal{A}\colon \forall \mathcal{B}\colon \mathcal{A} \dot{\Rightarrow} \mathcal{B} \dot{\Rightarrow} \mathcal{A}$]

[S **rule** A2: $\forall \mathcal{A}\colon \forall \mathcal{B}\colon \forall \mathcal{C}\colon (\mathcal{A} \dot{\Rightarrow} \mathcal{B} \dot{\Rightarrow} \mathcal{C}) \dot{\Rightarrow} (\mathcal{A} \dot{\Rightarrow} \mathcal{B}) \dot{\Rightarrow} \mathcal{A} \dot{\Rightarrow} \mathcal{C}$]

[S **rule** A3: $(\dot{\neg}\mathcal{B} \dot{\Rightarrow} \dot{\neg}\mathcal{A}) \dot{\Rightarrow} (\dot{\neg}\mathcal{B} \dot{\Rightarrow} \mathcal{A}) \dot{\Rightarrow} \mathcal{B}$]

The order of quantifiers in the following axiom is such that [$\mathcal{C}$] which the current conclusion tactic cannot guess comes first. This allows to supply a value for [$\mathcal{C}$] without having to supply values for the other meta-variables.

[S **rule** A4: $\forall \mathcal{C}\colon \forall \mathcal{A}\colon \forall \mathcal{X}\colon \forall \mathcal{B}\colon \lceil \mathcal{A} \rceil \equiv \langle \lceil \mathcal{B} \rceil | \lceil \mathcal{X} \rceil := \lceil \mathcal{C} \rceil \rangle \Vdash \dot{\forall} \mathcal{X}\colon \mathcal{B} \dot{\Rightarrow} \mathcal{A}$]

[S **rule** A5: $\forall \mathcal{X}\colon \forall \mathcal{A}\colon \forall \mathcal{B}\colon \dot{\mathrm{nonfree}}(\mathcal{X}, \mathcal{A}) \Vdash \dot{\forall} \mathcal{X}\colon (\mathcal{A} \dot{\Rightarrow} \mathcal{B}) \dot{\Rightarrow} \mathcal{A} \dot{\Rightarrow} \dot{\forall} \mathcal{X}\colon \mathcal{B}$]

[S **rule** MP: $\forall \mathcal{A}\colon \forall \mathcal{B}\colon \mathcal{A} \dot{\Rightarrow} \mathcal{B} \vdash \mathcal{A} \vdash \mathcal{B}$]

[S **rule** Gen: $\forall \mathcal{X}\colon \forall \mathcal{A}\colon \mathcal{A} \vdash \dot{\forall} \mathcal{X}\colon \mathcal{A}$]

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[24] [S $\stackrel{\text{pyk}}{=}$ "system s"]

[25] [A1 $\stackrel{\text{pyk}}{=}$ "axiom a one"]

[26] [A2 $\stackrel{\text{pyk}}{=}$ "axiom a two"]

[27] [A3 $\stackrel{\text{pyk}}{=}$ "axiom a three"]

[28] [A4 $\stackrel{\text{pyk}}{=}$ "axiom a four"]

[29] [A5 $\stackrel{\text{pyk}}{=}$ "axiom a five"]

[30] [MP $\stackrel{\text{pyk}}{=}$ "rule mp"]

[31] [Gen $\stackrel{\text{pyk}}{=}$ "rule gen"]

[32] [S1 $\stackrel{\text{pyk}}{=}$ "axiom s one"]

[33] [S2 $\stackrel{\text{pyk}}{=}$ "axiom s two"]

[34] [S3 $\stackrel{\text{pyk}}{=}$ "axiom s three"]

[35] [S4 $\stackrel{\text{pyk}}{=}$ "axiom s four"]

[36] [S5 $\stackrel{\text{pyk}}{=}$ "axiom s five"]

[37] [S6 $\stackrel{\text{pyk}}{=}$ "axiom s six"]

[38] [S7 $\stackrel{\text{pyk}}{=}$ "axiom s seven"]

[39] [S8 $\stackrel{\text{pyk}}{=}$ "axiom s eight"]

[40] [S9 $\stackrel{\text{pyk}}{=}$ "axiom s nine"]

Axiom [S1] to [S8] are stated as Mendelson [2] does. This is done here to test certain parts of the Logiweb system. Serious users of Peano arithmetic are advised to take Mendelson's Lemma 3.1 as axioms instead.

[S **rule** S1: $\dot{a} \stackrel{\text{p}}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{\text{p}}{=} \dot{c} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{\text{p}}{=} \dot{c}$]

[S **rule** S2: $\dot{a} \stackrel{\text{p}}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a}' \stackrel{\text{p}}{=} \dot{b}'$]

[S **rule** S3: $\dot{\neg} \dot{0} \stackrel{\text{p}}{=} \dot{a}'$]

[S **rule** S4: $\dot{a}' \stackrel{\text{p}}{=} \dot{b}' \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{\text{p}}{=} \dot{b}$]

[S **rule** S5: $\dot{a} \mathbin{\dot{+}} \dot{0} \stackrel{\text{p}}{=} \dot{a}$]

[S **rule** S6: $\dot{a} \mathbin{\dot{+}} \dot{b}' \stackrel{\text{p}}{=} (\dot{a} \mathbin{\dot{+}} \dot{b})'$]

[S **rule** S7: $\dot{a} \mathbin{\dot{\cdot}} \dot{0} \stackrel{\text{p}}{=} \dot{0}$]

[S **rule** S8: $\dot{a} \mathbin{\dot{\cdot}} (\dot{b}') \stackrel{\text{p}}{=} (\dot{a} \mathbin{\dot{\cdot}} \dot{b}) \mathbin{\dot{+}} \dot{a}$]

[S **rule** S9: $\forall \mathcal{A} \colon \forall \mathcal{B} \colon \forall \mathcal{C} \colon \forall \mathcal{X} \colon$
$\mathcal{B} \equiv \langle \mathcal{A} | \mathcal{X} := \dot{0} \rangle \Vdash \mathcal{C} \equiv \langle \mathcal{A} | \mathcal{X} := \mathcal{X}' \rangle \Vdash$
$\mathcal{B} \mathrel{\dot{\Rightarrow}} \dot{\forall} \mathcal{X} \colon (\mathcal{A} \mathrel{\dot{\Rightarrow}} \mathcal{C}) \mathrel{\dot{\Rightarrow}} \dot{\forall} \mathcal{X} \colon \mathcal{A}$]

1.4 A lemma and a proof

We now prove Lemma [L3.2(a)][41] which is an instance of the corresponding proposition in Mendelson [2]:

[S **lemma** L3.2(a): $\dot{x} \stackrel{\text{p}}{=} \dot{x}$]

S **proof of** L3.2(a):

