Peano arithmetic

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Contents

1 Peano arithmetic

This Logiweb page [1] defines Peano arithmetic.

1.1 The constructs of Peano arithmetic

Terms of Peano arithmetic are constructed from zero $[\dot{0}]$[1], successor $[x']$[2], plus $[x \dot{+} y]$[3], and times $[x \dot{\cdot} y]$[4].

[1] $[\dot{0} \stackrel{\text{pyk}}{=}$ "peano zero"$]$

[2] $[x' \stackrel{\text{pyk}}{=}$ "* peano succ"$]$

[3] $[x \dot{+} y \stackrel{\text{pyk}}{=}$ "* peano plus *"$]$

[4] $[x \dot{\cdot} y \stackrel{\text{pyk}}{=}$ "* peano times *"$]$

Formulas of Peano arithmeric are constructed from equality $[\mathsf{x} \stackrel{\mathrm{p}}{=} \mathsf{y}]$[5], negation $[\dot{\neg}\,\mathsf{x}]$[6], implication $[\mathsf{x} \dot{\Rightarrow} \mathsf{y}]$[7], and universal quantification $[\dot{\forall}\mathsf{x}\colon \mathsf{y}]$[8].

From these constructs we macro define one $[\dot{1}]$[9], two $[\dot{2}]$[10], conjunction $[\mathsf{x} \dot{\wedge} \mathsf{y}]$[11], disjunction $[\mathsf{x} \dot{\vee} \mathsf{y}]$[12], biimplication $[\mathsf{x} \dot{\Leftrightarrow} \mathsf{y}]$[13], and existential quantification $[\dot{\exists}\mathsf{x}\colon \mathsf{y}]$[14]:

$$[\dot{1} \mathrel{\ddot{=}} \dot{0}']$$

$$[\dot{2} \mathrel{\ddot{=}} \dot{1}']$$

$$[\mathsf{x} \dot{\wedge} \mathsf{y} \mathrel{\ddot{=}} \dot{\neg}\,(\mathsf{x} \dot{\Rightarrow} \dot{\neg}\,\mathsf{y})]$$

$$[\mathsf{x} \dot{\vee} \mathsf{y} \mathrel{\ddot{=}} \dot{\neg}\,\mathsf{x} \dot{\Rightarrow} \mathsf{y}]$$

$$[\mathsf{x} \dot{\Leftrightarrow} \mathsf{y} \mathrel{\ddot{=}} (\mathsf{x} \dot{\Rightarrow} \mathsf{y}) \dot{\wedge} (\mathsf{y} \dot{\Rightarrow} \mathsf{x})]$$

$$[\dot{\exists}\mathsf{x}\colon \mathsf{y} \mathrel{\ddot{=}} \dot{\neg}\,\dot{\forall}\mathsf{x}\colon \dot{\neg}\,\mathsf{y}]$$

1.2 Variables

We now introduce the unary operator $[\dot{\mathsf{x}}]$[15] and define that a term is a *Peano variable* (i.e. a variable of Peano arithmetic) if it has the $[\dot{\mathsf{x}}]$ operator in its root. $[\mathsf{x}^{\mathcal{P}}]$[16] is true if $[\mathsf{x}]$ is a Peano variable:

$$[\mathsf{x}^{\mathcal{P}} \mathrel{\dot{=}} \mathsf{x} \stackrel{\mathrm{r}}{=} \lceil \dot{\mathsf{x}} \rceil]$$

We macro define $[\dot{a}]$[17] to be a Peano variable:

$$[\dot{a} \mathrel{\ddot{=}} \dot{\mathsf{a}}]$$

Appendix A.2 defines Peano variables for the other letters of the English alphabet.

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[5] $[\mathsf{x} \stackrel{\mathrm{p}}{=} \mathsf{y} \stackrel{\mathrm{pyk}}{=}$ "∗ peano is ∗"$]$

[6] $[\dot{\neg}\,\mathsf{x} \stackrel{\mathrm{pyk}}{=}$ "peano not ∗"$]$

[7] $[\mathsf{x} \dot{\Rightarrow} \mathsf{y} \stackrel{\mathrm{pyk}}{=}$ "∗ peano imply ∗"$]$

[8] $[\dot{\forall}\mathsf{x}\colon \mathsf{y} \stackrel{\mathrm{pyk}}{=}$ "peano all ∗ indeed ∗"$]$

[9] $[\dot{1} \stackrel{\mathrm{pyk}}{=}$ "peano one"$]$

[10] $[\dot{2} \stackrel{\mathrm{pyk}}{=}$ "peano two"$]$

[11] $[\mathsf{x} \dot{\wedge} \mathsf{y} \stackrel{\mathrm{pyk}}{=}$ "∗ peano and ∗"$]$

[12] $[\mathsf{x} \dot{\vee} \mathsf{y} \stackrel{\mathrm{pyk}}{=}$ "∗ peano or ∗"$]$

[13] $[\mathsf{x} \dot{\Leftrightarrow} \mathsf{y} \stackrel{\mathrm{pyk}}{=}$ "∗ peano iff ∗"$]$

[14] $[\dot{\exists}\mathsf{x}\colon \mathsf{y} \stackrel{\mathrm{pyk}}{=}$ "peano exist ∗ indeed ∗"$]$

[15] $[\dot{\mathsf{x}} \stackrel{\mathrm{pyk}}{=}$ "∗ peano var"$]$

[16] $[\mathsf{x}^{\mathcal{P}} \stackrel{\mathrm{pyk}}{=}$ "∗ is peano var"$]$

[17] $[\dot{a} \stackrel{\mathrm{pyk}}{=}$ "peano a"$]$

$[\dot{\mathrm{nonfree}}(\mathsf{x},\mathsf{y})]$[18] is true if the Peano variable $[\mathsf{x}]$ does not occur free in the Peano term/formula $[\mathsf{y}]$. $[\dot{\mathrm{nonfree}}^{*}(\mathsf{x},\mathsf{y})]$[19] is true if the Peano variable $[\mathsf{x}]$ does not occur free in the list $[\mathsf{y}]$ of Peano terms/formulas.

$$[\dot{\mathrm{nonfree}}(\mathsf{x},\mathsf{y}) \doteq$$
$$\textbf{if } \mathsf{y}^{\mathcal{P}} \textbf{ then } \neg\mathsf{x} \stackrel{\mathrm{t}}{=} \mathsf{y} \textbf{ else}$$
$$\textbf{if } \neg\mathsf{y} \stackrel{\mathrm{r}}{=} \lceil\dot\forall\mathsf{x}\colon\mathsf{y}\rceil \textbf{ then } \dot{\mathrm{nonfree}}^{*}(\mathsf{x},\mathsf{y}^{\mathrm{t}}) \textbf{ else}$$
$$\textbf{if } \mathsf{x} \stackrel{\mathrm{t}}{=} \mathsf{y}^{1} \textbf{ then } \mathsf{T} \textbf{ else } \dot{\mathrm{nonfree}}(\mathsf{x},\mathsf{y}^{2})]$$

$$[\dot{\mathrm{nonfree}}^{*}(\mathsf{x},\mathsf{y}) \doteq \mathsf{x}!\mathrm{If}(\mathsf{y},\mathsf{T},\dot{\mathrm{nonfree}}(\mathsf{x},\mathsf{y}^{\mathrm{h}}) \ddot{\wedge} \dot{\mathrm{nonfree}}^{*}(\mathsf{x},\mathsf{y}^{\mathrm{t}}))]$$

$[\dot{\mathrm{free}}\langle\mathsf{a}|\mathsf{x} := \mathsf{b}\rangle]$[20] is true if the substitution $[\langle\mathsf{a}\,|\mathsf{x}{:=}\,\mathsf{b}\rangle]$ is free. $[\dot{\mathrm{free}}^{*}\langle\mathsf{a}|\mathsf{x} := \mathsf{b}\rangle]$[21] is the version where $[\mathsf{a}]$ is a list of terms.

