Peano arithmetic

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Contents

1 Peano arithmetic **1**
1.1 The constructs of Peano arithmetic 1
1.2 Variables 2
1.3 Mendelsons system S 4
1.4 A lemma and a proof 5
1.5 An alternative axiomatic system 6
1.6 Restatement of lemma and a proof 7

A Chores **8**
A.1 The name of the page 8
A.2 Variables of Peano arithmetic 8
A.3 TEX definitions 9
A.4 Test 14
A.5 Priority table 15

B Index **18**

C Bibliography **22**

1 Peano arithmetic

This Logiweb page [1] defines Peano arithmetic.

1.1 The constructs of Peano arithmetic

Terms of Peano arithmetic are constructed from zero $[\dot{0}]$[1], successor $[x']$[2], plus $[x \dot{+} y]$[3], and times $[x \dot{\cdot} y]$[4].

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[1] $[\dot{0} \stackrel{\text{pyk}}{=}$ "peano zero"$]$
[2] $[x' \stackrel{\text{pyk}}{=}$ "* peano succ"$]$
[3] $[x \dot{+} y \stackrel{\text{pyk}}{=}$ "* peano plus *"$]$
[4] $[x \dot{\cdot} y \stackrel{\text{pyk}}{=}$ "* peano times *"$]$

Formulas of Peano arithmeric are constructed from equality $[\mathsf{x} \stackrel{\mathrm{p}}{=} \mathsf{y}]$[5], negation $[\dot{\neg}\,\mathsf{x}]$[6], implication $[\mathsf{x} \mathrel{\dot{\Rightarrow}} \mathsf{y}]$[7], and universal quantification $[\dot{\forall}\mathsf{x}\colon \mathsf{y}]$[8].

From these constructs we macro define one $[\dot{1}]$[9], two $[\dot{2}]$[10], conjunction $[\mathsf{x} \mathrel{\dot{\wedge}} \mathsf{y}]$[11], disjunction $[\mathsf{x} \mathrel{\dot{\vee}} \mathsf{y}]$[12], biimplication $[\mathsf{x} \mathrel{\dot{\Leftrightarrow}} \mathsf{y}]$[13], and existential quantification $[\dot{\exists}\mathsf{x}\colon \mathsf{y}]$[14]:

$$[\dot{1} \mathrel{\ddot{=}} \dot{0}']$$

$$[\dot{2} \mathrel{\ddot{=}} \dot{1}']$$

$$[\mathsf{x} \mathrel{\dot{\wedge}} \mathsf{y} \mathrel{\ddot{=}} \dot{\neg}\,(\mathsf{x} \mathrel{\dot{\Rightarrow}} \dot{\neg}\,\mathsf{y})]$$

$$[\mathsf{x} \mathrel{\dot{\vee}} \mathsf{y} \mathrel{\ddot{=}} \dot{\neg}\,\mathsf{x} \mathrel{\dot{\Rightarrow}} \mathsf{y}]$$

$$[\mathsf{x} \mathrel{\dot{\Leftrightarrow}} \mathsf{y} \mathrel{\ddot{=}} (\mathsf{x} \mathrel{\dot{\Rightarrow}} \mathsf{y}) \mathrel{\dot{\wedge}} (\mathsf{y} \mathrel{\dot{\Rightarrow}} \mathsf{x})]$$

$$[\dot{\exists}\mathsf{x}\colon \mathsf{y} \mathrel{\ddot{=}} \dot{\neg}\,\dot{\forall}\mathsf{x}\colon \dot{\neg}\,\mathsf{y}]$$

1.2 Variables

We now introduce the unary operator $[\dot{\mathsf{x}}]$[15] and define that a term is a *Peano variable* (i.e. a variable of Peano arithmetic) if it has the $[\dot{\mathsf{x}}]$ operator in its root. $[\mathsf{x}^{\mathcal{P}}]$[16] is true if $[\mathsf{x}]$ is a Peano variable:

$$[\mathsf{x}^{\mathcal{P}} \mathrel{\dot{=}} \mathsf{x} \stackrel{\mathrm{r}}{=} \lceil \dot{\mathsf{x}} \rceil]$$

We macro define $[\dot{a}]$[17] to be a Peano variable:

$$[\dot{a} \mathrel{\ddot{=}} \dot{\mathsf{a}}]$$

Appendix A.2 defines Peano variables for the other letters of the English alphabet.

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[5] $[\mathsf{x} \stackrel{\mathrm{p}}{=} \mathsf{y} \stackrel{\mathrm{pyk}}{=}$ "∗ peano is ∗"$]$

[6] $[\dot{\neg}\,\mathsf{x} \stackrel{\mathrm{pyk}}{=}$ "peano not ∗"$]$

[7] $[\mathsf{x} \mathrel{\dot{\Rightarrow}} \mathsf{y} \stackrel{\mathrm{pyk}}{=}$ "∗ peano imply ∗"$]$

[8] $[\dot{\forall}\mathsf{x}\colon \mathsf{y} \stackrel{\mathrm{pyk}}{=}$ "peano all ∗ indeed ∗"$]$

[9] $[\dot{1} \stackrel{\mathrm{pyk}}{=}$ "peano one"$]$

[10] $[\dot{2} \stackrel{\mathrm{pyk}}{=}$ "peano two"$]$

[11] $[\mathsf{x} \mathrel{\dot{\wedge}} \mathsf{y} \stackrel{\mathrm{pyk}}{=}$ "∗ peano and ∗"$]$

[12] $[\mathsf{x} \mathrel{\dot{\vee}} \mathsf{y} \stackrel{\mathrm{pyk}}{=}$ "∗ peano or ∗"$]$

[13] $[\mathsf{x} \mathrel{\dot{\Leftrightarrow}} \mathsf{y} \stackrel{\mathrm{pyk}}{=}$ "∗ peano iff ∗"$]$

[14] $[\dot{\exists}\mathsf{x}\colon \mathsf{y} \stackrel{\mathrm{pyk}}{=}$ "peano exist ∗ indeed ∗"$]$

[15] $[\dot{\mathsf{x}} \stackrel{\mathrm{pyk}}{=}$ "∗ peano var"$]$

[16] $[\mathsf{x}^{\mathcal{P}} \stackrel{\mathrm{pyk}}{=}$ "∗ is peano var"$]$

[17] $[\dot{a} \stackrel{\mathrm{pyk}}{=}$ "peano a"$]$

$[\mathrm{non\dot{f}ree}(\mathsf{x},\mathsf{y})]$[18] is true if the Peano variable $[\mathsf{x}]$ does not occur free in the Peano term/formula $[\mathsf{y}]$. $[\mathrm{non\dot{f}ree}^{*}(\mathsf{x},\mathsf{y})]$[19] is true if the Peano variable $[\mathsf{x}]$ does not occur free in the list $[\mathsf{y}]$ of Peano terms/formulas.

$$
\begin{array}{l}
[\mathrm{non\dot{f}ree}(\mathsf{x},\mathsf{y}) \doteq \\
\textbf{if } \mathsf{y}^{\mathcal{P}} \textbf{ then } \neg\mathsf{x} \stackrel{\mathrm{t}}{=} \mathsf{y} \textbf{ else} \\
\textbf{if } \neg\mathsf{y} \stackrel{\mathrm{r}}{=} \lceil\dot{\forall}\mathsf{x}\colon\mathsf{y}\rceil \textbf{ then } \mathrm{non\dot{f}ree}^{*}(\mathsf{x},\mathsf{y}^{\mathrm{t}}) \textbf{ else} \\
\textbf{if } \mathsf{x} \stackrel{\mathrm{t}}{=} \mathsf{y}^{1} \textbf{ then } \mathsf{T} \textbf{ else } \mathrm{non\dot{f}ree}(\mathsf{x},\mathsf{y}^{2})]
\end{array}
$$

$$[\mathrm{non\dot{f}ree}^{*}(\mathsf{x},\mathsf{y}) \doteq \mathsf{x}!\mathrm{If}(\mathsf{y},\mathsf{T},\mathrm{non\dot{f}ree}(\mathsf{x},\mathsf{y}^{\mathrm{h}}) \ddot{\wedge} \mathrm{non\dot{f}ree}^{*}(\mathsf{x},\mathsf{y}^{\mathrm{t}}))]$$

$[\mathrm{\dot{f}ree}\langle\mathsf{a}|\mathsf{x}:=\mathsf{b}\rangle]$[20] is true if the substitution $[\langle\mathsf{a}\,|\mathsf{x}{:=}\mathsf{b}\rangle]$ is free. $[\mathrm{\dot{f}ree}^{*}\langle\mathsf{a}|\mathsf{x}:=\mathsf{b}\rangle]$[21] is the version where $[\mathsf{a}]$ is a list of terms.

$$
\begin{array}{l}
[\mathrm{\dot{f}ree}\langle\mathsf{a}|\mathsf{x}:=\mathsf{b}\rangle \doteq \mathsf{x}!\mathsf{b}! \\
\textbf{if } \mathsf{a}^{\mathcal{P}} \textbf{ then } \mathsf{T} \textbf{ else} \\
\textbf{if } \neg\mathsf{a} \stackrel{\mathrm{r}}{=} \lceil\dot{\forall}\mathsf{u}\colon\mathsf{v}\rceil \textbf{ then } \mathrm{\dot{f}ree}^{*}\langle\mathsf{a}^{\mathrm{t}}|\mathsf{x}:=\mathsf{b}\rangle \textbf{ else} \\
\textbf{if } \mathsf{a}^{1} \stackrel{\mathrm{t}}{=} \mathsf{x} \textbf{ then } \mathsf{T} \textbf{ else} \\
\textbf{if } \mathrm{non\dot{f}ree}(\mathsf{x},\mathsf{a}^{2}) \textbf{ then } \mathsf{T} \textbf{ else} \\
\textbf{if } \neg\mathrm{non\dot{f}ree}(\mathsf{a}^{1},\mathsf{b}) \textbf{ then } \mathsf{F} \textbf{ else} \\
\mathrm{\dot{f}ree}\langle\mathsf{a}^{2}|\mathsf{x}:=\mathsf{b}\rangle]
\end{array}
$$

$$[\mathrm{\dot{f}ree}^{*}\langle\mathsf{a}|\mathsf{x}:=\mathsf{b}\rangle \doteq \mathsf{x}!\mathsf{b}!\mathrm{If}(\mathsf{a},\mathsf{T},\mathrm{\dot{f}ree}\langle\mathsf{a}^{\mathrm{h}}|\mathsf{x}:=\mathsf{b}\rangle \ddot{\wedge} \mathrm{\dot{f}ree}^{*}\langle\mathsf{a}^{\mathrm{t}}|\mathsf{x}:=\mathsf{b}\rangle)]$$

