

Peano arithmetic

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GRD-2005-06-20.UTC:11:21:10.729935

Contents

1	Peano arithmetic	1
1.1	The constructs of Peano arithmetic	1
1.2	Variables	2
1.3	Mendelsons system S	4
1.4	A lemma and a proof	5
1.5	An alternative axiomatic system	6
1.6	Restatement of lemma and a proof	7
A	Chores	8
A.1	The name of the page	8
A.2	Variables of Peano arithmetic	8
A.3	T _E X definitions	9
A.4	Test	14
A.5	Priority table	15
B	Index	18
C	Bibliography	22

1 Peano arithmetic

This Logiweb page [1] defines Peano arithmetic.

1.1 The constructs of Peano arithmetic

Terms of Peano arithmetic are constructed from zero $[0]$ ¹, successor $[x']$ ², plus $[x \dot{+} y]$ ³, and times $[x \dot{:} y]$ ⁴.

¹ $[0] \stackrel{\text{pyk}}{=} \text{"peano zero"}$

² $[x'] \stackrel{\text{pyk}}{=} \text{"* peano succ"}$

³ $[x \dot{+} y] \stackrel{\text{pyk}}{=} \text{"* peano plus *"}$

⁴ $[x \dot{:} y] \stackrel{\text{pyk}}{=} \text{"* peano times *"}$

Formulas of Peano arithmeric are constructed from equality $[x \stackrel{p}{=} y]$ ⁵, negation $[\neg x]$ ⁶, implication $[x \Rightarrow y]$ ⁷, and universal quantification $[\forall x: y]$ ⁸.

From these constructs we macro define one $[1]$ ⁹, two $[2]$ ¹⁰, conjunction $[x \wedge y]$ ¹¹, disjunction $[x \vee y]$ ¹², biimplication $[x \Leftrightarrow y]$ ¹³, and existential quantification $[\exists x: y]$ ¹⁴:

$$[1 \doteq 0']$$

$$[2 \doteq 1']$$

$$[x \wedge y \doteq \neg (x \Rightarrow \neg y)]$$

$$[x \vee y \doteq \neg x \Rightarrow y]$$

$$[x \Leftrightarrow y \doteq (x \Rightarrow y) \wedge (y \Rightarrow x)]$$

$$[\exists x: y \doteq \neg \forall x: \neg y]$$

1.2 Variables

We now introduce the unary operator $[\dot{x}]$ ¹⁵ and define that a term is a *Peano variable* (i.e. a variable of Peano arithmetic) if it has the $[\dot{x}]$ operator in its root. $[x^{\mathcal{P}}]$ ¹⁶ is true if $[x]$ is a Peano variable:

$$[x^{\mathcal{P}} \doteq x \stackrel{r}{=} [\dot{x}]]$$

We macro define $[\dot{a}]$ ¹⁷ to be a Peano variable:

$$[\dot{a} \doteq \dot{a}]$$

Appendix A.2 defines Peano variables for the other letters of the English alphabet.

⁵ $[x \stackrel{p}{=} y \stackrel{\text{pyk}}{=} \text{"* peano is *"}]$

⁶ $[\neg x \stackrel{\text{pyk}}{=} \text{"peano not *"}]$

⁷ $[x \Rightarrow y \stackrel{\text{pyk}}{=} \text{"* peano imply *"}]$

⁸ $[\forall x: y \stackrel{\text{pyk}}{=} \text{"peano all * indeed *"}]$

⁹ $[1 \stackrel{\text{pyk}}{=} \text{"peano one"}]$

¹⁰ $[2 \stackrel{\text{pyk}}{=} \text{"peano two"}]$

¹¹ $[x \wedge y \stackrel{\text{pyk}}{=} \text{"* peano and *"}]$

¹² $[x \vee y \stackrel{\text{pyk}}{=} \text{"* peano or *"}]$

¹³ $[x \Leftrightarrow y \stackrel{\text{pyk}}{=} \text{"* peano iff *"}]$

¹⁴ $[\exists x: y \stackrel{\text{pyk}}{=} \text{"peano exist * indeed *"}]$

¹⁵ $[\dot{x} \stackrel{\text{pyk}}{=} \text{"* peano var"}]$

¹⁶ $[x^{\mathcal{P}} \stackrel{\text{pyk}}{=} \text{"* is peano var"}]$

¹⁷ $[\dot{a} \stackrel{\text{pyk}}{=} \text{"peano a"}]$

$[\text{nonfree}(x, y)]^{18}$ is true if the Peano variable $[x]$ does not occur free in the Peano term/formula $[y]$. $[\text{nonfree}^*(x, y)]^{19}$ is true if the Peano variable $[x]$ does not occur free in the list $[y]$ of Peano terms/formulas.

$[\text{nonfree}(x, y) \doteq$
if $y^{\mathcal{P}}$ **then** $\neg x \stackrel{t}{=} y$ **else**
if $\neg y \stackrel{r}{=} [\forall x: y]$ **then** $\text{nonfree}^*(x, y^t)$ **else**
if $x \stackrel{t}{=} y^1$ **then** $\top$ **else** $\text{nonfree}(x, y^2)]$

$[\text{nonfree}^*(x, y) \doteq x! \text{If}(y, \top, \text{nonfree}(x, y^h) \wedge \text{nonfree}^*(x, y^t))]$

$[\text{free}\langle a|x := b \rangle]^{20}$ is true if the substitution $[\langle a|x := b \rangle]$ is free. $[\text{free}^*\langle a|x := b \rangle]^{21}$ is the version where $[a]$ is a list of terms.

$[\text{free}\langle a|x := b \rangle \doteq x!b!$
if $a^{\mathcal{P}}$ **then** $\top$ **else**
if $\neg a \stackrel{r}{=} [\forall u: v]$ **then** $\text{free}^*\langle a^t|x := b \rangle$ **else**
if $a^1 \stackrel{t}{=} x$ **then** $\top$ **else**
if $\text{nonfree}(x, a^2)$ **then** $\top$ **else**
if $\neg \text{nonfree}(a^1, b)$ **then** F **else**
 $\text{free}\langle a^2|x := b \rangle]$

$[\text{free}^*\langle a|x := b \rangle \doteq x!b! \text{If}(a, \top, \text{free}\langle a^h|x := b \rangle \wedge \text{free}^*\langle a^t|x := b \rangle)]$

$[a \equiv \langle b|x := c \rangle]^{22}$ is true if $[a]$ equals $[\langle b|x := c \rangle]$. $[a \equiv \langle *b|x := c \rangle]^{23}$ is the version where $[a]$ and $[b]$ are lists.

$[a \equiv \langle b|x := c \rangle \doteq a!x!c!$
if $b \stackrel{r}{=} [\forall u: v] \wedge b^1 \stackrel{t}{=} x$ **then** $a \stackrel{t}{=} b$ **else**
if $b^{\mathcal{P}} \wedge b \stackrel{t}{=} x$ **then** $a \stackrel{t}{=} c$ **else**
 $a \stackrel{r}{=} b \wedge a^t \equiv \langle *b^t|x := c \rangle]$

$[a \equiv \langle *b|x := c \rangle \doteq b!x!c! \text{If}(a, \top, a^h \equiv \langle b^h|x := c \rangle \wedge a^t \equiv \langle *b^t|x := c \rangle)]$

¹⁸ $[\text{nonfree}(x, y)] \stackrel{\text{pyk}}{=} \text{"peano nonfree * in * end nonfree"}$

¹⁹ $[\text{nonfree}^*(x, y)] \stackrel{\text{pyk}}{=} \text{"peano nonfree star * in * end nonfree"}$

²⁰ $[\text{free}\langle a|x := b \rangle] \stackrel{\text{pyk}}{=} \text{"peano free * set * to * end free"}$

²¹ $[\text{free}^*\langle a|x := b \rangle] \stackrel{\text{pyk}}{=} \text{"peano free star * set * to * end free"}$

²² $[a \equiv \langle b|x := c \rangle] \stackrel{\text{pyk}}{=} \text{"peano sub * is * where * is * end sub"}$

²³ $[a \equiv \langle *b|x := c \rangle] \stackrel{\text{pyk}}{=} \text{"peano sub star * is * where * is * end sub"}$