$$[\dot{\mathrm{free}}\langle\mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \doteq \mathsf{x}!\mathsf{b}!$$
$$\textbf{if } \mathsf{a}^{\mathcal{P}} \textbf{ then } \mathsf{T} \textbf{ else}$$
$$\textbf{if } \neg\mathsf{a} \stackrel{\mathrm{r}}{=} \lceil\dot\forall\mathsf{u}\colon\mathsf{v}\rceil \textbf{ then } \dot{\mathrm{free}}^{*}\langle\mathsf{a}^{\mathrm{t}}|\mathsf{x} := \mathsf{b}\rangle \textbf{ else}$$
$$\textbf{if } \mathsf{a}^{1} \stackrel{\mathrm{t}}{=} \mathsf{x} \textbf{ then } \mathsf{T} \textbf{ else}$$
$$\textbf{if } \dot{\mathrm{nonfree}}(\mathsf{x},\mathsf{a}^{2}) \textbf{ then } \mathsf{T} \textbf{ else}$$
$$\textbf{if } \neg\dot{\mathrm{nonfree}}(\mathsf{a}^{1},\mathsf{b}) \textbf{ then } \mathsf{F} \textbf{ else}$$
$$\dot{\mathrm{free}}\langle\mathsf{a}^{2}|\mathsf{x} := \mathsf{b}\rangle]$$

$$[\dot{\mathrm{free}}^{*}\langle\mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \doteq \mathsf{x}!\mathsf{b}!\mathrm{If}(\mathsf{a},\mathsf{T},\dot{\mathrm{free}}\langle\mathsf{a}^{\mathrm{h}}|\mathsf{x} := \mathsf{b}\rangle \ddot{\wedge} \dot{\mathrm{free}}^{*}\langle\mathsf{a}^{\mathrm{t}}|\mathsf{x} := \mathsf{b}\rangle)]$$

$[\mathsf{a}{\equiv}\langle\mathsf{b}|\mathsf{x} := \mathsf{c}\rangle]$[22] is true if $[\mathsf{a}]$ equals $[\langle\mathsf{b}\,|\mathsf{x}{:=}\,\mathsf{c}\rangle]$. $[\mathsf{a}{\equiv}\langle^{*}\mathsf{b}|\mathsf{x} := \mathsf{c}\rangle]$[23] is the version where $[\mathsf{a}]$ and $[\mathsf{b}]$ are lists.

$$[\mathsf{a}{\equiv}\langle\mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \doteq \mathsf{a}!\mathsf{x}!\mathsf{c}!$$
$$\textbf{if } \mathsf{b} \stackrel{\mathrm{r}}{=} \lceil\dot\forall\mathsf{u}\colon\mathsf{v}\rceil \ddot{\wedge} \mathsf{b}^{1} \stackrel{\mathrm{t}}{=} \mathsf{x} \textbf{ then } \mathsf{a} \stackrel{\mathrm{t}}{=} \mathsf{b} \textbf{ else}$$
$$\textbf{if } \mathsf{b}^{\mathcal{P}} \wedge \mathsf{b} \stackrel{\mathrm{t}}{=} \mathsf{x} \textbf{ then } \mathsf{a} \stackrel{\mathrm{t}}{=} \mathsf{c} \textbf{ else}$$
$$\mathsf{a} \stackrel{\mathrm{r}}{=} \mathsf{b} \ddot{\wedge} \mathsf{a}^{\mathrm{t}}{\equiv}\langle^{*}\mathsf{b}^{\mathrm{t}}|\mathsf{x} := \mathsf{c}\rangle]$$

$$[\mathsf{a}{\equiv}\langle^{*}\mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \doteq \mathsf{b}!\mathsf{x}!\mathsf{c}!\mathrm{If}(\mathsf{a},\mathsf{T},\mathsf{a}^{\mathrm{h}}{\equiv}\langle\mathsf{b}^{\mathrm{h}}|\mathsf{x} := \mathsf{c}\rangle \ddot{\wedge} \mathsf{a}^{\mathrm{t}}{\equiv}\langle^{*}\mathsf{b}^{\mathrm{t}}|\mathsf{x} := \mathsf{c}\rangle)]$$

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18 $[\dot{\mathrm{nonfree}}(\mathsf{x},\mathsf{y}) \stackrel{\mathrm{pyk}}{=}$ "peano nonfree ∗ in ∗ end nonfree"$]$

19 $[\dot{\mathrm{nonfree}}^{*}(\mathsf{x},\mathsf{y}) \stackrel{\mathrm{pyk}}{=}$ "peano nonfree star ∗ in ∗ end nonfree"$]$

20 $[\dot{\mathrm{free}}\langle\mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \stackrel{\mathrm{pyk}}{=}$ "peano free ∗ set ∗ to ∗ end free"$]$

21 $[\dot{\mathrm{free}}^{*}\langle\mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \stackrel{\mathrm{pyk}}{=}$ "peano free star ∗ set ∗ to ∗ end free"$]$

22 $[\mathsf{a}{\equiv}\langle\mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \stackrel{\mathrm{pyk}}{=}$ "peano sub ∗ is ∗ where ∗ is ∗ end sub"$]$

23 $[\mathsf{a}{\equiv}\langle^{*}\mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \stackrel{\mathrm{pyk}}{=}$ "peano sub star ∗ is ∗ where ∗ is ∗ end sub"$]$

1.3 Mendelsons system S

System [S][24] of Mendelson [2] expresses Peano arithmetic. It comprises the axioms [A1][25], [A2][26], [A3][27], [A4][28], and [A5][29] and inference rules [MP][30] and [Gen][31] of first order predicate calculus. Furthermore, it comprises the proper axioms [S1][32], [S2][33], [S3][34], [S4][35], [S5][36], [S6][37], [S7][38], [S8][39], and [S9][40]. System [S] is defined thus:

[**Theory** S]

[S **rule** A1: $\forall \mathcal{A}\colon \forall \mathcal{B}\colon \mathcal{A} \dot\Rightarrow \mathcal{B} \dot\Rightarrow \mathcal{A}$]

[S **rule** A2: $\forall \mathcal{A}\colon \forall \mathcal{B}\colon \forall \mathcal{C}\colon (\mathcal{A} \dot\Rightarrow \mathcal{B} \dot\Rightarrow \mathcal{C}) \dot\Rightarrow (\mathcal{A} \dot\Rightarrow \mathcal{B}) \dot\Rightarrow \mathcal{A} \dot\Rightarrow \mathcal{C}$]

[S **rule** A3: $\forall \mathcal{A}\colon \forall \mathcal{B}\colon (\dot\neg \mathcal{B} \dot\Rightarrow \dot\neg \mathcal{A}) \dot\Rightarrow (\dot\neg \mathcal{B} \dot\Rightarrow \mathcal{A}) \dot\Rightarrow \mathcal{B}$]

The order of quantifiers in the following axiom is such that [$\mathcal{C}$] which the current conclusion tactic cannot guess comes first. This allows to supply a value for [$\mathcal{C}$] without having to supply values for the other meta-variables.

[S **rule** A4: $\forall \mathcal{C}\colon \forall \mathcal{A}\colon \forall \mathcal{X}\colon \forall \mathcal{B}\colon \lceil \mathcal{A} \rceil \equiv \langle \lceil \mathcal{B} \rceil | \lceil \mathcal{X} \rceil := \lceil \mathcal{C} \rceil \rangle \Vdash \dot\forall \mathcal{X}\colon \mathcal{B} \dot\Rightarrow \mathcal{A}$]

[S **rule** A5: $\forall \mathcal{X}\colon \forall \mathcal{A}\colon \forall \mathcal{B}\colon \dot{\text{nonfree}}(\mathcal{X}, \mathcal{A}) \Vdash \dot\forall \mathcal{X}\colon (\mathcal{A} \dot\Rightarrow \mathcal{B}) \dot\Rightarrow \mathcal{A} \dot\Rightarrow \dot\forall \mathcal{X}\colon \mathcal{B}$]

[S **rule** MP: $\forall \mathcal{A}\colon \forall \mathcal{B}\colon \mathcal{A} \dot\Rightarrow \mathcal{B} \vdash \mathcal{A} \vdash \mathcal{B}$]

[S **rule** Gen: $\forall \mathcal{X}\colon \forall \mathcal{A}\colon \mathcal{A} \vdash \dot\forall \mathcal{X}\colon \mathcal{A}$]

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[24] [S $\overset{\text{pyk}}{=}$ "system s"]

[25] [A1 $\overset{\text{pyk}}{=}$ "axiom a one"]

[26] [A2 $\overset{\text{pyk}}{=}$ "axiom a two"]

[27] [A3 $\overset{\text{pyk}}{=}$ "axiom a three"]

[28] [A4 $\overset{\text{pyk}}{=}$ "axiom a four"]

[29] [A5 $\overset{\text{pyk}}{=}$ "axiom a five"]

[30] [MP $\overset{\text{pyk}}{=}$ "rule mp"]

[31] [Gen $\overset{\text{pyk}}{=}$ "rule gen"]

[32] [S1 $\overset{\text{pyk}}{=}$ "axiom s one"]

[33] [S2 $\overset{\text{pyk}}{=}$ "axiom s two"]

[34] [S3 $\overset{\text{pyk}}{=}$ "axiom s three"]

[35] [S4 $\overset{\text{pyk}}{=}$ "axiom s four"]

[36] [S5 $\overset{\text{pyk}}{=}$ "axiom s five"]

[37] [S6 $\overset{\text{pyk}}{=}$ "axiom s six"]

[38] [S7 $\overset{\text{pyk}}{=}$ "axiom s seven"]

[39] [S8 $\overset{\text{pyk}}{=}$ "axiom s eight"]

[40] [S9 $\overset{\text{pyk}}{=}$ "axiom s nine"]

Axiom [S1] to [S8] are stated as Mendelson [2] does. This is done here to test certain parts of the Logiweb system. Serious users of Peano arithmetic are advised to take Mendelson's Lemma 3.1 as axioms instead.

