A κ-denotational semantics for Map Theory in $ZFC + SI$

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April 15, 1996

Abstract

Map theory, or MT for short, has been designed as an "integrated" foundation for mathematics, logic and computer science. By this we mean that most primitives and tools are designed from the beginning to bear the three intended meanings: logical, computational, and set-theoretic.

MT was originally introduced in [18]. It is based on λ-calculus instead of logic and sets, and it fulfills Church's original aim of introducing λ-calculus. In particular, it embodies all of ZFC set theory, including classical propositional and classical first order predicate calculus. MT also embodies the unrestricted, untyped lambda calculus including unrestricted abstraction and unrestricted use of the fixed point operator. MT is an equational theory.

We present here a semantic proof of the consistency of map theory within $ZFC + SI$, where SI asserts the existence of an inaccessible cardinal. The proof is in the spirit of denotational semantics and relies on mathematical tools which reflect faithfully, and in a transparent way, the intuitions behind map theory. This gives a consistency proof, but also for the first time gives a clear presentation of the semantics of map theory in a traditional framework. Furthermore, the proof seems to indicate that all "big" models of (a very weak extension of) λ-calculus can be expanded to models of MT.

From the metamathematical point of view the strength of MT lies somewhere between ZFC and $ZFC + SI$. The lower bound is proved in [18] by means of a syntactical translation of ZFC (including classical propositional and predicate calculus) into map theory, and the upper bound by building an (exceedingly complex) model of map theory within $ZFC + SI$. The present paper confirms the upper bound by providing much simpler models, the "canonical models" of the paper, which are in fact the paradigm of a large class of quite natural models of MT.

That all these models interpret a model of ZFC is a consequence of the syntactic translation, which is a difficult theorem of [18]. We can however

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give here a direct proof of a stronger result, namely that they interpret some $(V_\sigma, \in)$, where σ is an inaccessible cardinal.

Finally we return to the "canonical" models and show that they are adequate for the notion of computation which underlies MT.

Contents

1 Introduction

1.1 Presentation of MT

Since all of part I of [18] is a presentation of the semantics and computational ideas behind the present axiomatisation of MT, we will just present here the most simple intuitions. In particular, we will just consider 3-valued first order predicate calculus and usual *ZFC* set theory, and we will not say much about the computational aspects.

The syntax and axioms of MT is recalled in Appendix C. The syntax and variables, terms and well-formed formulas reads:

$$\begin{array}{lcl} \mathsf{variable} & ::= & x \mid y \mid z \mid \ldots \\ \mathsf{term} & ::= & \mathsf{variable} \mid \lambda\mathsf{variable.term} \mid (\mathsf{term}\,\mathsf{term}) \mid \mathsf{T} \mid \bot \mid \mathsf{if} \mid \varepsilon \mid \phi \\ \mathsf{wff} & ::= & \mathsf{term} = \mathsf{term} \end{array}$$

A proof in MT is a sequence of well-formed formulas where each formula is an axiom or follows from previous formulas by an inference rule.

As an equational theory, MT is an extension of the theory of β-equivalence. Its language is very simple since it is that of untyped λ-calculus, namely abstraction and application, augmented by five constants if, T, $\bot$, ϕ and ε. The word "equational" means, as is usual in the λ-calculus community that the axiom schemes are equations or inference rules for deriving equations. Such a theory T is consistent if one cannot derive $A = B$ for all terms A and B. An equation is inconsistent with T if adding it to the axioms produces an inconsistent theory.

The underlying notion of computation is obtained, as in λ-calculus, by turning some of the axioms into rewriting rules, in orienting them from left to right. The model stated later is faithful to the computational aspect of MT as follows: Let $\mathcal{A}$ be a term of MT which does not contain ϕ and ε. Now $\mathcal{A} = \bot$ in the model iff straightforward normal order reduction using the rewrite rules of $\mathcal{A}$ never terminates. If $\mathcal{A} \neq \bot$ in the model, then normal order reduction can decide whether or not $\mathcal{A} = \mathsf{T}$ in the model.

1.2 Intended meaning of constants

As mentioned in the abstract, most primitives and tools are designed from the beginning to bear the three intended meanings: logical, computational, and set-theoretic.

T represents "Truth" (and not "Top") in the logical world, the empty set in the set-theoretic world, and the empty list in the computational world.

$\bot$ represents "undefinedness" in the logical world, non-termination in the computational world, and nothing in the set-theoretical world.

if is the McCarthy's conditional ([26], p. 54). In the logical world, if allows to define the usual logical connectives, in the computational world, if represents forking and the pairing operator, and in the set-theoretic world, if represents the pair set operator. When defining logical connectives, if is used in conjunction with truth T and falsehood F. Falsehood F could be defined as any abstraction $\lambda x.\mathcal{A}$ or as any term provably equal in MT to an abstraction, but the canonical choice $\mathsf{F} \equiv \lambda x.\mathsf{T}$ will be used here. In the computational context, F is the constant function that always returns truth and in the set-theoretic context, F happens to be $\{\emptyset\}$ (i.e. the ordinal 1).

ε is (a strict version of) Hilbert's choice operator [20]. In the set-theoretic world, ε implies the axiom of choice. In the logical world, ε allows to define existential and universal quantification. ε is not computable by machine.

ϕ represents something like the "characteristic function" of the class of all sets (here characteristic functions take values T and $\bot$). In the logical world, ϕ effectively allows to restrict free variables to range over exactly the same collection that quantifiers range over. In the set-theoretic world it allows to restrict free variables to range over sets only.

Notation. We write $\boxed{\mathcal{A} =_{MT} \mathcal{B}}$ if $\mathcal{A} = \mathcal{B}$ is derivable (or provable) in MT. All words in italics and all formulas in boxes appear in the index at the end of the paper.

A term $\mathcal{A}$ will be called *well-founded* if $\phi\mathcal{A} = \mathsf{T}$ holds in MT (i.e. provably well-founded if $\phi\mathcal{A} =_{MT} \mathsf{T}$ and well-founded in a model if $\phi\mathcal{A} = \mathsf{T}$ holds in that model).

1.3 Intuitive description of the axioms

The axioms are listed in Appendix C, where they are divided into four groups: λ-calculus, propositional calculus, predicate calculus, and set-theory.

The λ-calculus axioms express that $=_{MT}$ is a contextual equivalence relation which contains β-equivalence; they also dictate the applicative behaviour of $\bot$, T and if. When orienting from left to right the "apply" and "select" axioms in Appendix C, one obtains the rewrite rules mentioned in Section 1.1.

Note that the applicative behaviour of T and F, (respectively of $\perp$ and $\lambda x.\perp$) will be the same, but that $\mathsf{T} = \mathsf{F}$ (respectively $\perp = \lambda x.\perp$) is inconsistent with the λ-calculus axioms. This is coherent with the logical and set-theoretic meaning of T and F. Note also that a term $\mathcal{B}$ is equal to an abstraction $\lambda x.\mathcal{A}$ iff $\mathcal{B} = \mathsf{F}'\mathcal{B}$ where $\mathsf{F}' \equiv \lambda f \lambda x.fx$.

In the logical world, the λ-calculus axioms allow to interpret strict three-valued propositional calculus, where "strict" means that all connectives return $\perp$ as soon as one of their arguments is $\perp$. This corresponds to Kleene's weak connectives. This choice differs from that of most authors, to begin with Kleene himself (c.f. [10], p.87), it is neither the choice retained by Scott in [34]. For a treatment of two-valued propositional calculus see Section 3.4.

The propositional calculus group of axioms merely contains one axiom (a rule of inference, actually). The rule is called QND$'$ (for Quartum Non Datur) and expresses that any object of MT can be given a meaning in three-valued logic in MT. For a connective like $\dot{\wedge}$ (defined in Appendix C), the lambda calculus axioms allow to prove statements like $\mathsf{T}\dot{\wedge}\mathsf{F} = \mathsf{F}$ whereas QND$'$ allows to prove more general tautologies like $x\dot{\wedge}y = y\dot{\wedge}x$. A strong way to ensure QND$'$ is of course to find a model of λ-calculus where all elements x different from T and $\perp$ satisfy $\mathsf{F}'x = x$. This condition will later on be called SQND, for Strong QND, and will be satisfied by our model.

The predicate calculus axioms express the semantics of the three quantifiers ε, $\forall$ and $\exists$. Like in Hilberts approach [20], $\forall$ and $\exists$ are defined from ε (c.f. Section 4.2 and Appendix C). As noticed by Honsell the three quantifiers operate on abstractions, so that there is only one notion of substitution, namely that of λ-calculus; this is also the case for example in [34, 1, 13] for $\forall$ and $\exists$, and the approach goes back to Church [8].

The choice of ε as the primitive for quantifications gives the axiom of choice for free in the set-theoretic world. In the world of computation, ε would be more convenient than $\forall$ and $\exists$ since it returns a richer structure than $\forall$ and $\exists$ that can merely return T, F and $\perp$. This is somewhat hypothetical, though, since none of the quantifiers are computable by machine. Furthermore, ε is more natural in the computational world since it gives less bias towards the three particular values of three-valued logic. Also in the set-theoretical world, ε is more convenient than $\forall$ and $\exists$ because of the richness of the structures it returns. This richness is not merely used in proving the axiom of choice. The need for a choice operator pops up in such an unexpected place as the proof of the axiom of comprehension in [18]. This is tightly connected to the scheme of representation of sets introduced in [18].

Let very provisionally $\boxed{\Phi}$ be the set of well-founded objects of MT. Then the axioms express first that quantification is relative to Φ, and second that the quantifiers (including ε) are strict: $\forall f$ is undefined as soon as fx is undefined for some $x \in \Phi$. Once more this choice differs from those quoted in ([10], p. 96) and from [34].

At this point the axioms do not yet ensure that the interpretation of predicate calculus will be satisfactory. In particular Φ could be "finite-like", which would be poor from the logical point of view, and it would be consistent to put $\lambda x.x$ in Φ, which would amount to have a set of all sets (c.f. Section 4.3).

It is the role of the last group of axioms, the "set theory axioms" or "well-foundedness axioms" to give *MT* the power of *ZFC* (including the well-foundedness axiom AF) and, as a side-effect, to ensure that the interpretation of predicate calculus is satisfactory. The price to pay for the entrance in the wide world of Set Theory is that ε and the derived quantifiers $\forall$ and $\exists$ definitely become uncomputable. The quantifiers still have an interpretation in the world of computation. As an example, $\forall f$ computes fx in parallel for all well-founded x and bases its result on the results of all those parallel processes. Since there are as many well-founded maps as there are sets in the universe of *ZFC*, this gives rise to infinite parallelism far beyond what makes sense in the world of computation.

The direct meaning of the well-foundedness axioms, especially that of Well 2, C-M1 and C-M2, is not at all obvious at first sight, but in Section 5.2 it is shown that the inference rule of induction is an approximation to a "Strong Induction Principle" (SIP) and that the other axioms are all special cases of a property that will be referred to as the "Generic Closure Property" (GCP). The "totality theorem" in [18], p.51, defines a large syntax class Σ and states that all terms in Σ denote well-founded maps (well-founded maps were called "total maps" in early versions of [18], which explains the name of the theorem). In the development of *MT*, the totality theorem was stated first, based on the intuition formalised here by GCP, and axioms supporting the totality theorem were formulated afterwards. For that reason, the well-foundedness axioms are just echos of the syntax class Σ, and the individual axioms in general and C-M1 and C-M2 in particular do not make much sense when seen in isolation. It is the hope that the formulation of GCP given in the present paper may help in giving new formulations to the axioms.

