

Peano arithmetic

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GRD-2005-06-22.UTC:11:16:56.809226

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1 Peano arithmetic

This Logiweb page [1] defines Peano arithmetic.

1.1 The constructs of Peano arithmetic

Terms of Peano arithmetic are constructed from zero $[0]^1$, successor $[x']^2$, plus $[x + y]^3$, and times $[x : y]^4$.

Formulas of Peano arithmetic are constructed from equality $[x \stackrel{P}{=} y]^5$, negation $[\neg x]^6$, implication $[x \Rightarrow y]^7$, and universal quantification $[\forall x: y]^8$.

From these constructs we macro define one $[1]^9$, two $[2]^{10}$, conjunction $[x \wedge y]^{11}$, disjunction $[x \vee y]^{12}$, biimplication $[x \Leftrightarrow y]^{13}$, and existential quantification $[\exists x: y]^{14}$:

$$[1 \equiv 0']$$

$$[2 \equiv 1']$$

$$[x \wedge y \equiv \neg (x \Rightarrow \neg y)]$$

$$[x \vee y \equiv \neg x \Rightarrow y]$$

$$[x \Leftrightarrow y \equiv (x \Rightarrow y) \wedge (y \Rightarrow x)]$$

$$[\exists x: y \equiv \neg \forall x: \neg y]$$

$$^1[0 \stackrel{\text{pyk}}{=} \text{"peano zero"}]$$

$$^2[x' \stackrel{\text{pyk}}{=} \text{"* peano succ"}]$$

$$^3[x + y \stackrel{\text{pyk}}{=} \text{"* peano plus *"}]$$

$$^4[x : y \stackrel{\text{pyk}}{=} \text{"* peano times *"}]$$

$$^5[x \stackrel{P}{=} y \stackrel{\text{pyk}}{=} \text{"* peano is *"}]$$

$$^6[\neg x \stackrel{\text{pyk}}{=} \text{"peano not *"}]$$

$$^7[x \Rightarrow y \stackrel{\text{pyk}}{=} \text{"* peano imply *"}]$$

$$^8[\forall x: y \stackrel{\text{pyk}}{=} \text{"peano all * indeed *"}]$$

$$^9[1 \stackrel{\text{pyk}}{=} \text{"peano one"}]$$

$$^{10}[2 \stackrel{\text{pyk}}{=} \text{"peano two"}]$$

$$^{11}[x \wedge y \stackrel{\text{pyk}}{=} \text{"* peano and *"}]$$

$$^{12}[x \vee y \stackrel{\text{pyk}}{=} \text{"* peano or *"}]$$

$$^{13}[x \Leftrightarrow y \stackrel{\text{pyk}}{=} \text{"* peano iff *"}]$$

$$^{14}[\exists x: y \stackrel{\text{pyk}}{=} \text{"peano exist * indeed *"}]$$

1.2 Variables

We now introduce the unary operator $[\dot{x}]^{15}$ and define that a term is a *Peano variable* (i.e. a variable of Peano arithmetic) if it has the $[\dot{x}]$ operator in its root. $[\dot{x}^P]^{16}$ is true if $[\dot{x}]$ is a Peano variable:

$$[\dot{x}^P \doteq x \doteq [\dot{x}]]$$

We macro define $[\dot{a}]^{17}$ to be a Peano variable:

$$[\dot{a} \doteq \dot{a}]$$

Appendix A.2 defines Peano variables for the other letters of the English alphabet.

$[\text{nonfree}(x, y)]^{18}$ is true if the Peano variable $[x]$ does not occur free in the Peano term/formula $[y]$. $[\text{nonfree}^*(x, y)]^{19}$ is true if the Peano variable $[x]$ does not occur free in the list $[y]$ of Peano terms/formulas.

$$\begin{aligned} &[\text{nonfree}(x, y) \doteq \\ &\text{if } y^P \text{ then } \neg x \stackrel{t}{=} y \text{ else} \\ &\text{if } \neg y \stackrel{r}{=} [\dot{\forall}x: y] \text{ then } \text{nonfree}^*(x, y^t) \text{ else} \\ &\text{if } x \stackrel{t}{=} y^1 \text{ then } \top \text{ else } \text{nonfree}(x, y^2)] \end{aligned}$$

$$[\text{nonfree}^*(x, y) \doteq x! \text{If}(y, \top, \text{nonfree}(x, y^h) \wedge \text{nonfree}^*(x, y^t))]$$

$[\text{free}(a|x := b)]^{20}$ is true if the substitution $[\langle a|x := b \rangle]$ is free. $[\text{free}^*(a|x := b)]^{21}$ is the version where $[a]$ is a list of terms.

$$\begin{aligned} &[\text{free}(a|x := b) \doteq x!b! \\ &\text{if } a^P \text{ then } \top \text{ else} \\ &\text{if } \neg a \stackrel{r}{=} [\dot{\forall}u: v] \text{ then } \text{free}^*(a^t|x := b) \text{ else} \\ &\text{if } a^1 \stackrel{t}{=} x \text{ then } \top \text{ else} \\ &\text{if } \text{nonfree}(x, a^2) \text{ then } \top \text{ else} \\ &\text{if } \neg \text{nonfree}(a^1, b) \text{ then } \top \text{ else} \\ &\text{free}(a^2|x := b)] \end{aligned}$$

$$[\text{free}^*(a|x := b) \doteq x!b! \text{If}(a, \top, \text{free}(a^h|x := b) \wedge \text{free}^*(a^t|x := b))]$$

¹⁵ $[\dot{x} \stackrel{\text{pyk}}{=} \text{"* peano var"}]$

¹⁶ $[\dot{x}^P \stackrel{\text{pyk}}{=} \text{"* is peano var"}]$

¹⁷ $[\dot{a} \stackrel{\text{pyk}}{=} \text{"peano a"}]$

¹⁸ $[\text{nonfree}(x, y) \stackrel{\text{pyk}}{=} \text{"peano nonfree * in * end nonfree"}]$

¹⁹ $[\text{nonfree}^*(x, y) \stackrel{\text{pyk}}{=} \text{"peano nonfree star * in * end nonfree"}]$

²⁰ $[\text{free}(a|x := b) \stackrel{\text{pyk}}{=} \text{"peano free * set * to * end free"}]$

²¹ $[\text{free}^*(a|x := b) \stackrel{\text{pyk}}{=} \text{"peano free star * set * to * end free"}]$

$[a \equiv \langle b | x := c \rangle]$ ²² is true if $[a]$ equals $[\langle b | x := c \rangle]$. $[a \equiv \langle *b | x := c \rangle]$ ²³ is the version where $[a]$ and $[b]$ are lists.

$[a \equiv \langle b | x := c \rangle] \doteq a!x!c!$
if $b \stackrel{r}{=} [\forall u: v] \wedge b^1 \stackrel{t}{=} x$ **then** $a \stackrel{t}{=} b$ **else**
if $b^p \wedge b \stackrel{t}{=} x$ **then** $a \stackrel{t}{=} c$ **else**
 $a \stackrel{r}{=} b \wedge a^t \equiv \langle *b^t | x := c \rangle]$

$[a \equiv \langle *b | x := c \rangle] \doteq b!x!c! \text{If}(a, T, a^h \equiv \langle b^h | x := c \rangle \wedge a^t \equiv \langle *b^t | x := c \rangle)]$

1.3 Mendelsons system S

System $[S]$ ²⁴ of Mendelson [2] expresses Peano arithmetic. It comprises the axioms $[A1]$ ²⁵, $[A2]$ ²⁶, $[A3]$ ²⁷, $[A4]$ ²⁸, and $[A5]$ ²⁹ and inference rules $[MP]$ ³⁰ and $[Gen]$ ³¹ of first order predicate calculus. Furthermore, it comprises the proper axioms $[S1]$ ³², $[S2]$ ³³, $[S3]$ ³⁴, $[S4]$ ³⁵, $[S5]$ ³⁶, $[S6]$ ³⁷, $[S7]$ ³⁸, $[S8]$ ³⁹, and $[S9]$ ⁴⁰. System $[S]$ is defined thus:

[Theory S]

[S rule A1: $\forall \mathcal{A}: \forall \mathcal{B}: \mathcal{A} \Rightarrow \mathcal{B} \Rightarrow \mathcal{A}$]

[S rule A2: $\forall \mathcal{A}: \forall \mathcal{B}: \forall \mathcal{C}: (\mathcal{A} \Rightarrow \mathcal{B} \Rightarrow \mathcal{C}) \Rightarrow (\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow \mathcal{A} \Rightarrow \mathcal{C}$]

[S rule A3: $\forall \mathcal{A}: \forall \mathcal{B}: (\neg \mathcal{B} \Rightarrow \neg \mathcal{A}) \Rightarrow (\neg \mathcal{B} \Rightarrow \mathcal{A}) \Rightarrow \mathcal{B}$]

²² $[a \equiv \langle b | x := c \rangle] \stackrel{\text{pyk}}{=} \text{"peano sub * is * where * is * end sub"}$

²³ $[a \equiv \langle *b | x := c \rangle] \stackrel{\text{pyk}}{=} \text{"peano sub star * is * where * is * end sub"}$

