Peano arithmetic

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1 Peano arithmetic

This Logiweb page [1] defines Peano arithmetic.

1.1 The constructs of Peano arithmetic

Terms of Peano arithmetic are constructed from zero $[\dot{0} \stackrel{\text{pyk}}{\to}$ "peano zero"$][\dot{0} \stackrel{\text{tex}}{\to}$ "
\dot{0}"], successor $[x' \stackrel{\text{pyk}}{\to}$ "* peano succ"$][x' \stackrel{\text{tex}}{\to}$ "#1.'"], plus $[x \dot{+} y \stackrel{\text{pyk}}{\to}$ "* peano plus *"$][x \dot{+} y \stackrel{\text{tex}}{\to}$ "#1.
\mathop{\dot{+}} #2."], and times $[x \dot{\cdot} y \stackrel{\text{pyk}}{\to}$ "* peano times *"$][x \dot{\cdot} y \stackrel{\text{tex}}{\to}$ "#1.
\mathop{\dot{\cdot}} #2."].

Formulas of Peano arithmeric are constructed from equality $[x \stackrel{p}{=} y \stackrel{\text{pyk}}{\to}$ "* peano is *"$][x \stackrel{p}{=} y \stackrel{\text{tex}}{\to}$ "#1.
\stackrel{p}{=} #2."], negation $[\dot{\neg} x \stackrel{\text{pyk}}{\to}$ "peano not *"$][\dot{\neg} x \stackrel{\text{tex}}{\to}$ "
\dot{\neg}\, {#1.}"], implication $[x \dot{\Rightarrow} y \stackrel{\text{pyk}}{\to}$ "* peano imply *"$][x \dot{\Rightarrow} y \stackrel{\text{tex}}{\to}$ "#1.
\mathrel{\dot{\Rightarrow}} #2."], and universal quantification $[\dot{\forall} x \colon y \stackrel{\text{pyk}}{\to}$ "peano all * indeed *"$][\dot{\forall} x \colon y \stackrel{\text{tex}}{\to}$ "
\dot{\forall} #1.
\colon #2."].

From these constructs we macro define one $[\dot{1} \stackrel{\text{pyk}}{\to}$ "peano one"$][\dot{1} \stackrel{\text{tex}}{\to}$ "
\dot{1}"], two $[\dot{2} \stackrel{\text{pyk}}{\to}$ "peano two"$][\dot{2} \stackrel{\text{tex}}{\to}$ "
\dot{2}"], conjunction $[x \dot{\wedge} y \stackrel{\text{pyk}}{\to}$ "* peano and *"$][x \dot{\wedge} y \stackrel{\text{tex}}{\to}$ "#1.
\mathrel{\dot{\wedge}} #2."], disjunction $[x \dot{\vee} y \stackrel{\text{pyk}}{\to}$ "* peano or *"$][x \dot{\vee} y \stackrel{\text{tex}}{\to}$ "#1.
\mathrel{\dot{\vee}} #2."], biimplication $[x \dot{\Leftrightarrow} y \stackrel{\text{pyk}}{\to}$ "* peano iff *"$][x \dot{\Leftrightarrow} y \stackrel{\text{tex}}{\to}$ "#1.
\mathrel{\dot{\Leftrightarrow}} #2."], and existential quantification $[\dot{\exists} x \colon y \stackrel{\text{pyk}}{\to}$ "peano exist * indeed *"$][\dot{\exists} x \colon y \stackrel{\text{tex}}{\to}$ "
\dot{\exists} #1.
\colon #2."]:

$$[\dot{1} \stackrel{\text{macro}}{\to} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, \lceil[\dot{1} \doteq \dot{0}']\rceil)]$$

$$[\dot{2} \stackrel{\text{macro}}{\to} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, \lceil[\dot{2} \doteq \dot{1}']\rceil)]$$

$[\mathsf{x} \dot{\wedge} \mathsf{y} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \dot{\wedge} \mathsf{y} \ddot{=} \dot{\neg}(\mathsf{x} \dot{\Rightarrow} \dot{\neg}\mathsf{y})]\rceil)]$

$[\mathsf{x} \dot{\vee} \mathsf{y} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \dot{\vee} \mathsf{y} \ddot{=} \dot{\neg}\mathsf{x} \dot{\Rightarrow} \mathsf{y}]\rceil)]$

$[\mathsf{x} \dot{\Leftrightarrow} \mathsf{y} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \dot{\Leftrightarrow} \mathsf{y} \ddot{=} (\mathsf{x} \dot{\Rightarrow} \mathsf{y}) \dot{\wedge} (\mathsf{y} \dot{\Rightarrow} \mathsf{x})]\rceil)]$

$[\dot{\exists}\mathsf{x}\colon \mathsf{y} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{\exists}\mathsf{x}\colon \mathsf{y} \ddot{=} \dot{\neg}\dot{\forall}\mathsf{x}\colon \dot{\neg}\mathsf{y}]\rceil)]$

1.2 Variables

We now introduce the unary operator $[\dot{\mathsf{x}} \stackrel{\text{pyk}}{\to}$ "∗ peano var"$][\dot{\mathsf{x}} \stackrel{\text{tex}}{\to}$ "
\dot{#1.
}"] and define that a term is a *Peano variable* (i.e. a variable of Peano arithmetic) if it has the $[\dot{\mathsf{x}}]$ operator in its root. $[\mathsf{x}^{\mathcal{P}} \stackrel{\text{pyk}}{\to}$ "∗ is peano var"$][\mathsf{x}^{\mathcal{P}} \stackrel{\text{tex}}{\to}$ "#1.
{} ^ {\cal P}"] is true if $[\mathsf{x}]$ is a Peano variable:

$$[\mathsf{x}^{\mathcal{P}} \stackrel{\text{val}}{\to} \mathsf{x} \stackrel{\text{r}}{=} \lceil \dot{\mathsf{x}} \rceil]$$

We macro define $[\dot{a} \stackrel{\text{pyk}}{\to}$ "peano a"$][\dot{a} \stackrel{\text{tex}}{\to}$ "
\dot{\mathit{a}}"] to be a Peano variable:

$$[\dot{a} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{a} \ddot{=} \dot{\mathsf{a}}]\rceil)]$$

Appendix A.2 defines Peano variables for the other letters of the English alphabet.
$[\dot{\text{nonfree}}(\mathsf{x},\mathsf{y}) \stackrel{\text{pyk}}{\to}$ "peano nonfree ∗ in ∗ end nonfree"$][\dot{\text{nonfree}}(\mathsf{x},\mathsf{y}) \stackrel{\text{tex}}{\to}$ "
\dot{nonfree}(#1.
, #2.
)"] is true if the Peano variable $[\mathsf{x}]$ does not occur free in the Peano term/formula $[\mathsf{y}]$. $[\dot{\text{nonfree}}^*(\mathsf{x},\mathsf{y}) \stackrel{\text{pyk}}{\to}$ "peano nonfree star ∗ in ∗ end nonfree"$][\dot{\text{nonfree}}^*(\mathsf{x},\mathsf{y}) \stackrel{\text{tex}}{\to}$ "
\dot{nonfree}^*(#1.
, #2.
)"] is true if the Peano variable $[\mathsf{x}]$ does not occur free in the list $[\mathsf{y}]$ of Peano terms/formulas.

$$\begin{aligned}&[\dot{\text{nonfree}}(\mathsf{x},\mathsf{y}) \stackrel{\text{val}}{\to}\\ &\text{If}(\mathsf{y}^{\mathcal{P}}, \neg \mathsf{x} \stackrel{\text{t}}{=} \mathsf{y},\\ &\text{If}(\neg \mathsf{y} \stackrel{\text{r}}{=} \lceil \dot{\forall}\mathsf{x}\colon \mathsf{y}\rceil, \dot{\text{nonfree}}^*(\mathsf{x},\mathsf{y}^{\text{t}}),\\ &\text{If}(\mathsf{x} \stackrel{\text{t}}{=} \mathsf{y}^1, \mathsf{T}, \dot{\text{nonfree}}(\mathsf{x},\mathsf{y}^2))))]\end{aligned}$$

$$[\dot{\text{nonfree}}^*(\mathsf{x},\mathsf{y}) \stackrel{\text{val}}{\to} \mathsf{x}!\text{If}(\mathsf{y},\mathsf{T},\text{If}(\dot{\text{nonfree}}(\mathsf{x},\mathsf{y}^{\text{h}}),\dot{\text{nonfree}}^*(\mathsf{x},\mathsf{y}^{\text{t}}),\mathsf{F}))]$$

[$\dot{\mathrm{free}}\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \stackrel{\text{pyk}}{\to}$ "peano free * set * to * end free"][$\dot{\mathrm{free}}\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \stackrel{\text{tex}}{\to}$ "
\dot{free}\langle #1.
| #2.
:= #3.
\rangle"] is true if the substitution [$\langle \mathsf{a}\,|\mathsf{x}{:=}\,\mathsf{b}\rangle$] is free.

[$\dot{\mathrm{free}}^*\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \stackrel{\text{pyk}}{\to}$ "peano free star * set * to * end free"][$\dot{\mathrm{free}}^*\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \stackrel{\text{tex}}{\to}$ "
\dot{free}{}^*\langle #1.
| #2.
:= #3.
\rangle"] is the version where [a] is a list of terms.

$$
\begin{aligned}
&[\dot{\mathrm{free}}\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \stackrel{\text{val}}{\to} \mathsf{x!b!}\\
&\mathrm{If}(\mathsf{a}^{\mathcal{P}}, \mathsf{T},\\
&\mathrm{If}(\neg \mathsf{a} \stackrel{\text{r}}{=} \lceil \dot{\forall} \mathsf{u}\colon \mathsf{v}\rceil, \dot{\mathrm{free}}^*\langle \mathsf{a}^{\mathrm{t}}|\mathsf{x} := \mathsf{b}\rangle,\\
&\mathrm{If}(\mathsf{a}^1 \stackrel{\text{t}}{=} \mathsf{x}, \mathsf{T},\\
&\mathrm{If}(\mathrm{non}\dot{\mathrm{free}}(\mathsf{x}, \mathsf{a}^2), \mathsf{T},\\
&\mathrm{If}(\neg \mathrm{non}\dot{\mathrm{free}}(\mathsf{a}^1, \mathsf{b}), \mathsf{F},\\
&\dot{\mathrm{free}}\langle \mathsf{a}^2|\mathsf{x} := \mathsf{b}\rangle))))))]
\end{aligned}
$$

$$
[\dot{\mathrm{free}}^*\langle \mathsf{a}|\mathsf{x} := \mathsf{b}\rangle \stackrel{\text{val}}{\to} \mathsf{x!b!}\mathrm{If}(\mathsf{a}, \mathsf{T}, \mathrm{If}(\dot{\mathrm{free}}\langle \mathsf{a}^{\mathrm{h}}|\mathsf{x} := \mathsf{b}\rangle, \dot{\mathrm{free}}^*\langle \mathsf{a}^{\mathrm{t}}|\mathsf{x} := \mathsf{b}\rangle, \mathsf{F}))]
$$

[$\mathsf{a}{\equiv}\langle \mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \stackrel{\text{pyk}}{\to}$ "peano sub * is * where * is * end sub"][$\mathsf{a}{\equiv}\langle \mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \stackrel{\text{tex}}{\to}$ "#1.
{\equiv}\langle #2.
|#3.
:=#4.
\rangle"] is true if [a] equals [$\langle \mathsf{b}\,|\mathsf{x}{:=}\,\mathsf{c}\rangle$]. [$\mathsf{a}{\equiv}\langle^* \mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \stackrel{\text{pyk}}{\to}$ "peano sub star * is * where * is * end sub"][$\mathsf{a}{\equiv}\langle^* \mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \stackrel{\text{tex}}{\to}$ "#1.
{\equiv}\langle^* #2.
|#3.
:=#4.
\rangle"] is the version where [a] and [b] are lists.

$$
\begin{aligned}
&[\mathsf{a}{\equiv}\langle \mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \stackrel{\text{val}}{\to} \mathsf{a!x!c!}\\
&\mathrm{If}(\mathrm{If}(\mathsf{b} \stackrel{\text{r}}{=} \lceil \dot{\forall} \mathsf{u}\colon \mathsf{v}\rceil, \mathsf{b}^1 \stackrel{\text{t}}{=} \mathsf{x}, \mathsf{F}), \mathsf{a} \stackrel{\text{t}}{=} \mathsf{b},\\
&\mathrm{If}(\mathsf{b}^{\mathcal{P}} \wedge \mathsf{b} \stackrel{\text{t}}{=} \mathsf{x}, \mathsf{a} \stackrel{\text{t}}{=} \mathsf{c}, \mathrm{If}(\\
&\mathsf{a} \stackrel{\text{r}}{=} \mathsf{b}, \mathsf{a}^{\mathrm{t}}{\equiv}\langle^* \mathsf{b}^{\mathrm{t}}|\mathsf{x} := \mathsf{c}\rangle, \mathsf{F})))]
\end{aligned}
$$

$$
[\mathsf{a}{\equiv}\langle^* \mathsf{b}|\mathsf{x} := \mathsf{c}\rangle \stackrel{\text{val}}{\to} \mathsf{b!x!c!}\mathrm{If}(\mathsf{a}, \mathsf{T}, \mathrm{If}(\mathsf{a}^{\mathrm{h}}{\equiv}\langle \mathsf{b}^{\mathrm{h}}|\mathsf{x} := \mathsf{c}\rangle, \mathsf{a}^{\mathrm{t}}{\equiv}\langle^* \mathsf{b}^{\mathrm{t}}|\mathsf{x} := \mathsf{c}\rangle, \mathsf{F}))]
$$

