

Peano arithmetic

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1 Peano arithmetic

This Logiweb page [1] defines Peano arithmetic.

1.1 The constructs of Peano arithmetic

Terms of Peano arithmetic are constructed from zero $[\dot{0} \xrightarrow{\text{pyk}} \text{“peano zero”}][\dot{0} \xrightarrow{\text{tex}} \text{“}$

$\backslash\text{dot}\{0\}”]$, successor $[x' \xrightarrow{\text{pyk}} \text{“* peano succ”}][x' \xrightarrow{\text{tex}} \text{“}\#1.$

”], plus $[x \dot{+} y \xrightarrow{\text{pyk}} \text{“* peano plus *”}][x \dot{+} y \xrightarrow{\text{tex}} \text{“}\#1.$

$\backslash\text{mathop}\{\backslash\text{dot}\{+\}\} \#2.”]$, and times $[x \cdot y \xrightarrow{\text{pyk}} \text{“* peano times *”}][x \cdot y \xrightarrow{\text{tex}} \text{“}\#1.$
 $\backslash\text{mathop}\{\backslash\text{dot}\{\backslash\text{cdot}\}\} \#2.”]$.

$\backslash\mathrm{stackrel{p}{p}}\{=\} \#2.$ ”, negation $[\neg x \xrightarrow{\mathrm{pyk}} \text{“peano not } * \text{”}][\neg x \xrightarrow{\mathrm{tex}} \text{“} \backslash\mathrm{dot}\{\neg\} \backslash, \#1.$ ”, implication $[x \Rightarrow y \xrightarrow{\mathrm{pyk}} \text{“} * \text{ peano imply } * \text{”}][x \Rightarrow y \xrightarrow{\mathrm{tex}} \text{“} \#1.$
 $\backslash\mathrm{mathrel}\{\backslash\mathrm{dot}\{\rightarrow\}\} \#2.$ ”, and universal quantification
 $[\forall x: y \xrightarrow{\mathrm{pyk}} \text{“peano all } * \text{ indeed } * \text{”}][\forall x: y \xrightarrow{\mathrm{tex}} \text{“} \backslash\mathrm{dot}\{\forall\} \#1.$
 $\backslash\mathrm{colon} \#2.$ ”.

From these constructs we macro define one $[i \xrightarrow{\text{pyk}} \text{“peano one”}][i \xrightarrow{\text{tex}} \text{“}\backslash\text{dot}\{1\}”]$, two $[2 \xrightarrow{\text{pyk}} \text{“peano two”}][2 \xrightarrow{\text{tex}} \text{“}\backslash\text{dot}\{2\}”]$, conjunction $[x \dot{\wedge} y \xrightarrow{\text{pyk}} \text{“* peano and *}”][x \dot{\wedge} y \xrightarrow{\text{tex}} \text{“}\#1.$
 $\backslash\text{mathrel}\{\backslash\text{dot}\{\backslash\text{wedge}\}\} \#2.”]$, disjunction $[x \dot{\vee} y \xrightarrow{\text{pyk}} \text{“* peano or$
 $*”][x \dot{\vee} y \xrightarrow{\text{tex}} \text{“}\#1.$
 $\backslash\text{mathrel}\{\backslash\text{dot}\{\backslash\text{vee}\}\} \#2.”]$, biimplication $[x \dot{\Leftrightarrow} y \xrightarrow{\text{pyk}} \text{“* peano iff$
 $*”][x \dot{\Leftrightarrow} y \xrightarrow{\text{tex}} \text{“}\#1.$
 $\backslash\text{mathrel}\{\backslash\text{dot}\{\backslash\text{Leftrightarrow}\}\} \#2.”]$, and existential quantification
 $[\dot{\exists}x: y \xrightarrow{\text{pyk}} \text{“peano exist * indeed *}”][\dot{\exists}x: y \xrightarrow{\text{tex}} \text{“}$
 $\backslash\text{dot}\{\backslash\text{exists}\} \#1.$
 $\backslash\text{colon} \#2.”]:$

$$\begin{aligned} & [\dot{1} \xrightarrow{\text{macro}} \lambda t. \lambda s. \lambda c. \tilde{\mathcal{M}}_4(t, s, c, [[1 \dot{=} 0']])] \\ & [\dot{2} \xrightarrow{\text{macro}} \lambda t. \lambda s. \lambda c. \tilde{\mathcal{M}}_4(t, s, c, [[2 \dot{=} 1']])] \\ & [x \dot{\wedge} y \xrightarrow{\text{macro}} \lambda t. \lambda s. \lambda c. \tilde{\mathcal{M}}_4(t, s, c, [[x \dot{\wedge} y \dot{=} \dot{\neg} (x \dot{\Rightarrow} \dot{\neg} y)])]) \\ & [x \dot{\vee} y \xrightarrow{\text{macro}} \lambda t. \lambda s. \lambda c. \tilde{\mathcal{M}}_4(t, s, c, [[x \dot{\vee} y \dot{=} \dot{\neg} x \dot{\Rightarrow} y]])] \\ & [x \dot{\Leftrightarrow} y \xrightarrow{\text{macro}} \lambda t. \lambda s. \lambda c. \tilde{\mathcal{M}}_4(t, s, c, [[x \dot{\Leftrightarrow} y \dot{=} (x \dot{\Rightarrow} y) \dot{\wedge} (y \dot{\Rightarrow} x)])]) \\ & [\dot{\exists} x: y \xrightarrow{\text{macro}} \lambda t. \lambda s. \lambda c. \tilde{\mathcal{M}}_4(t, s, c, [[\dot{\exists} x: y \dot{=} \dot{\neg} \dot{\forall} x: \dot{\neg} y]])] \end{aligned}$$

1.2 Variables

We now introduce the unary operator $[\dot{x} \xrightarrow{\text{pyk}} \text{"* peano var"}][\dot{x} \xrightarrow{\text{tex}} \text{"\dot{\#1}}]$ and define that a term is a *Peano variable* (i.e. a variable of Peano arithmetic) if it has the $[\dot{x}]$ operator in its root. $[\dot{x}^{\mathcal{P}} \xrightarrow{\text{pyk}} \text{"* is peano var"}][\dot{x}^{\mathcal{P}} \xrightarrow{\text{tex}} \text{"\#1}}]$ is true if $[x]$ is a Peano variable:

$$[X^{\mathcal{P}} \xrightarrow{\text{val}} X \stackrel{\text{r}}{=} [\dot{X}]]$$

We macro define $[\dot{a} \xrightarrow{\text{pyk}} \text{“peano a”}][\dot{a} \xrightarrow{\text{tex}} \text{“} \backslash \dot{\text{mathit}}\{\text{a}\} \text{”}]$ to be a Peano variable:

$$[\dot{a} \xrightarrow{\text{macro}} \lambda t. \lambda s. \lambda c. \tilde{\mathcal{M}}_4(t, s, c, [[\dot{a} \doteq \dot{a}]])]$$

Appendix A.2 defines Peano variables for the other letters of the English alphabet.

$$[\text{nonfree}(x, y) \xrightarrow{\text{pyk}} \text{“peano nonfree * in * end nonfree”}][\text{nonfree}(x, y) \xrightarrow{\text{tex}} \text{“} \backslash \dot{\text{mathit}}\{\text{nonfree}\}(\#1.$$

, #2.

)”] is true if the Peano variable $[x]$ does not occur free in the Peano

term/formula $[y]$. $[\text{nonfree}^*(x, y) \xrightarrow{\text{pyk}} \text{“peano nonfree star * in * end$

$$\text{nonfree”}][\text{nonfree}^*(x, y) \xrightarrow{\text{tex}} \text{“} \backslash \dot{\text{mathit}}\{\text{nonfree}\}^*(\#1.$$

, #2.