²⁴ $[S] \stackrel{\text{pyk}}{=} \text{"system s"}$

²⁵ $[A1] \stackrel{\text{pyk}}{=} \text{"axiom a one"}$

²⁶ $[A2] \stackrel{\text{pyk}}{=} \text{"axiom a two"}$

²⁷ $[A3] \stackrel{\text{pyk}}{=} \text{"axiom a three"}$

²⁸ $[A4] \stackrel{\text{pyk}}{=} \text{"axiom a four"}$

²⁹ $[A5] \stackrel{\text{pyk}}{=} \text{"axiom a five"}$

³⁰ $[MP] \stackrel{\text{pyk}}{=} \text{"rule mp"}$

³¹ $[Gen] \stackrel{\text{pyk}}{=} \text{"rule gen"}$

³² $[S1] \stackrel{\text{pyk}}{=} \text{"axiom s one"}$

³³ $[S2] \stackrel{\text{pyk}}{=} \text{"axiom s two"}$

³⁴ $[S3] \stackrel{\text{pyk}}{=} \text{"axiom s three"}$

³⁵ $[S4] \stackrel{\text{pyk}}{=} \text{"axiom s four"}$

³⁶ $[S5] \stackrel{\text{pyk}}{=} \text{"axiom s five"}$

³⁷ $[S6] \stackrel{\text{pyk}}{=} \text{"axiom s six"}$

³⁸ $[S7] \stackrel{\text{pyk}}{=} \text{"axiom s seven"}$

³⁹ $[S8] \stackrel{\text{pyk}}{=} \text{"axiom s eight"}$

⁴⁰ $[S9] \stackrel{\text{pyk}}{=} \text{"axiom s nine"}$

The order of quantifiers in the following axiom is such that $[C]$ which the current conclusion tactic cannot guess comes first. This allows to supply a value for $[C]$ without having to supply values for the other meta-variables.

$$[\text{S rule A4: } \forall C: \forall A: \forall X: \forall B: [\mathcal{A}] \equiv \langle [\mathcal{B}] \mid [\mathcal{X}] \rangle := [\mathcal{C}] \rangle \Vdash \dot{\forall} X: B \Rightarrow A]$$

$$[\text{S rule A5: } \forall X: \forall A: \forall B: \text{nonfree}(X, A) \Vdash \dot{\forall} X: (A \Rightarrow B) \Rightarrow A \Rightarrow \dot{\forall} X: B]$$

$$[\text{S rule MP: } \forall A: \forall B: A \Rightarrow B \vdash A \vdash B]$$

$$[\text{S rule Gen: } \forall X: \forall A: A \vdash \dot{\forall} X: A]$$

Axiom [S1] to [S8] are stated as Mendelson [2] does. This is done here to test certain parts of the Logiweb system. Serious users of Peano arithmetic are advised to take Mendelson's Lemma 3.1 as axioms instead.

$$[\text{S rule S1: } a \stackrel{P}{=} b \Rightarrow a \stackrel{P}{=} c \Rightarrow b \stackrel{P}{=} c]$$

$$[\text{S rule S2: } a \stackrel{P}{=} b \Rightarrow a' \stackrel{P}{=} b']$$

$$[\text{S rule S3: } \neg 0 \stackrel{P}{=} a']$$

$$[\text{S rule S4: } a' \stackrel{P}{=} b' \Rightarrow a \stackrel{P}{=} b]$$

$$[\text{S rule S5: } a + 0 \stackrel{P}{=} a]$$

$$[\text{S rule S6: } a + b' \stackrel{P}{=} (a + b)']$$

$$[\text{S rule S7: } a : 0 \stackrel{P}{=} 0]$$

$$[\text{S rule S8: } a : (b') \stackrel{P}{=} (a : b) + a]$$

$$[\text{S rule S9: } \forall A: \forall B: \forall C: \forall X: \\ B \equiv \langle A \mid X := 0 \rangle \Vdash C \equiv \langle A \mid X := X' \rangle \Vdash \\ B \Rightarrow \dot{\forall} X: (A \Rightarrow C) \Rightarrow \dot{\forall} X: A]$$

1.4 A lemma and a proof

We now prove Lemma [L3.2(a)]⁴¹ which is an instance of the corresponding proposition in Mendelson [2]:

$$[\text{S lemma L3.2(a): } x \stackrel{P}{=} x]$$

⁴¹ [L3.2(a)]^{pyk} "lemma 1 three two a"]

S proof of L3.2(a):

L01:	S5 $\gg$	$\dot{a} \dot{+} \dot{0} \stackrel{P}{=} \dot{a}$	;
L02:	Gen $\triangleright$ L01 $\gg$	$\forall \dot{a}: \dot{a} \dot{+} \dot{0} \stackrel{P}{=} \dot{a}$	;
L03:	A4 @ $\dot{x}$ $\gg$	$\forall \dot{a}: \dot{a} \dot{+} \dot{0} \stackrel{P}{=} \dot{a} \Rightarrow \dot{x} \dot{+} \dot{0} \stackrel{P}{=} \dot{x}$	;
L04:	MP $\triangleright$ L03 $\triangleright$ L02 $\gg$	$\dot{x} \dot{+} \dot{0} \stackrel{P}{=} \dot{x}$	;
L05:	S1 $\gg$	$\dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{c} \Rightarrow \dot{b} \stackrel{P}{=} \dot{c}$	;
L06:	Gen $\triangleright$ L05 $\gg$	$\forall \dot{c}: (\dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{c} \Rightarrow \dot{b} \stackrel{P}{=} \dot{c})$	;
L07:	A4 @ $\dot{x}$ $\gg$	$\forall \dot{c}: (\dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{c} \Rightarrow \dot{b} \stackrel{P}{=} \dot{c}) \Rightarrow \dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{b} \stackrel{P}{=} \dot{x}$	;
L08:	MP $\triangleright$ L07 $\triangleright$ L06 $\gg$	$\dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{b} \stackrel{P}{=} \dot{x}$	;
L09:	Gen $\triangleright$ L08 $\gg$	$\forall \dot{b}: (\dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{b} \stackrel{P}{=} \dot{x})$	;
L10:	A4 @ $\dot{x}$ $\gg$	$\forall \dot{b}: (\dot{a} \stackrel{P}{=} \dot{b} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{b} \stackrel{P}{=} \dot{x}) \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x}$	;
L11:	MP $\triangleright$ L10 $\triangleright$ L09 $\gg$	$\dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x}$	;
L12:	Gen $\triangleright$ L11 $\gg$	$\forall \dot{a}: (\dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x})$	;
L13:	A4 @ $\dot{x} \dot{+} \dot{0}$ $\gg$	$\forall \dot{a}: (\dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x}) \Rightarrow \dot{x} \dot{+} \dot{0} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \dot{+} \dot{0} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x}$	;
L14:	MP $\triangleright$ L13 $\triangleright$ L12 $\gg$	$\dot{x} \dot{+} \dot{0} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \dot{+} \dot{0} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x}$	;
L15:	MP $\triangleright$ L14 $\triangleright$ L04 $\gg$	$\dot{x} \dot{+} \dot{0} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x}$	;
L16:	MP $\triangleright$ L15 $\triangleright$ L04 $\gg$	$\dot{x} \stackrel{P}{=} \dot{x}$	□

1.5 An alternative axiomatic system

System [S']⁴² is system [S] in which the proper axioms are taken from Lemma 3.1 in Mendelson [2]. It comprises the axioms [A1']⁴³, [A2']⁴⁴, [A3']⁴⁵, [A4']⁴⁶, and [A5']⁴⁷ and inference rules [MP']⁴⁸ and [Gen']⁴⁹ of first order predicate calculus. Furthermore, it comprises the proper axioms [S1']⁵⁰, [S2']⁵¹, [S3']⁵²,

⁴²[S']^{pyk} ≡ “system prime s”]

⁴³[A1']^{pyk} ≡ “axiom prime a one”]

⁴⁴[A2']^{pyk} ≡ “axiom prime a two”]

⁴⁵[A3']^{pyk} ≡ “axiom prime a three”]

⁴⁶[A4']^{pyk} ≡ “axiom prime a four”]

⁴⁷[A5']^{pyk} ≡ “axiom prime a five”]

⁴⁸[MP']^{pyk} ≡ “rule prime mp”]

⁴⁹[Gen']^{pyk} ≡ “rule prime gen”]

⁵⁰[S1']^{pyk} ≡ “axiom prime s one”]

⁵¹[S2']^{pyk} ≡ “axiom prime s two”]

⁵²[S3']^{pyk} ≡ “axiom prime s three”]

[S4']⁵³, [S5']⁵⁴, [S6']⁵⁵, [S7']⁵⁶, [S8']⁵⁷, and [S9']⁵⁸.

System [S'] is defined thus:

[Theory S']

[S' rule A1': $\forall A: \forall B: \mathcal{A} \Rightarrow B \Rightarrow \mathcal{A}$]

[S' rule A2': $\forall A: \forall B: \forall C: (\mathcal{A} \Rightarrow B \Rightarrow C) \Rightarrow (\mathcal{A} \Rightarrow B) \Rightarrow \mathcal{A} \Rightarrow C$]

[S' rule A3': $\forall A: \forall B: (\neg B \Rightarrow \neg A) \Rightarrow (\neg B \Rightarrow \mathcal{A}) \Rightarrow B$]

[S' rule A4': $\forall C: \forall A: \forall \mathcal{X}: \forall B: [\mathcal{A}] \equiv \langle [\mathcal{B}] \mid [\mathcal{X}] \rangle := [\mathcal{C}] \rangle \Vdash \dot{\forall} \mathcal{X}: B \Rightarrow \mathcal{A}$]

[S' rule A5': $\forall \mathcal{X}: \forall A: \forall B: \text{nonfree}([\mathcal{X}], [\mathcal{A}]) \Vdash \dot{\forall} \mathcal{X}: (\mathcal{A} \Rightarrow B) \Rightarrow \mathcal{A} \Rightarrow \dot{\forall} \mathcal{X}: B$]

[S' rule MP': $\forall A: \forall B: \mathcal{A} \Rightarrow B \vdash \mathcal{A} \vdash B$]

[S' rule Gen': $\forall \mathcal{X}: \forall A: \mathcal{A} \vdash \dot{\forall} \mathcal{X}: \mathcal{A}$]

