

Propositional Calculus

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[propositional calculus $\stackrel{\text{pyk}}{=}$ “propositional calculus”]

Define theory: [**Theory** T_{Prop}]¹

Implication

$[x \Rightarrow y \doteq \neg x \vee y]$ ²

Axioms

[T_{Prop} **rule** A1: $\forall \mathcal{A}: \forall \mathcal{B}: \mathcal{A} \Rightarrow \mathcal{B} \Rightarrow \mathcal{A}$]³

[T_{Prop} **rule** A2: $\forall \mathcal{A}: \forall \mathcal{B}: \forall \mathcal{C}: (\mathcal{A} \Rightarrow \mathcal{B} \Rightarrow \mathcal{C}) \Rightarrow (\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow \mathcal{A} \Rightarrow \mathcal{C}$]⁴

[T_{Prop} **rule** A3: $\forall \mathcal{A}: \forall \mathcal{B}: (\neg \mathcal{A} \Rightarrow \neg \mathcal{B}) \Rightarrow (\neg \mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow \mathcal{A}$]⁵

[T_{Prop} **rule** A4: $\forall \mathcal{A}: \forall \mathcal{B}: \forall \mathcal{C}: \mathcal{C}$ free for $\mathcal{B}$ in $\mathcal{A} \Vdash \forall \mathcal{B}: \mathcal{A} \Rightarrow \langle \mathcal{A} | \mathcal{B} := \mathcal{C} \rangle$]⁶

[T_{Prop} **rule** A5: $\forall \mathcal{A}: \forall \mathcal{B}: \forall \mathcal{C}: \neg \mathcal{C}$ free in $\mathcal{A} \Vdash (\forall \mathcal{C}: \mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow \mathcal{A} \Rightarrow \forall \mathcal{C}: \mathcal{B}$]⁷

[T_{Prop} **rule** MP: $\forall \mathcal{A}: \forall \mathcal{B}: \mathcal{A} \vdash \mathcal{A} \Rightarrow \mathcal{B} \vdash \mathcal{B}$]⁸

[T_{Prop} **rule** Gen: $\forall \mathcal{A}: \forall \mathcal{B}: \mathcal{A} \vdash \forall \mathcal{B}: \mathcal{A}$]⁹

Proof

[T_{Prop} **lemma** Mendelson 1.8: $\forall \mathcal{A}: \mathcal{A} \Rightarrow \mathcal{A}$]¹⁰

T_{Prop} **proof of** Mendelson 1.8:

L01:	Arbitrary $\gg$	$\mathcal{A}$	;
L02:	A2 $\gg$	$(\mathcal{A} \Rightarrow (\mathcal{A} \Rightarrow \mathcal{A}) \Rightarrow \mathcal{A}) \Rightarrow (\mathcal{A} \Rightarrow$	
		$\mathcal{A} \Rightarrow \mathcal{A}) \Rightarrow \mathcal{A} \Rightarrow \mathcal{A}$	;

¹[T_{Prop} $\stackrel{\text{pyk}}{=}$ “propositional theory”]

² $[x \Rightarrow y \stackrel{\text{pyk}}{=} “* \text{ imply } *”]$

³[A1 $\stackrel{\text{pyk}}{=}$ “axiom one”]

⁴[A2 $\stackrel{\text{pyk}}{=}$ “axiom two”]

⁵[A3 $\stackrel{\text{pyk}}{=}$ “axiom three”]

⁶[A4 $\stackrel{\text{pyk}}{=}$ “axiom four”]

⁷[A5 $\stackrel{\text{pyk}}{=}$ “axiom five”]

⁸[MP $\stackrel{\text{pyk}}{=}$ “axiom mp”]

⁹[Gen $\stackrel{\text{pyk}}{=}$ “axiom gen”]

¹⁰[Mendelson 1.8 $\stackrel{\text{pyk}}{=}$ “mendelson lemma one eight”]