L01:	S5 $\gg$	$\dot{a} \mathbin{\dot{+}} \dot{0} \stackrel{\text{p}}{=} \dot{a}$	;
L02:	Gen $\rhd$ L01 $\gg$	$\dot{\forall} \dot{a} \colon \dot{a} \mathbin{\dot{+}} \dot{0} \stackrel{\text{p}}{=} \dot{a}$	;
L03:	A4 @ $\dot{x}$ $\gg$	$\dot{\forall} \dot{a} \colon \dot{a} \mathbin{\dot{+}} \dot{0} \stackrel{\text{p}}{=} \dot{a} \mathrel{\dot{\Rightarrow}} \dot{x} \mathbin{\dot{+}} \dot{0} \stackrel{\text{p}}{=} \dot{x}$	;
L04:	MP $\rhd$ L03 $\rhd$ L02 $\gg$	$\dot{x} \mathbin{\dot{+}} \dot{0} \stackrel{\text{p}}{=} \dot{x}$	;
L05:	S1 $\gg$	$\dot{a} \stackrel{\text{p}}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{\text{p}}{=} \dot{c} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{\text{p}}{=} \dot{c}$	;
L06:	Gen $\rhd$ L05 $\gg$	$\dot{\forall} \dot{c} \colon (\dot{a} \stackrel{\text{p}}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{\text{p}}{=} \dot{c} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{\text{p}}{=} \dot{c})$	;
L07:	A4 @ $\dot{x}$ $\gg$	$\dot{\forall} \dot{c} \colon (\dot{a} \stackrel{\text{p}}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{\text{p}}{=} \dot{c} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{\text{p}}{=} \dot{c}) \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{\text{p}}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{\text{p}}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{\text{p}}{=} \dot{x}$	;
L08:	MP $\rhd$ L07 $\rhd$ L06 $\gg$	$\dot{a} \stackrel{\text{p}}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{\text{p}}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{\text{p}}{=} \dot{x}$	;
L09:	Gen $\rhd$ L08 $\gg$	$\dot{\forall} \dot{b} \colon (\dot{a} \stackrel{\text{p}}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{\text{p}}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{\text{p}}{=} \dot{x})$	;
L10:	A4 @ $\dot{x}$ $\gg$	$\dot{\forall} \dot{b} \colon (\dot{a} \stackrel{\text{p}}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{\text{p}}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{\text{p}}{=} \dot{x}) \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{\text{p}}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{\text{p}}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \stackrel{\text{p}}{=} \dot{x}$	;

[41] [L3.2(a) $\stackrel{\text{pyk}}{=}$ "lemma l three two a"]

L11: MP $\rhd$ L10 $\rhd$ L09 $\gg$ $\dot{a} \overset{\text{p}}{=} \dot{x} \Rightarrow \dot{a} \overset{\text{p}}{=} \dot{x} \Rightarrow \dot{x} \overset{\text{p}}{=} \dot{x}$;

L12: Gen $\rhd$ L11 $\gg$ $\dot{\forall} \dot{a}\colon (\dot{a} \overset{\text{p}}{=} \dot{x} \Rightarrow \dot{a} \overset{\text{p}}{=} \dot{x} \Rightarrow \dot{x} \overset{\text{p}}{=} \dot{x})$;

L13: A4 @ $\dot{x} \dot{+} \dot{0}$ $\gg$ $\dot{\forall} \dot{a}\colon (\dot{a} \overset{\text{p}}{=} \dot{x} \Rightarrow \dot{a} \overset{\text{p}}{=} \dot{x} \Rightarrow \dot{x} \overset{\text{p}}{=} \dot{x}) \dot{\Rightarrow} \dot{x} \dot{+} \dot{0} \overset{\text{p}}{=} \dot{x} \Rightarrow \dot{x} \dot{+} \dot{0} \overset{\text{p}}{=} \dot{x} \Rightarrow \dot{x} \overset{\text{p}}{=} \dot{x}$;

L14: MP $\rhd$ L13 $\rhd$ L12 $\gg$ $\dot{x} \dot{+} \dot{0} \overset{\text{p}}{=} \dot{x} \Rightarrow \dot{x} \dot{+} \dot{0} \overset{\text{p}}{=} \dot{x} \Rightarrow \dot{x} \overset{\text{p}}{=} \dot{x}$;

L15: MP $\rhd$ L14 $\rhd$ L04 $\gg$ $\dot{x} \dot{+} \dot{0} \overset{\text{p}}{=} \dot{x} \Rightarrow \dot{x} \overset{\text{p}}{=} \dot{x}$;

L16: MP $\rhd$ L15 $\rhd$ L04 $\gg$ $\dot{x} \overset{\text{p}}{=} \dot{x}$ $\square$

1.5 An alternative axiomatic system

System [S′][42] is system [S] in which the proper axioms are taken from Lemma 3.1 in Mendelson [2]. It comprises the axioms [A1′][43], [A2′][44], [A3′][45], [A4′][46], and [A5′][47] and inference rules [MP′][48] and [Gen′][49] of first order predicate calculus. Furthermore, it comprises the proper axioms [S1′][50], [S2′][51], [S3′][52], [S4′][53], [S5′][54], [S6′][55], [S7′][56], [S8′][57], and [S9′][58].

System [S′] is defined thus:

[**Theory** S′]

[S′ **rule** A1′: $\forall \mathcal{A}\colon \forall \mathcal{B}\colon \mathcal{A} \dot{\Rightarrow} \mathcal{B} \dot{\Rightarrow} \mathcal{A}$]

[S′ **rule** A2′: $\forall \mathcal{A}\colon \forall \mathcal{B}\colon \forall \mathcal{C}\colon (\mathcal{A} \dot{\Rightarrow} \mathcal{B} \dot{\Rightarrow} \mathcal{C}) \dot{\Rightarrow} (\mathcal{A} \dot{\Rightarrow} \mathcal{B}) \dot{\Rightarrow} \mathcal{A} \dot{\Rightarrow} \mathcal{C}$]

[S′ **rule** A3′: $(\dot{\neg} \mathcal{B} \dot{\Rightarrow} \dot{\neg} \mathcal{A}) \dot{\Rightarrow} (\dot{\neg} \mathcal{B} \dot{\Rightarrow} \mathcal{A}) \dot{\Rightarrow} \mathcal{B}$]

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[42] [S′ $\overset{\text{pyk}}{=}$ "system prime s"]

[43] [A1′ $\overset{\text{pyk}}{=}$ "axiom prime a one"]

[44] [A2′ $\overset{\text{pyk}}{=}$ "axiom prime a two"]

[45] [A3′ $\overset{\text{pyk}}{=}$ "axiom prime a three"]

[46] [A4′ $\overset{\text{pyk}}{=}$ "axiom prime a four"]

[47] [A5′ $\overset{\text{pyk}}{=}$ "axiom prime a five"]

[48] [MP′ $\overset{\text{pyk}}{=}$ "rule prime mp"]

[49] [Gen′ $\overset{\text{pyk}}{=}$ "rule prime gen"]

[50] [S1′ $\overset{\text{pyk}}{=}$ "axiom prime s one"]

[51] [S2′ $\overset{\text{pyk}}{=}$ "axiom prime s two"]

[52] [S3′ $\overset{\text{pyk}}{=}$ "axiom prime s three"]

[53] [S4′ $\overset{\text{pyk}}{=}$ "axiom prime s four"]

[54] [S5′ $\overset{\text{pyk}}{=}$ "axiom prime s five"]

[55] [S6′ $\overset{\text{pyk}}{=}$ "axiom prime s six"]

[56] [S7′ $\overset{\text{pyk}}{=}$ "axiom prime s seven"]

[57] [S8′ $\overset{\text{pyk}}{=}$ "axiom prime s eight"]

[58] [S9′ $\overset{\text{pyk}}{=}$ "axiom prime s nine"]

[S′ **rule** A4′: $\forall \mathcal{C} \colon \forall \mathcal{A} \colon \forall \mathcal{X} \colon \forall \mathcal{B} \colon \lceil \mathcal{A} \rceil \equiv \langle \lceil \mathcal{B} \rceil | \lceil \mathcal{X} \rceil := \lceil \mathcal{C} \rceil \rangle \Vdash \dot{\forall} \mathcal{X} \colon \mathcal{B} \mathrel{\dot{\Rightarrow}} \mathcal{A}$]