[S' rule S1': $\forall A: \forall B: \forall C: \mathcal{A} \stackrel{P}{\Rightarrow} B \Rightarrow \mathcal{A} \stackrel{P}{\Rightarrow} C \Rightarrow B \stackrel{P}{\Rightarrow} C$]

[S' rule S2': $\forall A: \forall B: \mathcal{A} \stackrel{P}{\Rightarrow} B \Rightarrow \mathcal{A}' \stackrel{P}{\Rightarrow} B'$]

[S' rule S3': $\forall A: \neg \dot{0} \stackrel{P}{\Rightarrow} \mathcal{A}'$]

[S' rule S4': $\forall A: \forall B: \mathcal{A}' \stackrel{P}{\Rightarrow} B' \Rightarrow \mathcal{A} \stackrel{P}{\Rightarrow} B$]

[S' rule S5': $\forall A: \mathcal{A} \dot{+} \dot{0} \stackrel{P}{\Rightarrow} \mathcal{A}$]

[S' rule S6': $\forall A: \forall B: \mathcal{A} \dot{+} B' \stackrel{P}{\Rightarrow} (\mathcal{A} \dot{+} B)'$]

[S' rule S7': $\forall A: \mathcal{A} \dot{:} \dot{0} \stackrel{P}{\Rightarrow} \dot{0}$]

[S' rule S8': $\forall A: \forall B: \mathcal{A} \dot{:} (B') \stackrel{P}{\Rightarrow} (\mathcal{A} \dot{:} B) \dot{+} \mathcal{A}$]

[S' rule S9': $\forall A: \forall B: \forall C: \forall \mathcal{X}: \mathcal{B} \equiv \langle \mathcal{A} \mid \mathcal{X} \rangle := \dot{0} \rangle \Vdash \mathcal{C} \equiv \langle \mathcal{A} \mid \mathcal{X} \rangle := \mathcal{X}' \rangle \Vdash \mathcal{B} \Rightarrow \dot{\forall} \mathcal{X}: (\mathcal{A} \Rightarrow \mathcal{C}) \Rightarrow \dot{\forall} \mathcal{X}: \mathcal{A}$]

Note that [A1] and [A1'] are distinct. The former says $[S \vdash \forall A: \forall B: \mathcal{A} \Rightarrow B \Rightarrow \mathcal{A}]$ and the latter says $[S' \vdash \forall A: \forall B: \mathcal{A} \Rightarrow B \Rightarrow \mathcal{A}]$.

⁵³[S4']^{pyk} “axiom prime s four”]

⁵⁴[S5']^{pyk} “axiom prime s five”]

⁵⁵[S6']^{pyk} “axiom prime s six”]

⁵⁶[S7']^{pyk} “axiom prime s seven”]

⁵⁷[S8']^{pyk} “axiom prime s eight”]

⁵⁸[S9']^{pyk} “axiom prime s nine”]

1.6 Restatement of lemma and proof

We now prove Lemma [L3.2(a)] once again under the name of [L3.2(a)]⁵⁹:

[S' lemma L3.2(a)': $\forall \mathcal{A}: \mathcal{A} \stackrel{p}{=} \mathcal{A}$]

S' proof of L3.2(a)':

L01:	Arbitrary $\gg$	$\mathcal{A}$	;
L02:	$S5' \gg$	$\mathcal{A} \dot{+} \dot{0} \stackrel{p}{=} \mathcal{A}$	;
L03:	$S1' \gg$	$\mathcal{A} \dot{+} \dot{0} \stackrel{p}{=} \mathcal{A} \Rightarrow$	
		$\mathcal{A} \dot{+} \dot{0} \stackrel{p}{=} \mathcal{A} \Rightarrow \mathcal{A} \stackrel{p}{=} \mathcal{A}$	;
L04:	$MP' \triangleright L03 \triangleright L02 \gg$	$\mathcal{A} \dot{+} \dot{0} \stackrel{p}{=} \mathcal{A} \Rightarrow \mathcal{A} \stackrel{p}{=} \mathcal{A}$	;
L05:	$MP' \triangleright L04 \triangleright L02 \gg$	$\mathcal{A} \stackrel{p}{=} \mathcal{A}$	□

2 Formal development

2.1 Propositional calculus

2.1.1 Modus ponens

We use $[x \supseteq y]$ ⁶⁰ as shorthand for modus ponens:

$$[x \supseteq y \doteq MP' \triangleright x \triangleright y]$$

2.1.2 Lemma M1.7

Lemma [M1.7]⁶¹ (i.e. Lemma 1.7 in Mendelson [2]) reads:

[S' lemma M1.7: $\forall \mathcal{B}: \mathcal{B} \Rightarrow \mathcal{B}$]

S' proof of M1.7:

L01:	Arbitrary $\gg$	$\mathcal{B}$	;
L02:	$A1' \gg$	$\mathcal{B} \Rightarrow (\mathcal{B} \Rightarrow \mathcal{B}) \Rightarrow \mathcal{B}$	;
L03:	$A2' \gg$	$(\mathcal{B} \Rightarrow (\mathcal{B} \Rightarrow \mathcal{B}) \Rightarrow \mathcal{B}) \Rightarrow$	
		$(\mathcal{B} \Rightarrow \mathcal{B} \Rightarrow \mathcal{B}) \Rightarrow \mathcal{B} \Rightarrow \mathcal{B}$	;
L04:	$L03 \supseteq L02 \gg$	$(\mathcal{B} \Rightarrow \mathcal{B} \Rightarrow \mathcal{B}) \Rightarrow \mathcal{B} \Rightarrow \mathcal{B}$	;
L05:	$A1' \gg$	$\mathcal{B} \Rightarrow \mathcal{B} \Rightarrow \mathcal{B}$	;
L06:	$L04 \supseteq L05 \gg$	$\mathcal{B} \Rightarrow \mathcal{B}$	□

2.1.3 Hypothetical modus ponens

The hypothetical version $[MP'_h]$ ⁶² of modus ponens MP' has a hypothesis $\mathcal{H}$ on each premise and on the conclusion:

⁵⁹[L3.2(a)]^{pyk} “lemma prime l three two a”]

⁶⁰ $[x \supseteq y]$ ^{pyk} “* macro modus ponens *”]

⁶¹[M1.7]^{pyk} “mendelson one seven”]

⁶² $[MP'_h]$ ^{pyk} “hypothetical rule prime mp”]

[S' lemma MP'_{h} : $\forall \mathcal{H}: \forall \mathcal{A}: \forall \mathcal{B}: \mathcal{H} \Rightarrow \mathcal{A} \Rightarrow \mathcal{B} \vdash \mathcal{H} \Rightarrow \mathcal{A} \vdash \mathcal{H} \Rightarrow \mathcal{B}$]

S' proof of MP'_{h} :

L01:	Arbitrary $\gg$	$\mathcal{H}$	;
L02:	Arbitrary $\gg$	$\mathcal{A}$	;
L03:	Arbitrary $\gg$	$\mathcal{B}$	;
L04:	Premise $\gg$	$\mathcal{H} \Rightarrow \mathcal{A} \Rightarrow \mathcal{B}$	;
L05:	Premise $\gg$	$\mathcal{H} \Rightarrow \mathcal{A}$	;
L06:	$\text{A2}' \gg$	$(\mathcal{H} \Rightarrow \mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow (\mathcal{H} \Rightarrow \mathcal{A}) \Rightarrow$ $\mathcal{H} \Rightarrow \mathcal{B}$	;
L07:	$\text{L06} \sqsupseteq \text{L04} \gg$	$(\mathcal{H} \Rightarrow \mathcal{A}) \Rightarrow \mathcal{H} \Rightarrow \mathcal{B}$	;
L08:	$\text{L07} \sqsupseteq \text{L05} \gg$	$\mathcal{H} \Rightarrow \mathcal{B}$	□

We use $[\text{x} \sqsupseteq_{\text{h}} \text{y}]$ ⁶³ as shorthand for hypothetical modus ponens:

$$[\text{x} \sqsupseteq_{\text{h}} \text{y} \ddot{=} \text{MP}'_{\text{h}} \triangleright \text{x} \triangleright \text{y}]$$

2.1.4 Turning lemmas to hypothetical lemmas

Lemma [Hypothesize]⁶⁴ turns a lemma with no premises into one that assumes the hypothesis $[\mathcal{H}]$ to hold:

[S' lemma Hypothesize: $\forall \mathcal{H}: \forall \mathcal{A}: \mathcal{A} \vdash \mathcal{H} \Rightarrow \mathcal{A}$]

S' proof of Hypothesize:

L01:	Arbitrary $\gg$	$\mathcal{H}$	;
L02:	Arbitrary $\gg$	$\mathcal{A}$	;
L03:	Premise $\gg$	$\mathcal{A}$	;
L04:	$\text{A1}' \gg$	$\mathcal{A} \Rightarrow \mathcal{H} \Rightarrow \mathcal{A}$	;
L05:	$\text{L04} \sqsupseteq \text{L03} \gg$	$\mathcal{H} \Rightarrow \mathcal{A}$	□

2.2 First order predicate calculus

2.2.1 Hypothetical generalization

The hypothetical version $[\text{Gen}'_{\text{h}}]$ ⁶⁵ of generalisation Gen' has a hypothesis $\mathcal{H}$ on premise and conclusion:

[S' lemma Gen'_{h} : $\forall \mathcal{H}: \forall \mathcal{X}: \forall \mathcal{A}: \text{nonfree}([\mathcal{X}], [\mathcal{H}]) \vdash \mathcal{H} \Rightarrow \mathcal{A} \vdash \mathcal{H} \Rightarrow \dot{\forall} \mathcal{X}: \mathcal{A}$]

S' proof of Gen'_{h} :