)”] is true if the Peano variable $[x]$ does not occur free in the list $[y]$ of Peano terms/formulas.

$$\begin{aligned} &[\text{nonfree}(x, y) \xrightarrow{\text{val}} \\ &\text{If}(y^{\mathcal{P}}, \neg x \doteq y, \\ &\text{If}(\neg y \doteq [\dot{\forall} x: y], \text{nonfree}^*(x, y^t), \\ &\text{If}(x \doteq y^1, \top, \text{nonfree}(x, y^2)))] \end{aligned}$$

$$[\text{nonfree}^*(x, y) \xrightarrow{\text{val}} x! \text{If}(y, \top, \text{If}(\text{nonfree}(x, y^h), \text{nonfree}^*(x, y^t), \text{F}))]$$

$$[\text{free}(a|x := b) \xrightarrow{\text{pyk}} \text{“peano free * set * to * end free”}][\text{free}(a|x := b) \xrightarrow{\text{tex}} \text{“} \backslash \dot{\text{mathit}}\{\text{free}\} \backslash \text{angle} \#1.$$

| #2.

:= #3.

$\backslash \text{angle}$ ”] is true if the substitution $[\langle a | x := b \rangle]$ is free.

$$[\text{free}^*(a|x := b) \xrightarrow{\text{pyk}} \text{“peano free star * set * to * end free”}][\text{free}^*(a|x := b) \xrightarrow{\text{tex}} \text{“} \backslash \dot{\text{mathit}}\{\text{free}\} \{ \}^* \backslash \text{angle} \#1.$$

| #2.

:= #3.

$\backslash \text{angle}$ ”] is the version where $[a]$ is a list of terms.

$$\begin{aligned} &[\text{free}(a|x := b) \xrightarrow{\text{val}} x!b! \\ &\text{If}(a^{\mathcal{P}}, \top, \\ &\text{If}(\neg a \doteq [\dot{\forall} u: v], \text{free}^*(a^t|x := b), \\ &\text{If}(a^1 \doteq x, \top, \\ &\text{If}(\text{nonfree}(x, a^2), \top, \\ &\text{If}(\neg \text{nonfree}(a^1, b), \text{F}, \\ &\text{free}(a^2|x := b)))))) \end{aligned}$$

$$[\text{free}^* \langle a | x := b \rangle \xrightarrow{\text{val}} x!b! \text{If}(a, T, \text{If}(\text{free}^* \langle a^h | x := b \rangle, \text{free}^* \langle a^t | x := b \rangle, F))]$$

$[a \equiv \langle b | x := c \rangle \xrightarrow{\text{pyk}} \text{"peano sub } * \text{ is } * \text{ where } * \text{ is } * \text{ end sub"}][a \equiv \langle b | x := c \rangle \xrightarrow{\text{tex}} \text{"\#1. \{\backslash\text{equiv}\} \backslash \text{angle} \#2.}$

$|\#3.$

$:= \#4.$

$\backslash \text{angle}"]$ is true if $[a]$ equals $[\langle b | x := c \rangle]$. $[a \equiv \langle *b | x := c \rangle \xrightarrow{\text{pyk}} \text{"peano sub star } * \text{ is } * \text{ where } * \text{ is } * \text{ end sub"}][a \equiv \langle *b | x := c \rangle \xrightarrow{\text{tex}} \text{"\#1. \{\backslash\text{equiv}\} \backslash \text{angle}^* \#2.}$

$|\#3.$

$:= \#4.$

$\backslash \text{angle}"]$ is the version where $[a]$ and $[b]$ are lists.

$$[a \equiv \langle b | x := c \rangle \xrightarrow{\text{val}} a!x!c!$$

$$\text{If}(\text{If}(b \stackrel{r}{=} \lceil \forall u: v \rceil, b^1 \stackrel{t}{=} x, F), a \stackrel{t}{=} b,$$

$$\text{If}(b^p \wedge b \stackrel{t}{=} x, a \stackrel{t}{=} c, \text{If}(\text{If}(b^p \wedge b \stackrel{t}{=} x, a \stackrel{t}{=} c, F)))$$

$$a \stackrel{r}{=} b, a^t \equiv \langle *b^t | x := c \rangle, F)))]$$

$$[a \equiv \langle *b | x := c \rangle \xrightarrow{\text{val}} b!x!c! \text{If}(a, T, \text{If}(a^h \equiv \langle b^h | x := c \rangle, a^t \equiv \langle *b^t | x := c \rangle, F)))]$$

1.3 Mendelsons system S

System $[S \xrightarrow{\text{pyk}} \text{"system s"}][S \xrightarrow{\text{tex}} \text{"S"}]$ of Mendelson [2] expresses Peano arithmetic. It comprises the axioms $[A1 \xrightarrow{\text{pyk}} \text{"axiom a one"}][A1 \xrightarrow{\text{tex}} \text{"A1"}]$, $[A2 \xrightarrow{\text{pyk}} \text{"axiom a two"}][A2 \xrightarrow{\text{tex}} \text{"A2"}]$, $[A3 \xrightarrow{\text{pyk}} \text{"axiom a three"}][A3 \xrightarrow{\text{tex}} \text{"A3"}]$, $[A4 \xrightarrow{\text{pyk}} \text{"axiom a four"}][A4 \xrightarrow{\text{tex}} \text{"A4"}]$, and $[A5 \xrightarrow{\text{pyk}} \text{"axiom a five"}][A5 \xrightarrow{\text{tex}} \text{"A5"}]$ and inference rules $[MP \xrightarrow{\text{pyk}} \text{"rule mp"}][MP \xrightarrow{\text{tex}} \text{"MP"}]$ and $[Gen \xrightarrow{\text{pyk}} \text{"rule gen"}][Gen \xrightarrow{\text{tex}} \text{"Gen"}]$ of first order predicate calculus. Furthermore, it comprises the proper axioms $[S1 \xrightarrow{\text{pyk}} \text{"axiom s one"}][S1 \xrightarrow{\text{tex}} \text{"S1"}]$, $[S2 \xrightarrow{\text{pyk}} \text{"axiom s two"}][S2 \xrightarrow{\text{tex}} \text{"S2"}]$, $[S3 \xrightarrow{\text{pyk}} \text{"axiom s three"}][S3 \xrightarrow{\text{tex}} \text{"S3"}]$, $[S4 \xrightarrow{\text{pyk}} \text{"axiom s four"}][S4 \xrightarrow{\text{tex}} \text{"S4"}]$, $[S5 \xrightarrow{\text{pyk}} \text{"axiom s five"}][S5 \xrightarrow{\text{tex}} \text{"S5"}]$, $[S6 \xrightarrow{\text{pyk}} \text{"axiom s six"}][S6 \xrightarrow{\text{tex}} \text{"S6"}]$, $[S7 \xrightarrow{\text{pyk}} \text{"axiom s seven"}][S7 \xrightarrow{\text{tex}} \text{"S7"}]$, $[S8 \xrightarrow{\text{pyk}} \text{"axiom s eight"}][S8 \xrightarrow{\text{tex}} \text{"S8"}]$

S8”], and $[S9 \xrightarrow{\text{pyk}} \text{“axiom s nine”}][S9 \xrightarrow{\text{tex}} \text{“S9”}]$. System [S] is defined thus:

$$\begin{aligned}
& [S \xrightarrow{\text{stmt}} \dot{a} + \dot{b}' \stackrel{P}{=} \dot{a} + \dot{b}' \oplus \dot{b} \Rightarrow \dot{a} \Rightarrow \dot{b} \Rightarrow \underline{a} \Rightarrow \underline{b} \oplus \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a}' \stackrel{P}{=} \dot{b}' \oplus \\
& \forall \underline{a}: \forall \underline{b}: \underline{a} \Rightarrow \underline{b} \vdash \underline{a} \vdash \underline{b} \oplus \dot{a}' \stackrel{P}{=} \dot{b}' \Rightarrow \dot{a} \stackrel{P}{=} \dot{b} \oplus \forall \underline{a}: \forall \underline{b}: \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a} \oplus \\
& \forall \underline{x}: \forall \underline{a}: \forall \underline{b}: \text{nonfree}(\underline{x}, \underline{a}) \Vdash \dot{\forall \underline{x}}: \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a} \Rightarrow \dot{\forall \underline{x}}: \underline{b} \oplus \dot{a}: \dot{b}' \stackrel{P}{=} \dot{a}: \dot{b} + \dot{a} \oplus \\
& \dot{a} + \dot{0} \stackrel{P}{=} \dot{a} \oplus \forall \underline{a}: \forall \underline{b}: \forall \underline{c}: \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a} \Rightarrow \underline{c} \oplus \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{c} \Rightarrow \\
& \dot{b} \stackrel{P}{=} \dot{c} \oplus \forall \underline{a}: \forall \underline{b}: \forall \underline{c}: \forall \underline{x}: \underline{b} \equiv \langle \underline{a} | \underline{x} := \dot{0} \rangle \Vdash \underline{c} \equiv \langle \underline{a} | \underline{x} := \underline{x}' \rangle \Vdash \underline{b} \Rightarrow \dot{\forall \underline{x}}: \underline{a} \Rightarrow \underline{c} \Rightarrow \\
& \dot{\forall \underline{x}}: \underline{a} \oplus \neg \dot{0} \stackrel{P}{=} \dot{a}' \oplus \forall \underline{x}: \forall \underline{a}: \underline{a} \vdash \dot{\forall \underline{x}}: \underline{a} \oplus \forall \underline{c}: \forall \underline{a}: \forall \underline{x}: \forall \underline{b}: [\underline{a}] \equiv \langle [\underline{b}] | [\underline{x}] := [\underline{c}] \rangle \Vdash \\
& \dot{\forall \underline{x}}: \underline{b} \Rightarrow \underline{a} \oplus \dot{a}: \dot{0} \stackrel{P}{=} \dot{0}]
\end{aligned}$$