1.3 Mendelsons system S

System [S $\stackrel{\text{pyk}}{\to}$ "system s"][S $\stackrel{\text{tex}}{\to}$ "
S"] of Mendelson [2] expresses Peano arithmetic. It comprises the axioms [A1 $\stackrel{\text{pyk}}{\to}$ "axiom a one"][A1 $\stackrel{\text{tex}}{\to}$ "

A1"], [A2 $\stackrel{\text{pyk}}{\rightarrow}$ "axiom a two"][A2 $\stackrel{\text{tex}}{\rightarrow}$ "
A2"], [A3 $\stackrel{\text{pyk}}{\rightarrow}$ "axiom a three"][A3 $\stackrel{\text{tex}}{\rightarrow}$ "
A3"], [A4 $\stackrel{\text{pyk}}{\rightarrow}$ "axiom a four"][A4 $\stackrel{\text{tex}}{\rightarrow}$ "
A4"], and [A5 $\stackrel{\text{pyk}}{\rightarrow}$ "axiom a five"][A5 $\stackrel{\text{tex}}{\rightarrow}$ "
A5"] and inference rules [MP $\stackrel{\text{pyk}}{\rightarrow}$ "rule mp"][MP $\stackrel{\text{tex}}{\rightarrow}$ "
MP"] and [Gen $\stackrel{\text{pyk}}{\rightarrow}$ "rule gen"][Gen $\stackrel{\text{tex}}{\rightarrow}$ "
Gen"] of first order predicate calculus. Furthermore, it comprises the proper axioms [S1 $\stackrel{\text{pyk}}{\rightarrow}$ "axiom s one"][S1 $\stackrel{\text{tex}}{\rightarrow}$ "
S1"], [S2 $\stackrel{\text{pyk}}{\rightarrow}$ "axiom s two"][S2 $\stackrel{\text{tex}}{\rightarrow}$ "
S2"], [S3 $\stackrel{\text{pyk}}{\rightarrow}$ "axiom s three"][S3 $\stackrel{\text{tex}}{\rightarrow}$ "
S3"], [S4 $\stackrel{\text{pyk}}{\rightarrow}$ "axiom s four"][S4 $\stackrel{\text{tex}}{\rightarrow}$ "
S4"], [S5 $\stackrel{\text{pyk}}{\rightarrow}$ "axiom s five"][S5 $\stackrel{\text{tex}}{\rightarrow}$ "
S5"], [S6 $\stackrel{\text{pyk}}{\rightarrow}$ "axiom s six"][S6 $\stackrel{\text{tex}}{\rightarrow}$ "
S6"], [S7 $\stackrel{\text{pyk}}{\rightarrow}$ "axiom s seven"][S7 $\stackrel{\text{tex}}{\rightarrow}$ "
S7"], [S8 $\stackrel{\text{pyk}}{\rightarrow}$ "axiom s eight"][S8 $\stackrel{\text{tex}}{\rightarrow}$ "
S8"], and [S9 $\stackrel{\text{pyk}}{\rightarrow}$ "axiom s nine"][S9 $\stackrel{\text{tex}}{\rightarrow}$ "
S9"]. System [S] is defined thus:

$$[\mathrm{S} \stackrel{\text{stmt}}{\rightarrow} \dot{a} \dot{+} \dot{b}' \stackrel{p}{=} \dot{a} \dot{+} \dot{b}' \oplus \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg}\, \underline{b} \dot{\Rightarrow} \dot{\neg}\, \underline{a} \dot{\Rightarrow} \dot{\neg}\, \underline{b} \Rightarrow \underline{a} \dot{\Rightarrow} \underline{b} \oplus \dot{a} \stackrel{p}{=} \dot{b} \dot{\Rightarrow} \dot{a}' \stackrel{p}{=} \dot{b}' \oplus \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \dot{\Rightarrow} \underline{b} \vdash \underline{a} \vdash \underline{b} \oplus \dot{a}' \stackrel{p}{=} \dot{b}' \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{b} \oplus \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \oplus \forall \underline{x}\colon \forall \underline{a}\colon \forall \underline{b}\colon \mathrm{nonfree}(\underline{x}, \underline{a}) \Vdash \dot{\forall} \underline{x}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \dot{\forall} \underline{x}\colon \underline{b} \oplus \dot{a} \mathbin{\dot{\cdot}} \dot{b}' \stackrel{p}{=} \dot{a} \mathbin{\dot{\cdot}} \dot{b} \dot{+} \dot{a} \oplus \dot{a} \dot{+} \dot{0} \stackrel{p}{=} \dot{a} \oplus \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{c} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{c} \oplus \dot{a} \stackrel{p}{=} \dot{b} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{c} \dot{\Rightarrow} \dot{b} \stackrel{p}{=} \dot{c} \oplus \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{x}\colon \underline{b} \equiv \langle \underline{a} | \underline{x} := \dot{0} \rangle \Vdash \underline{c} \equiv \langle \underline{a} | \underline{x} := \underline{x}' \rangle \Vdash \underline{b} \dot{\Rightarrow} \dot{\forall} \underline{x}\colon \underline{a} \dot{\Rightarrow} \underline{c} \dot{\Rightarrow} \dot{\forall} \underline{x}\colon \underline{a} \oplus \neg \dot{0} \stackrel{p}{=} \dot{a}' \oplus \forall \underline{x}\colon \forall \underline{a}\colon \underline{a} \vdash \dot{\forall} \underline{x}\colon \underline{a} \oplus \forall \underline{c}\colon \forall \underline{a}\colon \forall \underline{x}\colon \forall \underline{b}\colon \lceil \underline{a} \rceil \equiv \langle \lceil \underline{b} \rceil | \lceil \underline{x} \rceil := \lceil \underline{c} \rceil \rangle \Vdash \dot{\forall} \underline{x}\colon \underline{b} \dot{\Rightarrow} \underline{a} \oplus \dot{a} \mathbin{\dot{\cdot}} \dot{0} \stackrel{p}{=} \dot{0}]$$

[A1 $\stackrel{\text{stmt}}{\rightarrow}$ $\mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a}$][A1 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[A2 $\stackrel{\text{stmt}}{\rightarrow}$ $\mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{c} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{c}$][A2 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[A3 $\stackrel{\text{stmt}}{\rightarrow}$ $\mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg}\, \underline{b} \dot{\Rightarrow} \dot{\neg}\, \underline{a} \dot{\Rightarrow} \dot{\neg}\, \underline{b} \Rightarrow \underline{a} \dot{\Rightarrow} \underline{b}$][A3 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

The order of quantifiers in the following axiom is such that [$\underline{c}$] which the current conclusion tactic cannot guess comes first. This allows to supply a value for [$\underline{c}$] without having to supply values for the other meta-variables.

[A4 $\stackrel{\text{stmt}}{\rightarrow}$ $\mathrm{S} \vdash \forall \underline{c}\colon \forall \underline{a}\colon \forall \underline{x}\colon \forall \underline{b}\colon \lceil \underline{a} \rceil \equiv \langle \lceil \underline{b} \rceil | \lceil \underline{x} \rceil := \lceil \underline{c} \rceil \rangle \Vdash \dot{\forall} \underline{x}\colon \underline{b} \dot{\Rightarrow} \underline{a}$][A4 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[A5 $\stackrel{\text{stmt}}{\rightarrow}$ $\mathrm{S} \vdash \forall \underline{x}\colon \forall \underline{a}\colon \forall \underline{b}\colon \mathrm{nonfree}(\underline{x}, \underline{a}) \Vdash \dot{\forall} \underline{x}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \dot{\forall} \underline{x}\colon \underline{b}$][A5 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[MP $\stackrel{\text{stmt}}{\rightarrow}$ $\mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \dot{\Rightarrow} \underline{b} \vdash \underline{a} \vdash \underline{b}$][MP $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[Gen $\stackrel{\text{stmt}}{\rightarrow}$ $\mathrm{S} \vdash \forall \underline{x}\colon \forall \underline{a}\colon \underline{a} \vdash \dot{\forall} \underline{x}\colon \underline{a}$][Gen $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

Axiom [S1] to [S8] are stated as Mendelson [2] does. This is done here to test certain parts of the Logiweb system. Serious users of Peano arithmetic are advised to take Mendelson's Lemma 3.1 as axioms instead.

[S1 $\stackrel{\text{stmt}}{\rightarrow}$ $\mathrm{S} \vdash \dot{a} \stackrel{p}{=} \dot{b} \Rightarrow \dot{a} \stackrel{p}{=} \dot{c} \dot{\Rightarrow} \dot{b} \stackrel{p}{=} \dot{c}$][S1 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[S2 $\stackrel{\text{stmt}}{\rightarrow}$ $\mathrm{S} \vdash \dot{a} \stackrel{p}{=} \dot{b} \Rightarrow \dot{a}' \stackrel{p}{=} \dot{b}'$][S2 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[S3 $\stackrel{\text{stmt}}{\rightarrow}$ $\mathrm{S} \vdash \neg \dot{0} \stackrel{p}{=} \dot{a}'$][S3 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[S4 $\stackrel{\text{stmt}}{\rightarrow}$ $\mathrm{S} \vdash \dot{a}' \stackrel{p}{=} \dot{b}' \Rightarrow \dot{a} \stackrel{p}{=} \dot{b}$][S4 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[S5 $\stackrel{\text{stmt}}{\rightarrow}$ $\mathrm{S} \vdash \dot{a} \dot{+} \dot{0} \stackrel{p}{=} \dot{a}$][S5 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[S6 $\stackrel{\text{stmt}}{\rightarrow}$ $\mathrm{S} \vdash \dot{a} \dot{+} \dot{b}' \stackrel{p}{=} \dot{a} \dot{+} \dot{b}'$][S6 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[S7 $\stackrel{\text{stmt}}{\rightarrow}$ $\mathrm{S} \vdash \dot{a} \dot{\cdot} \dot{0} \stackrel{p}{=} \dot{0}$][S7 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[S8 $\stackrel{\text{stmt}}{\rightarrow}$ $\mathrm{S} \vdash \dot{a} \dot{\cdot} \dot{b}' \stackrel{p}{=} \dot{a} \dot{\cdot} \dot{b} \dot{+} \dot{a}$][S8 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[S9 $\stackrel{\text{stmt}}{\rightarrow}$ $\mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{x}\colon \underline{b} \equiv \langle \underline{a} | \underline{x} := \dot{0} \rangle \Vdash \underline{c} \equiv \langle \underline{a} | \underline{x} := \underline{x}' \rangle \Vdash \underline{b} \dot{\Rightarrow} \dot{\forall} \underline{x}\colon \underline{a} \Rightarrow \underline{c} \dot{\Rightarrow} \dot{\forall} \underline{x}\colon \underline{a}$][S9 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

1.4 A lemma and a proof

We now prove Lemma [L3.2(a) $\stackrel{\text{pyk}}{\rightarrow}$ "lemma l three two a"][L3.2(a) $\stackrel{\text{tex}}{\rightarrow}$ " L3.2(a)"] which is an instance of the corresponding proposition in Mendelson [2]:

[L3.2(a) $\stackrel{\text{stmt}}{\rightarrow}$ $\mathrm{S} \vdash \dot{x} \stackrel{p}{=} \dot{x}$]