1.4 Links to set theory

Concerning the link to usual set theory let us recall from [18] that there are two combinators (closed terms) $\doteq$ and $\dot\in$ which make $(\Phi/\doteq, \dot\in)$ look like a model of *ZFC* ($\doteq$ is an equivalence relation over Φ and $\Phi/\doteq$ is an ordinary quotient). Since *MT* is based on λ-calculus rather than logic, it is not quite enough to give a definition of $\dot\in$ to establish a model. It is also necessary to simulate logic in *MT*, i.e. to give definitions of $\dot\forall$, $\dot\Rightarrow$ and $\dot\neg$ such that $\dot\forall$ quantifies over all sets and such that $\dot\Rightarrow$ and $\dot\neg$ express implication and negation, respectively. Since all sets are represented by well-founded maps and all well-founded maps represent sets, the construct $\forall$ defined in Section 4.2 may serve as $\dot\forall$. The constructs $\dot\Rightarrow$ and $\dot\neg$ are treated in Section 3.4.

The view of set theory in *MT* in general and the definition of $\dot\in$ in particular differs from the traditional translation of set theory into λ-calculus. The idea behind $\dot\in$ is that for any u and v in Φ, $v \neq \mathsf{T}$ (T represents the empty set), we will have that $u \dot\in v$ equals T iff there is an x in Φ such that $u \doteq vx$. This $\dot\in$ differs from $\tilde\in$ defined by $x \tilde\in S = Sx$ which is used e.g. in [34, 1, 11, 12, 13] and corresponds to the view of propositions as functions, which comes back to Frege

[14] and Schönfinkel [31].

For a comparison of $\dot{\in}$ and $\tilde{\in}$ let for a moment $\perp$ represent falsehood, define the 'domain' of a map f as the set of x for which $fx \neq \perp$, and define the 'range' of a map f as the set of fx for which x ranges over all well-founded maps (as an exception, the range of T is empty). With these conventions, $x \tilde{\in} f$ states that x belongs to the domain of f and $x \dot{\in} f$ states that x belongs to the range of f. Hence, using $\tilde{\in}$, a set S is represented by a truth valued map whose domain is S; using $\dot{\in}$, S is represented by a map whose range is S. In this respect, $\dot{\in}$ and $\tilde{\in}$ may be thought of as dual concepts.

In *MT*, $\dot{\in}$ is used to represent set membership, but $\tilde{\in}$ is also used implicitly a few places. As an example, a map x is well-founded iff $x \tilde{\in} \phi = \mathsf{T}$. In the present paper, however, we write ϕx instead of $x \tilde{\in} \phi$, e.g. in Appendix C.

The proof that $(\Phi/\dot{=}, \dot{\in})$ forms a model of *ZFC* is indeed a difficult theorem in [18]. Actually, looking at the axioms of *MT* it is far from obvious that *MT* contains *ZFC*. This is so because *MT* is really *based* on λ-calculus and, even though it has the power of *ZFC*, it is fundamentally different and distant from *ZFC*.

As noted in Appendix A.3, if SQND holds, then $(\Phi/\dot{=}, \dot{\in})$ is a model of *ZFC* in the traditional sense, and if SQND does not hold, then the model may have some pathological properties. It should be noted that it is still open whether or not *MT* is strictly stronger than *ZFC* even though $(\Phi/\dot{=}, \dot{\in})$ is a model of *ZFC* inside *MT* in some sense.

The relation between *MT* and *ZFC* is somewhat like the relation between different programming languages. One could think of *MT* as the machine language (foundation) and *ZFC* as a specialised high level language particularly suited to deal with non-computable aspects of mathematics. In this view, *ZFC* can be compiled into *MT* by replacing $\in$, $\forall$, $\Rightarrow$ and $\neg$ of *ZFC* by $\dot{\in}$, $\forall$, $\dot{\Rightarrow}$ and $\dot{\neg}$ defined in *MT*. Compilation the other way is also possible, just more difficult, and such a "compilation" from *MT* to *ZFC* (actually *ZFC*+ *SI*) is exactly the contents of the present paper. The aim of the present paper is to define each primitive construct of *MT* in *ZFC*+ *SI* in a way that is faithful to the intuitions behind *MT*, and to prove that the constructs so defined form a model of *MT*.

1.5 Description of the paper

As is clear when reading [18], *MT* has been designed from semantic intuitions (based on computational requirements), such as the principle that maps should be monotonous for some partial order. As a matter of fact, *MT* is the equational approximation of this semantic view.

The aim of this paper is to show that (a variation of) Scott's denotational semantics is indeed adequate to realise all the semantic ideas behind the (present) axiomatisation of *MT*. More precisely that one can find a model M of *MT*, in the spirit of denotational semantics, inside every model of *ZFC*+ *SI*, where *SI* asserts the existence of an inaccessible cardinal. We will thus get a semantical consistency proof of *MT* which is conceptually much simpler than the more syntactic one in [18]. The model M, whose elements are called *maps*, will in particular model usual λ-calculus. The model will also satisfy the *Strong Quartum Non Datur* (*SQND*) which asserts that any map is true ($= \mathsf{T}$), bottom ($= \perp$) or false (equal to some term of form $\lambda x.\mathcal{A}$). This, together with the fact that T should be incompatible with any proper map in any non-trivial model, leads us to solve the recursive domain equation:

$$\mathcal{D} \cong [\mathcal{D} \to \mathcal{D}] \oplus_\perp \{\mathsf{T}\} \tag{1}$$

in a suitable Cartesian closed category (ccc) of domains. Here $\{\mathsf{T}\}$ is the trivial domain which has T as unique element and $[\mathcal{D} \to \mathcal{D}]$ is the domain of morphisms from $\mathcal{D}$ to $\mathcal{D}$. Domains are in particular partially ordered sets (p.o's) with a least element (called $\perp$) and $\oplus_\perp$ means that we take as resulting p.o the disjoint union of the two p.o's and add a (new) common least element below. This equation can be solved, for example, in the ccc of Scott domains with continuous functions, by taking $\mathcal{D}$ to be the inverse limit of a suitable projective system of adequate domains, a method due to Scott [32] and well understood now, c.f. [4], p.477.

This is the starting point for the model construction. Now we have to interpret in $\mathcal{D}$ the constants ϕ and ε, which are subject to axioms which prevent them from being continuous. The reason why ε cannot be continuous is linked to the fact that $\forall$ represents a very strong form of parallelism. Since maps are intended to act as monotonous functions we are lead to introduce weaker notions of continuity.

In Section 2 we present the κ-denotational semantics (κ any regular cardinal) and the ccc of κ-cpos and κ-continuous functions, which is a generalisation of Scott's one (which is the case $\kappa = \omega$): the κ-continuous functions are those monotonous functions which commute with all sups of κ-directed sets. In a sense this is a straightforward generalisation of the ω-case to any regular κ. We do make a precise presentation anyway in order to make the paper accessible to the readers interested in foundations, but with no knowledge of domain theory and lambda calculus. A second reason why making a precise exposition is necessary is that a weaker notion with the same name, which requires only commutation with sups of κ-*chains*, also occurs in the literature (e.g. [27, 15]). For "small" domains the two notions coincide (c.f. the remark at the end of Section 2.2). Weak ω_1-continuity was introduced by Plotkin (as mentioned already in [35]) to model countable non-determinism (in small models). The present choice is more convenient for general treatments. It coincides with that of [11, 13] (which was introduced also for consistency purposes), and with the case $\lambda = 0$ of the κ-λ-topologies mentioned in [28].

In Section 3 we define "κ-continuous premodels" as the solutions of (1) in the κ-ccc and show that, modulo the obvious interpretations of the constants $\perp$, T and if, they satisfy the λ-calculus axioms of *MT* and the QND inference rule (see Appendix C for the list of λ-calculus axioms).

The core of the paper is to show that any κ-continuous premodel may be expanded to a model of *MT*, provided there is some inaccessible σ such that $\sigma < \kappa$ (as an example we may take $\kappa = \sigma^+$). This is done in Sections 4 to 8 where we show that it is possible to choose a κ-open set Φ in such a way that the characteristic function ϕ of Φ and some adequate choice function ε over Φ are suitable interpretations of the remaining constants of MT.

In Section 4 we show that it is easy to satisfy the first-order predicate calculus axioms of MT (c.f. Appendix C), even without assuming the existence of an inaccessible ordinal. As noted in Section 4.3, the first-order predicate calculus axioms in MT do not in themselves ensure a faithful representation of first-order predicate calculus since they allow the domain Φ of quantification to be finite.

In Section 5 we introduce two semantic conditions on Φ, namely the *Strong Induction Principle* (*SIP*) and the *Generic Closure Property* (*GCP*), and show that their satisfaction implies satisfaction of the well-foundedness axioms of MT (c.f. Appendix C). Elements of Φ will be called *well-founded maps*.

To express SIP and GCP, we first introduce some auxiliary concepts:

- Let M be a premodel and recall that elements of M are called "maps".
- Let G and H be arbitrary sets of maps (i.e. let $G, H \subseteq M$). Elements of G well be referred to as "G-maps" in the following.
- A map f is said to be *well-founded w.r.t. G* iff, for any infinite sequence $x_1, x_2, \ldots$ of G-maps there exists an n such that $f x_1 x_2 \cdots x_n = \mathsf{T}$.
- The *dual* of G, $\boxed{G^\circ}$, is the set of maps that are well-founded w.r.t. G.
- SIP simply asserts $\Phi \subseteq \Phi^\circ$.
- $G \to H \equiv \{f{\in}M \mid \forall x{\in}H\colon fx \in G\}$ is the arrow operator attributed to Scott e.g. by [9].
- GCP asserts that Φ equals the union of $G^\circ \to \Phi$ over a "suitable" collection of G's, where "suitable" has something with limitation of size to do. The exact formulations are given in Section 5

The axiom of induction in Appendix C is an approximation of SIP whereas all the other set theory axioms in Appendix C together form an approximation of GCP. Section 5.2 verifies that all these axioms follow from SIP and GCP.

It remains to prove the existence of a Φ satisfying SIP and GCP in any κ-premodel. A prerequisite for this is to develop the mathematical properties of the dual and the arrow operators with particular emphasis on the nature and size of $G^\circ \to G$. This development is done in Section 6, via the study of a further notion from [18], namely that of the *type over G* of any element u of M, denoted $t(u/G)$ or u^G. The notions of duality, type and GCP are closely linked to the intuitive (and purely semantic) notion of the *inner range* of a map. In fact the GCP expresses that the elements of Φ are exactly those maps f which admit an inner range H of cardinality less than the inaccessible σ (i.e. there is a small $H \subseteq \Phi$ such that, for any x in Φ (or H°), $t(fx/\Phi)$ only depends on $t(x/H)$; a concrete example is given in Appendix A.3).

This notion of type, which clearly has nothing to do with the usual notions of types in typed λ-calculus, may on the other hand be related to the various notions of types which occur in (general) Model Theory (c.f. [7, 30]) and in the model theoretic study of algebraic structures. The type of u over H may be seen indeed as the set of formulas with parameters in H and of a given shape, which are satisfied by u in M.

Section 7 proves that if there is an inaccessible ordinal below κ, then any κ-premodel contains a κ-open set Φ which satisfies SIP and GCP. We prove first the existence of a set Ψ of κ-compact elements which satisfies properties analogous to those of Φ but is more accessible to fixed point arguments, and take for Φ the open subset generated by Ψ. Finally we prove in Appendix A.1 some further properties of Φ concerning the size of Φ and embeddings of *ZFC* into Φ. In particular we prove that for the models presented in this paper (defined inside *ZFC+SI*), there is an isomorphism (in *ZFC+SI*) between $(\Phi/\doteq, \dot{\in})$ and $(V_\sigma, \in)$, and we support the conjecture that *MT* is stronger than *ZFC* by showing that all models of *MT* which satisfy SQND interpret a model of *ZFC*, and with a different and direct proof that our κ-continuous models interpret an ω-model of *ZFC*.

It remains to exhibit a κ-continuous premodel, that is a κ-domain which is a solution of (1) in the κ-ccc. Furthermore we want to get a model as easy to handle as possible (for further investigations of *MT*).

A classical way to solve recursive domain equations in the ω-case is to use Scott's inverse limit construction. The problem with this method is that if $\kappa > \omega$ there are limit ordinals $\alpha < \kappa$. This adds theoretical difficulties (as pointed out in [15]), and considerably increases the technical complexity of the construction even if one succeeds in writing out a direct construction. Finally it is always technically difficult to work concretely with models which are presented as inverse limits, since the notations are very heavy, especially in the κ-case.