$[\mathsf{a}{\equiv}\langle\mathsf{b}|\mathsf{x}:=\mathsf{c}\rangle]$[22] is true if $[\mathsf{a}]$ equals $[\langle\mathsf{b}\,|\mathsf{x}{:=}\mathsf{c}\rangle]$. $[\mathsf{a}{\equiv}\langle^{*}\mathsf{b}|\mathsf{x}:=\mathsf{c}\rangle]$[23] is the version where $[\mathsf{a}]$ and $[\mathsf{b}]$ are lists.

$$
\begin{array}{l}
[\mathsf{a}{\equiv}\langle\mathsf{b}|\mathsf{x}:=\mathsf{c}\rangle \doteq \mathsf{a}!\mathsf{x}!\mathsf{c}! \\
\textbf{if } \mathsf{b} \stackrel{\mathrm{r}}{=} \lceil\dot{\forall}\mathsf{u}\colon\mathsf{v}\rceil \ddot{\wedge} \mathsf{b}^{1} \stackrel{\mathrm{t}}{=} \mathsf{x} \textbf{ then } \mathsf{a} \stackrel{\mathrm{t}}{=} \mathsf{b} \textbf{ else} \\
\textbf{if } \mathsf{b}^{\mathcal{P}} \wedge \mathsf{b} \stackrel{\mathrm{t}}{=} \mathsf{x} \textbf{ then } \mathsf{a} \stackrel{\mathrm{t}}{=} \mathsf{c} \textbf{ else} \\
\mathsf{a} \stackrel{\mathrm{r}}{=} \mathsf{b} \ddot{\wedge} \mathsf{a}^{\mathrm{t}}{\equiv}\langle^{*}\mathsf{b}^{\mathrm{t}}|\mathsf{x}:=\mathsf{c}\rangle]
\end{array}
$$

$$[\mathsf{a}{\equiv}\langle^{*}\mathsf{b}|\mathsf{x}:=\mathsf{c}\rangle \doteq \mathsf{b}!\mathsf{x}!\mathsf{c}!\mathrm{If}(\mathsf{a},\mathsf{T},\mathsf{a}^{\mathrm{h}}{\equiv}\langle\mathsf{b}^{\mathrm{h}}|\mathsf{x}:=\mathsf{c}\rangle \ddot{\wedge} \mathsf{a}^{\mathrm{t}}{\equiv}\langle^{*}\mathsf{b}^{\mathrm{t}}|\mathsf{x}:=\mathsf{c}\rangle)]$$

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[18] $[\mathrm{non\dot{f}ree}(\mathsf{x},\mathsf{y}) \stackrel{\mathrm{pyk}}{=}$ "peano nonfree $*$ in $*$ end nonfree"$]$

[19] $[\mathrm{non\dot{f}ree}^{*}(\mathsf{x},\mathsf{y}) \stackrel{\mathrm{pyk}}{=}$ "peano nonfree star $*$ in $*$ end nonfree"$]$

[20] $[\mathrm{\dot{f}ree}\langle\mathsf{a}|\mathsf{x}:=\mathsf{b}\rangle \stackrel{\mathrm{pyk}}{=}$ "peano free $*$ set $*$ to $*$ end free"$]$

[21] $[\mathrm{\dot{f}ree}^{*}\langle\mathsf{a}|\mathsf{x}:=\mathsf{b}\rangle \stackrel{\mathrm{pyk}}{=}$ "peano free star $*$ set $*$ to $*$ end free"$]$

[22] $[\mathsf{a}{\equiv}\langle\mathsf{b}|\mathsf{x}:=\mathsf{c}\rangle \stackrel{\mathrm{pyk}}{=}$ "peano sub $*$ is $*$ where $*$ is $*$ end sub"$]$

[23] $[\mathsf{a}{\equiv}\langle^{*}\mathsf{b}|\mathsf{x}:=\mathsf{c}\rangle \stackrel{\mathrm{pyk}}{=}$ "peano sub star $*$ is $*$ where $*$ is $*$ end sub"$]$

1.3 Mendelsons system S

System [S][24] of Mendelson [2] expresses Peano arithmetic. It comprises the axioms [A1][25], [A2][26], [A3][27], [A4][28], and [A5][29] and inference rules [MP][30] and [Gen][31] of first order predicate calculus. Furthermore, it comprises the proper axioms [S1][32], [S2][33], [S3][34], [S4][35], [S5][36], [S6][37], [S7][38], [S8][39], and [S9][40]. System [S] is defined thus:

[**Theory** S]

[S **rule** A1: $\forall \mathcal{A} \colon \forall \mathcal{B} \colon \mathcal{A} \mathrel{\dot{\Rightarrow}} \mathcal{B} \mathrel{\dot{\Rightarrow}} \mathcal{A}$]

[S **rule** A2: $\forall \mathcal{A} \colon \forall \mathcal{B} \colon \forall \mathcal{C} \colon (\mathcal{A} \mathrel{\dot{\Rightarrow}} \mathcal{B} \mathrel{\dot{\Rightarrow}} \mathcal{C}) \mathrel{\dot{\Rightarrow}} (\mathcal{A} \mathrel{\dot{\Rightarrow}} \mathcal{B}) \mathrel{\dot{\Rightarrow}} \mathcal{A} \mathrel{\dot{\Rightarrow}} \mathcal{C}$]

[S **rule** A3: $(\dot{\neg} \mathcal{B} \mathrel{\dot{\Rightarrow}} \dot{\neg} \mathcal{A}) \mathrel{\dot{\Rightarrow}} (\dot{\neg} \mathcal{B} \mathrel{\dot{\Rightarrow}} \mathcal{A}) \mathrel{\dot{\Rightarrow}} \mathcal{B}$]

The order of quantifiers in the following axiom is such that [$\mathcal{C}$] which the current conclusion tactic cannot guess comes first. This allows to supply a value for [$\mathcal{C}$] without having to supply values for the other meta-variables.

[S **rule** A4: $\forall \mathcal{C} \colon \forall \mathcal{A} \colon \forall \mathcal{X} \colon \forall \mathcal{B} \colon \lceil \mathcal{A} \rceil \equiv \langle \lceil \mathcal{B} \rceil | \lceil \mathcal{X} \rceil := \lceil \mathcal{C} \rceil \rangle \Vdash \dot{\forall} \mathcal{X} \colon \mathcal{B} \mathrel{\dot{\Rightarrow}} \mathcal{A}$]

[S **rule** A5: $\forall \mathcal{X} \colon \forall \mathcal{A} \colon \forall \mathcal{B} \colon \dot{\mathrm{nonfree}}(\mathcal{X}, \mathcal{A}) \Vdash \dot{\forall} \mathcal{X} \colon (\mathcal{A} \mathrel{\dot{\Rightarrow}} \mathcal{B}) \mathrel{\dot{\Rightarrow}} \mathcal{A} \mathrel{\dot{\Rightarrow}} \dot{\forall} \mathcal{X} \colon \mathcal{B}$]

[S **rule** MP: $\forall \mathcal{A} \colon \forall \mathcal{B} \colon \mathcal{A} \mathrel{\dot{\Rightarrow}} \mathcal{B} \vdash \mathcal{A} \vdash \mathcal{B}$]

[S **rule** Gen: $\forall \mathcal{X} \colon \forall \mathcal{A} \colon \mathcal{A} \vdash \dot{\forall} \mathcal{X} \colon \mathcal{A}$]

---

[24] [S $\stackrel{\text{pyk}}{=}$ "system s"]

[25] [A1 $\stackrel{\text{pyk}}{=}$ "axiom a one"]

[26] [A2 $\stackrel{\text{pyk}}{=}$ "axiom a two"]

[27] [A3 $\stackrel{\text{pyk}}{=}$ "axiom a three"]

[28] [A4 $\stackrel{\text{pyk}}{=}$ "axiom a four"]

[29] [A5 $\stackrel{\text{pyk}}{=}$ "axiom a five"]

[30] [MP $\stackrel{\text{pyk}}{=}$ "rule mp"]

[31] [Gen $\stackrel{\text{pyk}}{=}$ "rule gen"]

[32] [S1 $\stackrel{\text{pyk}}{=}$ "axiom s one"]

[33] [S2 $\stackrel{\text{pyk}}{=}$ "axiom s two"]

[34] [S3 $\stackrel{\text{pyk}}{=}$ "axiom s three"]

[35] [S4 $\stackrel{\text{pyk}}{=}$ "axiom s four"]

[36] [S5 $\stackrel{\text{pyk}}{=}$ "axiom s five"]

[37] [S6 $\stackrel{\text{pyk}}{=}$ "axiom s six"]

[38] [S7 $\stackrel{\text{pyk}}{=}$ "axiom s seven"]

[39] [S8 $\stackrel{\text{pyk}}{=}$ "axiom s eight"]

[40] [S9 $\stackrel{\text{pyk}}{=}$ "axiom s nine"]

Axiom [S1] to [S8] are stated as Mendelson [2] does. This is done here to test certain parts of the Logiweb system. Serious users of Peano arithmetic are advised to take Mendelson's Lemma 3.1 as axioms instead.