1.3 Mendelsons system S

System [S]²⁴ of Mendelson [2] expresses Peano arithmetic. It comprises the axioms [A1]²⁵, [A2]²⁶, [A3]²⁷, [A4]²⁸, and [A5]²⁹ and inference rules [MP]³⁰ and [Gen]³¹ of first order predicate calculus. Furthermore, it comprises the proper axioms [S1]³², [S2]³³, [S3]³⁴, [S4]³⁵, [S5]³⁶, [S6]³⁷, [S7]³⁸, [S8]³⁹, and [S9]⁴⁰. System [S] is defined thus:

[**Theory S**]

[S **rule** A1: $\forall \mathcal{A}: \forall \mathcal{B}: \mathcal{A} \Rightarrow \mathcal{B} \Rightarrow \mathcal{A}$]

[S **rule** A2: $\forall \mathcal{A}: \forall \mathcal{B}: \forall \mathcal{C}: (\mathcal{A} \Rightarrow \mathcal{B} \Rightarrow \mathcal{C}) \Rightarrow (\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow \mathcal{A} \Rightarrow \mathcal{C}$]

[S **rule** A3: $\forall \mathcal{A}: \forall \mathcal{B}: (\neg \mathcal{B} \Rightarrow \neg \mathcal{A}) \Rightarrow (\neg \mathcal{B} \Rightarrow \mathcal{A}) \Rightarrow \mathcal{B}$]

The order of quantifiers in the following axiom is such that [C] which the current conclusion tactic cannot guess comes first. This allows to supply a value for [C] without having to supply values for the other meta-variables.

[S **rule** A4: $\forall \mathcal{C}: \forall \mathcal{A}: \forall \mathcal{X}: \forall \mathcal{B}: [\mathcal{A}] \equiv ([\mathcal{B}] || [\mathcal{X}]) := [\mathcal{C}] \Vdash \dot{\forall} \mathcal{X}: \mathcal{B} \Rightarrow \mathcal{A}$]

[S **rule** A5: $\forall \mathcal{X}: \forall \mathcal{A}: \forall \mathcal{B}: \text{nonfree}(\mathcal{X}, \mathcal{A}) \Vdash \dot{\forall} \mathcal{X}: (\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow \mathcal{A} \Rightarrow \dot{\forall} \mathcal{X}: \mathcal{B}$]

[S **rule** MP: $\forall \mathcal{A}: \forall \mathcal{B}: \mathcal{A} \Rightarrow \mathcal{B} \vdash \mathcal{A} \vdash \mathcal{B}$]

[S **rule** Gen: $\forall \mathcal{X}: \forall \mathcal{A}: \mathcal{A} \vdash \dot{\forall} \mathcal{X}: \mathcal{A}$]

²⁴[S $\stackrel{\text{pyk}}{=} \text{"system s"}]$

²⁵[A1 $\stackrel{\text{pyk}}{=} \text{"axiom a one"}]$

²⁶[A2 $\stackrel{\text{pyk}}{=} \text{"axiom a two"}]$

²⁷[A3 $\stackrel{\text{pyk}}{=} \text{"axiom a three"}]$

²⁸[A4 $\stackrel{\text{pyk}}{=} \text{"axiom a four"}]$

²⁹[A5 $\stackrel{\text{pyk}}{=} \text{"axiom a five"}]$

³⁰[MP $\stackrel{\text{pyk}}{=} \text{"rule mp"}]$

³¹[Gen $\stackrel{\text{pyk}}{=} \text{"rule gen"}]$

³²[S1 $\stackrel{\text{pyk}}{=} \text{"axiom s one"}]$

³³[S2 $\stackrel{\text{pyk}}{=} \text{"axiom s two"}]$

³⁴[S3 $\stackrel{\text{pyk}}{=} \text{"axiom s three"}]$

³⁵[S4 $\stackrel{\text{pyk}}{=} \text{"axiom s four"}]$

³⁶[S5 $\stackrel{\text{pyk}}{=} \text{"axiom s five"}]$

³⁷[S6 $\stackrel{\text{pyk}}{=} \text{"axiom s six"}]$

³⁸[S7 $\stackrel{\text{pyk}}{=} \text{"axiom s seven"}]$

³⁹[S8 $\stackrel{\text{pyk}}{=} \text{"axiom s eight"}]$

⁴⁰[S9 $\stackrel{\text{pyk}}{=} \text{"axiom s nine"}]$

Axiom [S1] to [S8] are stated as Mendelson [2] does. This is done here to test certain parts of the Logiweb system. Serious users of Peano arithmetic are advised to take Mendelson's Lemma 3.1 as axioms instead.

$$[\text{S rule S1: } a \stackrel{P}{=} b \Rightarrow a \stackrel{P}{=} c \Rightarrow b \stackrel{P}{=} c]$$

$$[\text{S rule S2: } a \stackrel{P}{=} b \Rightarrow a' \stackrel{P}{=} b']$$

$$[\text{S rule S3: } \neg 0 \stackrel{P}{=} a']$$

$$[\text{S rule S4: } a' \stackrel{P}{=} b' \Rightarrow a \stackrel{P}{=} b]$$

$$[\text{S rule S5: } a + 0 \stackrel{P}{=} a]$$

$$[\text{S rule S6: } a + b' \stackrel{P}{=} (a + b)']$$

$$[\text{S rule S7: } a : 0 \stackrel{P}{=} 0]$$

$$[\text{S rule S8: } a : (b') \stackrel{P}{=} (a : b) + a]$$

$$\begin{aligned} &[\text{S rule S9: } \forall \mathcal{A}: \forall \mathcal{B}: \forall \mathcal{C}: \forall \mathcal{X}: \\ &\mathcal{B} \equiv \langle \mathcal{A} | \mathcal{X} := 0 \rangle \Vdash \mathcal{C} \equiv \langle \mathcal{A} | \mathcal{X} := \mathcal{X}' \rangle \Vdash \\ &\mathcal{B} \Rightarrow \forall \mathcal{X}: (\mathcal{A} \Rightarrow \mathcal{C}) \Rightarrow \forall \mathcal{X}: \mathcal{A}] \end{aligned}$$

1.4 A lemma and a proof

We now prove Lemma [L3.2(a)]⁴¹ which is an instance of the corresponding proposition in Mendelson [2]:

$$[\text{S lemma L3.2(a): } x \stackrel{P}{=} x]$$

S proof of L3.2(a):

L01:	S5 $\gg$	$a + 0 \stackrel{P}{=} a$	;
L02:	Gen $\triangleright$ L01 $\gg$	$\forall a: a + 0 \stackrel{P}{=} a$	;
L03:	A4 @ $x \gg$	$\forall a: a + 0 \stackrel{P}{=} a \Rightarrow x + 0 \stackrel{P}{=} x$	;
L04:	MP $\triangleright$ L03 $\triangleright$ L02 $\gg$	$x + 0 \stackrel{P}{=} x$	;
L05:	S1 $\gg$	$a \stackrel{P}{=} b \Rightarrow a \stackrel{P}{=} c \Rightarrow b \stackrel{P}{=} c$	;
L06:	Gen $\triangleright$ L05 $\gg$	$\forall c: (a \stackrel{P}{=} b \Rightarrow a \stackrel{P}{=} c \Rightarrow b \stackrel{P}{=} c)$	;
L07:	A4 @ $x \gg$	$\forall c: (a \stackrel{P}{=} b \Rightarrow a \stackrel{P}{=} c \Rightarrow b \stackrel{P}{=} c) \Rightarrow a \stackrel{P}{=} b \Rightarrow a \stackrel{P}{=} x \Rightarrow b \stackrel{P}{=} x$	;
L08:	MP $\triangleright$ L07 $\triangleright$ L06 $\gg$	$a \stackrel{P}{=} b \Rightarrow a \stackrel{P}{=} x \Rightarrow b \stackrel{P}{=} x$	;
L09:	Gen $\triangleright$ L08 $\gg$	$\forall b: (a \stackrel{P}{=} b \Rightarrow a \stackrel{P}{=} x \Rightarrow b \stackrel{P}{=} x)$	;
L10:	A4 @ $x \gg$	$\forall b: (a \stackrel{P}{=} b \Rightarrow a \stackrel{P}{=} x \Rightarrow b \stackrel{P}{=} x) \Rightarrow a \stackrel{P}{=} x \Rightarrow a \stackrel{P}{=} x \Rightarrow x \stackrel{P}{=} x$	;

