Peano arithmetic

Klaus Grue

GRD-2005-06-20.UTC:11:21:10.729935

Contents

1 Peano arithmetic

This Logiweb page [1] defines Peano arithmetic.

1.1 The constructs of Peano arithmetic

Terms of Peano arithmetic are constructed from zero $[\dot{0} \stackrel{\text{pyk}}{\to}$ "peano zero"$][\dot{0} \stackrel{\text{tex}}{\to}$ "
\dot{0}"], successor $[\mathsf{x}' \stackrel{\text{pyk}}{\to}$ "∗ peano succ"$][\mathsf{x}' \stackrel{\text{tex}}{\to}$ "#1.'"], plus $[\mathsf{x} \mathbin{\dot{+}} \mathsf{y} \stackrel{\text{pyk}}{\to}$ "∗ peano plus ∗"$][\mathsf{x} \mathbin{\dot{+}} \mathsf{y} \stackrel{\text{tex}}{\to}$ "#1.
\mathop{\dot{+}} #2."], and times $[\mathsf{x} \mathbin{\dot{\cdot}} \mathsf{y} \stackrel{\text{pyk}}{\to}$ "∗ peano times ∗"$][\mathsf{x} \mathbin{\dot{\cdot}} \mathsf{y} \stackrel{\text{tex}}{\to}$ "#1.
\mathop{\dot{\cdot}} #2."].

Formulas of Peano arithmeric are constructed from equality [$\mathsf{x} \stackrel{p}{=} \mathsf{y} \stackrel{\text{pyk}}{\to}$ “* peano is *”][$\mathsf{x} \stackrel{p}{=} \mathsf{y} \stackrel{\text{tex}}{\to}$ “#1.
\stackrel{p}{=} #2.”], negation [$\dot{\neg}\,\mathsf{x} \stackrel{\text{pyk}}{\to}$ “peano not *”][$\dot{\neg}\,\mathsf{x} \stackrel{\text{tex}}{\to}$ “
\dot{\neg}\, #1.”], implication [$\mathsf{x} \mathrel{\dot{\Rightarrow}} \mathsf{y} \stackrel{\text{pyk}}{\to}$ “* peano imply *”][$\mathsf{x} \mathrel{\dot{\Rightarrow}} \mathsf{y} \stackrel{\text{tex}}{\to}$ “#1.
\mathrel{\dot{\Rightarrow}} #2.”], and universal quantification
[$\dot{\forall}\mathsf{x}\colon \mathsf{y} \stackrel{\text{pyk}}{\to}$ “peano all * indeed *”][$\dot{\forall}\mathsf{x}\colon \mathsf{y} \stackrel{\text{tex}}{\to}$ “
\dot{\forall} #1.
\colon #2.”].

From these constructs we macro define one [$\dot{1} \stackrel{\text{pyk}}{\to}$ “peano one”][$\dot{1} \stackrel{\text{tex}}{\to}$ “
\dot{1}”], two [$\dot{2} \stackrel{\text{pyk}}{\to}$ “peano two”][$\dot{2} \stackrel{\text{tex}}{\to}$ “
\dot{2}”], conjunction [$\mathsf{x} \mathrel{\dot{\wedge}} \mathsf{y} \stackrel{\text{pyk}}{\to}$ “* peano and *”][$\mathsf{x} \mathrel{\dot{\wedge}} \mathsf{y} \stackrel{\text{tex}}{\to}$ “#1.
\mathrel{\dot{\wedge}} #2.”], disjunction [$\mathsf{x} \mathrel{\dot{\vee}} \mathsf{y} \stackrel{\text{pyk}}{\to}$ “* peano or *”][$\mathsf{x} \mathrel{\dot{\vee}} \mathsf{y} \stackrel{\text{tex}}{\to}$ “#1.
\mathrel{\dot{\vee}} #2.”], biimplication [$\mathsf{x} \mathrel{\dot{\Leftrightarrow}} \mathsf{y} \stackrel{\text{pyk}}{\to}$ “* peano iff *”][$\mathsf{x} \mathrel{\dot{\Leftrightarrow}} \mathsf{y} \stackrel{\text{tex}}{\to}$ “#1.
\mathrel{\dot{\Leftrightarrow}} #2.”], and existential quantification
[$\dot{\exists}\mathsf{x}\colon \mathsf{y} \stackrel{\text{pyk}}{\to}$ “peano exist * indeed *”][$\dot{\exists}\mathsf{x}\colon \mathsf{y} \stackrel{\text{tex}}{\to}$ “
\dot{\exists} #1.
\colon #2.”]:

$$[\dot{1} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{1} \stackrel{..}{=} \dot{0}']\rceil)]$$

$$[\dot{2} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{2} \stackrel{..}{=} \dot{1}']\rceil)]$$

$$[\mathsf{x} \mathrel{\dot{\wedge}} \mathsf{y} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \mathrel{\dot{\wedge}} \mathsf{y} \stackrel{..}{=} \dot{\neg}\,(\mathsf{x} \mathrel{\dot{\Rightarrow}} \dot{\neg}\,\mathsf{y})]\rceil)]$$

$$[\mathsf{x} \mathrel{\dot{\vee}} \mathsf{y} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \mathrel{\dot{\vee}} \mathsf{y} \stackrel{..}{=} \dot{\neg}\,\mathsf{x} \mathrel{\dot{\Rightarrow}} \mathsf{y}]\rceil)]$$

$$[\mathsf{x} \mathrel{\dot{\Leftrightarrow}} \mathsf{y} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \mathrel{\dot{\Leftrightarrow}} \mathsf{y} \stackrel{..}{=} (\mathsf{x} \mathrel{\dot{\Rightarrow}} \mathsf{y}) \mathrel{\dot{\wedge}} (\mathsf{y} \mathrel{\dot{\Rightarrow}} \mathsf{x})]\rceil)]$$

$$[\dot{\exists}\mathsf{x}\colon \mathsf{y} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{\exists}\mathsf{x}\colon \mathsf{y} \stackrel{..}{=} \dot{\neg}\,\dot{\forall}\mathsf{x}\colon \dot{\neg}\,\mathsf{y}]\rceil)]$$

1.2 Variables

We now introduce the unary operator [$\dot{\mathsf{x}} \stackrel{\text{pyk}}{\to}$ “* peano var”][$\dot{\mathsf{x}} \stackrel{\text{tex}}{\to}$ “
\dot{#1.
}”] and define that a term is a *Peano variable* (i.e. a variable of Peano arithmetic) if it has the [$\dot{\mathsf{x}}$] operator in its root. [$\mathsf{x}^{\mathcal{P}} \stackrel{\text{pyk}}{\to}$ “* is peano var”][$\mathsf{x}^{\mathcal{P}} \stackrel{\text{tex}}{\to}$ “#1.
{} ^ {\cal P}”] is true if [x] is a Peano variable:

$$[\mathsf{x}^{\mathcal{P}} \stackrel{\text{val}}{\to} \mathsf{x} \stackrel{r}{=} \lceil \dot{\mathsf{x}} \rceil]$$

We macro define $[\dot{a} \stackrel{\text{pyk}}{\to}$ "peano a"$][\dot{a} \stackrel{\text{tex}}{\to}$ "
\dot{\mathit{a}}"] to be a Peano variable:

$$[\dot{a} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil [\dot{a} \doteq \dot{\mathsf{a}}] \rceil)]$$

Appendix A.2 defines Peano variables for the other letters of the English alphabet.
$[\dot{\text{nonfree}}(\mathsf{x}, \mathsf{y}) \stackrel{\text{pyk}}{\to}$ "peano nonfree ∗ in ∗ end nonfree"$][\dot{\text{nonfree}}(\mathsf{x}, \mathsf{y}) \stackrel{\text{tex}}{\to}$ "
\dot{nonfree}(#1.
, #2.
)"] is true if the Peano variable [x] does not occur free in the Peano term/formula [y]. $[\dot{\text{nonfree}}^*(\mathsf{x}, \mathsf{y}) \stackrel{\text{pyk}}{\to}$ "peano nonfree star ∗ in ∗ end nonfree"$][\dot{\text{nonfree}}^*(\mathsf{x}, \mathsf{y}) \stackrel{\text{tex}}{\to}$ "
\dot{nonfree}ˆ∗(#1.
, #2.
)"] is true if the Peano variable [x] does not occur free in the list [y] of Peano terms/formulas.

$$\begin{array}{l}[\dot{\text{nonfree}}(\mathsf{x}, \mathsf{y}) \stackrel{\text{val}}{\to} \\ \text{If}(\mathsf{y}^{\mathcal{P}}, \neg \mathsf{x} \stackrel{\text{t}}{=} \mathsf{y}, \\ \text{If}(\neg \mathsf{y} \stackrel{\text{r}}{=} \lceil \dot{\forall} \mathsf{x} \colon \mathsf{y} \rceil, \dot{\text{nonfree}}^*(\mathsf{x}, \mathsf{y}^{\text{t}}), \\ \text{If}(\mathsf{x} \stackrel{\text{t}}{=} \mathsf{y}^1, \mathsf{T}, \dot{\text{nonfree}}(\mathsf{x}, \mathsf{y}^2))))]\end{array}$$

$$[\dot{\text{nonfree}}^*(\mathsf{x}, \mathsf{y}) \stackrel{\text{val}}{\to} \mathsf{x}!\text{If}(\mathsf{y}, \mathsf{T}, \text{If}(\dot{\text{nonfree}}(\mathsf{x}, \mathsf{y}^{\text{h}}), \dot{\text{nonfree}}^*(\mathsf{x}, \mathsf{y}^{\text{t}}), \mathsf{F}))]$$