There is a way to overcome all these difficulties: it is to work only with domains $\mathcal{D}$ uniformly built from "webs" D (which are simple relational structures of a fixed signature), and to replace the inverse limit construction on domains by an increasing union of webs. Not only do we get a presentation of the models which is now much more easy to handle, even in the ω case, but it is now almost trivial to deal with limit ordinal stages.

This general method of solving recursive equations on domains is presented in [25] for ω-Scott domains; there the webs are Scott's information systems [36].

The most simple classes of webbed-domains where one can solve recursive equations are Girard's coherent spaces [16], and Krivine's spaces of initial segments [23, 24]. In the first case the webs are of the shape $(D, \sim)$ where D is a set and $\sim$ is a reflexive and symmetric relation, and the domain $\mathcal{D}$ is the set of "coherent subsets" of D, namely those subsets whose elements are pairwise related by $\sim$. In the second case the webs are preordered sets $(D, \leq)$ and $\mathcal{D}$ is the set of "initial segments" of D, namely those subsets which are downward closed.

There is however no solution of (1) within Girard's or Krivine's spaces since the first class is not closed under lifting and the second consists only of complete lattices, but by merging these two classes we obtain a third simple class where (1) can be solved. All three classes may be viewed as particular cases of information systems, since even when working on coherent spaces it is the continuous semantics we are interested in here, not the stable one. Note however that there seems to be no problem in giving a κ-stable semantics to the present axiomatisation of *MT*, along the lines of the present paper.

In Section 8 we introduce the class of preordered coherent spaces, which will be the webs of our domains, and give an explicit construction of a solution of the κ-version of (1) in this class, the simplest one in fact.

Appendix A states further properties of κ-continuous models and studies models of *ZFC* inside such models. Appendix B proves that the models defined in this paper are faithful to the computational aspect of *MT*. Appendix C gives a summary of *MT*. An index is included after Appendix C.

In summary, the sections of this paper serve the following purposes:

Section 8 proves the existence of solutions to (1) in the κ-ccc.

Section 7 proves that for any such solution there is a Φ that satisfies SIP and GCP.

Section 5 proves that any Φ which satisfies SIP and GCP also satisfies the set theory axioms in Appendix C.

Section 4 proves that any Φ whatsoever satisfies the predicate calculus axioms in Appendix C.

Section 3 proves that any solution to (1) satisfies the remaining axioms of Appendix C.

Section 2 and 6 give necessary background.

Appendix A states further properties κ-continuous models and studies models of *ZFC* inside such models.

Appendix B proves that the models defined in this paper are faithful to the computational aspect of *MT*.

Appendix C gives a summary of *MT*.

The order of presentation is chosen to give a natural progression. A bottom up presentation may be found in the first version of the present paper [6].

1.6 Comparison with Flagg-Myhill's system

Introduction

We will end this introduction with some elements of comparison between our work and that of [13] which was pointed out by one of the referees. The system *EFL** of [13] and *MT* were designed (independently) from very different points of view and behave very differently on the syntactical level, but they both aim at combining *ZFC*-power with λ-calculus, and the consistency proofs of both systems are formulated in similar κ-frameworks.

The two systems are different in spirit in several places: *EFL** is more syntactical of nature whereas *MT* is more semantical; the role of λ-calculus is different; and *MT* has a stronger computational motivation of concepts.

Though the two systems are obviously very distant there are, at least, two reasons why we can be interested in a comparison:

A clear common point is the introduction of the κ-continuous framework by Flagg-Myhill and by us for consistency purposes. But this is rather superficial since in both cases it was naturally introduced to keep what could be kept of Scott's continuous semantics.

The real point is to trace some resemblance at a deep level. Indeed the syntactical tools used by Flagg an Myhill to inject Set Theory at the level of axioms, have echos in semantic technical tools which are used in intermediate steps in our consistency proof (c.f. the section labelled "monotonicity" below).

A brief survey of Flagg-Myhill's system

System *EFL** is the third in an increasing sequence of 4 systems: *FL* $\subseteq$ *EFL* $\subseteq$ *EFL** $\subseteq$ *EFL***, which is issued from Frege's work and [1]. The consistency of *FL* is essentially due to Aczel and is rather similar to the consistency result that already appears in [34]. System *EFL* adds to *FL* a comprehension rule for discrete classes and contains second-order arithmetic with full comprehension scheme. System *EFL** adds, at the level of constants and axioms, a well-ordering of the universe and an inaccessible cardinal; there is a syntactic translation of *ZFC* in *EFL**. Finally, *EFL*** adds to *EFL** the requirement that "discrete classes are closed under direct images". The consistency of *EFL*** was left open by Flagg and Myhill, but was reduced via a general model-theoretic argument to a question of Friedman that Plotkin solved in his recent preprint [29], namely that of the existence of a model of $\lambda\eta$-calculus in which all finite sets are separable.

The consistency proofs in Flagg-Myhill, Aczel and Scott [1, 11, 12, 13, 34], follow the following pattern: take a model $\mathcal{P}$ of λ-calculus, then interpret all constants in the model in a semi-Gödel-like fashion ("semi" takes into account that the interpretation of quantifiers is done via λ-abstraction); then define by mutual induction two disjoint predicates of Truth and Falsity ($\mathcal{T}$ and $\mathcal{F}$, on $\mathcal{P}$, in such a way that the interpretations of all provable terms are true, and finally prove that it is not possible that A and $\neg A$ are simultaneously true. Such a triple $(\mathcal{P}, \mathcal{T}, \mathcal{F})$ was called a *Frege structure* by Aczel. With the exception of *EFL***, which needs Plotkin's model, all $\mathcal{P}$'s are chosen as Scott's solution of $\mathcal{P} \cong [\mathcal{P} \to \mathcal{P}]$ in the ccc of complete lattices and continuous functions (κ-continuous functions for *EFL**).

From now on we will only be concerned with *EFL**.

Syntactic versus semantic nature

The following observations support the informal assertion that *EFL** is more syntactic and *MT* more semantic of nature; some of them will be elaborated below.

1. Most logical and set theoretic concepts are primitive constants in *EFL**, while they are defined concepts in *MT*.
2. Their behaviour is axiomatised in a natural deduction style, while *MT* is an equational extension of $\cong_\beta$ (where the extension has a semantic definition rather than one based on conversion, c.f. the section titled "extensionality" below).
3. Models of *EFL** (Frege structures) are based on models of λ-calculus but need furthermore Truth and Falsity predicates, which are defined by mutual ordinal induction following the structural rules of the system.
4. The interpretation of *EFL** terms in Frege structures relies on a semi-Gödel-like encoding of constants, while the interpretation of (the few) *MT* constants follows semantic intuitions.
5. *EFL** admits monotonous (even κ-continuous) models, but monotonicity is of no use in *EFL**. Indeed relevant monotonicity is ruled out in *EFL** at the level of axiomatisation. Also Truth (and Falsity) are non-monotonic concepts in the semantics of *EFL**, while they are monotonic in that of *MT*.
6. Finally one can also say that extensionality has a more semantic and deeper meaning in *MT* than in *EFL**.

The language of *EFL**

The terms of *EFL** are those of λ-calculus with the following primitive constants added: $=$, N, P, $\wedge$, $\vee$, $\Rightarrow$, $\forall$, $\exists$, $\prec$, and κ. Note that $=$ is really a term in *EFL**. The intended meaning of the non-logical constants is: N is the class of integers, P the class of propositions, $\prec$ a well-ordering of the universe, and κ is an inaccessible cardinal. The behaviour of constants is axiomatised by rules in a natural deduction style. In what follows, 0 and 1 are Church integers.

Derivability and provability

*EFL** derives or "proves" terms. Let us call all terms of form $= AB$ "term-equations". We denote them $[A = B]$ in infix notation. Similarly there are (term-)inequations $[A \neq B]$, where $\neq$ is the term $\lambda x.\lambda y.([x = y] \Rightarrow [0 = 1])$. Thus *EFL** is able to prove term-equations $[A = B]$ as well as term-inequations $[A \neq B]$; in the first case we will say that A and B are provably equal and in the second that they are provably unequal.

MT derives equations $A = B$ between terms of *MT*, but neither $=$ nor $A = B$ are terms in *MT*, and inequations and contradictions do not belong to the scope of *MT*. Among others, *MT* is able to derive equations of the form $A = \mathsf{T}$; one will say that the corresponding A's are the "provable" or "provably true" terms of *MT*.

It is obvious from this latter definition (by the transitivity of $=$ in *MT*) that any two provable terms of *MT* are provably equal.

By way of contrast it is easy to find, as follows, two *EFL**-terms A and B which are provable, and provably unequal:

Given any provable term A, e.g. $[0 = 0]$, we may choose any term B such that the term-equation $[B = [A \neq B]]$ is provable: since $\neq$ is a definable term of *EFL**, and since $=$ is an internalised $\beta\eta$-equivalence, such a term can be found via a fixed point combinator. It is rather easy to show from the rules of *EFL** that B and $[A \neq B]$ are provable.

The role of λ-calculus in *EFL**

Besides the management of logical substitution via the encoding of quantifiers in a semi-Gödel encoding way, λ-calculus is used for the definition of class-membership, in the traditional way recalled in Section 1.3, and for class formation.

As already mentioned, there exists a syntactic translation of *ZFC* into *EFL**. There, set-membership is a defined concept as in *MT*: two terms $\in^V$ and $=^V$ are defined as double fixed points of *EFL**-combinators (c.f. [13], p.89). However, the idea underlying $\in^V$ is the traditional view of set-membership as application and, hence, is dual to that of *MT*.

The formalisation of set membership in *EFL** is much heavier than in *MT* since the ordinal inductive definition of V (including a name for κ) is part of the definition of $\in^V$ and $=^V$. The role of V in *EFL** is similar to the role of Φ in *MT*.

Computational motivation

All terms in *MT* have a computational motivation. The computational motivation of λ-abstraction, functional application, T and if are obvious as these constructs are directly implementable on machine. The computational motivation of $\perp$ is also obvious in the sense that it represents infinite looping. Machine implementations of λ-abstraction, functional application, T and if are "complete" in the following sense: If A is a term built up from the above then $A = \perp$ in the canonical models built in Section 8 iff computation of A never ends (c.f. Appendix B). Furthermore, if $A \neq \perp$ then computation of A will in finite time determine whether or not $A = \mathsf{T}$ (i.e. whether or not A is "true"). The remaining constructs ϕ and ε of *MT* are not computable by machine but their properties are still motivated in a computational setting. Both ϕ and ε can be thought of as parallel operators that start up infinitely many processes in parallel. This is of course impossible on a finite computer, but still motivates the properties assigned to these operators.

In particular, the semantics of all terms of *MT* are born monotonic.

In contrast, *EFL** has several constructs whose semantics is alien to computation, to begin with $=$ and N which are interpreted by total predicates in Frege structures, which makes it impossible to them to be monotonic.

It is worth noting that λ-calculus plays a very active role in *MT* and that surprisingly many concepts can be represented by computable functions (i.e. without using ε and ϕ). As an example, the set ω of positive integers is represented by a computable function in A.3. Subsets of ω are representable by computable functions iff they are recursively enumerable. Also some sets of larger cardinality such as the power set of ω is representable by computable functions. In [18], the union set axiom is proved using a computable function as union set operator, and a computable power set operator is deviced though it is not used in the actual proof of the power set axiom.

The role of monotonicity

As a matter of fact monotonicity is of no real use for *EFL**; this is confirmed by the fact that *EFL** can be modelled by a Frege structure based on Plotkin's model [29], which is anti-monotonic by essence (any non-trivial partial order on it contradicts the monotonicity of application since any two elements of the model are separable, which implies that they can be exchanged by a representable function). In fact, monotonicity is ruled out already at the level of the axiomatisation of *EFL**, since discrete classes, which are the basic concept of the set theoretical axioms, can only be interpreted by separable sets of elements, and hence in any monotonous semantics, by sets of incompatible elements.