⁴¹[L3.2(a)] $\stackrel{\text{Pyk}}{=} \text{"lemma 1 three two a"}$

L11:	MP $\triangleright$ L10 $\triangleright$ L09 $\gg$	$\dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x}$	;
L12:	Gen $\triangleright$ L11 $\gg$	$\forall \dot{a}: (\dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x})$	;
L13:	A4 @ $\dot{x} \dot{+} \dot{0} \gg$	$\forall \dot{a}: (\dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{a} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x}) \Rightarrow \dot{x} \dot{+} \dot{0} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \dot{+} \dot{0} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x}$	;
L14:	MP $\triangleright$ L13 $\triangleright$ L12 $\gg$	$\dot{x} \dot{+} \dot{0} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \dot{+} \dot{0} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x}$	;
L15:	MP $\triangleright$ L14 $\triangleright$ L04 $\gg$	$\dot{x} \dot{+} \dot{0} \stackrel{P}{=} \dot{x} \Rightarrow \dot{x} \stackrel{P}{=} \dot{x}$	;
L16:	MP $\triangleright$ L15 $\triangleright$ L04 $\gg$	$\dot{x} \stackrel{P}{=} \dot{x}$	□

1.5 An alternative axiomatic system

System [S']⁴² is system [S] in which the proper axioms are taken from Lemma 3.1 in Mendelson [2]. It comprises the axioms [A1']⁴³, [A2']⁴⁴, [A3']⁴⁵, [A4']⁴⁶, and [A5']⁴⁷ and inference rules [MP']⁴⁸ and [Gen']⁴⁹ of first order predicate calculus. Furthermore, it comprises the proper axioms [S1']⁵⁰, [S2']⁵¹, [S3']⁵², [S4']⁵³, [S5']⁵⁴, [S6']⁵⁵, [S7']⁵⁶, [S8']⁵⁷, and [S9']⁵⁸.

System [S'] is defined thus:

[Theory S']

[S' rule A1': $\forall \mathcal{A}: \forall \mathcal{B}: \mathcal{A} \Rightarrow \mathcal{B} \Rightarrow \mathcal{A}$]

[S' rule A2': $\forall \mathcal{A}: \forall \mathcal{B}: \forall \mathcal{C}: (\mathcal{A} \Rightarrow \mathcal{B} \Rightarrow \mathcal{C}) \Rightarrow (\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow \mathcal{A} \Rightarrow \mathcal{C}$]

[S' rule A3': $\forall \mathcal{A}: \forall \mathcal{B}: (\neg \mathcal{B} \Rightarrow \neg \mathcal{A}) \Rightarrow (\neg \mathcal{B} \Rightarrow \mathcal{A}) \Rightarrow \mathcal{B}$]

⁴²[S' $\stackrel{\text{pyk}}{=}$ "system prime s"]

⁴³[A1' $\stackrel{\text{pyk}}{=}$ "axiom prime a one"]

⁴⁴[A2' $\stackrel{\text{pyk}}{=}$ "axiom prime a two"]

⁴⁵[A3' $\stackrel{\text{pyk}}{=}$ "axiom prime a three"]

⁴⁶[A4' $\stackrel{\text{pyk}}{=}$ "axiom prime a four"]

⁴⁷[A5' $\stackrel{\text{pyk}}{=}$ "axiom prime a five"]

⁴⁸[MP' $\stackrel{\text{pyk}}{=}$ "rule prime mp"]

⁴⁹[Gen' $\stackrel{\text{pyk}}{=}$ "rule prime gen"]

⁵⁰[S1' $\stackrel{\text{pyk}}{=}$ "axiom prime s one"]

⁵¹[S2' $\stackrel{\text{pyk}}{=}$ "axiom prime s two"]

⁵²[S3' $\stackrel{\text{pyk}}{=}$ "axiom prime s three"]

⁵³[S4' $\stackrel{\text{pyk}}{=}$ "axiom prime s four"]

⁵⁴[S5' $\stackrel{\text{pyk}}{=}$ "axiom prime s five"]

⁵⁵[S6' $\stackrel{\text{pyk}}{=}$ "axiom prime s six"]

⁵⁶[S7' $\stackrel{\text{pyk}}{=}$ "axiom prime s seven"]

⁵⁷[S8' $\stackrel{\text{pyk}}{=}$ "axiom prime s eight"]

⁵⁸[S9' $\stackrel{\text{pyk}}{=}$ "axiom prime s nine"]

$$[S' \text{ rule A4}': \forall \mathcal{C}: \forall \mathcal{A}: \forall \mathcal{X}: \forall \mathcal{B}: [\mathcal{A}] \equiv \langle [\mathcal{B}] | [\mathcal{X}] \rangle := [\mathcal{C}]] \vdash \dot{\forall} \mathcal{X}: \mathcal{B} \Rightarrow \mathcal{A}]$$

$$[S' \text{ rule A5}': \forall \mathcal{X}: \forall \mathcal{A}: \forall \mathcal{B}: \text{nonfree}(\mathcal{X}, \mathcal{A}) \vdash \dot{\forall} \mathcal{X}: (\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow \mathcal{A} \Rightarrow \dot{\forall} \mathcal{X}: \mathcal{B}]$$

$$[S' \text{ rule MP}': \forall \mathcal{A}: \forall \mathcal{B}: \mathcal{A} \Rightarrow \mathcal{B} \vdash \mathcal{A} \vdash \mathcal{B}]$$

$$[S' \text{ rule Gen}': \forall \mathcal{X}: \forall \mathcal{A}: \mathcal{A} \vdash \dot{\forall} \mathcal{X}: \mathcal{A}]$$

$$[S' \text{ rule S1}': \forall \mathcal{A}: \forall \mathcal{B}: \forall \mathcal{C}: \mathcal{A} \stackrel{P}{=} \mathcal{B} \Rightarrow \mathcal{A} \stackrel{P}{=} \mathcal{C} \Rightarrow \mathcal{B} \stackrel{P}{=} \mathcal{C}]$$

$$[S' \text{ rule S2}': \forall \mathcal{A}: \forall \mathcal{B}: \mathcal{A} \stackrel{P}{=} \mathcal{B} \Rightarrow \mathcal{A}' \stackrel{P}{=} \mathcal{B}']$$

$$[S' \text{ rule S3}': \forall \mathcal{A}: \neg \dot{0} \stackrel{P}{=} \mathcal{A}']$$

$$[S' \text{ rule S4}': \forall \mathcal{A}: \forall \mathcal{B}: \mathcal{A}' \stackrel{P}{=} \mathcal{B}' \Rightarrow \mathcal{A} \stackrel{P}{=} \mathcal{B}]$$

$$[S' \text{ rule S5}': \forall \mathcal{A}: \mathcal{A} \dot{+} \dot{0} \stackrel{P}{=} \mathcal{A}]$$

$$[S' \text{ rule S6}': \forall \mathcal{A}: \forall \mathcal{B}: \mathcal{A} \dot{+} \mathcal{B}' \stackrel{P}{=} (\mathcal{A} \dot{+} \mathcal{B})']$$

$$[S' \text{ rule S7}': \forall \mathcal{A}: \mathcal{A} \dot{:} \dot{0} \stackrel{P}{=} \dot{0}]$$

$$[S' \text{ rule S8}': \forall \mathcal{A}: \forall \mathcal{B}: \mathcal{A} \dot{:} (\mathcal{B}') \stackrel{P}{=} (\mathcal{A} \dot{:} \mathcal{B}) \dot{+} \mathcal{A}]$$

$$\begin{aligned} [S' \text{ rule S9}': \forall \mathcal{A}: \forall \mathcal{B}: \forall \mathcal{C}: \forall \mathcal{X}: \\ \mathcal{B} \equiv \langle \mathcal{A} | \mathcal{X} := \dot{0} \rangle \vdash \mathcal{C} \equiv \langle \mathcal{A} | \mathcal{X} := \mathcal{X}' \rangle \vdash \\ \mathcal{B} \Rightarrow \dot{\forall} \mathcal{X}: (\mathcal{A} \Rightarrow \mathcal{C}) \Rightarrow \dot{\forall} \mathcal{X}: \mathcal{A}] \end{aligned}$$

Note that [A1] and [A1'] are distinct. The former says $[S \vdash \forall \mathcal{A}: \forall \mathcal{B}: \mathcal{A} \Rightarrow \mathcal{B} \Rightarrow \mathcal{A}]$ and the latter says $[S' \vdash \forall \mathcal{A}: \forall \mathcal{B}: \mathcal{A} \Rightarrow \mathcal{B} \Rightarrow \mathcal{A}]$.