$[\dot{\text{free}}\langle \mathsf{a} | \mathsf{x} := \mathsf{b} \rangle \stackrel{\text{pyk}}{\to}$ "peano free ∗ set ∗ to ∗ end free"$][\dot{\text{free}}\langle \mathsf{a} | \mathsf{x} := \mathsf{b} \rangle \stackrel{\text{tex}}{\to}$ "
\dot{free}\langle #1.
| #2.
:= #3.
\rangle"] is true if the substitution $[\langle \mathsf{a} | \mathsf{x} {:=} \mathsf{b} \rangle]$ is free.
$[\dot{\text{free}}^*\langle \mathsf{a} | \mathsf{x} := \mathsf{b} \rangle \stackrel{\text{pyk}}{\to}$ "peano free star ∗ set ∗ to ∗ end free"$][\dot{\text{free}}^*\langle \mathsf{a} | \mathsf{x} := \mathsf{b} \rangle \stackrel{\text{tex}}{\to}$ "
\dot{free}{}ˆ∗\langle #1.
| #2.
:= #3.
\rangle"] is the version where [a] is a list of terms.

$$\begin{array}{l}[\dot{\text{free}}\langle \mathsf{a} | \mathsf{x} := \mathsf{b} \rangle \stackrel{\text{val}}{\to} \mathsf{x}!\mathsf{b}! \\ \text{If}(\mathsf{a}^{\mathcal{P}}, \mathsf{T}, \\ \text{If}(\neg \mathsf{a} \stackrel{\text{r}}{=} \lceil \dot{\forall} \mathsf{u} \colon \mathsf{v} \rceil, \dot{\text{free}}^*\langle \mathsf{a}^{\text{t}} | \mathsf{x} := \mathsf{b} \rangle, \\ \text{If}(\mathsf{a}^1 \stackrel{\text{t}}{=} \mathsf{x}, \mathsf{T}, \\ \text{If}(\dot{\text{nonfree}}(\mathsf{x}, \mathsf{a}^2), \mathsf{T}, \\ \text{If}(\neg \dot{\text{nonfree}}(\mathsf{a}^1, \mathsf{b}), \mathsf{F}, \\ \dot{\text{free}}\langle \mathsf{a}^2 | \mathsf{x} := \mathsf{b} \rangle))))))]\end{array}$$

$[\dot{\text{free}}^*\langle a|x := b\rangle \stackrel{\text{val}}{\to} x!b!\text{If}(a, T, \text{If}(\dot{\text{free}}\langle a^h|x := b\rangle, \dot{\text{free}}^*\langle a^t|x := b\rangle, F))]$

$[a{\equiv}\langle b|x := c\rangle \stackrel{\text{pyk}}{\to}$ "peano sub * is * where * is * end sub"$][a{\equiv}\langle b|x := c\rangle \stackrel{\text{tex}}{\to}$ "#1.
{\equiv}\langle #2.
|#3.
:=#4.
\rangle"] is true if [a] equals [⟨b |x:= c⟩]. $[a{\equiv}\langle^* b|x := c\rangle \stackrel{\text{pyk}}{\to}$ "peano sub star * is * where * is * end sub"$][a{\equiv}\langle^* b|x := c\rangle \stackrel{\text{tex}}{\to}$ "#1.
{\equiv}\langleˆ* #2.
|#3.
:=#4.
\rangle"] is the version where [a] and [b] are lists.

$$[a{\equiv}\langle b|x := c\rangle \stackrel{\text{val}}{\to} a!x!c!$$
$$\text{If}(\text{If}(b \stackrel{r}{=} \lceil \dot{\forall} u\colon v\rceil, b^1 \stackrel{t}{=} x, F), a \stackrel{t}{=} b,$$
$$\text{If}(b^{\mathcal{P}} \wedge b \stackrel{t}{=} x, a \stackrel{t}{=} c, \text{If}($$
$$a \stackrel{r}{=} b, a^t{\equiv}\langle^* b^t|x := c\rangle, F)))]$$

$$[a{\equiv}\langle^* b|x := c\rangle \stackrel{\text{val}}{\to} b!x!c!\text{If}(a, T, \text{If}(a^h{\equiv}\langle b^h|x := c\rangle, a^t{\equiv}\langle^* b^t|x := c\rangle, F))]$$

1.3 Mendelsons system S

System [S $\stackrel{\text{pyk}}{\to}$ "system s"][S $\stackrel{\text{tex}}{\to}$ "
S"] of Mendelson [2] expresses Peano arithmetic. It comprises the axioms
[A1 $\stackrel{\text{pyk}}{\to}$ "axiom a one"][A1 $\stackrel{\text{tex}}{\to}$ "
A1"], [A2 $\stackrel{\text{pyk}}{\to}$ "axiom a two"][A2 $\stackrel{\text{tex}}{\to}$ "
A2"], [A3 $\stackrel{\text{pyk}}{\to}$ "axiom a three"][A3 $\stackrel{\text{tex}}{\to}$ "
A3"], [A4 $\stackrel{\text{pyk}}{\to}$ "axiom a four"][A4 $\stackrel{\text{tex}}{\to}$ "
A4"], and [A5 $\stackrel{\text{pyk}}{\to}$ "axiom a five"][A5 $\stackrel{\text{tex}}{\to}$ "
A5"] and inference rules [MP $\stackrel{\text{pyk}}{\to}$ "rule mp"][MP $\stackrel{\text{tex}}{\to}$ "
MP"] and [Gen $\stackrel{\text{pyk}}{\to}$ "rule gen"][Gen $\stackrel{\text{tex}}{\to}$ "
Gen"] of first order predicate calculus. Furthermore, it comprises the proper axioms [S1 $\stackrel{\text{pyk}}{\to}$ "axiom s one"][S1 $\stackrel{\text{tex}}{\to}$ "
S1"], [S2 $\stackrel{\text{pyk}}{\to}$ "axiom s two"][S2 $\stackrel{\text{tex}}{\to}$ "
S2"], [S3 $\stackrel{\text{pyk}}{\to}$ "axiom s three"][S3 $\stackrel{\text{tex}}{\to}$ "
S3"], [S4 $\stackrel{\text{pyk}}{\to}$ "axiom s four"][S4 $\stackrel{\text{tex}}{\to}$ "
S4"], [S5 $\stackrel{\text{pyk}}{\to}$ "axiom s five"][S5 $\stackrel{\text{tex}}{\to}$ "
S5"], [S6 $\stackrel{\text{pyk}}{\to}$ "axiom s six"][S6 $\stackrel{\text{tex}}{\to}$ "
S6"], [S7 $\stackrel{\text{pyk}}{\to}$ "axiom s seven"][S7 $\stackrel{\text{tex}}{\to}$ "
S7"], [S8 $\stackrel{\text{pyk}}{\to}$ "axiom s eight"][S8 $\stackrel{\text{tex}}{\to}$ "

S8”], and [S9 $\stackrel{\text{pyk}}{\rightarrow}$ “axiom s nine”][S9 $\stackrel{\text{tex}}{\rightarrow}$ “ S9”]. System [S] is defined thus:

[S $\stackrel{\text{stmt}}{\rightarrow}$ $\dot{a} \dot{+} \dot{b}' \stackrel{p}{=} \dot{a} \dot{+} \dot{b}' \oplus \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg} \underline{b} \dot{\Rightarrow} \dot{\neg} \underline{a} \dot{\Rightarrow} \dot{\neg} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{b} \oplus \dot{a} \stackrel{p}{=} \dot{b} \Rightarrow \dot{a}' \stackrel{p}{=} \dot{b}' \oplus \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \dot{\Rightarrow} \underline{b} \vdash \underline{a} \vdash \underline{b} \oplus \dot{a}' \stackrel{p}{=} \dot{b}' \Rightarrow \dot{a} \stackrel{p}{=} \dot{b} \oplus \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \oplus \forall \underline{x}\colon \forall \underline{a}\colon \forall \underline{b}\colon \dot{\text{nonfree}}(\underline{x}, \underline{a}) \Vdash \dot{\forall} \underline{x}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \dot{\forall} \underline{x}\colon \underline{b} \oplus \dot{a} \dot{\cdot} \dot{b}' \stackrel{p}{=} \dot{a} \dot{\cdot} \dot{b} \dot{+} \dot{a} \oplus \dot{a} \dot{+} \dot{0} \stackrel{p}{=} \dot{a} \oplus \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{c} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{c} \oplus \dot{a} \stackrel{p}{=} \dot{b} \Rightarrow \dot{a} \stackrel{p}{=} \dot{c} \dot{\Rightarrow} \dot{b} \stackrel{p}{=} \dot{c} \oplus \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{x}\colon \underline{b} \equiv \langle \underline{a} | \underline{x} := \dot{0} \rangle \Vdash \underline{c} \equiv \langle \underline{a} | \underline{x} := \underline{x}' \rangle \Vdash \underline{b} \dot{\Rightarrow} \dot{\forall} \underline{x}\colon \underline{a} \dot{\Rightarrow} \underline{c} \dot{\Rightarrow} \dot{\forall} \underline{x}\colon \underline{a} \oplus \neg \dot{0} \stackrel{p}{=} \dot{a}' \oplus \forall \underline{x}\colon \forall \underline{a}\colon \underline{a} \vdash \dot{\forall} \underline{x}\colon \underline{a} \oplus \forall \underline{c}\colon \forall \underline{a}\colon \forall \underline{x}\colon \forall \underline{b}\colon \lceil \underline{a} \rceil \equiv \langle \lceil \underline{b} \rceil | \lceil \underline{x} \rceil := \lceil \underline{c} \rceil \rangle \Vdash \dot{\forall} \underline{x}\colon \underline{b} \dot{\Rightarrow} \underline{a} \oplus \dot{a} \dot{\cdot} \dot{0} \stackrel{p}{=} \dot{0}$]

[A1 $\stackrel{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a}$][A1 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[A2 $\stackrel{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{c} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{c}$][A2 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[A3 $\stackrel{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg} \underline{b} \dot{\Rightarrow} \dot{\neg} \underline{a} \dot{\Rightarrow} \dot{\neg} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{b}$][A3 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

The order of quantifiers in the following axiom is such that [$\underline{c}$] which the current conclusion tactic cannot guess comes first. This allows to supply a value for [$\underline{c}$] without having to supply values for the other meta-variables.