[L3.2(a) $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{S} \vdash \mathrm{S5} \gg \dot{a} \dot{+} \dot{0} \stackrel{p}{=} \dot{a}; \mathrm{Gen} \rhd \dot{a} \dot{+} \dot{0} \stackrel{p}{=} \dot{a} \gg \dot{\forall} \dot{a}\colon \dot{a} \dot{+} \dot{0} \stackrel{p}{=} \dot{a}; \mathrm{A4} @ \dot{x} \gg \dot{\forall} \dot{a}\colon \dot{a} \dot{+} \dot{0} \stackrel{p}{=} \dot{a} \Rightarrow \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x}; \mathrm{MP} \rhd \dot{\forall} \dot{a}\colon \dot{a} \dot{+} \dot{0} \stackrel{p}{=} \dot{a} \Rightarrow \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \rhd \dot{\forall} \dot{a}\colon \dot{a} \dot{+} \dot{0} \stackrel{p}{=} \dot{a} \gg \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x}; \mathrm{S1} \gg \dot{a} \stackrel{p}{=} \dot{b} \Rightarrow \dot{a} \stackrel{p}{=} \dot{c} \Rightarrow \dot{b} \stackrel{p}{=} \dot{c}; \mathrm{Gen} \rhd \dot{a} \stackrel{p}{=} \dot{b} \Rightarrow \dot{a} \stackrel{p}{=} \dot{c} \Rightarrow \dot{b} \stackrel{p}{=} \dot{c} \gg \dot{\forall} \dot{c}\colon \dot{a} \stackrel{p}{=} \dot{b} \Rightarrow \dot{a} \stackrel{p}{=} \dot{c} \Rightarrow \dot{b} \stackrel{p}{=} \dot{c}; \mathrm{A4} @ \dot{x} \gg \dot{\forall} \dot{c}\colon \dot{a} \stackrel{p}{=} \dot{b} \Rightarrow \dot{a} \stackrel{p}{=} \dot{c} \Rightarrow \dot{b} \stackrel{p}{=} \dot{c} \Rightarrow \dot{a} \stackrel{p}{=} \dot{b} \Rightarrow \dot{a} \stackrel{p}{=} \dot{x} \Rightarrow \dot{b} \stackrel{p}{=} \dot{x}; \mathrm{MP} \rhd \dot{\forall} \dot{c}\colon \dot{a} \stackrel{p}{=} \dot{b} \Rightarrow \dot{a} \stackrel{p}{=} \dot{c} \Rightarrow \dot{b} \stackrel{p}{=} \dot{c} \Rightarrow \dot{a} \stackrel{p}{=} \dot{b} \Rightarrow \dot{a} \stackrel{p}{=} \dot{x} \Rightarrow \dot{b} \stackrel{p}{=} \dot{x} \rhd \dot{\forall} \dot{c}\colon \dot{a} \stackrel{p}{=} \dot{b} \Rightarrow \dot{a} \stackrel{p}{=} \dot{c} \Rightarrow \dot{b} \stackrel{p}{=} \dot{c} \gg \dot{a} \stackrel{p}{=} \dot{b} \Rightarrow \dot{a} \stackrel{p}{=} \dot{x} \Rightarrow \dot{b} \stackrel{p}{=} \dot{x}; \mathrm{Gen} \rhd \dot{a} \stackrel{p}{=} \dot{b} \Rightarrow \dot{a} \stackrel{p}{=} \dot{x} \Rightarrow \dot{b} \stackrel{p}{=} \dot{x} \gg \dot{\forall} \dot{b}\colon \dot{a} \stackrel{p}{=} \dot{b} \Rightarrow \dot{a} \stackrel{p}{=} \dot{x} \Rightarrow \dot{b} \stackrel{p}{=} \dot{x}; \mathrm{A4} @ \dot{x} \gg \dot{\forall} \dot{b}\colon \dot{a} \stackrel{p}{=} \dot{b} \Rightarrow \dot{a} \stackrel{p}{=} \dot{x} \Rightarrow \dot{b} \stackrel{p}{=} \dot{x} \Rightarrow \dot{a} \stackrel{p}{=} \dot{x} \Rightarrow \dot{a} \stackrel{p}{=} \dot{x} \Rightarrow \dot{x} \stackrel{p}{=} \dot{x}; \mathrm{MP} \rhd \dot{\forall} \dot{b}\colon \dot{a} \stackrel{p}{=} \dot{b} \Rightarrow \dot{a} \stackrel{p}{=} \dot{x} \Rightarrow \dot{b} \stackrel{p}{=} \dot{x} \Rightarrow \dot{a} \stackrel{p}{=} \dot{x} \Rightarrow \dot{a} \stackrel{p}{=} \dot{x} \Rightarrow \dot{x} \stackrel{p}{=} \dot{x} \rhd \dot{\forall} \dot{b}\colon \dot{a} \stackrel{p}{=} \dot{b} \Rightarrow \dot{a} \stackrel{p}{=} \dot{x} \Rightarrow \dot{b} \stackrel{p}{=} \dot{x} \gg \dot{a} \stackrel{p}{=} \dot{x} \Rightarrow \dot{a} \stackrel{p}{=} \dot{x} \Rightarrow \dot{x} \stackrel{p}{=}$

$\dot{x}$; Gen $\rhd \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{p}{=} \dot{x} \gg \dot{\forall}\dot{a}\colon \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{p}{=} \dot{x}$; A4 $@ \dot{x} \dot{+} \dot{0} \gg \dot{\forall}\dot{a}\colon \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{p}{=} \dot{x}$; MP $\rhd \dot{\forall}\dot{a}\colon \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{p}{=} \dot{x} \rhd \dot{\forall}\dot{a}\colon \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{p}{=} \dot{x} \gg \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{p}{=} \dot{x}$; MP $\rhd \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{p}{=} \dot{x} \rhd \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \gg \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{p}{=} \dot{x}$; MP $\rhd \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \dot{\Rightarrow} \dot{x} \stackrel{p}{=} \dot{x} \rhd \dot{x} \dot{+} \dot{0} \stackrel{p}{=} \dot{x} \gg \dot{x} \stackrel{p}{=} \dot{x}$], p_0, c)]

1.5 An alternative axiomatic system

System [S′ $\stackrel{\text{pyk}}{\to}$ “system prime s”][S′ $\stackrel{\text{tex}}{\to}$ “
S’”] is system [S] in which the proper axioms are taken from Lemma 3.1 in Mendelson [2]. It comprises the axioms [A1′ $\stackrel{\text{pyk}}{\to}$ “axiom prime a one”][A1′ $\stackrel{\text{tex}}{\to}$ “
A1’”], [A2′ $\stackrel{\text{pyk}}{\to}$ “axiom prime a two”][A2′ $\stackrel{\text{tex}}{\to}$ “
A2’”], [A3′ $\stackrel{\text{pyk}}{\to}$ “axiom prime a three”][A3′ $\stackrel{\text{tex}}{\to}$ “
A3’”], [A4′ $\stackrel{\text{pyk}}{\to}$ “axiom prime a four”][A4′ $\stackrel{\text{tex}}{\to}$ “
A4’”], and [A5′ $\stackrel{\text{pyk}}{\to}$ “axiom prime a five”][A5′ $\stackrel{\text{tex}}{\to}$ “
A5’”] and inference rules [MP′ $\stackrel{\text{pyk}}{\to}$ “rule prime mp”][MP′ $\stackrel{\text{tex}}{\to}$ “
MP’”] and [Gen′ $\stackrel{\text{pyk}}{\to}$ “rule prime gen”][Gen′ $\stackrel{\text{tex}}{\to}$ “
Gen’”] of first order predicate calculus. Furthermore, it comprises the proper axioms [S1′ $\stackrel{\text{pyk}}{\to}$ “axiom prime s one”][S1′ $\stackrel{\text{tex}}{\to}$ “
S1’”], [S2′ $\stackrel{\text{pyk}}{\to}$ “axiom prime s two”][S2′ $\stackrel{\text{tex}}{\to}$ “
S2’”], [S3′ $\stackrel{\text{pyk}}{\to}$ “axiom prime s three”][S3′ $\stackrel{\text{tex}}{\to}$ “
S3’”], [S4′ $\stackrel{\text{pyk}}{\to}$ “axiom prime s four”][S4′ $\stackrel{\text{tex}}{\to}$ “
S4’”], [S5′ $\stackrel{\text{pyk}}{\to}$ “axiom prime s five”][S5′ $\stackrel{\text{tex}}{\to}$ “
S5’”], [S6′ $\stackrel{\text{pyk}}{\to}$ “axiom prime s six”][S6′ $\stackrel{\text{tex}}{\to}$ “
S6’”], [S7′ $\stackrel{\text{pyk}}{\to}$ “axiom prime s seven”][S7′ $\stackrel{\text{tex}}{\to}$ “
S7’”], [S8′ $\stackrel{\text{pyk}}{\to}$ “axiom prime s eight”][S8′ $\stackrel{\text{tex}}{\to}$ “
S8’”], and [S9′ $\stackrel{\text{pyk}}{\to}$ “axiom prime s nine”][S9′ $\stackrel{\text{tex}}{\to}$ “
S9’”].
System [S′] is defined thus:

$$[S' \stackrel{\text{stmt}}{\to} \forall\underline{a}\colon \forall\underline{b}\colon \underline{a}' \stackrel{p}{=} \underline{b}' \dot{\Rightarrow} \underline{a} \stackrel{p}{=} \underline{b} \oplus \forall\underline{h}\colon \forall\underline{t}\colon \forall\underline{r}\colon \forall\underline{s}\colon \underline{h} \dot{\Rightarrow} \underline{t} \stackrel{p}{=} \underline{r} \vdash \underline{h} \dot{\Rightarrow} \underline{t} \stackrel{p}{=} \underline{s} \vdash \underline{h} \dot{\Rightarrow} \underline{r} \stackrel{p}{=} \underline{s} \oplus \forall\underline{a}\colon \forall\underline{b}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \oplus \forall\underline{x}\colon \forall\underline{a}\colon \forall\underline{b}\colon \mathrm{nonfree}(\lceil\underline{x}\rceil, \lceil\underline{a}\rceil) \Vdash \dot{\forall}\underline{x}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \dot{\forall}\underline{x}\colon \underline{b} \oplus \forall\underline{h}\colon \forall\underline{t}\colon \forall\underline{r}\colon \underline{h} \dot{\Rightarrow} \underline{t} \dot{+} \underline{r}' \stackrel{p}{=} \underline{t} \dot{+} \underline{r}' \oplus \forall\underline{a}\colon \forall\underline{b}\colon \underline{a} \dot{\cdot} \underline{b}' \stackrel{p}{=} \underline{a} \dot{\cdot} \underline{b} \dot{+} \underline{a} \oplus \forall\underline{a}\colon \forall\underline{b}\colon \underline{a} \stackrel{p}{=} \underline{b} \dot{\Rightarrow} \underline{a}' \stackrel{p}{=} \underline{b}' \oplus \forall\underline{a}\colon \forall\underline{b}\colon \underline{a} \dot{\Rightarrow} \underline{b} \vdash \underline{a} \vdash \underline{b} \oplus \forall\underline{a}\colon \forall\underline{b}\colon \dot{\neg}\underline{b} \dot{\Rightarrow} \dot{\neg}\underline{a} \dot{\Rightarrow} \dot{\neg}\underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{b} \oplus \forall\underline{h}\colon \forall\underline{t}\colon \forall\underline{r}\colon \underline{h} \dot{\Rightarrow} \underline{t} \stackrel{p}{=} \underline{r} \vdash \underline{h} \dot{\Rightarrow} \underline{t}' \stackrel{p}{=} \underline{r}' \oplus \forall\underline{a}\colon \forall\underline{b}\colon \underline{a} \dot{+} \underline{b}' \stackrel{p}{=} \underline{a} \dot{+} \underline{b}' \oplus \forall\underline{a}\colon \neg\dot{0} \stackrel{p}{=} \underline{a}' \oplus \forall\underline{x}\colon \forall\underline{a}\colon \underline{a} \vdash \dot{\forall}\underline{x}\colon \underline{a} \oplus \forall\underline{c}\colon \forall\underline{a}\colon \forall\underline{x}\colon \forall\underline{b}\colon \lceil\underline{a}\rceil \equiv \langle\lceil\underline{b}\rceil | \lceil\underline{x}\rceil := \lceil\underline{c}\rceil\rangle \Vdash \dot{\forall}\underline{x}\colon \underline{b} \dot{\Rightarrow} \underline{a} \oplus \forall\underline{h}\colon \forall\underline{t}\colon \underline{h} \dot{\Rightarrow} \underline{t} \dot{+} \dot{0} \stackrel{p}{=} \underline{t} \oplus \forall\underline{a}\colon \underline{a} \dot{\cdot} \dot{0} \stackrel{p}{=} \dot{0} \oplus \forall\underline{a}\colon \forall\underline{b}\colon \forall\underline{c}\colon \underline{a} \stackrel{p}{=} \underline{b} \dot{\Rightarrow} \underline{a} \stackrel{p}{=} \underline{c} \dot{\Rightarrow} \underline{b} \stackrel{p}{=} \underline{c} \oplus \forall\underline{t}\colon \underline{t} \stackrel{p}{=} \underline{t} \oplus \forall\underline{a}\colon \forall\underline{b}\colon \forall\underline{c}\colon \forall\underline{x}\colon \underline{b} \equiv \langle\underline{a} | \underline{x} := \dot{0}\rangle \Vdash \underline{c} \equiv \langle\underline{a} | \underline{x} := \underline{x}'\rangle \Vdash \underline{b} \dot{\Rightarrow} \dot{\forall}\underline{x}\colon \underline{a} \dot{\Rightarrow} \underline{c} \dot{\Rightarrow} \dot{\forall}\underline{x}\colon \underline{a} \oplus \forall\underline{a}\colon \forall\underline{b}\colon \forall\underline{c}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{c} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{c} \oplus \forall\underline{h}\colon \forall\underline{t}\colon \forall\underline{r}\colon \forall\underline{s}\colon \underline{h} \dot{\Rightarrow} \underline{t} \stackrel{p}{=} \underline{r} \vdash \underline{h} \dot{\Rightarrow} \underline{r} \stackrel{p}{=} \underline{s} \vdash \underline{h} \dot{\Rightarrow} \underline{t} \stackrel{p}{=} \underline{s} \oplus \forall\underline{a}\colon \underline{a} \dot{+} \dot{0} \stackrel{p}{=} \underline{a}]$$