L01:	Arbitrary $\gg$	$\mathcal{H}$	;
L02:	Arbitrary $\gg$	$\mathcal{X}$	;
L03:	Arbitrary $\gg$	$\mathcal{A}$	;
L04:	Side-condition $\gg$	$\text{nonfree}([\mathcal{X}], [\mathcal{H}])$	;

⁶³ $[\text{x} \sqsupseteq_{\text{h}} \text{y}] \stackrel{\text{pyk}}{=} \text{"* hypothetical modus ponens *"}]$

⁶⁴ $[\text{Hypothesize}] \stackrel{\text{pyk}}{=} \text{"hypothesize"}]$

⁶⁵ $[\text{Gen}'_{\text{h}}] \stackrel{\text{pyk}}{=} \text{"hypothetical rule prime gen"}]$

L05:	Premise $\gg$	$\mathcal{H} \Rightarrow \mathcal{A}$	;
L06:	$A5' \triangleright L04 \gg$	$\forall \mathcal{X}: (\mathcal{H} \Rightarrow \mathcal{A}) \Rightarrow \mathcal{H} \Rightarrow \forall \mathcal{X}: \mathcal{A}$	;
L07:	$Gen' \triangleright L05 \gg$	$\forall \mathcal{X}: (\mathcal{H} \Rightarrow \mathcal{A})$	;
L08:	$L06 \sqsubseteq L07 \gg$	$\mathcal{H} \Rightarrow \forall \mathcal{X}: \mathcal{A}$	$\square$

2.3 Peano arithmetic

2.3.1 Lemma M3.2(a)

Lemma [M3.2(a)]⁶⁶ and the associated hypothetical lemma [M3.2(a)_h]⁶⁷ read:

[S' rule M3.2(a): $\forall \mathcal{T}: \mathcal{T} \stackrel{p}{=} \mathcal{T}$]

[S' lemma M3.2(a)_h: $\forall \mathcal{H}: \forall \mathcal{T}: \mathcal{H} \Rightarrow \mathcal{T} \stackrel{p}{=} \mathcal{T}$]

Above we cheat in stating M3.2(a) as a rule and not as a lemma. A reasonable way to construct a large proof is to start stating everything as rules and then changing the rules to lemmas one at a time until only the rules of the theory are left.

S' proof of M3.2(a)_h:

L01:	Arbitrary $\gg$	$\mathcal{H}$	;
L02:	Arbitrary $\gg$	$\mathcal{T}$	;
L03:	M3.2(a) $\gg$	$\mathcal{T} \stackrel{p}{=} \mathcal{T}$	;
L04:	Hypothesize $\triangleright L03 \gg$	$\mathcal{H} \Rightarrow \mathcal{T} \stackrel{p}{=} \mathcal{T}$	$\square$

2.3.2 Lemma M3.2(b)

Lemma [M3.2(b)_h]⁶⁸ reads:

[S' rule M3.2(b)_h: $\forall \mathcal{H}: \forall \mathcal{T}: \forall \mathcal{R}: \mathcal{H} \Rightarrow \mathcal{T} \stackrel{p}{=} \mathcal{R} \vdash \mathcal{H} \Rightarrow \mathcal{R} \stackrel{p}{=} \mathcal{T}$]

S' proof of M3.2(b)_h:

L01:	Arbitrary $\gg$	$\mathcal{H}$	;
L02:	Arbitrary $\gg$	$\mathcal{T}$	;
L03:	Arbitrary $\gg$	$\mathcal{R}$	;
L04:	Premise $\gg$	$\mathcal{H} \Rightarrow \mathcal{T} \stackrel{p}{=} \mathcal{R}$	;
L05:	$S1' \gg$	$\mathcal{T} \stackrel{p}{=} \mathcal{R} \Rightarrow \mathcal{T} \stackrel{p}{=} \mathcal{T} \Rightarrow \mathcal{R} \stackrel{p}{=} \mathcal{T}$	;
L06:	Hypothesize $\triangleright L05 \gg$	$\mathcal{H} \Rightarrow \mathcal{T} \stackrel{p}{=} \mathcal{R} \Rightarrow \mathcal{T} \stackrel{p}{=} \mathcal{T} \Rightarrow \mathcal{R} \stackrel{p}{=} \mathcal{T}$	;
L07:	$L06 \sqsubseteq_h L04 \gg$	$\mathcal{H} \Rightarrow \mathcal{T} \stackrel{p}{=} \mathcal{T} \Rightarrow \mathcal{R} \stackrel{p}{=} \mathcal{T}$	;
L08:	M3.2(a) _h $\gg$	$\mathcal{H} \Rightarrow \mathcal{T} \stackrel{p}{=} \mathcal{T}$	;
L09:	$L07 \sqsubseteq_h L08 \gg$	$\mathcal{H} \Rightarrow \mathcal{R} \stackrel{p}{=} \mathcal{T}$	$\square$

⁶⁶[M3.2(a)]^{pyk} = “mendelson three two a”]

⁶⁷[M3.2(a)_h]^{pyk} = “hypothetical three two a”]

⁶⁸[M3.2(b)_h]^{pyk} = “hypothetical three two b”]

2.3.3 Lemma M3.1(S1)

Lemma [M3.1(S1')_h]⁶⁹ is the hypothetical version of Mendelson Lemma 3.1(S1'):

$$[S' \text{ rule M3.1(S1')}_h: \forall \mathcal{H}: \forall \mathcal{T}: \forall \mathcal{R}: \forall \mathcal{S}: \mathcal{H} \Rightarrow \mathcal{T} \stackrel{P}{=} \mathcal{R} \vdash \mathcal{H} \Rightarrow \mathcal{T} \stackrel{P}{=} \mathcal{S} \vdash \mathcal{H} \Rightarrow \mathcal{R} \stackrel{P}{=} \mathcal{S}]$$

2.3.4 Lemma M3.2(c)

Lemma [M3.2(c)_h]⁷⁰ is the hypothetical version of Mendelson Lemma 3.2(c) which expresses ordinary transitivity:

$$[S' \text{ rule M3.2(c)}_h: \forall \mathcal{H}: \forall \mathcal{T}: \forall \mathcal{R}: \forall \mathcal{S}: \mathcal{H} \Rightarrow \mathcal{T} \stackrel{P}{=} \mathcal{R} \vdash \mathcal{H} \Rightarrow \mathcal{R} \stackrel{P}{=} \mathcal{S} \vdash \mathcal{H} \Rightarrow \mathcal{T} \stackrel{P}{=} \mathcal{S}]$$

2.3.5 Lemma M3.1(S2)

Lemma [M3.1(S2')_h]⁷¹ is the hypothetical version of Mendelson Lemma 3.1(S2'):

$$[S' \text{ rule M3.1(S2')}_h: \forall \mathcal{H}: \forall \mathcal{T}: \forall \mathcal{R}: \mathcal{H} \Rightarrow \mathcal{T} \stackrel{P}{=} \mathcal{R} \vdash \mathcal{H} \Rightarrow \mathcal{T}' \stackrel{P}{=} \mathcal{R}']$$

2.3.6 Lemma M3.1(S5)

Lemma [M3.1(S5')_h]⁷² is the hypothetical version of Mendelson Lemma 3.1(S5'):

$$[S' \text{ rule M3.1(S5')}_h: \forall \mathcal{H}: \forall \mathcal{T}: \mathcal{H} \Rightarrow \mathcal{T} \dot{+} \dot{0} \stackrel{P}{=} \mathcal{T}]$$

2.3.7 Lemma M3.1(S6)

Lemma [M3.1(S6')_h]⁷³ is the hypothetical version of Mendelson Lemma 3.1(S6'):

$$[S' \text{ rule M3.1(S6')}_h: \forall \mathcal{H}: \forall \mathcal{T}: \forall \mathcal{R}: \mathcal{H} \Rightarrow \mathcal{T} \dot{+} \mathcal{R}' \stackrel{P}{=} (\mathcal{T} \dot{+} \mathcal{R})']$$

2.3.8 Lemma M3.2(f)

Lemma [M3.2(f)]⁷⁴ is the closure of Mendelson Lemma 3.2(f) for the concrete variable $[t]$:

$$[S' \text{ lemma M3.2(f)}: \dot{\forall} t: t \stackrel{P}{=} \dot{0} \dot{+} t]$$

The proof below uses local macro definitions.

⁶⁹[M3.1(S1')_h]^{pyk} ≡ “hypothetical three one s one”]

⁷⁰[M3.2(c)_h]^{pyk} ≡ “hypothetical three two c”]

⁷¹[M3.1(S2')_h]^{pyk} ≡ “hypothetical three one s two”]

⁷²[M3.1(S5')_h]^{pyk} ≡ “hypothetical three one s five”]

⁷³[M3.1(S6')_h]^{pyk} ≡ “hypothetical three one s six”]

⁷⁴[M3.2(f)]^{pyk} ≡ “mendelson three two f”]

S' proof of M3.2(f):