[S′ **rule** A5′: $\forall \mathcal{X} \colon \forall \mathcal{A} \colon \forall \mathcal{B} \colon \dot{\text{nonfree}}(\mathcal{X}, \mathcal{A}) \Vdash \dot{\forall} \mathcal{X} \colon (\mathcal{A} \mathrel{\dot{\Rightarrow}} \mathcal{B}) \mathrel{\dot{\Rightarrow}} \mathcal{A} \mathrel{\dot{\Rightarrow}} \dot{\forall} \mathcal{X} \colon \mathcal{B}$]

[S′ **rule** MP′: $\forall \mathcal{A} \colon \forall \mathcal{B} \colon \mathcal{A} \mathrel{\dot{\Rightarrow}} \mathcal{B} \vdash \mathcal{A} \vdash \mathcal{B}$]

[S′ **rule** Gen′: $\forall \mathcal{X} \colon \forall \mathcal{A} \colon \mathcal{A} \vdash \dot{\forall} \mathcal{X} \colon \mathcal{A}$]

[S′ **rule** S1′: $\forall \mathcal{A} \colon \forall \mathcal{B} \colon \forall \mathcal{C} \colon \mathcal{A} \stackrel{\mathrm{p}}{=} \mathcal{B} \mathrel{\dot{\Rightarrow}} \mathcal{A} \stackrel{\mathrm{p}}{=} \mathcal{C} \mathrel{\dot{\Rightarrow}} \mathcal{B} \stackrel{\mathrm{p}}{=} \mathcal{C}$]

[S′ **rule** S2′: $\forall \mathcal{A} \colon \forall \mathcal{B} \colon \mathcal{A} \stackrel{\mathrm{p}}{=} \mathcal{B} \mathrel{\dot{\Rightarrow}} \mathcal{A}' \stackrel{\mathrm{p}}{=} \mathcal{B}'$]

[S′ **rule** S3′: $\forall \mathcal{A} \colon \neg \dot{0} \stackrel{\mathrm{p}}{=} \mathcal{A}'$]

[S′ **rule** S4′: $\forall \mathcal{A} \colon \forall \mathcal{B} \colon \mathcal{A}' \stackrel{\mathrm{p}}{=} \mathcal{B}' \mathrel{\dot{\Rightarrow}} \mathcal{A} \stackrel{\mathrm{p}}{=} \mathcal{B}$]

[S′ **rule** S5′: $\forall \mathcal{A} \colon \mathcal{A} \mathbin{\dot{+}} \dot{0} \stackrel{\mathrm{p}}{=} \mathcal{A}$]

[S′ **rule** S6′: $\forall \mathcal{A} \colon \forall \mathcal{B} \colon \mathcal{A} \mathbin{\dot{+}} \mathcal{B}' \stackrel{\mathrm{p}}{=} (\mathcal{A} \mathbin{\dot{+}} \mathcal{B})'$]

[S′ **rule** S7′: $\forall \mathcal{A} \colon \mathcal{A} \mathbin{\dot{\cdot}} \dot{0} \stackrel{\mathrm{p}}{=} \dot{0}$]

[S′ **rule** S8′: $\forall \mathcal{A} \colon \forall \mathcal{B} \colon \mathcal{A} \mathbin{\dot{\cdot}} (\mathcal{B}') \stackrel{\mathrm{p}}{=} (\mathcal{A} \mathbin{\dot{\cdot}} \mathcal{B}) \mathbin{\dot{+}} \mathcal{A}$]

[S′ **rule** S9′: $\forall \mathcal{A} \colon \forall \mathcal{B} \colon \forall \mathcal{C} \colon \forall \mathcal{X} \colon$
$\mathcal{B} \equiv \langle \mathcal{A} | \mathcal{X} := \dot{0} \rangle \Vdash \mathcal{C} \equiv \langle \mathcal{A} | \mathcal{X} := \mathcal{X}' \rangle \Vdash$
$\mathcal{B} \mathrel{\dot{\Rightarrow}} \dot{\forall} \mathcal{X} \colon (\mathcal{A} \mathrel{\dot{\Rightarrow}} \mathcal{C}) \mathrel{\dot{\Rightarrow}} \dot{\forall} \mathcal{X} \colon \mathcal{A}$]

Note that [A1] and [A1′] are distinct. The former says [S $\vdash \forall \mathcal{A} \colon \forall \mathcal{B} \colon \mathcal{A} \mathrel{\dot{\Rightarrow}} \mathcal{B} \mathrel{\dot{\Rightarrow}} \mathcal{A}$] and the latter says [S′ $\vdash \forall \mathcal{A} \colon \forall \mathcal{B} \colon \mathcal{A} \mathrel{\dot{\Rightarrow}} \mathcal{B} \mathrel{\dot{\Rightarrow}} \mathcal{A}$].

1.6 Restatement of lemma and a proof

We now prove Lemma [L3.2(a)] once again under the name of [L3.2(a)′][59]:

[S′ **lemma** L3.2(a)′: $\forall \mathcal{A} \colon \mathcal{A} \stackrel{\mathrm{p}}{=} \mathcal{A}$]

S′ **proof of** L3.2(a)′:

L01:	Arbitrary $\gg$	$\mathcal{A}$	;
L02:	S5′ $\gg$	$\mathcal{A} \mathbin{\dot{+}} \dot{0} \stackrel{\mathrm{p}}{=} \mathcal{A}$	;
L03:	S1′ $\gg$	$\mathcal{A} \mathbin{\dot{+}} \dot{0} \stackrel{\mathrm{p}}{=} \mathcal{A} \mathrel{\dot{\Rightarrow}}$ $\mathcal{A} \mathbin{\dot{+}} \dot{0} \stackrel{\mathrm{p}}{=} \mathcal{A} \mathrel{\dot{\Rightarrow}} \mathcal{A} \stackrel{\mathrm{p}}{=} \mathcal{A}$	;
L04:	MP′ $\rhd$ L03 $\rhd$ L02 $\gg$	$\mathcal{A} \mathbin{\dot{+}} \dot{0} \stackrel{\mathrm{p}}{=} \mathcal{A} \mathrel{\dot{\Rightarrow}} \mathcal{A} \stackrel{\mathrm{p}}{=} \mathcal{A}$	;
L05:	MP′ $\rhd$ L04 $\rhd$ L02 $\gg$	$\mathcal{A} \stackrel{\mathrm{p}}{=} \mathcal{A}$	$\square$

[59] [L3.2(a)′ $\stackrel{\mathrm{pyk}}{=}$ "lemma prime l three two a"]

A Chores

A.1 The name of the page

This defines the name of the page:

$$[\text{peano axioms} \stackrel{\text{pyk}}{=} \text{“peano axioms”}]$$

A.2 Variables of Peano arithmetic

We use $[\dot{b}]$[60], $[\dot{c}]$[61], $[\dot{d}]$[62], $[\dot{e}]$[63], $[\dot{f}]$[64], $[\dot{g}]$[65], $[\dot{h}]$[66], $[\dot{i}]$[67], $[\dot{j}]$[68], $[\dot{k}]$[69], $[\dot{l}]$[70], $[\dot{m}]$[71], $[\dot{n}]$[72], $[\dot{o}]$[73], $[\dot{p}]$[74], $[\dot{q}]$[75], $[\dot{r}]$[76], $[\dot{s}]$[77], $[\dot{t}]$[78], $[\dot{u}]$[79], $[\dot{v}]$[80], $[\dot{w}]$[81], $[\dot{x}]$[82], $[\dot{y}]$[83], and $[\dot{z}]$[84] to denote variables of Peano arithmetic:

$[\dot{b} \ddot{=} \dot{\mathsf{b}}]$, $[\dot{c} \ddot{=} \dot{\mathsf{c}}]$, $[\dot{d} \ddot{=} \dot{\mathsf{d}}]$, $[\dot{e} \ddot{=} \dot{\mathsf{e}}]$, $[\dot{f} \ddot{=} \dot{\mathsf{f}}]$, $[\dot{g} \ddot{=} \dot{\mathsf{g}}]$, $[\dot{h} \ddot{=} \dot{\mathsf{h}}]$, $[\dot{i} \ddot{=} \dot{\mathsf{i}}]$, $[\dot{j} \ddot{=} \dot{\mathsf{j}}]$, $[\dot{k} \ddot{=} \dot{\mathsf{k}}]$, $[\dot{l} \ddot{=} \dot{\mathsf{l}}]$, $[\dot{m} \ddot{=} \dot{\mathsf{m}}]$, $[\dot{n} \ddot{=} \dot{\mathsf{n}}]$, $[\dot{o} \ddot{=} \dot{\mathsf{o}}]$, $[\dot{p} \ddot{=} \dot{\mathsf{p}}]$, $[\dot{q} \ddot{=} \dot{\mathsf{q}}]$, $[\dot{r} \ddot{=} \dot{\mathsf{r}}]$, $[\dot{s} \ddot{=} \dot{\mathsf{s}}]$, $[\dot{t} \ddot{=} \dot{\mathsf{t}}]$, $[\dot{u} \ddot{=} \dot{\mathsf{u}}]$, $[\dot{v} \ddot{=} \dot{\mathsf{v}}]$, $[\dot{w} \ddot{=} \dot{\mathsf{w}}]$, $[\dot{x} \ddot{=} \dot{\mathsf{x}}]$, $[\dot{y} \ddot{=} \dot{\mathsf{y}}]$, and $[\dot{z} \ddot{=} \dot{\mathsf{z}}]$.

---

60 $[\dot{b} \stackrel{\text{pyk}}{=} \text{“peano b”}]$

61 $[\dot{c} \stackrel{\text{pyk}}{=} \text{“peano c”}]$

62 $[\dot{d} \stackrel{\text{pyk}}{=} \text{“peano d”}]$

63 $[\dot{e} \stackrel{\text{pyk}}{=} \text{“peano e”}]$

64 $[\dot{f} \stackrel{\text{pyk}}{=} \text{“peano f”}]$

65 $[\dot{g} \stackrel{\text{pyk}}{=} \text{“peano g”}]$

66 $[\dot{h} \stackrel{\text{pyk}}{=} \text{“peano h”}]$

67 $[\dot{i} \stackrel{\text{pyk}}{=} \text{“peano i”}]$

68 $[\dot{j} \stackrel{\text{pyk}}{=} \text{“peano j”}]$

69 $[\dot{k} \stackrel{\text{pyk}}{=} \text{“peano k”}]$

70 $[\dot{l} \stackrel{\text{pyk}}{=} \text{“peano l”}]$

71 $[\dot{m} \stackrel{\text{pyk}}{=} \text{“peano m”}]$

72 $[\dot{n} \stackrel{\text{pyk}}{=} \text{“peano n”}]$

73 $[\dot{o} \stackrel{\text{pyk}}{=} \text{“peano o”}]$

74 $[\dot{p} \stackrel{\text{pyk}}{=} \text{“peano p”}]$

75 $[\dot{q} \stackrel{\text{pyk}}{=} \text{“peano q”}]$

76 $[\dot{r} \stackrel{\text{pyk}}{=} \text{“peano r”}]$

77 $[\dot{s} \stackrel{\text{pyk}}{=} \text{“peano s”}]$

78 $[\dot{t} \stackrel{\text{pyk}}{=} \text{“peano t”}]$

79 $[\dot{u} \stackrel{\text{pyk}}{=} \text{“peano u”}]$

80 $[\dot{v} \stackrel{\text{pyk}}{=} \text{“peano v”}]$

81 $[\dot{w} \stackrel{\text{pyk}}{=} \text{“peano w”}]$

82 $[\dot{x} \stackrel{\text{pyk}}{=} \text{“peano x”}]$

83 $[\dot{y} \stackrel{\text{pyk}}{=} \text{“peano y”}]$

84 $[\dot{z} \stackrel{\text{pyk}}{=} \text{“peano z”}]$

A.3 TeX definitions

[$\dot{0} \stackrel{\text{tex}}{=}$ "
\dot{0}"]

[$\mathsf{x}' \stackrel{\text{tex}}{=}$ "#1.
'"]

[$\mathsf{x} \mathbin{\dot{+}} \mathsf{y} \stackrel{\text{tex}}{=}$ "#1.
\mathop{\dot{+}} #2."]

[$\mathsf{x} \mathbin{\dot{\cdot}} \mathsf{y} \stackrel{\text{tex}}{=}$ "#1.
\mathop{\dot{\cdot}} #2."]

[$\mathsf{x} \stackrel{\text{p}}{=} \mathsf{y} \stackrel{\text{tex}}{=}$ "#1.
\stackrel{p}{=} #2."]

[$\dot{\neg}\, \mathsf{x} \stackrel{\text{tex}}{=}$ "
\dot{\neg}\, #1."]

[$\mathsf{x} \mathrel{\dot{\Rightarrow}} \mathsf{y} \stackrel{\text{tex}}{=}$ "#1.
\mathrel{\dot{\Rightarrow}} #2."]

[$\dot{\forall} \mathsf{x} \colon \mathsf{y} \stackrel{\text{tex}}{=}$ "
\dot{\forall} #1.
\colon #2."]

[$\dot{1} \stackrel{\text{tex}}{=}$ "
\dot{1}"]

[$\dot{2} \stackrel{\text{tex}}{=}$ "
\dot{2}"]

[$\mathsf{x} \mathrel{\dot{\wedge}} \mathsf{y} \stackrel{\text{tex}}{=}$ "#1.
\mathrel{\dot{\wedge}} #2."]

[$\mathsf{x} \mathrel{\dot{\vee}} \mathsf{y} \stackrel{\text{tex}}{=}$ "#1.
\mathrel{\dot{\vee}} #2."]

[$\mathsf{x} \mathrel{\dot{\Leftrightarrow}} \mathsf{y} \stackrel{\text{tex}}{=}$ "#1.
\mathrel{\dot{\Leftrightarrow}} #2."]

[$\dot{\exists} \mathsf{x} \colon \mathsf{y} \stackrel{\text{tex}}{=}$ "
\dot{\exists} #1.
\colon #2."]