[S **rule** S1: $\dot{a} \stackrel{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{c} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{p}{=} \dot{c}$]

[S **rule** S2: $\dot{a} \stackrel{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a}' \stackrel{p}{=} \dot{b}'$]

[S **rule** S3: $\dot{\neg} \dot{0} \stackrel{p}{=} \dot{a}'$]

[S **rule** S4: $\dot{a}' \stackrel{p}{=} \dot{b}' \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{b}$]

[S **rule** S5: $\dot{a} \mathbin{\dot{+}} \dot{0} \stackrel{p}{=} \dot{a}$]

[S **rule** S6: $\dot{a} \mathbin{\dot{+}} \dot{b}' \stackrel{p}{=} (\dot{a} \mathbin{\dot{+}} \dot{b})'$]

[S **rule** S7: $\dot{a} \mathbin{\dot{\cdot}} \dot{0} \stackrel{p}{=} \dot{0}$]

[S **rule** S8: $\dot{a} \mathbin{\dot{\cdot}} (\dot{b}') \stackrel{p}{=} (\dot{a} \mathbin{\dot{\cdot}} \dot{b}) \mathbin{\dot{+}} \dot{a}$]

[S **rule** S9: $\forall \mathcal{A}\colon \forall \mathcal{B}\colon \forall \mathcal{C}\colon \forall \mathcal{X}\colon$
$\mathcal{B} \equiv \langle \mathcal{A} | \mathcal{X} := \dot{0} \rangle \Vdash \mathcal{C} \equiv \langle \mathcal{A} | \mathcal{X} := \mathcal{X}' \rangle \Vdash$
$\mathcal{B} \mathrel{\dot{\Rightarrow}} \dot{\forall} \mathcal{X}\colon (\mathcal{A} \mathrel{\dot{\Rightarrow}} \mathcal{C}) \mathrel{\dot{\Rightarrow}} \dot{\forall} \mathcal{X}\colon \mathcal{A}$]

1.4 A lemma and a proof

We now prove Lemma [L3.2(a)][41] which is an instance of the corresponding proposition in Mendelson [2]:

[S **lemma** L3.2(a): $\dot{x} \stackrel{p}{=} \dot{x}$]

S **proof of** L3.2(a):

L01:	S5 $\gg$	$\dot{a} \mathbin{\dot{+}} \dot{0} \stackrel{p}{=} \dot{a}$	;
L02:	Gen $\rhd$ L01 $\gg$	$\dot{\forall} \dot{a}\colon \dot{a} \mathbin{\dot{+}} \dot{0} \stackrel{p}{=} \dot{a}$	;
L03:	A4 @ $\dot{x}$ $\gg$	$\dot{\forall} \dot{a}\colon \dot{a} \mathbin{\dot{+}} \dot{0} \stackrel{p}{=} \dot{a} \mathrel{\dot{\Rightarrow}} \dot{x} \mathbin{\dot{+}} \dot{0} \stackrel{p}{=} \dot{x}$	;
L04:	MP $\rhd$ L03 $\rhd$ L02 $\gg$	$\dot{x} \mathbin{\dot{+}} \dot{0} \stackrel{p}{=} \dot{x}$	;
L05:	S1 $\gg$	$\dot{a} \stackrel{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{c} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{p}{=} \dot{c}$	;
L06:	Gen $\rhd$ L05 $\gg$	$\dot{\forall} \dot{c}\colon (\dot{a} \stackrel{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{c} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{p}{=} \dot{c})$	;
L07:	A4 @ $\dot{x}$ $\gg$	$\dot{\forall} \dot{c}\colon (\dot{a} \stackrel{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{c} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{p}{=} \dot{c}) \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{p}{=} \dot{x}$	;
L08:	MP $\rhd$ L07 $\rhd$ L06 $\gg$	$\dot{a} \stackrel{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{p}{=} \dot{x}$	;
L09:	Gen $\rhd$ L08 $\gg$	$\dot{\forall} \dot{b}\colon (\dot{a} \stackrel{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{p}{=} \dot{x})$	;
L10:	A4 @ $\dot{x}$ $\gg$	$\dot{\forall} \dot{b}\colon (\dot{a} \stackrel{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{b} \stackrel{p}{=} \dot{x}) \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{a} \stackrel{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \stackrel{p}{=} \dot{x}$	;

[41] [L3.2(a) $\stackrel{pyk}{=}$ "lemma l three two a"]

L11:	MP $\rhd$ L10 $\rhd$ L09 $\gg$	$\dot{a} \stackrel{\mathrm{p}}{=} \dot{x} \dot{\Rightarrow} \dot{a} \stackrel{\mathrm{p}}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{\mathrm{p}}{=} \dot{x}$	;
L12:	Gen $\rhd$ L11 $\gg$	$\dot{\forall} \dot{a}\colon (\dot{a} \stackrel{\mathrm{p}}{=} \dot{x} \dot{\Rightarrow} \dot{a} \stackrel{\mathrm{p}}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{\mathrm{p}}{=} \dot{x})$	;
L13:	A4 @ $\dot{x} \dot{+} \dot{0} \gg$	$\dot{\forall} \dot{a}\colon (\dot{a} \stackrel{\mathrm{p}}{=} \dot{x} \dot{\Rightarrow} \dot{a} \stackrel{\mathrm{p}}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{\mathrm{p}}{=} \dot{x}) \dot{\Rightarrow} \dot{x} \dot{+} \dot{0} \stackrel{\mathrm{p}}{=} \dot{x} \dot{\Rightarrow} \dot{x} \dot{+} \dot{0} \stackrel{\mathrm{p}}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{\mathrm{p}}{=} \dot{x}$	;
L14:	MP $\rhd$ L13 $\rhd$ L12 $\gg$	$\dot{x} \dot{+} \dot{0} \stackrel{\mathrm{p}}{=} \dot{x} \dot{\Rightarrow} \dot{x} \dot{+} \dot{0} \stackrel{\mathrm{p}}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{\mathrm{p}}{=} \dot{x}$	;
L15:	MP $\rhd$ L14 $\rhd$ L04 $\gg$	$\dot{x} \dot{+} \dot{0} \stackrel{\mathrm{p}}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{\mathrm{p}}{=} \dot{x}$	;
L16:	MP $\rhd$ L15 $\rhd$ L04 $\gg$	$\dot{x} \stackrel{\mathrm{p}}{=} \dot{x}$	$\square$

1.5 An alternative axiomatic system

System [S′][42] is system [S] in which the proper axioms are taken from Lemma 3.1 in Mendelson [2]. It comprises the axioms [A1′][43], [A2′][44], [A3′][45], [A4′][46], and [A5′][47] and inference rules [MP′][48] and [Gen′][49] of first order predicate calculus. Furthermore, it comprises the proper axioms [S1′][50], [S2′][51], [S3′][52], [S4′][53], [S5′][54], [S6′][55], [S7′][56], [S8′][57], and [S9′][58].

System [S′] is defined thus:

[**Theory** S′]

[S′ **rule** A1′: $\forall \mathcal{A}\colon \forall \mathcal{B}\colon \mathcal{A} \dot{\Rightarrow} \mathcal{B} \dot{\Rightarrow} \mathcal{A}$]

[S′ **rule** A2′: $\forall \mathcal{A}\colon \forall \mathcal{B}\colon \forall \mathcal{C}\colon (\mathcal{A} \dot{\Rightarrow} \mathcal{B} \dot{\Rightarrow} \mathcal{C}) \dot{\Rightarrow} (\mathcal{A} \dot{\Rightarrow} \mathcal{B}) \dot{\Rightarrow} \mathcal{A} \dot{\Rightarrow} \mathcal{C}$]

[S′ **rule** A3′: $(\dot{\neg} \mathcal{B} \dot{\Rightarrow} \dot{\neg} \mathcal{A}) \dot{\Rightarrow} (\dot{\neg} \mathcal{B} \dot{\Rightarrow} \mathcal{A}) \dot{\Rightarrow} \mathcal{B}$]

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[42] [S′ $\stackrel{\text{pyk}}{=}$ "system prime s"]

[43] [A1′ $\stackrel{\text{pyk}}{=}$ "axiom prime a one"]

[44] [A2′ $\stackrel{\text{pyk}}{=}$ "axiom prime a two"]

[45] [A3′ $\stackrel{\text{pyk}}{=}$ "axiom prime a three"]

[46] [A4′ $\stackrel{\text{pyk}}{=}$ "axiom prime a four"]

[47] [A5′ $\stackrel{\text{pyk}}{=}$ "axiom prime a five"]

[48] [MP′ $\stackrel{\text{pyk}}{=}$ "rule prime mp"]

[49] [Gen′ $\stackrel{\text{pyk}}{=}$ "rule prime gen"]

[50] [S1′ $\stackrel{\text{pyk}}{=}$ "axiom prime s one"]

[51] [S2′ $\stackrel{\text{pyk}}{=}$ "axiom prime s two"]

[52] [S3′ $\stackrel{\text{pyk}}{=}$ "axiom prime s three"]

[53] [S4′ $\stackrel{\text{pyk}}{=}$ "axiom prime s four"]

[54] [S5′ $\stackrel{\text{pyk}}{=}$ "axiom prime s five"]

[55] [S6′ $\stackrel{\text{pyk}}{=}$ "axiom prime s six"]

[56] [S7′ $\stackrel{\text{pyk}}{=}$ "axiom prime s seven"]

[57] [S8′ $\stackrel{\text{pyk}}{=}$ "axiom prime s eight"]

[58] [S9′ $\stackrel{\text{pyk}}{=}$ "axiom prime s nine"]

[S′ **rule** A4′: $\forall \mathcal{C}\colon \forall \mathcal{A}\colon \forall \mathcal{X}\colon \forall \mathcal{B}\colon \lceil \mathcal{A} \rceil \equiv \langle \lceil \mathcal{B} \rceil | \lceil \mathcal{X} \rceil := \lceil \mathcal{C} \rceil \rangle \Vdash \dot{\forall} \mathcal{X}\colon \mathcal{B} \dot{\Rightarrow} \mathcal{A}$]

[S′ **rule** A5′: $\forall \mathcal{X}\colon \forall \mathcal{A}\colon \forall \mathcal{B}\colon \dot{\text{nonfree}}(\mathcal{X}, \mathcal{A}) \Vdash \dot{\forall} \mathcal{X}\colon (\mathcal{A} \dot{\Rightarrow} \mathcal{B}) \dot{\Rightarrow} \mathcal{A} \dot{\Rightarrow} \dot{\forall} \mathcal{X}\colon \mathcal{B}$]

[S′ **rule** MP′: $\forall \mathcal{A}\colon \forall \mathcal{B}\colon \mathcal{A} \dot{\Rightarrow} \mathcal{B} \vdash \mathcal{A} \vdash \mathcal{B}$]

[S′ **rule** Gen′: $\forall \mathcal{X}\colon \forall \mathcal{A}\colon \mathcal{A} \vdash \dot{\forall} \mathcal{X}\colon \mathcal{A}$]

[S′ **rule** S1′: $\forall \mathcal{A}\colon \forall \mathcal{B}\colon \forall \mathcal{C}\colon \mathcal{A} \stackrel{\mathrm{p}}{=} \mathcal{B} \dot{\Rightarrow} \mathcal{A} \stackrel{\mathrm{p}}{=} \mathcal{C} \dot{\Rightarrow} \mathcal{B} \stackrel{\mathrm{p}}{=} \mathcal{C}$]

[S′ **rule** S2′: $\forall \mathcal{A}\colon \forall \mathcal{B}\colon \mathcal{A} \stackrel{\mathrm{p}}{=} \mathcal{B} \dot{\Rightarrow} \mathcal{A}' \stackrel{\mathrm{p}}{=} \mathcal{B}'$]