L01:	Local $\gg$	$\mathcal{Z} = x \Rightarrow x \Rightarrow x$	;
L02:	A1' $\gg$	$\mathcal{Z}$	;
L03:	M3.1(S5') _h $\gg$	$\mathcal{Z} \Rightarrow \dot{0} \dot{+} \dot{0} \stackrel{P}{=} \dot{0}$	;
L04:	M3.2(b) _h $\triangleright$ L03 $\gg$	$\mathcal{Z} \Rightarrow \dot{0} \stackrel{P}{=} \dot{0} \dot{+} \dot{0}$	;
L05:	L04 $\sqsubseteq$ L02 $\gg$	$\dot{0} \stackrel{P}{=} \dot{0} \dot{+} \dot{0}$	;
L06:	Local $\gg$	$\mathcal{H} = \dot{t} \stackrel{P}{=} \dot{0} \dot{+} \dot{t}$	;
L07:	M1.7 $\gg$	$\mathcal{H} \Rightarrow \dot{t} \stackrel{P}{=} \dot{0} \dot{+} \dot{t}$	;
L08:	M3.1(S2') _h $\triangleright$ L07 $\gg$	$\mathcal{H} \Rightarrow \dot{t}' \stackrel{P}{=} (\dot{0} \dot{+} \dot{t})'$	;
L09:	M3.1(S6') _h $\gg$	$\mathcal{H} \Rightarrow \dot{0} \dot{+} \dot{t}' \stackrel{P}{=} (\dot{0} \dot{+} \dot{t})'$	;
L10:	M3.2(b) _h $\triangleright$ L09 $\gg$	$\mathcal{H} \Rightarrow (\dot{0} \dot{+} \dot{t})' \stackrel{P}{=} \dot{0} \dot{+} \dot{t}'$	;
L11:	M3.2(c) _h $\triangleright$ L08 $\triangleright$ L10 $\gg$	$\mathcal{H} \Rightarrow \dot{t}' \stackrel{P}{=} \dot{0} \dot{+} \dot{t}'$	;
L12:	Gen' $\triangleright$ L11 $\gg$	$\forall \dot{t}: (\mathcal{H} \Rightarrow \dot{t}' \stackrel{P}{=} \dot{0} \dot{+} \dot{t}')$	;
L13:	S9' $\gg$	$\dot{0} \stackrel{P}{=} \dot{0} \dot{+} \dot{0} \Rightarrow \forall \dot{t}: (\dot{t} \stackrel{P}{=} \dot{0} \dot{+} \dot{t} \Rightarrow$ $\dot{t}' \stackrel{P}{=} \dot{0} \dot{+} \dot{t}') \Rightarrow \forall \dot{t}: \dot{t} \stackrel{P}{=} \dot{0} \dot{+} \dot{t}$	;
L14:	L13 $\sqsubseteq$ L05 $\gg$	$\forall \dot{t}: (\mathcal{H} \Rightarrow \dot{t}' \stackrel{P}{=} \dot{0} \dot{+} \dot{t}') \Rightarrow \forall \dot{t}: \dot{t} \stackrel{P}{=} \dot{0} \dot{+} \dot{t}$	;
L15:	L14 $\sqsubseteq$ L12 $\gg$	$\forall \dot{t}: \dot{t} \stackrel{P}{=} \dot{0} \dot{+} \dot{t}$	□

A Chores

A.1 The name of the page

This defines the name of the page:

$$[\text{peano} \stackrel{\text{pyk}}{=} \text{“peano”}]$$

A.2 Variables of Peano arithmetic

We use $[\dot{b}]^{75}$, $[\dot{c}]^{76}$, $[\dot{d}]^{77}$, $[\dot{e}]^{78}$, $[\dot{f}]^{79}$, $[\dot{g}]^{80}$, $[\dot{h}]^{81}$, $[\dot{i}]^{82}$, $[\dot{j}]^{83}$, $[\dot{k}]^{84}$, $[\dot{l}]^{85}$, $[\dot{m}]^{86}$, $[\dot{n}]^{87}$, $[\dot{o}]^{88}$, $[\dot{p}]^{89}$, $[\dot{q}]^{90}$, $[\dot{r}]^{91}$, $[\dot{s}]^{92}$, $[\dot{t}]^{93}$, $[\dot{u}]^{94}$, $[\dot{v}]^{95}$, $[\dot{w}]^{96}$, $[\dot{x}]^{97}$, $[\dot{y}]^{98}$, and $[\dot{z}]^{99}$ to denote variables of Peano arithmetic:

$[\dot{b} \doteq \dot{b}]$, $[\dot{c} \doteq \dot{c}]$, $[\dot{d} \doteq \dot{d}]$, $[\dot{e} \doteq \dot{e}]$, $[\dot{f} \doteq \dot{f}]$, $[\dot{g} \doteq \dot{g}]$, $[\dot{h} \doteq \dot{h}]$, $[\dot{i} \doteq \dot{i}]$, $[\dot{j} \doteq \dot{j}]$, $[\dot{k} \doteq \dot{k}]$, $[\dot{l} \doteq \dot{l}]$, $[\dot{m} \doteq \dot{m}]$, $[\dot{n} \doteq \dot{n}]$, $[\dot{o} \doteq \dot{o}]$, $[\dot{p} \doteq \dot{p}]$, $[\dot{q} \doteq \dot{q}]$, $[\dot{r} \doteq \dot{r}]$, $[\dot{s} \doteq \dot{s}]$, $[\dot{t} \doteq \dot{t}]$, $[\dot{u} \doteq \dot{u}]$, $[\dot{v} \doteq \dot{v}]$, $[\dot{w} \doteq \dot{w}]$, $[\dot{x} \doteq \dot{x}]$, $[\dot{y} \doteq \dot{y}]$, and $[\dot{z} \doteq \dot{z}]$.

A.3 $\text{\TeX}$ definitions

$[\dot{0}^{\text{tex}} \equiv \text{"\dot{0}"}]$
 $[\text{x}'^{\text{tex}} \equiv \text{"\#1."}]$
 $[\text{x} \dot{+} \text{y}^{\text{tex}} \equiv \text{"\#1.\mathop{\dot{+}} \text{\#2."}]$

$^{75}[\dot{b}^{\text{pyk}} \equiv \text{"peano b"}]$
 $^{76}[\dot{c}^{\text{pyk}} \equiv \text{"peano c"}]$
 $^{77}[\dot{d}^{\text{pyk}} \equiv \text{"peano d"}]$
 $^{78}[\dot{e}^{\text{pyk}} \equiv \text{"peano e"}]$
 $^{79}[\dot{f}^{\text{pyk}} \equiv \text{"peano f"}]$
 $^{80}[\dot{g}^{\text{pyk}} \equiv \text{"peano g"}]$
 $^{81}[\dot{h}^{\text{pyk}} \equiv \text{"peano h"}]$
 $^{82}[\dot{i}^{\text{pyk}} \equiv \text{"peano i"}]$
 $^{83}[\dot{j}^{\text{pyk}} \equiv \text{"peano j"}]$
 $^{84}[\dot{k}^{\text{pyk}} \equiv \text{"peano k"}]$
 $^{85}[\dot{l}^{\text{pyk}} \equiv \text{"peano l"}]$
 $^{86}[\dot{m}^{\text{pyk}} \equiv \text{"peano m"}]$
 $^{87}[\dot{n}^{\text{pyk}} \equiv \text{"peano n"}]$
 $^{88}[\dot{o}^{\text{pyk}} \equiv \text{"peano o"}]$
 $^{89}[\dot{p}^{\text{pyk}} \equiv \text{"peano p"}]$
 $^{90}[\dot{q}^{\text{pyk}} \equiv \text{"peano q"}]$
 $^{91}[\dot{r}^{\text{pyk}} \equiv \text{"peano r"}]$
 $^{92}[\dot{s}^{\text{pyk}} \equiv \text{"peano s"}]$
 $^{93}[\dot{t}^{\text{pyk}} \equiv \text{"peano t"}]$
 $^{94}[\dot{u}^{\text{pyk}} \equiv \text{"peano u"}]$
 $^{95}[\dot{v}^{\text{pyk}} \equiv \text{"peano v"}]$
 $^{96}[\dot{w}^{\text{pyk}} \equiv \text{"peano w"}]$
 $^{97}[\dot{x}^{\text{pyk}} \equiv \text{"peano x"}]$
 $^{98}[\dot{y}^{\text{pyk}} \equiv \text{"peano y"}]$
 $^{99}[\dot{z}^{\text{pyk}} \equiv \text{"peano z"}]$

[$\dot{x} \cdot y \stackrel{\text{tex}}{=} \text{"\#1.}$
 $\backslash\mathop{\dot{\{\}\cdot\}} \text{"\#2.}"$]

[$x \stackrel{\text{p}}{=} y \stackrel{\text{tex}}{=} \text{"\#1.}$
 $\backslash\stackrel{\text{p}}{\text{rel}}{\{p\}}{\{=\}} \text{"\#2.}"$]

[$\dot{\neg} x \stackrel{\text{tex}}{=} \text{"}$
 $\backslash\dot{\{\neg\}} \backslash, \{\#1.\}$ "]

[$x \Rightarrow y \stackrel{\text{tex}}{=} \text{"\#1.}$
 $\backslash\mathrel{\dot{\{\rightarrow\}}} \text{"\#2.}"$]

[$\dot{\forall} x: y \stackrel{\text{tex}}{=} \text{"}$
 $\backslash\dot{\{\forall\}} \text{"\#1.}$
 $\backslash\text{colon} \text{"\#2.}"$]

[$\dot{1} \stackrel{\text{tex}}{=} \text{"}$
 $\backslash\dot{\{1\}}$ "]

[$\dot{2} \stackrel{\text{tex}}{=} \text{"}$
 $\backslash\dot{\{2\}}$ "]

[$x \dot{\wedge} y \stackrel{\text{tex}}{=} \text{"\#1.}$
 $\backslash\mathrel{\dot{\{\wedge\}}} \text{"\#2.}"$]

[$x \dot{\vee} y \stackrel{\text{tex}}{=} \text{"\#1.}$
 $\backslash\mathrel{\dot{\{\vee\}}} \text{"\#2.}"$]

[$x \Leftrightarrow y \stackrel{\text{tex}}{=} \text{"\#1.}$
 $\backslash\mathrel{\dot{\{\Leftrightarrow\}}} \text{"\#2.}"$]

[$\dot{\exists} x: y \stackrel{\text{tex}}{=} \text{"}$
 $\backslash\dot{\{\exists\}} \text{"\#1.}$
 $\backslash\text{colon} \text{"\#2.}"$]