1.6 Restatement of lemma and a proof

We now prove Lemma [L3.2(a)] once again under the name of [L3.2(a)]⁵⁹:

$$[S' \text{ lemma L3.2(a)}': \forall \mathcal{A}: \mathcal{A} \stackrel{P}{=} \mathcal{A}]$$

S' proof of L3.2(a)':

L01:	Arbitrary $\gg$	$\mathcal{A}$	;
L02:	S5' $\gg$	$\mathcal{A} \dot{+} \dot{0} \stackrel{P}{=} \mathcal{A}$	;
L03:	S1' $\gg$	$\mathcal{A} \dot{+} \dot{0} \stackrel{P}{=} \mathcal{A} \Rightarrow$	
		$\mathcal{A} \dot{+} \dot{0} \stackrel{P}{=} \mathcal{A} \Rightarrow \mathcal{A} \stackrel{P}{=} \mathcal{A}$	;
L04:	MP' $\triangleright$ L03 $\triangleright$ L02 $\gg$	$\mathcal{A} \dot{+} \dot{0} \stackrel{P}{=} \mathcal{A} \Rightarrow \mathcal{A} \stackrel{P}{=} \mathcal{A}$	;
L05:	MP' $\triangleright$ L04 $\triangleright$ L02 $\gg$	$\mathcal{A} \stackrel{P}{=} \mathcal{A}$	□

⁵⁹[L3.2(a)]^{pyk} “lemma prime l three two a”]

A Chores

A.1 The name of the page

This defines the name of the page:

$$[\text{peano} \stackrel{\text{pyk}}{=} \text{“peano”}]$$

A.2 Variables of Peano arithmetic

We use $[\dot{b}]^{60}$, $[\dot{c}]^{61}$, $[\dot{d}]^{62}$, $[\dot{e}]^{63}$, $[\dot{f}]^{64}$, $[\dot{g}]^{65}$, $[\dot{h}]^{66}$, $[\dot{i}]^{67}$, $[\dot{j}]^{68}$, $[\dot{k}]^{69}$, $[\dot{l}]^{70}$, $[\dot{m}]^{71}$, $[\dot{n}]^{72}$, $[\dot{o}]^{73}$, $[\dot{p}]^{74}$, $[\dot{q}]^{75}$, $[\dot{r}]^{76}$, $[\dot{s}]^{77}$, $[\dot{t}]^{78}$, $[\dot{u}]^{79}$, $[\dot{v}]^{80}$, $[\dot{w}]^{81}$, $[\dot{x}]^{82}$, $[\dot{y}]^{83}$, and $[\dot{z}]^{84}$ to denote variables of Peano arithmetic:

$[\dot{b} \doteq \dot{b}]$, $[\dot{c} \doteq \dot{c}]$, $[\dot{d} \doteq \dot{d}]$, $[\dot{e} \doteq \dot{e}]$, $[\dot{f} \doteq \dot{f}]$, $[\dot{g} \doteq \dot{g}]$, $[\dot{h} \doteq \dot{h}]$, $[\dot{i} \doteq \dot{i}]$, $[\dot{j} \doteq \dot{j}]$, $[\dot{k} \doteq \dot{k}]$, $[\dot{l} \doteq \dot{l}]$, $[\dot{m} \doteq \dot{m}]$, $[\dot{n} \doteq \dot{n}]$, $[\dot{o} \doteq \dot{o}]$, $[\dot{p} \doteq \dot{p}]$, $[\dot{q} \doteq \dot{q}]$, $[\dot{r} \doteq \dot{r}]$, $[\dot{s} \doteq \dot{s}]$, $[\dot{t} \doteq \dot{t}]$, $[\dot{u} \doteq \dot{u}]$, $[\dot{v} \doteq \dot{v}]$, $[\dot{w} \doteq \dot{w}]$, $[\dot{x} \doteq \dot{x}]$, $[\dot{y} \doteq \dot{y}]$, and $[\dot{z} \doteq \dot{z}]$.