[A4 $\stackrel{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{c}\colon \forall \underline{a}\colon \forall \underline{x}\colon \forall \underline{b}\colon \lceil \underline{a} \rceil \equiv \langle \lceil \underline{b} \rceil | \lceil \underline{x} \rceil := \lceil \underline{c} \rceil \rangle \Vdash \dot{\forall} \underline{x}\colon \underline{b} \dot{\Rightarrow} \underline{a}$][A4 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[A5 $\stackrel{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{x}\colon \forall \underline{a}\colon \forall \underline{b}\colon \dot{\text{nonfree}}(\underline{x}, \underline{a}) \Vdash \dot{\forall} \underline{x}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \dot{\forall} \underline{x}\colon \underline{b}$][A5 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[MP $\stackrel{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \dot{\Rightarrow} \underline{b} \vdash \underline{a} \vdash \underline{b}$][MP $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[Gen $\stackrel{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{x}\colon \forall \underline{a}\colon \underline{a} \vdash \dot{\forall} \underline{x}\colon \underline{a}$][Gen $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

Axiom [S1] to [S8] are stated as Mendelson [2] does. This is done here to test certain parts of the Logiweb system. Serious users of Peano arithmetic are advised to take Mendelson’s Lemma 3.1 as axioms instead.

[S1 $\stackrel{\text{stmt}}{\rightarrow}$ $S \vdash \dot{a} \stackrel{p}{=} \dot{b} \Rightarrow \dot{a} \stackrel{p}{=} \dot{c} \dot{\Rightarrow} \dot{b} \stackrel{p}{=} \dot{c}$][S1 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[S2 $\stackrel{\text{stmt}}{\rightarrow}$ $S \vdash \dot{a} \stackrel{p}{=} \dot{b} \Rightarrow \dot{a}' \stackrel{p}{=} \dot{b}'$][S2 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[S3 $\stackrel{\text{stmt}}{\rightarrow}$ $S \vdash \neg \dot{0} \stackrel{p}{=} \dot{a}'$][S3 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[S4 $\stackrel{\text{stmt}}{\rightarrow}$ $S \vdash \dot{a}' \stackrel{p}{=} \dot{b}' \dot{\Rightarrow} \dot{a} \stackrel{p}{=} \dot{b}$][S4 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[S5 $\stackrel{\text{stmt}}{\rightarrow}$ $S \vdash \dot{a} \dot{+} \dot{0} \stackrel{p}{=} \dot{a}$][S5 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[S6 $\overset{\text{stmt}}{\rightarrow}$ S $\vdash \dot{a} \mathbin{\dot{+}} \dot{b}' \overset{p}{=} \dot{a} \mathbin{\dot{+}} \dot{b}'$][S6 $\overset{\text{proof}}{\rightarrow}$ Rule tactic]

[S7 $\overset{\text{stmt}}{\rightarrow}$ S $\vdash \dot{a} \mathbin{\dot{\cdot}} \dot{0} \overset{p}{=} \dot{0}$][S7 $\overset{\text{proof}}{\rightarrow}$ Rule tactic]

[S8 $\overset{\text{stmt}}{\rightarrow}$ S $\vdash \dot{a} \mathbin{\dot{\cdot}} \dot{b}' \overset{p}{=} \dot{a} \mathbin{\dot{\cdot}} \dot{b} \mathbin{\dot{+}} \dot{a}$][S8 $\overset{\text{proof}}{\rightarrow}$ Rule tactic]

[S9 $\overset{\text{stmt}}{\rightarrow}$ S $\vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{x}\colon \underline{b} \equiv \langle \underline{a} | \underline{x} := \dot{0} \rangle \Vdash \underline{c} \equiv \langle \underline{a} | \underline{x} := \underline{x}' \rangle \Vdash \underline{b} \mathrel{\dot{\Rightarrow}} \dot{\forall} \underline{x}\colon \underline{a} \mathrel{\dot{\Rightarrow}} \underline{c} \mathrel{\dot{\Rightarrow}} \dot{\forall} \underline{x}\colon \underline{a}$][S9 $\overset{\text{proof}}{\rightarrow}$ Rule tactic]

1.4 A lemma and a proof

We now prove Lemma [L3.2(a) $\overset{\text{pyk}}{\rightarrow}$ "lemma l three two a"][L3.2(a) $\overset{\text{tex}}{\rightarrow}$ " L3.2(a)"] which is an instance of the corresponding proposition in Mendelson [2]:

[L3.2(a) $\overset{\text{stmt}}{\rightarrow}$ S $\vdash \dot{x} \overset{p}{=} \dot{x}$]

[L3.2(a) $\overset{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil S \vdash S5 \gg \dot{a} \mathbin{\dot{+}} \dot{0} \overset{p}{=} \dot{a}; \text{Gen} \rhd \dot{a} \mathbin{\dot{+}} \dot{0} \overset{p}{=} \dot{a} \gg \dot{\forall}\dot{a}\colon \dot{a} \mathbin{\dot{+}} \dot{0} \overset{p}{=} \dot{a}; \text{A4} @ \dot{x} \gg \dot{\forall}\dot{a}\colon \dot{a} \mathbin{\dot{+}} \dot{0} \overset{p}{=} \dot{a} \mathrel{\dot{\Rightarrow}} \dot{x} \mathbin{\dot{+}} \dot{0} \overset{p}{=} \dot{x}; \text{MP} \rhd \dot{\forall}\dot{a}\colon \dot{a} \mathbin{\dot{+}} \dot{0} \overset{p}{=} \dot{a} \mathrel{\dot{\Rightarrow}} \dot{x} \mathbin{\dot{+}} \dot{0} \overset{p}{=} \dot{x} \rhd \dot{\forall}\dot{a}\colon \dot{a} \mathbin{\dot{+}} \dot{0} \overset{p}{=} \dot{a} \gg \dot{x} \mathbin{\dot{+}} \dot{0} \overset{p}{=} \dot{x}; S1 \gg \dot{a} \overset{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \overset{p}{=} \dot{c} \mathrel{\dot{\Rightarrow}} \dot{b} \overset{p}{=} \dot{c}; \text{Gen} \rhd \dot{a} \overset{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \overset{p}{=} \dot{c} \mathrel{\dot{\Rightarrow}} \dot{b} \overset{p}{=} \dot{c} \gg \dot{\forall}\dot{c}\colon \dot{a} \overset{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \overset{p}{=} \dot{c} \mathrel{\dot{\Rightarrow}} \dot{b} \overset{p}{=} \dot{c}; \text{A4} @ \dot{x} \gg \dot{\forall}\dot{c}\colon \dot{a} \overset{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \overset{p}{=} \dot{c} \mathrel{\dot{\Rightarrow}} \dot{b} \overset{p}{=} \dot{c} \mathrel{\dot{\Rightarrow}} \dot{a} \overset{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{b} \overset{p}{=} \dot{x}; \text{MP} \rhd \dot{\forall}\dot{c}\colon \dot{a} \overset{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \overset{p}{=} \dot{c} \mathrel{\dot{\Rightarrow}} \dot{b} \overset{p}{=} \dot{c} \mathrel{\dot{\Rightarrow}} \dot{a} \overset{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{b} \overset{p}{=} \dot{x} \rhd \dot{\forall}\dot{c}\colon \dot{a} \overset{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \overset{p}{=} \dot{c} \mathrel{\dot{\Rightarrow}} \dot{b} \overset{p}{=} \dot{c} \gg \dot{a} \overset{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{b} \overset{p}{=} \dot{x}; \text{Gen} \rhd \dot{a} \overset{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{b} \overset{p}{=} \dot{x} \gg \dot{\forall}\dot{b}\colon \dot{a} \overset{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{b} \overset{p}{=} \dot{x}; \text{A4} @ \dot{x} \gg \dot{\forall}\dot{b}\colon \dot{a} \overset{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{b} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{a} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{a} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \overset{p}{=} \dot{x}; \text{MP} \rhd \dot{\forall}\dot{b}\colon \dot{a} \overset{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{b} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{a} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{a} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \overset{p}{=} \dot{x} \rhd \dot{\forall}\dot{b}\colon \dot{a} \overset{p}{=} \dot{b} \mathrel{\dot{\Rightarrow}} \dot{a} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{b} \overset{p}{=} \dot{x} \gg \dot{a} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{a} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \overset{p}{=} \dot{x}; \text{Gen} \rhd \dot{a} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{a} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \overset{p}{=} \dot{x} \gg \dot{\forall}\dot{a}\colon \dot{a} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{a} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \overset{p}{=} \dot{x}; \text{A4} @ \dot{x} \mathbin{\dot{+}} \dot{0} \gg \dot{\forall}\dot{a}\colon \dot{a} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{a} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \mathbin{\dot{+}} \dot{0} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \mathbin{\dot{+}} \dot{0} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \overset{p}{=} \dot{x}; \text{MP} \rhd \dot{\forall}\dot{a}\colon \dot{a} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{a} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \mathbin{\dot{+}} \dot{0} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \mathbin{\dot{+}} \dot{0} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \overset{p}{=} \dot{x} \rhd \dot{\forall}\dot{a}\colon \dot{a} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{a} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \overset{p}{=} \dot{x} \gg \dot{x} \mathbin{\dot{+}} \dot{0} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \mathbin{\dot{+}} \dot{0} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \overset{p}{=} \dot{x}; \text{MP} \rhd \dot{x} \mathbin{\dot{+}} \dot{0} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \mathbin{\dot{+}} \dot{0} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \overset{p}{=} \dot{x} \rhd \dot{x} \mathbin{\dot{+}} \dot{0} \overset{p}{=} \dot{x} \gg \dot{x} \mathbin{\dot{+}} \dot{0} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \overset{p}{=} \dot{x}; \text{MP} \rhd \dot{x} \mathbin{\dot{+}} \dot{0} \overset{p}{=} \dot{x} \mathrel{\dot{\Rightarrow}} \dot{x} \overset{p}{=} \dot{x} \rhd \dot{x} \mathbin{\dot{+}} \dot{0} \overset{p}{=} \dot{x} \gg \dot{x} \overset{p}{=} \dot{x} \rceil, p_0, c)$]