There is some flavour of this too in our semantics of *MT* since the $\delta(G^\circ)$, which are also sets of incompatible elements, happen to be important tools. However they do not appear at all at the level of syntax, even in an implicit way. Furthermore, at the semantic level they appear only as tools in the proof that there exists an open set which is a solution of the GCP. The same remark applies to our strict arrow $\rightarrow_G$, which can be viewed as the semantic *MT*-counterpart of the syntactic *EFL**-arrow $\rightarrow^*$.

Extensionality

Equality in *ZFC* is semantically defined thus: Two sets are equal if they contain the same elements. In other words, two sets A and B are equal if the truth value of $x \in A$ equals the truth value of $x \in B$ for all x. This is often referred to as extensionality. The important thing to note here is that equality of sets is defined from the simpler concept of equality of truth values.

Equality in *MT* is defined semantically in much the same way, but the definition is complicated by two things: first, *MT* includes a third truth value $\perp$, second, *MT* treats truth values as defined rather than as fundamental concepts. In the following, a map will be said to have the truth value "true" if it equals T, "undefined" if it equals $\perp$, and "false" in all other cases. Two maps U and V will be said to satisfy $U \leftrightarrow V$ if they have the same truth value. Now, two sets A and B were equal if

$$x \in A \Leftrightarrow x \in B$$

for all sets x. Similarly, two maps f and g are equal if

$$fx_1 \cdots x_n \leftrightarrow gx_1 \cdots x_n$$

for all $n \geq 0$ and all maps $x_1, \ldots, x_n$. In conclusion, equality in *MT* is conceptually based on a semantic notion of extensionality.

In λ-calculus and *EFL** there is also a property called extensionality, namely the property that if $Ax = Bx$ for all x then $A = B$. This holds both in $\lambda\eta$-calculus and in *EFL** (and it almost holds in *MT*; in *MT* it is necessary to assume that A and B differ from T and $\perp$). This kind of extensionality is different from that of *ZFC*, however. The extensionality of *ZFC* links equality of sets with the simpler concept of equality of truth values, and thereby defines equality of sets from a simpler concept. The extensionality of λ-calculus and *EFL** links function equality with function equality itself, so this kind of extensionality does not define function equality from a simpler concept.

The equality in *EFL** resembles the syntactic β-equivalence more than the semantic *ZFC*-equality. One place this shows up is in the example with the provable terms A and B in *EFL** which are provably unequal.

In *EFL**, the second kind of extensionality is axiomatised by $Ax = Bx \vdash A = B$. In *MT*, it is axiomatised by $A = B \vdash \lambda x.A = \lambda x.B$. In $\lambda\eta$-calculus, it is typically axiomatised by one of these two formulations. The present paper considers the version of *MT* presented in [18], and in that version there is no formalisation of the first kind of extensionality. The first kind of extensionality has been formalised in [17].

Final remark

That the intended semantics of *MT* is monotonous from the beginning does not rule out the possibility that *MT* could admit a non-monotonic one. In particular one could ask whether the methods of Plotkin [29] could be exploited to provide a model of *MT*, as they can be to give a model to Flagg-Myhill's systems. The QND inference rule is the first obstacle to be passed (and might be the only one), while the stronger extensionality rule of *EFL** gives for free a Church-Rosser conversion underlying λ-calculus, namely $\lambda\eta$-conversion.

2 The κ-denotational semantics

From now on κ is a regular cardinal $\geq \omega$ (c.f. [7] for definitions of "regular" and "inaccessible"). For any cardinal χ, χ-*small* will mean: non-empty and of cardinality strictly less than χ; we will only use it for $\chi = \kappa$ and $\chi = \sigma$ where σ is an inaccessible below κ. $\alpha, \beta, \ldots$ denote ordinals. (Note that all concepts written in italics occur in the index).

$\boxed{\mathcal{P}_\chi(E)}$ will denote the set of χ-small subsets of the set E; if χ is regular, $\mathcal{P}_\chi(E)$ is closed under unions of χ-small families.

A p.o $\mathcal{D}$ is a partially ordered set, $(D, \leq)$. We use $x \in \mathcal{D}$ and $x \subseteq \mathcal{D}$ as shorthand for $x \in D$ and $x \subseteq D$, respectively. Elements and subsets of $\mathcal{D}$ will be denoted by the letters u, v and A, B, U, G, H, respectively. A set of *compatible* elements is just a subset of $\mathcal{D}$ which has an upper bound in $\mathcal{D}$, such a set is also called *consistent* or a *bounded* subset of $\mathcal{D}$ in the literature.

Further notations: $\boxed{\uparrow u}$ means $\{v \mid v \geq u\}$, $\boxed{\downarrow u}$ means $\{v \mid v \leq u\}$, $\boxed{\uparrow G}$ means $\bigcup\{\uparrow u \mid u \in G\}$ and $\boxed{\downarrow G}$ means $\bigcup\{\downarrow u \mid u \in G\}$. (Note that all boxed entities occur in the index).

2.1 κ-cpo's and κ-continuous functions

$A \subseteq \mathcal{D}$ is *κ-directed* if $A \neq \emptyset$ and every κ-small $B \subseteq A$ is bounded by an element of A. A is a *strict κ-directed* set if moreover it has no maximal element (note that a κ-directed set has at most one maximal element).

$\mathcal{D}$ is a *κ-cpo* (a 'κ-complete p.o') if it has a bottom (denoted $\boxed{\perp_{\mathcal{D}}}$, or simply $\perp$ if there is no ambiguity) and every κ-directed A has a sup; it is a *κ-ccpo* (for 'consistently κ-complete p.o') if moreover every bounded A has a sup. It follows immediately from the definition that in a κ-ccpo every non-empty set A has an inf.

The *κ-topology* is the topology over $\mathcal{D}$ whose open sets are the subsets U of $\mathcal{D}$ such that (i) $U = \uparrow U$ and (ii) $\sup A \in U$ implies $A \cap U \neq \emptyset$, for all κ-directed A.

Fact 2.1.1 Any intersection of a κ-small family of open sets is open.

Thus, the ω-topology is the usual Scott topology. The κ-continuous functions that we are going to define now are exactly the continuous functions for the κ-topology.

Let $\mathcal{D}$ and $\mathcal{E}$ be κ-cpos, then $f : \mathcal{D} \to \mathcal{E}$ is *κ-continuous* iff $f(\sup A) = \sup f(A)$, for all non-empty κ-directed A; $[\mathcal{D} \to \mathcal{E}]_\kappa$ will denote the space of all κ-continuous functions on $\mathcal{D}$, endowed with the pointwise ordering of functions, while $\mathcal{D} \times \mathcal{E}$ is the Cartesian product (with coordinate-wise partial ordering). It is easy to see that both are κ-cpo's, and that, as in the ω-case, we are working here with a 'Cartesian closed category (ccc) with enough points', in particular, for any three cpos $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ the canonical function from $[\mathcal{D} \times \mathcal{E} \to \mathcal{F}]_\kappa$ to $[\mathcal{D} \to [\mathcal{E} \to \mathcal{F}]_\kappa]_\kappa$ is a κ-isomorphism, i.e. a bijective κ-continuous function the inverse of which is κ-continuous too. Note that a κ-isomorphism is nothing more than an order-isomorphism between cpos. A few categorical words will be employed, either for ease of terminology, or to make links with some standard framework, but no knowledge of category theory is really needed here.

Remarks:

1. $f : \mathcal{D} \to \mathcal{E}$ is κ-continuous iff it is monotone and $f(\sup A) \leq \sup f(A)$ for all non-empty κ-directed A.

2. If $\kappa \leq \kappa'$ and $\mathcal{D}$ is a p.o, then all κ'-directed $A \subseteq \mathcal{D}$ are κ-directed, so, if $\mathcal{D}$ is a κ-cpo, then $\mathcal{D}$ is a κ'-cpo, and κ-continuous functions of $\mathcal{D}$ are κ'-continuous; in particular ω-continuous functions are κ-continuous for

all κ. This has to be contrasted with the framework of [27, 15], where ω- and ω_1-continuity are independent notions.

3. If $\mathcal{D}$ has no strict κ-directed subset, which is in particular the case if $\mid \mathcal{D} \mid < \kappa$, then the κ-continuous functions are exactly the monotone functions.

The notion of a premodel of map theory that we will define in Section 3 will rely on the more restricted class of κ-Scott domains. This will enable us to keep a control on the width of the open subsets needed to model ϕ, and on ϕ itself. This limited size of ϕ will in turn enable us to model ε; we will also need sups of bounded subsets. So the notions of κ-compact elements, and of κ-algebraic and κ-Scott domains, are essential for our purpose. This is not the case for that of prime elements, κ-prime algebraic domains, and traces of κ-continuous functions, which, from a purely deductive point of view, could as well have been omitted; their presence below is due to the fact that they enlighten the construction of the premodel in Section 8 (which is indeed a κ-prime algebraic domain), and, of course, as their classical analogues, they are basic tools for further developments.

2.2 κ-compact elements and κ-Scott domains

A *κ-compact* element of a κ-cpo $\mathcal{D}$ is an element $u \in \mathcal{D}$ such that, for all κ-directed A, $u \leq \sup(A)$ implies $u \leq v$ for some $v \in A$ (thus the κ-compact elements are exactly the elements u of D such that $\uparrow u$ is open). $\boxed{\mathcal{D}_c}$ is the set of κ-compact elements and $\boxed{\downarrow_c u} = \downarrow u \cap \mathcal{D}_c$. We will constantly use the following easy fact:

Fact 2.2.1 $\mathcal{D}_c$ is closed under sups of κ-small subsets.

Note that we do not intend here that such sups always exist.

A cpo $\mathcal{D}$ is *κ-algebraic* if, for all $u \in \mathcal{D}$, $\downarrow_c u$ is κ-directed and $u = \sup \downarrow_c u$. As in the ω-case we have:

Fact 2.2.2 If $\mathcal{D}$ is κ-algebraic, then f is κ-continuous iff $f(u) = \sup f(\downarrow_c u)$.

Fact 2.2.3 If $\mathcal{D}$ is κ-algebraic, then the following are equivalent for any subset G of $\mathcal{D}$:

$$G \text{ is open} \tag{2}$$

$$G = \uparrow G \text{ and } \forall u{\in}G : G \cap \downarrow_c u \neq \emptyset \tag{3}$$

$$\exists H{\subseteq}\mathcal{D}_c : G = \uparrow H \text{ (so } H \subseteq G \cap \mathcal{D}_c \text{ and } G = \uparrow(G \cap \mathcal{D}_c)) \tag{4}$$

Moreover, even if the cpo $\mathcal{D}$ is not algebraic, G is open iff, for all fixed $v \neq \bot$

$$\chi_G \text{ is continuous,} \tag{5}$$

where $\chi_G : \mathcal{D} \to \mathcal{D}$ is the function which takes value v on G and $\perp$ elsewhere.

A κ-ccpo $\mathcal{D}$ is a *κ-Scott domain* iff for all $u \in \mathcal{D}$, $u = \sup \downarrow_c u$.

Since in any κ-ccpo $\downarrow_c u$ is κ-directed (and is strictly directed iff $u \notin \mathcal{D}_c$), the κ-Scott domains are exactly the κ-algebraic κ-ccpo's. κ-Scott domains are closed under products or spaces of κ-continuous functions, and form also a ccc.

Remark. If $\mathcal{D}$ is κ-algebraic and if $|\mathcal{D}_c| \leq \kappa$, then every strict κ-directed $A \subseteq \mathcal{D}$ has a cofinal κ-chain (of compact elements) and in such a $\mathcal{D}$ it is enough to define κ-continuity via a commutation with sups of κ-chains. Here a κ-chain S is a monotone sequence indexed by κ, and S is cofinal to some κ-directed A if $A \subseteq \downarrow S$ and $\sup S = \sup A$ (we do not ask for $S \subseteq A$). Such a presentation has been chosen e.g. in [24] (ω-topology) and [27] (ω_1-topology). The model we build in Section 8 (κ-topology) satisfies also this strong hypothesis.