[S′ **rule** S3′: $\forall \mathcal{A}\colon \neg \dot{0} \stackrel{\mathrm{p}}{=} \mathcal{A}'$]

[S′ **rule** S4′: $\forall \mathcal{A}\colon \forall \mathcal{B}\colon \mathcal{A}' \stackrel{\mathrm{p}}{=} \mathcal{B}' \dot{\Rightarrow} \mathcal{A} \stackrel{\mathrm{p}}{=} \mathcal{B}$]

[S′ **rule** S5′: $\forall \mathcal{A}\colon \mathcal{A} \dot{+} \dot{0} \stackrel{\mathrm{p}}{=} \mathcal{A}$]

[S′ **rule** S6′: $\forall \mathcal{A}\colon \forall \mathcal{B}\colon \mathcal{A} \dot{+} \mathcal{B}' \stackrel{\mathrm{p}}{=} (\mathcal{A} \dot{+} \mathcal{B})'$]

[S′ **rule** S7′: $\forall \mathcal{A}\colon \mathcal{A} \dot{\cdot} \dot{0} \stackrel{\mathrm{p}}{=} \dot{0}$]

[S′ **rule** S8′: $\forall \mathcal{A}\colon \forall \mathcal{B}\colon \mathcal{A} \dot{\cdot} (\mathcal{B}') \stackrel{\mathrm{p}}{=} (\mathcal{A} \dot{\cdot} \mathcal{B}) \dot{+} \mathcal{A}$]

[S′ **rule** S9′: $\forall \mathcal{A}\colon \forall \mathcal{B}\colon \forall \mathcal{C}\colon \forall \mathcal{X}\colon$
$\mathcal{B} \equiv \langle \mathcal{A} | \mathcal{X} := \dot{0} \rangle \Vdash \mathcal{C} \equiv \langle \mathcal{A} | \mathcal{X} := \mathcal{X}' \rangle \Vdash$
$\mathcal{B} \dot{\Rightarrow} \dot{\forall} \mathcal{X}\colon (\mathcal{A} \dot{\Rightarrow} \mathcal{C}) \dot{\Rightarrow} \dot{\forall} \mathcal{X}\colon \mathcal{A}$]

Note that [A1] and [A1′] are distinct. The former says [S $\vdash \forall \mathcal{A}\colon \forall \mathcal{B}\colon \mathcal{A} \dot{\Rightarrow} \mathcal{B} \dot{\Rightarrow} \mathcal{A}$] and the latter says [S′ $\vdash \forall \mathcal{A}\colon \forall \mathcal{B}\colon \mathcal{A} \dot{\Rightarrow} \mathcal{B} \dot{\Rightarrow} \mathcal{A}$].

1.6 Restatement of lemma and a proof

We now prove Lemma [L3.2(a)] once again under the name of [L3.2(a)′][59]:

[S′ **lemma** L3.2(a)′: $\forall \mathcal{A}\colon \mathcal{A} \stackrel{\mathrm{p}}{=} \mathcal{A}$]

S′ **proof of** L3.2(a)′:

L01:	Arbitrary ≫	$\mathcal{A}$	;
L02:	S5′ ≫	$\mathcal{A} \dot{+} \dot{0} \stackrel{\mathrm{p}}{=} \mathcal{A}$	;
L03:	S1′ ≫	$\mathcal{A} \dot{+} \dot{0} \stackrel{\mathrm{p}}{=} \mathcal{A} \dot{\Rightarrow}$ $\mathcal{A} \dot{+} \dot{0} \stackrel{\mathrm{p}}{=} \mathcal{A} \dot{\Rightarrow} \mathcal{A} \stackrel{\mathrm{p}}{=} \mathcal{A}$	;
L04:	MP′ ⊳ L03 ⊳ L02 ≫	$\mathcal{A} \dot{+} \dot{0} \stackrel{\mathrm{p}}{=} \mathcal{A} \dot{\Rightarrow} \mathcal{A} \stackrel{\mathrm{p}}{=} \mathcal{A}$	;
L05:	MP′ ⊳ L04 ⊳ L02 ≫	$\mathcal{A} \stackrel{\mathrm{p}}{=} \mathcal{A}$	□

[59] [L3.2(a)′ $\stackrel{\text{pyk}}{=}$ "lemma prime l three two a"]

A Chores

A.1 The name of the page

This defines the name of the page:

$$[\text{peano} \stackrel{\text{pyk}}{=} \text{“peano”}]$$

A.2 Variables of Peano arithmetic

We use $[\dot{b}]$[60], $[\dot{c}]$[61], $[\dot{d}]$[62], $[\dot{e}]$[63], $[\dot{f}]$[64], $[\dot{g}]$[65], $[\dot{h}]$[66], $[\dot{i}]$[67], $[\dot{j}]$[68], $[\dot{k}]$[69], $[\dot{l}]$[70], $[\dot{m}]$[71], $[\dot{n}]$[72], $[\dot{o}]$[73], $[\dot{p}]$[74], $[\dot{q}]$[75], $[\dot{r}]$[76], $[\dot{s}]$[77], $[\dot{t}]$[78], $[\dot{u}]$[79], $[\dot{v}]$[80], $[\dot{w}]$[81], $[\dot{x}]$[82], $[\dot{y}]$[83], and $[\dot{z}]$[84] to denote variables of Peano arithmetic:

$[\dot{b} \ddot{=} \dot{\mathsf{b}}]$, $[\dot{c} \ddot{=} \dot{\mathsf{c}}]$, $[\dot{d} \ddot{=} \dot{\mathsf{d}}]$, $[\dot{e} \ddot{=} \dot{\mathsf{e}}]$, $[\dot{f} \ddot{=} \dot{\mathsf{f}}]$, $[\dot{g} \ddot{=} \dot{\mathsf{g}}]$, $[\dot{h} \ddot{=} \dot{\mathsf{h}}]$, $[\dot{i} \ddot{=} \dot{\mathsf{i}}]$, $[\dot{j} \ddot{=} \dot{\mathsf{j}}]$, $[\dot{k} \ddot{=} \dot{\mathsf{k}}]$, $[\dot{l} \ddot{=} \dot{\mathsf{l}}]$, $[\dot{m} \ddot{=} \dot{\mathsf{m}}]$, $[\dot{n} \ddot{=} \dot{\mathsf{n}}]$, $[\dot{o} \ddot{=} \dot{\mathsf{o}}]$, $[\dot{p} \ddot{=} \dot{\mathsf{p}}]$, $[\dot{q} \ddot{=} \dot{\mathsf{q}}]$, $[\dot{r} \ddot{=} \dot{\mathsf{r}}]$, $[\dot{s} \ddot{=} \dot{\mathsf{s}}]$, $[\dot{t} \ddot{=} \dot{\mathsf{t}}]$, $[\dot{u} \ddot{=} \dot{\mathsf{u}}]$, $[\dot{v} \ddot{=} \dot{\mathsf{v}}]$, $[\dot{w} \ddot{=} \dot{\mathsf{w}}]$, $[\dot{x} \ddot{=} \dot{\mathsf{x}}]$, $[\dot{y} \ddot{=} \dot{\mathsf{y}}]$, and $[\dot{z} \ddot{=} \dot{\mathsf{z}}]$.

---

[60] $[\dot{b} \stackrel{\text{pyk}}{=} \text{“peano b”}]$

[61] $[\dot{c} \stackrel{\text{pyk}}{=} \text{“peano c”}]$

[62] $[\dot{d} \stackrel{\text{pyk}}{=} \text{“peano d”}]$

[63] $[\dot{e} \stackrel{\text{pyk}}{=} \text{“peano e”}]$

[64] $[\dot{f} \stackrel{\text{pyk}}{=} \text{“peano f”}]$

[65] $[\dot{g} \stackrel{\text{pyk}}{=} \text{“peano g”}]$

[66] $[\dot{h} \stackrel{\text{pyk}}{=} \text{“peano h”}]$

[67] $[\dot{i} \stackrel{\text{pyk}}{=} \text{“peano i”}]$

[68] $[\dot{j} \stackrel{\text{pyk}}{=} \text{“peano j”}]$

[69] $[\dot{k} \stackrel{\text{pyk}}{=} \text{“peano k”}]$

[70] $[\dot{l} \stackrel{\text{pyk}}{=} \text{“peano l”}]$

[71] $[\dot{m} \stackrel{\text{pyk}}{=} \text{“peano m”}]$

[72] $[\dot{n} \stackrel{\text{pyk}}{=} \text{“peano n”}]$

[73] $[\dot{o} \stackrel{\text{pyk}}{=} \text{“peano o”}]$

[74] $[\dot{p} \stackrel{\text{pyk}}{=} \text{“peano p”}]$

[75] $[\dot{q} \stackrel{\text{pyk}}{=} \text{“peano q”}]$

[76] $[\dot{r} \stackrel{\text{pyk}}{=} \text{“peano r”}]$

[77] $[\dot{s} \stackrel{\text{pyk}}{=} \text{“peano s”}]$

[78] $[\dot{t} \stackrel{\text{pyk}}{=} \text{“peano t”}]$

[79] $[\dot{u} \stackrel{\text{pyk}}{=} \text{“peano u”}]$

[80] $[\dot{v} \stackrel{\text{pyk}}{=} \text{“peano v”}]$

[81] $[\dot{w} \stackrel{\text{pyk}}{=} \text{“peano w”}]$

[82] $[\dot{x} \stackrel{\text{pyk}}{=} \text{“peano x”}]$

[83] $[\dot{y} \stackrel{\text{pyk}}{=} \text{“peano y”}]$

[84] $[\dot{z} \stackrel{\text{pyk}}{=} \text{“peano z”}]$

A.3 TEX definitions

[$\dot{0} \stackrel{\text{tex}}{=}$ "
\dot{0}"]

[$\mathsf{x}' \stackrel{\text{tex}}{=}$ "#1.
'"]

[$\mathsf{x} \mathop{\dot{+}} \mathsf{y} \stackrel{\text{tex}}{=}$ "#1.
\mathop{\dot{+}} #2."]

[$\mathsf{x} \mathop{\dot{\cdot}} \mathsf{y} \stackrel{\text{tex}}{=}$ "#1.
\mathop{\dot{\cdot}} #2."]

[$\mathsf{x} \stackrel{p}{=} \mathsf{y} \stackrel{\text{tex}}{=}$ "#1.
\stackrel{p}{=} #2."]

[$\dot{\neg}\, \mathsf{x} \stackrel{\text{tex}}{=}$ "
\dot{\neg}\, #1."]