1.5 An alternative axiomatic system

System [S′ $\overset{\text{pyk}}{\rightarrow}$ "system prime s"][S′ $\overset{\text{tex}}{\rightarrow}$ " S′"] is system [S] in which the proper axioms are taken from Lemma 3.1 in Mendelson [2]. It comprises the axioms [A1′ $\overset{\text{pyk}}{\rightarrow}$ "axiom prime a one"][A1′ $\overset{\text{tex}}{\rightarrow}$ " A1′"], [A2′ $\overset{\text{pyk}}{\rightarrow}$ "axiom prime a two"][A2′ $\overset{\text{tex}}{\rightarrow}$ " A2′"], [A3′ $\overset{\text{pyk}}{\rightarrow}$ "axiom prime a three"][A3′ $\overset{\text{tex}}{\rightarrow}$ " A3′"], [A4′ $\overset{\text{pyk}}{\rightarrow}$ "axiom prime a four"][A4′ $\overset{\text{tex}}{\rightarrow}$ " A4′"], and [A5′ $\overset{\text{pyk}}{\rightarrow}$ "axiom prime a five"][A5′ $\overset{\text{tex}}{\rightarrow}$ " A5′"] and inference rules [MP′ $\overset{\text{pyk}}{\rightarrow}$ "rule prime mp"][MP′ $\overset{\text{tex}}{\rightarrow}$ " MP′"] and [Gen′ $\overset{\text{pyk}}{\rightarrow}$ "rule prime gen"][Gen′ $\overset{\text{tex}}{\rightarrow}$ "

Gen”’] of first order predicate calculus. Furthermore, it comprises the proper axioms [S1′ $\stackrel{\text{pyk}}{\rightarrow}$ “axiom prime s one”][S1′ $\stackrel{\text{tex}}{\rightarrow}$ “ S1”’], [S2′ $\stackrel{\text{pyk}}{\rightarrow}$ “axiom prime s two”][S2′ $\stackrel{\text{tex}}{\rightarrow}$ “ S2”’], [S3′ $\stackrel{\text{pyk}}{\rightarrow}$ “axiom prime s three”][S3′ $\stackrel{\text{tex}}{\rightarrow}$ “ S3”’], [S4′ $\stackrel{\text{pyk}}{\rightarrow}$ “axiom prime s four”][S4′ $\stackrel{\text{tex}}{\rightarrow}$ “ S4”’], [S5′ $\stackrel{\text{pyk}}{\rightarrow}$ “axiom prime s five”][S5′ $\stackrel{\text{tex}}{\rightarrow}$ “ S5”’], [S6′ $\stackrel{\text{pyk}}{\rightarrow}$ “axiom prime s six”][S6′ $\stackrel{\text{tex}}{\rightarrow}$ “ S6”’], [S7′ $\stackrel{\text{pyk}}{\rightarrow}$ “axiom prime s seven”][S7′ $\stackrel{\text{tex}}{\rightarrow}$ “ S7”’], [S8′ $\stackrel{\text{pyk}}{\rightarrow}$ “axiom prime s eight”][S8′ $\stackrel{\text{tex}}{\rightarrow}$ “ S8”’], and [S9′ $\stackrel{\text{pyk}}{\rightarrow}$ “axiom prime s nine”][S9′ $\stackrel{\text{tex}}{\rightarrow}$ “ S9”’].

System [S′] is defined thus:

$$[\mathrm{S}' \stackrel{\text{stmt}}{\rightarrow} \forall \underline{a}\colon \forall \underline{b}\colon \underline{a}' \stackrel{p}{=} \underline{b}' \dot{\Rightarrow} \underline{a} \stackrel{p}{=} \underline{b} \oplus \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \oplus \forall \underline{x}\colon \forall \underline{a}\colon \forall \underline{b}\colon \mathrm{nonfree}(\underline{x}, \underline{a}) \Vdash \dot{\forall} \underline{x}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \dot{\forall} \underline{x}\colon \underline{b} \oplus \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \dot{\cdot} \underline{b}' \stackrel{p}{=} \underline{a} \dot{\cdot} \underline{b} \dot{+} \underline{a} \oplus \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \stackrel{p}{=} \underline{b} \dot{\Rightarrow} \underline{a}' \stackrel{p}{=} \underline{b}' \oplus \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \dot{\Rightarrow} \underline{b} \vdash \underline{a} \vdash \underline{b} \oplus \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg} \underline{b} \dot{\Rightarrow} \dot{\neg} \underline{a} \dot{\Rightarrow} \dot{\neg} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{b} \oplus \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \dot{+} \underline{b}' \stackrel{p}{=} \underline{a} \dot{+} \underline{b}' \oplus \forall \underline{a}\colon \neg \dot{0} \stackrel{p}{=} \underline{a}' \oplus \forall \underline{x}\colon \forall \underline{a}\colon \underline{a} \vdash \dot{\forall} \underline{x}\colon \underline{a} \oplus \forall \underline{c}\colon \forall \underline{a}\colon \forall \underline{x}\colon \forall \underline{b}\colon \lceil \underline{a} \rceil \equiv \langle \lceil \underline{b} \rceil | \lceil \underline{x} \rceil := \lceil \underline{c} \rceil \rangle \Vdash \dot{\forall} \underline{x}\colon \underline{b} \dot{\Rightarrow} \underline{a} \oplus \forall \underline{a}\colon \underline{a} \dot{\cdot} \dot{0} \stackrel{p}{=} \dot{0} \oplus \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} \stackrel{p}{=} \underline{b} \dot{\Rightarrow} \underline{a} \stackrel{p}{=} \underline{c} \dot{\Rightarrow} \underline{b} \stackrel{p}{=} \underline{c} \oplus \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{x}\colon \underline{b} \equiv \langle \underline{a} | \underline{x} := \dot{0} \rangle \Vdash \underline{c} \equiv \langle \underline{a} | \underline{x} := \underline{x}' \rangle \Vdash \underline{b} \dot{\Rightarrow} \dot{\forall} \underline{x}\colon \underline{a} \dot{\Rightarrow} \underline{c} \dot{\Rightarrow} \dot{\forall} \underline{x}\colon \underline{a} \oplus \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{c} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{c} \oplus \forall \underline{a}\colon \underline{a} \dot{+} \dot{0} \stackrel{p}{=} \underline{a}]$$

$$[\mathrm{A1}' \stackrel{\text{stmt}}{\rightarrow} \mathrm{S}' \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a}][\mathrm{A1}' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$$

$$[\mathrm{A2}' \stackrel{\text{stmt}}{\rightarrow} \mathrm{S}' \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{c} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{c}][\mathrm{A2}' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$$

$$[\mathrm{A3}' \stackrel{\text{stmt}}{\rightarrow} \mathrm{S}' \vdash \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg} \underline{b} \dot{\Rightarrow} \dot{\neg} \underline{a} \dot{\Rightarrow} \dot{\neg} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \underline{b}][\mathrm{A3}' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$$

$$[\mathrm{A4}' \stackrel{\text{stmt}}{\rightarrow} \mathrm{S}' \vdash \forall \underline{c}\colon \forall \underline{a}\colon \forall \underline{x}\colon \forall \underline{b}\colon \lceil \underline{a} \rceil \equiv \langle \lceil \underline{b} \rceil | \lceil \underline{x} \rceil := \lceil \underline{c} \rceil \rangle \Vdash \dot{\forall} \underline{x}\colon \underline{b} \dot{\Rightarrow} \underline{a}][\mathrm{A4}' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$$

$$[\mathrm{A5}' \stackrel{\text{stmt}}{\rightarrow} \mathrm{S}' \vdash \forall \underline{x}\colon \forall \underline{a}\colon \forall \underline{b}\colon \mathrm{nonfree}(\underline{x}, \underline{a}) \Vdash \dot{\forall} \underline{x}\colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a} \dot{\Rightarrow} \dot{\forall} \underline{x}\colon \underline{b}][\mathrm{A5}' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$$

$$[\mathrm{MP}' \stackrel{\text{stmt}}{\rightarrow} \mathrm{S}' \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \dot{\Rightarrow} \underline{b} \vdash \underline{a} \vdash \underline{b}][\mathrm{MP}' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$$

$$[\mathrm{Gen}' \stackrel{\text{stmt}}{\rightarrow} \mathrm{S}' \vdash \forall \underline{x}\colon \forall \underline{a}\colon \underline{a} \vdash \dot{\forall} \underline{x}\colon \underline{a}][\mathrm{Gen}' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$$

$$[\mathrm{S1}' \stackrel{\text{stmt}}{\rightarrow} \mathrm{S}' \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} \stackrel{p}{=} \underline{b} \dot{\Rightarrow} \underline{a} \stackrel{p}{=} \underline{c} \dot{\Rightarrow} \underline{b} \stackrel{p}{=} \underline{c}][\mathrm{S1}' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$$

$$[\mathrm{S2}' \stackrel{\text{stmt}}{\rightarrow} \mathrm{S}' \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \stackrel{p}{=} \underline{b} \dot{\Rightarrow} \underline{a}' \stackrel{p}{=} \underline{b}'][\mathrm{S2}' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$$