$[A1' \stackrel{\text{stmt}}{\rightarrow} S' \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \mathrel{\dot\Rightarrow} \underline{b} \mathrel{\dot\Rightarrow} \underline{a}][A1' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[A2' \stackrel{\text{stmt}}{\rightarrow} S' \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} \mathrel{\dot\Rightarrow} \underline{b} \mathrel{\dot\Rightarrow} \underline{c} \mathrel{\dot\Rightarrow} \underline{a} \mathrel{\dot\Rightarrow} \underline{b} \mathrel{\dot\Rightarrow} \underline{a} \mathrel{\dot\Rightarrow} \underline{c}][A2' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[A3' \stackrel{\text{stmt}}{\rightarrow} S' \vdash \forall \underline{a}\colon \forall \underline{b}\colon \dot\neg\, \underline{b} \mathrel{\dot\Rightarrow} \dot\neg\, \underline{a} \mathrel{\dot\Rightarrow} \dot\neg\, \underline{b} \mathrel{\dot\Rightarrow} \underline{a} \mathrel{\dot\Rightarrow} \underline{b}][A3' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[A4' \stackrel{\text{stmt}}{\rightarrow} S' \vdash \forall \underline{c}\colon \forall \underline{a}\colon \forall \underline{x}\colon \forall \underline{b}\colon \lceil \underline{a} \rceil \equiv \langle \lceil \underline{b} \rceil | \lceil \underline{x} \rceil := \lceil \underline{c} \rceil \rangle \Vdash \dot\forall \underline{x}\colon \underline{b} \mathrel{\dot\Rightarrow} \underline{a}][A4' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[A5' \stackrel{\text{stmt}}{\rightarrow} S' \vdash \forall \underline{x}\colon \forall \underline{a}\colon \forall \underline{b}\colon \dot{\mathrm{nonfree}}(\lceil \underline{x} \rceil, \lceil \underline{a} \rceil) \Vdash \dot\forall \underline{x}\colon \underline{a} \mathrel{\dot\Rightarrow} \underline{b} \mathrel{\dot\Rightarrow} \underline{a} \mathrel{\dot\Rightarrow} \dot\forall \underline{x}\colon \underline{b}][A5' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[MP' \stackrel{\text{stmt}}{\rightarrow} S' \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \mathrel{\dot\Rightarrow} \underline{b} \vdash \underline{a} \vdash \underline{b}][MP' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[Gen' \stackrel{\text{stmt}}{\rightarrow} S' \vdash \forall \underline{x}\colon \forall \underline{a}\colon \underline{a} \vdash \dot\forall \underline{x}\colon \underline{a}][Gen' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[S1' \stackrel{\text{stmt}}{\rightarrow} S' \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} \stackrel{\text{p}}{=} \underline{b} \mathrel{\dot\Rightarrow} \underline{a} \stackrel{\text{p}}{=} \underline{c} \mathrel{\dot\Rightarrow} \underline{b} \stackrel{\text{p}}{=} \underline{c}][S1' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[S2' \stackrel{\text{stmt}}{\rightarrow} S' \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \stackrel{\text{p}}{=} \underline{b} \mathrel{\dot\Rightarrow} \underline{a}' \stackrel{\text{p}}{=} \underline{b}'][S2' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[S3' \stackrel{\text{stmt}}{\rightarrow} S' \vdash \forall \underline{a}\colon \neg \dot{0} \stackrel{\text{p}}{=} \underline{a}'][S3' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[S4' \stackrel{\text{stmt}}{\rightarrow} S' \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a}' \stackrel{\text{p}}{=} \underline{b}' \mathrel{\dot\Rightarrow} \underline{a} \stackrel{\text{p}}{=} \underline{b}][S4' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[S5' \stackrel{\text{stmt}}{\rightarrow} S' \vdash \forall \underline{a}\colon \underline{a} \mathbin{\dot+} \dot{0} \stackrel{\text{p}}{=} \underline{a}][S5' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[S6' \stackrel{\text{stmt}}{\rightarrow} S' \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \mathbin{\dot+} \underline{b}' \stackrel{\text{p}}{=} \underline{a} \mathbin{\dot+} \underline{b}'][S6' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[S7' \stackrel{\text{stmt}}{\rightarrow} S' \vdash \forall \underline{a}\colon \underline{a} \mathbin{\dot\cdot} \dot{0} \stackrel{\text{p}}{=} \dot{0}][S7' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[S8' \stackrel{\text{stmt}}{\rightarrow} S' \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \mathbin{\dot\cdot} \underline{b}' \stackrel{\text{p}}{=} \underline{a} \mathbin{\dot\cdot} \underline{b} \mathbin{\dot+} \underline{a}][S8' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[S9' \stackrel{\text{stmt}}{\rightarrow} S' \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{x}\colon \underline{b} \equiv \langle \underline{a} | \underline{x} := \dot{0} \rangle \Vdash \underline{c} \equiv \langle \underline{a} | \underline{x} := \underline{x}' \rangle \Vdash \underline{b} \mathrel{\dot\Rightarrow} \dot\forall \underline{x}\colon \underline{a} \mathrel{\dot\Rightarrow} \underline{c} \mathrel{\dot\Rightarrow} \dot\forall \underline{x}\colon \underline{a}][S9' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

Note that [A1] and [A1′] are distinct. The former says $[S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \mathrel{\dot\Rightarrow} \underline{b} \mathrel{\dot\Rightarrow} \underline{a}]$ and the latter says $[S' \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \mathrel{\dot\Rightarrow} \underline{b} \mathrel{\dot\Rightarrow} \underline{a}]$.

1.6 Restatement of lemma and proof

We now prove Lemma [L3.2(a)] once again under the name of [L3.2(a)′ $\stackrel{\text{pyk}}{\rightarrow}$ “lemma prime l three two a”][L3.2(a)′ $\stackrel{\text{tex}}{\rightarrow}$ “ L3.2(a)’”]:

$$[\mathrm{L3.2(a)}' \stackrel{\text{stmt}}{\rightarrow} \mathrm{S}' \vdash \forall \underline{\mathsf{a}} \colon \underline{\mathsf{a}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{a}}]$$

$$[\mathrm{L3.2(a)}' \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{S}' \vdash \forall \underline{\mathsf{a}} \colon \mathrm{S5}' \gg \underline{\mathsf{a}} \dot{+} \dot{0} \stackrel{\mathrm{p}}{=} \underline{\mathsf{a}}; \mathrm{S1}' \gg \underline{\mathsf{a}} \dot{+} \dot{0} \stackrel{\mathrm{p}}{=} \underline{\mathsf{a}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{a}} \dot{+} \dot{0} \stackrel{\mathrm{p}}{=} \underline{\mathsf{a}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{a}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{a}}; \mathrm{MP}' \rhd \underline{\mathsf{a}} \dot{+} \dot{0} \stackrel{\mathrm{p}}{=} \underline{\mathsf{a}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{a}} \dot{+} \dot{0} \stackrel{\mathrm{p}}{=} \underline{\mathsf{a}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{a}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{a}} \rhd \underline{\mathsf{a}} \dot{+} \dot{0} \stackrel{\mathrm{p}}{=} \underline{\mathsf{a}} \gg \underline{\mathsf{a}} \dot{+} \dot{0} \stackrel{\mathrm{p}}{=} \underline{\mathsf{a}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{a}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{a}}; \mathrm{MP}' \rhd \underline{\mathsf{a}} \dot{+} \dot{0} \stackrel{\mathrm{p}}{=} \underline{\mathsf{a}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{a}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{a}} \rhd \underline{\mathsf{a}} \dot{+} \dot{0} \stackrel{\mathrm{p}}{=} \underline{\mathsf{a}} \gg \underline{\mathsf{a}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{a}} \rceil, \mathrm{p}_0, \mathsf{c})]$$

2 Formal development

2.1 Propositional calculus

2.1.1 Modus ponens

We use [$\mathsf{x} \unrhd \mathsf{y}$ $\stackrel{\text{pyk}}{\rightarrow}$ “$*$ macro modus ponens $*$”][$\mathsf{x} \unrhd \mathsf{y}$ $\stackrel{\text{tex}}{\rightarrow}$ “#1. \unrhd #2.”] as shorthand for modus ponens:

$$[\mathsf{x} \unrhd \mathsf{y} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil [\mathsf{x} \unrhd \mathsf{y} \mathrel{\ddot{=}} \mathrm{MP}' \rhd \mathsf{x} \rhd \mathsf{y}] \rceil)]$$

2.1.2 Lemma M1.7

Lemma [M1.7 $\stackrel{\text{pyk}}{\rightarrow}$ “mendelson one seven”][M1.7 $\stackrel{\text{tex}}{\rightarrow}$ “ M1.7”] (i.e. Lemma 1.7 in Mendelson [2]) reads:

$$[\mathrm{M1.7} \stackrel{\text{stmt}}{\rightarrow} \mathrm{S}' \vdash \forall \underline{\mathsf{b}} \colon \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}}]$$

$$[\mathrm{M1.7} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{S}' \vdash \forall \underline{\mathsf{b}} \colon \mathrm{A1}' \gg \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}}; \mathrm{A2}' \gg \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}}; \mathrm{MP}' \rhd \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \rhd \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \gg \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}}; \mathrm{A1}' \gg \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}}; \mathrm{MP}' \rhd \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \rhd \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \gg \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \rceil, \mathrm{p}_0, \mathsf{c})]$$

2.1.3 Hypothetical modus ponens

The hypothetical version [$\mathrm{MP}'_{\mathrm{h}}$ $\stackrel{\text{pyk}}{\rightarrow}$ “hypothetical rule prime mp”][$\mathrm{MP}'_{\mathrm{h}}$ $\stackrel{\text{tex}}{\rightarrow}$ “ MP’_h”] of modus ponens MP′ has a hypothesis $\underline{\mathsf{h}}$ on each premise and on the conclusion:

$$[\mathrm{MP}'_{\mathrm{h}} \stackrel{\text{stmt}}{\rightarrow} \mathrm{S}' \vdash \forall \underline{\mathsf{h}} \colon \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{h}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{a}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \vdash \underline{\mathsf{h}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{a}} \vdash \underline{\mathsf{h}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}}]$$

$$[\mathrm{MP}'_{\mathrm{h}} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{S}' \vdash \forall \underline{\mathsf{h}} \colon \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{h}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{a}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \vdash \underline{\mathsf{h}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{a}} \vdash \mathrm{A2}' \gg \underline{\mathsf{h}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{a}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{h}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{a}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{h}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}}; \mathrm{MP}' \rhd \underline{\mathsf{h}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{a}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{h}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{a}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{h}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \rhd \underline{\mathsf{h}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{a}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \gg \underline{\mathsf{h}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{a}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{h}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}}; \mathrm{MP}' \rhd \underline{\mathsf{h}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{a}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{h}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \rhd \underline{\mathsf{h}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{a}} \gg \underline{\mathsf{h}} \mathrel{\dot{\Rightarrow}} \underline{\mathsf{b}} \rceil, \mathrm{p}_0, \mathsf{c})]$$

We use $[\mathsf{x} \unrhd_{\mathrm{h}} \mathsf{y} \overset{\text{pyk}}{\to}$ "* hypothetical modus ponens *"$][\mathsf{x} \unrhd_{\mathrm{h}} \mathsf{y} \overset{\text{tex}}{\to}$ "#1. \unrhd_h #2."] as shorthand for hypothetical modus ponens:

$$[\mathsf{x} \unrhd_{\mathrm{h}} \mathsf{y} \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil[\mathsf{x} \unrhd_{\mathrm{h}} \mathsf{y} \ddot{=} \mathrm{MP}'_{\mathrm{h}} \rhd \mathsf{x} \rhd \mathsf{y}]\rceil)]$$

2.1.4 Turning lemmas to hypothetical lemmas

Lemma [Hypothesize $\overset{\text{pyk}}{\to}$ "hypothesize"][Hypothesize $\overset{\text{tex}}{\to}$ " Hypothesize"] turns a lemma with no premises into one that assumes the hypothesis $[\underline{\mathsf{h}}]$ to hold:

$[\text{Hypothesize} \overset{\text{stmt}}{\to} \mathrm{S}' \vdash \forall \underline{\mathsf{h}} \colon \forall \underline{\mathsf{a}} \colon \underline{\mathsf{a}} \vdash \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{a}}]$

$[\text{Hypothesize} \overset{\text{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{S}' \vdash \forall \underline{\mathsf{h}} \colon \forall \underline{\mathsf{a}} \colon \underline{\mathsf{a}} \vdash \mathrm{A1}' \gg \underline{\mathsf{a}} \dot{\Rightarrow} \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{a}}; \mathrm{MP}' \rhd \underline{\mathsf{a}} \dot{\Rightarrow} \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{a}} \rhd \underline{\mathsf{a}} \gg \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{a}} \rceil, \mathrm{p}_0, \mathsf{c})]$

2.2 First order predicate calculus

2.2.1 Hypothetical generalization

The hypothetical version $[\mathrm{Gen}'_{\mathrm{h}} \overset{\text{pyk}}{\to}$ "hypothetical rule prime gen"$][\mathrm{Gen}'_{\mathrm{h}} \overset{\text{tex}}{\to}$ " Gen'_h"] of generalisation Gen′ has a hypothesis $\underline{\mathsf{h}}$ on premise and conclusion:

$[\mathrm{Gen}'_{\mathrm{h}} \overset{\text{stmt}}{\to} \mathrm{S}' \vdash \forall \underline{\mathsf{h}} \colon \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{a}} \colon \dot{\mathrm{nonfree}}(\lceil \underline{\mathsf{x}} \rceil, \lceil \underline{\mathsf{h}} \rceil) \Vdash \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{a}} \vdash \underline{\mathsf{h}} \dot{\Rightarrow} \dot{\forall} \underline{\mathsf{x}} \colon \underline{\mathsf{a}}]$

$[\mathrm{Gen}'_{\mathrm{h}} \overset{\text{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{S}' \vdash \forall \underline{\mathsf{h}} \colon \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{a}} \colon \dot{\mathrm{nonfree}}(\lceil \underline{\mathsf{x}} \rceil, \lceil \underline{\mathsf{h}} \rceil) \Vdash \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{a}} \vdash \mathrm{A5}' \trianglerighteq \dot{\mathrm{nonfree}}(\lceil \underline{\mathsf{x}} \rceil, \lceil \underline{\mathsf{h}} \rceil) \gg \dot{\forall} \underline{\mathsf{x}} \colon \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{a}} \dot{\Rightarrow} \underline{\mathsf{h}} \dot{\Rightarrow} \dot{\forall} \underline{\mathsf{x}} \colon \underline{\mathsf{a}}; \mathrm{Gen}' \rhd \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{a}} \gg \dot{\forall} \underline{\mathsf{x}} \colon \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{a}}; \mathrm{MP}' \rhd \dot{\forall} \underline{\mathsf{x}} \colon \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{a}} \dot{\Rightarrow} \underline{\mathsf{h}} \dot{\Rightarrow} \dot{\forall} \underline{\mathsf{x}} \colon \underline{\mathsf{a}} \rhd \dot{\forall} \underline{\mathsf{x}} \colon \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{a}} \gg \underline{\mathsf{h}} \dot{\Rightarrow} \dot{\forall} \underline{\mathsf{x}} \colon \underline{\mathsf{a}} \rceil, \mathrm{p}_0, \mathsf{c})]$

2.3 Peano arithmetic

2.3.1 Lemma M3.2(a)

Lemma [M3.2(a) $\overset{\text{pyk}}{\to}$ "mendelson three two a"][M3.2(a) $\overset{\text{tex}}{\to}$ " M3.2(a)"] and the associated hypothetical lemma $[\mathrm{M3.2(a)}_{\mathrm{h}} \overset{\text{pyk}}{\to}$ "hypothetical three two a"$][\mathrm{M3.2(a)}_{\mathrm{h}} \overset{\text{tex}}{\to}$ " M3.2(a)_h"] read:

$[\mathrm{M3.2(a)} \overset{\text{stmt}}{\to} \mathrm{S}' \vdash \forall \underline{\mathsf{t}} \colon \underline{\mathsf{t}} \overset{\mathrm{p}}{=} \underline{\mathsf{t}}][\mathrm{M3.2(a)} \overset{\text{proof}}{\to} \text{Rule tactic}]$

$[\mathrm{M3.2(a)}_{\mathrm{h}} \overset{\text{stmt}}{\to} \mathrm{S}' \vdash \forall \underline{\mathsf{h}} \colon \forall \underline{\mathsf{t}} \colon \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}} \overset{\mathrm{p}}{=} \underline{\mathsf{t}}]$

Above we cheat in stating M3.2(a) as a rule and not as a lemma. A reasonable way to construct a large proof is to start stating everything as rules and then changing the rules to lemmas one at a time until only the rules of the theory are left.

$[\mathrm{M3.2(a)_h} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{S}' \vdash \forall \underline{\mathsf{h}}\colon \forall \underline{\mathsf{t}}\colon \mathrm{M3.2(a)} \gg \underline{\mathsf{t}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{t}}; \mathrm{Hypothesize} \rhd \underline{\mathsf{t}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{t}} \gg \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{t}}\rceil, \mathrm{p}_0, \mathsf{c})]$

2.3.2 Lemma M3.2(b)

Lemma $[\mathrm{M3.2(b)_h} \stackrel{\mathrm{pyk}}{\rightarrow}$ "hypothetical three two b"$][\mathrm{M3.2(b)_h} \stackrel{\mathrm{tex}}{\rightarrow}$ " M3.2(b)_h"] reads:

$[\mathrm{M3.2(b)_h} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S}' \vdash \forall \underline{\mathsf{h}}\colon \forall \underline{\mathsf{t}}\colon \forall \underline{\mathsf{r}}\colon \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{r}} \vdash \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{r}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{t}}][\mathrm{M3.2(b)_h} \stackrel{\mathrm{proof}}{\rightarrow} \text{Rule tactic}]$

$[\mathrm{M3.2(b)_h} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{S}' \vdash \forall \underline{\mathsf{h}}\colon \forall \underline{\mathsf{t}}\colon \forall \underline{\mathsf{r}}\colon \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{r}} \vdash \mathrm{S1}' \gg \underline{\mathsf{t}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{r}} \dot{\Rightarrow} \underline{\mathsf{t}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{t}} \dot{\Rightarrow} \underline{\mathsf{r}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{t}}; \mathrm{Hypothesize} \rhd \underline{\mathsf{t}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{r}} \dot{\Rightarrow} \underline{\mathsf{t}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{t}} \dot{\Rightarrow} \underline{\mathsf{r}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{t}} \gg \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{r}} \dot{\Rightarrow} \underline{\mathsf{t}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{t}} \dot{\Rightarrow} \underline{\mathsf{r}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{t}}; \mathrm{MP}'_{\mathrm{h}} \rhd \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{r}} \dot{\Rightarrow} \underline{\mathsf{t}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{t}} \dot{\Rightarrow} \underline{\mathsf{r}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{t}} \rhd \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{r}} \gg \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{t}} \dot{\Rightarrow} \underline{\mathsf{r}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{t}}; \mathrm{M3.2(a)_h} \gg \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{t}}; \mathrm{MP}'_{\mathrm{h}} \rhd \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{t}} \dot{\Rightarrow} \underline{\mathsf{r}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{t}} \rhd \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{t}} \gg \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{r}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{t}}\rceil, \mathrm{p}_0, \mathsf{c})]$

2.3.3 Lemma M3.1(S1)

Lemma $[\mathrm{M3.1(S1')_h} \stackrel{\mathrm{pyk}}{\rightarrow}$ "hypothetical three one s one"$][\mathrm{M3.1(S1')_h} \stackrel{\mathrm{tex}}{\rightarrow}$ " M3.1(S1')_h"] is the hypothetical version of Mendelson Lemma 3.1(S1′):

$[\mathrm{M3.1(S1')_h} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S}' \vdash \forall \underline{\mathsf{h}}\colon \forall \underline{\mathsf{t}}\colon \forall \underline{\mathsf{r}}\colon \forall \underline{\mathsf{s}}\colon \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{r}} \vdash \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{s}} \vdash \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{r}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{s}}][\mathrm{M3.1(S1')_h} \stackrel{\mathrm{proof}}{\rightarrow} \text{Rule tactic}]$

2.3.4 Lemma M3.2(c)

Lemma $[\mathrm{M3.2(c)_h} \stackrel{\mathrm{pyk}}{\rightarrow}$ "hypothetical three two c"$][\mathrm{M3.2(c)_h} \stackrel{\mathrm{tex}}{\rightarrow}$ " M3.2(c)_h"] is the hypothetical version of Mendelson Lemma 3.2(c) which expresses ordinary transitivity:

$[\mathrm{M3.2(c)_h} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S}' \vdash \forall \underline{\mathsf{h}}\colon \forall \underline{\mathsf{t}}\colon \forall \underline{\mathsf{r}}\colon \forall \underline{\mathsf{s}}\colon \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{r}} \vdash \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{r}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{s}} \vdash \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{s}}][\mathrm{M3.2(c)_h} \stackrel{\mathrm{proof}}{\rightarrow} \text{Rule tactic}]$

2.3.5 Lemma M3.1(S2)

Lemma $[\mathrm{M3.1(S2')_h} \stackrel{\mathrm{pyk}}{\rightarrow}$ "hypothetical three one s two"$][\mathrm{M3.1(S2')_h} \stackrel{\mathrm{tex}}{\rightarrow}$ " M3.1(S2')_h"] is the hypothetical version of Mendelson Lemma 3.1(S2′):

$[\mathrm{M3.1(S2')_h} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S}' \vdash \forall \underline{\mathsf{h}}\colon \forall \underline{\mathsf{t}}\colon \forall \underline{\mathsf{r}}\colon \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}} \stackrel{\mathrm{p}}{=} \underline{\mathsf{r}} \vdash \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}}' \stackrel{\mathrm{p}}{=} \underline{\mathsf{r}}'][\mathrm{M3.1(S2')_h} \stackrel{\mathrm{proof}}{\rightarrow} \text{Rule tactic}]$

2.3.6 Lemma M3.1(S5)

Lemma $[\mathrm{M3.1(S5')_h} \stackrel{\mathrm{pyk}}{\rightarrow}$ "hypothetical three one s five"$][\mathrm{M3.1(S5')_h} \stackrel{\mathrm{tex}}{\rightarrow}$ " M3.1(S5')_h"] is the hypothetical version of Mendelson Lemma 3.1(S5′):

$[\mathrm{M3.1(S5')_h} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S}' \vdash \forall \underline{\mathsf{h}}\colon \forall \underline{\mathsf{t}}\colon \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}} \dot{+} \dot{0} \stackrel{\mathrm{p}}{=} \underline{\mathsf{t}}][\mathrm{M3.1(S5')_h} \stackrel{\mathrm{proof}}{\rightarrow} \text{Rule tactic}]$

2.3.7 Lemma M3.1(S6)

Lemma [M3.1(S6′)$_\mathrm{h}$ $\stackrel{\mathrm{pyk}}{\rightarrow}$ "hypothetical three one s six"][M3.1(S6′)$_\mathrm{h}$ $\stackrel{\mathrm{tex}}{\rightarrow}$ " M3.1(S6')_h"] is the hypothetical version of Mendelson Lemma 3.1(S6′):

[M3.1(S6′)$_\mathrm{h}$ $\stackrel{\mathrm{stmt}}{\rightarrow}$ $\mathrm{S}' \vdash \forall \underline{\mathsf{h}}\colon \forall \underline{\mathsf{t}}\colon \forall \underline{\mathsf{r}}\colon \underline{\mathsf{h}} \dot{\Rightarrow} \underline{\mathsf{t}} \dot{+} \underline{\mathsf{r}}' \stackrel{\mathrm{p}}{=} \underline{\mathsf{t}} \dot{+} \underline{\mathsf{r}}'$][M3.1(S6′)$_\mathrm{h}$ $\stackrel{\mathrm{proof}}{\rightarrow}$ Rule tactic]

2.3.8 Lemma M3.2(f)

Lemma [M3.2(f) $\stackrel{\mathrm{pyk}}{\rightarrow}$ "mendelson three two f"][M3.2(f) $\stackrel{\mathrm{tex}}{\rightarrow}$ " M3.2(f)"] is the closure of Mendelson Lemma 3.2(f) for the concrete variable [$\dot{\mathsf{t}}$]:

[M3.2(f) $\stackrel{\mathrm{stmt}}{\rightarrow}$ $\mathrm{S}' \vdash \dot{\forall} \dot{\mathsf{t}}\colon \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}$]

The proof below uses local macro definitions.