[$\mathsf{x} \mathrel{\dot{\Rightarrow}} \mathsf{y} \stackrel{\text{tex}}{=}$ "#1.
\mathrel{\dot{\Rightarrow}} #2."]

[$\dot{\forall} \mathsf{x} \colon \mathsf{y} \stackrel{\text{tex}}{=}$ "
\dot{\forall} #1.
\colon #2."]

[$\dot{1} \stackrel{\text{tex}}{=}$ "
\dot{1}"]

[$\dot{2} \stackrel{\text{tex}}{=}$ "
\dot{2}"]

[$\mathsf{x} \mathrel{\dot{\wedge}} \mathsf{y} \stackrel{\text{tex}}{=}$ "#1.
\mathrel{\dot{\wedge}} #2."]

[$\mathsf{x} \mathrel{\dot{\vee}} \mathsf{y} \stackrel{\text{tex}}{=}$ "#1.
\mathrel{\dot{\vee}} #2."]

[$\mathsf{x} \mathrel{\dot{\Leftrightarrow}} \mathsf{y} \stackrel{\text{tex}}{=}$ "#1.
\mathrel{\dot{\Leftrightarrow}} #2."]

[$\dot{\exists} \mathsf{x} \colon \mathsf{y} \stackrel{\text{tex}}{=}$ "
\dot{\exists} #1.
\colon #2."]

[$\dot{\mathsf{x}} \stackrel{\text{tex}}{=}$ "
\dot{#1.
}"]

$[\mathsf{x}^{\mathcal{P}} \stackrel{\text{tex}}{=}$ “#1.
{} ˆ {\cal P}”]

$[\dot{a} \stackrel{\text{tex}}{=}$ “
\dot{\mathit{a}}”]

$[\dot{\text{nonfree}}(\mathsf{x}, \mathsf{y}) \stackrel{\text{tex}}{=}$ “
\dot{nonfree}(#1.
, #2.
)”]

$[\dot{\text{nonfree}}^{*}(\mathsf{x}, \mathsf{y}) \stackrel{\text{tex}}{=}$ “
\dot{nonfree}ˆ∗(#1.
, #2.
)”]

$[\dot{\text{free}}\langle \mathsf{a} | \mathsf{x} := \mathsf{b} \rangle \stackrel{\text{tex}}{=}$ “
\dot{free}\langle #1.
| #2.
:= #3.
\rangle”]

$[\dot{\text{free}}^{*}\langle \mathsf{a} | \mathsf{x} := \mathsf{b} \rangle \stackrel{\text{tex}}{=}$ “
\dot{free}{}ˆ∗\langle #1.
| #2.
:= #3.
\rangle”]

$[\mathsf{a} {\equiv} \langle \mathsf{b} | \mathsf{x} := \mathsf{c} \rangle \stackrel{\text{tex}}{=}$ “#1.
{\equiv}\langle #2.
|#3.
:=#4.
\rangle”]

$[\mathsf{a} {\equiv} \langle^{*} \mathsf{b} | \mathsf{x} := \mathsf{c} \rangle \stackrel{\text{tex}}{=}$ “#1.
{\equiv}\langleˆ∗ #2.
|#3.
:=#4.
\rangle”]

$[\mathrm{S} \stackrel{\text{tex}}{=}$ “
S”]

$[\mathrm{A1} \stackrel{\text{tex}}{=}$ “
A1”]

$[\mathrm{A2} \stackrel{\text{tex}}{=}$ “
A2”]

[A3 $\stackrel{\text{tex}}{=}$ “

A3”]

[A4 $\stackrel{\text{tex}}{=}$ “

A4”]

[A5 $\stackrel{\text{tex}}{=}$ “

A5”]

[MP $\stackrel{\text{tex}}{=}$ “

MP”]

[Gen $\stackrel{\text{tex}}{=}$ “

Gen”]

[S1 $\stackrel{\text{tex}}{=}$ “

S1”]

[S2 $\stackrel{\text{tex}}{=}$ “

S2”]

[S3 $\stackrel{\text{tex}}{=}$ “

S3”]

[S4 $\stackrel{\text{tex}}{=}$ “

S4”]

[S5 $\stackrel{\text{tex}}{=}$ “

S5”]

[S6 $\stackrel{\text{tex}}{=}$ “

S6”]

[S7 $\stackrel{\text{tex}}{=}$ “

S7”]

[S8 $\stackrel{\text{tex}}{=}$ “

S8”]

[S9 $\stackrel{\text{tex}}{=}$ “

S9”]

[L3.2(a) $\stackrel{\text{tex}}{=}$ “

L3.2(a)”]

[S′ $\stackrel{\text{tex}}{=}$ “

S’”]

[A1′ $\stackrel{\text{tex}}{=}$ “
A1”’]

[A2′ $\stackrel{\text{tex}}{=}$ “
A2”’]

[A3′ $\stackrel{\text{tex}}{=}$ “
A3”’]

[A4′ $\stackrel{\text{tex}}{=}$ “
A4”’]

[A5′ $\stackrel{\text{tex}}{=}$ “
A5”’]

[MP′ $\stackrel{\text{tex}}{=}$ “
MP”’]

[Gen′ $\stackrel{\text{tex}}{=}$ “
Gen”’]

[S1′ $\stackrel{\text{tex}}{=}$ “
S1”’]

[S2′ $\stackrel{\text{tex}}{=}$ “
S2”’]

[S3′ $\stackrel{\text{tex}}{=}$ “
S3”’]

[S4′ $\stackrel{\text{tex}}{=}$ “
S4”’]

[S5′ $\stackrel{\text{tex}}{=}$ “
S5”’]

[S6′ $\stackrel{\text{tex}}{=}$ “
S6”’]

[S7′ $\stackrel{\text{tex}}{=}$ “
S7”’]

[S8′ $\stackrel{\text{tex}}{=}$ “
S8”’]

[S9′ $\stackrel{\text{tex}}{=}$ “
S9”’]

[L3.2(a)′ $\stackrel{\text{tex}}{=}$ "
L3.2(a)'"]

[$\dot{b}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{b}}"]

[$\dot{c}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{c}}"]

[$\dot{d}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{d}}"]

[$\dot{e}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{e}}"]

[$\dot{f}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{f}}"]

[$\dot{g}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{g}}"]

[$\dot{h}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{h}}"]

[$\dot{i}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{i}}"]

[$\dot{j}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{j}}"]

[$\dot{k}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{k}}"]

[$\dot{l}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{l}}"]

[$\dot{m}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{m}}"]

[$\dot{n}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{n}}"]

[$\dot{o}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{o}}"]

[$\dot{p}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{p}}"]

$[\dot{q} \stackrel{\text{tex}}{=}$ "

\dot{\mathit{q}}"]

$[\dot{r} \stackrel{\text{tex}}{=}$ "

\dot{\mathit{r}}"]

$[\dot{s} \stackrel{\text{tex}}{=}$ "

\dot{\mathit{s}}"]

$[\dot{t} \stackrel{\text{tex}}{=}$ "

\dot{\mathit{t}}"]

$[\dot{u} \stackrel{\text{tex}}{=}$ "

\dot{\mathit{u}}"]

$[\dot{v} \stackrel{\text{tex}}{=}$ "

\dot{\mathit{v}}"]

$[\dot{w} \stackrel{\text{tex}}{=}$ "

\dot{\mathit{w}}"]

$[\dot{x} \stackrel{\text{tex}}{=}$ "

\dot{\mathit{x}}"]

$[\dot{y} \stackrel{\text{tex}}{=}$ "

\dot{\mathit{y}}"]

$[\dot{z} \stackrel{\text{tex}}{=}$ "

\dot{\mathit{z}}"]

A.4 Test

$[\lceil \dot{a} \rceil^{\mathcal{P}}]^{\cdot}$

$[\lceil \mathsf{a} \rceil^{\mathcal{P}}]^{-}$

$[\dot{\text{nonfree}}(\lceil \dot{x} \rceil, \lceil \dot{y} \stackrel{\text{p}}{=} \dot{z} \mathrel{\dot{\Rightarrow}} \dot{\forall} \dot{x} \colon \dot{x} \stackrel{\text{p}}{=} \dot{y} \rceil)]^{\cdot}$

$[\dot{\text{nonfree}}(\lceil \dot{x} \rceil, \lceil \dot{x} \stackrel{\text{p}}{=} \dot{z} \mathrel{\dot{\Rightarrow}} \dot{\forall} \dot{x} \colon \dot{x} \stackrel{\text{p}}{=} \dot{y} \rceil)]^{-}$

$[\dot{\text{nonfree}}(\lceil \dot{x} \rceil, \lceil \dot{y} \stackrel{\text{p}}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{\forall} \dot{x} \colon \dot{x} \stackrel{\text{p}}{=} \dot{y} \rceil)]^{-}$

$[\dot{\text{nonfree}}(\lceil \dot{x} \rceil, \lceil \dot{y} \stackrel{\text{p}}{=} \dot{z} \mathrel{\dot{\Rightarrow}} \dot{\forall} \dot{y} \colon \dot{x} \stackrel{\text{p}}{=} \dot{y} \rceil)]^{-}$

$[\dot{\text{free}}\langle \lceil \dot{\forall} \dot{x} \colon \mathsf{b} :: \dot{x} :: \mathsf{c} \rceil | \lceil \dot{x} \rceil := \lceil \mathsf{x} :: \dot{y} :: \mathsf{z} \rceil \rangle]^{\cdot}$

$[\dot{\text{free}}\langle \lceil \dot{\forall} \dot{y} \colon \mathsf{b} :: \dot{x} :: \mathsf{c} \rceil | \lceil \dot{x} \rceil := \lceil \mathsf{x} :: \dot{y} :: \mathsf{z} \rceil \rangle]^{-}$

$[\dot{\mathrm{free}}\langle\lceil\dot\forall\dot x\colon \mathsf{b} :: \dot x :: \mathsf{c}\rceil|\lceil\dot y\rceil := \lceil\mathsf{x} :: \dot y :: \mathsf{z}\rceil\rangle]^{\cdot}$

$[\dot{\mathrm{free}}\langle\lceil\dot\forall\dot y\colon \mathsf{b} :: \dot x :: \mathsf{c}\rceil|\lceil\dot y\rceil := \lceil\mathsf{x} :: \dot y :: \mathsf{z}\rceil\rangle]^{\cdot}$

$[\dot a\equiv\langle\dot a|\dot b := \dot c\rangle]^{\cdot}$

$[\dot c\equiv\langle\dot b|\dot b := \dot c\rangle]^{\cdot}$

$[\forall\dot a\colon \dot a \stackrel{\mathrm{p}}{=} \dot b\equiv\langle\forall\dot a\colon \dot a \stackrel{\mathrm{p}}{=} \dot b|\dot a := \dot c\rangle]^{\cdot}$