[S **rule** S1: $\dot{a} \stackrel{p}{=} \dot{b} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{c} \dot{\Rightarrow} \dot{b} \stackrel{p}{=} \dot{c}$]

[S **rule** S2: $\dot{a} \stackrel{p}{=} \dot{b} \dot{\Rightarrow} \dot{a}' \stackrel{p}{=} \dot{b}'$]

[S **rule** S3: $\dot{\neg} \dot{0} \stackrel{p}{=} \dot{a}'$]

[S **rule** S4: $\dot{a}' \stackrel{p}{=} \dot{b}' \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{b}$]

[S **rule** S5: $\dot{a} \dot{+} \dot{0} \stackrel{p}{=} \dot{a}$]

[S **rule** S6: $\dot{a} \dot{+} \dot{b}' \stackrel{p}{=} (\dot{a} \dot{+} \dot{b})'$]

[S **rule** S7: $\dot{a} \dot{\cdot} \dot{0} \stackrel{p}{=} \dot{0}$]

[S **rule** S8: $\dot{a} \dot{\cdot} (\dot{b}') \stackrel{p}{=} (\dot{a} \dot{\cdot} \dot{b}) \dot{+} \dot{a}$]

[S **rule** S9: $\forall \mathcal{A} \colon \forall \mathcal{B} \colon \forall \mathcal{C} \colon \forall \mathcal{X} \colon$
$\mathcal{B} \equiv \langle \mathcal{A} | \mathcal{X} := \dot{0} \rangle \Vdash \mathcal{C} \equiv \langle \mathcal{A} | \mathcal{X} := \mathcal{X}' \rangle \Vdash$
$\mathcal{B} \dot{\Rightarrow} \dot{\forall} \mathcal{X} \colon (\mathcal{A} \dot{\Rightarrow} \mathcal{C}) \dot{\Rightarrow} \dot{\forall} \mathcal{X} \colon \mathcal{A}$]

1.4 An alternative axiomatic system

System [S′][41] is system [S] in which the proper axioms are taken from Lemma 3.1 in Mendelson [2]. It comprises the axioms [A1′][42], [A2′][43], [A3′][44], [A4′][45], and [A5′][46] and inference rules [MP′][47] and [Gen′][48] of first order predicate calculus. Furthermore, it comprises the proper axioms [S1′][49], [S2′][50], [S3′][51], [S4′][52], [S5′][53], [S6′][54], [S7′][55], [S8′][56], and [S9′][57].

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[41] [S′ $\stackrel{\text{pyk}}{=}$ "system prime s"]

[42] [A1′ $\stackrel{\text{pyk}}{=}$ "axiom prime a one"]

[43] [A2′ $\stackrel{\text{pyk}}{=}$ "axiom prime a two"]

[44] [A3′ $\stackrel{\text{pyk}}{=}$ "axiom prime a three"]

[45] [A4′ $\stackrel{\text{pyk}}{=}$ "axiom prime a four"]

[46] [A5′ $\stackrel{\text{pyk}}{=}$ "axiom prime a five"]

[47] [MP′ $\stackrel{\text{pyk}}{=}$ "rule prime mp"]

[48] [Gen′ $\stackrel{\text{pyk}}{=}$ "rule prime gen"]

[49] [S1′ $\stackrel{\text{pyk}}{=}$ "axiom prime s one"]

[50] [S2′ $\stackrel{\text{pyk}}{=}$ "axiom prime s two"]

[51] [S3′ $\stackrel{\text{pyk}}{=}$ "axiom prime s three"]

[52] [S4′ $\stackrel{\text{pyk}}{=}$ "axiom prime s four"]

[53] [S5′ $\stackrel{\text{pyk}}{=}$ "axiom prime s five"]

[54] [S6′ $\stackrel{\text{pyk}}{=}$ "axiom prime s six"]

[55] [S7′ $\stackrel{\text{pyk}}{=}$ "axiom prime s seven"]

[56] [S8′ $\stackrel{\text{pyk}}{=}$ "axiom prime s eight"]

[57] [S9′ $\stackrel{\text{pyk}}{=}$ "axiom prime s nine"]

System [S′] is defined thus:

[**Theory** S′]

[S′ **rule** A1′: $\forall\mathcal{A}{:}\,\forall\mathcal{B}{:}\,\mathcal{A} \mathrel{\dot{\Rightarrow}} \mathcal{B} \mathrel{\dot{\Rightarrow}} \mathcal{A}$]

[S′ **rule** A2′: $\forall\mathcal{A}{:}\,\forall\mathcal{B}{:}\,\forall\mathcal{C}{:}\,(\mathcal{A} \mathrel{\dot{\Rightarrow}} \mathcal{B} \mathrel{\dot{\Rightarrow}} \mathcal{C}) \mathrel{\dot{\Rightarrow}} (\mathcal{A} \mathrel{\dot{\Rightarrow}} \mathcal{B}) \mathrel{\dot{\Rightarrow}} \mathcal{A} \mathrel{\dot{\Rightarrow}} \mathcal{C}$]

[S′ **rule** A3′: $\forall\mathcal{A}{:}\,\forall\mathcal{B}{:}\,(\dot{\neg}\,\mathcal{B} \mathrel{\dot{\Rightarrow}} \dot{\neg}\,\mathcal{A}) \mathrel{\dot{\Rightarrow}} (\dot{\neg}\,\mathcal{B} \mathrel{\dot{\Rightarrow}} \mathcal{A}) \mathrel{\dot{\Rightarrow}} \mathcal{B}$]

[S′ **rule** A4′: $\forall\mathcal{C}{:}\,\forall\mathcal{A}{:}\,\forall\mathcal{X}{:}\,\forall\mathcal{B}{:}\,\lceil\mathcal{A}\rceil {\equiv} \langle\lceil\mathcal{B}\rceil|\lceil\mathcal{X}\rceil := \lceil\mathcal{C}\rceil\rangle \Vdash \dot{\forall}\mathcal{X}{:}\,\mathcal{B} \mathrel{\dot{\Rightarrow}} \mathcal{A}$]

[S′ **rule** A5′: $\forall\mathcal{X}{:}\,\forall\mathcal{A}{:}\,\forall\mathcal{B}{:}\,\mathrm{nonfree}(\lceil\mathcal{X}\rceil, \lceil\mathcal{A}\rceil) \Vdash \dot{\forall}\mathcal{X}{:}\,(\mathcal{A} \mathrel{\dot{\Rightarrow}} \mathcal{B}) \mathrel{\dot{\Rightarrow}} \mathcal{A} \mathrel{\dot{\Rightarrow}} \dot{\forall}\mathcal{X}{:}\,\mathcal{B}$]

[S′ **rule** MP′: $\forall\mathcal{A}{:}\,\forall\mathcal{B}{:}\,\mathcal{A} \mathrel{\dot{\Rightarrow}} \mathcal{B} \vdash \mathcal{A} \vdash \mathcal{B}$]

[S′ **rule** Gen′: $\forall\mathcal{X}{:}\,\forall\mathcal{A}{:}\,\mathcal{A} \vdash \dot{\forall}\mathcal{X}{:}\,\mathcal{A}$]

[S′ **rule** S1′: $\forall\mathcal{A}{:}\,\forall\mathcal{B}{:}\,\forall\mathcal{C}{:}\,\mathcal{A} \stackrel{\mathrm{p}}{=} \mathcal{B} \mathrel{\dot{\Rightarrow}} \mathcal{A} \stackrel{\mathrm{p}}{=} \mathcal{C} \mathrel{\dot{\Rightarrow}} \mathcal{B} \stackrel{\mathrm{p}}{=} \mathcal{C}$]