$$[A1 \xrightarrow{\text{stmt}} S \vdash \forall \underline{a}: \forall \underline{b}: \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a}][A1 \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[A2 \xrightarrow{\text{stmt}} S \vdash \forall \underline{a}: \forall \underline{b}: \forall \underline{c}: \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a} \Rightarrow \underline{c}][A2 \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[A3 \xrightarrow{\text{stmt}} S \vdash \dot{a} \Rightarrow \dot{a} \Rightarrow \dot{b} \Rightarrow \underline{a} \Rightarrow \underline{b}][A3 \xrightarrow{\text{proof}} \text{Rule tactic}]$$

The order of quantifiers in the following axiom is such that $[\underline{c}]$ which the current conclusion tactic cannot guess comes first. This allows to supply a value for $[\underline{c}]$ without having to supply values for the other meta-variables.

$$[A4 \xrightarrow{\text{stmt}} S \vdash \forall \underline{c}: \forall \underline{a}: \forall \underline{x}: \forall \underline{b}: [\underline{a}] \equiv \langle [\underline{b}] | [\underline{x}] := [\underline{c}] \rangle \Vdash \dot{\forall \underline{x}}: \underline{b} \Rightarrow \underline{a}][A4 \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[A5 \xrightarrow{\text{stmt}} S \vdash \forall \underline{x}: \forall \underline{a}: \forall \underline{b}: \text{nonfree}(\underline{x}, \underline{a}) \Vdash \dot{\forall \underline{x}}: \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a} \Rightarrow \dot{\forall \underline{x}}: \underline{b}][A5 \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[MP \xrightarrow{\text{stmt}} S \vdash \forall \underline{a}: \forall \underline{b}: \underline{a} \Rightarrow \underline{b} \vdash \underline{a} \vdash \underline{b}][MP \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[Gen \xrightarrow{\text{stmt}} S \vdash \forall \underline{x}: \forall \underline{a}: \underline{a} \vdash \dot{\forall \underline{x}}: \underline{a}][Gen \xrightarrow{\text{proof}} \text{Rule tactic}]$$

Axiom [S1] to [S8] are stated as Mendelson [2] does. This is done here to test certain parts of the Logiweb system. Serious users of Peano arithmetic are advised to take Mendelson’s Lemma 3.1 as axioms instead.

$$[S1 \xrightarrow{\text{stmt}} S \vdash \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{c} \Rightarrow \dot{b} \stackrel{P}{=} \dot{c}][S1 \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[S2 \xrightarrow{\text{stmt}} S \vdash \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a}' \stackrel{P}{=} \dot{b}'][S2 \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[S3 \xrightarrow{\text{stmt}} S \vdash \neg \dot{0} \stackrel{P}{=} \dot{a}'][S3 \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[S4 \xrightarrow{\text{stmt}} S \vdash \dot{a}' \stackrel{P}{=} \dot{b}' \Rightarrow \dot{a} \stackrel{P}{=} \dot{b}][S4 \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[S5 \xrightarrow{\text{stmt}} S \vdash \dot{a} + \dot{0} \stackrel{P}{=} \dot{a}][S5 \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[S6 \xrightarrow{\text{stmt}} S \vdash \dot{a} \dot{+} \dot{b}' \stackrel{P}{=} \dot{a} \dot{+} \dot{b}'] [S6 \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[S7 \xrightarrow{\text{stmt}} S \vdash \dot{a} \dot{:} \dot{0} \stackrel{P}{=} \dot{0}] [S7 \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[S8 \xrightarrow{\text{stmt}} S \vdash \dot{a} \dot{:} \dot{b}' \stackrel{P}{=} \dot{a} \dot{:} \dot{b} \dot{+} \dot{a}] [S8 \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[S9 \xrightarrow{\text{stmt}} S \vdash \forall \underline{a}: \forall \underline{b}: \forall \underline{c}: \forall \underline{x}: \underline{b} \equiv \langle \underline{a} | \underline{x} := \dot{0} \rangle \vdash \underline{c} \equiv \langle \underline{a} | \underline{x} := \underline{x}' \rangle \vdash \underline{b} \Rightarrow \dot{\forall} \underline{x}: \underline{a} \Rightarrow \underline{c} \Rightarrow \dot{\forall} \underline{x}: \underline{a}] [S9 \xrightarrow{\text{proof}} \text{Rule tactic}]$$

1.4 A lemma and a proof

We now prove Lemma [L3.2(a) $\xrightarrow{\text{pyk}}$ “lemma 1 three two a”][L3.2(a) $\xrightarrow{\text{tex}}$ “L3.2(a)”] which is an instance of the corresponding proposition in Mendelson [2]:

$$[L3.2(a) \xrightarrow{\text{stmt}} S \vdash \dot{x} \stackrel{P}{=} \dot{x}]$$

$$\begin{aligned} & [L3.2(a) \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([S \vdash S5 \gg \dot{a} \dot{+} \dot{0} \stackrel{P}{=} \dot{a}; \text{Gen} \triangleright \dot{a} \dot{+} \dot{0} \stackrel{P}{=} \dot{a} \gg \dot{\forall} \dot{a}: \dot{a} \dot{+} \dot{0} \stackrel{P}{=} \dot{a}; A4 @ \dot{x} \gg \dot{\forall} \dot{a}: \dot{a} \dot{+} \dot{0} \stackrel{P}{=} \dot{a} \Rightarrow \dot{x} \dot{+} \dot{0} \stackrel{P}{=} \dot{x}; MP \triangleright \dot{\forall} \dot{a}: \dot{a} \dot{+} \dot{0} \stackrel{P}{=} \dot{a} \Rightarrow \dot{x} \dot{+} \dot{0} \stackrel{P}{=} \dot{x} \\ & \dot{x} \triangleright \dot{\forall} \dot{a}: \dot{a} \dot{+} \dot{0} \stackrel{P}{=} \dot{a} \gg \dot{x} \dot{+} \dot{0} \stackrel{P}{=} \dot{x}; S1 \gg \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{c} \Rightarrow \dot{b} \stackrel{P}{=} \dot{c}; \text{Gen} \triangleright \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{c} \Rightarrow \dot{b} \stackrel{P}{=} \dot{c} \gg \dot{\forall} \dot{c}: \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{c} \Rightarrow \dot{b} \stackrel{P}{=} \dot{c}; A4 @ \dot{x} \gg \dot{\forall} \dot{c}: \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{c} \Rightarrow \dot{b} \stackrel{P}{=} \dot{c} \\ & \dot{c} \Rightarrow \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{b} \stackrel{P}{=} \dot{x}; MP \triangleright \dot{\forall} \dot{c}: \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{c} \Rightarrow \dot{b} \stackrel{P}{=} \dot{c} \Rightarrow \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{b} \stackrel{P}{=} \dot{x} \triangleright \dot{\forall} \dot{c}: \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{c} \Rightarrow \dot{b} \stackrel{P}{=} \dot{c} \gg \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{b} \stackrel{P}{=} \dot{x}; \text{Gen} \triangleright \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{b} \stackrel{P}{=} \dot{x} \gg \dot{\forall} \dot{b}: \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{b} \stackrel{P}{=} \dot{x}; A4 @ \dot{x} \gg \dot{\forall} \dot{b}: \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{b} \stackrel{P}{=} \dot{x} \\ & \dot{x} \Rightarrow \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x}; MP \triangleright \dot{\forall} \dot{b}: \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{b} \stackrel{P}{=} \dot{x} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x} \triangleright \dot{\forall} \dot{b}: \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{b} \stackrel{P}{=} \dot{x} \gg \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{b} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x} \\ & \dot{x}; \text{Gen} \triangleright \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x} \gg \dot{\forall} \dot{a}: \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x}; A4 @ \dot{x} \dot{+} \dot{0} \gg \dot{\forall} \dot{a}: \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \dot{+} \dot{0} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \dot{+} \dot{0} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x}; MP \triangleright \dot{\forall} \dot{a}: \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \dot{+} \dot{0} \stackrel{P}{=} \dot{x} \\ & \dot{x} \gg \dot{x} \dot{+} \dot{0} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x}; MP \triangleright \dot{x} \dot{+} \dot{0} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x} \triangleright \dot{x} \dot{+} \dot{0} \stackrel{P}{=} \dot{x} \gg \dot{x} \stackrel{P}{=} \dot{x}], p_0, c)] \end{aligned}$$