L03:	A1 $\gg$	$\mathcal{A} \Rightarrow (\mathcal{A} \Rightarrow \mathcal{A}) \Rightarrow \mathcal{A}$	;
L04:	MP $\triangleright$ L03 $\triangleright$ L02 $\gg$	$(\mathcal{A} \Rightarrow \mathcal{A} \Rightarrow \mathcal{A}) \Rightarrow \mathcal{A} \Rightarrow \mathcal{A}$	;
L05:	A1 $\gg$	$\mathcal{A} \Rightarrow \mathcal{A} \Rightarrow \mathcal{A}$	;
L06:	MP $\triangleright$ L05 $\triangleright$ L04 $\gg$	$\mathcal{A} \Rightarrow \mathcal{A}$	□

Experimental

[T_{Prop} **lemma Mendelson 1.10a**: $\forall \mathcal{E}: \forall \mathcal{D}: \forall \mathcal{F}: \mathcal{D} \Rightarrow \mathcal{E} \vdash \mathcal{E} \Rightarrow \mathcal{F} \vdash \mathcal{D} \Rightarrow \mathcal{F}$] ¹¹

T_{Prop} **proof of Mendelson 1.10a**:

L01:	Arbitrary $\gg$	$\mathcal{E}$	;
L02:	Arbitrary $\gg$	$\mathcal{D}$	;
L03:	Arbitrary $\gg$	$\mathcal{F}$	;
L04:	Premise $\gg$	$\mathcal{D} \Rightarrow \mathcal{E}$	;
L05:	Premise $\gg$	$\mathcal{E} \Rightarrow \mathcal{F}$	;
L06:	A2 $\gg$	$(\mathcal{D} \Rightarrow \mathcal{E} \Rightarrow \mathcal{F}) \Rightarrow (\mathcal{D} \Rightarrow \mathcal{E}) \Rightarrow$ $\mathcal{D} \Rightarrow \mathcal{F}$	;
L07:	A1 $\gg$	$(\mathcal{E} \Rightarrow \mathcal{F}) \Rightarrow \mathcal{D} \Rightarrow (\mathcal{E} \Rightarrow \mathcal{F})$	;
L08:	MP $\triangleright$ L05 $\triangleright$ L07 $\gg$	$\mathcal{D} \Rightarrow (\mathcal{E} \Rightarrow \mathcal{F})$	;
L09:	MP $\triangleright$ L08 $\triangleright$ L06 $\gg$	$(\mathcal{D} \Rightarrow \mathcal{E}) \Rightarrow \mathcal{D} \Rightarrow \mathcal{F}$	;
L10:	MP $\triangleright$ L04 $\triangleright$ L09 $\gg$	$\mathcal{D} \Rightarrow \mathcal{F}$	□

[T_{Prop} **lemma Mendelson 1.10b**: $\forall \mathcal{D}: \forall \mathcal{E}: \forall \mathcal{F}: \mathcal{D} \Rightarrow \mathcal{E} \Rightarrow \mathcal{F} \vdash \mathcal{E} \vdash \mathcal{D} \Rightarrow \mathcal{F}$] ¹²

T_{Prop} **proof of Mendelson 1.10b**:

L01:	Arbitrary $\gg$	$\mathcal{D}$	;
L02:	Arbitrary $\gg$	$\mathcal{E}$	;
L03:	Arbitrary $\gg$	$\mathcal{F}$	;
L04:	Premise $\gg$	$\mathcal{D} \Rightarrow \mathcal{E} \Rightarrow \mathcal{F}$	;
L05:	Premise $\gg$	$\mathcal{E}$	;
L06:	A2 $\gg$	$(\mathcal{D} \Rightarrow \mathcal{E} \Rightarrow \mathcal{F}) \Rightarrow (\mathcal{D} \Rightarrow \mathcal{E}) \Rightarrow$ $\mathcal{D} \Rightarrow \mathcal{F}$	;
L07:	A1 $\gg$	$\mathcal{E} \Rightarrow \mathcal{D} \Rightarrow \mathcal{E}$	;
L08:	MP $\triangleright$ L05 $\triangleright$ L07 $\gg$	$\mathcal{D} \Rightarrow \mathcal{E}$	;
L09:	MP $\triangleright$ L04 $\triangleright$ L06 $\gg$	$(\mathcal{D} \Rightarrow \mathcal{E}) \Rightarrow \mathcal{D} \Rightarrow \mathcal{F}$	;
L10:	MP $\triangleright$ L08 $\triangleright$ L09 $\gg$	$\mathcal{D} \Rightarrow \mathcal{F}$	□