2.3 Pointwise sups and infs

A step in proving that $[\mathcal{D} \to \mathcal{E}]_\kappa$ is a κ-Scott domain if $\mathcal{D}$ and $\mathcal{E}$ are is to show:

Lemma 2.3.1 Let $\mathcal{D}$ and $\mathcal{E}$ be κ-Scott domains. Then the pointwise sup of any bounded subset B of $[\mathcal{D} \to \mathcal{E}]_\kappa$ is κ-continuous and hence is the sup of B in $[\mathcal{D} \to \mathcal{E}]_\kappa$.

The analogue for infs is:

Lemma 2.3.2 If B is a κ-small non-empty subset of $[\mathcal{D} \to \mathcal{E}]_\kappa$ then the pointwise inf, $\inf B$, of the elements of B is κ-continuous, and thus is the inf of B in $[D \to E]_\kappa$.

Proof. Let $f = \inf B$ and $a = \sup A$ where A is a κ-directed subset of $\mathcal{D}$. We have to prove that $f(a) \leq \sup\{f(v) \mid v \in A\}$; and for this it is sufficient to prove that for any κ-compact $u \leq f(a)$ there is a $v \in A$ such that $u \leq f(v)$. Now $u \leq g(a)$ for all $g \in B$; since the g's are continuous there are $v_g \in A$ such that $u \leq g(v_g)$; now, the set of all v_g's is κ-small, hence bounded by a $v \in A$; by monotonicity of the g's we have $u \leq g(v)$ for all g, hence $u \leq f(v)$. $\diamondsuit$

2.4 κ-prime algebraic domains

A *prime* element is an element $u \in \mathcal{D}$ such that, for all bounded A, $u \leq \sup A$ implies $u \leq v$ for some $v \in A$. Thus prime elements of κ-cpo's are κ-compact, and $\perp$ is prime. Exercise: An element $p \in \mathcal{D}$ is prime iff it is κ-compact and, for all bounded κ-small $B \subseteq \mathcal{D}$, we have $u \leq \sup B \Rightarrow u \leq v$ for some $v \in B$.

Remark. Prime elements need not be minimal in $\mathcal{D} - \{\perp\}$, nor incomparable.

Notations. $\boxed{\mathcal{D}_p}$ is the set of prime elements of $\mathcal{D}$, and $\boxed{\downarrow_p u} = \downarrow u \cap \mathcal{D}_p$.

Definition. A κ-Scott domain is *κ-prime algebraic* if for all $u \in \mathcal{D}$, $u = \sup \downarrow_p u$.

It is easy to see that in κ-prime algebraic domains the κ-compact elements are exactly the sups of the κ-small bounded sets of prime elements. (The non-trivial direction uses the fact that if $\mathcal{D}$ is κ-Scott and $A \subseteq \mathcal{D}$ is bounded, then $\sup A = \sup d(A)$ where $d(A)$ is the κ-directed set whose elements are the sups of κ-small families of elements of A).

Preordered coherent spaces (pcs's), as defined in Section 8, will convey the significant part of the structure of the $\mathcal{D}_p$'s associated with simple κ-prime algebraic domains $\mathcal{D}$.

Remarks. If $\kappa \leq \kappa'$ and if $\mathcal{D}$ is a κ-cpo then $\mathcal{D}$ is a κ'-cpo, as we have already seen. Now κ-compact elements of $\mathcal{D}$ are obviously κ'-compact and the κ'-compact elements of $\mathcal{D}$ are the sups of κ'-small sets of compatible κ-compact elements. This ensures that the set of κ'-compact elements below some $u \in \mathcal{D}$ is κ'-directed. Hence, if $\mathcal{D}$ is κ-Scott, then $\mathcal{D}$ is κ'-Scott. In particular, for every κ, every compact element is κ-compact, every Scott domain is a κ-Scott domain and every prime-algebraic domain is a κ-prime algebraic domain.

2.5 Traces of continuous functions

Suppose that $\mathcal{D}$ is a κ-Scott-domain. From the fact that any function $f : \mathcal{D} \to \mathcal{D}$ is indeed a graph and that $\mathcal{D}$ is κ-algebraic, we have that any function f is determined by:

$$T_1(f) = \{(u, v) \in \mathcal{D} \times \mathcal{D}_c \mid v \leq f(u)\}$$

If f is κ-continuous, and since the v's are compact, it is sufficient to consider:

$$T_2(f) = \{(u, v) \in \mathcal{D}_c \times \mathcal{D}_c \mid v \leq f(u)\}$$

If moreover $\mathcal{D}$ is κ-prime algebraic, then it is sufficient to know:

$$T(f) = \{(u, v) \in \mathcal{D}_c \times \mathcal{D}_p \mid v \leq f(u)\}$$

The pairs (u, v) are in one–one correspondence with the 'step functions' $\varepsilon_{u,v}$ ($\varepsilon_{u,v}(x) = v$ if $x \geq u$ and $\varepsilon_{u,v}(x) = \perp$ otherwise). The trace of $\varepsilon_{u,v}$, ordered by $(u', v') \leq (u'', v'')$ iff $u' \geq u''$ and $v' \leq v''$ (which corresponds to the pointwise ordering of step functions $\varepsilon_{u',v'}$, $\varepsilon_{u'',v''}$), contains (u, v) as maximal element.

2.6 Reflexive κ-cpo's and the interpretation of λ-calculus

As we already mentioned, κ-Scott domains (or κ-cpo's) and κ-continuous functions form a ccc (with 'enough points'), say the κ-ccc. Now, any reflexive object of such a ccc (and not only solutions of (1)) can model pure (i.e. untyped) λ-calculus; let us tell what reflexive means in the case of the κ-ccc (the general definition can easily be extrapolated).

$\mathcal{D}$ is a reflexive object of the κ-ccc if there are two κ-continuous functions:

$$\boxed{A} : \mathcal{D} \to [\mathcal{D} \to \mathcal{D}]_\kappa \tag{6}$$

$$\boxed{\lambda} : [\mathcal{D} \to \mathcal{D}]_\kappa \to \mathcal{D} \tag{7}$$

such that

$$A \circ \lambda = \mathrm{id} \tag{8}$$

In particular λ is injective and A is surjective.

Then there is a standard way to interpret terms of λ-calculus, where parameters in $\mathcal{D}$ are allowed (sketched in Section 3, c.f. also [4], Chapter 5, Paragraph 4), and (8) ensures that any two β-equivalent terms get the same interpretation.

If furthermore $\lambda \circ A = \mathrm{id}$, then $\mathcal{D} \cong [\mathcal{D} \to \mathcal{D}]_\kappa$, the model is called *extensional*, and any two η-equivalent terms get the same interpretation. Solving (1) amounts to finding an *almost extensional* reflexive model of λ-calculus, 'almost' getting here a totally accurate meaning if we take into consideration the two small elementary axioms of map theory which specify the applicative behaviour of $\perp$ and T.

The coding of unary continuous functions in $\mathcal{D}$ by means of λ generalises to n-ary κ-continuous functions as follows: we define $\lambda^n : [\mathcal{D}^n \to \mathcal{D}]_\kappa \to \mathcal{D}$ by induction on n: $\lambda^1 = \lambda$, and $\lambda^{n+1}(f[x_1, \ldots, x_{n+1}]) = \lambda(u_1 \mapsto \lambda^n(f[u_1, x_2, \ldots, x_{n+1}]))$. Indeed we can easily prove, by induction, that λ^n is well-defined and κ-continuous (using that a k-ary function is κ-continuous iff it is component-wise κ-continuous).

Notations. Finite sequences of elements of $\mathcal{D}$ are denoted by $\bar{u}$, $\bar{v}$, etc and the length of $\bar{u}$ by $\ell(\bar{u})$. We will use the following simplified notation of application:

$$\begin{array}{lcll}
uv & = & A(u)(v) & \\
u\bar{v} & = & uv_1 \cdots v_n & \text{if } \bar{v} = (v_1, \ldots v_n) \text{ and } n \geq 1 \\
u\bar{v} & = & u & \text{if } \ell(\bar{v}) = 0 \\
uw\bar{v} & = & (uw)\bar{v} &
\end{array}$$

With this notation, and a repeated use of (8) we get:

Fact 2.6.1 For any $f \in [\mathcal{D}^n \to \mathcal{D}]_\kappa$ and any $\bar{u} \in \mathcal{D}^n$ we have

$$\lambda^n(f)u_1 \cdots u_n = f(u_1, \ldots, u_n)$$

3 κ-continuous premodels

$\tilde{\mathsf{T}}$ and $\tilde{\perp}$ are two objects of the universe (sets), which are not sets of pairs, and hence are not functions (graphs).

Definition. For any κ-Scott domain $(\mathcal{E}, \leq)$ such that $\tilde{\mathsf{T}}, \tilde{\perp} \notin \mathcal{E}$ we denote by $\boxed{\mathcal{E} \oplus_{\tilde{\perp}} \{\tilde{\mathsf{T}}\}}$ the κ-Scott domain $(\mathcal{E}', \leq')$ such that $\mathcal{E}' = \mathcal{E} \cup \{\tilde{\mathsf{T}}, \tilde{\perp}\}$ and $x \leq' y$ iff: $x = \tilde{\perp}$ or $x = y = \tilde{\mathsf{T}}$ or ($x, y \in \mathcal{E}$ and $x \leq y$).

3.1 Premodels

Definition. A *κ-continuous premodel* of map theory, or simply a *premodel*, is a triple $\mathcal{P} = (\mathcal{M}, \tilde{A}, \tilde{\lambda})$ where $\mathcal{M}$ is a κ-Scott domain and

$$\mathcal{M} \underset{\tilde{\lambda}}{\overset{\tilde{A}}{\rightleftarrows}} [\mathcal{M} \to \mathcal{M}]_\kappa \oplus_{\tilde{\perp}} \{\tilde{\mathsf{T}}\}$$

are two inverse order-isomorphisms.

Notation. $\boxed{\perp} \equiv \perp_{\mathcal{M}}$, hence $\perp = \tilde{\lambda}(\tilde{\perp})$; $\boxed{\mathsf{T}} \equiv \tilde{\lambda}(\tilde{\mathsf{T}})$; $\boxed{\mathcal{F}} \equiv \mathcal{M} \setminus \{\perp, \mathsf{T}\}$. Elements of $\mathcal{M}$ are called *maps*, and elements of $\mathcal{F}$ *proper maps*.

The *root function* $\boxed{r} : \mathcal{M} \to \mathcal{M}$ is defined by $r(\perp) = \perp$, $r(\mathsf{T}) = \mathsf{T}$, and $r(u) = \boxed{\mathsf{F}} \equiv \lambda x.\mathsf{T}$ if u is a proper map. Using Fact 3.1.1 below it is easy to prove that r is κ-continuous and that r commutes with all existing sups and with infs of κ-small bounded subsets. The definition of r was a key one in the original consistency proof where r operates on terms of map theory and where the definition of r takes up all of page 95 in [18]. The model construction in the present paper is, structurally, considerably more simple and conceptual than the original one among other because it is based on κ-denotational semantics instead of a syntactic definition of r.

The following fact will be used constantly.