[S′ **rule** S2′: $\forall\mathcal{A}{:}\,\forall\mathcal{B}{:}\,\mathcal{A} \stackrel{\mathrm{p}}{=} \mathcal{B} \mathrel{\dot{\Rightarrow}} \mathcal{A}' \stackrel{\mathrm{p}}{=} \mathcal{B}'$]

[S′ **rule** S3′: $\forall\mathcal{A}{:}\,\neg\dot{0} \stackrel{\mathrm{p}}{=} \mathcal{A}'$]

[S′ **rule** S4′: $\forall\mathcal{A}{:}\,\forall\mathcal{B}{:}\,\mathcal{A}' \stackrel{\mathrm{p}}{=} \mathcal{B}' \mathrel{\dot{\Rightarrow}} \mathcal{A} \stackrel{\mathrm{p}}{=} \mathcal{B}$]

[S′ **rule** S5′: $\forall\mathcal{A}{:}\,\mathcal{A} \mathbin{\dot{+}} \dot{0} \stackrel{\mathrm{p}}{=} \mathcal{A}$]

[S′ **rule** S6′: $\forall\mathcal{A}{:}\,\forall\mathcal{B}{:}\,\mathcal{A} \mathbin{\dot{+}} \mathcal{B}' \stackrel{\mathrm{p}}{=} (\mathcal{A} \mathbin{\dot{+}} \mathcal{B})'$]

[S′ **rule** S7′: $\forall\mathcal{A}{:}\,\mathcal{A} \mathbin{\dot{\cdot}} \dot{0} \stackrel{\mathrm{p}}{=} \dot{0}$]

[S′ **rule** S8′: $\forall\mathcal{A}{:}\,\forall\mathcal{B}{:}\,\mathcal{A} \mathbin{\dot{\cdot}} (\mathcal{B}') \stackrel{\mathrm{p}}{=} (\mathcal{A} \mathbin{\dot{\cdot}} \mathcal{B}) \mathbin{\dot{+}} \mathcal{A}$]

[S′ **rule** S9′: $\forall\mathcal{A}{:}\,\forall\mathcal{B}{:}\,\forall\mathcal{C}{:}\,\forall\mathcal{X}{:}$
$\mathcal{B}{\equiv}\langle\mathcal{A}|\mathcal{X} := \dot{0}\rangle \Vdash \mathcal{C}{\equiv}\langle\mathcal{A}|\mathcal{X} := \mathcal{X}'\rangle \Vdash$
$\mathcal{B} \mathrel{\dot{\Rightarrow}} \dot{\forall}\mathcal{X}{:}\,(\mathcal{A} \mathrel{\dot{\Rightarrow}} \mathcal{C}) \mathrel{\dot{\Rightarrow}} \dot{\forall}\mathcal{X}{:}\,\mathcal{A}$]

Note that [A1] and [A1′] are distinct. The former says [S $\vdash \forall\mathcal{A}{:}\,\forall\mathcal{B}{:}\,\mathcal{A} \mathrel{\dot{\Rightarrow}} \mathcal{B} \mathrel{\dot{\Rightarrow}} \mathcal{A}$] and the latter says [S′ $\vdash \forall\mathcal{A}{:}\,\forall\mathcal{B}{:}\,\mathcal{A} \mathrel{\dot{\Rightarrow}} \mathcal{B} \mathrel{\dot{\Rightarrow}} \mathcal{A}$].

A Chores

A.1 The name of the page

This defines the name of the page:

$$[\text{peano} \overset{\text{pyk}}{=} \text{“peano”}]$$

A.2 Variables of Peano arithmetic

We use $[\dot{b}]$[58], $[\dot{c}]$[59], $[\dot{d}]$[60], $[\dot{e}]$[61], $[\dot{f}]$[62], $[\dot{g}]$[63], $[\dot{h}]$[64], $[\dot{i}]$[65], $[\dot{j}]$[66], $[\dot{k}]$[67], $[\dot{l}]$[68], $[\dot{m}]$[69], $[\dot{n}]$[70], $[\dot{o}]$[71], $[\dot{p}]$[72], $[\dot{q}]$[73], $[\dot{r}]$[74], $[\dot{s}]$[75], $[\dot{t}]$[76], $[\dot{u}]$[77], $[\dot{v}]$[78], $[\dot{w}]$[79], $[\dot{x}]$[80], $[\dot{y}]$[81], and $[\dot{z}]$[82] to denote variables of Peano arithmetic:

$[\dot{b} \ddot{=} \dot{\mathsf{b}}]$, $[\dot{c} \ddot{=} \dot{\mathsf{c}}]$, $[\dot{d} \ddot{=} \dot{\mathsf{d}}]$, $[\dot{e} \ddot{=} \dot{\mathsf{e}}]$, $[\dot{f} \ddot{=} \dot{\mathsf{f}}]$, $[\dot{g} \ddot{=} \dot{\mathsf{g}}]$, $[\dot{h} \ddot{=} \dot{\mathsf{h}}]$, $[\dot{i} \ddot{=} \dot{\mathsf{i}}]$, $[\dot{j} \ddot{=} \dot{\mathsf{j}}]$, $[\dot{k} \ddot{=} \dot{\mathsf{k}}]$, $[\dot{l} \ddot{=} \dot{\mathsf{l}}]$, $[\dot{m} \ddot{=} \dot{\mathsf{m}}]$, $[\dot{n} \ddot{=} \dot{\mathsf{n}}]$, $[\dot{o} \ddot{=} \dot{\mathsf{o}}]$, $[\dot{p} \ddot{=} \dot{\mathsf{p}}]$, $[\dot{q} \ddot{=} \dot{\mathsf{q}}]$, $[\dot{r} \ddot{=} \dot{\mathsf{r}}]$, $[\dot{s} \ddot{=} \dot{\mathsf{s}}]$, $[\dot{t} \ddot{=} \dot{\mathsf{t}}]$, $[\dot{u} \ddot{=} \dot{\mathsf{u}}]$, $[\dot{v} \ddot{=} \dot{\mathsf{v}}]$, $[\dot{w} \ddot{=} \dot{\mathsf{w}}]$, $[\dot{x} \ddot{=} \dot{\mathsf{x}}]$, $[\dot{y} \ddot{=} \dot{\mathsf{y}}]$, and $[\dot{z} \ddot{=} \dot{\mathsf{z}}]$.

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[58] $[\dot{b} \overset{\text{pyk}}{=} \text{“peano b”}]$

[59] $[\dot{c} \overset{\text{pyk}}{=} \text{“peano c”}]$

[60] $[\dot{d} \overset{\text{pyk}}{=} \text{“peano d”}]$

[61] $[\dot{e} \overset{\text{pyk}}{=} \text{“peano e”}]$

[62] $[\dot{f} \overset{\text{pyk}}{=} \text{“peano f”}]$

[63] $[\dot{g} \overset{\text{pyk}}{=} \text{“peano g”}]$

[64] $[\dot{h} \overset{\text{pyk}}{=} \text{“peano h”}]$

[65] $[\dot{i} \overset{\text{pyk}}{=} \text{“peano i”}]$

[66] $[\dot{j} \overset{\text{pyk}}{=} \text{“peano j”}]$

[67] $[\dot{k} \overset{\text{pyk}}{=} \text{“peano k”}]$

[68] $[\dot{l} \overset{\text{pyk}}{=} \text{“peano l”}]$

[69] $[\dot{m} \overset{\text{pyk}}{=} \text{“peano m”}]$

[70] $[\dot{n} \overset{\text{pyk}}{=} \text{“peano n”}]$

[71] $[\dot{o} \overset{\text{pyk}}{=} \text{“peano o”}]$

[72] $[\dot{p} \overset{\text{pyk}}{=} \text{“peano p”}]$

[73] $[\dot{q} \overset{\text{pyk}}{=} \text{“peano q”}]$

[74] $[\dot{r} \overset{\text{pyk}}{=} \text{“peano r”}]$

[75] $[\dot{s} \overset{\text{pyk}}{=} \text{“peano s”}]$

[76] $[\dot{t} \overset{\text{pyk}}{=} \text{“peano t”}]$

[77] $[\dot{u} \overset{\text{pyk}}{=} \text{“peano u”}]$

[78] $[\dot{v} \overset{\text{pyk}}{=} \text{“peano v”}]$

[79] $[\dot{w} \overset{\text{pyk}}{=} \text{“peano w”}]$

[80] $[\dot{x} \overset{\text{pyk}}{=} \text{“peano x”}]$

[81] $[\dot{y} \overset{\text{pyk}}{=} \text{“peano y”}]$

[82] $[\dot{z} \overset{\text{pyk}}{=} \text{“peano z”}]$

A.3 TEX definitions

[$\dot{0} \stackrel{\text{tex}}{=}$ “
\dot{0}”]

[$\mathsf{x}' \stackrel{\text{tex}}{=}$ “#1.’”]