[T_{Prop} **lemma Mendelson 1.11a**: $\forall \mathcal{G}: \neg \neg \mathcal{G} \Rightarrow \mathcal{G}$] ¹³

T_{Prop} **proof of Mendelson 1.11a**:

L01:	Arbitrary $\gg$	$\mathcal{G}$	;
L02:	A3 $\gg$	$(\neg \mathcal{G} \Rightarrow \neg \neg \mathcal{G}) \Rightarrow (\neg \mathcal{G} \Rightarrow \neg \mathcal{G}) \Rightarrow$ $\mathcal{G}$	;
L03:	Mendelson 1.8 $\gg$	$\neg \mathcal{G} \Rightarrow \neg \mathcal{G}$	;
L04:	A1 $\gg$	$\neg \neg \mathcal{G} \Rightarrow (\neg \mathcal{G} \Rightarrow \neg \neg \mathcal{G})$	;
L05:	Mendelson 1.10b $\triangleright$ L02 $\triangleright$ L03 $\gg$	$(\neg \mathcal{G} \Rightarrow \neg \neg \mathcal{G}) \Rightarrow \mathcal{G}$	;

¹¹[Mendelson 1.10a $\stackrel{\text{pyk}}{=}$ “mendelson corollary one ten a”]

¹²[Mendelson 1.10b $\stackrel{\text{pyk}}{=}$ “mendelson corollary one ten b”]

¹³[Mendelson 1.11a $\stackrel{\text{pyk}}{=}$ “mendelson lemma one eleven a”]

A $\text{\TeX}$ definitions

$[\text{T}_{\text{Prop}} \stackrel{\text{tex}}{=} \text{“T_{\text{Prop}}”}]$

$[\text{x} \Rightarrow \text{y} \stackrel{\text{tex}}{=} \text{“\#1. \Rightarrow\{} \#2.”}]$

$[\text{A1} \stackrel{\text{tex}}{=} \text{“A1”}]$

$[\text{A2} \stackrel{\text{tex}}{=} \text{“A2”}]$

$[\text{A3} \stackrel{\text{tex}}{=} \text{“A3”}]$

$[\text{A4} \stackrel{\text{tex}}{=} \text{“A4”}]$

$[\text{A5} \stackrel{\text{tex}}{=} \text{“A5”}]$

$[\text{MP} \stackrel{\text{tex}}{=} \text{“MP”}]$

$[\text{Gen} \stackrel{\text{tex}}{=} \text{“Gen”}]$

$[\text{Mendelson 1.8} \stackrel{\text{tex}}{=} \text{“Mendelson\ \textbf{\{1.8\}}”}]$

$[\text{Mendelson 1.10a} \stackrel{\text{tex}}{=} \text{“Mendelson\ \textbf{\{1.10\} a}”}]$

$[\text{Mendelson 1.10b} \stackrel{\text{tex}}{=} \text{“Mendelson\ \textbf{\{1.10\} b}”}]$

$[\text{Mendelson 1.11a} \stackrel{\text{tex}}{=} \text{“Mendelson\ \textbf{\{1.11\} a}”}]$