[$\dot{\mathsf{x}} \stackrel{\text{tex}}{=}$ "
\dot{#1.
}"]

[$\mathsf{x}^{\mathcal{P}} \stackrel{\text{tex}}{=}$ "#1.
{} ^ {\cal P}"]

[$\dot{a} \stackrel{\text{tex}}{=}$ "
\dot{\mathit{a}}"]

[$\dot{\text{nonfree}}(\mathsf{x},\mathsf{y}) \stackrel{\text{tex}}{=}$ "
\dot{nonfree}(#1.
, #2.
)"]

[$\dot{\text{nonfree}}^{*}(\mathsf{x},\mathsf{y}) \stackrel{\text{tex}}{=}$ "
\dot{nonfree}^*(#1.
, #2.
)"]

[$\dot{\text{free}}\langle\mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \stackrel{\text{tex}}{=}$ "
\dot{free}\langle #1.
| #2.
:= #3.
\rangle"]

[$\dot{\text{free}}^{*}\langle\mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \stackrel{\text{tex}}{=}$ "
\dot{free}{}^*\langle #1.
| #2.
:= #3.
\rangle"]

[$\mathsf{a}{\equiv}\langle\mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \stackrel{\text{tex}}{=}$ "#1.
{\equiv}\langle #2.
|#3.
:=#4.
\rangle"]

[$\mathsf{a}{\equiv}\langle^{*}\mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \stackrel{\text{tex}}{=}$ "#1.
{\equiv}\langle^* #2.
|#3.
:=#4.
\rangle"]

[S $\stackrel{\text{tex}}{=}$ "
S"]

[A1 $\stackrel{\text{tex}}{=}$ "
A1"]

[A2 $\stackrel{\text{tex}}{=}$ "
A2"]

[A3 $\stackrel{\text{tex}}{=}$ "

A3"]

[A4 $\stackrel{\text{tex}}{=}$ "

A4"]

[A5 $\stackrel{\text{tex}}{=}$ "

A5"]

[MP $\stackrel{\text{tex}}{=}$ "

MP"]

[Gen $\stackrel{\text{tex}}{=}$ "

Gen"]

[S1 $\stackrel{\text{tex}}{=}$ "

S1"]

[S2 $\stackrel{\text{tex}}{=}$ "

S2"]

[S3 $\stackrel{\text{tex}}{=}$ "

S3"]

[S4 $\stackrel{\text{tex}}{=}$ "

S4"]

[S5 $\stackrel{\text{tex}}{=}$ "

S5"]

[S6 $\stackrel{\text{tex}}{=}$ "

S6"]

[S7 $\stackrel{\text{tex}}{=}$ "

S7"]

[S8 $\stackrel{\text{tex}}{=}$ "

S8"]

[S9 $\stackrel{\text{tex}}{=}$ "

S9"]

[L3.2(a) $\stackrel{\text{tex}}{=}$ "

L3.2(a)"]

[S′ $\stackrel{\text{tex}}{=}$ "

S'"]

[A1′ $\stackrel{\text{tex}}{=}$ “
A1”’]

[A2′ $\stackrel{\text{tex}}{=}$ “
A2”’]

[A3′ $\stackrel{\text{tex}}{=}$ “
A3”’]

[A4′ $\stackrel{\text{tex}}{=}$ “
A4”’]

[A5′ $\stackrel{\text{tex}}{=}$ “
A5”’]

[MP′ $\stackrel{\text{tex}}{=}$ “
MP”’]

[Gen′ $\stackrel{\text{tex}}{=}$ “
Gen”’]

[S1′ $\stackrel{\text{tex}}{=}$ “
S1”’]

[S2′ $\stackrel{\text{tex}}{=}$ “
S2”’]

[S3′ $\stackrel{\text{tex}}{=}$ “
S3”’]

[S4′ $\stackrel{\text{tex}}{=}$ “
S4”’]

[S5′ $\stackrel{\text{tex}}{=}$ “
S5”’]

[S6′ $\stackrel{\text{tex}}{=}$ “
S6”’]

[S7′ $\stackrel{\text{tex}}{=}$ “
S7”’]

[S8′ $\stackrel{\text{tex}}{=}$ “
S8”’]

[S9′ $\stackrel{\text{tex}}{=}$ “
S9”’]

[L3.2(a)$' \stackrel{\text{tex}}{=}$ "
L3.2(a)'"]

[$\dot{b} \stackrel{\text{tex}}{=}$ "
\dot{\mathit{b}}"]

[$\dot{c} \stackrel{\text{tex}}{=}$ "
\dot{\mathit{c}}"]

[$\dot{d} \stackrel{\text{tex}}{=}$ "
\dot{\mathit{d}}"]

[$\dot{e} \stackrel{\text{tex}}{=}$ "
\dot{\mathit{e}}"]

[$\dot{f} \stackrel{\text{tex}}{=}$ "
\dot{\mathit{f}}"]

[$\dot{g} \stackrel{\text{tex}}{=}$ "
\dot{\mathit{g}}"]

[$\dot{h} \stackrel{\text{tex}}{=}$ "
\dot{\mathit{h}}"]

[$\dot{i} \stackrel{\text{tex}}{=}$ "
\dot{\mathit{i}}"]

[$\dot{j} \stackrel{\text{tex}}{=}$ "
\dot{\mathit{j}}"]

[$\dot{k} \stackrel{\text{tex}}{=}$ "
\dot{\mathit{k}}"]

[$\dot{l} \stackrel{\text{tex}}{=}$ "
\dot{\mathit{l}}"]

[$\dot{m} \stackrel{\text{tex}}{=}$ "
\dot{\mathit{m}}"]

[$\dot{n} \stackrel{\text{tex}}{=}$ "
\dot{\mathit{n}}"]

[$\dot{o} \stackrel{\text{tex}}{=}$ "
\dot{\mathit{o}}"]

[$\dot{p} \stackrel{\text{tex}}{=}$ "
\dot{\mathit{p}}"]

$[\dot{q} \stackrel{\text{tex}}{=}$ "

\dot{\mathit{q}}"]

$[\dot{r} \stackrel{\text{tex}}{=}$ "

\dot{\mathit{r}}"]

$[\dot{s} \stackrel{\text{tex}}{=}$ "

\dot{\mathit{s}}"]

$[\dot{t} \stackrel{\text{tex}}{=}$ "

\dot{\mathit{t}}"]

$[\dot{u} \stackrel{\text{tex}}{=}$ "

\dot{\mathit{u}}"]

$[\dot{v} \stackrel{\text{tex}}{=}$ "

\dot{\mathit{v}}"]

$[\dot{w} \stackrel{\text{tex}}{=}$ "

\dot{\mathit{w}}"]

$[\dot{x} \stackrel{\text{tex}}{=}$ "

\dot{\mathit{x}}"]

$[\dot{y} \stackrel{\text{tex}}{=}$ "

\dot{\mathit{y}}"]

$[\dot{z} \stackrel{\text{tex}}{=}$ "

\dot{\mathit{z}}"]

A.4 Test

$[\lceil \dot{a} \rceil^{\mathcal{P}}]^{\cdot}$

$[\lceil \mathsf{a} \rceil^{\mathcal{P}}]^{-}$

$[\dot{\text{nonfree}}(\lceil \dot{x} \rceil, \lceil \dot{y} \stackrel{\text{p}}{=} \dot{z} \mathrel{\dot{\Rightarrow}} \dot{\forall} \dot{x}\colon \dot{x} \stackrel{\text{p}}{=} \dot{y} \rceil)]^{\cdot}$

$[\dot{\text{nonfree}}(\lceil \dot{x} \rceil, \lceil \dot{x} \stackrel{\text{p}}{=} \dot{z} \mathrel{\dot{\Rightarrow}} \dot{\forall} \dot{x}\colon \dot{x} \stackrel{\text{p}}{=} \dot{y} \rceil)]^{-}$

$[\dot{\text{nonfree}}(\lceil \dot{x} \rceil, \lceil \dot{y} \stackrel{\text{p}}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{\forall} \dot{x}\colon \dot{x} \stackrel{\text{p}}{=} \dot{y} \rceil)]^{-}$

$[\dot{\text{nonfree}}(\lceil \dot{x} \rceil, \lceil \dot{y} \stackrel{\text{p}}{=} \dot{z} \mathrel{\dot{\Rightarrow}} \dot{\forall} \dot{y}\colon \dot{x} \stackrel{\text{p}}{=} \dot{y} \rceil)]^{-}$