[$\mathsf{x} \mathbin{\dot{+}} \mathsf{y} \stackrel{\text{tex}}{=}$ “#1.
\mathop{\dot{+}} #2.”]

[$\mathsf{x} \mathbin{\dot{\cdot}} \mathsf{y} \stackrel{\text{tex}}{=}$ “#1.
\mathop{\dot{\cdot}} #2.”]

[$\mathsf{x} \stackrel{\text{p}}{=} \mathsf{y} \stackrel{\text{tex}}{=}$ “#1.
\stackrel{p}{=} #2.”]

[$\dot{\neg}\, \mathsf{x} \stackrel{\text{tex}}{=}$ “
\dot{\neg}\, {#1.}”]

[$\mathsf{x} \mathrel{\dot{\Rightarrow}} \mathsf{y} \stackrel{\text{tex}}{=}$ “#1.
\mathrel{\dot{\Rightarrow}} #2.”]

[$\dot{\forall} \mathsf{x} \colon \mathsf{y} \stackrel{\text{tex}}{=}$ “
\dot{\forall} #1.
\colon #2.”]

[$\dot{1} \stackrel{\text{tex}}{=}$ “
\dot{1}”]

[$\dot{2} \stackrel{\text{tex}}{=}$ “
\dot{2}”]

[$\mathsf{x} \mathrel{\dot{\wedge}} \mathsf{y} \stackrel{\text{tex}}{=}$ “#1.
\mathrel{\dot{\wedge}} #2.”]

[$\mathsf{x} \mathrel{\dot{\vee}} \mathsf{y} \stackrel{\text{tex}}{=}$ “#1.
\mathrel{\dot{\vee}} #2.”]

[$\mathsf{x} \mathrel{\dot{\Leftrightarrow}} \mathsf{y} \stackrel{\text{tex}}{=}$ “#1.
\mathrel{\dot{\Leftrightarrow}} #2.”]

[$\dot{\exists} \mathsf{x} \colon \mathsf{y} \stackrel{\text{tex}}{=}$ “
\dot{\exists} #1.
\colon #2.”]

[$\dot{\mathsf{x}} \stackrel{\text{tex}}{=}$ “
\dot{#1.
}”]

[$\mathsf{x}^{\mathcal{P}} \stackrel{\text{tex}}{=}$ “#1.
{} ^ {\cal P}”]

[$\dot{a} \stackrel{\text{tex}}{=}$ “
\dot{\mathit{a}}”]

[$\dot{\text{nonfree}}(\mathsf{x}, \mathsf{y}) \stackrel{\text{tex}}{=}$ “
\dot{nonfree}(#1.
, #2.
)”]

[$\dot{\text{nonfree}}^{*}(\mathsf{x}, \mathsf{y}) \stackrel{\text{tex}}{=}$ “
\dot{nonfree}ˆ∗(#1.
, #2.
)”]

[$\dot{\text{free}}\langle \mathsf{a} | \mathsf{x} := \mathsf{b} \rangle \stackrel{\text{tex}}{=}$ “
\dot{free}\langle #1.
| #2.
:= #3.
\rangle”]

[$\dot{\text{free}}^{*}\langle \mathsf{a} | \mathsf{x} := \mathsf{b} \rangle \stackrel{\text{tex}}{=}$ “
\dot{free}{}ˆ∗\langle #1.
| #2.
:= #3.
\rangle”]

[$\mathsf{a} \equiv \langle \mathsf{b} | \mathsf{x} := \mathsf{c} \rangle \stackrel{\text{tex}}{=}$ “#1.
{\equiv}\langle #2.
|#3.
:=#4.
\rangle”]

[$\mathsf{a} \equiv \langle^{*} \mathsf{b} | \mathsf{x} := \mathsf{c} \rangle \stackrel{\text{tex}}{=}$ “#1.
{\equiv}\langleˆ∗ #2.
|#3.
:=#4.
\rangle”]

[S $\stackrel{\text{tex}}{=}$ “
S”]

[A1 $\stackrel{\text{tex}}{=}$ “
A1”]

[A2 $\stackrel{\text{tex}}{=}$ “
A2”]

[A3 $\stackrel{\text{tex}}{=}$ "
A3"]

[A4 $\stackrel{\text{tex}}{=}$ "
A4"]

[A5 $\stackrel{\text{tex}}{=}$ "
A5"]

[MP $\stackrel{\text{tex}}{=}$ "
MP"]

[Gen $\stackrel{\text{tex}}{=}$ "
Gen"]

[S1 $\stackrel{\text{tex}}{=}$ "
S1"]

[S2 $\stackrel{\text{tex}}{=}$ "
S2"]

[S3 $\stackrel{\text{tex}}{=}$ "
S3"]

[S4 $\stackrel{\text{tex}}{=}$ "
S4"]

[S5 $\stackrel{\text{tex}}{=}$ "
S5"]

[S6 $\stackrel{\text{tex}}{=}$ "
S6"]

[S7 $\stackrel{\text{tex}}{=}$ "
S7"]

[S8 $\stackrel{\text{tex}}{=}$ "
S8"]

[S9 $\stackrel{\text{tex}}{=}$ "
S9"]

[S′ $\stackrel{\text{tex}}{=}$ "
S'"]

[A1′ $\stackrel{\text{tex}}{=}$ "
A1'"]

[A2′ $\stackrel{\text{tex}}{=}$ "
A2'"]

[A3′ $\stackrel{\text{tex}}{=}$ "
A3'"]

[A4′ $\stackrel{\text{tex}}{=}$ "
A4'"]

[A5′ $\stackrel{\text{tex}}{=}$ "
A5'"]

[MP′ $\stackrel{\text{tex}}{=}$ "
MP'"]

[Gen′ $\stackrel{\text{tex}}{=}$ "
Gen'"]

[S1′ $\stackrel{\text{tex}}{=}$ "
S1'"]

[S2′ $\stackrel{\text{tex}}{=}$ "
S2'"]

[S3′ $\stackrel{\text{tex}}{=}$ "
S3'"]

[S4′ $\stackrel{\text{tex}}{=}$ "
S4'"]

[S5′ $\stackrel{\text{tex}}{=}$ "
S5'"]

[S6′ $\stackrel{\text{tex}}{=}$ "
S6'"]

[S7′ $\stackrel{\text{tex}}{=}$ "
S7'"]

[S8′ $\stackrel{\text{tex}}{=}$ "
S8'"]

[S9′ $\stackrel{\text{tex}}{=}$ "
S9'"]

[$\dot{b}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{b}}"]

[$\dot{c}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{c}}"]

[$\dot{d}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{d}}"]

[$\dot{e}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{e}}"]

[$\dot{f}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{f}}"]

[$\dot{g}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{g}}"]

[$\dot{h}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{h}}"]

[$\dot{i}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{i}}"]

[$\dot{j}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{j}}"]

[$\dot{k}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{k}}"]

[$\dot{l}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{l}}"]

[$\dot{m}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{m}}"]

[$\dot{n}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{n}}"]

[$\dot{o}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{o}}"]

[$\dot{p}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{p}}"]

[$\dot{q}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{q}}"]

[$\dot{r}$ $\stackrel{\text{tex}}{=}$ "
\dot{\mathit{r}}"]

$[\dot{s} \stackrel{\text{tex}}{=}$ “

\dot{\mathit{s}}”]

$[\dot{t} \stackrel{\text{tex}}{=}$ “

\dot{\mathit{t}}”]

$[\dot{u} \stackrel{\text{tex}}{=}$ “

\dot{\mathit{u}}”]

$[\dot{v} \stackrel{\text{tex}}{=}$ “

\dot{\mathit{v}}”]

$[\dot{w} \stackrel{\text{tex}}{=}$ “

\dot{\mathit{w}}”]

$[\dot{x} \stackrel{\text{tex}}{=}$ “

\dot{\mathit{x}}”]

$[\dot{y} \stackrel{\text{tex}}{=}$ “

\dot{\mathit{y}}”]

$[\dot{z} \stackrel{\text{tex}}{=}$ “

\dot{\mathit{z}}”]