$[\forall\dot a\colon \dot a \stackrel{\mathrm{p}}{=} \dot c\equiv\langle\forall\dot a\colon \dot a \stackrel{\mathrm{p}}{=} \dot b|\dot b := \dot c\rangle]^{\cdot}$

$[\dot\forall\dot a\colon \dot a \stackrel{\mathrm{p}}{=} \dot 0 \dot+ \dot a \dot\Rightarrow \dot c \dot{:} \dot d \stackrel{\mathrm{p}}{=} \dot 0 \dot+ \dot c \dot{:} \dot d\equiv\langle\dot\forall\dot a\colon \dot a \stackrel{\mathrm{p}}{=} \dot 0 \dot+ \dot a \dot\Rightarrow \dot b \stackrel{\mathrm{p}}{=} \dot 0 \dot+ \dot b|\dot b := \dot c \dot{:} \dot d\rangle]^{\cdot}$

$[\dot\forall\dot a\colon \dot a \stackrel{\mathrm{p}}{=} \dot 0 \dot+ \dot a \dot\Rightarrow \dot b \stackrel{\mathrm{p}}{=} \dot 0 \dot+ \dot b\equiv\langle\dot\forall\dot a\colon \dot a \stackrel{\mathrm{p}}{=} \dot 0 \dot+ \dot a \dot\Rightarrow \dot b \stackrel{\mathrm{p}}{=} \dot 0 \dot+ \dot b|\dot a := \dot c\rangle]^{\cdot}$

A.5 Priority table

Priority table
Preassociative
[peano], [base], [bracket $*$ end bracket], [big bracket $*$ end bracket], [math $*$ end math], [**flush left** [$*$]], [x], [y], [z], [[$* \bowtie *$]], [[$* \stackrel{*}{\rightarrow} *$]], [pyk], [tex], [name], [prio], [$*$], [T], [if($*,*,*$)], [[$* \stackrel{*}{\Rightarrow} *$]], [val], [claim], [$\bot$], [f($*$)], [$(*)^{\mathrm{I}}$], [F], [$\underline{0}$], [$\underline{1}$], [$\underline{2}$], [$\underline{3}$], [$\underline{4}$], [$\underline{5}$], [$\underline{6}$], [$\underline{7}$], [$\underline{8}$], [$\underline{9}$], [0], [1], [2], [3], [4], [5], [6], [7], [8], [9], [a], [b], [c], [d], [e], [f], [g], [h], [i], [j], [k], [l], [m], [n], [o], [p], [q], [r], [s], [t], [u], [v], [w], [$(*)^{\mathrm{M}}$], [If($*,*,*$)], [array{$*$} $*$ end array], [l], [c], [r], [empty], [$\langle * \mid * := * \rangle$], [$\mathcal{M}(*)$], [$\tilde{\mathcal{U}}(*)$], [$\mathcal{U}(*)$], [$\mathcal{U}^{\mathrm{M}}(*)$], [**apply**($*,*$)], [$\mathbf{apply}_1(*,*)$], [identifier($*$)], [$\mathrm{identifier}_1(*,*)$], [array-plus($*,*$)], [array-remove($*,*,*$)], [array-put($*,*,*,*$)], [array-add($*,*,*,*,*$)], [bit($*,*$)], [$\mathrm{bit}_1(*,*)$], [rack], ["vector"], ["bibliography"], ["dictionary"], ["body"], ["codex"], ["expansion"], ["code"], ["cache"], ["diagnose"], ["pyk"], ["tex"], ["texname"], ["value"], ["message"], ["macro"], ["definition"], ["unpack"], ["claim"], ["priority"], ["lambda"], ["apply"], ["true"], ["if"], ["quote"], ["proclaim"], ["define"], ["introduce"], ["hide"], ["pre"], ["post"], [$\mathcal{E}(*,*,*)$], [$\mathcal{E}_2(*,*,*,*,*)$], [$\mathcal{E}_3(*,*,*,*)$], [$\mathcal{E}_4(*,*,*,*)$], [**lookup**($*,*,*$)], [**abstract**($*,*,*,*$)], [[$\lceil *\rceil$]], [$\mathcal{M}(*,*,*)$], [$\mathcal{M}_2(*,*,*,*)$], [$\mathcal{M}^*(*,*,*)$], [macro], [s_0], [**zip**($*,*$)], [$\mathbf{assoc}_1(*,*,*)$], [$(*)^{\mathbf{P}}$], [self], [[$* \ddot{=} *$]], [[$* \dot{=} *$]], [[$* \acute{=} *$]], [[$* \stackrel{\mathrm{pyk}}{=} *$]], [[$* \stackrel{\mathrm{tex}}{=} *$]], [[$* \stackrel{\mathrm{name}}{=} *$]], [**Priority table**[$*$]], [$\tilde{\mathcal{M}}_1$], [$\tilde{\mathcal{M}}_2(*)$], [$\tilde{\mathcal{M}}_3(*)$], [$\tilde{\mathcal{M}}_4(*,*,*,*)$], [$\tilde{\mathcal{M}}(*,*,*)$], [$\tilde{\mathcal{Q}}(*,*,*)$], [$\tilde{\mathcal{Q}}_2(*,*,*)$], [$\tilde{\mathcal{Q}}_3(*,*,*,*)$], [$\tilde{\mathcal{Q}}^*(*,*,*)$], [($*$)], [**aspect**($*,*$)], [**aspect**($*,*,*$)], [$\langle * \rangle$], [$\mathbf{tuple}_1(*)$], [$\mathbf{tuple}_2(*)$], [$\mathrm{let}_2(*,*)$], [$\mathrm{let}_1(*,*)$], [[$* \stackrel{\mathrm{claim}}{=} *$]], [checker], [**check**($*,*$)], [$\mathbf{check}_2(*,*,*)$], [$\mathbf{check}_3(*,*,*)$], [$\mathbf{check}^*(*,*)$], [$\mathbf{check}^*_2(*,*,*)$], [$[*]^{\cdot}$], [$[*]^{-}$], [$[*]^{\circ}$], [msg], [[$* \stackrel{\mathrm{msg}}{=} *$]], [<stmt>], [stmt], [[$* \stackrel{\mathrm{stmt}}{=} *$]], [HeadNil′], [HeadPair′], [Transitivity′], [$\bot\!\!\!\bot$], [Contra′], [T'_{E}], [L_1], [$\underline{*}$], [$\mathcal{A}$], [$\mathcal{B}$], [$\mathcal{C}$], [$\mathcal{D}$], [$\mathcal{E}$], [$\mathcal{F}$], [$\mathcal{G}$], [$\mathcal{H}$], [$\mathcal{I}$], [$\mathcal{J}$], [$\mathcal{K}$], [$\mathcal{L}$], [$\mathcal{M}$], [$\mathcal{N}$], [$\mathcal{O}$], [$\mathcal{P}$], [$\mathcal{Q}$], [$\mathcal{R}$], [$\mathcal{S}$], [$\mathcal{T}$], [$\mathcal{U}$], [$\mathcal{V}$], [$\mathcal{W}$], [$\mathcal{X}$], [$\mathcal{Y}$], [$\mathcal{Z}$], [$\langle * \mid * := * \rangle$], [$\langle^* * \mid * := * \rangle$], [$\emptyset$], [Remainder], [$(*)^{\mathbf{v}}$], [intro($*,*,*,*$)], [intro($*,*,*$)], [error($*,*$)], [$\mathrm{error}_2(*,*)$], [proof($*,*,*$)],