1.5 An alternative axiomatic system

System $[S' \xrightarrow{\text{pyk}}$ “system prime s”][$S' \xrightarrow{\text{tex}}$ “S”] is system $[S]$ in which the proper axioms are taken from Lemma 3.1 in Mendelson [2]. It comprises the axioms $[A1' \xrightarrow{\text{pyk}}$ “axiom prime a one”][$A1' \xrightarrow{\text{tex}}$ “A1”], $[A2' \xrightarrow{\text{pyk}}$ “axiom prime a two”][$A2' \xrightarrow{\text{tex}}$ “A2”], $[A3' \xrightarrow{\text{pyk}}$ “axiom prime a three”][$A3' \xrightarrow{\text{tex}}$ “A3”], $[A4' \xrightarrow{\text{pyk}}$ “axiom prime a four”][$A4' \xrightarrow{\text{tex}}$ “A4”], and $[A5' \xrightarrow{\text{pyk}}$ “axiom prime a five”][$A5' \xrightarrow{\text{tex}}$ “A5”] and inference rules $[MP' \xrightarrow{\text{pyk}}$ “rule prime mp”][$MP' \xrightarrow{\text{tex}}$ “MP”] and $[Gen' \xrightarrow{\text{pyk}}$ “rule prime gen”][$Gen' \xrightarrow{\text{tex}}$ “Gen”]

Gen"] of first order predicate calculus. Furthermore, it comprises the proper

axioms $[S1' \xrightarrow{\text{pyk}} \text{"axiom prime s one"}][S1' \xrightarrow{\text{tex}} \text{"S1"}]$, $[S2' \xrightarrow{\text{pyk}} \text{"axiom prime s two"}][S2' \xrightarrow{\text{tex}} \text{"S2"}]$, $[S3' \xrightarrow{\text{pyk}} \text{"axiom prime s three"}][S3' \xrightarrow{\text{tex}} \text{"S3"}]$, $[S4' \xrightarrow{\text{pyk}} \text{"axiom prime s four"}][S4' \xrightarrow{\text{tex}} \text{"S4"}]$, $[S5' \xrightarrow{\text{pyk}} \text{"axiom prime s five"}][S5' \xrightarrow{\text{tex}} \text{"S5"}]$, $[S6' \xrightarrow{\text{pyk}} \text{"axiom prime s six"}][S6' \xrightarrow{\text{tex}} \text{"S6"}]$, $[S7' \xrightarrow{\text{pyk}} \text{"axiom prime s seven"}][S7' \xrightarrow{\text{tex}} \text{"S7"}]$, $[S8' \xrightarrow{\text{pyk}} \text{"axiom prime s eight"}][S8' \xrightarrow{\text{tex}} \text{"S8"}]$, and $[S9' \xrightarrow{\text{pyk}} \text{"axiom prime s nine"}][S9' \xrightarrow{\text{tex}} \text{"S9"}]$.

System $[S']$ is defined thus:

$$\begin{aligned} & [S' \xrightarrow{\text{stmt}} \forall \underline{a}: \forall \underline{b}: \underline{a} \stackrel{P}{=} \underline{b} \Rightarrow \underline{a} \stackrel{P}{=} \underline{b} \oplus \forall \underline{a}: \forall \underline{b}: \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a} \oplus \\ & \forall \underline{x}: \forall \underline{a}: \forall \underline{b}: \text{nonfree}(\underline{x}, \underline{a}) \Vdash \dot{\forall} \underline{x}: \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a} \Rightarrow \dot{\forall} \underline{x}: \underline{b} \oplus \forall \underline{a}: \forall \underline{b}: \underline{a} \stackrel{P}{=} \underline{b} \\ & \underline{a} \stackrel{P}{=} \underline{b} \oplus \underline{a} \oplus \forall \underline{a}: \forall \underline{b}: \underline{a} \stackrel{P}{=} \underline{b} \Rightarrow \underline{a}' \stackrel{P}{=} \underline{b}' \oplus \forall \underline{a}: \forall \underline{b}: \underline{a} \Rightarrow \underline{b} \vdash \underline{b} \oplus \dot{\neg} \underline{b} \Rightarrow \dot{\neg} \underline{a} \Rightarrow \\ & \dot{\neg} \underline{b} \Rightarrow \underline{a} \Rightarrow \underline{b} \oplus \forall \underline{a}: \forall \underline{b}: \underline{a} \dot{+} \underline{b}' \stackrel{P}{=} \underline{a} \dot{+} \underline{b}' \oplus \forall \underline{a}: \neg \dot{0} \stackrel{P}{=} \underline{a}' \oplus \forall \underline{x}: \forall \underline{a}: \underline{a} \vdash \dot{\forall} \underline{x}: \underline{a} \oplus \\ & \forall \underline{c}: \forall \underline{a}: \forall \underline{x}: \forall \underline{b}: [\underline{a}] \equiv \langle [\underline{b}] | [\underline{x}] := [\underline{c}] \rangle \Vdash \dot{\forall} \underline{x}: \underline{b} \Rightarrow \underline{a} \oplus \forall \underline{a}: \underline{a} \cdot \dot{0} \stackrel{P}{=} \dot{0} \oplus \\ & \forall \underline{a}: \forall \underline{b}: \forall \underline{c}: \underline{a} \stackrel{P}{=} \underline{b} \Rightarrow \underline{a} \stackrel{P}{=} \underline{c} \Rightarrow \underline{b} \stackrel{P}{=} \underline{c} \oplus \forall \underline{a}: \forall \underline{b}: \forall \underline{c}: \underline{b} \equiv \langle \underline{a} | \underline{x} := \dot{0} \rangle \Vdash \\ & \underline{c} \equiv \langle \underline{a} | \underline{x} := \underline{x}' \rangle \Vdash \underline{b} \Rightarrow \dot{\forall} \underline{x}: \underline{a} \Rightarrow \underline{c} \Rightarrow \dot{\forall} \underline{x}: \underline{a} \oplus \forall \underline{a}: \forall \underline{b}: \forall \underline{c}: \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{a} \Rightarrow \\ & \underline{b} \Rightarrow \underline{a} \Rightarrow \underline{c} \oplus \forall \underline{a}: \underline{a} \dot{+} \dot{0} \stackrel{P}{=} \underline{a}] \end{aligned}$$