[M3.2(f) $\stackrel{\mathrm{proof}}{\rightarrow}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{S}' \vdash \mathrm{A1}' \gg \mathsf{x} \dot{\Rightarrow} \mathsf{x} \dot{\Rightarrow} \mathsf{x}; \mathrm{M3.1(S5')_h} \gg \mathsf{x} \dot{\Rightarrow} \mathsf{x} \dot{\Rightarrow} \mathsf{x} \dot{\Rightarrow} \dot{0} \dot{+} \dot{0} \stackrel{\mathrm{p}}{=} \dot{0}; \mathrm{M3.2(b)_h} \rhd \mathsf{x} \dot{\Rightarrow} \mathsf{x} \dot{\Rightarrow} \mathsf{x} \dot{\Rightarrow} \dot{0} \dot{+} \dot{0} \stackrel{\mathrm{p}}{=} \dot{0} \gg \mathsf{x} \dot{\Rightarrow} \mathsf{x} \dot{\Rightarrow} \mathsf{x} \dot{\Rightarrow} \dot{0} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{0}; \mathrm{MP}' \rhd \mathsf{x} \dot{\Rightarrow} \mathsf{x} \dot{\Rightarrow} \mathsf{x} \dot{\Rightarrow} \dot{0} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{0} \rhd \mathsf{x} \dot{\Rightarrow} \mathsf{x} \dot{\Rightarrow} \mathsf{x} \gg \dot{0} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{0}; \mathrm{M1.7} \gg \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \dot{\Rightarrow} \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}; \mathrm{M3.1(S2')_h} \rhd \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \dot{\Rightarrow} \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \gg \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \dot{\Rightarrow} \dot{\mathsf{t}}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}'; \mathrm{M3.1(S6')_h} \gg \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \dot{\Rightarrow} \dot{0} \dot{+} \dot{\mathsf{t}}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}'; \mathrm{M3.2(b)_h} \rhd \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \dot{\Rightarrow} \dot{0} \dot{+} \dot{\mathsf{t}}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}' \gg \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \dot{\Rightarrow} \dot{0} \dot{+} \dot{\mathsf{t}}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}'; \mathrm{M3.2(c)_h} \rhd \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \dot{\Rightarrow} \dot{\mathsf{t}}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}' \rhd \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \dot{\Rightarrow} \dot{0} \dot{+} \dot{\mathsf{t}}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}' \gg \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \dot{\Rightarrow} \dot{\mathsf{t}}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}'; \mathrm{Gen}' \rhd \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \dot{\Rightarrow} \dot{\mathsf{t}}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}' \gg \dot{\forall} \dot{\mathsf{t}}\colon \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \dot{\Rightarrow} \dot{\mathsf{t}}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}'; \mathrm{S9}' \gg \dot{0} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{0} \dot{\Rightarrow} \dot{\forall} \dot{\mathsf{t}}\colon \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \dot{\Rightarrow} \dot{\mathsf{t}}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}' \dot{\Rightarrow} \dot{\forall} \dot{\mathsf{t}}\colon \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}; \mathrm{MP}' \rhd \dot{0} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{0} \dot{\Rightarrow} \dot{\forall} \dot{\mathsf{t}}\colon \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \dot{\Rightarrow} \dot{\mathsf{t}}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}' \dot{\Rightarrow} \dot{\forall} \dot{\mathsf{t}}\colon \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \rhd \dot{0} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{0} \gg \dot{\forall} \dot{\mathsf{t}}\colon \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \dot{\Rightarrow} \dot{\mathsf{t}}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}' \dot{\Rightarrow} \dot{\forall} \dot{\mathsf{t}}\colon \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}; \mathrm{MP}' \rhd \dot{\forall} \dot{\mathsf{t}}\colon \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \dot{\Rightarrow} \dot{\mathsf{t}}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}' \dot{\Rightarrow} \dot{\forall} \dot{\mathsf{t}}\colon \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \rhd \dot{\forall} \dot{\mathsf{t}}\colon \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \dot{\Rightarrow} \dot{\mathsf{t}}' \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}}' \gg \dot{\forall} \dot{\mathsf{t}}\colon \dot{\mathsf{t}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{t}} \rceil, \mathrm{p}_0, \mathsf{c})$]

A Chores

A.1 The name of the page

This defines the name of the page:

[peano $\stackrel{\mathrm{pyk}}{\rightarrow}$ "peano"]

A.2 Variables of Peano arithmetic

We use [$\dot{b}$ $\stackrel{\mathrm{pyk}}{\rightarrow}$ "peano b"][$\dot{b}$ $\stackrel{\mathrm{tex}}{\rightarrow}$ " \dot{\mathit{b}}"], [$\dot{c}$ $\stackrel{\mathrm{pyk}}{\rightarrow}$ "peano c"][$\dot{c}$ $\stackrel{\mathrm{tex}}{\rightarrow}$ " \dot{\mathit{c}}"], [$\dot{d}$ $\stackrel{\mathrm{pyk}}{\rightarrow}$ "peano d"][$\dot{d}$ $\stackrel{\mathrm{tex}}{\rightarrow}$ " \dot{\mathit{d}}"], [$\dot{e}$ $\stackrel{\mathrm{pyk}}{\rightarrow}$ "peano e"][$\dot{e}$ $\stackrel{\mathrm{tex}}{\rightarrow}$ " \dot{\mathit{e}}"], [$\dot{f}$ $\stackrel{\mathrm{pyk}}{\rightarrow}$ "peano f"][$\dot{f}$ $\stackrel{\mathrm{tex}}{\rightarrow}$ "

\dot{\mathit{f}}”], [$\dot{g} \stackrel{\text{pyk}}{\rightarrow}$ “peano g”][$\dot{g} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{g}}”], [$\dot{h} \stackrel{\text{pyk}}{\rightarrow}$ “peano h”][$\dot{h} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{h}}”], [$\dot{i} \stackrel{\text{pyk}}{\rightarrow}$ “peano i”][$\dot{i} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{i}}”], [$\dot{j} \stackrel{\text{pyk}}{\rightarrow}$ “peano j”][$\dot{j} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{j}}”], [$\dot{k} \stackrel{\text{pyk}}{\rightarrow}$ “peano k”][$\dot{k} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{k}}”], [$\dot{l} \stackrel{\text{pyk}}{\rightarrow}$ “peano l”][$\dot{l} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{l}}”], [$\dot{m} \stackrel{\text{pyk}}{\rightarrow}$ “peano m”][$\dot{m} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{m}}”], [$\dot{n} \stackrel{\text{pyk}}{\rightarrow}$ “peano n”][$\dot{n} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{n}}”], [$\dot{o} \stackrel{\text{pyk}}{\rightarrow}$ “peano o”][$\dot{o} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{o}}”], [$\dot{p} \stackrel{\text{pyk}}{\rightarrow}$ “peano p”][$\dot{p} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{p}}”], [$\dot{q} \stackrel{\text{pyk}}{\rightarrow}$ “peano q”][$\dot{q} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{q}}”], [$\dot{r} \stackrel{\text{pyk}}{\rightarrow}$ “peano r”][$\dot{r} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{r}}”], [$\dot{s} \stackrel{\text{pyk}}{\rightarrow}$ “peano s”][$\dot{s} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{s}}”], [$\dot{t} \stackrel{\text{pyk}}{\rightarrow}$ “peano t”][$\dot{t} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{t}}”], [$\dot{u} \stackrel{\text{pyk}}{\rightarrow}$ “peano u”][$\dot{u} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{u}}”], [$\dot{v} \stackrel{\text{pyk}}{\rightarrow}$ “peano v”][$\dot{v} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{v}}”], [$\dot{w} \stackrel{\text{pyk}}{\rightarrow}$ “peano w”][$\dot{w} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{w}}”], [$\dot{x} \stackrel{\text{pyk}}{\rightarrow}$ “peano x”][$\dot{x} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{x}}”], [$\dot{y} \stackrel{\text{pyk}}{\rightarrow}$ “peano y”][$\dot{y} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{y}}”], and [$\dot{z} \stackrel{\text{pyk}}{\rightarrow}$ “peano z”][$\dot{z} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{z}}”] to denote variables of Peano arithmetic:
[$\dot{b} \stackrel{\text{macro}}{\rightarrow} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{b} \ddot{=} \dot{\mathsf{b}}]\rceil)$], [$\dot{c} \stackrel{\text{macro}}{\rightarrow} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{c} \ddot{=} \dot{\mathsf{c}}]\rceil)$], [$\dot{d} \stackrel{\text{macro}}{\rightarrow} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{d} \ddot{=} \dot{\mathsf{d}}]\rceil)$], [$\dot{e} \stackrel{\text{macro}}{\rightarrow} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{e} \ddot{=} \dot{\mathsf{e}}]\rceil)$], [$\dot{f} \stackrel{\text{macro}}{\rightarrow} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{f} \ddot{=} \dot{\mathsf{f}}]\rceil)$], [$\dot{g} \stackrel{\text{macro}}{\rightarrow} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{g} \ddot{=} \dot{\mathsf{g}}]\rceil)$], [$\dot{h} \stackrel{\text{macro}}{\rightarrow} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{h} \ddot{=} \dot{\mathsf{h}}]\rceil)$], [$\dot{i} \stackrel{\text{macro}}{\rightarrow} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{i} \ddot{=} \dot{\mathsf{i}}]\rceil)$], [$\dot{j} \stackrel{\text{macro}}{\rightarrow} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{j} \ddot{=} \dot{\mathsf{j}}]\rceil)$], [$\dot{k} \stackrel{\text{macro}}{\rightarrow} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{k} \ddot{=} \dot{\mathsf{k}}]\rceil)$], [$\dot{l} \stackrel{\text{macro}}{\rightarrow} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{l} \ddot{=} \dot{\mathsf{l}}]\rceil)$], [$\dot{m} \stackrel{\text{macro}}{\rightarrow} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{m} \ddot{=} \dot{\mathsf{m}}]\rceil)$], [$\dot{n} \stackrel{\text{macro}}{\rightarrow} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{n} \ddot{=} \dot{\mathsf{n}}]\rceil)$], [$\dot{o} \stackrel{\text{macro}}{\rightarrow} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{o} \ddot{=} \dot{\mathsf{o}}]\rceil)$], [$\dot{p} \stackrel{\text{macro}}{\rightarrow} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{p} \ddot{=} \dot{\mathsf{p}}]\rceil)$], [$\dot{q} \stackrel{\text{macro}}{\rightarrow} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{q} \ddot{=} \dot{\mathsf{q}}]\rceil)$], [$\dot{r} \stackrel{\text{macro}}{\rightarrow} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{r} \ddot{=} \dot{\mathsf{r}}]\rceil)$], [$\dot{s} \stackrel{\text{macro}}{\rightarrow} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{s} \ddot{=} \dot{\mathsf{s}}]\rceil)$], [$\dot{t} \stackrel{\text{macro}}{\rightarrow} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{t} \ddot{=} \dot{\mathsf{t}}]\rceil)$], [$\dot{u} \stackrel{\text{macro}}{\rightarrow} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{u} \ddot{=} \dot{\mathsf{u}}]\rceil)$], [$\dot{v} \stackrel{\text{macro}}{\rightarrow} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{v} \ddot{=} \dot{\mathsf{v}}]\rceil)$], [$\dot{w} \stackrel{\text{macro}}{\rightarrow} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{w} \ddot{=} \dot{\mathsf{w}}]\rceil)$], [$\dot{x} \stackrel{\text{macro}}{\rightarrow} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{x} \ddot{=} \dot{\mathsf{x}}]\rceil)$], [$\dot{y} \stackrel{\text{macro}}{\rightarrow} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{y} \ddot{=} \dot{\mathsf{y}}]\rceil)$], and [$\dot{z} \stackrel{\text{macro}}{\rightarrow} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{z} \ddot{=} \dot{\mathsf{z}}]\rceil)$].