$[S3' \stackrel{\text{stmt}}{\rightarrow} S' \vdash \forall \underline{a} \colon \neg \dot{0} \stackrel{p}{=} \underline{a}'][S3' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[S4' \stackrel{\text{stmt}}{\rightarrow} S' \vdash \forall \underline{a} \colon \forall \underline{b} \colon \underline{a}' \stackrel{p}{=} \underline{b}' \dot{\Rightarrow} \underline{a} \stackrel{p}{=} \underline{b}][S4' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[S5' \stackrel{\text{stmt}}{\rightarrow} S' \vdash \forall \underline{a} \colon \underline{a} \dot{+} \dot{0} \stackrel{p}{=} \underline{a}][S5' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[S6' \stackrel{\text{stmt}}{\rightarrow} S' \vdash \forall \underline{a} \colon \forall \underline{b} \colon \underline{a} \dot{+} \underline{b}' \stackrel{p}{=} \underline{a} \dot{+} \underline{b}'][S6' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[S7' \stackrel{\text{stmt}}{\rightarrow} S' \vdash \forall \underline{a} \colon \underline{a} \dot{\cdot} \dot{0} \stackrel{p}{=} \dot{0}][S7' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[S8' \stackrel{\text{stmt}}{\rightarrow} S' \vdash \forall \underline{a} \colon \forall \underline{b} \colon \underline{a} \dot{\cdot} \underline{b}' \stackrel{p}{=} \underline{a} \dot{\cdot} \underline{b} \dot{+} \underline{a}][S8' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[S9' \stackrel{\text{stmt}}{\rightarrow} S' \vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \forall \underline{x} \colon \underline{b} \equiv \langle \underline{a} | \underline{x} := \dot{0} \rangle \Vdash \underline{c} \equiv \langle \underline{a} | \underline{x} := \underline{x}' \rangle \Vdash \underline{b} \dot{\Rightarrow} \dot{\forall} \underline{x} \colon \underline{a} \dot{\Rightarrow} \underline{c} \dot{\Rightarrow} \dot{\forall} \underline{x} \colon \underline{a}][S9' \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

Note that [A1] and [A1′] are distinct. The former says $[S \vdash \forall \underline{a} \colon \forall \underline{b} \colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a}]$ and the latter says $[S' \vdash \forall \underline{a} \colon \forall \underline{b} \colon \underline{a} \dot{\Rightarrow} \underline{b} \dot{\Rightarrow} \underline{a}]$.

1.6 Restatement of lemma and a proof

We now prove Lemma [L3.2(a)] once again under the name of $[L3.2(a)' \stackrel{\text{pyk}}{\rightarrow}$ "lemma prime l three two a"$][L3.2(a)' \stackrel{\text{tex}}{\rightarrow}$ " L3.2(a)'"]:

$[L3.2(a)' \stackrel{\text{stmt}}{\rightarrow} S' \vdash \forall \underline{a} \colon \underline{a} \stackrel{p}{=} \underline{a}]$

$[L3.2(a)' \stackrel{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil S' \vdash \forall \underline{a} \colon S5' \gg \underline{a} \dot{+} \dot{0} \stackrel{p}{=} \underline{a}; S1' \gg \underline{a} \dot{+} \dot{0} \stackrel{p}{=} \underline{a} \dot{\Rightarrow} \underline{a} \dot{+} \dot{0} \stackrel{p}{=} \underline{a} \dot{\Rightarrow} \underline{a} \stackrel{p}{=} \underline{a}; MP' \rhd \underline{a} \dot{+} \dot{0} \stackrel{p}{=} \underline{a} \dot{\Rightarrow} \underline{a} \dot{+} \dot{0} \stackrel{p}{=} \underline{a} \dot{\Rightarrow} \underline{a} \stackrel{p}{=} \underline{a} \rhd \underline{a} \dot{+} \dot{0} \stackrel{p}{=} \underline{a} \gg \underline{a} \dot{+} \dot{0} \stackrel{p}{=} \underline{a} \dot{\Rightarrow} \underline{a} \stackrel{p}{=} \underline{a}; MP' \rhd \underline{a} \dot{+} \dot{0} \stackrel{p}{=} \underline{a} \dot{\Rightarrow} \underline{a} \stackrel{p}{=} \underline{a} \rhd \underline{a} \dot{+} \dot{0} \stackrel{p}{=} \underline{a} \gg \underline{a} \stackrel{p}{=} \underline{a}\rceil, p_0, c)]$

A Chores

A.1 The name of the page

This defines the name of the page:

$[\text{peano} \stackrel{\text{pyk}}{\rightarrow}$ "peano"]

A.2 Variables of Peano arithmetic

We use $[\dot{b} \stackrel{\text{pyk}}{\rightarrow}$ "peano b"$][\dot{b} \stackrel{\text{tex}}{\rightarrow}$ " \dot{\mathit{b}}"], $[\dot{c} \stackrel{\text{pyk}}{\rightarrow}$ "peano c"$][\dot{c} \stackrel{\text{tex}}{\rightarrow}$ " \dot{\mathit{c}}"], $[\dot{d} \stackrel{\text{pyk}}{\rightarrow}$ "peano d"$][\dot{d} \stackrel{\text{tex}}{\rightarrow}$ " \dot{\mathit{d}}"], $[\dot{e} \stackrel{\text{pyk}}{\rightarrow}$ "peano e"$][\dot{e} \stackrel{\text{tex}}{\rightarrow}$ "

\dot{\mathit{e}}”], [$\dot{f} \stackrel{\text{pyk}}{\rightarrow}$ “peano f”][$\dot{f} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{f}}”], [$\dot{g} \stackrel{\text{pyk}}{\rightarrow}$ “peano g”][$\dot{g} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{g}}”], [$\dot{h} \stackrel{\text{pyk}}{\rightarrow}$ “peano h”][$\dot{h} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{h}}”], [$\dot{i} \stackrel{\text{pyk}}{\rightarrow}$ “peano i”][$\dot{i} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{i}}”], [$\dot{j} \stackrel{\text{pyk}}{\rightarrow}$ “peano j”][$\dot{j} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{j}}”], [$\dot{k} \stackrel{\text{pyk}}{\rightarrow}$ “peano k”][$\dot{k} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{k}}”], [$\dot{l} \stackrel{\text{pyk}}{\rightarrow}$ “peano l”][$\dot{l} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{l}}”], [$\dot{m} \stackrel{\text{pyk}}{\rightarrow}$ “peano m”][$\dot{m} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{m}}”], [$\dot{n} \stackrel{\text{pyk}}{\rightarrow}$ “peano n”][$\dot{n} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{n}}”], [$\dot{o} \stackrel{\text{pyk}}{\rightarrow}$ “peano o”][$\dot{o} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{o}}”], [$\dot{p} \stackrel{\text{pyk}}{\rightarrow}$ “peano p”][$\dot{p} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{p}}”], [$\dot{q} \stackrel{\text{pyk}}{\rightarrow}$ “peano q”][$\dot{q} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{q}}”], [$\dot{r} \stackrel{\text{pyk}}{\rightarrow}$ “peano r”][$\dot{r} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{r}}”], [$\dot{s} \stackrel{\text{pyk}}{\rightarrow}$ “peano s”][$\dot{s} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{s}}”], [$\dot{t} \stackrel{\text{pyk}}{\rightarrow}$ “peano t”][$\dot{t} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{t}}”], [$\dot{u} \stackrel{\text{pyk}}{\rightarrow}$ “peano u”][$\dot{u} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{u}}”], [$\dot{v} \stackrel{\text{pyk}}{\rightarrow}$ “peano v”][$\dot{v} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{v}}”], [$\dot{w} \stackrel{\text{pyk}}{\rightarrow}$ “peano w”][$\dot{w} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{w}}”], [$\dot{x} \stackrel{\text{pyk}}{\rightarrow}$ “peano x”][$\dot{x} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{x}}”], [$\dot{y} \stackrel{\text{pyk}}{\rightarrow}$ “peano y”][$\dot{y} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{y}}”], and [$\dot{z} \stackrel{\text{pyk}}{\rightarrow}$ “peano z”][$\dot{z} \stackrel{\text{tex}}{\rightarrow}$ “
\dot{\mathit{z}}”] to denote variables of Peano arithmetic:
$[\dot{b} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{b} \ddot{=} \dot{\mathsf{b}}]\rceil)]$, $[\dot{c} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{c} \ddot{=} \dot{\mathsf{c}}]\rceil)]$,
$[\dot{d} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{d} \ddot{=} \dot{\mathsf{d}}]\rceil)]$, $[\dot{e} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{e} \ddot{=} \dot{\mathsf{e}}]\rceil)]$,
$[\dot{f} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{f} \ddot{=} \dot{\mathsf{f}}]\rceil)]$, $[\dot{g} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{g} \ddot{=} \dot{\mathsf{g}}]\rceil)]$,
$[\dot{h} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{h} \ddot{=} \dot{\mathsf{h}}]\rceil)]$, $[\dot{i} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{i} \ddot{=} \dot{\mathsf{i}}]\rceil)]$,
$[\dot{j} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{j} \ddot{=} \dot{\mathsf{j}}]\rceil)]$, $[\dot{k} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{k} \ddot{=} \dot{\mathsf{k}}]\rceil)]$,
$[\dot{l} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{l} \ddot{=} \dot{\mathsf{l}}]\rceil)]$, $[\dot{m} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{m} \ddot{=} \dot{\mathsf{m}}]\rceil)]$,
$[\dot{n} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{n} \ddot{=} \dot{\mathsf{n}}]\rceil)]$, $[\dot{o} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{o} \ddot{=} \dot{\mathsf{o}}]\rceil)]$,
$[\dot{p} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{p} \ddot{=} \dot{\mathsf{p}}]\rceil)]$, $[\dot{q} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{q} \ddot{=} \dot{\mathsf{q}}]\rceil)]$,
$[\dot{r} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{r} \ddot{=} \dot{\mathsf{r}}]\rceil)]$, $[\dot{s} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{s} \ddot{=} \dot{\mathsf{s}}]\rceil)]$,
$[\dot{t} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{t} \ddot{=} \dot{\mathsf{t}}]\rceil)]$, $[\dot{u} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{u} \ddot{=} \dot{\mathsf{u}}]\rceil)]$,
$[\dot{v} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{v} \ddot{=} \dot{\mathsf{v}}]\rceil)]$, $[\dot{w} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{w} \ddot{=} \dot{\mathsf{w}}]\rceil)]$,
$[\dot{x} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{x} \ddot{=} \dot{\mathsf{x}}]\rceil)]$, $[\dot{y} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{y} \ddot{=} \dot{\mathsf{y}}]\rceil)]$,
and $[\dot{z} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\dot{z} \ddot{=} \dot{\mathsf{z}}]\rceil)]$.