$$[A1' \xrightarrow{\text{stmt}} S' \vdash \forall \underline{a}: \forall \underline{b}: \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a}] [A1' \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[A2' \xrightarrow{\text{stmt}} S' \vdash \forall \underline{a}: \forall \underline{b}: \forall \underline{c}: \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a} \Rightarrow \underline{c}] [A2' \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[A3' \xrightarrow{\text{stmt}} S' \vdash \dot{\neg} \underline{b} \Rightarrow \dot{\neg} \underline{a} \Rightarrow \dot{\neg} \underline{b} \Rightarrow \underline{a} \Rightarrow \underline{b}] [A3' \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[A4' \xrightarrow{\text{stmt}} S' \vdash \forall \underline{c}: \forall \underline{a}: \forall \underline{x}: \forall \underline{b}: [\underline{a}] \equiv \langle [\underline{b}] | [\underline{x}] := [\underline{c}] \rangle \Vdash \dot{\forall} \underline{x}: \underline{b} \Rightarrow \underline{a}] [A4' \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[A5' \xrightarrow{\text{stmt}} S' \vdash \forall \underline{x}: \forall \underline{a}: \forall \underline{b}: \text{nonfree}(\underline{x}, \underline{a}) \Vdash \dot{\forall} \underline{x}: \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a} \Rightarrow \dot{\forall} \underline{x}: \underline{b}] [A5' \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[MP' \xrightarrow{\text{stmt}} S' \vdash \forall \underline{a}: \forall \underline{b}: \underline{a} \Rightarrow \underline{b} \vdash \underline{a} \vdash \underline{b}] [MP' \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[Gen' \xrightarrow{\text{stmt}} S' \vdash \forall \underline{x}: \forall \underline{a}: \underline{a} \vdash \dot{\forall} \underline{x}: \underline{a}] [Gen' \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[S1' \xrightarrow{\text{stmt}} S' \vdash \forall \underline{a}: \forall \underline{b}: \forall \underline{c}: \underline{a} \stackrel{P}{=} \underline{b} \Rightarrow \underline{a} \stackrel{P}{=} \underline{c} \Rightarrow \underline{b} \stackrel{P}{=} \underline{c}] [S1' \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[S2' \xrightarrow{\text{stmt}} S' \vdash \forall \underline{a}: \forall \underline{b}: \underline{a} \stackrel{P}{=} \underline{b} \Rightarrow \underline{a}' \stackrel{P}{=} \underline{b}'] [S2' \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[S3' \xrightarrow{\text{stmt}} S' \vdash \forall \underline{a}: \neg \dot{0} \stackrel{P}{=} \underline{a}'] [S3' \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[S4' \xrightarrow{\text{stmt}} S' \vdash \forall \underline{a}: \forall \underline{b}: \underline{a}' \stackrel{P}{=} \underline{b}' \Rightarrow \underline{a} \stackrel{P}{=} \underline{b}] [S4' \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[S5' \xrightarrow{\text{stmt}} S' \vdash \forall \underline{a}: \underline{a} \dot{+} \dot{0} \stackrel{P}{=} \underline{a}] [S5' \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[S6' \xrightarrow{\text{stmt}} S' \vdash \forall \underline{a}: \forall \underline{b}: \underline{a} \dot{+} \underline{b}' \stackrel{P}{=} \underline{a} \dot{+} \underline{b}'] [S6' \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[S7' \xrightarrow{\text{stmt}} S' \vdash \forall \underline{a}: \underline{a} \dot{:} \dot{0} \stackrel{P}{=} \dot{0}] [S7' \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[S8' \xrightarrow{\text{stmt}} S' \vdash \forall \underline{a}: \forall \underline{b}: \underline{a} \dot{:} \underline{b}' \stackrel{P}{=} \underline{a} \dot{:} \underline{b} \dot{+} \underline{a}] [S8' \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[S9' \xrightarrow{\text{stmt}} S' \vdash \forall \underline{a}: \forall \underline{b}: \forall \underline{c}: \forall \underline{x}: \underline{b} \equiv \langle \underline{a} | \underline{x} := \dot{0} \rangle \Vdash \underline{c} \equiv \langle \underline{a} | \underline{x} := \underline{x}' \rangle \Vdash \underline{b} \Rightarrow \dot{\forall} \underline{x}: \underline{a} \Rightarrow \underline{c} \Rightarrow \dot{\forall} \underline{x}: \underline{a}] [S9' \xrightarrow{\text{proof}} \text{Rule tactic}]$$

Note that [A1] and [A1'] are distinct. The former says $[S \vdash \forall \underline{a}: \forall \underline{b}: \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a}]$ and the latter says $[S' \vdash \forall \underline{a}: \forall \underline{b}: \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a}]$.

1.6 Restatement of lemma and a proof

We now prove Lemma [L3.2(a)] once again under the name of

[L3.2(a)' $\xrightarrow{\text{pyk}}$ “lemma prime l three two a”] [L3.2(a)' $\xrightarrow{\text{tex}}$ “L3.2(a)”]:

$$[L3.2(a)' \xrightarrow{\text{stmt}} S' \vdash \forall \underline{a}: \underline{a} \stackrel{P}{=} \underline{a}]$$

$$[L3.2(a)' \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([S' \vdash \forall \underline{a}: S5' \gg \underline{a} \dot{+} \dot{0} \stackrel{P}{=} \underline{a}; S1' \gg \underline{a} \dot{+} \dot{0} \stackrel{P}{=} \underline{a} \Rightarrow \underline{a} \dot{+} \dot{0} \stackrel{P}{=} \underline{a} \Rightarrow \underline{a} \stackrel{P}{=} \underline{a}; MP' \triangleright \underline{a} \dot{+} \dot{0} \stackrel{P}{=} \underline{a} \Rightarrow \underline{a} \dot{+} \dot{0} \stackrel{P}{=} \underline{a} \Rightarrow \underline{a} \stackrel{P}{=} \underline{a} \triangleright \underline{a} \dot{+} \dot{0} \stackrel{P}{=} \underline{a} \gg \underline{a} \dot{+} \dot{0} \stackrel{P}{=} \underline{a} \Rightarrow \underline{a} \stackrel{P}{=} \underline{a}; MP' \triangleright \underline{a} \dot{+} \dot{0} \stackrel{P}{=} \underline{a} \Rightarrow \underline{a} \stackrel{P}{=} \underline{a} \triangleright \underline{a} \dot{+} \dot{0} \stackrel{P}{=} \underline{a} \gg \underline{a} \stackrel{P}{=} \underline{a}], p_0, c)]$$

A Chores

A.1 The name of the page

This defines the name of the page:

$$[\text{peano} \xrightarrow{\text{pyk}} \text{“peano”}]$$

A.2 Variables of Peano arithmetic

We use $[\dot{b} \xrightarrow{\text{pyk}} \text{“peano b”}] [\dot{b} \xrightarrow{\text{tex}} \text{“}$

$\dot{\mathit{b}}$ ”], $[\dot{c} \xrightarrow{\text{pyk}} \text{“peano c”}] [\dot{c} \xrightarrow{\text{tex}} \text{“}$

$\dot{\mathit{c}}$ ”], $[\dot{d} \xrightarrow{\text{pyk}} \text{“peano d”}] [\dot{d} \xrightarrow{\text{tex}} \text{“}$

$\dot{\mathit{d}}$ ”], $[\dot{e} \xrightarrow{\text{pyk}} \text{“peano e”}] [\dot{e} \xrightarrow{\text{tex}} \text{“}$

A.3 T_EX definitions

A.4 Test

$[[\dot{\mathbf{a}}]^{\mathcal{P}}]$.

$[[\mathbf{a}]^{\mathcal{P}}]^{-}$

$[\text{nonfree}(\lceil \dot{\mathbf{x}} \rceil, [\dot{\mathbf{y}} \stackrel{\mathcal{P}}{=} \dot{\mathbf{z}} \Rightarrow \dot{\mathbf{v}}\dot{\mathbf{x}}:\dot{\mathbf{x}} \stackrel{\mathcal{P}}{=} \dot{\mathbf{y}}])]$.