$[\dot{\text{free}}\langle \lceil \dot{\forall} \dot{x}\colon \mathsf{b} :: \dot{x} :: \mathsf{c} \rceil | \lceil \dot{x} \rceil := \lceil \mathsf{x} :: \dot{y} :: \mathsf{z} \rceil \rangle]^{\cdot}$

$[\dot{\text{free}}\langle \lceil \dot{\forall} \dot{y}\colon \mathsf{b} :: \dot{x} :: \mathsf{c} \rceil | \lceil \dot{x} \rceil := \lceil \mathsf{x} :: \dot{y} :: \mathsf{z} \rceil \rangle]^{-}$

$[\dot{\mathrm{free}}\langle \lceil \dot{\forall}\dot{x}\colon \mathsf{b} :: \dot{x} :: \mathsf{c}\rceil | \lceil \dot{y}\rceil := \lceil \mathsf{x} :: \dot{y} :: \mathsf{z}\rceil \rangle]^{\cdot}$

$[\dot{\mathrm{free}}\langle \lceil \dot{\forall}\dot{y}\colon \mathsf{b} :: \dot{x} :: \mathsf{c}\rceil | \lceil \dot{y}\rceil := \lceil \mathsf{x} :: \dot{y} :: \mathsf{z}\rceil \rangle]^{\cdot}$

$[\dot{a}\equiv\langle \dot{a}|\dot{b} := \dot{c}\rangle]^{\cdot}$

$[\dot{c}\equiv\langle \dot{b}|\dot{b} := \dot{c}\rangle]^{\cdot}$

$[\forall\dot{a}\colon \dot{a} \stackrel{\mathrm{p}}{=} \dot{b}\equiv\langle \forall\dot{a}\colon \dot{a} \stackrel{\mathrm{p}}{=} \dot{b}|\dot{a} := \dot{c}\rangle]^{\cdot}$

$[\forall\dot{a}\colon \dot{a} \stackrel{\mathrm{p}}{=} \dot{c}\equiv\langle \forall\dot{a}\colon \dot{a} \stackrel{\mathrm{p}}{=} \dot{b}|\dot{b} := \dot{c}\rangle]^{\cdot}$

$[\dot{\forall}\dot{a}\colon \dot{a} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{a} \dot{\Rightarrow} \dot{c} \dot{\because} \dot{d} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{c} \dot{\because} \dot{d}\equiv\langle \dot{\forall}\dot{a}\colon \dot{a} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{a} \dot{\Rightarrow} \dot{b} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{b}|\dot{b} := \dot{c} \dot{\because} \dot{d}\rangle]^{\cdot}$

$[\dot{\forall}\dot{a}\colon \dot{a} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{a} \dot{\Rightarrow} \dot{b} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{b}\equiv\langle \dot{\forall}\dot{a}\colon \dot{a} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{a} \dot{\Rightarrow} \dot{b} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{b}|\dot{a} := \dot{c}\rangle]^{\cdot}$

A.5 Priority table

Priority table
Preassociative
[peano axioms], [base], [bracket $*$ end bracket], [big bracket $*$ end bracket], [math $*$ end math], [**flush left** [$*$]], [x], [y], [z], [[$* \bowtie *$]], [[$* \stackrel{*}{\to} *$]], [pyk], [tex], [name], [prio], [$*$], [T], [if($*, *, *$)], [[$* \stackrel{*}{\Rightarrow} *$]], [val], [claim], [$\bot$], [f($*$)], [$(*)^{\mathrm{I}}$], [$\mathsf{F}$], [$\underline{0}$], [$\underline{1}$], [$\underline{2}$], [$\underline{3}$], [$\underline{4}$], [$\underline{5}$], [$\underline{6}$], [$\underline{7}$], [$\underline{8}$], [$\underline{9}$], [0], [1], [2], [3], [4], [5], [6], [7], [8], [9], [a], [b], [c], [d], [e], [f], [g], [h], [i], [j], [k], [l], [m], [n], [o], [p], [q], [r], [s], [t], [u], [v], [w], [$(*)^{\mathrm{M}}$], [If($*, *, *$)], [array{$*$} $*$ end array], [l], [c], [r], [empty], [$\langle * \mid * := * \rangle$], [$\mathcal{M}(*)$], [$\tilde{\mathcal{U}}(*)$], [$\mathcal{U}(*)$], [$\mathcal{U}^{\mathrm{M}}(*)$], [**apply**($*, *$)], [$\mathbf{apply}_1(*, *)$], [identifier($*$)], [$\mathrm{identifier}_1(*, *)$], [array-plus($*, *$)], [array-remove($*, *, *$)], [array-put($*, *, *, *$)], [array-add($*, *, *, *, *$)], [bit($*, *$)], [$\mathrm{bit}_1(*, *)$], [rack], ["vector"], ["bibliography"], ["dictionary"], ["body"], ["codex"], ["expansion"], ["code"], ["cache"], ["diagnose"], ["pyk"], ["tex"], ["texname"], ["value"], ["message"], ["macro"], ["definition"], ["unpack"], ["claim"], ["priority"], ["lambda"], ["apply"], ["true"], ["if"], ["quote"], ["proclaim"], ["define"], ["introduce"], ["hide"], ["pre"], ["post"], [$\mathcal{E}(*, *, *)$], [$\mathcal{E}_2(*, *, *, *, *)$], [$\mathcal{E}_3(*, *, *, *)$], [$\mathcal{E}_4(*, *, *, *)$], [**lookup**($*, *, *$)], [**abstract**($*, *, *, *$)], [[$\lceil * \rceil$]], [$\mathcal{M}(*, *, *)$], [$\mathcal{M}_2(*, *, *, *)$], [$\mathcal{M}^*(*, *, *)$], [macro], [$\mathrm{s}_0$], [**zip**($*, *$)], [$\mathbf{assoc}_1(*, *, *)$], [$(*)^{\mathbf{P}}$], [self], [[$* \stackrel{..}{=} *$]], [[$* \doteq *$]], [[$* \stackrel{'}{=} *$]], [[$* \stackrel{\mathrm{pyk}}{=} *$]], [[$* \stackrel{\mathrm{tex}}{=} *$]], [[$* \stackrel{\mathrm{name}}{=} *$]], [**Priority table**[$*$]], [$\tilde{\mathcal{M}}_1$], [$\tilde{\mathcal{M}}_2(*)$], [$\tilde{\mathcal{M}}_3(*)$], [$\tilde{\mathcal{M}}_4(*, *, *, *)$], [$\tilde{\mathcal{M}}(*, *, *)$], [$\tilde{\mathcal{Q}}(*, *, *)$], [$\tilde{\mathcal{Q}}_2(*, *, *)$], [$\tilde{\mathcal{Q}}_3(*, *, *, *)$], [$\tilde{\mathcal{Q}}^*(*, *, *)$], [($*$)], [**aspect**($*, *$)], [**aspect**($*, *, *$)], [$\langle * \rangle$], [$\mathbf{tuple}_1(*)$], [$\mathbf{tuple}_2(*)$], [$\mathrm{let}_2(*, *)$], [$\mathrm{let}_1(*, *)$], [[$* \stackrel{\mathrm{claim}}{=} *$]], [checker], [**check**($*, *$)], [$\mathbf{check}_2(*, *, *)$], [$\mathbf{check}_3(*, *, *)$], [$\mathbf{check}^*(*, *)$], [$\mathbf{check}^*_2(*, *, *)$], [$[*]^{\cdot}$], [$[*]^{-}$], [$[*]^{\circ}$], [msg], [[$* \stackrel{\mathrm{msg}}{=} *$]], [<stmt>], [stmt], [[$* \stackrel{\mathrm{stmt}}{=} *$]], [HeadNil$'$], [HeadPair$'$], [Transitivity$'$], [$\perp\!\!\!\perp$], [Contra$'$], [$\mathrm{T}'_{\mathrm{E}}$], [$\mathrm{L}_1$], [$\underline{*}$], [$\mathcal{A}$], [$\mathcal{B}$], [$\mathcal{C}$], [$\mathcal{D}$], [$\mathcal{E}$], [$\mathcal{F}$], [$\mathcal{G}$], [$\mathcal{H}$], [$\mathcal{I}$], [$\mathcal{J}$], [$\mathcal{K}$], [$\mathcal{L}$], [$\mathcal{M}$], [$\mathcal{N}$], [$\mathcal{O}$], [$\mathcal{P}$], [$\mathcal{Q}$], [$\mathcal{R}$], [$\mathcal{S}$], [$\mathcal{T}$], [$\mathcal{U}$], [$\mathcal{V}$], [$\mathcal{W}$], [$\mathcal{X}$], [$\mathcal{Y}$], [$\mathcal{Z}$], [$\langle * \mid * := * \rangle$], [$\langle^* * \mid * := * \rangle$], [$\emptyset$], [Remainder], [$(*)^{\mathbf{v}}$], [intro($*, *, *, *$)], [intro($*, *, *$)], [error($*, *$)], [$\mathrm{error}_2(*, *)$], [proof($*, *, *$)],