[$\dot{x} \stackrel{\text{tex}}{=} \text{"}$
 $\backslash\dot{\{\#1.}}$
 $\} \text{"}$]

[$x^{\mathcal{P}} \stackrel{\text{tex}}{=} \text{"\#1.}$
 $\{\} \wedge \{\backslash\text{cal P}\}$ "]

[$\dot{a} \stackrel{\text{tex}}{=} \text{"}$
 $\backslash\dot{\{\mathit{a}\}}$ "]

[$\text{nonfree}(x, y) \stackrel{\text{tex}}{=} \text{"}$
 $\backslash\dot{\{\text{nonfree}\}}(\text{"\#1.}$
 $\text{"\#2.}$
 $\text{"})"$]

$$[\text{MP} \stackrel{\text{tex}}{=} \text{“MP”}]$$

$$[\text{Gen} \stackrel{\text{tex}}{=} \text{“Gen”}]$$

$$[\text{S1} \stackrel{\text{tex}}{=} \text{“S1”}]$$

$$[\text{S2} \stackrel{\text{tex}}{=} \text{“S2”}]$$

$$[\text{S3} \stackrel{\text{tex}}{=} \text{“S3”}]$$

$$[\text{S4} \stackrel{\text{tex}}{=} \text{“S4”}]$$

$$[\text{S5} \stackrel{\text{tex}}{=} \text{“S5”}]$$

$$[\text{S6} \stackrel{\text{tex}}{=} \text{“S6”}]$$

$$[\text{S7} \stackrel{\text{tex}}{=} \text{“S7”}]$$

$$[\text{S8} \stackrel{\text{tex}}{=} \text{“S8”}]$$

$$[\text{S9} \stackrel{\text{tex}}{=} \text{“S9”}]$$

$$[\text{L3.2(a)} \stackrel{\text{tex}}{=} \text{“L3.2(a)”}]$$

$$[\text{S'} \stackrel{\text{tex}}{=} \text{“S”}]$$

$$[\text{A1'} \stackrel{\text{tex}}{=} \text{“A1”}]$$

$$[\text{A2'} \stackrel{\text{tex}}{=} \text{“A2”}]$$

$$[\text{A3'} \stackrel{\text{tex}}{=} \text{“A3”}]$$

$$[A4' \stackrel{\text{tex}}{=} \text{“} \qquad \qquad A4' \text{”}]$$

$$[A5' \stackrel{\text{tex}}{=} \text{“} \qquad \qquad A5' \text{”}]$$

$$[MP' \stackrel{\text{tex}}{=} \text{“} \qquad \qquad MP' \text{”}]$$

$$[Gen' \stackrel{\text{tex}}{=} \text{“} \qquad \qquad Gen' \text{”}]$$

$$[S1' \stackrel{\text{tex}}{=} \text{“} \qquad \qquad S1' \text{”}]$$

$$[S2' \stackrel{\text{tex}}{=} \text{“} \qquad \qquad S2' \text{”}]$$

$$[S3' \stackrel{\text{tex}}{=} \text{“} \qquad \qquad S3' \text{”}]$$

$$[S4' \stackrel{\text{tex}}{=} \text{“} \qquad \qquad S4' \text{”}]$$

$$[S5' \stackrel{\text{tex}}{=} \text{“} \qquad \qquad S5' \text{”}]$$

$$[S6' \stackrel{\text{tex}}{=} \text{“} \qquad \qquad S6' \text{”}]$$

$$[S7' \stackrel{\text{tex}}{=} \text{“} \qquad \qquad S7' \text{”}]$$

$$[S8' \stackrel{\text{tex}}{=} \text{“} \qquad \qquad S8' \text{”}]$$

$$[S9' \stackrel{\text{tex}}{=} \text{“} \qquad \qquad S9' \text{”}]$$

$$[L3.2(a)' \stackrel{\text{tex}}{=} \text{“} \qquad \qquad L3.2(a)' \text{”}]$$

$$[x \supseteq y \stackrel{\text{tex}}{=} \text{“}\#1. \qquad \qquad \backslash\text{unrhd } \#2.\text{”}]$$

$$[M1.7 \stackrel{\text{tex}}{=} \text{“} \qquad \qquad M1.7 \text{”}]$$

$$[\text{MP}'_{\text{h}} \stackrel{\text{tex}}{=} \text{“MP}'_{\text{h}}”}]$$

$$[\text{x} \triangleright_{\text{h}} \text{y} \stackrel{\text{tex}}{=} \text{“}\#1.\backslash\text{unrhd}_{\text{h}} \#2.”}]$$

$$[\text{Hypothesize} \stackrel{\text{tex}}{=} \text{“Hypothesize”}]$$

$$[\text{Gen}'_{\text{h}} \stackrel{\text{tex}}{=} \text{“Gen}'_{\text{h}}”}]$$

$$[\text{M3.2(a)} \stackrel{\text{tex}}{=} \text{“M3.2(a)”}]$$

$$[\text{M3.2(a)}_{\text{h}} \stackrel{\text{tex}}{=} \text{“M3.2(a)}_{\text{h}}”}]$$

$$[\text{M3.2(b)}_{\text{h}} \stackrel{\text{tex}}{=} \text{“M3.2(b)}_{\text{h}}”}]$$

$$[\text{M3.1(S1')}_{\text{h}} \stackrel{\text{tex}}{=} \text{“M3.1(S1')}_{\text{h}}”}]$$

$$[\text{M3.2(c)}_{\text{h}} \stackrel{\text{tex}}{=} \text{“M3.2(c)}_{\text{h}}”}]$$

$$[\text{M3.1(S2')}_{\text{h}} \stackrel{\text{tex}}{=} \text{“M3.1(S2')}_{\text{h}}”}]$$

$$[\text{M3.1(S5')}_{\text{h}} \stackrel{\text{tex}}{=} \text{“M3.1(S5')}_{\text{h}}”}]$$

$$[\text{M3.1(S6')}_{\text{h}} \stackrel{\text{tex}}{=} \text{“M3.1(S6')}_{\text{h}}”}]$$

$$[\text{M3.2(f)} \stackrel{\text{tex}}{=} \text{“M3.2(f)”}]$$

$$[\dot{b} \stackrel{\text{tex}}{=} \text{“}\dot{\{\mathit{b}\}}”]$$

$$[\dot{c} \stackrel{\text{tex}}{=} \text{“}\dot{\{\mathit{c}\}}”]$$

$$[\dot{d} \stackrel{\text{tex}}{=} \text{“}\dot{\{\mathit{d}\}}”]$$

$\dot{e} \stackrel{\text{tex}}{=} “$
 $\dot{\mathit{e}}”]$
 $\dot{f} \stackrel{\text{tex}}{=} “$
 $\dot{\mathit{f}}”]$
 $\dot{g} \stackrel{\text{tex}}{=} “$
 $\dot{\mathit{g}}”]$
 $\dot{h} \stackrel{\text{tex}}{=} “$
 $\dot{\mathit{h}}”]$
 $\dot{i} \stackrel{\text{tex}}{=} “$
 $\dot{\mathit{i}}”]$
 $\dot{j} \stackrel{\text{tex}}{=} “$
 $\dot{\mathit{j}}”]$
 $\dot{k} \stackrel{\text{tex}}{=} “$
 $\dot{\mathit{k}}”]$
 $\dot{l} \stackrel{\text{tex}}{=} “$
 $\dot{\mathit{l}}”]$
 $\dot{m} \stackrel{\text{tex}}{=} “$
 $\dot{\mathit{m}}”]$
 $\dot{n} \stackrel{\text{tex}}{=} “$
 $\dot{\mathit{n}}”]$
 $\dot{o} \stackrel{\text{tex}}{=} “$
 $\dot{\mathit{o}}”]$
 $\dot{p} \stackrel{\text{tex}}{=} “$
 $\dot{\mathit{p}}”]$
 $\dot{q} \stackrel{\text{tex}}{=} “$
 $\dot{\mathit{q}}”]$
 $\dot{r} \stackrel{\text{tex}}{=} “$
 $\dot{\mathit{r}}”]$
 $\dot{s} \stackrel{\text{tex}}{=} “$
 $\dot{\mathit{s}}”]$
 $\dot{t} \stackrel{\text{tex}}{=} “$
 $\dot{\mathit{t}}”]$

$$[\dot{u} \stackrel{\text{tex}}{=} “ \backslash\dot{\{\mathit{u}\}} ”]$$

$$[\dot{v} \stackrel{\text{tex}}{=} “ \backslash\dot{\{\mathit{v}\}} ”]$$

$$[\dot{w} \stackrel{\text{tex}}{=} “ \backslash\dot{\{\mathit{w}\}} ”]$$

$$[\dot{x} \stackrel{\text{tex}}{=} “ \backslash\dot{\{\mathit{x}\}} ”]$$

$$[\dot{y} \stackrel{\text{tex}}{=} “ \backslash\dot{\{\mathit{y}\}} ”]$$

$$[\dot{z} \stackrel{\text{tex}}{=} “ \backslash\dot{\{\mathit{z}\}} ”]$$

A.4 Test

$$[[\dot{a}]^{\mathcal{P}}]$$

$$[[\mathbf{a}]^{\mathcal{P}}]^{-}$$

$$[\text{nonfree}(\lceil \dot{x} \rceil, [\dot{y} \stackrel{\text{P}}{=} \dot{z} \Rightarrow \dot{\forall} \dot{x} : \dot{x} \stackrel{\text{P}}{=} \dot{y}])] \cdot$$

$$[\text{nonfree}(\lceil \dot{x} \rceil, [\dot{x} \stackrel{\text{P}}{=} \dot{z} \Rightarrow \dot{\forall} \dot{x} : \dot{x} \stackrel{\text{P}}{=} \dot{y}])]^{-}$$