A.3 TeX definitions

A.4 Test

$[\lceil \dot{\mathsf{a}} \rceil^{\mathcal{P}}]^{\cdot}$

$[\lceil \mathsf{a} \rceil^{\mathcal{P}}]^{-}$

$[\dot{\mathrm{nonfree}}(\lceil \dot{\mathsf{x}} \rceil, \lceil \dot{\mathsf{y}} \stackrel{\mathrm{p}}{=} \dot{\mathsf{z}} \dot{\Rightarrow} \dot{\forall} \dot{\mathsf{x}} \colon \dot{\mathsf{x}} \stackrel{\mathrm{p}}{=} \dot{\mathsf{y}} \rceil)]^{\cdot}$

$[\dot{\mathrm{nonfree}}(\lceil \dot{\mathsf{x}} \rceil, \lceil \dot{\mathsf{x}} \stackrel{\mathrm{p}}{=} \dot{\mathsf{z}} \dot{\Rightarrow} \dot{\forall} \dot{\mathsf{x}} \colon \dot{\mathsf{x}} \stackrel{\mathrm{p}}{=} \dot{\mathsf{y}} \rceil)]^{-}$

$[\dot{\mathrm{nonfree}}(\lceil \dot{\mathsf{x}} \rceil, \lceil \dot{\mathsf{y}} \stackrel{\mathrm{p}}{=} \dot{\mathsf{x}} \dot{\Rightarrow} \dot{\forall} \dot{\mathsf{x}} \colon \dot{\mathsf{x}} \stackrel{\mathrm{p}}{=} \dot{\mathsf{y}} \rceil)]^{-}$

$[\dot{\mathrm{nonfree}}(\lceil \dot{\mathsf{x}} \rceil, \lceil \dot{\mathsf{y}} \stackrel{\mathrm{p}}{=} \dot{\mathsf{z}} \dot{\Rightarrow} \dot{\forall} \dot{\mathsf{y}} \colon \dot{\mathsf{x}} \stackrel{\mathrm{p}}{=} \dot{\mathsf{y}} \rceil)]^{-}$

$[\dot{\mathrm{free}}\langle \lceil \dot{\forall} \dot{\mathsf{x}} \colon \mathsf{b} :: \dot{\mathsf{x}} :: \mathsf{c} \rceil | \lceil \dot{\mathsf{x}} \rceil := \lceil \mathsf{x} :: \dot{\mathsf{y}} :: \mathsf{z} \rceil \rangle]^{\cdot}$

$[\dot{\mathrm{free}}\langle \lceil \dot{\forall} \dot{\mathsf{y}} \colon \mathsf{b} :: \dot{\mathsf{x}} :: \mathsf{c} \rceil | \lceil \dot{\mathsf{x}} \rceil := \lceil \mathsf{x} :: \dot{\mathsf{y}} :: \mathsf{z} \rceil \rangle]^{-}$

$[\dot{\mathrm{free}}\langle \lceil \dot{\forall} \dot{\mathsf{x}} \colon \mathsf{b} :: \dot{\mathsf{x}} :: \mathsf{c} \rceil | \lceil \dot{\mathsf{y}} \rceil := \lceil \mathsf{x} :: \dot{\mathsf{y}} :: \mathsf{z} \rceil \rangle]^{\cdot}$

$[\dot{\mathrm{free}}\langle \lceil \dot{\forall} \dot{\mathsf{y}} \colon \mathsf{b} :: \dot{\mathsf{x}} :: \mathsf{c} \rceil | \lceil \dot{\mathsf{y}} \rceil := \lceil \mathsf{x} :: \dot{\mathsf{y}} :: \mathsf{z} \rceil \rangle]^{\cdot}$

$[\dot{\mathsf{a}} \equiv \langle \dot{\mathsf{a}} | \dot{\mathsf{b}} := \dot{\mathsf{c}} \rangle]^{\cdot}$

$[\dot{\mathsf{c}} \equiv \langle \dot{\mathsf{b}} | \dot{\mathsf{b}} := \dot{\mathsf{c}} \rangle]^{\cdot}$

$[\forall \dot{\mathsf{a}} \colon \dot{\mathsf{a}} \stackrel{\mathrm{p}}{=} \dot{\mathsf{b}} \equiv \langle \forall \dot{\mathsf{a}} \colon \dot{\mathsf{a}} \stackrel{\mathrm{p}}{=} \dot{\mathsf{b}} | \dot{\mathsf{a}} := \dot{\mathsf{c}} \rangle]^{\cdot}$

$[\forall \dot{\mathsf{a}} \colon \dot{\mathsf{a}} \stackrel{\mathrm{p}}{=} \dot{\mathsf{c}} \equiv \langle \forall \dot{\mathsf{a}} \colon \dot{\mathsf{a}} \stackrel{\mathrm{p}}{=} \dot{\mathsf{b}} | \dot{\mathsf{b}} := \dot{\mathsf{c}} \rangle]^{\cdot}$

$[\dot{\forall} \dot{\mathsf{a}} \colon \dot{\mathsf{a}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{a}} \dot{\Rightarrow} \dot{\mathsf{c}} \dot{:} \dot{\mathsf{d}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{c}} \dot{:} \dot{\mathsf{d}} \equiv \langle \dot{\forall} \dot{\mathsf{a}} \colon \dot{\mathsf{a}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{a}} \dot{\Rightarrow} \dot{\mathsf{b}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{b}} | \dot{\mathsf{b}} := \dot{\mathsf{c}} \dot{:} \dot{\mathsf{d}} \rangle]^{\cdot}$

$[\dot{\forall} \dot{\mathsf{a}} \colon \dot{\mathsf{a}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{a}} \dot{\Rightarrow} \dot{\mathsf{b}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{b}} \equiv \langle \dot{\forall} \dot{\mathsf{a}} \colon \dot{\mathsf{a}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{a}} \dot{\Rightarrow} \dot{\mathsf{b}} \stackrel{\mathrm{p}}{=} \dot{0} \dot{+} \dot{\mathsf{b}} | \dot{\mathsf{a}} := \dot{\mathsf{c}} \rangle]^{\cdot}$

A.5 Priority table

[peano $\stackrel{\mathrm{prio}}{\rightarrow}$
Preassociative
[peano], [base], [bracket $*$ end bracket], [big bracket $*$ end bracket], [math $*$ end math], [**flush left** [$*$]], [x], [y], [z], [[$* \bowtie *$]], [[$* \stackrel{*}{\rightarrow} *$]], [pyk], [tex], [name], [prio], [$*$], [T], [if($*, *, *$)], [[$* \stackrel{*}{\Rightarrow} *$]], [val], [claim], [$\bot$], [f($*$)], [$(*)^{\mathrm{I}}$], [$\mathsf{F}$], [$\underline{0}$], [$\underline{1}$], [$\underline{2}$], [$\underline{3}$], [$\underline{4}$], [$\underline{5}$], [$\underline{6}$], [$\underline{7}$], [$\underline{8}$], [$\underline{9}$], [0], [1], [2], [3], [4], [5], [6], [7], [8], [9], [a], [b], [c], [d], [e], [f], [g], [h], [i], [j], [k], [l], [m], [n], [o], [p], [q], [r], [s], [t], [u], [v], [w], [$(*)^{\mathrm{M}}$], [If($*, *, *$)], [array{$*$} $*$ end array], [l], [c], [r], [empty], [$\langle * \mid * := * \rangle$], [$\mathcal{M}(*)$], [$\tilde{\mathcal{U}}(*)$], [$\mathcal{U}(*)$],