$[\text{nonfree}(\lceil \dot{\mathbf{x}} \rceil, [\dot{\mathbf{x}} \stackrel{\mathcal{P}}{=} \dot{\mathbf{z}} \Rightarrow \dot{\mathbf{v}}\dot{\mathbf{x}}:\dot{\mathbf{x}} \stackrel{\mathcal{P}}{=} \dot{\mathbf{y}}])^{-}$

$[\text{nonfree}(\lceil \dot{\mathbf{x}} \rceil, [\dot{\mathbf{y}} \stackrel{\mathcal{P}}{=} \dot{\mathbf{x}} \Rightarrow \dot{\mathbf{v}}\dot{\mathbf{x}}:\dot{\mathbf{x}} \stackrel{\mathcal{P}}{=} \dot{\mathbf{y}}])^{-}$

$[\text{nonfree}(\lceil \dot{\mathbf{x}} \rceil, [\dot{\mathbf{y}} \stackrel{\mathcal{P}}{=} \dot{\mathbf{z}} \Rightarrow \dot{\mathbf{v}}\dot{\mathbf{y}}:\dot{\mathbf{x}} \stackrel{\mathcal{P}}{=} \dot{\mathbf{y}}])^{-}$

$[\text{free}(\langle \lceil \dot{\mathbf{v}}\dot{\mathbf{x}}:\mathbf{b} :: \dot{\mathbf{x}} :: \mathbf{c} \rceil \lceil \dot{\mathbf{x}} \rceil := \lceil \mathbf{x} :: \dot{\mathbf{y}} :: \mathbf{z} \rceil \rangle)]$.

$[\text{free}(\langle \lceil \dot{\mathbf{v}}\dot{\mathbf{y}}:\mathbf{b} :: \dot{\mathbf{x}} :: \mathbf{c} \rceil \lceil \dot{\mathbf{x}} \rceil := \lceil \mathbf{x} :: \dot{\mathbf{y}} :: \mathbf{z} \rceil \rangle)^{-}$

$[\text{free}(\langle \lceil \dot{\mathbf{v}}\dot{\mathbf{x}}:\mathbf{b} :: \dot{\mathbf{x}} :: \mathbf{c} \rceil \lceil \dot{\mathbf{y}} \rceil := \lceil \mathbf{x} :: \dot{\mathbf{y}} :: \mathbf{z} \rceil \rangle)]$.

$[\text{free}(\langle \lceil \dot{\mathbf{v}}\dot{\mathbf{y}}:\mathbf{b} :: \dot{\mathbf{x}} :: \mathbf{c} \rceil \lceil \dot{\mathbf{y}} \rceil := \lceil \mathbf{x} :: \dot{\mathbf{y}} :: \mathbf{z} \rceil \rangle)^{-}$

$[\dot{\mathbf{a}} \equiv \langle \dot{\mathbf{a}} | \dot{\mathbf{b}} := \dot{\mathbf{c}} \rangle]$.

$[\dot{\mathbf{c}} \equiv \langle \dot{\mathbf{b}} | \dot{\mathbf{b}} := \dot{\mathbf{c}} \rangle]$.

$[\forall \dot{\mathbf{a}}:\dot{\mathbf{a}} \stackrel{\mathcal{P}}{=} \dot{\mathbf{b}} \equiv \langle \forall \dot{\mathbf{a}}:\dot{\mathbf{a}} \stackrel{\mathcal{P}}{=} \dot{\mathbf{b}} | \dot{\mathbf{a}} := \dot{\mathbf{c}} \rangle]$.

$[\forall \dot{\mathbf{a}}:\dot{\mathbf{a}} \stackrel{\mathcal{P}}{=} \dot{\mathbf{c}} \equiv \langle \forall \dot{\mathbf{a}}:\dot{\mathbf{a}} \stackrel{\mathcal{P}}{=} \dot{\mathbf{b}} | \dot{\mathbf{b}} := \dot{\mathbf{c}} \rangle]$.

$[\forall \dot{\mathbf{a}}:\dot{\mathbf{a}} \stackrel{\mathcal{P}}{=} \dot{\mathbf{0}} \dot{+} \dot{\mathbf{a}} \Rightarrow \dot{\mathbf{c}}:\dot{\mathbf{d}} \stackrel{\mathcal{P}}{=} \dot{\mathbf{0}} \dot{+} \dot{\mathbf{c}}:\dot{\mathbf{d}} \equiv \langle \forall \dot{\mathbf{a}}:\dot{\mathbf{a}} \stackrel{\mathcal{P}}{=} \dot{\mathbf{0}} \dot{+} \dot{\mathbf{a}} \Rightarrow \dot{\mathbf{b}} \stackrel{\mathcal{P}}{=} \dot{\mathbf{0}} \dot{+} \dot{\mathbf{b}} | \dot{\mathbf{a}} := \dot{\mathbf{c}}:\dot{\mathbf{d}} \rangle]$.

$[\forall \dot{\mathbf{a}}:\dot{\mathbf{a}} \stackrel{\mathcal{P}}{=} \dot{\mathbf{0}} \dot{+} \dot{\mathbf{a}} \Rightarrow \dot{\mathbf{b}} \stackrel{\mathcal{P}}{=} \dot{\mathbf{0}} \dot{+} \dot{\mathbf{b}} \equiv \langle \forall \dot{\mathbf{a}}:\dot{\mathbf{a}} \stackrel{\mathcal{P}}{=} \dot{\mathbf{0}} \dot{+} \dot{\mathbf{a}} \Rightarrow \dot{\mathbf{b}} \stackrel{\mathcal{P}}{=} \dot{\mathbf{0}} \dot{+} \dot{\mathbf{b}} | \dot{\mathbf{a}} := \dot{\mathbf{c}} \rangle]$.

A.5 Priority table

$[\text{peano} \xrightarrow{\text{prio}}$

Preassociative

$[\text{peano}], [\text{base}], [\text{bracket} * \text{end bracket}], [\text{big bracket} * \text{end bracket}],$

$[\text{math} * \text{end math}], [\text{flush left} *], [\mathbf{x}], [\mathbf{y}], [\mathbf{z}], [[* \bowtie *]], [[* \xrightarrow{*} *]], [\text{pyk}], [\text{tex}],$

$[\text{name}], [\text{prio}], [*, [\mathbf{T}], [\text{if}(*, *, *)], [[* \xrightarrow{*} *]], [\text{val}], [\text{claim}], [\perp], [\text{f}(*)], [(*)^I], [\mathbf{F}], [\underline{0}],$

$[\underline{1}], [\underline{2}], [\underline{3}], [\underline{4}], [\underline{5}], [\underline{6}], [\underline{7}], [\underline{8}], [\underline{9}], [0], [1], [2], [3], [4], [5], [6], [7], [8], [9], [\mathbf{a}], [\mathbf{b}], [\mathbf{c}], [\mathbf{d}],$

$[\mathbf{e}], [\mathbf{f}], [\mathbf{g}], [\mathbf{h}], [\mathbf{i}], [\mathbf{j}], [\mathbf{k}], [\mathbf{l}], [\mathbf{m}], [\mathbf{n}], [\mathbf{o}], [\mathbf{p}], [\mathbf{q}], [\mathbf{r}], [\mathbf{s}], [\mathbf{t}], [\mathbf{u}], [\mathbf{v}], [\mathbf{w}], [(*)^M], [\text{If}(*, *,$

$*)], [\text{array}\{*\} * \text{end array}], [\mathbf{l}], [\mathbf{c}], [\mathbf{r}], [\text{empty}], [(* | * := *)], [\mathcal{M}(*)], [\mathcal{U}(*)], [\mathcal{U}(*)],$