A.3 TEX definitions

A.4 Test

$[\lceil \dot{\mathsf{a}} \rceil^{\mathcal{P}}]^{\cdot}$

$[\lceil \mathsf{a} \rceil^{\mathcal{P}}]^{-}$

$[\dot{\mathrm{nonfree}}(\lceil \dot{\mathsf{x}} \rceil, \lceil \dot{\mathsf{y}} \stackrel{\mathrm{p}}{=} \dot{\mathsf{z}} \dot{\Rightarrow} \dot{\forall}\dot{\mathsf{x}}\colon \dot{\mathsf{x}} \stackrel{\mathrm{p}}{=} \dot{\mathsf{y}} \rceil)]^{\cdot}$

$[\dot{\mathrm{nonfree}}(\lceil \dot{\mathsf{x}} \rceil, \lceil \dot{\mathsf{x}} \stackrel{\mathrm{p}}{=} \dot{\mathsf{z}} \dot{\Rightarrow} \dot{\forall}\dot{\mathsf{x}}\colon \dot{\mathsf{x}} \stackrel{\mathrm{p}}{=} \dot{\mathsf{y}} \rceil)]^{-}$

$[\dot{\mathrm{nonfree}}(\lceil \dot{\mathsf{x}} \rceil, \lceil \dot{\mathsf{y}} \stackrel{\mathrm{p}}{=} \dot{\mathsf{x}} \dot{\Rightarrow} \dot{\forall}\dot{\mathsf{x}}\colon \dot{\mathsf{x}} \stackrel{\mathrm{p}}{=} \dot{\mathsf{y}} \rceil)]^{-}$

$[\dot{\mathrm{nonfree}}(\lceil \dot{\mathsf{x}} \rceil, \lceil \dot{\mathsf{y}} \stackrel{\mathrm{p}}{=} \dot{\mathsf{z}} \dot{\Rightarrow} \dot{\forall}\dot{\mathsf{y}}\colon \dot{\mathsf{x}} \stackrel{\mathrm{p}}{=} \dot{\mathsf{y}} \rceil)]^{-}$

$[\dot{\mathrm{free}}\langle \lceil \dot{\forall}\dot{\mathsf{x}}\colon \mathsf{b} :: \dot{\mathsf{x}} :: \mathsf{c} \rceil | \lceil \dot{\mathsf{x}} \rceil := \lceil \mathsf{x} :: \dot{\mathsf{y}} :: \mathsf{z} \rceil \rangle]^{\cdot}$

$[\dot{\mathrm{free}}\langle \lceil \dot{\forall}\dot{\mathsf{y}}\colon \mathsf{b} :: \dot{\mathsf{x}} :: \mathsf{c} \rceil | \lceil \dot{\mathsf{x}} \rceil := \lceil \mathsf{x} :: \dot{\mathsf{y}} :: \mathsf{z} \rceil \rangle]^{-}$

$[\dot{\mathrm{free}}\langle \lceil \dot{\forall}\dot{\mathsf{x}}\colon \mathsf{b} :: \dot{\mathsf{x}} :: \mathsf{c} \rceil | \lceil \dot{\mathsf{y}} \rceil := \lceil \mathsf{x} :: \dot{\mathsf{y}} :: \mathsf{z} \rceil \rangle]^{\cdot}$

$[\dot{\mathrm{free}}\langle \lceil \dot{\forall}\dot{\mathsf{y}}\colon \mathsf{b} :: \dot{\mathsf{x}} :: \mathsf{c} \rceil | \lceil \dot{\mathsf{y}} \rceil := \lceil \mathsf{x} :: \dot{\mathsf{y}} :: \mathsf{z} \rceil \rangle]^{\cdot}$

$[\dot{\mathsf{a}} \equiv \langle \dot{\mathsf{a}} | \dot{\mathsf{b}} := \dot{\mathsf{c}} \rangle]^{\cdot}$

$[\dot{\mathsf{c}} \equiv \langle \dot{\mathsf{b}} | \dot{\mathsf{b}} := \dot{\mathsf{c}} \rangle]^{\cdot}$

$[\forall \dot{\mathsf{a}}\colon \dot{\mathsf{a}} \stackrel{\mathrm{p}}{=} \dot{\mathsf{b}} \equiv \langle \forall \dot{\mathsf{a}}\colon \dot{\mathsf{a}} \stackrel{\mathrm{p}}{=} \dot{\mathsf{b}} | \dot{\mathsf{a}} := \dot{\mathsf{c}} \rangle]^{\cdot}$

$[\forall \dot{\mathsf{a}}\colon \dot{\mathsf{a}} \stackrel{\mathrm{p}}{=} \dot{\mathsf{c}} \equiv \langle \forall \dot{\mathsf{a}}\colon \dot{\mathsf{a}} \stackrel{\mathrm{p}}{=} \dot{\mathsf{b}} | \dot{\mathsf{b}} := \dot{\mathsf{c}} \rangle]^{\cdot}$

$[\dot{\forall} \dot{\mathsf{a}}\colon \dot{\mathsf{a}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{a}} \dot{\Rightarrow} \dot{\mathsf{c}} \dot{\cdot} \dot{\mathsf{d}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{c}} \dot{\cdot} \dot{\mathsf{d}} \equiv \langle \dot{\forall} \dot{\mathsf{a}}\colon \dot{\mathsf{a}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{a}} \dot{\Rightarrow} \dot{\mathsf{b}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{b}} | \dot{\mathsf{b}} := \dot{\mathsf{c}} \dot{\cdot} \dot{\mathsf{d}} \rangle]^{\cdot}$

$[\dot{\forall} \dot{\mathsf{a}}\colon \dot{\mathsf{a}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{a}} \dot{\Rightarrow} \dot{\mathsf{b}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{b}} \equiv \langle \dot{\forall} \dot{\mathsf{a}}\colon \dot{\mathsf{a}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{a}} \dot{\Rightarrow} \dot{\mathsf{b}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{b}} | \dot{\mathsf{a}} := \dot{\mathsf{c}} \rangle]^{\cdot}$

A.5 Priority table

[peano $\stackrel{\mathrm{prio}}{\rightarrow}$
Preassociative
[peano], [base], [bracket $*$ end bracket], [big bracket $*$ end bracket], [math $*$ end math], [**flush left** [$*$]], [x], [y], [z], [[$* \bowtie *$]], [[$* \stackrel{*}{\rightarrow} *$]], [pyk], [tex], [name], [prio], [$*$], [T], [if($*, *, *$)], [[$* \stackrel{*}{\Rightarrow} *$]], [val], [claim], [$\perp$], [f($*$)], [$(*)^{\mathrm{I}}$], [$\mathsf{F}$], [$\underline{0}$], [$\underline{1}$], [$\underline{2}$], [$\underline{3}$], [$\underline{4}$], [$\underline{5}$], [$\underline{6}$], [$\underline{7}$], [$\underline{8}$], [$\underline{9}$], [0], [1], [2], [3], [4], [5], [6], [7], [8], [9], [a], [b], [c], [d], [e], [f], [g], [h], [i], [j], [k], [l], [m], [n], [o], [p], [q], [r], [s], [t], [u], [v], [w], [$(*)^{\mathrm{M}}$], [If($*, *, *$)], [array{$*$} $*$ end array], [l], [c], [r], [empty], [$\langle * \mid * := * \rangle$], [$\mathcal{M}(*)$], [$\tilde{\mathcal{U}}(*)$], [$\mathcal{U}(*)$], [$\mathcal{U}^{\mathrm{M}}(*)$], [**apply**($*, *$)], [$\mathbf{apply}_1$($*, *$)], [identifier($*$)], [$\mathrm{identifier}_1$($*, *$)], [array-plus($*, *$)], [array-remove($*, *, *$)], [array-put($*, *, *, *$)], [array-add($*, *, *, *, *$)], [bit($*, *$)], [$\mathrm{bit}_1$($*, *$)], [rack], ["vector"], ["bibliography"], ["dictionary"], ["body"], ["codex"], ["expansion"], ["code"], ["cache"], ["diagnose"], ["pyk"],