A.4 Test

$[\lceil \dot{a} \rceil^{\mathcal{P}}]^{\cdot}$

$[\lceil \mathsf{a} \rceil^{\mathcal{P}}]^{-}$

$[\dot{\text{nonfree}}(\lceil \dot{x} \rceil, \lceil \dot{y} \stackrel{\text{p}}{=} \dot{z} \mathrel{\dot{\Rightarrow}} \dot{\forall} \dot{x}\colon \dot{x} \stackrel{\text{p}}{=} \dot{y} \rceil)]^{\cdot}$

$[\dot{\text{nonfree}}(\lceil \dot{x} \rceil, \lceil \dot{x} \stackrel{\text{p}}{=} \dot{z} \mathrel{\dot{\Rightarrow}} \dot{\forall} \dot{x}\colon \dot{x} \stackrel{\text{p}}{=} \dot{y} \rceil)]^{-}$

$[\dot{\text{nonfree}}(\lceil \dot{x} \rceil, \lceil \dot{y} \stackrel{\text{p}}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{\forall} \dot{x}\colon \dot{x} \stackrel{\text{p}}{=} \dot{y} \rceil)]^{-}$

$[\dot{\text{nonfree}}(\lceil \dot{x} \rceil, \lceil \dot{y} \stackrel{\text{p}}{=} \dot{z} \mathrel{\dot{\Rightarrow}} \dot{\forall} \dot{y}\colon \dot{x} \stackrel{\text{p}}{=} \dot{y} \rceil)]^{-}$

$[\dot{\text{free}}\langle \lceil \dot{\forall} \dot{x}\colon \mathsf{b} :: \dot{x} :: \mathsf{c} \rceil | \lceil \dot{x} \rceil := \lceil \mathsf{x} :: \dot{y} :: \mathsf{z} \rceil \rangle]^{\cdot}$

$[\dot{\text{free}}\langle \lceil \dot{\forall} \dot{y}\colon \mathsf{b} :: \dot{x} :: \mathsf{c} \rceil | \lceil \dot{x} \rceil := \lceil \mathsf{x} :: \dot{y} :: \mathsf{z} \rceil \rangle]^{-}$

$[\dot{\text{free}}\langle \lceil \dot{\forall} \dot{x}\colon \mathsf{b} :: \dot{x} :: \mathsf{c} \rceil | \lceil \dot{y} \rceil := \lceil \mathsf{x} :: \dot{y} :: \mathsf{z} \rceil \rangle]^{\cdot}$

$[\dot{\text{free}}\langle \lceil \dot{\forall} \dot{y}\colon \mathsf{b} :: \dot{x} :: \mathsf{c} \rceil | \lceil \dot{y} \rceil := \lceil \mathsf{x} :: \dot{y} :: \mathsf{z} \rceil \rangle]^{\cdot}$

$[\dot{a} \equiv \langle \dot{a} | \dot{b} := \dot{c} \rangle]^{\cdot}$

$[\dot{c} \equiv \langle \dot{b} | \dot{b} := \dot{c} \rangle]^{\cdot}$

$[\forall \dot{a}\colon \dot{a} \stackrel{\text{p}}{=} \dot{b} \equiv \langle \forall \dot{a}\colon \dot{a} \stackrel{\text{p}}{=} \dot{b} | \dot{a} := \dot{c} \rangle]^{\cdot}$

$[\forall \dot{a}\colon \dot{a} \stackrel{\text{p}}{=} \dot{c} \equiv \langle \forall \dot{a}\colon \dot{a} \stackrel{\text{p}}{=} \dot{b} | \dot{b} := \dot{c} \rangle]^{\cdot}$

$[\dot{\forall} \dot{a}\colon \dot{a} \stackrel{\text{p}}{=} \dot{0} \dot{+} \dot{a} \dot{\Rightarrow} \dot{c} \dot{:} \dot{d} \stackrel{\text{p}}{=} \dot{0} \dot{+} \dot{c} \dot{:} \dot{d} \equiv \langle \dot{\forall} \dot{a}\colon \dot{a} \stackrel{\text{p}}{=} \dot{0} \dot{+} \dot{a} \dot{\Rightarrow} \dot{b} \stackrel{\text{p}}{=} \dot{0} \dot{+} \dot{b} | \dot{b} := \dot{c} \dot{:} \dot{d} \rangle]^{\cdot}$

$[\dot{\forall} \dot{a}\colon \dot{a} \stackrel{\text{p}}{=} \dot{0} \dot{+} \dot{a} \dot{\Rightarrow} \dot{b} \stackrel{\text{p}}{=} \dot{0} \dot{+} \dot{b} \equiv \langle \dot{\forall} \dot{a}\colon \dot{a} \stackrel{\text{p}}{=} \dot{0} \dot{+} \dot{a} \dot{\Rightarrow} \dot{b} \stackrel{\text{p}}{=} \dot{0} \dot{+} \dot{b} | \dot{a} := \dot{c} \rangle]^{\cdot}$