$$60[\dot{b} \stackrel{\text{pyk}}{=} \text{“peano b”}]$$

$$61[\dot{c} \stackrel{\text{pyk}}{=} \text{“peano c”}]$$

$$62[\dot{d} \stackrel{\text{pyk}}{=} \text{“peano d”}]$$

$$63[\dot{e} \stackrel{\text{pyk}}{=} \text{“peano e”}]$$

$$64[\dot{f} \stackrel{\text{pyk}}{=} \text{“peano f”}]$$

$$65[\dot{g} \stackrel{\text{pyk}}{=} \text{“peano g”}]$$

$$66[\dot{h} \stackrel{\text{pyk}}{=} \text{“peano h”}]$$

$$67[\dot{i} \stackrel{\text{pyk}}{=} \text{“peano i”}]$$

$$68[\dot{j} \stackrel{\text{pyk}}{=} \text{“peano j”}]$$

$$69[\dot{k} \stackrel{\text{pyk}}{=} \text{“peano k”}]$$

$$70[\dot{l} \stackrel{\text{pyk}}{=} \text{“peano l”}]$$

$$71[\dot{m} \stackrel{\text{pyk}}{=} \text{“peano m”}]$$

$$72[\dot{n} \stackrel{\text{pyk}}{=} \text{“peano n”}]$$

$$73[\dot{o} \stackrel{\text{pyk}}{=} \text{“peano o”}]$$

$$74[\dot{p} \stackrel{\text{pyk}}{=} \text{“peano p”}]$$

$$75[\dot{q} \stackrel{\text{pyk}}{=} \text{“peano q”}]$$

$$76[\dot{r} \stackrel{\text{pyk}}{=} \text{“peano r”}]$$

$$77[\dot{s} \stackrel{\text{pyk}}{=} \text{“peano s”}]$$

$$78[\dot{t} \stackrel{\text{pyk}}{=} \text{“peano t”}]$$

$$79[\dot{u} \stackrel{\text{pyk}}{=} \text{“peano u”}]$$

$$80[\dot{v} \stackrel{\text{pyk}}{=} \text{“peano v”}]$$

$$81[\dot{w} \stackrel{\text{pyk}}{=} \text{“peano w”}]$$

$$82[\dot{x} \stackrel{\text{pyk}}{=} \text{“peano x”}]$$

$$83[\dot{y} \stackrel{\text{pyk}}{=} \text{“peano y”}]$$

$$84[\dot{z} \stackrel{\text{pyk}}{=} \text{“peano z”}]$$

$$[A3 \stackrel{\text{tex}}{=} \text{“} \qquad A3\text{”}]$$

$$[A4 \stackrel{\text{tex}}{=} \text{“} \qquad A4\text{”}]$$

$$[A5 \stackrel{\text{tex}}{=} \text{“} \qquad A5\text{”}]$$

$$[MP \stackrel{\text{tex}}{=} \text{“} \qquad MP\text{”}]$$

$$[Gen \stackrel{\text{tex}}{=} \text{“} \qquad Gen\text{”}]$$

$$[S1 \stackrel{\text{tex}}{=} \text{“} \qquad S1\text{”}]$$

$$[S2 \stackrel{\text{tex}}{=} \text{“} \qquad S2\text{”}]$$

$$[S3 \stackrel{\text{tex}}{=} \text{“} \qquad S3\text{”}]$$

$$[S4 \stackrel{\text{tex}}{=} \text{“} \qquad S4\text{”}]$$

$$[S5 \stackrel{\text{tex}}{=} \text{“} \qquad S5\text{”}]$$

$$[S6 \stackrel{\text{tex}}{=} \text{“} \qquad S6\text{”}]$$

$$[S7 \stackrel{\text{tex}}{=} \text{“} \qquad S7\text{”}]$$

$$[S8 \stackrel{\text{tex}}{=} \text{“} \qquad S8\text{”}]$$

$$[S9 \stackrel{\text{tex}}{=} \text{“} \qquad S9\text{”}]$$

$$[L3.2(a) \stackrel{\text{tex}}{=} \text{“} \qquad L3.2(a)\text{”}]$$

$$[S' \stackrel{\text{tex}}{=} \text{“} \qquad S'\text{”}]$$

$$[A1' \stackrel{\text{tex}}{=} \text{“} \quad A1\text{”}]$$

$$[A2' \stackrel{\text{tex}}{=} \text{“} \quad A2\text{”}]$$

$$[A3' \stackrel{\text{tex}}{=} \text{“} \quad A3\text{”}]$$

$$[A4' \stackrel{\text{tex}}{=} \text{“} \quad A4\text{”}]$$

$$[A5' \stackrel{\text{tex}}{=} \text{“} \quad A5\text{”}]$$

$$[MP' \stackrel{\text{tex}}{=} \text{“} \quad MP\text{”}]$$

$$[Gen' \stackrel{\text{tex}}{=} \text{“} \quad Gen\text{”}]$$

$$[S1' \stackrel{\text{tex}}{=} \text{“} \quad S1\text{”}]$$

$$[S2' \stackrel{\text{tex}}{=} \text{“} \quad S2\text{”}]$$

$$[S3' \stackrel{\text{tex}}{=} \text{“} \quad S3\text{”}]$$

$$[S4' \stackrel{\text{tex}}{=} \text{“} \quad S4\text{”}]$$

$$[S5' \stackrel{\text{tex}}{=} \text{“} \quad S5\text{”}]$$

$$[S6' \stackrel{\text{tex}}{=} \text{“} \quad S6\text{”}]$$

$$[S7' \stackrel{\text{tex}}{=} \text{“} \quad S7\text{”}]$$

$$[S8' \stackrel{\text{tex}}{=} \text{“} \quad S8\text{”}]$$

$$[S9' \stackrel{\text{tex}}{=} \text{“} \quad S9\text{”}]$$

$$\begin{aligned}
&[\text{L3.2(a)}' \stackrel{\text{tex}}{=} \text{“} \\
&\qquad\qquad\qquad \text{L3.2(a)}' \text{”}] \\
&[\dot{b} \stackrel{\text{tex}}{=} \text{“} \\
&\qquad\qquad\qquad \dot{\text{\texttt{\textbackslash mathit\{b\}}}} \text{”}] \\
&[\dot{c} \stackrel{\text{tex}}{=} \text{“} \\
&\qquad\qquad\qquad \dot{\text{\texttt{\textbackslash mathit\{c\}}}} \text{”}] \\
&[\dot{d} \stackrel{\text{tex}}{=} \text{“} \\
&\qquad\qquad\qquad \dot{\text{\texttt{\textbackslash mathit\{d\}}}} \text{”}] \\
&[\dot{e} \stackrel{\text{tex}}{=} \text{“} \\
&\qquad\qquad\qquad \dot{\text{\texttt{\textbackslash mathit\{e\}}}} \text{”}] \\
&[\dot{f} \stackrel{\text{tex}}{=} \text{“} \\
&\qquad\qquad\qquad \dot{\text{\texttt{\textbackslash mathit\{f\}}}} \text{”}] \\
&[\dot{g} \stackrel{\text{tex}}{=} \text{“} \\
&\qquad\qquad\qquad \dot{\text{\texttt{\textbackslash mathit\{g\}}}} \text{”}] \\
&[\dot{h} \stackrel{\text{tex}}{=} \text{“} \\
&\qquad\qquad\qquad \dot{\text{\texttt{\textbackslash mathit\{h\}}}} \text{”}] \\
&[\dot{i} \stackrel{\text{tex}}{=} \text{“} \\
&\qquad\qquad\qquad \dot{\text{\texttt{\textbackslash mathit\{i\}}}} \text{”}] \\
&[\dot{j} \stackrel{\text{tex}}{=} \text{“} \\
&\qquad\qquad\qquad \dot{\text{\texttt{\textbackslash mathit\{j\}}}} \text{”}] \\
&[\dot{k} \stackrel{\text{tex}}{=} \text{“} \\
&\qquad\qquad\qquad \dot{\text{\texttt{\textbackslash mathit\{k\}}}} \text{”}] \\
&[\dot{l} \stackrel{\text{tex}}{=} \text{“} \\
&\qquad\qquad\qquad \dot{\text{\texttt{\textbackslash mathit\{l\}}}} \text{”}] \\
&[\dot{m} \stackrel{\text{tex}}{=} \text{“} \\
&\qquad\qquad\qquad \dot{\text{\texttt{\textbackslash mathit\{m\}}}} \text{”}] \\
&[\dot{n} \stackrel{\text{tex}}{=} \text{“} \\
&\qquad\qquad\qquad \dot{\text{\texttt{\textbackslash mathit\{n\}}}} \text{”}] \\
&[\dot{o} \stackrel{\text{tex}}{=} \text{“} \\
&\qquad\qquad\qquad \dot{\text{\texttt{\textbackslash mathit\{o\}}}} \text{”}] \\
&[\dot{p} \stackrel{\text{tex}}{=} \text{“} \\
&\qquad\qquad\qquad \dot{\text{\texttt{\textbackslash mathit\{p\}}}} \text{”}]
\end{aligned}$$

$$[\dot{q} \stackrel{\text{tex}}{=} “ \dot{\mathit{q}} ”]$$

$$[\dot{r} \stackrel{\text{tex}}{=} “ \dot{\mathit{r}} ”]$$

$$[\dot{s} \stackrel{\text{tex}}{=} “ \dot{\mathit{s}} ”]$$

$$[\dot{t} \stackrel{\text{tex}}{=} “ \dot{\mathit{t}} ”]$$

$$[\dot{u} \stackrel{\text{tex}}{=} “ \dot{\mathit{u}} ”]$$

$$[\dot{v} \stackrel{\text{tex}}{=} “ \dot{\mathit{v}} ”]$$

$$[\dot{w} \stackrel{\text{tex}}{=} “ \dot{\mathit{w}} ”]$$

$$[\dot{x} \stackrel{\text{tex}}{=} “ \dot{\mathit{x}} ”]$$

$$[\dot{y} \stackrel{\text{tex}}{=} “ \dot{\mathit{y}} ”]$$

$$[\dot{z} \stackrel{\text{tex}}{=} “ \dot{\mathit{z}} ”]$$

A.4 Test

$$[[\dot{a}]^{\mathcal{P}}]$$

$$[[\mathbf{a}]^{\mathcal{P}}]^{-}$$

$$[\text{nonfree}(\lceil \dot{x} \rceil, [\dot{y} \stackrel{\text{P}}{=} z \Rightarrow \dot{\forall} x: \dot{x} \stackrel{\text{P}}{=} \dot{y}])]$$

$$[\text{nonfree}(\lceil \dot{x} \rceil, [\dot{x} \stackrel{\text{P}}{=} z \Rightarrow \dot{\forall} x: \dot{x} \stackrel{\text{P}}{=} \dot{y}])^{-}$$

$$[\text{nonfree}(\lceil \dot{x} \rceil, [\dot{y} \stackrel{\text{P}}{=} \dot{x} \Rightarrow \dot{\forall} x: \dot{x} \stackrel{\text{P}}{=} \dot{y}])^{-}$$

$$[\text{nonfree}(\lceil \dot{x} \rceil, [\dot{y} \stackrel{\text{P}}{=} z \Rightarrow \dot{\forall} y: \dot{x} \stackrel{\text{P}}{=} \dot{y}])^{-}$$

$$[\text{free}(\lceil \dot{\forall} x: \mathbf{b} :: \dot{x} :: \mathbf{c} \rceil \lceil \dot{x} \rceil := \lceil \mathbf{x} :: \dot{y} :: \mathbf{z} \rceil)]$$

$$[\text{free}(\lceil \dot{\forall} y: \mathbf{b} :: \dot{x} :: \mathbf{c} \rceil \lceil \dot{x} \rceil := \lceil \mathbf{x} :: \dot{y} :: \mathbf{z} \rceil)]^{-}$$