[$\mathcal{U}^{\mathrm{M}}(*)$], [**apply**$(*,*)$], [**apply**$_1(*,*)$], [identifier$(*)$], [identifier$_1(*,*)$], [array-plus$(*,*)$], [array-remove$(*,*,*)$], [array-put$(*,*,*,*)$], [array-add$(*,*,*,*,*)$], [bit$(*,*)$], [bit$_1(*,*)$], [rack], ["vector"], ["bibliography"], ["dictionary"], ["body"], ["codex"], ["expansion"], ["code"], ["cache"], ["diagnose"], ["pyk"], ["tex"], ["texname"], ["value"], ["message"], ["macro"], ["definition"], ["unpack"], ["claim"], ["priority"], ["lambda"], ["apply"], ["true"], ["if"], ["quote"], ["proclaim"], ["define"], ["introduce"], ["hide"], ["pre"], ["post"], [$\mathcal{E}(*,*,*)$], [$\mathcal{E}_2(*,*,*,*,*)$], [$\mathcal{E}_3(*,*,*,*)$], [$\mathcal{E}_4(*,*,*,*)$], [**lookup**$(*,*,*)$], [**abstract**$(*,*,*,*)$], [$\lceil * \rceil$], [$\mathcal{M}(*,*,*)$], [$\mathcal{M}_2(*,*,*,*)$], [$\mathcal{M}^*(*,*,*)$], [macro], [s_0], [**zip**$(*,*)$], [**assoc**$_1(*,*,*)$], [$(*)^{\mathbf{P}}$], [self], [[$* \ddot{=} *$]], [[$* \dot{=} *$]], [[$* \acute{=} *$]], [[$* \stackrel{\text{pyk}}{=} *$]], [[$* \stackrel{\text{tex}}{=} *$]], [[$* \stackrel{\text{name}}{=} *$]], [**Priority table**[$*$]], [$\tilde{\mathcal{M}}_1$], [$\tilde{\mathcal{M}}_2(*)$], [$\tilde{\mathcal{M}}_3(*)$], [$\tilde{\mathcal{M}}_4(*,*,*,*)$], [$\tilde{\mathcal{M}}(*,*,*)$], [$\tilde{\mathcal{Q}}(*,*,*)$], [$\tilde{\mathcal{Q}}_2(*,*,*)$], [$\tilde{\mathcal{Q}}_3(*,*,*,*)$], [$\tilde{\mathcal{Q}}^*(*,*,*)$], [$(*)$], [**aspect**$(*,*)$], [**aspect**$(*,*,*)$], [$\langle * \rangle$], [**tuple**$_1(*)$], [**tuple**$_2(*)$], [$\mathrm{let}_2(*,*)$], [$\mathrm{let}_1(*,*)$], [[$* \stackrel{\text{claim}}{=} *$]], [checker], [**check**$(*,*)$], [**check**$_2(*,*,*)$], [**check**$_3(*,*,*)$], [**check**$^*(*,*)$], [**check**$^*_2(*,*,*)$], [[$*$]$^{\cdot}$], [[$*$]$^{-}$], [[$*$]$^{\circ}$], [msg], [[$* \stackrel{\text{msg}}{=} *$]], [<stmt>], [stmt], [[$* \stackrel{\text{stmt}}{=} *$]], [HeadNil′], [HeadPair′], [Transitivity′], [$\perp\!\!\!\perp$], [Contra′], [T'_{E}], [L_1], [$\underline{*}$], [$\mathcal{A}$], [$\mathcal{B}$], [$\mathcal{C}$], [$\mathcal{D}$], [$\mathcal{E}$], [$\mathcal{F}$], [$\mathcal{G}$], [$\mathcal{H}$], [$\mathcal{I}$], [$\mathcal{J}$], [$\mathcal{K}$], [$\mathcal{L}$], [$\mathcal{M}$], [$\mathcal{N}$], [$\mathcal{O}$], [$\mathcal{P}$], [$\mathcal{Q}$], [$\mathcal{R}$], [$\mathcal{S}$], [$\mathcal{T}$], [$\mathcal{U}$], [$\mathcal{V}$], [$\mathcal{W}$], [$\mathcal{X}$], [$\mathcal{Y}$], [$\mathcal{Z}$], [$\langle * \mid * := * \rangle$], [$\langle^* * \mid * := * \rangle$], [$\emptyset$], [Remainder], [$(*)^{\mathbf{v}}$], [$\mathrm{intro}(*,*,*,*)$], [$\mathrm{intro}(*,*,*)$], [$\mathrm{error}(*,*)$], [$\mathrm{error}_2(*,*)$], [$\mathrm{proof}(*,*,*)$], [$\mathrm{proof}_2(*,*)$], [$\mathcal{S}(*,*)$], [$\mathcal{S}^{\mathrm{I}}(*,*)$], [$\mathcal{S}^{\rhd}(*,*)$], [$\mathcal{S}^{\rhd}_1(*,*,*)$], [$\mathcal{S}^{\mathrm{E}}(*,*)$], [$\mathcal{S}^{\mathrm{E}}_1(*,*,*)$], [$\mathcal{S}^{+}(*,*)$], [$\mathcal{S}^{+}_1(*,*,*)$], [$\mathcal{S}^{-}(*,*)$], [$\mathcal{S}^{-}_1(*,*,*)$], [$\mathcal{S}^{*}(*,*)$], [$\mathcal{S}^{*}_1(*,*,*)$], [$\mathcal{S}^{*}_2(*,*,*,*)$], [$\mathcal{S}^{@}(*,*)$], [$\mathcal{S}^{@}_1(*,*,*)$], [$\mathcal{S}^{\vdash}(*,*)$], [$\mathcal{S}^{\vdash}_1(*,*,*,*)$], [$\mathcal{S}^{\Vdash}(*,*)$], [$\mathcal{S}^{\Vdash}_1(*,*,*,*)$], [$\mathcal{S}^{\text{i.e.}}(*,*)$], [$\mathcal{S}^{\text{i.e.}}_1(*,*,*,*)$], [$\mathcal{S}^{\text{i.e.}}_2(*,*,*,*,*)$], [$\mathcal{S}^{\forall}(*,*)$], [$\mathcal{S}^{\forall}_1(*,*,*,*)$], [$\mathcal{S}^{;}(*,*)$], [$\mathcal{S}^{;}_1(*,*,*)$], [$\mathcal{S}^{;}_2(*,*,*,*)$], [$\mathcal{T}(*)$], [$\mathrm{claims}(*,*,*)$], [$\mathrm{claims}_2(*,*,*)$], [<proof>], [proof], [[**Lemma** $*$: $*$]], [[**Proof of** $*$: $*$]], [[$*$ **lemma** $*$: $*$]], [[$*$ **antilemma** $*$: $*$]], [[$*$ **rule** $*$: $*$]], [[$*$ **antirule** $*$: $*$]], [verifier], [$\mathcal{V}_1(*)$], [$\mathcal{V}_2(*,*)$], [$\mathcal{V}_3(*,*,*,*)$], [$\mathcal{V}_4(*,*)$], [$\mathcal{V}_5(*,*,*,*)$], [$\mathcal{V}_6(*,*,*,*)$], [$\mathcal{V}_7(*,*,*,*)$], [$\mathrm{Cut}(*,*)$], [$\mathrm{Head}_{\oplus}(*)$], [$\mathrm{Tail}_{\oplus}(*)$], [$\mathrm{rule}_1(*,*)$], [$\mathrm{rule}(*,*)$], [Rule tactic], [$\mathrm{Plus}(*,*)$], [[**Theory** $*$]], [$\mathrm{theory}_2(*,*)$], [$\mathrm{theory}_3(*,*)$], [$\mathrm{theory}_4(*,*,*)$], [HeadNil″], [HeadPair″], [Transitivity″], [Contra″], [HeadNil], [HeadPair], [Transitivity], [Contra], [T_{E}], [ragged right], [ragged right expansion], [$\mathrm{parm}(*,*,*)$], [$\mathrm{parm}^*(*,*,*)$], [$\mathrm{inst}(*,*)$], [$\mathrm{inst}^*(*,*)$], [$\mathrm{occur}(*,*,*)$], [$\mathrm{occur}^*(*,*,*)$], [$\mathrm{unify}(* = *,*)$], [$\mathrm{unify}^*(* = *,*)$], [$\mathrm{unify}_2(* = *,*)$], [$\mathrm{L_a}$], [$\mathrm{L_b}$], [$\mathrm{L_c}$], [$\mathrm{L_d}$], [$\mathrm{L_e}$], [$\mathrm{L_f}$], [$\mathrm{L_g}$], [$\mathrm{L_h}$], [$\mathrm{L_i}$], [$\mathrm{L_j}$], [$\mathrm{L_k}$], [$\mathrm{L_l}$], [$\mathrm{L_m}$], [$\mathrm{L_n}$], [$\mathrm{L_o}$], [$\mathrm{L_p}$], [$\mathrm{L_q}$], [$\mathrm{L_r}$], [$\mathrm{L_s}$], [$\mathrm{L_t}$], [$\mathrm{L_u}$], [$\mathrm{L_v}$], [$\mathrm{L_w}$], [$\mathrm{L_x}$], [$\mathrm{L_y}$], [$\mathrm{L_z}$], [$\mathrm{L_A}$], [$\mathrm{L_B}$], [$\mathrm{L_C}$], [$\mathrm{L_D}$], [$\mathrm{L_E}$], [$\mathrm{L_F}$], [$\mathrm{L_G}$], [$\mathrm{L_H}$], [$\mathrm{L_I}$], [$\mathrm{L_J}$], [$\mathrm{L_K}$], [$\mathrm{L_L}$], [$\mathrm{L_M}$], [$\mathrm{L_N}$], [$\mathrm{L_O}$], [$\mathrm{L_P}$], [$\mathrm{L_Q}$], [$\mathrm{L_R}$], [$\mathrm{L_S}$], [$\mathrm{L_T}$], [$\mathrm{L_U}$], [$\mathrm{L_V}$], [$\mathrm{L_W}$], [$\mathrm{L_X}$], [$\mathrm{L_Y}$], [$\mathrm{L_Z}$], [$\mathrm{L_?}$], [Reflexivity], [$\mathrm{Reflexivity}_1$], [Commutativity], [$\mathrm{Commutativity}_1$], [<tactic>], [tactic], [[$* \stackrel{\text{tactic}}{=} *$]], [$\mathcal{P}(*,*,*)$], [$\mathcal{P}^*(*,*,*)$], [$\mathrm{p}_0$], [$\mathrm{conclude}_1(*,*)$], [$\mathrm{conclude}_2(*,*,*)$], [$\mathrm{conclude}_3(*,*,*,*)$], [$\dot{0}$], [$\dot{1}$], [$\dot{2}$], [$\dot{a}$], [$\dot{b}$], [$\dot{c}$], [$\dot{d}$], [$\dot{e}$], [$\dot{f}$], [$\dot{g}$], [$\dot{h}$], [$\dot{i}$], [$\dot{j}$], [$\dot{k}$], [$\dot{l}$], [$\dot{m}$], [$\dot{n}$], [$\dot{o}$], [$\dot{p}$], [$\dot{q}$], [$\dot{r}$], [$\dot{s}$], [$\dot{t}$], [$\dot{u}$], [$\dot{v}$], [$\dot{w}$], [$\dot{x}$], [$\dot{y}$], [$\dot{z}$], [$\mathrm{nonfree}(*,*)$], [$\mathrm{nonfree}^*(*,*)$], [$\mathrm{free}\langle * | * := * \rangle$], [$\mathrm{free}^*\langle * | * := * \rangle$], [$* \equiv \langle * | * := * \rangle$], [$* \equiv \langle^* * | * := * \rangle$], [S], [A1], [A2], [A3], [A4], [A5], [S1], [S2], [S3], [S4], [S5], [S6], [S7], [S8], [S9], [MP], [Gen], [L3.2(a)], [S′], [A1′], [A2′], [A3′], [A4′], [A5′], [S1′], [S2′], [S3′], [S4′], [S5′], [S6′], [S7′], [S8′], [S9′], [MP′],