[proof$_2$(∗, ∗)], [$\mathcal{S}$(∗, ∗)], [$\mathcal{S}^{\mathrm{I}}$(∗, ∗)], [$\mathcal{S}^{\rhd}$(∗, ∗)], [$\mathcal{S}_1^{\rhd}$(∗, ∗, ∗)], [$\mathcal{S}^{\mathrm{E}}$(∗, ∗)], [$\mathcal{S}_1^{\mathrm{E}}$(∗, ∗, ∗)],
[$\mathcal{S}^{+}$(∗, ∗)], [$\mathcal{S}_1^{+}$(∗, ∗, ∗)], [$\mathcal{S}^{-}$(∗, ∗)], [$\mathcal{S}_1^{-}$(∗, ∗, ∗)], [$\mathcal{S}^{*}$(∗, ∗)], [$\mathcal{S}_1^{*}$(∗, ∗, ∗)],
[$\mathcal{S}_2^{*}$(∗, ∗, ∗, ∗)], [$\mathcal{S}^{@}$(∗, ∗)], [$\mathcal{S}_1^{@}$(∗, ∗, ∗)], [$\mathcal{S}^{\vdash}$(∗, ∗)], [$\mathcal{S}_1^{\vdash}$(∗, ∗, ∗, ∗)], [$\mathcal{S}^{\Vdash}$(∗, ∗)],
[$\mathcal{S}_1^{\Vdash}$(∗, ∗, ∗, ∗)], [$\mathcal{S}^{\mathrm{i.e.}}$(∗, ∗)], [$\mathcal{S}_1^{\mathrm{i.e.}}$(∗, ∗, ∗, ∗)], [$\mathcal{S}_2^{\mathrm{i.e.}}$(∗, ∗, ∗, ∗, ∗)], [$\mathcal{S}^{\forall}$(∗, ∗)],
[$\mathcal{S}_1^{\forall}$(∗, ∗, ∗, ∗)], [$\mathcal{S}^{;}$(∗, ∗)], [$\mathcal{S}_1^{;}$(∗, ∗, ∗)], [$\mathcal{S}_2^{;}$(∗, ∗, ∗, ∗)], [$\mathcal{T}$(∗)], [claims(∗, ∗, ∗)],
[claims$_2$(∗, ∗, ∗)], [<proof>], [proof], [[**Lemma** ∗: ∗]], [[**Proof of** ∗: ∗]],
[[∗ **lemma** ∗: ∗]], [[∗ **antilemma** ∗: ∗]], [[∗ **rule** ∗: ∗]], [[∗ **antirule** ∗: ∗]],
[verifier], [$\mathcal{V}_1$(∗)], [$\mathcal{V}_2$(∗, ∗)], [$\mathcal{V}_3$(∗, ∗, ∗, ∗)], [$\mathcal{V}_4$(∗, ∗)], [$\mathcal{V}_5$(∗, ∗, ∗, ∗)], [$\mathcal{V}_6$(∗, ∗, ∗, ∗)],
[$\mathcal{V}_7$(∗, ∗, ∗, ∗)], [Cut(∗, ∗)], [Head$_\oplus$(∗)], [Tail$_\oplus$(∗)], [rule$_1$(∗, ∗)], [rule(∗, ∗)],
[Rule tactic], [Plus(∗, ∗)], [[**Theory** ∗]], [theory$_2$(∗, ∗)], [theory$_3$(∗, ∗)],
[theory$_4$(∗, ∗, ∗)], [HeadNil″], [HeadPair″], [Transitivity″], [Contra″], [HeadNil],
[HeadPair], [Transitivity], [Contra], [T$_{\mathrm{E}}$], [ragged right],
[ragged right expansion], [parm(∗, ∗, ∗)], [parm*(∗, ∗, ∗)], [inst(∗, ∗)],
[inst*(∗, ∗)], [occur(∗, ∗, ∗)], [occur*(∗, ∗, ∗)], [unify(∗ = ∗, ∗)], [unify*(∗ = ∗, ∗)],
[unify$_2$(∗ = ∗, ∗)], [L$_{\mathrm{a}}$], [L$_{\mathrm{b}}$], [L$_{\mathrm{c}}$], [L$_{\mathrm{d}}$], [L$_{\mathrm{e}}$], [L$_{\mathrm{f}}$], [L$_{\mathrm{g}}$], [L$_{\mathrm{h}}$], [L$_{\mathrm{i}}$], [L$_{\mathrm{j}}$], [L$_{\mathrm{k}}$], [L$_{\mathrm{l}}$], [L$_{\mathrm{m}}$],
[L$_{\mathrm{n}}$], [L$_{\mathrm{o}}$], [L$_{\mathrm{p}}$], [L$_{\mathrm{q}}$], [L$_{\mathrm{r}}$], [L$_{\mathrm{s}}$], [L$_{\mathrm{t}}$], [L$_{\mathrm{u}}$], [L$_{\mathrm{v}}$], [L$_{\mathrm{w}}$], [L$_{\mathrm{x}}$], [L$_{\mathrm{y}}$], [L$_{\mathrm{z}}$], [L$_{\mathrm{A}}$], [L$_{\mathrm{B}}$], [L$_{\mathrm{C}}$],
[L$_{\mathrm{D}}$], [L$_{\mathrm{E}}$], [L$_{\mathrm{F}}$], [L$_{\mathrm{G}}$], [L$_{\mathrm{H}}$], [L$_{\mathrm{I}}$], [L$_{\mathrm{J}}$], [L$_{\mathrm{K}}$], [L$_{\mathrm{L}}$], [L$_{\mathrm{M}}$], [L$_{\mathrm{N}}$], [L$_{\mathrm{O}}$], [L$_{\mathrm{P}}$], [L$_{\mathrm{Q}}$], [L$_{\mathrm{R}}$],
[L$_{\mathrm{S}}$], [L$_{\mathrm{T}}$], [L$_{\mathrm{U}}$], [L$_{\mathrm{V}}$], [L$_{\mathrm{W}}$], [L$_{\mathrm{X}}$], [L$_{\mathrm{Y}}$], [L$_{\mathrm{Z}}$], [L$_{?}$], [Reflexivity], [Reflexivity$_1$],
[Commutativity], [Commutativity$_1$], [<tactic>], [tactic], [[∗ $\stackrel{\text{tactic}}{=}$ ∗]], [$\mathcal{P}$(∗, ∗, ∗)],
[$\mathcal{P}^*$(∗, ∗, ∗)], [p$_0$], [conclude$_1$(∗, ∗)], [conclude$_2$(∗, ∗, ∗)], [conclude$_3$(∗, ∗, ∗, ∗)], [$\dot{0}$],
[$\dot{1}$], [$\dot{2}$], [$\dot{a}$], [$\dot{b}$], [$\dot{c}$], [$\dot{d}$], [$\dot{e}$], [$\dot{f}$], [$\dot{g}$], [$\dot{h}$], [$\dot{i}$], [$\dot{j}$], [$\dot{k}$], [$\dot{l}$], [$\dot{m}$], [$\dot{n}$], [$\dot{o}$], [$\dot{p}$], [$\dot{q}$], [$\dot{r}$], [$\dot{s}$], [$\dot{t}$],
[$\dot{u}$], [$\dot{v}$], [$\dot{w}$], [$\dot{x}$], [$\dot{y}$], [$\dot{z}$], [nonfree(∗, ∗)], [nonfree*(∗, ∗)], [free⟨∗|∗ := ∗⟩],
[free*⟨∗|∗ := ∗⟩], [∗≡⟨∗|∗ := ∗⟩], [∗≡⟨*∗|∗ := ∗⟩], [S], [A1], [A2], [A3], [A4], [A5],
[S1], [S2], [S3], [S4], [S5], [S6], [S7], [S8], [S9], [MP], [Gen], [L3.2(a)], [S′], [A1′],
[A2′], [A3′], [A4′], [A5′], [S1′], [S2′], [S3′], [S4′], [S5′], [S6′], [S7′], [S8′], [S9′], [MP′],
[Gen′], [L3.2(a)′];

Preassociative

[∗_{∗}], [∗′], [∗[∗]], [∗[∗→∗]], [∗[∗⇒∗]], [$\dot{*}$];

Preassociative

[“ ∗ ”], [], [(∗)$^{\mathbf{t}}$], [string(∗) + ∗], [string(∗) ++ ∗], [
∗], [∗], [!∗], [”∗], [#∗], [$∗], [%∗], [&∗], [’∗], [(∗], [)∗], [∗∗], [+∗], [, ∗], [-∗], [.∗], [/∗],
[0∗], [1∗], [2∗], [3∗], [4∗], [5∗], [6∗], [7∗], [8∗], [9∗], [:∗], [; ∗], [<∗], [=∗], [>∗], [?∗],
[@∗], [A∗], [B∗], [C∗], [D∗], [E∗], [F∗], [G∗], [H∗], [I∗], [J∗], [K∗], [L∗], [M∗], [N∗],
[O∗], [P∗], [Q∗], [R∗], [S∗], [T∗], [U∗], [V∗], [W∗], [X∗], [Y∗], [Z∗], [[∗], [\∗], []∗], [ˆ∗],
[_∗], [‘∗], [a∗], [b∗], [c∗], [d∗], [e∗], [f∗], [g∗], [h∗], [i∗], [j∗], [k∗], [l∗], [m∗], [n∗], [o∗],
[p∗], [q∗], [r∗], [s∗], [t∗], [u∗], [v∗], [w∗], [x∗], [y∗], [z∗], [{∗], [|∗], [}∗], [˜∗],
[**Preassociative** ∗; ∗], [**Postassociative** ∗; ∗], [[∗], ∗], [priority ∗ end],
[newline ∗], [macro newline ∗];

Preassociative

[∗0], [∗1], [0b], [∗-color(∗)], [∗-color*(∗)];

Preassociative

[∗ ’ ∗], [∗ ‘ ∗];

Preassociative

[∗$^{\mathrm{H}}$], [∗$^{\mathrm{T}}$], [∗$^{\mathrm{U}}$], [∗$^{\mathrm{h}}$], [∗$^{\mathrm{t}}$], [∗$^{\mathrm{s}}$], [∗$^{\mathrm{c}}$], [∗$^{\mathrm{d}}$], [∗$^{\mathrm{a}}$], [∗$^{\mathrm{C}}$], [∗$^{\mathrm{M}}$], [∗$^{\mathrm{B}}$], [∗$^{\mathrm{r}}$], [∗$^{\mathrm{i}}$], [∗$^{\mathrm{d}}$], [∗$^{\mathrm{R}}$], [∗^{0}],
[∗^{1}], [∗^{2}], [∗^{3}], [∗^{4}], [∗^{5}], [∗^{6}], [∗^{7}], [∗^{8}], [∗^{9}], [∗$^{\mathrm{E}}$], [∗$^{\mathcal{V}}$], [∗$^{\mathcal{C}}$], [∗$^{\mathcal{C}^*}$], [∗′];

Preassociative

$[* \cdot *], [* \cdot_0 *], [* \mathrel{\dot{\cdot}} *];$

Preassociative

$[* + *], [* +_0 *], [* +_1 *], [* - *], [* -_0 *], [* -_1 *], [* \dot{+} *];$

Preassociative

$[* \cup \{*\}], [* \cup *], [* \backslash \{*\}];$

Postassociative

$[* \mathrel{\therefore} *], [* \underline{\therefore} *], [* \underline{::} *], [* \underline{+2*} *], [* :: *], [* +2* *];$

Postassociative

$[*, *];$

Preassociative

$[* \stackrel{\mathrm{B}}{\approx} *], [* \stackrel{\mathrm{D}}{\approx} *], [* \stackrel{\mathrm{C}}{\approx} *], [* \stackrel{\mathrm{P}}{\approx} *], [* \approx *], [* = *], [* \stackrel{+}{\rightarrow} *], [* \stackrel{\mathrm{t}}{=} *], [* \stackrel{\mathrm{t}^*}{=} *], [* \stackrel{\mathrm{r}}{=} *],$
$[* \in_{\mathrm{t}} *], [* \subseteq_{\mathrm{T}} *], [* \stackrel{\mathrm{T}}{=} *], [* \stackrel{\mathrm{s}}{=} *], [* \text{ free in } *], [* \text{ free in}^* *], [* \text{ free for } * \text{ in } *],$
$[* \text{ free for}^* * \text{ in } *], [* \in_{\mathrm{c}} *], [* < *], [* <' *], [* \leq' *], [* \stackrel{\mathrm{p}}{=} *], [*^{\mathcal{P}}];$

Preassociative

$[\neg *], [\dot{\neg} *];$

Preassociative

$[* \wedge *], [* \ddot{\wedge} *], [* \tilde{\wedge} *], [* \wedge_{\mathrm{c}} *], [* \dot{\wedge} *];$

Preassociative

$[* \vee *], [* \parallel *], [* \ddot{\vee} *], [* \dot{\vee} *];$

Preassociative

$[\dot{\forall} * \colon *], [\dot{\exists} * \colon *];$

Postassociative

$[* \ddot{\Rightarrow} *], [* \dot{\Rightarrow} *], [* \dot{\Leftrightarrow} *];$

Postassociative

$[* : *], [*!*];$

Preassociative

$[* \begin{cases} * \\ * \end{cases}];$

Preassociative

$[\lambda * . *], [\Lambda *], [\mathbf{if}\ *\ \mathbf{then}\ *\ \mathbf{else}\ *], [\mathbf{let}\ * = *\ \mathbf{in}\ *], [\mathbf{let}\ * \stackrel{..}{=} *\ \text{ in }\ *];$