$$\begin{aligned}
& [\text{free}(\langle \dot{\forall} \dot{x} : \mathbf{b} :: \dot{x} :: \mathbf{c} \rangle \mid \dot{y}) := \langle \mathbf{x} :: \dot{y} :: \mathbf{z} \rangle)] \cdot \\
& [\text{free}(\langle \dot{\forall} \dot{y} : \mathbf{b} :: \dot{x} :: \mathbf{c} \rangle \mid \dot{y}) := \langle \mathbf{x} :: \dot{y} :: \mathbf{z} \rangle)] \cdot \\
& [\dot{a} \equiv \langle \dot{a} \mid \dot{b} := \dot{c} \rangle] \cdot \\
& [\dot{c} \equiv \langle \dot{b} \mid \dot{b} := \dot{c} \rangle] \cdot \\
& [\forall \dot{a} : \dot{a} \stackrel{\mathbb{P}}{=} \dot{b} \equiv \langle \forall \dot{a} : \dot{a} \stackrel{\mathbb{P}}{=} \dot{b} \mid \dot{a} := \dot{c} \rangle] \cdot \\
& [\forall \dot{a} : \dot{a} \stackrel{\mathbb{P}}{=} \dot{c} \equiv \langle \forall \dot{a} : \dot{a} \stackrel{\mathbb{P}}{=} \dot{b} \mid \dot{b} := \dot{c} \rangle] \cdot \\
& [\forall \dot{a} : \dot{a} \stackrel{\mathbb{P}}{=} \dot{0} \dot{+} \dot{a} \Rightarrow \dot{c} : \dot{d} \stackrel{\mathbb{P}}{=} \dot{0} \dot{+} \dot{c} : \dot{d} \equiv \langle \forall \dot{a} : \dot{a} \stackrel{\mathbb{P}}{=} \dot{0} \dot{+} \dot{a} \Rightarrow \dot{b} \stackrel{\mathbb{P}}{=} \dot{0} \dot{+} \dot{b} \mid \dot{b} := \dot{c} : \dot{d} \rangle] \cdot \\
& [\forall \dot{a} : \dot{a} \stackrel{\mathbb{P}}{=} \dot{0} \dot{+} \dot{a} \Rightarrow \dot{b} \stackrel{\mathbb{P}}{=} \dot{0} \dot{+} \dot{b} \equiv \langle \forall \dot{a} : \dot{a} \stackrel{\mathbb{P}}{=} \dot{0} \dot{+} \dot{a} \Rightarrow \dot{b} \stackrel{\mathbb{P}}{=} \dot{0} \dot{+} \dot{b} \mid \dot{a} := \dot{c} \rangle] \cdot
\end{aligned}$$

A.5 Priority table

Priority table

Preassociative

[peano], [base], [bracket * end bracket], [big bracket * end bracket],
 [math * end math], [**flush left** [*], [x], [y], [z], [[* $\bowtie$ *]], [[* $\rightarrow$ *]], [pyk], [tex],
 [name], [prio], [*, [T], [if(*, *, *)], [[* $\xrightarrow{*}$ *]], [val], [claim], [$\perp$], [f(*)], [(*)^I], [F], [0],
 [1], [2], [3], [4], [5], [6], [7], [8], [9], [0], [1], [2], [3], [4], [5], [6], [7], [8], [9], [a], [b], [c], [d],
 [e], [f], [g], [h], [i], [j], [k], [l], [m], [n], [o], [p], [q], [r], [s], [t], [u], [v], [w], [(*)^M], [If(*, *,
)], [array{} * end array], [l], [c], [r], [empty], [(< * | * := *)], [$\mathcal{M}(*)$], [$\mathcal{U}(*)$], [$\mathcal{U}(*)$],
 [$\mathcal{U}^M(*)$], [**apply**(*, *)], [**apply**₁(*, *)], [identifier(*)], [identifier₁(*, *)], [array-
 plus(*, *)], [array-remove(*, *, *)], [array-put(*, *, *, *)], [array-add(*, *, *, *, *)],
 [bit(*, *)], [bit₁(*, *)], [rack], ["vector"], ["bibliography"], ["dictionary"],
 ["body"], ["codex"], ["expansion"], ["code"], ["cache"], ["diagnose"], ["pyk"],
 ["tex"], ["texname"], ["value"], ["message"], ["macro"], ["definition"],
 ["unpack"], ["claim"], ["priority"], ["lambda"], ["apply"], ["true"], ["if"],
 ["quote"], ["proclaim"], ["define"], ["introduce"], ["hide"], ["pre"], ["post"],
 [$\mathcal{E}(*, *, *)$], [$\mathcal{E}_2(*, *, *, *, *)$], [$\mathcal{E}_3(*, *, *, *, *)$], [$\mathcal{E}_4(*, *, *, *, *)$], [**lookup**(*, *, *)],
 [**abstract**(*, *, *, *)], [[*]], [$\mathcal{M}(*, *, *)$], [$\mathcal{M}_2(*, *, *, *)$], [$\mathcal{M}^*(*, *, *)$], [macro],
 [s₀], [**zip**(*, *)], [**assoc**₁(*, *, *)], [(*)^P], [self], [[* $\doteq$ *]], [[* $\dot{=}$ *]], [[* $\dot{=}$ *]],
 [[* $\stackrel{\text{pyk}}{=}$ *]], [[* $\stackrel{\text{tex}}{=}$ *]], [[* $\stackrel{\text{name}}{=}$ *]], [**Priority table**[*]], [$\tilde{\mathcal{M}}_1$], [$\tilde{\mathcal{M}}_2(*)$], [$\tilde{\mathcal{M}}_3(*)$],
 [$\tilde{\mathcal{M}}_4(*, *, *, *, *)$], [$\mathcal{M}(*, *, *)$], [$\mathcal{Q}(*, *, *)$], [$\tilde{\mathcal{Q}}_2(*, *, *)$], [$\tilde{\mathcal{Q}}_3(*, *, *, *)$], [$\tilde{\mathcal{Q}}^*(*, *, *, *)$],
 [(*)], [**aspect**(*, *)], [**aspect**(*, *, *)], [(< >)], [**tuple**₁(*)], [**tuple**₂(*)], [let₂(*, *)],
 [let₁(*, *)], [[* $\stackrel{\text{claim}}{=}$ *]], [checker], [**check**(*, *)], [**check**₂(*, *, *)], [**check**₃(*, *, *)],
 [**check**^{*}(*, *)], [**check**₂^{*}(*, *, *)], [[*]'], [[*]⁻], [[*]°], [msg], [[* $\stackrel{\text{msg}}{=}$ *]], [<stmt>],
 [stmt], [[* $\stackrel{\text{stmt}}{=}$ *]], [HeadNil'], [HeadPair'], [Transitivity'], [$\perp$], [Contra'], [T'_E],
 [L₁], [*, [A], [B], [C], [D], [E], [F], [G], [H], [I], [J], [K], [L], [M], [N], [O], [P], [Q],
 [R], [S], [T], [U], [V], [W], [X], [Y], [Z], [(< * | * := *)], [($\leftarrow$ * | * := *)], [∅], [Remainder],
 [(*)^v], [intro(*, *, *, *)], [intro(*, *, *)], [error(*, *)], [error₂(*, *)], [proof(*, *, *)],