["tex"], ["texname"], ["value"], ["message"], ["macro"], ["definition"],
["unpack"], ["claim"], ["priority"], ["lambda"], ["apply"], ["true"], ["if"],
["quote"], ["proclaim"], ["define"], ["introduce"], ["hide"], ["pre"], ["post"],
[$\mathcal{E}(*,*,*)$], [$\mathcal{E}_2(*,*,*,*,*)$], [$\mathcal{E}_3(*,*,*,*)$], [$\mathcal{E}_4(*,*,*,*)$], [$\mathbf{lookup}(*,*,*)$],
[$\mathbf{abstract}(*,*,*,*)$], [$[\lceil *]$], [$\mathcal{M}(*,*,*)$], [$\mathcal{M}_2(*,*,*,*)$], [$\mathcal{M}^*(*,*,*)$], [macro],
[s_0], [$\mathbf{zip}(*,*)$], [$\mathbf{assoc}_1(*,*,*)$], [$(*)^{\mathbf{p}}$], [self], [$[* \ddot{=} *]$], [$[* \dot{=} *]$], [$[* \acute{=} *]$],
[$[* \overset{\text{pyk}}{=} *]$], [$[* \overset{\text{tex}}{=} *]$], [$[* \overset{\text{name}}{=} *]$], [$\mathbf{Priority\ table}[*]$], [$\tilde{\mathcal{M}}_1$], [$\tilde{\mathcal{M}}_2(*)$], [$\tilde{\mathcal{M}}_3(*)$],
[$\tilde{\mathcal{M}}_4(*,*,*,*)$], [$\tilde{\mathcal{M}}(*,*,*)$], [$\tilde{\mathcal{Q}}(*,*,*)$], [$\tilde{\mathcal{Q}}_2(*,*,*)$], [$\tilde{\mathcal{Q}}_3(*,*,*,*)$], [$\tilde{\mathcal{Q}}^*(*,*,*)$],
[$(*)$], [$\mathbf{aspect}(*,*)$], [$\mathbf{aspect}(*,*,*)$], [$\langle * \rangle$], [$\mathbf{tuple}_1(*)$], [$\mathbf{tuple}_2(*)$], [$\mathrm{let}_2(*,*)$],
[$\mathrm{let}_1(*,*)$], [$[* \overset{\text{claim}}{=} *]$], [checker], [$\mathbf{check}(*,*)$], [$\mathbf{check}_2(*,*,*)$], [$\mathbf{check}_3(*,*,*)$],
[$\mathbf{check}^*(*,*)$], [$\mathbf{check}_2^*(*,*,*)$], [$[*]^{\cdot}$], [$[*]^{-}$], [$[*]^{\circ}$], [msg], [$[* \overset{\text{msg}}{=} *]$], [<stmt>],
[stmt], [$[* \overset{\text{stmt}}{=} *]$], [HeadNil′], [HeadPair′], [Transitivity′], [$\perp\!\!\!\perp$], [Contra′], [T'_{E}],
[L_1], [$\underline{*}$], [$\mathcal{A}$], [$\mathcal{B}$], [$\mathcal{C}$], [$\mathcal{D}$], [$\mathcal{E}$], [$\mathcal{F}$], [$\mathcal{G}$], [$\mathcal{H}$], [$\mathcal{I}$], [$\mathcal{J}$], [$\mathcal{K}$], [$\mathcal{L}$], [$\mathcal{M}$], [$\mathcal{N}$], [$\mathcal{O}$], [$\mathcal{P}$], [$\mathcal{Q}$],
[$\mathcal{R}$], [$\mathcal{S}$], [$\mathcal{T}$], [$\mathcal{U}$], [$\mathcal{V}$], [$\mathcal{W}$], [$\mathcal{X}$], [$\mathcal{Y}$], [$\mathcal{Z}$], [$\langle * \mid * := * \rangle$], [$\langle^* * \mid * := * \rangle$], [$\emptyset$], [Remainder],
[$(*)^{\mathbf{v}}$], [$\mathrm{intro}(*,*,*,*)$], [$\mathrm{intro}(*,*,*)$], [$\mathrm{error}(*,*)$], [$\mathrm{error}_2(*,*)$], [$\mathrm{proof}(*,*,*)$],
[$\mathrm{proof}_2(*,*)$], [$\mathcal{S}(*,*)$], [$\mathcal{S}^{\mathrm{I}}(*,*)$], [$\mathcal{S}^{\rhd}(*,*)$], [$\mathcal{S}_1^{\rhd}(*,*,*)$], [$\mathcal{S}^{\mathrm{E}}(*,*)$], [$\mathcal{S}_1^{\mathrm{E}}(*,*,*)$],
[$\mathcal{S}^{+}(*,*)$], [$\mathcal{S}_1^{+}(*,*,*)$], [$\mathcal{S}^{-}(*,*)$], [$\mathcal{S}_1^{-}(*,*,*)$], [$\mathcal{S}^{*}(*,*)$], [$\mathcal{S}_1^{*}(*,*,*)$],
[$\mathcal{S}_2^{*}(*,*,*,*)$], [$\mathcal{S}^{@}(*,*)$], [$\mathcal{S}_1^{@}(*,*,*)$], [$\mathcal{S}^{\vdash}(*,*)$], [$\mathcal{S}_1^{\vdash}(*,*,*,*)$], [$\mathcal{S}^{\Vdash}(*,*)$],
[$\mathcal{S}_1^{\Vdash}(*,*,*,*)$], [$\mathcal{S}^{\mathrm{i.e.}}(*,*)$], [$\mathcal{S}_1^{\mathrm{i.e.}}(*,*,*,*)$], [$\mathcal{S}_2^{\mathrm{i.e.}}(*,*,*,*,*)$], [$\mathcal{S}^{\forall}(*,*)$],
[$\mathcal{S}_1^{\forall}(*,*,*,*)$], [$\mathcal{S}^{;}(*,*)$], [$\mathcal{S}_1^{;}(*,*,*)$], [$\mathcal{S}_2^{;}(*,*,*,*)$], [$\mathcal{T}(*)$], [$\mathrm{claims}(*,*,*)$],
[$\mathrm{claims}_2(*,*,*)$], [<proof>], [proof], [[**Lemma** $*\colon *$]], [[**Proof of** $*\colon *$]],
[[$*$ **lemma** $*\colon *$]], [[$*$ **antilemma** $*\colon *$]], [[$*$ **rule** $*\colon *$]], [[$*$ **antirule** $*\colon *$]],
[verifier], [$\mathcal{V}_1(*)$], [$\mathcal{V}_2(*,*)$], [$\mathcal{V}_3(*,*,*,*)$], [$\mathcal{V}_4(*,*)$], [$\mathcal{V}_5(*,*,*,*)$], [$\mathcal{V}_6(*,*,*,*)$],
[$\mathcal{V}_7(*,*,*,*)$], [$\mathrm{Cut}(*,*)$], [$\mathrm{Head}_{\oplus}(*)$], [$\mathrm{Tail}_{\oplus}(*)$], [$\mathrm{rule}_1(*,*)$], [$\mathrm{rule}(*,*)$],
[Rule tactic], [$\mathrm{Plus}(*,*)$], [[**Theory** $*$]], [$\mathrm{theory}_2(*,*)$], [$\mathrm{theory}_3(*,*)$],
[$\mathrm{theory}_4(*,*,*)$], [HeadNil″], [HeadPair″], [Transitivity″], [Contra″], [HeadNil],
[HeadPair], [Transitivity], [Contra], [T_{E}], [ragged right],
[ragged right expansion], [$\mathrm{parm}(*,*,*)$], [$\mathrm{parm}^*(*,*,*)$], [$\mathrm{inst}(*,*)$],
[$\mathrm{inst}^*(*,*)$], [$\mathrm{occur}(*,*,*)$], [$\mathrm{occur}^*(*,*,*)$], [$\mathrm{unify}(* = *,*)$], [$\mathrm{unify}^*(* = *,*)$],
[$\mathrm{unify}_2(* = *,*)$], [$\mathrm{L_a}$], [$\mathrm{L_b}$], [$\mathrm{L_c}$], [$\mathrm{L_d}$], [$\mathrm{L_e}$], [$\mathrm{L_f}$], [$\mathrm{L_g}$], [$\mathrm{L_h}$], [$\mathrm{L_i}$], [$\mathrm{L_j}$], [$\mathrm{L_k}$], [$\mathrm{L_l}$], [$\mathrm{L_m}$],
[$\mathrm{L_n}$], [$\mathrm{L_o}$], [$\mathrm{L_p}$], [$\mathrm{L_q}$], [$\mathrm{L_r}$], [$\mathrm{L_s}$], [$\mathrm{L_t}$], [$\mathrm{L_u}$], [$\mathrm{L_v}$], [$\mathrm{L_w}$], [$\mathrm{L_x}$], [$\mathrm{L_y}$], [$\mathrm{L_z}$], [$\mathrm{L_A}$], [$\mathrm{L_B}$], [$\mathrm{L_C}$],
[$\mathrm{L_D}$], [$\mathrm{L_E}$], [$\mathrm{L_F}$], [$\mathrm{L_G}$], [$\mathrm{L_H}$], [$\mathrm{L_I}$], [$\mathrm{L_J}$], [$\mathrm{L_K}$], [$\mathrm{L_L}$], [$\mathrm{L_M}$], [$\mathrm{L_N}$], [$\mathrm{L_O}$], [$\mathrm{L_P}$], [$\mathrm{L_Q}$], [$\mathrm{L_R}$],
[$\mathrm{L_S}$], [$\mathrm{L_T}$], [$\mathrm{L_U}$], [$\mathrm{L_V}$], [$\mathrm{L_W}$], [$\mathrm{L_X}$], [$\mathrm{L_Y}$], [$\mathrm{L_Z}$], [$\mathrm{L_?}$], [Reflexivity], [$\mathrm{Reflexivity}_1$],
[Commutativity], [$\mathrm{Commutativity}_1$], [<tactic>], [tactic], [$[* \overset{\text{tactic}}{=} *]$], [$\mathcal{P}(*,*,*)$],
[$\mathcal{P}^*(*,*,*)$], [p_0], [$\mathrm{conclude}_1(*,*)$], [$\mathrm{conclude}_2(*,*,*)$], [$\mathrm{conclude}_3(*,*,*,*)$],
[$\mathrm{conclude}_4(*,*)$], [$\dot{0}$], [$\dot{1}$], [$\dot{2}$], [$\dot{a}$], [$\dot{b}$], [$\dot{c}$], [$\dot{d}$], [$\dot{e}$], [$\dot{f}$], [$\dot{g}$], [$\dot{h}$], [$\dot{i}$], [$\dot{j}$], [$\dot{k}$], [$\dot{l}$], [$\dot{m}$], [$\dot{n}$],
[$\dot{o}$], [$\dot{p}$], [$\dot{q}$], [$\dot{r}$], [$\dot{s}$], [$\dot{t}$], [$\dot{u}$], [$\dot{v}$], [$\dot{w}$], [$\dot{x}$], [$\dot{y}$], [$\dot{z}$], [$\mathrm{nonfree}(*,*)$], [$\mathrm{nonfree}^*(*,*)$],
[$\mathrm{free}\langle * | * := * \rangle$], [$\mathrm{free}^*\langle * | * := * \rangle$], [$*\equiv\langle * | * := * \rangle$], [$*\equiv\langle^* * | * := * \rangle$], [S], [A1], [A2],
[A3], [A4], [A5], [S1], [S2], [S3], [S4], [S5], [S6], [S7], [S8], [S9], [MP], [Gen],
[L3.2(a)], [S′], [A1′], [A2′], [A3′], [A4′], [A5′], [S1′], [S2′], [S3′], [S4′], [S5′], [S6′],
[S7′], [S8′], [S9′], [MP′], [Gen′], [L3.2(a)′], [M1.7], [$\mathrm{MP}'_{\mathrm{h}}$], [Hypothesize], [$\mathrm{Gen}'_{\mathrm{h}}$],
[M3.2(a)], [$\mathrm{M3.2(a)_h}$], [$\mathrm{M3.2(b)_h}$], [$\mathrm{M3.1(S1')_h}$], [$\mathrm{M3.2(c)_h}$], [$\mathrm{M3.1(S2')_h}$],
[$\mathrm{M3.1(S5')_h}$], [$\mathrm{M3.1(S6')_h}$], [M3.2(f)];

Preassociative

$[*\_\{*\}], [*'], [*[\,*\,]], [*[*{\rightarrow}*]], [*[*{\Rightarrow}*]], [\dot{*}]$;
Preassociative
[“ * ”], [], $[(*)^{\mathbf{t}}]$, [string(*) + *], [string(*) ++ *], [
*], [*], [!*], [”*], [#*], [$*], [%*], [&*], [’*], [(*], [)*], [**], [+*], [, *], [-*], [.*], [/*],
[0*], [1*], [2*], [3*], [4*], [5*], [6*], [7*], [8*], [9*], [:*], [; *], [<*], [=*], [>*], [?*],
[@*], [A*], [B*], [C*], [D*], [E*], [F*], [G*], [H*], [I*], [J*], [K*], [L*], [M*], [N*],
[O*], [P*], [Q*], [R*], [S*], [T*], [U*], [V*], [W*], [X*], [Y*], [Z*], [[*], [*], []*], [^*],
[_*], [‘*], [a*], [b*], [c*], [d*], [e*], [f*], [g*], [h*], [i*], [j*], [k*], [l*], [m*], [n*], [o*],
[p*], [q*], [r*], [s*], [t*], [u*], [v*], [w*], [x*], [y*], [z*], [{*], [|*], [}*], [~*],
[**Preassociative** *; *], [**Postassociative** *; *], [[*], *], [priority * end],
[newline *], [macro newline *];
Preassociative
[*0], [*1], [0b], [*-color(*)], [*-color*(*)];
Preassociative
[* ’ *], [* ‘ *];
Preassociative
$[*^{\mathrm{H}}], [*^{\mathrm{T}}], [*^{\mathrm{U}}], [*^{\mathrm{h}}], [*^{\mathrm{t}}], [*^{\mathrm{s}}], [*^{\mathrm{c}}], [*^{\mathrm{d}}], [*^{\mathrm{a}}], [*^{\mathrm{C}}], [*^{\mathrm{M}}], [*^{\mathrm{B}}], [*^{\mathrm{r}}], [*^{\mathrm{i}}], [*^{\mathrm{d}}], [*^{\mathrm{R}}], [*^{0}],$
$[*^{1}], [*^{2}], [*^{3}], [*^{4}], [*^{5}], [*^{6}], [*^{7}], [*^{8}], [*^{9}], [*^{\mathrm{E}}], [*^{\mathcal{V}}], [*^{\mathcal{C}}], [*^{\mathcal{C}^*}], [*']$;
Preassociative
$[* \cdot *], [* \cdot_0 *], [* \mathbin{\dot{:}} *]$;
Preassociative
$[* + *], [* +_0 *], [* +_1 *], [* - *], [* -_0 *], [* -_1 *], [* \mathbin{\dot{+}} *]$;
Preassociative
$[* \cup \{*\}], [* \cup *], [* \backslash \{*\}]$;
Postassociative
$[* \therefore *], [* \underline{\therefore} *], [* \underline{::} *], [* \underline{+2*} *], [* :: *], [* +2* *]$;
Postassociative
$[*, *]$;
Preassociative
$[* \stackrel{\mathrm{B}}{\approx} *], [* \stackrel{\mathrm{D}}{\approx} *], [* \stackrel{\mathrm{C}}{\approx} *], [* \stackrel{\mathrm{P}}{\approx} *], [* \approx *], [* = *], [* \stackrel{+}{\rightarrow} *], [* \stackrel{\mathrm{t}}{=} *], [* \stackrel{\mathrm{t}^*}{=} *], [* \stackrel{\mathrm{r}}{=} *],$
$[* \in_{\mathrm{t}} *], [* \subseteq_{\mathrm{T}} *], [* \stackrel{\mathrm{T}}{=} *], [* \stackrel{\mathrm{s}}{=} *], [* \text{ free in } *], [* \text{ free in}^* *], [* \text{ free for } * \text{ in } *],$
$[* \text{ free for}^* * \text{ in } *], [* \in_{\mathrm{c}} *], [* < *], [* <' *], [* \leq' *], [* \stackrel{\mathrm{p}}{=} *], [*^{\mathcal{P}}]$;
Preassociative
$[\neg *], [\dot{\neg} *]$;
Preassociative
$[* \wedge *], [* \ddot{\wedge} *], [* \tilde{\wedge} *], [* \wedge_{\mathrm{c}} *], [* \dot{\wedge} *]$;
Preassociative
$[* \vee *], [* \parallel *], [* \ddot{\vee} *], [* \dot{\vee} *]$;
Preassociative
$[\dot{\forall} * \colon *], [\dot{\exists} * \colon *]$;
Postassociative
$[* \mathrel{\ddot{\Rightarrow}} *], [* \mathrel{\dot{\Rightarrow}} *], [* \mathrel{\dot{\Leftrightarrow}} *]$;
Postassociative
$[* : *], [*!*]$;
Preassociative

$[* \left\{ \begin{array}{l} * \\ * \end{array} \right.];$

Preassociative

$[\lambda * .*], [\Lambda *], [\textbf{if } * \textbf{ then } * \textbf{ else } *], [\textbf{let } * = * \textbf{ in } *], [\textbf{let } * \ddot{=} * \text{ in } *];$

Preassociative

$[*^{\mathrm{I}}], [*^{\rhd}], [*^{\mathrm{V}}], [*^{+}], [*^{-}], [*^{*}];$

Preassociative

$[* @ *], [* \rhd *], [* \unrhd\!\!\!\rhd *], [* \gg *], [* \unrhd *], [* \unrhd_{\mathrm{h}} *];$

Postassociative

$[* \vdash *], [* \Vdash *], [* \text{ i.e. } *];$

Preassociative

$[\forall *: *];$

Postassociative

$[* \oplus *];$

Postassociative

$[*; *];$

Preassociative

$[* \text{ proves } *];$

Preassociative

$[* \textbf{ proof of } * : *], [\text{Line} * : * \gg *; *], [\text{Last line} * \gg * \square],$
$[\text{Line} * : \text{Premise} \gg *; *], [\text{Line} * : \text{Side-condition} \gg *; *], [\text{Arbitrary} \gg *; *],$
$[\text{Local} \gg * = *; *];$

Postassociative

$[* \text{ then } *], [*[*]*];$

Preassociative

$[* \& *];$

Preassociative

$[* \backslash\backslash *];]$

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