B Test

$[\text{T} \Rightarrow \text{T}] \cdot$

$[\text{T} \Rightarrow \text{F}]^-$

$[\text{F} \Rightarrow \text{T}] \cdot$

$[\text{F} \Rightarrow \text{F}] \cdot$

C Priority table

Priority table

Preassociative

[propositional calculus], [base], [bracket * end bracket],
 [big bracket * end bracket], [math * end math], [**flush left** *], [x], [y], [z],
 [[* $\bowtie$ *]], [[* $\xrightarrow{*}$ *]], [pyk], [tex], [name], [prio], [*], [T], [if(*, *, *)], [[* $\xrightarrow{*}$ *]], [val],
 [claim], [$\perp$], [f(*)], [(*)^I], [F], [$\underline{0}$], [$\underline{1}$], [$\underline{2}$], [$\underline{3}$], [$\underline{4}$], [$\underline{5}$], [$\underline{6}$], [$\underline{7}$], [$\underline{8}$], [$\underline{9}$], [$\underline{0}$], [$\underline{1}$], [$\underline{2}$], [$\underline{3}$], [$\underline{4}$],
 [$\underline{5}$], [$\underline{6}$], [$\underline{7}$], [$\underline{8}$], [$\underline{9}$], [a], [b], [c], [d], [e], [f], [g], [h], [i], [j], [k], [l], [m], [n], [o], [p], [q], [r],
 [s], [t], [u], [v], [w], [(*)^M], [If(*, *, *)], [array{*} * end array], [l], [c], [r], [empty],
 [<* := *>], [$\mathcal{M}(*)$], [$\mathcal{U}(*)$], [$\mathcal{U}(*)$], [$\mathcal{U}^M(*)$], [**apply**(*, *)], [**apply**₁(*, *)],
 [identifier(*)], [identifier₁(*, *)], [array-plus(*, *)], [array-remove(*, *, *)], [array-
 put(*, *, *, *)], [array-add(*, *, *, *, *)], [bit(*, *)], [bit₁(*, *)], [rack], ["vector"],
 ["bibliography"], ["dictionary"], ["body"], ["codex"], ["expansion"], ["code"],
 ["cache"], ["diagnose"], ["pyk"], ["tex"], ["texname"], ["value"], ["message"],
 ["macro"], ["definition"], ["unpack"], ["claim"], ["priority"], ["lambda"],
 ["apply"], ["true"], ["if"], ["quote"], ["proclaim"], ["define"], ["introduce"],
 ["hide"], ["pre"], ["post"], [$\mathcal{E}(*, *, *)$], [$\mathcal{E}_2(*, *, *, *, *)$], [$\mathcal{E}_3(*, *, *, *, *)$],
 [$\mathcal{E}_4(*, *, *, *, *)$], [**lookup**(*, *, *, *)], [**abstract**(*, *, *, *)], [(*)], [$\mathcal{M}(*, *, *)$],
 [$\mathcal{M}_2(*, *, *, *, *)$], [$\mathcal{M}^*(*, *, *, *)$], [macro], [s₀], [**zip**(*, *)], [**assoc**₁(*, *, *)], [(*)^P],
 [self], [[* $\doteq$ *]], [[* $\dot{=}$ *]], [[* $\dot{=}$ *]], [[* $\stackrel{\text{pyk}}{=}$ *]], [[* $\stackrel{\text{tex}}{=}$ *]], [[* $\stackrel{\text{name}}{=}$ *]],
 [**Priority table** *], [$\mathcal{M}_1$], [$\tilde{\mathcal{M}}_2(*)$], [$\tilde{\mathcal{M}}_3(*)$], [$\tilde{\mathcal{M}}_4(*, *, *, *)$], [$\mathcal{M}(*, *, *)$],
 [$\tilde{\mathcal{Q}}(*, *, *)$], [$\tilde{\mathcal{Q}}_2(*, *, *, *)$], [$\tilde{\mathcal{Q}}_3(*, *, *, *)$], [$\tilde{\mathcal{Q}}^*(*, *, *, *)$], [(*)], [**aspect**(*, *)],
 [**aspect**(*, *, *)], [(*)], [**tuple**₁(*)], [**tuple**₂(*)], [let₂(*, *)], [let₁(*, *)],
 [[* $\stackrel{\text{claim}}{=}$ *]], [checker], [**check**(*, *)], [**check**₂(*, *, *)], [**check**₃(*, *, *, *)],
 [**check**^{*}(*, *)], [**check**₂^{*}(*, *, *)], [[*]·], [[*]−], [[*]°], [msg], [[* $\stackrel{\text{msg}}{=}$ *]], [<stmt>],
 [stmt], [[* $\stackrel{\text{stmt}}{=}$ *]], [HeadNil'], [HeadPair'], [Transitivity'], [$\perp$], [Contra'], [T_E'],
 [L₁], [⋆], [A], [B], [C], [D], [E], [F], [G], [H], [I], [J], [K], [L], [M], [N], [O], [P], [Q],
 [R], [S], [T], [U], [V], [W], [X], [Y], [Z], [(*) := *], [(*) := *], [∅], [Remainder],
 [(*)[∨]], [error(*, *)], [error₂(*, *)], [proof(*, *, *)], [proof₂(*, *, *)], [S(*, *)], [S^I(*, *)],
 [S[∇](*, *)], [S₁[∇](*, *, *)], [S^E(*, *)], [S₁^E(*, *, *)], [S⁺(*, *)], [S₁⁺(*, *, *)],
 [S[−](*, *)], [S₁[−](*, *, *)], [S^{*}(*, *)], [S₁^{*}(*, *, *)], [S₂^{*}(*, *, *, *)], [S[⊗](*, *)],
 [S₁[⊗](*, *, *)], [S[⊖](*, *)], [S₁[⊖](*, *, *, *)], [S[⧻](*, *)], [S₁[⧻](*, *, *, *)], [S^{i.e.}(*, *)],
 [S₁^{i.e.}(*, *, *, *)], [S₂^{i.e.}(*, *, *, *)], [S[∇](*, *)], [S₁[∇](*, *, *, *)], [Sⁱ(*, *)],
 [S₁ⁱ(*, *, *)], [S₂ⁱ(*, *, *, *)], [T(*)], [claims(*, *, *)], [claims₂(*, *, *)], [<proof>],
 [proof], [[**Lemma** * : *]], [[**Proof of** * : *]], [[* **lemma** * : *]],
 [[* **antilemma** * : *]], [[* **rule** * : *]], [[* **antirule** * : *]], [verifier], [V₁(*)],
 [V₂(*, *)], [V₃(*, *, *, *)], [V₄(*, *)], [V₅(*, *, *, *)], [V₆(*, *, *, *)], [V₇(*, *, *, *)],
 [Cut(*, *)], [Head_⊕(*)], [Tail_⊕(*)], [rule₁(*, *)], [rule(*, *)], [Rule tactic],
 [Plus(*, *)], [[**Theory** *]], [theory₂(*, *)], [theory₃(*, *)], [theory₄(*, *, *)],
 [HeadNil''], [HeadPair''], [Transitivity''], [Contra''], [HeadNil], [HeadPair],
 [Transitivity], [Contra], [T_E], [ragged right], [ragged right expansion],
 [color(* : *)], [color^{*}(* : *)], [vars(*)], [vars^{*}(*)], [instantiate(* + * : *)],
 [instantiate^{*}(* + * : *)], [unify(* : * = * : * + *)], [unify^{*}(* : * = * : * + *)],
 [unify₁(* : * = * : * + *)], [unify₂(* + *)], [unify₃(* = * + *)], [check],