A.5 Priority table

Priority table
Preassociative
[peano], [base], [bracket ∗ end bracket], [big bracket ∗ end bracket], [math ∗ end math], [**flush left** [∗]], [x], [y], [z], [[∗ ⋈ ∗]], $[[* \xrightarrow{*} *]]$, [pyk], [tex], [name], [prio], [∗], [T], [if(∗, ∗, ∗)], $[[* \stackrel{*}{\Rightarrow} *]]$, [val], [claim], [⊥], [f(∗)], $[(*)^{\mathrm{I}}]$, [F], $[\underline{0}]$, $[\underline{1}]$, $[\underline{2}]$, $[\underline{3}]$, $[\underline{4}]$, $[\underline{5}]$, $[\underline{6}]$, $[\underline{7}]$, $[\underline{8}]$, $[\underline{9}]$, [0], [1], [2], [3], [4], [5], [6], [7], [8], [9], [a], [b], [c], [d], [e], [f], [g], [h], [i], [j], [k], [l], [m], [n], [o], [p], [q], [r], [s], [t], [u], [v], [w], $[(*)^{\mathrm{M}}]$, [If(∗, ∗, ∗)], [array{∗} ∗ end array], [l], [c], [r], [empty], [⟨∗ | ∗ := ∗⟩], $[\mathcal{M}(*)]$, $[\tilde{\mathcal{U}}(*)]$, $[\mathcal{U}(*)]$, $[\mathcal{U}^{\mathrm{M}}(*)]$, [**apply**(∗, ∗)], [$\mathbf{apply}_1$(∗, ∗)], [identifier(∗)], [identifier_1(∗, ∗)], [array-plus(∗, ∗)], [array-remove(∗, ∗, ∗)], [array-put(∗, ∗, ∗, ∗)], [array-add(∗, ∗, ∗, ∗, ∗)], [bit(∗, ∗)], [bit_1(∗, ∗)], [rack], ["vector"], ["bibliography"], ["dictionary"], ["body"], ["codex"], ["expansion"], ["code"], ["cache"], ["diagnose"], ["pyk"], ["tex"], ["texname"], ["value"], ["message"], ["macro"], ["definition"], ["unpack"], ["claim"], ["priority"], ["lambda"], ["apply"], ["true"], ["if"], ["quote"], ["proclaim"], ["define"], ["introduce"], ["hide"], ["pre"], ["post"], $[\mathcal{E}(*, *, *)]$, $[\mathcal{E}_2(*, *, *, *, *)]$, $[\mathcal{E}_3(*, *, *, *)]$, $[\mathcal{E}_4(*, *, *, *)]$, [**lookup**(∗, ∗, ∗)], [**abstract**(∗, ∗, ∗, ∗)], [⌈∗⌉], $[\mathcal{M}(*, *, *)]$, $[\mathcal{M}_2(*, *, *, *)]$, $[\mathcal{M}^*(*, *, *)]$, [macro], $[\mathrm{s}_0]$, [**zip**(∗, ∗)], [$\mathbf{assoc}_1$(∗, ∗, ∗)], $[(*)^{\mathbf{p}}]$, [self], $[[* \ddot{=} *]]$, $[[* \dot{=} *]]$, $[[* \acute{=} *]]$, $[[* \stackrel{\text{pyk}}{=} *]]$, $[[* \stackrel{\text{tex}}{=} *]]$, $[[* \stackrel{\text{name}}{=} *]]$, [**Priority table**[∗]], $[\tilde{\mathcal{M}}_1]$, $[\tilde{\mathcal{M}}_2(*)]$, $[\tilde{\mathcal{M}}_3(*)]$, $[\tilde{\mathcal{M}}_4(*, *, *, *)]$, $[\tilde{\mathcal{M}}(*, *, *)]$, $[\tilde{\mathcal{Q}}(*, *, *)]$, $[\tilde{\mathcal{Q}}_2(*, *, *)]$, $[\tilde{\mathcal{Q}}_3(*, *, *, *)]$, $[\tilde{\mathcal{Q}}^*(*, *, *)]$, [(∗)], [**aspect**(∗, ∗)], [**aspect**(∗, ∗, ∗)], [⟨∗⟩], [$\mathbf{tuple}_1$(∗)], [$\mathbf{tuple}_2$(∗)], [let_2(∗, ∗)], [let_1(∗, ∗)], $[[* \stackrel{\text{claim}}{=} *]]$, [checker], [**check**(∗, ∗)], [$\mathbf{check}_2$(∗, ∗, ∗)], [$\mathbf{check}_3$(∗, ∗, ∗)], [$\mathbf{check}^*$(∗, ∗)], [$\mathbf{check}_2^*$(∗, ∗, ∗)], $[[*]^{\cdot}]$, $[[*]^{-}]$, $[[*]^{\circ}]$, [msg], $[[* \stackrel{\text{msg}}{=} *]]$, [<stmt>], [stmt], $[[* \stackrel{\text{stmt}}{=} *]]$, [HeadNil′], [HeadPair′], [Transitivity′], [⊥⊥], [Contra′], $[\mathrm{T}'_{\mathrm{E}}]$, $[\mathrm{L}_1]$, $[\underline{*}]$, $[\mathcal{A}]$, $[\mathcal{B}]$, $[\mathcal{C}]$, $[\mathcal{D}]$, $[\mathcal{E}]$, $[\mathcal{F}]$, $[\mathcal{G}]$, $[\mathcal{H}]$, $[\mathcal{I}]$, $[\mathcal{J}]$, $[\mathcal{K}]$, $[\mathcal{L}]$, $[\mathcal{M}]$, $[\mathcal{N}]$, $[\mathcal{O}]$, $[\mathcal{P}]$, $[\mathcal{Q}]$, $[\mathcal{R}]$, $[\mathcal{S}]$, $[\mathcal{T}]$, $[\mathcal{U}]$, $[\mathcal{V}]$, $[\mathcal{W}]$, $[\mathcal{X}]$, $[\mathcal{Y}]$, $[\mathcal{Z}]$, [⟨∗ | ∗ := ∗⟩], [⟨*∗ | ∗ := ∗⟩], [∅], [Remainder], $[(*)^{\mathbf{v}}]$, [intro(∗, ∗, ∗, ∗)], [intro(∗, ∗, ∗)], [error(∗, ∗)], [error_2(∗, ∗)], [proof(∗, ∗, ∗)], [proof_2(∗, ∗)], $[\mathcal{S}(*, *)]$, $[\mathcal{S}^{\mathrm{I}}(*, *)]$, $[\mathcal{S}^{\rhd}(*, *)]$, $[\mathcal{S}_1^{\rhd}(*, *, *)]$, $[\mathcal{S}^{\mathrm{E}}(*, *)]$, $[\mathcal{S}_1^{\mathrm{E}}(*, *, *)]$, $[\mathcal{S}^{+}(*, *)]$, $[\mathcal{S}_1^{+}(*, *, *)]$, $[\mathcal{S}^{-}(*, *)]$, $[\mathcal{S}_1^{-}(*, *, *)]$, $[\mathcal{S}^{*}(*, *)]$, $[\mathcal{S}_1^{*}(*, *, *)]$, $[\mathcal{S}_2^{*}(*, *, *, *)]$, $[\mathcal{S}^{@}(*, *)]$, $[\mathcal{S}_1^{@}(*, *, *)]$, $[\mathcal{S}^{\vdash}(*, *)]$, $[\mathcal{S}_1^{\vdash}(*, *, *, *)]$, $[\mathcal{S}^{\Vdash}(*, *)]$, $[\mathcal{S}_1^{\Vdash}(*, *, *, *)]$, $[\mathcal{S}^{\text{i.e.}}(*, *)]$, $[\mathcal{S}_1^{\text{i.e.}}(*, *, *, *)]$, $[\mathcal{S}_2^{\text{i.e.}}(*, *, *, *, *)]$, $[\mathcal{S}^{\forall}(*, *)]$, $[\mathcal{S}_1^{\forall}(*, *, *, *)]$, $[\mathcal{S}^{;}(*, *)]$, $[\mathcal{S}_1^{;}(*, *, *)]$, $[\mathcal{S}_2^{;}(*, *, *, *)]$, $[\mathcal{T}(*)]$, [claims(∗, ∗, ∗)],

[$\mathrm{claims}_2(*,*,*)$], [<proof>], [proof], [[**Lemma** ∗: ∗]], [[**Proof of** ∗: ∗]], [[∗ **lemma** ∗: ∗]], [[∗ **antilemma** ∗: ∗]], [[∗ **rule** ∗: ∗]], [[∗ **antirule** ∗: ∗]], [verifier], [$\mathcal{V}_1(*)$], [$\mathcal{V}_2(*,*)$], [$\mathcal{V}_3(*,*,*,*)$], [$\mathcal{V}_4(*,*)$], [$\mathcal{V}_5(*,*,*,*)$], [$\mathcal{V}_6(*,*,*,*)$], [$\mathcal{V}_7(*,*,*,*)$], [$\mathrm{Cut}(*,*)$], [$\mathrm{Head}_\oplus(*)$], [$\mathrm{Tail}_\oplus(*)$], [$\mathrm{rule}_1(*,*)$], [$\mathrm{rule}(*,*)$], [Rule tactic], [$\mathrm{Plus}(*,*)$], [[**Theory** ∗]], [$\mathrm{theory}_2(*,*)$], [$\mathrm{theory}_3(*,*)$], [$\mathrm{theory}_4(*,*,*)$], [HeadNil″], [HeadPair″], [Transitivity″], [Contra″], [HeadNil], [HeadPair], [Transitivity], [Contra], [$\mathrm{T_E}$], [ragged right], [ragged right expansion], [$\mathrm{parm}(*,*,*)$], [$\mathrm{parm}^*(*,*,*)$], [$\mathrm{inst}(*,*)$], [$\mathrm{inst}^*(*,*)$], [$\mathrm{occur}(*,*,*)$], [$\mathrm{occur}^*(*,*,*)$], [$\mathrm{unify}(*=*,*)$], [$\mathrm{unify}^*(*=*,*)$], [$\mathrm{unify}_2(*=*,*)$], [$\mathrm{L_a}$], [$\mathrm{L_b}$], [$\mathrm{L_c}$], [$\mathrm{L_d}$], [$\mathrm{L_e}$], [$\mathrm{L_f}$], [$\mathrm{L_g}$], [$\mathrm{L_h}$], [$\mathrm{L_i}$], [$\mathrm{L_j}$], [$\mathrm{L_k}$], [$\mathrm{L_l}$], [$\mathrm{L_m}$], [$\mathrm{L_n}$], [$\mathrm{L_o}$], [$\mathrm{L_p}$], [$\mathrm{L_q}$], [$\mathrm{L_r}$], [$\mathrm{L_s}$], [$\mathrm{L_t}$], [$\mathrm{L_u}$], [$\mathrm{L_v}$], [$\mathrm{L_w}$], [$\mathrm{L_x}$], [$\mathrm{L_y}$], [$\mathrm{L_z}$], [$\mathrm{L_A}$], [$\mathrm{L_B}$], [$\mathrm{L_C}$], [$\mathrm{L_D}$], [$\mathrm{L_E}$], [$\mathrm{L_F}$], [$\mathrm{L_G}$], [$\mathrm{L_H}$], [$\mathrm{L_I}$], [$\mathrm{L_J}$], [$\mathrm{L_K}$], [$\mathrm{L_L}$], [$\mathrm{L_M}$], [$\mathrm{L_N}$], [$\mathrm{L_O}$], [$\mathrm{L_P}$], [$\mathrm{L_Q}$], [$\mathrm{L_R}$], [$\mathrm{L_S}$], [$\mathrm{L_T}$], [$\mathrm{L_U}$], [$\mathrm{L_V}$], [$\mathrm{L_W}$], [$\mathrm{L_X}$], [$\mathrm{L_Y}$], [$\mathrm{L_Z}$], [$\mathrm{L_?}$], [Reflexivity], [$\mathrm{Reflexivity}_1$], [Commutativity], [$\mathrm{Commutativity}_1$], [<tactic>], [tactic], [[$* \overset{\text{tactic}}{=} *$]], [$\mathcal{P}(*,*,*)$], [$\mathcal{P}^*(*,*,*)$], [$\mathrm{p_0}$], [$\mathrm{conclude}_1(*,*)$], [$\mathrm{conclude}_2(*,*,*)$], [$\mathrm{conclude}_3(*,*,*,*)$], [$\mathrm{conclude}_4(*,*)$], [$\dot{0}$], [$\dot{1}$], [$\dot{2}$], [$\dot{a}$], [$\dot{b}$], [$\dot{c}$], [$\dot{d}$], [$\dot{e}$], [$\dot{f}$], [$\dot{g}$], [$\dot{h}$], [$\dot{i}$], [$\dot{j}$], [$\dot{k}$], [$\dot{l}$], [$\dot{m}$], [$\dot{n}$], [$\dot{o}$], [$\dot{p}$], [$\dot{q}$], [$\dot{r}$], [$\dot{s}$], [$\dot{t}$], [$\dot{u}$], [$\dot{v}$], [$\dot{w}$], [$\dot{x}$], [$\dot{y}$], [$\dot{z}$], [$\dot{\mathrm{nonfree}}(*,*)$], [$\dot{\mathrm{nonfree}}^*(*,*)$], [$\mathrm{free}\langle *|*:=*\rangle$], [$\mathrm{free}^*\langle *|*:=*\rangle$], [$*\equiv\langle *|*:=*\rangle$], [$*\equiv\langle^* *|*:=*\rangle$], [S], [A1], [A2], [A3], [A4], [A5], [S1], [S2], [S3], [S4], [S5], [S6], [S7], [S8], [S9], [MP], [Gen], [S′], [A1′], [A2′], [A3′], [A4′], [A5′], [S1′], [S2′], [S3′], [S4′], [S5′], [S6′], [S7′], [S8′], [S9′], [MP′], [Gen′];