[proof₂(*,*), [S(*,*), [S^I(*,*), [S[▷](*,*), [S[▷]₁(*,*,*), [S^E(*,*), [S^F₁(*,*,*), [S⁺(*,*), [S⁺₁(*,*,*), [S⁻(*,*), [S⁻₁(*,*,*), [S^{*}(*,*), [S^{*}₁(*,*,*), [S^{*}₂(*,*,*,*), [S[@](*,*), [S[@]₁(*,*,*), [S⁺(*,*), [S⁺₁(*,*,*,*), [S⁺₂(*,*,*,*), [S⁺₁(*,*,*,*), [S^{i.e.}(*,*), [S^{i.e.}₁(*,*,*,*), [S^{i.e.}₂(*,*,*,*), [S[∇](*,*), [S[∇]₁(*,*,*,*), [Sⁱ(*,*), [Sⁱ₁(*,*,*,*), [Sⁱ₂(*,*,*,*), [T(*), [claims(*,*,*), [claims₂(*,*,*), [<proof>], [proof], [[**Lemma** *: *]], [[**Proof of** *: *]], [[* **lemma** *: *]], [[* **antilemma** *: *]], [[* **rule** *: *]], [[* **antirule** *: *]], [verifier], [V₁(*)], [V₂(*,*)], [V₃(*,*,*,*)], [V₄(*,*)], [V₅(*,*,*,*)], [V₆(*,*,*,*)], [V₇(*,*,*,*)], [Cut(*,*)], [Head_⊕(*)], [Tail_⊕(*)], [rule₁(*,*)], [rule(*,*)], [Rule tactic], [Plus(*,*)], [[**Theory** *]], [theory₂(*,*)], [theory₃(*,*)], [theory₄(*,*,*)], [HeadNil''], [HeadPair''], [Transitivity''], [Contra''], [HeadNil], [HeadPair], [Transitivity], [Contra], [T_E], [ragged right], [ragged right expansion], [parm(*,*,*)], [parm^{*}(*,*,*)], [inst(*,*)], [inst^{*}(*,*)], [occur(*,*,*)], [occur^{*}(*,*,*)], [unify(* = *,*)], [unify^{*}(* = *,*)], [unify₂(* = *,*)], [L_a], [L_b], [L_c], [L_d], [L_e], [L_f], [L_g], [L_h], [L_i], [L_j], [L_k], [L_l], [L_m], [L_n], [L_o], [L_p], [L_q], [L_r], [L_s], [L_t], [L_u], [L_v], [L_w], [L_x], [L_y], [L_z], [L_A], [L_B], [L_C], [L_D], [L_E], [L_F], [L_G], [L_H], [L_I], [L_J], [L_K], [L_L], [L_M], [L_N], [L_O], [L_P], [L_Q], [L_R], [L_S], [L_T], [L_U], [L_V], [L_W], [L_X], [L_Y], [L_Z], [L_?], [Reflexivity], [Reflexivity₁], [Commutativity], [Commutativity₁], [<tactic>], [tactic], [[* ^{tactic} *]], [P(*,*,*)], [P^{*}(*,*,*)], [p₀], [conclude₁(*,*)], [conclude₂(*,*,*)], [conclude₃(*,*,*,*)], [0], [1], [2], [ā], [b̄], [c̄], [d̄], [ē], [f̄], [ḡ], [h̄], [ī], [j̄], [k̄], [l̄], [m̄], [n̄], [ō], [p̄], [q̄], [r̄], [s̄], [t̄], [ū], [v̄], [w̄], [x̄], [ȳ], [z̄], [nonfree(*,*)], [nonfree^{*}(*,*)], [free(* := *)], [free^{*}(* := *)], [* ≡ (* := *)], [* ≡ (* := *)], [S], [A1], [A2], [A3], [A4], [A5], [S1], [S2], [S3], [S4], [S5], [S6], [S7], [S8], [S9], [MP], [Gen], [L3.2(a)], [S'], [A1'], [A2'], [A3'], [A4'], [A5'], [S1'], [S2'], [S3'], [S4'], [S5'], [S6'], [S7'], [S8'], [S9'], [MP'], [Gen'], [L3.2(a)']];

Preassociative

[*-{*}], [*'], [* *], [* *→*], [* *⇒*], [*];

Preassociative

[“ * ”], [], [(*)^t], [string(*) + *], [string(*) ++ *], [*, [*, [*, [" *], [# *], [\$ *], [% *], [& *], [' *], [(*), [) *], [* *], [+ *], [, *], [- *], [. *], [/ *], [0 *], [1 *], [2 *], [3 *], [4 *], [5 *], [6 *], [7 *], [8 *], [9 *], [: *], [; *], [< *], [= *], [> *], [? *], [@ *], [A *], [B *], [C *], [D *], [E *], [F *], [G *], [H *], [I *], [J *], [K *], [L *], [M *], [N *], [O *], [P *], [Q *], [R *], [S *], [T *], [U *], [V *], [W *], [X *], [Y *], [Z *], [[*], [\ *], [] *], [^ *], [_ *], [*], [a *], [b *], [c *], [d *], [e *], [f *], [g *], [h *], [i *], [j *], [k *], [l *], [m *], [n *], [o *], [p *], [q *], [r *], [s *], [t *], [u *], [v *], [w *], [x *], [y *], [z *], [{ *], [| *], [} *], [~ *], [**Preassociative** *; *], [**Postassociative** *; *], [[*], *], [priority * end], [newline *], [macro newline *];

Preassociative

[*0], [*1], [0b], [*-color(*)], [*-color^{*}(*)];

Preassociative

[* ' *], [* ' *];

Preassociative

[*^H], [*^T], [*^U], [*^h], [*^t], [*^s], [*^c], [*^d], [*^a], [*^C], [*^M], [*^B], [*^r], [*ⁱ], [*^d], [*^R], [*⁰], [*¹], [*²], [*³], [*⁴], [*⁵], [*⁶], [*⁷], [*⁸], [*⁹], [*^E], [*^V], [*^C], [*^{C*}], [*'];

Preassociative

$$[* \cdot *], [* \cdot_0 *], [* \dot{\cdot} *];$$
Preassociative

$$[* + *], [* +_0 *], [* +_1 *], [* - *], [* -_0 *], [* -_1 *], [* \dot{+} *];$$
Preassociative

$$[* \cup \{*\}], [* \cup *], [* \setminus \{*\}];$$
Postassociative

$$[* \dot{\cdot} *], [* \dot{\cdot} *], [* \dot{\cdot} :: *], [* \underline{+2} *], [* :: *], [* +2 * *];$$
Postassociative

$$[* , *];$$
Preassociative

$$[* \overset{B}{\approx} *], [* \overset{D}{\approx} *], [* \overset{C}{\approx} *], [* \overset{P}{\approx} *], [* \approx *], [* = *], [* \overset{+}{\rightarrow} *], [* \overset{t}{=} *], [* \overset{t^*}{=} *], [* \overset{r}{=} *],$$

$$[* \in_t *], [* \subseteq_T *], [* \overset{T}{=} *], [* \overset{s}{=} *], [* \text{ free in } *], [* \text{ free in }^* *], [* \text{ free for } * \text{ in } *],$$

$$[* \text{ free for }^* * \text{ in } *], [* \in_c *], [* < *], [* <' *], [* \leq' *], [* \overset{P}{=} *], [* \mathcal{P}];$$
Preassociative

$$[\neg *], [\dot{\neg} *];$$
Preassociative

$$[* \wedge *], [* \ddot{\wedge} *], [* \tilde{\wedge} *], [* \wedge_c *], [* \dot{\wedge} *];$$
Preassociative

$$[* \vee *], [* \parallel *], [* \ddot{\vee} *], [* \dot{\vee} *];$$
Preassociative

$$[\dot{\forall} * : *], [\dot{\exists} * : *];$$
Postassociative

$$[* \rightrightarrows *], [* \Rightarrow *], [* \Leftrightarrow *];$$
Postassociative

$$[* : *], [* ! *];$$
Preassociative

$$[* \left\{ \begin{array}{c} * \\ * \end{array} \right\}];$$
Preassociative

$$[\lambda * . *], [\Lambda *], [\text{if } * \text{ then } * \text{ else } *], [\text{let } * = * \text{ in } *], [\text{let } * \doteq * \text{ in } *];$$
Preassociative

$$[*^I], [*^\triangleright], [*^V], [*^+], [*^-], [*^*];$$
Preassociative

$$[* @ *], [* \triangleright *], [* \blacktriangleright *], [* \gg *];$$
Postassociative

$$[* \vdash *], [* \Vdash *], [* \text{ i.e. } *];$$
Preassociative

$$[\forall * : *];$$
Postassociative

$$[* \oplus *];$$
Postassociative

$$[* ; *];$$
Preassociative

$$[* \text{ proves } *];$$

Preassociative

[* **proof of** * : *], [Line * : * $\gg$ *; *], [Last line * $\gg$ * $\square$],
[Line * : Premise $\gg$ *; *], [Line * : Side-condition $\gg$ *; *], [Arbitrary $\gg$ *; *],
[Local $\gg$ * = *; *];

Postassociative

[* then *], [* [*]*];

Preassociative

[*&*];