[proof$_2(*,*)$], [$\mathcal{S}(*,*)$], [$\mathcal{S}^{\mathrm{I}}(*,*)$], [$\mathcal{S}^{\rhd}(*,*)$], [$\mathcal{S}_1^{\rhd}(*,*,*)$], [$\mathcal{S}^{\mathrm{E}}(*,*)$], [$\mathcal{S}_1^{\mathrm{E}}(*,*,*)$],
[$\mathcal{S}^{+}(*,*)$], [$\mathcal{S}_1^{+}(*,*,*)$], [$\mathcal{S}^{-}(*,*)$], [$\mathcal{S}_1^{-}(*,*,*)$], [$\mathcal{S}^{*}(*,*)$], [$\mathcal{S}_1^{*}(*,*,*)$],
[$\mathcal{S}_2^{*}(*,*,*,*)$], [$\mathcal{S}^{@}(*,*)$], [$\mathcal{S}_1^{@}(*,*,*)$], [$\mathcal{S}^{\vdash}(*,*)$], [$\mathcal{S}_1^{\vdash}(*,*,*,*)$], [$\mathcal{S}^{\Vdash}(*,*)$],
[$\mathcal{S}_1^{\Vdash}(*,*,*,*)$], [$\mathcal{S}^{\text{i.e.}}(*,*)$], [$\mathcal{S}_1^{\text{i.e.}}(*,*,*,*)$], [$\mathcal{S}_2^{\text{i.e.}}(*,*,*,*,*)$], [$\mathcal{S}^{\forall}(*,*)$],
[$\mathcal{S}_1^{\forall}(*,*,*,*)$], [$\mathcal{S}^{;}(*,*)$], [$\mathcal{S}_1^{;}(*,*,*)$], [$\mathcal{S}_2^{;}(*,*,*,*)$], [$\mathcal{T}(*)$], [claims$(*,*,*)$],
[claims$_2(*,*,*)$], [<proof>], [proof], [[**Lemma** $*\colon *$]], [[**Proof of** $*\colon *$]],
[[$*$ **lemma** $*\colon *$]], [[$*$ **antilemma** $*\colon *$]], [[$*$ **rule** $*\colon *$]], [[$*$ **antirule** $*\colon *$]],
[verifier], [$\mathcal{V}_1(*)$], [$\mathcal{V}_2(*,*)$], [$\mathcal{V}_3(*,*,*,*)$], [$\mathcal{V}_4(*,*)$], [$\mathcal{V}_5(*,*,*,*)$], [$\mathcal{V}_6(*,*,*,*)$],
[$\mathcal{V}_7(*,*,*,*)$], [Cut$(*,*)$], [Head$_\oplus(*)$], [Tail$_\oplus(*)$], [rule$_1(*,*)$], [rule$(*,*)$],
[Rule tactic], [Plus$(*,*)$], [[**Theory** $*$]], [theory$_2(*,*)$], [theory$_3(*,*)$],
[theory$_4(*,*,*)$], [HeadNil$''$], [HeadPair$''$], [Transitivity$''$], [Contra$''$], [HeadNil],
[HeadPair], [Transitivity], [Contra], [T$_\mathrm{E}$], [ragged right],
[ragged right expansion], [parm$(*,*,*)$], [parm$^*(*,*,*)$], [inst$(*,*)$],
[inst$^*(*,*)$], [occur$(*,*,*)$], [occur$^*(*,*,*)$], [unify$(*=*,*)$], [unify$^*(*=*,*)$],
[unify$_2(*=*,*)$], [$\mathrm{L_a}$], [$\mathrm{L_b}$], [$\mathrm{L_c}$], [$\mathrm{L_d}$], [$\mathrm{L_e}$], [$\mathrm{L_f}$], [$\mathrm{L_g}$], [$\mathrm{L_h}$], [$\mathrm{L_i}$], [$\mathrm{L_j}$], [$\mathrm{L_k}$], [$\mathrm{L_l}$], [$\mathrm{L_m}$],
[$\mathrm{L_n}$], [$\mathrm{L_o}$], [$\mathrm{L_p}$], [$\mathrm{L_q}$], [$\mathrm{L_r}$], [$\mathrm{L_s}$], [$\mathrm{L_t}$], [$\mathrm{L_u}$], [$\mathrm{L_v}$], [$\mathrm{L_w}$], [$\mathrm{L_x}$], [$\mathrm{L_y}$], [$\mathrm{L_z}$], [$\mathrm{L_A}$], [$\mathrm{L_B}$], [$\mathrm{L_C}$],
[$\mathrm{L_D}$], [$\mathrm{L_E}$], [$\mathrm{L_F}$], [$\mathrm{L_G}$], [$\mathrm{L_H}$], [$\mathrm{L_I}$], [$\mathrm{L_J}$], [$\mathrm{L_K}$], [$\mathrm{L_L}$], [$\mathrm{L_M}$], [$\mathrm{L_N}$], [$\mathrm{L_O}$], [$\mathrm{L_P}$], [$\mathrm{L_Q}$], [$\mathrm{L_R}$],
[$\mathrm{L_S}$], [$\mathrm{L_T}$], [$\mathrm{L_U}$], [$\mathrm{L_V}$], [$\mathrm{L_W}$], [$\mathrm{L_X}$], [$\mathrm{L_Y}$], [$\mathrm{L_Z}$], [$\mathrm{L_?}$], [Reflexivity], [Reflexivity$_1$],
[Commutativity], [Commutativity$_1$], [<tactic>], [tactic], [[$* \overset{\text{tactic}}{=} *$]], [$\mathcal{P}(*,*,*)$],
[$\mathcal{P}^*(*,*,*)$], [p$_0$], [conclude$_1(*,*)$], [conclude$_2(*,*)$], [conclude$_3(*,*,*,*)$], [$\dot{0}$],
[$\dot{1}$], [$\dot{2}$], [$\dot{a}$], [$\dot{b}$], [$\dot{c}$], [$\dot{d}$], [$\dot{e}$], [$\dot{f}$], [$\dot{g}$], [$\dot{h}$], [$\dot{i}$], [$\dot{j}$], [$\dot{k}$], [$\dot{l}$], [$\dot{m}$], [$\dot{n}$], [$\dot{o}$], [$\dot{p}$], [$\dot{q}$], [$\dot{r}$], [$\dot{s}$], [$\dot{t}$],
[$\dot{u}$], [$\dot{v}$], [$\dot{w}$], [$\dot{x}$], [$\dot{y}$], [$\dot{z}$], [$\dot{\text{nonfree}}(*,*)$], [$\dot{\text{nonfree}}^*(*,*)$], [free$\langle *|* := *\rangle$],
[free$^*\langle *|* := *\rangle$], [$*\equiv\langle *|* := *\rangle$], [$*\equiv\langle^* *|* := *\rangle$], [S], [A1], [A2], [A3], [A4], [A5],
[S1], [S2], [S3], [S4], [S5], [S6], [S7], [S8], [S9], [MP], [Gen], [L3.2(a)], [S$'$], [A1$'$],
[A2$'$], [A3$'$], [A4$'$], [A5$'$], [S1$'$], [S2$'$], [S3$'$], [S4$'$], [S5$'$], [S6$'$], [S7$'$], [S8$'$], [S9$'$], [MP$'$],
[Gen$'$], [L3.2(a)$'$];