Fact 3.1.1 (a) For all bounded, and hence for all κ-directed, $B \subseteq \mathcal{M}$, we have

$$\begin{array}{lll} \sup B = \mathsf{T} & \text{iff} & B = \{\mathsf{T}\} \text{ or } B = \{\mathsf{T}, \perp\} \\ \sup B = \perp & \text{iff} & B = \{\perp\} \text{ or } B = \emptyset \\ \sup B \in \mathcal{F} & \text{iff} & B \subseteq \mathcal{F} \cup \{\perp\} \text{ and } B \cap \mathcal{F} \neq \emptyset \end{array}$$

(b) For all non-empty $B \subseteq \mathcal{M}$, we have:

$$\begin{array}{lll} \inf B = \mathsf{T} & \text{iff} & B = \{\mathsf{T}\} \\ \inf B = \perp & \text{iff} & \perp \in B \text{ or } (\mathsf{T} \in B \text{ and } B \cap \mathcal{F} \neq \emptyset) \\ \inf B \in \mathcal{F} & \text{iff} & B \subseteq \mathcal{F} \end{array}$$

We now define

$$\mathcal{M} \underset{\boxed{\lambda}}{\overset{\boxed{A}}{\rightleftarrows}} [\mathcal{M} \to \mathcal{M}]_\kappa$$

by $A(u) = \tilde{A}(u)$ if u is a proper map, $A(\perp) = x \mapsto \perp$, and $A(\mathsf{T}) = x \mapsto \mathsf{T}$; $\lambda(f) = \tilde{\lambda}(f)$ for any κ-continuous function f. It is very easy to prove, using Fact 3.1.1 (a):

Lemma 3.1.2 $(\mathcal{M}, A, \lambda)$ is a reflexive κ-Scott domain and we also have:

Lemma 3.1.3 A and λ are inverse κ-isomorphisms between the p.o's $\mathcal{F}$ and $[\mathcal{M} \to \mathcal{M}]_\kappa$.

The premodel we build in Section 8 is a *strong premodel* in the sense that $\tilde{A}$ (or, equivalently, A, because of Fact 3.1.1) is additive (in the sense that it commutes with all existing sups) and commutes with infs of non-empty κ-small subsets. Strong premodels have some nice supplementary properties (c.f. the exercise in Section 6.1 for an example).

We now adopt the simplified notations introduced in Section 2.6: uv for $(A(u))(v)$ and its generalisation to $u\overline{v}$. $\lambda x.f[x]$ for $\lambda(f)$, and $\lambda x_1 \dots x_n.f[x_1, \dots, x_n]$ for $\lambda^n(f)$ if f is n-ary. $\boxed{F'} \equiv \lambda x.\lambda y.xy$, so for all $u \in \mathcal{M}$, $F'u = \lambda y.uy \in \mathcal{F}$.

Lemma 3.1.4 For all $u \in \mathcal{M}$, $\mathsf{T}u = \mathsf{T}$ and $\perp u = \perp$.

Lemma 3.1.5 (SQND) $\mathcal{F} = \{u \mid F'u = u\} = \{u \mid \exists v : F'v = u\}$

Proof. The two sets are included in $\mathcal{F}$ (already seen). Conversely, suppose $u \in \mathcal{F}$, then $F'u = \lambda y.uy = \lambda(y \mapsto A(u)(y)) = \lambda(A(u)) = u$. $\diamondsuit$

Lemma 3.1.6 (Weak extensionality) For all $u, v \in \mathcal{F}$, $u \leq v$ $(u = v)$ iff $\forall x \in \mathcal{M} : ux \leq vx$ $(\forall x \in \mathcal{M} : ux = vx)$.

From Lemmas 3.1.4 and 3.1.6 one can deduce the following easy facts, which are left as exercises.

Fact 3.1.7 (a) The sequence $\lambda x_1 \dots x_n.\perp$, $n \in \mathbf{N}$, is an increasing sequence of elements of $\mathcal{F}$.
(b) The elements $\lambda x_1 \dots x_n.\mathsf{T}$, $n \in \mathbf{N}$, are incompatible maximal elements of $\mathcal{F}$.
(c) $\lambda x_1 \dots x_m.\perp \leq \lambda x_1 \dots x_n.\mathsf{T}$ iff $m \leq n$.

For any $u, v \in \mathcal{M}$, $u \circ v \equiv \lambda x.u(vx)$. Thus $u \circ v$ is always a proper map; if moreover $u, v \in \mathcal{F}$ and are respectively the codes of the functions f and g, then $u \circ v$ is the code of $f \circ g$.

For $u \in \mathcal{M}$, $G \subseteq \mathcal{M}$ define $\boxed{uG} = \{ux \mid x \in G\}$ and $\boxed{u^{-1}G} = \{x \mid ux \in G\}$.

For $u \in \mathcal{M}$ and H κ-open, $H \subseteq \operatorname{dom} u$, the *restriction of u to H* is the code v of the κ-continuous function f defined by $f(x) = ux$ if $x \in H$ and $\perp$ otherwise.

For $G \subseteq \mathcal{M}$, the *characteristic function* of G is the function $\boxed{\chi_G} : \mathcal{M} \to \mathcal{M}$ which takes value T on G and $\perp$ elsewhere; as already mentioned in Section 2.2, χ_G is κ-continuous iff G is κ-open.

3.2 Modelling λ-calculus

The formalism we use here, namely to work with terms with parameters, is the usual one in model theory. It is more convenient for algebraic computations than the use of open terms within environments, which is usual in theoretical computer science. It also allows us to keep close to the notation in [18].

Language

The three following sets are supposed to be disjoint.

$\boxed{\mathcal{V}}$ is a countable set of variables $x, y, \ldots$
$\boxed{\mathcal{C}}$ is a set of constants $\underline{c}$
$\boxed{\mathcal{M}}$ is a κ-cpo

Finite sequences of variables (respectively of elements of $\mathcal{M}$) are denoted $\overline{x}$, $\overline{y}$ (respectively $\overline{u}$, $\overline{v}$) and include the empty one. When necessary they are identified with their underlying set or the corresponding tuple. We only consider sequences of distinct variables. "For all $\overline{x}$, $\overline{u}$" means "for all sequences $\overline{x}$ of distinct variables and all $\overline{u} \in \mathcal{M}^{(<\omega)}$ such that $\ell(\overline{u}) = \ell(\overline{x})$". Here, $\boxed{\ell(\overline{u})}$ denotes the length of the sequence $\overline{u}$.

In the case of map theory, $\boxed{\mathcal{C}} = \{\underline{\perp}, \underline{\mathsf{T}}, \underline{\mathsf{if}}, \underline{\phi}, \underline{\varepsilon}\}$ (the underlining will be omitted soon).

λ-terms

The set $\Lambda_{\mathcal{M},\mathcal{C}}$ of λ-terms with constants in $\mathcal{C}$ and parameters in $\mathcal{M}$, or simply "λ-terms" or even "terms", is defined inductively by:

$$\mathcal{A} ::= x \mid \underline{c} \mid u \mid (\mathcal{A}\,\mathcal{A}) \mid \lambda x.\mathcal{A}$$

Otherwise stated, $\Lambda_{\mathcal{M},\mathcal{C}}$ is the smallest set of terms containing $\mathcal{V} \cup \mathcal{C} \cup \mathcal{M}$ and closed under the usual operations of λ-calculus.

- $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ always denote elements of $\Lambda_{\mathcal{M},\mathcal{C}}$.
- $\mathrm{FV}(\mathcal{A})$ is the set of free variables in $\mathcal{A}$.
- $\Lambda_{\mathcal{C}}$ and Λ are the sets of those terms which do not contain elements of $\mathcal{M}$ and $\mathcal{M} \cup \mathcal{C}$, respectively. Elements of Λ are also called *pure λ-terms*.
- $\mathcal{A}\mathcal{B}_1 \cdots \mathcal{B}_m$ is shorthand for $(\cdots((\mathcal{A}\,\mathcal{B}_1)\,\mathcal{B}_2)\cdots\mathcal{B}_m)$.

Substitution

$[\mathcal{A}/x_1 := \mathcal{A}_1, \cdots, x_m := \mathcal{A}_m]$ is defined if the x_i are distinct and no free variables of any $\mathcal{A}_i$ occur bound in $\mathcal{A}$; it is then the term resulting from the simultaneous substitution of the $\mathcal{A}_i$'s to all free occurrences of the corresponding x_i. So $[\mathcal{A}/x_1 := u_1, \cdots, x_m := u_m]$ makes sense for all $\mathcal{A}$, and all $\overline{x}$, $\overline{u}$, and will be abbreviated

$$[\mathcal{A}/\overline{x} := \overline{u}]$$

Calculus

α (renaming) and β conversions (or equivalences) are defined as usual.

Interpretation of terms

For a given interpretation j of constants in $\mathcal{M}$, i.e. a function $j : \mathcal{C} \to \mathcal{M}$, we define the interpretation $\boxed{|[\mathcal{A}/\overline{x} := \overline{u}]|}$ of all closed λ-terms $[\mathcal{A}/\overline{x} := \overline{u}]$ by elements of $\mathcal{M}$ by induction in the structural complexity of $\mathcal{A}$. If $\mathrm{FV}(\mathcal{A}) \subseteq \overline{x}$ then

$$
\begin{array}{rcll}
|[\mathcal{A}/\overline{x} := \overline{u}]| & \equiv & j(\underline{c}) & \text{if } \mathcal{A} \equiv \underline{c} \in \mathcal{C} \\
 & \equiv & u_i & \text{if } \mathcal{A} \equiv x_i \in \overline{x} \\
 & \equiv & u & \text{if } \mathcal{A} \equiv u \in \mathcal{M} \\
 & \equiv & |[\mathcal{B}/\overline{x} := \overline{u}]||[\mathcal{C}/\overline{x} := \overline{u}]| & \text{if } \mathcal{A} \equiv \mathcal{B}\mathcal{C} \\
 & \equiv & \lambda(v \mapsto [[\mathcal{B}/y := v]/\overline{x} := \overline{u}]) & \text{if } \mathcal{A} \equiv \lambda y.\mathcal{B}
\end{array}
$$

Interpretation $|\bullet|$ (which should in fact be denoted $|\bullet|_j$) is well defined and assigns the same interpretation to terms that are α- or β-equivalent like in the ω-case. For a proof, mimic [4], Chapter V, §4. The point is to show:

Fact 3.2.1 $\forall \mathcal{A} \in \Lambda_{\mathcal{M},\mathcal{C}}$, $\forall \overline{x}, \overline{u}$ such that $\mathrm{FV}(\mathcal{A}) \subseteq \overline{x}$, $\overline{u} \mapsto [\mathcal{A}/\overline{x} := \overline{u}]$ is a κ-continuous function.

Notation for constants

If we adopt the simplified notations of Section 2.6 for the "semantic operators" λ and A, then the only difference between a closed term of $\Lambda_{\mathcal{M},\mathcal{C}}$ and its interpretation in $\mathcal{M}$ is that each $\underline{c} \in \mathcal{C}$ is replaced by $j(\underline{c})$. If we happen to keep the same notation for $\underline{c}$ and $j(\underline{c})$ (for example c) then the same expression will denote as well a closed term and its interpretation in $\mathcal{M}$. It will always be clear from the context what we are really meaning. We will do this in particular with map theory, whose constants are directly named $\perp$, T, if, ϕ and ε.

Equations and inference rules

$\mathcal{E}$ and $\mathcal{E}_i$ will denote equations between terms of $\Lambda_{\mathcal{M},\mathcal{C}}$ and $\bigwedge \mathcal{E}_i$ is the conjunction of the $\mathcal{E}_i$. If $\mathcal{E} \equiv \mathcal{A} = \mathcal{B}$ then $\mathrm{FV}(\mathcal{E}) = \mathrm{FV}(\mathcal{A}) \cup \mathrm{FV}(\mathcal{B})$, and for all $\overline{x}$, $\overline{u}$, $[\mathcal{E}/\overline{x} := \overline{u}] \equiv [\mathcal{A}/\overline{x} := \overline{u}] = [\mathcal{B}/\overline{x} := \overline{u}]$.

An inference rule is an object $\mathcal{R}$ of shape $\mathcal{E}_1, \ldots, \mathcal{E}_n \vdash \mathcal{E}$. $\mathrm{FV}(\mathcal{R})$ is the set of variables which are free in $\mathcal{E}_1, \ldots \mathcal{E}_n, \mathcal{E}$.

An equational theory is a set of equations and rules of inferences, where terms range over $\Lambda_{\mathcal{C}}$.