Preassociative

[$*_{\{*\}}$], [∗′], [∗[∗]], [∗[∗→∗]], [∗[∗⇒∗]], [$\dot{*}$];

Preassociative

[“ ∗ ”], [], [$(*)^{\mathbf{t}}$], [string(∗) + ∗], [string(∗) ++ ∗], [
∗], [∗], [!∗], [”∗], [#∗], [$∗], [%∗], [&∗], [’∗], [(∗], [)∗], [∗∗], [+∗], [, ∗], [-∗], [.∗], [/∗], [0∗], [1∗], [2∗], [3∗], [4∗], [5∗], [6∗], [7∗], [8∗], [9∗], [:∗], [; ∗], [<∗], [=∗], [>∗], [?∗], [@∗], [A∗], [B∗], [C∗], [D∗], [E∗], [F∗], [G∗], [H∗], [I∗], [J∗], [K∗], [L∗], [M∗], [N∗], [O∗], [P∗], [Q∗], [R∗], [S∗], [T∗], [U∗], [V∗], [W∗], [X∗], [Y∗], [Z∗], [[∗], [\∗], []∗], [ˆ∗], [_∗], [‘∗], [a∗], [b∗], [c∗], [d∗], [e∗], [f∗], [g∗], [h∗], [i∗], [j∗], [k∗], [l∗], [m∗], [n∗], [o∗], [p∗], [q∗], [r∗], [s∗], [t∗], [u∗], [v∗], [w∗], [x∗], [y∗], [z∗], [{∗], [|∗], [}∗], [˜∗], [**Preassociative** ∗; ∗], [**Postassociative** ∗; ∗], [[∗], ∗], [priority ∗ end], [newline ∗], [macro newline ∗];

Preassociative

[∗0], [∗1], [0b], [∗-color(∗)], [$*\text{-color}^*(*)$];

Preassociative

[∗ ’ ∗], [∗ ‘ ∗];

Preassociative

[$*^{\mathrm{H}}$], [$*^{\mathrm{T}}$], [$*^{\mathrm{U}}$], [$*^{\mathrm{h}}$], [$*^{\mathrm{t}}$], [$*^{\mathrm{s}}$], [$*^{\mathrm{c}}$], [$*^{\mathrm{d}}$], [$*^{\mathrm{a}}$], [$*^{\mathrm{C}}$], [$*^{\mathrm{M}}$], [$*^{\mathrm{B}}$], [$*^{\mathrm{r}}$], [$*^{\mathrm{i}}$], [$*^{\mathrm{d}}$], [$*^{\mathrm{R}}$], [$*^{0}$], [$*^{1}$], [$*^{2}$], [$*^{3}$], [$*^{4}$], [$*^{5}$], [$*^{6}$], [$*^{7}$], [$*^{8}$], [$*^{9}$], [$*^{\mathrm{E}}$], [$*^{\mathcal{V}}$], [$*^{\mathcal{C}}$], [$*^{\mathcal{C}^*}$], [∗′];

Preassociative

[$* \cdot *$], [$* \cdot_0 *$], [$* \stackrel{\cdot}{:} *$];

Preassociative

[$* + *$], [$* +_0 *$], [$* +_1 *$], [$* - *$], [$* -_0 *$], [$* -_1 *$], [$* \dot{+} *$];

Preassociative

$[* \cup \{*\}], [* \cup *], [* \backslash \{*\}];$

Postassociative

$[* \therefore *], [* \underline{\therefore} *], [* \underline{::} *], [* \underline{+2*} *], [* :: *], [* +2* *];$

Postassociative

$[*, *];$

Preassociative

$[* \stackrel{\mathrm{B}}{\approx} *], [* \stackrel{\mathrm{D}}{\approx} *], [* \stackrel{\mathrm{C}}{\approx} *], [* \stackrel{\mathrm{P}}{\approx} *], [* \approx *], [* = *], [* \stackrel{+}{\rightarrow} *], [* \stackrel{\mathrm{t}}{=} *], [* \stackrel{\mathrm{t}^*}{=} *], [* \stackrel{\mathrm{r}}{=} *],$
$[* \in_{\mathrm{t}} *], [* \subseteq_{\mathrm{T}} *], [* \stackrel{\mathrm{T}}{=} *], [* \stackrel{\mathrm{s}}{=} *], [* \text{ free in } *], [* \text{ free in}^* *], [* \text{ free for } * \text{ in } *],$
$[* \text{ free for}^* * \text{ in } *], [* \in_{\mathrm{c}} *], [* < *], [* <' *], [* \leq' *], [* \stackrel{\mathrm{p}}{=} *], [*^{\mathcal{P}}];$

Preassociative

$[\neg *], [\dot{\neg} *];$

Preassociative

$[* \wedge *], [* \ddot{\wedge} *], [* \tilde{\wedge} *], [* \wedge_{\mathrm{c}} *], [* \dot{\wedge} *];$

Preassociative

$[* \vee *], [* \parallel *], [* \ddot{\vee} *], [* \dot{\vee} *];$

Preassociative

$[\dot{\forall} * \colon *], [\dot{\exists} * \colon *];$

Postassociative

$[* \ddot{\Rightarrow} *], [* \dot{\Rightarrow} *], [* \dot{\Leftrightarrow} *];$

Postassociative

$[* : *], [* ! *];$

Preassociative

$[* \left\{ \begin{array}{c} * \\ * \end{array} \right.];$

Preassociative

$[\lambda * . *], [\Lambda *], [\textbf{if } * \textbf{ then } * \textbf{ else } *], [\textbf{let } * = * \textbf{ in } *], [\textbf{let } * \ddot{=} * \text{ in } *];$

Preassociative

$[*^{\mathrm{I}}], [*^{\triangleright}], [*^{\mathrm{V}}], [*^{+}], [*^{-}], [*^{*}];$

Preassociative

$[* @ *], [* \triangleright *], [* \trianglerighteq *], [* \gg *];$

Postassociative

$[* \vdash *], [* \Vdash *], [* \text{ i.e. } *];$

Preassociative

$[\forall * \colon *];$

Postassociative

$[* \oplus *];$

Postassociative

$[*; *];$

Preassociative

$[* \text{ proves } *];$

Preassociative

$[* \textbf{ proof of } * : *], [\text{Line } * : * \gg *; *], [\text{Last line } * \gg * \square],$
$[\text{Line } * : \text{Premise} \gg *; *], [\text{Line } * : \text{Side-condition} \gg *; *], [\text{Arbitrary} \gg *; *],$
$[\text{Local} \gg * = *; *];$

Postassociative

$[* \text{ then } *], [*[\ *\ ]*]$;
Preassociative
$[*\&*]$;
Preassociative
$[*\backslash\backslash*]$; **End table**

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