[Gen′], [L3.2(a)′];
Preassociative
[∗_{∗}], [∗′], [∗[∗]], [∗[∗→∗]], [∗[∗⇒∗]], [$\dot{∗}$];
Preassociative
[“ ∗ ”], [], [(∗)$^{\mathbf{t}}$], [string(∗) + ∗], [string(∗) ++ ∗], [
∗], [∗], [!∗], [”∗], [#∗], [$∗], [%∗], [&∗], [’∗], [(∗], [)∗], [∗∗], [+∗], [, ∗], [-∗], [.∗], [/∗],
[0∗], [1∗], [2∗], [3∗], [4∗], [5∗], [6∗], [7∗], [8∗], [9∗], [:∗], [; ∗], [<∗], [=∗], [>∗], [?∗],
[@∗], [A∗], [B∗], [C∗], [D∗], [E∗], [F∗], [G∗], [H∗], [I∗], [J∗], [K∗], [L∗], [M∗], [N∗],
[O∗], [P∗], [Q∗], [R∗], [S∗], [T∗], [U∗], [V∗], [W∗], [X∗], [Y∗], [Z∗], [[∗], [\∗], []∗], [ˆ∗],
[_∗], [‘∗], [a∗], [b∗], [c∗], [d∗], [e∗], [f∗], [g∗], [h∗], [i∗], [j∗], [k∗], [l∗], [m∗], [n∗], [o∗],
[p∗], [q∗], [r∗], [s∗], [t∗], [u∗], [v∗], [w∗], [x∗], [y∗], [z∗], [{∗], [|∗], [}∗], [˜∗],
[**Preassociative** ∗; ∗], [**Postassociative** ∗; ∗], [[∗], ∗], [priority ∗ end],
[newline ∗], [macro newline ∗];
Preassociative
[∗0], [∗1], [0b], [∗-color(∗)], [∗-color*(∗)];
Preassociative
[∗ ’ ∗], [∗ ‘ ∗];
Preassociative
[$*^{\mathrm{H}}$], [$*^{\mathrm{T}}$], [$*^{\mathrm{U}}$], [$*^{\mathrm{h}}$], [$*^{\mathrm{t}}$], [$*^{\mathrm{s}}$], [$*^{\mathrm{c}}$], [$*^{\mathrm{d}}$], [$*^{\mathrm{a}}$], [$*^{\mathrm{C}}$], [$*^{\mathrm{M}}$], [$*^{\mathrm{B}}$], [$*^{\mathrm{r}}$], [$*^{\mathrm{i}}$], [$*^{\mathrm{d}}$], [$*^{\mathrm{R}}$], [$*^{0}$],
[$*^{1}$], [$*^{2}$], [$*^{3}$], [$*^{4}$], [$*^{5}$], [$*^{6}$], [$*^{7}$], [$*^{8}$], [$*^{9}$], [$*^{\mathrm{E}}$], [$*^{\mathcal{V}}$], [$*^{\mathcal{C}}$], [$*^{\mathcal{C}^*}$], [∗′];
Preassociative
[$* \cdot *$], [$* \cdot_0 *$], [$* \dot{:} *$];
Preassociative
[$* + *$], [$* +_0 *$], [$* +_1 *$], [$* - *$], [$* -_0 *$], [$* -_1 *$], [$* \dot{+} *$];
Preassociative
[$* \cup \{*\}$], [$* \cup *$], [$* \setminus \{*\}$];
Postassociative
[$* \therefore *$], [$* \underline{\therefore} *$], [$* \underline{::} *$], [$* \underline{+2*} *$], [$* :: *$], [$* +2* *$];
Postassociative
[∗, ∗];
Preassociative
[$* \stackrel{\mathrm{B}}{\approx} *$], [$* \stackrel{\mathrm{D}}{\approx} *$], [$* \stackrel{\mathrm{C}}{\approx} *$], [$* \stackrel{\mathrm{P}}{\approx} *$], [$* \approx *$], [$* = *$], [$* \stackrel{+}{\rightarrow} *$], [$* \stackrel{\mathrm{t}}{=} *$], [$* \stackrel{\mathrm{t}^*}{=} *$], [$* \stackrel{\mathrm{r}}{=} *$],
[$* \in_{\mathrm{t}} *$], [$* \subseteq_{\mathrm{T}} *$], [$* \stackrel{\mathrm{T}}{=} *$], [$* \stackrel{\mathrm{s}}{=} *$], [∗ free in ∗], [∗ free in* ∗], [∗ free for ∗ in ∗],
[∗ free for* ∗ in ∗], [$* \in_{\mathrm{c}} *$], [$* < *$], [$* <' *$], [$* \leq' *$], [$* \stackrel{\mathrm{p}}{=} *$], [$*^{\mathcal{P}}$];
Preassociative
[$\neg *$], [$\dot{\neg} *$];
Preassociative
[$* \wedge *$], [$* \ddot{\wedge} *$], [$* \tilde{\wedge} *$], [$* \wedge_{\mathrm{c}} *$], [$* \dot{\wedge} *$];
Preassociative
[$* \vee *$], [$* \parallel *$], [$* \ddot{\vee} *$], [$* \dot{\vee} *$];
Preassociative
[$\dot{\forall} *\colon *$], [$\dot{\exists} *\colon *$];
Postassociative
[$* \ddot{\Rightarrow} *$], [$* \dot{\Rightarrow} *$], [$* \dot{\Leftrightarrow} *$];
Postassociative

$[* : *], [*!*]$;
Preassociative
$[* \left\{ \begin{array}{c} * \\ * \end{array} \right.]$;
Preassociative
$[\lambda * .*], [\Lambda *]$, [**if** $*$ **then** $*$ **else** $*$], [**let** $* = *$ **in** $*$], [**let** $* \ddot{=} *$ in $*$];
Preassociative
$[*^{\mathrm{I}}], [*^{\triangleright}], [*^{\mathrm{V}}], [*^{+}], [*^{-}], [*^{*}]$;
Preassociative
$[* @ *], [* \rhd *], [* \unrhd *], [* \gg *]$;
Postassociative
$[* \vdash *], [* \Vdash *]$, [$*$ i.e. $*$];
Preassociative
$[\forall * : *]$;
Postassociative
$[* \oplus *]$;
Postassociative
$[*; *]$;
Preassociative
[$*$ proves $*$];
Preassociative
[$*$ **proof of** $* : *$], [Line $* : * \gg *; *$], [Last line $* \gg * \square$],
[Line $*$: Premise $\gg *; *$], [Line $*$: Side-condition $\gg *; *$], [Arbitrary $\gg *; *$],
[Local $\gg * = *; *$];
Postassociative
[$*$ then $*$], $[*[*]*]$;
Preassociative
$[* \& *]$;
Preassociative
$[* \backslash\backslash *]$;]

B Index

C Bibliography

[1] K. Grue. Logiweb. In Fairouz Kamareddine, editor, *Mathematical Knowledge Management Symposium 2003*, volume 93 of *Electronic Notes in Theoretical Computer Science*, pages 70–101. Elsevier, 2004.

[2] E. Mendelson. *Introduction to Mathematical Logic*. Wadsworth and Brooks, 3. edition, 1987.