$[U^M(*), [\mathbf{apply}(*, *), [\mathbf{apply}_1(*, *), [\mathbf{identifier}(*), [\mathbf{identifier}_1(*, *), [\mathbf{array-plus}(*, *), [\mathbf{array-remove}(*, *, *), [\mathbf{array-put}(*, *, *, *), [\mathbf{array-add}(*, *, *, *, *),$
 $[\mathbf{bit}(*, *), [\mathbf{bit}_1(*, *), [\mathbf{rack}, ["\mathbf{vector}"], ["\mathbf{bibliography}"], ["\mathbf{dictionary}"],$
 $["\mathbf{body}"], ["\mathbf{codex}"], ["\mathbf{expansion}"], ["\mathbf{code}"], ["\mathbf{cache}"], ["\mathbf{diagnose}"], ["\mathbf{pyk}"],$
 $["\mathbf{tex}"], ["\mathbf{texname}"], ["\mathbf{value}"], ["\mathbf{message}"], ["\mathbf{macro}"], ["\mathbf{definition}"],$
 $["\mathbf{unpack}"], ["\mathbf{claim}"], ["\mathbf{priority}"], ["\mathbf{lambda}"], ["\mathbf{apply}"], ["\mathbf{true}"], ["\mathbf{if}"],$
 $["\mathbf{quote}"], ["\mathbf{proclaim}"], ["\mathbf{define}"], ["\mathbf{introduce}"], ["\mathbf{hide}"], ["\mathbf{pre}"], ["\mathbf{post}"],$
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 $[\mathbf{abstract}(*, *, *, *, *), [\mathbf{[*]}], [\mathcal{M}(*, *, *, *), [\mathcal{M}_2(*, *, *, *, *), [\mathcal{M}^*(*, *, *, *), [\mathbf{macro},$
 $[\mathbf{s}_0], [\mathbf{zip}(*, *, *), [\mathbf{assoc}_1(*, *, *, *), [(*)^P], [\mathbf{self}], [[* \doteq *], [[* \doteq *], [[* \doteq *],$
 $[[* \stackrel{\mathbf{pyk}}{=} *], [[* \stackrel{\mathbf{tex}}{=} *], [[* \stackrel{\mathbf{name}}{=} *], [\mathbf{Priority\ table}[*], [\tilde{\mathcal{M}}_1], [\tilde{\mathcal{M}}_2(*), [\tilde{\mathcal{M}}_3(*),$
 $[\tilde{\mathcal{M}}_4(*, *, *, *, *), [\mathcal{M}(*, *, *, *), [\mathcal{Q}(*, *, *, *), [\tilde{\mathcal{Q}}_2(*, *, *, *), [\tilde{\mathcal{Q}}_3(*, *, *, *, *), [\tilde{\mathcal{Q}}^*(*, *, *, *),$
 $[(*)], [\mathbf{aspect}(*, *), [\mathbf{aspect}(*, *, *), [\langle * \rangle], [\mathbf{tuple}_1(*), [\mathbf{tuple}_2(*), [\mathbf{let}_2(*, *),$
 $[\mathbf{let}_1(*, *), [[* \stackrel{\mathbf{claim}}{=} *], [\mathbf{checker}], [\mathbf{check}(*, *), [\mathbf{check}_2(*, *, *), [\mathbf{check}_3(*, *, *),$
 $[\mathbf{check}^*(*, *), [\mathbf{check}_2^*(*, *, *), [[*]'], [[*]^-], [[*]^\circ], [\mathbf{msg}], [[* \stackrel{\mathbf{msg}}{=} *], [\langle \mathbf{stmt} \rangle],$
 $[\mathbf{stmt}], [[* \stackrel{\mathbf{stmt}}{=} *], [\mathbf{HeadNil}'], [\mathbf{HeadPair}'], [\mathbf{Transitivity}'], [\perp], [\mathbf{Contra}'], [\mathbf{T}_E],$
 $[\mathbf{L}_1], [\underline{*}], [\mathcal{A}], [\mathcal{B}], [\mathcal{C}], [\mathcal{D}], [\mathcal{E}], [\mathcal{F}], [\mathcal{G}], [\mathcal{H}], [\mathcal{I}], [\mathcal{J}], [\mathcal{K}], [\mathcal{L}], [\mathcal{M}], [\mathcal{N}], [\mathcal{O}], [\mathcal{P}], [\mathcal{Q}],$
 $[\mathcal{R}], [\mathcal{S}], [\mathcal{T}], [\mathcal{U}], [\mathcal{V}], [\mathcal{W}], [\mathcal{X}], [\mathcal{Y}], [\mathcal{Z}], [(* | * := *)], [(* * | * := *)], [\emptyset], [\mathbf{Remainder}],$
 $[(*)^\vee], [\mathbf{intro}(*, *, *, *), [\mathbf{intro}(*, *, *), [\mathbf{error}(*, *), [\mathbf{error}_2(*, *), [\mathbf{proof}(*, *, *),$
 $[\mathbf{proof}_2(*, *), [\mathcal{S}(*, *), [\mathcal{S}^I(*, *), [\mathcal{S}^\triangleright(*, *), [\mathcal{S}_1^\triangleright(*, *, *), [\mathcal{S}^E(*, *), [\mathcal{S}_1^E(*, *, *),$
 $[\mathcal{S}^+(*, *), [\mathcal{S}_1^+(*, *, *), [\mathcal{S}^-(*, *), [\mathcal{S}_1^-(*, *, *), [\mathcal{S}^*(*, *), [\mathcal{S}_1^*(*, *, *),$
 $[\mathcal{S}_2^*(*, *, *, *), [\mathcal{S}^\oplus(*, *), [\mathcal{S}_1^\oplus(*, *, *), [\mathcal{S}^\vdash(*, *), [\mathcal{S}_1^\vdash(*, *, *, *), [\mathcal{S}^\mp(*, *),$
 $[\mathcal{S}_1^\mp(*, *, *, *), [\mathcal{S}^{\mathbf{i.e.}}(*, *), [\mathcal{S}_1^{\mathbf{i.e.}}(*, *, *, *), [\mathcal{S}_2^{\mathbf{i.e.}}(*, *, *, *, *), [\mathcal{S}^\vee(*, *),$
 $[\mathcal{S}_1^\vee(*, *, *, *), [\mathcal{S}^{\mathbf{i}}(*, *), [\mathcal{S}_1^{\mathbf{i}}(*, *, *), [\mathcal{S}_2^{\mathbf{i}}(*, *, *, *), [\mathcal{T}(*), [\mathbf{claims}(*, *, *),$
 $[\mathbf{claims}_2(*, *, *), [\langle \mathbf{proof} \rangle], [\mathbf{proof}], [[\mathbf{Lemma} * : *], [[\mathbf{Proof\ of} * : *],$
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 $[\mathbf{verifier}], [\mathcal{V}_1(*), [\mathcal{V}_2(*, *), [\mathcal{V}_3(*, *, *, *), [\mathcal{V}_4(*, *), [\mathcal{V}_5(*, *, *, *), [\mathcal{V}_6(*, *, *, *),$
 $[\mathcal{V}_7(*, *, *, *), [\mathbf{Cut}(*, *), [\mathbf{Head}_\oplus(*), [\mathbf{Tail}_\oplus(*), [\mathbf{rule}_1(*, *), [\mathbf{rule}(*, *),$
 $[\mathbf{Rule\ tactic}], [\mathbf{Plus}(*, *), [[\mathbf{Theory} *], [\mathbf{theory}_2(*, *), [\mathbf{theory}_3(*, *),$
 $[\mathbf{theory}_4(*, *, *), [\mathbf{HeadNil}''], [\mathbf{HeadPair}''], [\mathbf{Transitivity}''], [\mathbf{Contra}''], [\mathbf{HeadNil}],$
 $[\mathbf{HeadPair}], [\mathbf{Transitivity}], [\mathbf{Contra}], [\mathbf{T}_E], [\mathbf{ragged\ right}],$
 $[\mathbf{ragged\ right\ expansion}], [\mathbf{parm}(*, *, *), [\mathbf{parm}^*(*, *, *), [\mathbf{inst}(*, *),$
 $[\mathbf{inst}^*(*, *), [\mathbf{occur}(*, *, *), [\mathbf{occur}^*(*, *, *), [\mathbf{unify}(* = *, *), [\mathbf{unify}^*(* = *, *),$
 $[\mathbf{unify}_2(* = *, *), [\mathbf{L}_a], [\mathbf{L}_b], [\mathbf{L}_c], [\mathbf{L}_d], [\mathbf{L}_e], [\mathbf{L}_f], [\mathbf{L}_g], [\mathbf{L}_h], [\mathbf{L}_i], [\mathbf{L}_j], [\mathbf{L}_k], [\mathbf{L}_l], [\mathbf{L}_m],$
 $[\mathbf{L}_n], [\mathbf{L}_o], [\mathbf{L}_p], [\mathbf{L}_q], [\mathbf{L}_r], [\mathbf{L}_s], [\mathbf{L}_t], [\mathbf{L}_u], [\mathbf{L}_v], [\mathbf{L}_w], [\mathbf{L}_x], [\mathbf{L}_y], [\mathbf{L}_z], [\mathbf{L}_A], [\mathbf{L}_B], [\mathbf{L}_C],$
 $[\mathbf{L}_D], [\mathbf{L}_E], [\mathbf{L}_F], [\mathbf{L}_G], [\mathbf{L}_H], [\mathbf{L}_I], [\mathbf{L}_J], [\mathbf{L}_K], [\mathbf{L}_L], [\mathbf{L}_M], [\mathbf{L}_N], [\mathbf{L}_O], [\mathbf{L}_P], [\mathbf{L}_Q], [\mathbf{L}_R],$
 $[\mathbf{L}_S], [\mathbf{L}_T], [\mathbf{L}_U], [\mathbf{L}_V], [\mathbf{L}_W], [\mathbf{L}_X], [\mathbf{L}_Y], [\mathbf{L}_Z], [\mathbf{L}_?], [\mathbf{Reflexivity}], [\mathbf{Reflexivity}_1],$
 $[\mathbf{Commutativity}], [\mathbf{Commutativity}_1], [\langle \mathbf{tactic} \rangle], [\mathbf{tactic}], [[* \stackrel{\mathbf{tactic}}{=} *], [\mathcal{P}(*, *, *),$
 $[\mathcal{P}^*(*, *, *), [\mathbf{p}_0], [\mathbf{conclude}_1(*, *), [\mathbf{conclude}_2(*, *, *), [\mathbf{conclude}_3(*, *, *, *), [\mathbf{0}],$
 $[\mathbf{1}], [\mathbf{2}], [\mathbf{a}], [\mathbf{b}], [\mathbf{c}], [\mathbf{d}], [\mathbf{e}], [\mathbf{f}], [\mathbf{g}], [\mathbf{h}], [\mathbf{i}], [\mathbf{j}], [\mathbf{k}], [\mathbf{l}], [\mathbf{m}], [\mathbf{n}], [\mathbf{o}], [\mathbf{p}], [\mathbf{q}], [\mathbf{r}], [\mathbf{s}], [\mathbf{t}],$
 $[\mathbf{u}], [\mathbf{v}], [\mathbf{w}], [\mathbf{x}], [\mathbf{y}], [\mathbf{z}], [\mathbf{nonfree}(*, *), [\mathbf{nonfree}^*(*, *), [\mathbf{free}(* * := *)],$
 $[\mathbf{free}^*(* * := *)], [* \equiv (* * := *)], [* \equiv (* * := *)], [\mathbf{S}], [\mathbf{A1}], [\mathbf{A2}], [\mathbf{A3}], [\mathbf{A4}], [\mathbf{A5}],$
 $[\mathbf{S1}], [\mathbf{S2}], [\mathbf{S3}], [\mathbf{S4}], [\mathbf{S5}], [\mathbf{S6}], [\mathbf{S7}], [\mathbf{S8}], [\mathbf{S9}], [\mathbf{MP}], [\mathbf{Gen}], [\mathbf{L3.2(a)}], [\mathbf{S'}], [\mathbf{A1'}],$
 $[\mathbf{A2'}], [\mathbf{A3'}], [\mathbf{A4'}], [\mathbf{A5'}], [\mathbf{S1'}], [\mathbf{S2'}], [\mathbf{S3'}], [\mathbf{S4'}], [\mathbf{S5'}], [\mathbf{S6'}], [\mathbf{S7'}], [\mathbf{S8'}], [\mathbf{S9'}], [\mathbf{MP'}],$