Satisfaction

We are going to define $(\mathcal{P}, j) \models E$ where E is either an equation, a finite conjunction of equations, or an inference rule. This has to be read: "the premodel $\mathcal{P}$ satisfies E (w.r.t. $j : \mathcal{C} \to \mathcal{M}$)". Let us first assume that $(\mathcal{E}_i)_{i \leq n}$ is a set of equations between **closed** terms ($\mathcal{E}_i \equiv \mathcal{A}_i = \mathcal{B}_i$). Then

$$(\mathcal{P}, j) \models \bigwedge \mathcal{E}_i$$

means that $\forall i \leq n : |\mathcal{A}_i| = |\mathcal{B}_i|$ (in $\mathcal{M}$). In the general case we define:

$$(\mathcal{P}, j) \models \bigwedge \mathcal{E}_i$$

if $\forall \overline{x} \supseteq \mathrm{FV}(\bigwedge \mathcal{E}_i) \forall \overline{u} : (\mathcal{P}, j) \models \bigwedge[\mathcal{E}_i/\overline{x} := \overline{u}]$. Since the interpretation of a term depends only on the values given to its free variables, this definition is equivalent to the one obtained by replacing '$\forall \overline{x}$' by '$\exists \overline{x}$'. This equivalence is used implicitly in several places, e.g. to check that $(\mathcal{P}, j) \models$ Trans.

Suppose now $\mathcal{R}$ is the rule $\mathcal{E}_1, \ldots, \mathcal{E}_n \vdash \mathcal{E}$, then there are different ways in which $(\mathcal{P}, j)$ may satisfy $\mathcal{R}$: the weakest is the following:

$$(\mathcal{P}, j) \models \mathcal{R} \text{ if } (\mathcal{P}, j) \models \bigwedge \mathcal{E}_i \Rightarrow (\mathcal{P}, j) \models \mathcal{E}$$

the strongest is:

$$(\mathcal{P}, j) \models_s \mathcal{R} \text{ if } \forall \overline{x} \supseteq \mathrm{FV}(\mathcal{R}) \forall \overline{u} : ((\mathcal{P}, j) \models \bigwedge[\mathcal{E}_i/\overline{x}{:=}\overline{u}] \Rightarrow (\mathcal{P}, j) \models [\mathcal{E}/\overline{x}{:=}\overline{u}]$$

Of course there is no difference between the two notions if the premisses of $\mathcal{R}$ are closed.

It is clear that to prove the consistency of map theory it is enough to prove that some κ-continuous premodel $\mathcal{P}$ satisfies all axioms, and weakly satisfies all rules (since e.g. $\mathcal{P} \not\models \perp = \mathsf{T}$). Strong satisfaction has a much more semantic flavour but restriction to weak satisfaction (or intermediate versions like in [4], p.100) is forced upon us by the rules Sub2 and Induction (c.f. Appendix C).

Non-monotonic implication

An important shorthand used in map theory is the *non-monotonic implication* $\boxed{\rightarrow}$. For any terms $\mathcal{A}_1, \ldots, \mathcal{A}_n, \mathcal{B}, \mathcal{C}$ let $\mathcal{A}_1, \ldots, \mathcal{A}_n \rightarrow (\mathcal{B} = \mathcal{C})$ denote the equation which is defined inductively by

$$\mathcal{A} \rightarrow (\mathcal{B} = \mathcal{C}) \equiv \mathsf{if}\, \mathcal{A}\, \mathcal{B}\, \mathsf{T} = \mathsf{if}\, \mathcal{A}\, \mathcal{C}\, \mathsf{T}$$

and

$$\mathcal{A}_1, \ldots, \mathcal{A}_n \rightarrow (\mathcal{B} = \mathcal{C}) \equiv \mathcal{A}_1, \ldots, \mathcal{A}_{n-1} \rightarrow (\mathcal{A}_n \rightarrow (\mathcal{B} = \mathcal{C}))$$

if $n > 1$. Furthermore,

$$\mathcal{A}_1, \ldots, \mathcal{A}_n \rightarrow \mathcal{C} \equiv \mathcal{A}_1, \ldots \mathcal{A}_n \rightarrow (\mathcal{C} = \mathsf{T})$$

It is easy to see:

Fact 3.2.2 Let $\mathcal{P}$ be any premodel, $\mathcal{A}_1, \ldots, \mathcal{A}_n, \mathcal{C} \in \Lambda_{\mathcal{M},\mathcal{C}}$ and consider the following assertions:

(a) $\mathcal{P} \models \mathcal{A}_1, \ldots, \mathcal{A}_n \rightarrow \mathcal{C}$.

(b) $\forall \overline{x} \supseteq \mathrm{FV}(\bigwedge \mathcal{A}_i, \mathcal{C}) \forall \overline{u} : (\mathcal{P} \models \bigwedge[\mathcal{A}_i/\overline{x}{:=}\overline{u}] = \mathsf{T} \Rightarrow \mathcal{P} \models [\mathcal{C}/\overline{x}{:=}\overline{u}] = \mathsf{T})$.

(c) $\mathcal{P} \models \bigwedge \mathcal{A}_i = \mathsf{T} \Rightarrow \mathcal{P} \models \mathcal{C} = \mathsf{T}$.

(d) $\mathcal{P} \models \bigwedge \mathcal{A}_i = \mathsf{T} \vdash \mathcal{C} = \mathsf{T}$.

Then (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c); (d) is a reformulation of (c) in case there is no parameters and finally (a) $\Leftrightarrow$ (c) if all $\mathcal{A}_i$ are closed terms of $\Lambda_{\mathcal{M},\mathcal{C}}$.

As a corollary we have for the parameter free case:

Fact 3.2.3 For all $\mathcal{A}_1, \ldots, \mathcal{A}_n, \mathcal{C} \in \Lambda_{\mathcal{C}}$ the following are equivalent:

(a) $\mathcal{P} \models \mathcal{A}_1, \ldots, \mathcal{A}_n \to \mathcal{C}$

(b) $\mathcal{P} \models_s \bigwedge(\mathcal{A}_i = \mathsf{T}) \vdash \mathcal{C} = \mathsf{T}$

3.3 Modelling the λ-calculus axioms of map theory and the QND$'$-principle

The fact that $\mathcal{P}$ is a model of λ-calculus as seen (but not formulated) in Section 3.2, can be rephrased by saying that, whatever j we will chose, $(\mathcal{P}, j)$ will satisfy all those λ-calculus axioms of map theory which contain no explicit mention of the constants. These axioms (Trans, Sub1,2, Apply2 and Rename [c.f. Appendix C]), are of course the usual λ-calculus axioms if we restrict $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ to range over pure λ-terms.

Suppose now we choose j such that $j(\underline{\bot}) = \bot$ and $j(\underline{\mathsf{T}}) = \mathsf{T}$. Then it is easy to check that the axioms Apply1 and Apply3 are satisfied and that the inference rule QND$'$ is strongly satisfied. For the latter we write out the (trivial) proof only for the case where $\mathrm{FV}(\mathcal{R}) = \{x\}$.

We have to show that for all $u \in \mathcal{M}$, if $(\mathcal{P}, j)$ satisfies :

$$\begin{array}{ll} [[\mathcal{A} = \mathcal{B}/x := \mathsf{T}]/x := u], & \text{i.e. } [\mathcal{A} = \mathcal{B}/x := \mathsf{T}] \\ [[\mathcal{A} = \mathcal{B}/x := \bot]/x := u], & \text{i.e. } [\mathcal{A} = \mathcal{B}/x := \bot] \\ [[\mathcal{A} = \mathcal{B}/x := \mathsf{F}'(x)]/x := u], & \text{i.e. } [\mathcal{A} = \mathcal{B}/x := \mathsf{F}'(u)] \end{array}$$

then $(\mathcal{P}, j)$ satisfies : $[\mathcal{A} = \mathcal{B}/x := u]$. But this is immediate from SQND.

To satisfy the Select1,2,3 axioms it is clearly sufficient to interpret if by the code $\mathsf{if} \equiv \lambda^3(\mathrm{If})$ of the ternary κ-continuous function $\mathrm{If} : \mathcal{M}^3 \to \mathcal{M}$ defined by:

$$\mathrm{If}(u, v, w) = \begin{cases} v & \text{if } u = \mathsf{T} \\ w & \text{if } u \in \mathcal{F} \\ \bot & \text{if } u = \bot \end{cases}$$

To prove that If is κ-continuous it is sufficient to check it w.r.t. each component. For the first component this follows from Fact 3.1.1. For the other two it is clear; indeed, when two components, including the first one, are fixed, then If acts on the last as the identity or as a constant map. In conclusion we have:

Lemma 3.3.1 Any premodel $\mathcal{P}$ can be expanded to a model of the λ-calculus axioms of map theory and of QND$'$.

3.4 Interpretation of propositional calculus in a premodel

The simple definitions below are transparent if we consider that T codes "truth", $\perp$ "undefinedness", and any proper map, in particular $\mathsf{F} \equiv \lambda x.\mathsf{T}$, represents "falsehood".

The usual propositional connectives $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$, are in map theory translated into terms $\dot{\neg}$, $\dot{\wedge}$, $\dot{\vee}$, $\dot{\Rightarrow}$, and $\dot{\Leftrightarrow}$, respectively, of $\Lambda_{\mathsf{if},\mathsf{T}}$ (c.f. [18], p.16). We are only interested in those which occur in the axioms, namely

$$\begin{array}{lcl} \dot{\wedge} & \equiv & \lambda x.\lambda y.\mathsf{if}\, x\, (\mathsf{if}\, y\, \mathsf{T}\, \mathsf{F})(\mathsf{if}\, y\, \mathsf{F}\, \mathsf{F}) \\ \dot{\neg} & \equiv & \lambda x.\mathsf{if}\, x\, \mathsf{F}\, \mathsf{T} \end{array}$$

The interpretation of these two terms are κ-continuous functions which are strict in all arguments (f is strict in x if $x = \perp$ implies $f(x) = \perp$); they behave as expected on T and F, and make no difference between F and other elements of $\mathcal{F}$. Two other terms that occur in the axioms are:

$$\begin{array}{lcl} \approx & \equiv & \lambda x.\mathsf{if}\, x\, \mathsf{T}\, \mathsf{F} \\ ! & \equiv & \lambda x.\mathsf{if}\, x\, \mathsf{T}\, \mathsf{T} \end{array}$$

The interpretation of $\approx$ and $!$ are the functions "root" and the characteristic function of $\mathcal{M} \setminus \{\perp\}$, respectively.

Remark. Let Map_1 consist of the λ-calculus axioms and QND$'$, and let $\dot{p}$ be the term of $\Lambda_{\mathsf{if},\perp,\mathsf{T}}$ obtained by replacing all connectives in the formula p of (two-valued) propositional calculus by the corresponding term of map theory ($\neg$ by $\dot{\neg}$, $\wedge$ by $\dot{\wedge}$ etc and propositional variables are viewed in $\dot{p}$ as λ-term variables). It is easy to prove:

- $\dot{p}$ is strict w.r.t. all its free variables.
- If p and q have exactly the same free variables, then $p \Leftrightarrow q$ is a tautology in propositional calculus iff $\mathrm{Map}_1 \vdash \approx\dot{p} = \approx\dot{q}$.
- If p is not a propositional variable, then $\mathrm{Map}_1 \vdash \approx\dot{p} = \dot{p}$.

Example: The following formulas are provable in Map_1 (and map theory in general): $x\dot{\wedge}y = y\dot{\wedge}x$, $x\dot{\wedge}x = \approx x$, $\dot{\neg}\dot{\neg}x = \approx x$, $\dot{\neg}(x\dot{\wedge}y) = \dot{\neg}x\dot{\vee}\dot{\neg}y$ etc and, hence, $\mathcal{A}\dot{\wedge}\mathcal{B} = \mathcal{B}\dot{\wedge}\mathcal{A}$, $\dot{\neg}\dot{\neg}\mathcal{A} = \approx\mathcal{A}$ etc are also provable, for any terms $\mathcal{A}$ and $\mathcal{B}$ or $\Lambda_{\mathcal{C}}$ (or $\Lambda_{\mathcal{M},\mathcal{C}}$ if we work with a premodel).