Preassociative

[**]; **End table**

B Index

* is peano var	$[x^P]$, 2
* peano and *	$[x \wedge y]$, 2
* peano iff *	$[x \Leftrightarrow y]$, 2
* peano imply *	$[x \Rightarrow y]$, 2
* peano is *	$[x \stackrel{P}{=} y]$, 2
* peano or *	$[x \vee y]$, 2
* peano plus *	$[x \dot{+} y]$, 1
* peano succ	$[x']$, 1
* peano times *	$[x \cdot y]$, 1
* peano var	$[\dot{x}]$, 2
$[x \wedge y]$	* peano and *, 2
$[x \Leftrightarrow y]$	* peano iff *, 2
$[x \Rightarrow y]$	* peano imply *, 2
$[x \vee y]$	* peano or *, 2
$[x \dot{+} y]$	* peano plus *, 1
$[x']$	* peano succ, 1
$[x \cdot y]$	* peano times *, 1
$[\dot{x}]$	* peano var, 2
$[\forall x: y]$	peano all * indeed *, 2
$[\exists x: y]$	peano exist * indeed *, 2
$[\dot{\neg} x]$	peano not *, 2
$[\dot{1}]$	peano one, 2
$[\dot{2}]$	peano two, 2
$[\dot{0}]$	peano zero, 1

A1':	$[A1']$ axiom prime a one, 6
A1:	$[A1]$ axiom a one, 4
A2':	$[A2']$ axiom prime a two, 6
A2:	$[A2]$ axiom a two, 4
A3':	$[A3']$ axiom prime a three, 6
A3:	$[A3]$ axiom a three, 4
A4':	$[A4']$ axiom prime a four, 6

A4: [A4] axiom a four, 4
 A5': [A5'] axiom prime a five, 6
 A5: [A5] axiom a five, 4
 a: [$\dot{a}$] peano a, 2
 axiom a five [A5], 4
 axiom a four [A4], 4
 axiom a one [A1], 4
 axiom a three [A3], 4
 axiom a two [A2], 4
 axiom prime a five [A5'], 6
 axiom prime a four [A4'], 6
 axiom prime a one [A1'], 6
 axiom prime a three [A3'], 6
 axiom prime a two [A2'], 6
 axiom prime s eight [S8'], 6
 axiom prime s five [S5'], 6
 axiom prime s four [S4'], 6
 axiom prime s nine [S9'], 6
 axiom prime s one [S1'], 6
 axiom prime s seven [S7'], 6
 axiom prime s six [S6'], 6
 axiom prime s three [S3'], 6
 axiom prime s two [S2'], 6
 axiom s eight [S8], 4
 axiom s five [S5], 4
 axiom s four [S4], 4
 axiom s nine [S9], 4
 axiom s one [S1], 4
 axiom s seven [S7], 4
 axiom s six [S6], 4
 axiom s three [S3], 4
 axiom s two [S2], 4

 b: [$\dot{b}$] peano b, 8

 c: [$\dot{c}$] peano c, 8

 d: [$\dot{d}$] peano d, 8

 e: [$\dot{e}$] peano e, 8

 f: [$\dot{f}$] peano f, 8
 free*: [$\text{free}^*\langle a, x := b \rangle$] peano free star * set * to * end free3
 free: [$\text{free}\langle a, x := b \rangle$] peano free * set * to * end free3

 g: [$\dot{g}$] peano g, 8
 Gen': [Gen'] rule prime gen, 6

Gen: [Gen] rule gen, 4

h: [$\dot{h}$] peano h, 8

i: [$\dot{i}$] peano i, 8

j: [$\dot{j}$] peano j, 8

k: [$\dot{k}$] peano k, 8

L3.2(a)': [L3.2(a)'] lemma prime l three two a, 7

L3.2(a): [L3.2(a)] lemma l three two a, 5

l: [$\dot{l}$] peano l, 8

lemma l three two a [L3.2(a)], 5

lemma prime l three two a [L3.2(a)'], 7

m: [$\dot{m}$] peano m, 8

MP': [MP'] rule prime mp, 6

MP: [MP] rule mp, 4

n: [$\dot{n}$] peano n, 8

nonfree*: [nonfree*(x,y)] peano nonfree star * in * end nonfree, 3

nonfree: [nonfree(x,y)] peano nonfree * in * end nonfree, 2

o: [$\dot{o}$] peano o, 8

P: [x^P] * is peano var, 2

p: [$x \stackrel{P}{=} y$] * peano is *, 2

p: [$\dot{p}$] peano p, 8

peano a [$\dot{a}$], 2

peano all * indeed * [$\dot{\forall}x:y$], 2

peano b [$\dot{b}$], 8

peano c [$\dot{c}$], 8

peano d [$\dot{d}$], 8

peano e [$\dot{e}$], 8

peano exist * indeed * [$\dot{\exists}x:y$], 2

peano f [$\dot{f}$], 8

peano free * set * to * end free [$\dot{\text{free}}(a, x := b)$]

peano free star * set * to * end free [$\dot{\text{free}}^*(a, x := b)$]

peano g [$\dot{g}$], 8

peano h [$\dot{h}$], 8

peano i [$\dot{i}$], 8

peano j [$\dot{j}$], 8

peano k [$\dot{k}$], 8

peano l [$\dot{l}$], 8

peano m [$\dot{m}$], 8

peano n [$\dot{n}$], 8

peano nonfree $*$ in $*$ end nonfree $[\text{nonfree}(x, y)]$, 2
 peano nonfree star $*$ in $*$ end nonfree $[\text{nonfree}^*(x, y)]$, 3
 peano not $*$ $[\neg x]$, 2
 peano o $[\dot{o}]$, 8
 peano one $[\dot{1}]$, 2
 peano p $[\dot{p}]$, 8
 peano q $[\dot{q}]$, 8
 peano r $[\dot{r}]$, 8
 peano s $[\dot{s}]$, 8
 peano sub $*$ is $*$ where $*$ is $*$ end sub $[a \equiv \langle b, x := c \rangle]$ 3
 peano sub star $*$ is $*$ where $*$ is $*$ end sub $[a \equiv \langle *b, x := c \rangle]$ 3
 peano t $[\dot{t}]$, 8
 peano two $[\dot{2}]$, 2
 peano u $[\dot{u}]$, 8
 peano v $[\dot{v}]$, 8
 Peano variable, 2
 peano w $[\dot{w}]$, 8
 peano x $[\dot{x}]$, 8
 peano y $[\dot{y}]$, 8
 peano z $[\dot{z}]$, 8
 peano zero $[\dot{0}]$, 1

 q: $[\dot{q}]$ peano q, 8

 r: $[\dot{r}]$ peano r, 8
 rule gen $[\text{Gen}]$, 4
 rule mp $[\text{MP}]$, 4
 rule prime gen $[\text{Gen}']$, 6
 rule prime mp $[\text{MP}']$, 6

 S': $[\text{S}']$ system prime s, 6
 S1': $[\text{S1}']$ axiom prime s one, 6
 S1: $[\text{S1}]$ axiom s one, 4
 S2': $[\text{S2}']$ axiom prime s two, 6
 S2: $[\text{S2}]$ axiom s two, 4
 S3': $[\text{S3}']$ axiom prime s three, 6
 S3: $[\text{S3}]$ axiom s three, 4
 S4': $[\text{S4}']$ axiom prime s four, 6
 S4: $[\text{S4}]$ axiom s four, 4
 S5': $[\text{S5}']$ axiom prime s five, 6
 S5: $[\text{S5}]$ axiom s five, 4
 S6': $[\text{S6}']$ axiom prime s six, 6
 S6: $[\text{S6}]$ axiom s six, 4
 S7': $[\text{S7}']$ axiom prime s seven, 6
 S7: $[\text{S7}]$ axiom s seven, 4
 S8': $[\text{S8}']$ axiom prime s eight, 6

S8: [S8] axiom s eight, 4
S9': [S9'] axiom prime s nine, 6
S9: [S9] axiom s nine, 4
s: [$\dot{s}$] peano s, 8
S: [S] system s, 4
system prime s [S'], 6
system s [S], 4

t: [$\dot{t}$] peano t, 8

u: [$\dot{u}$] peano u, 8

v: [$\dot{v}$] peano v, 8
variable, Peano, 2

w: [$\dot{w}$] peano w, 8

x: [$\dot{x}$] peano x, 8

y: [$\dot{y}$] peano y, 8

z: [$\dot{z}$] peano z, 8

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