[Gen'], [L3.2(a)'];

Preassociative

[*_{*}], [*'], [*[*]], [*[* $\rightarrow$ *]], [*[* $\Rightarrow$ *]], [*];

Preassociative

[“ * ”], [], [(*)^t], [string(*) + *], [string(*) ++ *], [
], [], [! *], [” *], [# *], [\$ *], [% *], [& *], [’ *], [(*), () *], [**], [+ *], [, *], [- *], [· *], [/ *],
[0 *], [1 *], [2 *], [3 *], [4 *], [5 *], [6 *], [7 *], [8 *], [9 *], [: *], [; *], [< *], [= *], [> *], [? *],
[@ *], [A *], [B *], [C *], [D *], [E *], [F *], [G *], [H *], [I *], [J *], [K *], [L *], [M *], [N *],
[O *], [P *], [Q *], [R *], [S *], [T *], [U *], [V *], [W *], [X *], [Y *], [Z *], [[*], [\ *], [] *], [^ *],
[_ *], [‘ *], [a *], [b *], [c *], [d *], [e *], [f *], [g *], [h *], [i *], [j *], [k *], [l *], [m *], [n *], [o *],
[p *], [q *], [r *], [s *], [t *], [u *], [v *], [w *], [x *], [y *], [z *], [{ *}, [| *}, [} *], [~ *],
[Preassociative *; *], [Postassociative *; *], [[*], *], [priority * end],
[newline *], [macro newline *];

Preassociative

[*0], [*1], [0b], [*-color(*)], [*-color^{*}(*)];

Preassociative

[*’ *], [*‘ *];

Preassociative

[*^H], [*^T], [*^U], [*^h], [*^t], [*^s], [*^c], [*^d], [*^a], [*^C], [*^M], [*^B], [*^r], [*ⁱ], [*^d], [*^R], [*⁰],
[*¹], [*²], [*³], [*⁴], [*⁵], [*⁶], [*⁷], [*⁸], [*⁹], [*^E], [*^V], [*^C], [*^{C*}], [*’];

Preassociative

[* · *], [* · 0 *], [* · *];

Preassociative

[* + *], [* + 0 *], [* + 1 *], [* − *], [* − 0 *], [* − 1 *], [* ÷ *];

Preassociative

[* ∪ { * }], [* ∪ *], [* \ { * }];

Postassociative

[* ·. *], [* ·. *], [* :: *], [* +2 *], [* :: *], [* +2 * *];

Postassociative

[*, *];

Preassociative

[* ^B ≈ *], [* ^D ≈ *], [* ^C ≈ *], [* ^P ≈ *], [* ≈ *], [* = *], [* $\stackrel{+}{\rightarrow}$ *], [* $\stackrel{t}{=}$ *], [* $\stackrel{t^*}{=}$ *], [* $\stackrel{r}{=}$ *],
[* $\in_t$ *], [* $\subseteq_T$ *], [* $\stackrel{T}{=}$ *], [* $\stackrel{s}{=}$ *], [* free in *], [* free in^{*} *], [* free for * in *],
[* free for^{*} * in *], [* $\in_c$ *], [* < *], [* <’ *], [* ≤’ *], [* $\stackrel{P}{=}$ *], [* ^P];

Preassociative

[¬ *], [÷ *];

Preassociative

[* ∧ *], [* $\tilde{\wedge}$ *], [* $\tilde{\wedge}$ *], [* ∧_c *], [* $\dot{\wedge}$ *];

Preassociative

[* ∨ *], [* || *], [* $\ddot{\vee}$ *], [* $\dot{\vee}$ *];

Preassociative

[$\dot{\vee}$ *; *], [$\dot{\exists}$ *; *];

Postassociative

[* $\Rightarrow$ *], [* $\Rightarrow$ *], [* $\Leftrightarrow$ *];

Postassociative

$[* : *], [*! *];$

Preassociative

$[* \begin{Bmatrix} * \\ * \end{Bmatrix}];$

Preassociative

$[\lambda * . *], [\Lambda *], [\text{if } * \text{ then } * \text{ else } *], [\text{let } * = * \text{ in } *], [\text{let } * \doteq * \text{ in } *];$

Preassociative

$[*^I], [*^\triangleright], [*^V], [*^+], [*^-], [*^*];$

Preassociative

$[* @ *], [* \triangleright *], [* \blacktriangleright *], [* \gg *];$

Postassociative

$[* \vdash *], [* \Vdash *], [* \text{ i.e. } *];$

Preassociative

$[\forall * : *];$

Postassociative

$[* \oplus *];$

Postassociative

$[* ; *];$

Preassociative

$[* \text{ proves } *];$

Preassociative

$[* \text{ proof of } * : *], [\text{Line } * : * \gg *; *], [\text{Last line } * \gg * \square],$
 $[\text{Line } * : \text{Premise } \gg *; *], [\text{Line } * : \text{Side-condition } \gg *; *], [\text{Arbitrary } \gg *; *],$
 $[\text{Local } \gg * = *; *];$

Postassociative

$[* \text{ then } *], [* [*] *];$

Preassociative

$[* \& *];$

Preassociative

$[* \setminus \setminus *];$

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