$[\neg *];$
Preassociative
 $[* \wedge *], [* \ddot{\wedge} *], [* \tilde{\wedge} *], [* \wedge_c *];$
Preassociative
 $[* \vee *], [* \parallel *], [* \ddot{\vee} *];$
Postassociative
 $[* \dot{\Rightarrow} *];$
Postassociative
 $[* : *], [*! *];$
Postassociative
 $[* \Rightarrow *];$
Preassociative
 $[* \left\{ \begin{array}{c} * \\ * \end{array} \right.];$
Preassociative
 $[\lambda * . *], [\Lambda *], [\text{if } * \text{ then } * \text{ else } *], [\text{let } * = * \text{ in } *], [\text{let } * \ddot{=} * \text{ in } *];$
Preassociative
 $[*^I], [*^\triangleright], [*^V], [*^+], [*^-], [*^*];$
Preassociative
 $[* @ *], [* \triangleright *], [* \blacktriangleright *], [* \gg *];$
Postassociative
 $[* \vdash *], [* \Vdash *], [* \text{i.e. } *];$
Preassociative
 $[\forall *: *];$
Postassociative
 $[* \oplus *];$
Postassociative
 $[*, *];$
Preassociative
 $[* \text{ proves } *];$
Preassociative
 $[* \text{ proof of } * : *], [\text{Line } * : * \gg *; *], [\text{Last line } * \gg *], [\text{Line } * : \text{Premise } \gg *; *],$
 $[\text{Line } * : \text{Side-condition } \gg *; *], [\text{Arbitrary } \gg *; *], [\text{Local } \gg * = *; *];$
Postassociative
 $[* \text{ then } *], [* [*] *];$
Preassociative
 $[* \& *];$
Preassociative
 $[* \backslash \backslash *];$ **End table**