Preassociative

$[*^{\mathrm{I}}], [*^{\triangleright}], [*^{\mathrm{V}}], [*^{+}], [*^{-}], [*^{*}];$

Preassociative

$[* @ *], [* \triangleright *], [* \,\|\!\!\triangleright *], [* \gg *];$

Postassociative

$[* \vdash *], [* \Vdash *], [* \text{ i.e. } *];$

Preassociative

$[\forall * \colon *];$

Postassociative

$[* \oplus *];$

Postassociative

$[*; *];$

Preassociative

$[* \text{ proves } *];$

Preassociative
[$*$ **proof of** $* : *$], [Line $* : * \gg *; *$], [Last line $* \gg * \square$],
[Line $*$: Premise $\gg *; *$], [Line $*$: Side-condition $\gg *; *$], [Arbitrary $\gg *; *$],
[Local $\gg * = *; *$];
Postassociative
[$*$ then $*$], [$*[*]*$];
Preassociative
[$*\&*$];
Preassociative
[$*\backslash\backslash*$]; **End table**

B Index

* is peano var [$\mathsf{x}^{\mathcal{P}}$], 2
* peano and * [$\mathsf{x} \dot{\wedge} \mathsf{y}$], 2
* peano iff * [$\mathsf{x} \dot{\Leftrightarrow} \mathsf{y}$], 2
* peano imply * [$\mathsf{x} \dot{\Rightarrow} \mathsf{y}$], 2
* peano is * [$\mathsf{x} \stackrel{\mathrm{p}}{=} \mathsf{y}$], 2
* peano or * [$\mathsf{x} \dot{\vee} \mathsf{y}$], 2
* peano plus * [$\mathsf{x} \dot{+} \mathsf{y}$], 1
* peano succ [x'], 1
* peano times * [$\mathsf{x} \dot{\cdot} \mathsf{y}$], 1
* peano var [$\dot{\mathsf{x}}$], 2
[$\mathsf{x} \dot{\wedge} \mathsf{y}$] * peano and *, 2
[$\mathsf{x} \dot{\Leftrightarrow} \mathsf{y}$] * peano iff *, 2
[$\mathsf{x} \dot{\Rightarrow} \mathsf{y}$] * peano imply *, 2
[$\mathsf{x} \dot{\vee} \mathsf{y}$] * peano or *, 2
[$\mathsf{x} \dot{+} \mathsf{y}$] * peano plus *, 1
[x'] * peano succ, 1
[$\mathsf{x} \dot{\cdot} \mathsf{y}$] * peano times *, 1
[$\dot{\mathsf{x}}$] * peano var, 2
[$\dot{\forall}\mathsf{x}\colon \mathsf{y}$] peano all * indeed *, 2
[$\dot{\exists}\mathsf{x}\colon \mathsf{y}$] peano exist * indeed *, 2
[$\dot{\neg}\, \mathsf{x}$] peano not *, 2
[$\dot{1}$] peano one, 2
[$\dot{2}$] peano two, 2
[$\dot{0}$] peano zero, 1

A1': [A1$'$] axiom prime a one, 6
A1: [A1] axiom a one, 4
A2': [A2$'$] axiom prime a two, 6
A2: [A2] axiom a two, 4
A3': [A3$'$] axiom prime a three, 6
A3: [A3] axiom a three, 4
A4': [A4$'$] axiom prime a four, 6

A4: [A4] axiom a four, 4
A5': [A5$'$] axiom prime a five, 6
A5: [A5] axiom a five, 4
a: [$\dot{a}$] peano a, 2
axiom a five [A5], 4
axiom a four [A4], 4
axiom a one [A1], 4
axiom a three [A3], 4
axiom a two [A2], 4
axiom prime a five [A5$'$], 6
axiom prime a four [A4$'$], 6
axiom prime a one [A1$'$], 6
axiom prime a three [A3$'$], 6
axiom prime a two [A2$'$], 6
axiom prime s eight [S8$'$], 6
axiom prime s five [S5$'$], 6
axiom prime s four [S4$'$], 6
axiom prime s nine [S9$'$], 6
axiom prime s one [S1$'$], 6
axiom prime s seven [S7$'$], 6
axiom prime s six [S6$'$], 6
axiom prime s three [S3$'$], 6
axiom prime s two [S2$'$], 6
axiom s eight [S8], 4
axiom s five [S5], 4
axiom s four [S4], 4
axiom s nine [S9], 4
axiom s one [S1], 4
axiom s seven [S7], 4
axiom s six [S6], 4
axiom s three [S3], 4
axiom s two [S2], 4

b: [$\dot{b}$] peano b, 8

c: [$\dot{c}$] peano c, 8

d: [$\dot{d}$] peano d, 8

e: [$\dot{e}$] peano e, 8

f: [$\dot{f}$] peano f, 8
free*: [$\dot{\text{free}}^*\langle \mathsf{a},\ \mathsf{x} := \mathsf{b}\rangle$] peano free star * set * to * end free3
free: [$\dot{\text{free}}\langle \mathsf{a},\ \mathsf{x} := \mathsf{b}\rangle$] peano free * set * to * end free3

g: [$\dot{g}$] peano g, 8
Gen': [Gen$'$] rule prime gen, 6

Gen: [Gen] rule gen, 4

h: [$\dot{h}$] peano h, 8

i: [$\dot{i}$] peano i, 8

j: [$\dot{j}$] peano j, 8

k: [$\dot{k}$] peano k, 8

L3.2(a)’: [L3.2(a)′] lemma prime l three two a, 7
L3.2(a): [L3.2(a)] lemma l three two a, 5
l: [$\dot{l}$] peano l, 8
lemma l three two a [L3.2(a)], 5
lemma prime l three two a [L3.2(a)′], 7

m: [$\dot{m}$] peano m, 8
MP’: [MP′] rule prime mp, 6
MP: [MP] rule mp, 4

n: [$\dot{n}$] peano n, 8
nonfree*: [$\dot{\mathrm{nonfree}}^{*}(\mathsf{x}, \mathsf{y})$] peano nonfree star * in * end nonfree, 3
nonfree: [$\dot{\mathrm{nonfree}}(\mathsf{x}, \mathsf{y})$] peano nonfree * in * end nonfree, 2

o: [$\dot{o}$] peano o, 8

P: [$\mathsf{x}^{\mathcal{P}}$] * is peano var, 2
p: [$\mathsf{x} \stackrel{\mathrm{p}}{=} \mathsf{y}$] * peano is *, 2
p: [$\dot{p}$] peano p, 8
peano a [$\dot{a}$], 2
peano all * indeed * [$\dot{\forall}\mathsf{x}\colon \mathsf{y}$], 2
peano b [$\dot{b}$], 8
peano c [$\dot{c}$], 8
peano d [$\dot{d}$], 8
peano e [$\dot{e}$], 8
peano exist * indeed * [$\dot{\exists}\mathsf{x}\colon \mathsf{y}$], 2
peano f [$\dot{f}$], 8
peano free * set * to * end free [$\dot{\mathrm{free}}\langle \mathsf{a},\ \mathsf{x} := \mathsf{b}\rangle$]3
peano free star * set * to * end free [$\dot{\mathrm{free}}^{*}\langle \mathsf{a},\ \mathsf{x} := \mathsf{b}\rangle$]3
peano g [$\dot{g}$], 8
peano h [$\dot{h}$], 8
peano i [$\dot{i}$], 8
peano j [$\dot{j}$], 8
peano k [$\dot{k}$], 8
peano l [$\dot{l}$], 8
peano m [$\dot{m}$], 8
peano n [$\dot{n}$], 8

peano nonfree * in * end nonfree [$\dot{\text{nonfree}}(\mathsf{x},\mathsf{y})$], 2
peano nonfree star * in * end nonfree [$\dot{\text{nonfree}}^{*}(\mathsf{x},\mathsf{y})$], 3
peano not * [$\dot{\neg}\,\mathsf{x}$], 2
peano o [$\dot{o}$], 8
peano one [$\dot{1}$], 2
peano p [$\dot{p}$], 8
peano q [$\dot{q}$], 8
peano r [$\dot{r}$], 8
peano s [$\dot{s}$], 8
peano sub * is * where * is * end sub [$\mathsf{a}{\equiv}\langle\mathsf{b},\ \mathsf{x} := \mathsf{c}\rangle$]3
peano sub star * is * where * is * end sub [$\mathsf{a}{\equiv}\langle^{*}\mathsf{b},\ \mathsf{x} := \mathsf{c}\rangle$]3
peano t [$\dot{t}$], 8
peano two [$\dot{2}$], 2
peano u [$\dot{u}$], 8
peano v [$\dot{v}$], 8
Peano variable, 2
peano w [$\dot{w}$], 8
peano x [$\dot{x}$], 8
peano y [$\dot{y}$], 8
peano z [$\dot{z}$], 8
peano zero [$\dot{0}$], 1

q: [$\dot{q}$] peano q, 8

r: [$\dot{r}$] peano r, 8
rule gen [Gen], 4
rule mp [MP], 4
rule prime gen [Gen′], 6
rule prime mp [MP′], 6

S’: [S′] system prime s, 6
S1’: [S1′] axiom prime s one, 6
S1: [S1] axiom s one, 4
S2’: [S2′] axiom prime s two, 6
S2: [S2] axiom s two, 4
S3’: [S3′] axiom prime s three, 6
S3: [S3] axiom s three, 4
S4’: [S4′] axiom prime s four, 6
S4: [S4] axiom s four, 4
S5’: [S5′] axiom prime s five, 6
S5: [S5] axiom s five, 4
S6’: [S6′] axiom prime s six, 6
S6: [S6] axiom s six, 4
S7’: [S7′] axiom prime s seven, 6
S7: [S7] axiom s seven, 4
S8’: [S8′] axiom prime s eight, 6

S8: [S8] axiom s eight, 4
S9': [S9′] axiom prime s nine, 6
S9: [S9] axiom s nine, 4
s: [$\dot{s}$] peano s, 8
S: [S] system s, 4
system prime s [S′], 6
system s [S], 4

t: [$\dot{t}$] peano t, 8

u: [$\dot{u}$] peano u, 8

v: [$\dot{v}$] peano v, 8
variable, Peano, 2

w: [$\dot{w}$] peano w, 8

x: [$\dot{x}$] peano x, 8

y: [$\dot{y}$] peano y, 8

z: [$\dot{z}$] peano z, 8

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