$$[\text{nonfree}(\lceil \dot{x} \rceil, [\dot{y} \stackrel{\text{P}}{=} \dot{x} \Rightarrow \dot{\forall} \dot{x} : \dot{x} \stackrel{\text{P}}{=} \dot{y}])]^{-}$$

$$[\text{nonfree}(\lceil \dot{x} \rceil, [\dot{y} \stackrel{\text{P}}{=} \dot{z} \Rightarrow \dot{\forall} \dot{y} : \dot{x} \stackrel{\text{P}}{=} \dot{y}])]^{-}$$

$$[\text{free}(\lceil \dot{\forall} \dot{x} : \mathbf{b} :: \dot{x} :: \mathbf{c} \rceil | \lceil \dot{x} \rceil := [\mathbf{x} :: \dot{y} :: \mathbf{z}])] \cdot$$

$$[\text{free}(\lceil \dot{\forall} \dot{y} : \mathbf{b} :: \dot{x} :: \mathbf{c} \rceil | \lceil \dot{x} \rceil := [\mathbf{x} :: \dot{y} :: \mathbf{z}])]^{-}$$

$$[\text{free}(\lceil \dot{\forall} \dot{x} : \mathbf{b} :: \dot{x} :: \mathbf{c} \rceil | \lceil \dot{y} \rceil := [\mathbf{x} :: \dot{y} :: \mathbf{z}])] \cdot$$

$$[\text{free}(\lceil \dot{\forall} \dot{y} : \mathbf{b} :: \dot{x} :: \mathbf{c} \rceil | \lceil \dot{y} \rceil := [\mathbf{x} :: \dot{y} :: \mathbf{z}])] \cdot$$

$$[\dot{a} \equiv \langle \dot{a} | \dot{b} := \dot{c} \rangle] \cdot$$

$$[\dot{c} \equiv \langle \dot{b} | \dot{b} := \dot{c} \rangle] \cdot$$

$$[\forall \dot{a} : \dot{a} \stackrel{\text{P}}{=} \dot{b} \equiv \langle \forall \dot{a} : \dot{a} \stackrel{\text{P}}{=} \dot{b} | \dot{a} := \dot{c} \rangle] \cdot$$

$$[\forall \dot{a} : \dot{a} \stackrel{\text{P}}{=} \dot{c} \equiv \langle \forall \dot{a} : \dot{a} \stackrel{\text{P}}{=} \dot{b} | \dot{b} := \dot{c} \rangle] \cdot$$

$$[\dot{\forall} \dot{a} : \dot{a} \stackrel{\text{P}}{=} \dot{0} + \dot{a} \Rightarrow \dot{c} : \dot{d} \stackrel{\text{P}}{=} \dot{0} + \dot{c} : \dot{d} \equiv \langle \dot{\forall} \dot{a} : \dot{a} \stackrel{\text{P}}{=} \dot{0} + \dot{a} \Rightarrow \dot{b} \stackrel{\text{P}}{=} \dot{0} + \dot{b} | \dot{b} := \dot{c} : \dot{d} \rangle] \cdot$$

$$\nabla\dot{a}: \dot{a} \stackrel{P}{=} \dot{0} + \dot{a} \Rightarrow \dot{b} \stackrel{P}{=} \dot{0} + \dot{b} \equiv (\nabla\dot{a}: \dot{a} \stackrel{P}{=} \dot{0} + \dot{a} \Rightarrow \dot{b} \stackrel{P}{=} \dot{0} + \dot{b} | \dot{a} := \dot{c})$$

A.5 Priority table

Priority table

Preassociative

[peano], [base], [bracket * end bracket], [big bracket * end bracket],
[math * end math], [**flush left** *], [x], [y], [z], [[* $\bowtie$ *]], [[* $\overset{*}{\rightarrow}$ *]], [pyk], [tex],
[name], [prio], [*], [T], [if(*, *, *)], [[* $\overset{*}{\Rightarrow}$ *]], [val], [claim], [$\perp$], [f(*)], [(*)^I], [F], [$\underline{0}$],
[1], [2], [3], [4], [5], [6], [7], [8], [9], [0], [1], [2], [3], [4], [5], [6], [7], [8], [9], [a], [b], [c], [d],
[e], [f], [g], [h], [i], [j], [k], [l], [m], [n], [o], [p], [q], [r], [s], [t], [u], [v], [w], [(*)^M], [If(*, *,
)], [array{} * end array], [l], [c], [r], [empty], [($\ast$ | $\ast$:= $\ast$)], [$\mathcal{M}(\ast)$], [$\mathcal{U}(\ast)$], [$\mathcal{U}(\ast)$],
[$\mathcal{U}^M(\ast)$], [**apply**(*, *)], [**apply**₁(*, *)], [identifier(*)], [identifier₁(*, *)], [array-
plus(*, *)], [array-remove(*, *, *)], [array-put(*, *, *, *)], [array-add(*, *, *, *, *)],
[bit(*, *)], [bit₁(*, *)], [rack], ["vector"], ["bibliography"], ["dictionary"],
["body"], ["codex"], ["expansion"], ["code"], ["cache"], ["diagnose"], ["pyk"],
["tex"], ["texname"], ["value"], ["message"], ["macro"], ["definition"],
["unpack"], ["claim"], ["priority"], ["lambda"], ["apply"], ["true"], ["if"],
["quote"], ["proclaim"], ["define"], ["introduce"], ["hide"], ["pre"], ["post"],
[$\mathcal{E}(*, *, *)$], [$\mathcal{E}_2(*, *, *, *)$], [$\mathcal{E}_3(*, *, *, *)$], [$\mathcal{E}_4(*, *, *, *)$], [**lookup**(*, *, *)],
[**abstract**(*, *, *, *)], [[*]], [$\mathcal{M}(*, *, *)$], [$\mathcal{M}_2(*, *, *, *)$], [$\mathcal{M}^*(*, *, *)$], [macro],
[s₀], [**zip**(*, *)], [**assoc**₁(*, *, *)], [(*)^P], [self], [[* $\dot{=}$ *]], [[* $\dot{=}$ *]], [[* $\dot{=}$ *]],
[[* ^{pyk} *]], [[* ^{tex} *]], [[* ^{name} *]], [**Priority table**[*]], [$\tilde{\mathcal{M}}_1$], [$\tilde{\mathcal{M}}_2(*)$], [$\tilde{\mathcal{M}}_3(*)$],
[$\mathcal{M}_4(*, *, *, *)$], [$\mathcal{M}(*, *, *)$], [$\mathcal{Q}(*, *, *)$], [$\tilde{\mathcal{Q}}_2(*, *, *)$], [$\tilde{\mathcal{Q}}_3(*, *, *, *)$], [$\tilde{\mathcal{Q}}^*(*, *, *)$],
[(*)], [**aspect**(*, *)], [**aspect**(*, *, *)], [($\ast$)], [**tuple**₁(*)], [**tuple**₂(*)], [let₂(*, *)],
[let₁(*, *)], [[* ^{claim} *]], [checker], [**check**(*, *)], [**check**₂(*, *, *)], [**check**₃(*, *, *)],
[**check**^{*}(*, *)], [**check**₂^{*}(*, *, *)], [[*]'], [[*]'], [[*]^o], [msg], [[* ^{msg} *]], [<stmt>],
[stmt], [[* ^{stmt} *]], [HeadNil'], [HeadPair'], [Transitivity'], [$\perp$], [Contra'], [T_E],
[L₁], [*], [$\mathcal{A}$], [$\mathcal{B}$], [$\mathcal{C}$], [$\mathcal{D}$], [$\mathcal{E}$], [$\mathcal{F}$], [$\mathcal{G}$], [$\mathcal{H}$], [$\mathcal{I}$], [$\mathcal{J}$], [$\mathcal{K}$], [$\mathcal{L}$], [$\mathcal{M}$], [$\mathcal{N}$], [$\mathcal{O}$], [$\mathcal{P}$], [$\mathcal{Q}$],
[$\mathcal{R}$], [$\mathcal{S}$], [$\mathcal{T}$], [$\mathcal{U}$], [$\mathcal{V}$], [$\mathcal{W}$], [$\mathcal{X}$], [$\mathcal{Y}$], [$\mathcal{Z}$], [($\ast$ | $\ast$:= $\ast$)], [($\ast$ | $\ast$:= $\ast$)], [$\emptyset$], [Remainder],
[($\ast$)^v], [intro(*, *, *, *)], [intro(*, *, *)], [error(*, *)], [error₂(*, *)], [proof(*, *, *)],
[proof₂(*, *)], [$\mathcal{S}(*, *)$], [$\mathcal{S}^I(*, *)$], [$\mathcal{S}^\triangleright(*, *)$], [$\mathcal{S}_1^\triangleright(*, *, *)$], [$\mathcal{S}^E(*, *)$], [$\mathcal{S}_1^E(*, *, *)$],
[$\mathcal{S}^+(*, *)$], [$\mathcal{S}_1^+(*, *, *)$], [$\mathcal{S}^-(*, *)$], [$\mathcal{S}_1^-(*, *, *)$], [$\mathcal{S}^*(*, *)$], [$\mathcal{S}_1^*(*, *, *)$],
[$\mathcal{S}_2^*(*, *, *, *)$], [$\mathcal{S}^\oplus(*, *)$], [$\mathcal{S}_1^\oplus(*, *, *)$], [$\mathcal{S}^+(*, *)$], [$\mathcal{S}_1^+(*, *, *, *)$], [$\mathcal{S}^{\#}(*, *)$],
[$\mathcal{S}_1^{\#}(*, *, *, *)$], [$\mathcal{S}^{i.e.}(*, *)$], [$\mathcal{S}_1^{i.e.}(*, *, *, *)$], [$\mathcal{S}_2^{i.e.}(*, *, *, *, *)$], [$\mathcal{S}^{\vee}(*, *)$],
[$\mathcal{S}_1^{\vee}(*, *, *, *)$], [$\mathcal{S}^i(*, *)$], [$\mathcal{S}_1^i(*, *, *)$], [$\mathcal{S}_2^i(*, *, *, *)$], [$\mathcal{T}(*, *)$], [claims(*, *, *)],
[claims₂(*, *, *)], [<proof>], [proof], [**Lemma** *: *], [**Proof of** *: *],
[[* **lemma** *: *]], [[* **antilemma** *: *]], [[* **rule** *: *]], [[* **antirule** *: *]],
[verifier], [$\mathcal{V}_1(*)$], [$\mathcal{V}_2(*, *)$], [$\mathcal{V}_3(*, *, *, *)$], [$\mathcal{V}_4(*, *)$], [$\mathcal{V}_5(*, *, *, *, *)$], [$\mathcal{V}_6(*, *, *, *, *)$],
[$\mathcal{V}_7(*, *, *, *, *)$], [Cut(*, *)], [Head_⊕(*)], [Tail_⊕(*)], [rule₁(*, *)], [rule(*, *)],
[Rule tactic], [Plus(*, *)], [**Theory** *], [theory₂(*, *)], [theory₃(*, *)],
[theory₄(*, *, *)], [HeadNil''], [HeadPair''], [Transitivity''], [Contra''], [HeadNil],
[HeadPair], [Transitivity], [Contra], [T_E], [ragged right],