Preassociative

[$*$_{$*$}], [$*'$], [$*[*]$], [$*[*\to*]$], [$*[*\Rightarrow*]$], [$\dot{*}$];

Preassociative

[“ * ”], [], [$(*)^{\mathbf{t}}$], [string$(*)+*$], [string$(*)$ ++ $*$], [
*], [*], [!*], [”*], [#*], [$\$*$], [%*], [&*], [’*], [(*], [)*], [**], [+*], [,*], [-*], [.*], [/*],
[0*], [1*], [2*], [3*], [4*], [5*], [6*], [7*], [8*], [9*], [:*], [;*], [<*], [=*], [>*], [?*],
[@*], [A*], [B*], [C*], [D*], [E*], [F*], [G*], [H*], [I*], [J*], [K*], [L*], [M*], [N*],
[O*], [P*], [Q*], [R*], [S*], [T*], [U*], [V*], [W*], [X*], [Y*], [Z*], [[*], [*], []*], [^*],
[_*], [‘*], [a*], [b*], [c*], [d*], [e*], [f*], [g*], [h*], [i*], [j*], [k*], [l*], [m*], [n*], [o*],
[p*], [q*], [r*], [s*], [t*], [u*], [v*], [w*], [x*], [y*], [z*], [{*], [|*], [}*], [~*],
[**Preassociative** $*;*$], [**Postassociative** $*;*$], [[*], *], [priority * end],
[newline *], [macro newline *];

Preassociative

[$*0$], [$*1$], [0b], [$*$-color$(*)$], [$*$-color$^*(*)$];

Preassociative

[* ’ *], [* ‘ *];

Preassociative

[$*^{\mathrm{H}}$], [$*^{\mathrm{T}}$], [$*^{\mathrm{U}}$], [$*^{\mathrm{h}}$], [$*^{\mathrm{t}}$], [$*^{\mathrm{s}}$], [$*^{\mathrm{c}}$], [$*^{\mathrm{d}}$], [$*^{\mathrm{a}}$], [$*^{\mathrm{C}}$], [$*^{\mathrm{M}}$], [$*^{\mathrm{B}}$], [$*^{\mathrm{r}}$], [$*^{\mathrm{i}}$], [$*^{\mathrm{d}}$], [$*^{\mathrm{R}}$], [$*^{0}$],
[$*^{1}$], [$*^{2}$], [$*^{3}$], [$*^{4}$], [$*^{5}$], [$*^{6}$], [$*^{7}$], [$*^{8}$], [$*^{9}$], [$*^{\mathrm{E}}$], [$*^{\mathcal{V}}$], [$*^{\mathcal{C}}$], [$*^{\mathcal{C}^*}$], [$*'$];

Preassociative
$[* \cdot *], [* \cdot_0 *], [* \mathbin{\dot{\cdot}} *];$
Preassociative
$[* + *], [* +_0 *], [* +_1 *], [* - *], [* -_0 *], [* -_1 *], [* \dot{+} *];$
Preassociative
$[* \cup \{*\}], [* \cup *], [* \backslash \{*\}];$
Postassociative
$[* \therefore *], [* \underline{\therefore} *], [* \underline{::} *], [* \underline{+2*} *], [* :: *], [* +2* *];$
Postassociative
$[*, *];$
Preassociative
$[* \stackrel{\mathrm{B}}{\approx} *], [* \stackrel{\mathrm{D}}{\approx} *], [* \stackrel{\mathrm{C}}{\approx} *], [* \stackrel{\mathrm{P}}{\approx} *], [* \approx *], [* = *], [* \stackrel{+}{\rightarrow} *], [* \stackrel{\mathrm{t}}{=} *], [* \stackrel{\mathrm{t}^*}{=} *], [* \stackrel{\mathrm{r}}{=} *],$
$[* \in_{\mathrm{t}} *], [* \subseteq_{\mathrm{T}} *], [* \stackrel{\mathrm{T}}{=} *], [* \stackrel{\mathrm{s}}{=} *], [* \text{ free in } *], [* \text{ free in}^* *], [* \text{ free for } * \text{ in } *],$
$[* \text{ free for}^* * \text{ in } *], [* \in_{\mathrm{c}} *], [* < *], [* <' *], [* \leq' *], [* \stackrel{\mathrm{p}}{=} *], [*^{\mathcal{P}}];$
Preassociative
$[\neg *], [\dot{\neg} *];$
Preassociative
$[* \wedge *], [* \ddot{\wedge} *], [* \tilde{\wedge} *], [* \wedge_{\mathrm{c}} *], [* \dot{\wedge} *];$
Preassociative
$[* \vee *], [* \parallel *], [* \ddot{\vee} *], [* \dot{\vee} *];$
Preassociative
$[\dot{\forall} *: *], [\dot{\exists} *: *];$
Postassociative
$[* \ddot{\Rightarrow} *], [* \dot{\Rightarrow} *], [* \dot{\Leftrightarrow} *];$
Postassociative
$[* : *], [*!*];$
Preassociative
$[* \left\{ \begin{array}{c} * \\ * \end{array} \right.];$
Preassociative
$[\lambda * .*], [\Lambda *], [\textbf{if } * \textbf{ then } * \textbf{ else } *], [\textbf{let } * = * \textbf{ in } *], [\textbf{let } * \ddot{=} * \text{ in } *];$
Preassociative
$[*^{\mathrm{I}}], [*^{\triangleright}], [*^{\mathrm{V}}], [*^{+}], [*^{-}], [*^{*}];$
Preassociative
$[* @ *], [* \rhd *], [* \mathrel{\triangleright\!\!\!\triangleright} *], [* \gg *];$
Postassociative
$[* \vdash *], [* \Vdash *], [* \text{ i.e. } *];$
Preassociative
$[\forall *: *];$
Postassociative
$[* \oplus *];$
Postassociative
$[*; *];$
Preassociative
$[* \text{ proves } *];$

Preassociative
[$*$ **proof of** $* : *$], [Line $* : * \gg *; *$], [Last line $* \gg * \square$],
[Line $*$: Premise $\gg *; *$], [Line $*$: Side-condition $\gg *; *$], [Arbitrary $\gg *; *$],
[Local $\gg * = *; *$];
Postassociative
[$*$ then $*$], [$*[*]*$];
Preassociative
[$*\&*$];
Preassociative
[$*\backslash\backslash*$]; **End table**

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C Bibliography

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