[ragged right expansion], [parm(*, *, *)], [parm*(*, *, *)], [inst(*, *)],
[inst*(*, *)], [occur(*, *, *)], [occur*(*, *, *)], [unify(* = *, *)], [unify*(*, *, *)],
[unify₂(* = *, *)], [L_a], [L_b], [L_c], [L_d], [L_e], [L_f], [L_g], [L_h], [L_i], [L_j], [L_k], [L_l], [L_m],
[L_n], [L_o], [L_p], [L_q], [L_r], [L_s], [L_t], [L_u], [L_v], [L_w], [L_x], [L_y], [L_z], [L_A], [L_B], [L_C],
[L_D], [L_E], [L_F], [L_G], [L_H], [L_I], [L_J], [L_K], [L_L], [L_M], [L_N], [L_O], [L_P], [L_Q], [L_R],
[L_S], [L_T], [L_U], [L_V], [L_W], [L_X], [L_Y], [L_Z], [L_?], [Reflexivity], [Reflexivity₁],
[Commutativity], [Commutativity₁], [<tactic>], [tactic], [[*^{tactic} = *]], [P(*, *, *)],
[P*(*, *, *)], [p₀], [conclude₁(*, *, *)], [conclude₂(*, *, *)], [conclude₃(*, *, *, *)],
[conclude₄(*, *, *)], [0̇], [1̇], [2̇], [ā], [ḃ], [ċ], [ḋ], [ē], [ḟ], [ġ], [ḣ], [ī], [j̇], [k̇], [l̇], [ṁ], [ṅ],
[ō], [ṗ], [q̇], [ṙ], [ṡ], [ṫ], [ū], [v̇], [ẇ], [ẋ], [ẏ], [ż], [nonfree(*, *)], [nonfree*(*, *)],
[free(*|* := *)], [free*(*)|* := *)], [*≡(*|* := *)], [*≡(*|* := *)], [S], [A1], [A2],
[A3], [A4], [A5], [S1], [S2], [S3], [S4], [S5], [S6], [S7], [S8], [S9], [MP], [Gen],
[L3.2(a)], [S'], [A1'], [A2'], [A3'], [A4'], [A5'], [S1'], [S2'], [S3'], [S4'], [S5'], [S6'],
[S7'], [S8'], [S9'], [MP'], [Gen'], [L3.2(a)'], [M1.7], [MP'_h], [Hypothesize], [Gen'_h],
[M3.2(a)], [M3.2(a)_h], [M3.2(b)_h], [M3.1(S1')_h], [M3.2(c)_h], [M3.1(S2')_h],
[M3.1(S5')_h], [M3.1(S6')_h], [M3.2(f)];

Preassociative

[*_{*}], [*'], [* [*]], [* [* → *]], [* [* ⇒ *]], [* :];

Preassociative

[“ * ”], [], [(*)^t], [string(*) + *], [string(*) ++ *], [
*, [*], [*], [*], [*], [# *], [\$ *], [% *], [& *], [' *], [(*), () *], [* *], [+ *], [, *], [- *], [. *], [/ *],
[0 *], [1 *], [2 *], [3 *], [4 *], [5 *], [6 *], [7 *], [8 *], [9 *], [: *], [; *], [< *], [= *], [> *], [? *],
[@ *], [A *], [B *], [C *], [D *], [E *], [F *], [G *], [H *], [I *], [J *], [K *], [L *], [M *], [N *],
[O *], [P *], [Q *], [R *], [S *], [T *], [U *], [V *], [W *], [X *], [Y *], [Z *], [[*], [\ *], [] *], [^ *],
[_ *], [` *], [a *], [b *], [c *], [d *], [e *], [f *], [g *], [h *], [i *], [j *], [k *], [l *], [m *], [n *], [o *],
[p *], [q *], [r *], [s *], [t *], [u *], [v *], [w *], [x *], [y *], [z *], [{ * }, [| *], [} *], [~ *],
[Preassociative *; *], [Postassociative *; *], [[*], *], [priority * end],
[newline *], [macro newline *];

Preassociative

[*0], [*1], [0b], [*-color(*)], [*-color*(*)];

Preassociative

[* ' *], [* ' *];

Preassociative

[*^H], [*^T], [*^U], [*^h], [*^t], [*^s], [*^c], [*^d], [*^a], [*^C], [*^M], [*^B], [*^r], [*ⁱ], [*^d], [*^R], [*⁰],
[*¹], [*²], [*³], [*⁴], [*⁵], [*⁶], [*⁷], [*⁸], [*⁹], [*^E], [*^ν], [*^C], [*^{C*}], [*^γ];

Preassociative

[* · *], [* · 0 *], [* : *];

Preassociative

[* + *], [* + 0 *], [* + 1 *], [* - *], [* - 0 *], [* - 1 *], [* ÷ *];

Preassociative

[* ∪ {*}], [* ∪ *], [* \ {*}];

Postassociative

[* ∴ *], [* ∴ *], [* ∴ *], [* + 2* *], [* ∴ *], [* + 2* *];

Postassociative

[*, *];

Preassociative

$[* \overset{B}{\approx} *], [* \overset{D}{\approx} *], [* \overset{C}{\approx} *], [* \overset{P}{\approx} *], [* \approx *], [* = *], [* \overset{+}{\rightarrow} *], [* \overset{t}{=} *], [* \overset{t^*}{=} *], [* \overset{r}{=} *],$
 $[* \in_t *], [* \subseteq_T *], [* \overset{T}{=} *], [* \overset{s}{=} *], [* \text{free in } *], [* \text{free in}^* *], [* \text{free for } * \text{ in } *],$
 $[* \text{free for}^* * \text{ in } *], [* \in_c *], [* < *], [* <' *], [* \leq' *], [* \overset{p}{=} *], [* \overset{\mathcal{P}}{=} *];$

Preassociative

$[\neg *], [\dot{\neg} *];$

Preassociative

$[* \wedge *], [* \ddot{\wedge} *], [* \tilde{\wedge} *], [* \wedge_c *], [* \dot{\wedge} *];$

Preassociative

$[* \vee *], [* \parallel *], [* \ddot{\vee} *], [* \dot{\vee} *];$

Preassociative

$[\dot{\forall} *: *], [\dot{\exists} *: *];$

Postassociative

$[* \rightrightarrows *], [* \Rightarrow *], [* \Leftrightarrow *];$

Postassociative

$[* : *], [*! *];$

Preassociative

$[* \left\{ \begin{smallmatrix} * \\ * \end{smallmatrix} \right.];$

Preassociative

$[\lambda * . *], [\Lambda *], [\text{if } * \text{ then } * \text{ else } *], [\text{let } * = * \text{ in } *], [\text{let } * \ddot{=} * \text{ in } *];$

Preassociative

$[*^I], [*^\triangleright], [*^V], [*^+], [*^-], [*^*];$

Preassociative

$[* @ *], [* \triangleright *], [* \blacktriangleright *], [* \gg *], [* \supseteq *], [* \supseteq_h *];$

Postassociative

$[* \vdash *], [* \Vdash *], [* \text{i.e. } *];$

Preassociative

$[\forall *: *];$

Postassociative

$[* \oplus *];$

Postassociative

$[* ; *];$

Preassociative

$[* \text{ proves } *];$

Preassociative

$[* \text{ proof of } * : *], [\text{Line } * : * \gg *; *], [\text{Last line } * \gg * \square],$
 $[\text{Line } * : \text{Premise } \gg *; *], [\text{Line } * : \text{Side-condition } \gg *; *], [\text{Arbitrary } \gg *; *],$
 $[\text{Local } \gg * = *; *];$

Postassociative

$[* \text{ then } *], [* [*] *];$

Preassociative

$[* \& *];$

Preassociative

$[* \\ *]; \text{End table}$

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 pyk: axiom prime a four $[A4']$, 6
 pyk: axiom prime a one $[A1']$, 6
 pyk: axiom prime a three $[A3']$, 6
 pyk: axiom prime a two $[A2']$, 6
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 pyk: axiom prime s five $[S5']$, 7
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 pyk: axiom prime s one $[S1']$, 6
 pyk: axiom prime s seven $[S7']$, 7
 pyk: axiom prime s six $[S6']$, 7
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 pyk: axiom prime s two $[S2']$, 6
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 pyk: axiom s five $[S5]$, 4
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