A formula like $x \vee \neg x \Leftrightarrow \mathsf{T}$ is an example of tautology where the two sides do not have exactly the same free variables and where the corresponding equation $x\dot{\vee}\dot{\neg}x = \mathsf{T}$ fails for $x = \perp$. To represent such formulas in map theory one may use e.g. the construct ! like in $x\dot{\vee}\dot{\neg}x = !x$. A more general result is: p is a tautology iff $\mathrm{Map}_1 \vdash \approx\dot{p} = !x_1\dot{\wedge}\cdots\dot{\wedge}x_n$ where $x_1, \ldots, x_n$ are the free variables of p.

Remark. Let $\mathcal{P}$ be any premodel, and let $\mathcal{A}, \mathcal{B}, \mathcal{A}_1, \ldots, \mathcal{A}_n \in \Lambda_{\mathcal{M},\mathcal{C}}$. Then the following is true in $\mathcal{P}$ (for the interpretations of terms).

$$
\begin{array}{lll}
\mathcal{A}\dot{\wedge}\mathcal{B} = \mathsf{T} & \text{iff} & \mathcal{A} = \mathsf{T} \text{ and } \mathcal{B} = \mathsf{T} \\
\mathcal{A}\dot{\wedge}\mathcal{B} = \bot & \text{iff} & \mathcal{A} = \bot \text{ or } \mathcal{B} = \bot \\
\mathcal{A}\dot{\wedge}\mathcal{B} = \mathsf{F} & & \text{otherwise} \\
\dot{\neg}\mathcal{A} = \mathsf{T} & \text{iff} & \mathcal{A} \in \mathcal{F} \\
\dot{\neg}\mathcal{A} = \mathsf{F} & \text{iff} & \mathcal{A} = \mathsf{T} \\
\dot{\neg}\mathcal{A} = \bot & \text{iff} & \mathcal{A} = \bot
\end{array}
$$

Remark. The embedding of propositional calculus presented above shows the semantic flavour of map theory compared to pure λ-calculus: No terms $\dot{\wedge}$, T, F of pure λ-calculus, $\{\mathsf{T}, \mathsf{F}\}$ separable, can satisfy $\mathsf{T}\dot{\wedge}\mathsf{T} = \mathsf{T}$, $\mathsf{T}\dot{\wedge}\mathsf{F} = \mathsf{F}\dot{\wedge}\mathsf{F} = \mathsf{F}$ and $x\dot{\wedge}y = y\dot{\wedge}x$. The ability to present propositional calculus this way in MT depends on the introduction of T and if.

Remark. Strict definitions for the logical connectives were chosen in order to make tautologies like $x\dot{\wedge}y = y\dot{\wedge}x$ carry over directly. However, non-strict logical connectives are useful in certain cases. As an example, Appendix C defines a non-strict logical "and", $x{:}y$, which is used indirectly in several axioms. Logical connectives like parallel or, $\tilde{\vee}$, which satisfy e.g. $\mathsf{T}\tilde{\vee}\bot = \mathsf{T}$, $\bot\tilde{\vee}\mathsf{T} = \mathsf{T}$ and $\mathsf{F}\tilde{\vee}\mathsf{F} = \mathsf{F}$ were avoided in MT for computational reasons. Even though a construct like $\tilde{\vee}$ is computable, it introduces a lot of trouble to include it in a programming language and it is virtually useless to the programmer. Parallel or, $\tilde{\vee}$, exists in the κ-continuous semantics, so it could be added to MT without loss of consistency.

4 Relative interpretation of ε and ϕ, and of predicate calculus in a premodel

Here κ is any regular cardinal $\geq \omega$. Let $\boxed{\Phi} \subseteq \mathcal{M}$ satisfy

$$\Phi = \uparrow\Phi,\ \Phi \text{ essentially } \kappa\text{-small},\ \bot \notin \Phi \tag{9}$$

Φ is essentially κ-small if there is a κ-small $\boxed{\Psi}$ such that $\Psi \subseteq \Phi = \uparrow\Psi$. Hence, we may assume

$$\Phi = \uparrow\Psi,\ \Psi\ \kappa\text{-small},\ \bot \notin \Psi \tag{10}$$

We define ε and ϕ relative to Φ and verify the predicate calculus axioms of map theory for arbitrary Φ satisfying (9) and $\mathsf{T} \in \Phi$. In Section 7 we fix Φ to obtain a model of all of map theory.

4.1 Interpretation of ε

Let p be a choice function on Φ, i.e. a function $p : \mathcal{P}(\Phi) \to \Phi$ such that $p(A) \in A$ for all non-empty subsets A of Φ. The existence of p follows from the axiom of choice. Define $e : \mathcal{M} \to \Phi \cup \{\bot, \mathsf{T}\}$ by:

$$\begin{array}{lll} e(u) = \bot & \text{iff} & \bot \in u\Phi \\ e(u) = \mathsf{T} & \text{iff} & u\Phi \subseteq \mathcal{F} \\ e(u) = p(\{x{\in}\Phi \mid ux = \mathsf{T}\}) & & \text{otherwise, i.e. if } \mathsf{T} \in u\Phi \not\ni \bot \end{array}$$

Lemma 4.1.1 Let $u, v \in \mathcal{M}$. If $e(u) \neq \bot$ and $u \leq v$ then $e(u) = e(v)$.

Proof. If $\bot \in u\Phi$ then $e(u) = \bot \leq e(v)$. If $u\Phi \subseteq \mathcal{F}$ then $v\Phi \subseteq \uparrow u\Phi \subseteq \mathcal{F}$ and $e(u) = e(v) = \mathsf{T}$. Now assume $\mathsf{T} \in u\Phi \not\ni \bot$. We have $ux = \mathsf{T} \Rightarrow vx = \mathsf{T}$ and $ux \in \mathcal{F} \Rightarrow vx \in \mathcal{F}$ so $\mathsf{T} \in v\Phi \not\ni \bot$, $ux = \mathsf{T} \Leftrightarrow vx = \mathsf{T}$ and $e(u) = e(v)$. $\diamondsuit$

Lemma 4.1.2 e is κ-continuous.

Proof. We first show that $\operatorname{dom} e = \{eu \mid u \neq \bot\}$ is open. Indeed $\operatorname{dom} e = \{u \mid \Phi \subseteq \operatorname{dom} u\} = \{u \mid \Psi \subseteq \operatorname{dom} u\}$ (since $\operatorname{dom} u = \uparrow\operatorname{dom} u$) $= \bigcap\{\mathcal{O}_x \mid x \in \Psi\}$, where $\mathcal{O}_x = \{u \mid ux \neq \bot\}$. Thus $\operatorname{dom} e$ appears as the intersection of a κ-small family of open sets, which is enough to conclude that $\operatorname{dom} e$ is open.

We now prove that e is κ-continuous [same argument as for Lemma 6.2.2 later on]. We have to show that $e(\sup B) \leq \sup eB$ for any κ-directed B. Without loss of generality we suppose $e(\sup B) \neq \bot$. Since $\operatorname{dom} e$ is open there is some $b \in B$ such that $eb \neq \bot$; since $b \leq \sup B$ we have $e(\sup B) = eb \leq \sup eB$. $\diamondsuit$

Definition. $\varepsilon \equiv \lambda(e)$ will be the interpretation in $\mathcal{M}$ of the constant ε in map theory (Hilbert's epsilon operator). Note that the definition of ε depends on Φ. ε will satisfy some axioms of map theory regardless of the choice of Φ. In Section 7 we fix Φ such that ε satisfies all axioms of map theory.

The following lemmas are direct consequences of the definition of ε:

Lemma 4.1.3 [Ackerman's axiom] For all $u, v \in \mathcal{M}$:

$$[\forall x \in \Phi : r(ux) = r(vx)] \Rightarrow \varepsilon u = \varepsilon v$$

Lemma 4.1.4 For all $u \in \mathcal{M}$:

$$\begin{array}{lll} u(\varepsilon u) = \bot & \text{iff} & \bot \in u\Phi \\ u(\varepsilon u) \in \mathcal{F} & \text{iff} & u\Phi \subseteq \mathcal{F} \\ u(\varepsilon u) = \mathsf{T} & \text{iff} & \mathsf{T} \in u\Phi \not\ni \bot \end{array}$$

4.2 Quantifiers

Definitions. $\dot\exists \equiv \lambda z.\approx(z(\varepsilon z))$ is a term of Λ_ε. For all terms $\mathcal{A} \in \Lambda_{\mathcal{M},\mathcal{C}}$ we define the terms $\varepsilon x.\mathcal{A}$, $\dot\exists x.\mathcal{A}$ and $\dot\forall x.\mathcal{A}$ of $\Lambda_{\mathcal{M},\mathcal{C}}$ as follows:

$$\begin{array}{lll} \varepsilon x.\mathcal{A} & \equiv & \varepsilon(\lambda x.\mathcal{A}) \\ \dot\exists x.\mathcal{A} & \equiv & \dot\exists(\lambda x.\mathcal{A}) \\ \dot\forall x.\mathcal{A} & \equiv & \dot\neg(\dot\exists x.\dot\neg\mathcal{A}) \end{array}$$

The following are provable using only the lambda-calculus axioms of map theory:

$$\begin{array}{lcl} \dot{\exists} x.\mathcal{A} & = & \approx[\mathcal{A}/x{:=}\varepsilon(\lambda x.\mathcal{A})] \\ \dot{\forall} x.\mathcal{A} & = & \approx[\mathcal{A}/x{:=}\varepsilon(\lambda x.\dot{\neg}\mathcal{A})] \end{array}$$

The following lemmas are straightforward consequences of Lemma 4.1.4:

Lemma 4.2.1 For all $\mathcal{A} \in \Lambda_{\mathcal{M},\mathcal{C}}$ such that $\mathrm{FV}(\mathcal{A}) \subseteq \{x\}$, $\dot{\exists} x.\mathcal{A}$ equals $\bot$, T or F in $\mathcal{M}$ and:

$$\begin{array}{lll} \mathcal{P} \models \dot{\exists} x.\mathcal{A} = \mathsf{T} & \text{iff} & \forall u \in \Phi.\mathcal{P} \models [\mathcal{A}/x{:=}u] \neq \bot \text{ and} \\ & & \exists u \in \Phi.\mathcal{P} \models [\mathcal{A}/x{:=}u] = \mathsf{T} \\ \mathcal{P} \models \dot{\exists} x.\mathcal{A} = \bot & \text{iff} & \exists u \in \Phi.\mathcal{P} \models [\mathcal{A}/x{:=}u] = \bot \\ \mathcal{P} \models \dot{\exists} x.\mathcal{A} = \mathsf{F} & \text{iff} & \forall u \in \Phi.\mathcal{P} \models [\mathcal{A}/x{:=}u] \in \mathcal{F} \end{array}$$

Lemma 4.2.2 For all $\mathcal{A} \in \Lambda_{\mathcal{M},\mathcal{C}}$ such that $\mathrm{FV}(\mathcal{A}) \subseteq \{x\}$, $\dot{\forall} x.\mathcal{A}$ equals $\bot$, T or F in $\mathcal{M}$ and:

$$\begin{array}{lll} \mathcal{P} \models \dot{\forall} x.\mathcal{A} = \mathsf{T} & \text{iff} & \forall u \in \Phi.\mathcal{P} \models [\mathcal{A}/x{:=}u] = \mathsf{T} \\ \mathcal{P} \models \dot{\forall} x.\mathcal{A} = \bot & \text{iff} & \exists u \in \Phi.\mathcal{P} \models [\mathcal{A}/x{:=}u] = \bot \\ \mathcal{P} \models \dot{\forall} x.\mathcal{A} = \mathsf{F} & \text{iff} & \forall u \in \Phi.\mathcal{P} \models [\mathcal{A}/x{:=}u] \neq \bot \text{ and} \\ & & \exists u \in \Phi.\mathcal{P} \models [\mathcal{A}/x{:=}u] \in \mathcal{F} \end{array}$$

4.3 Satisfaction of predicate calculus axioms

In addition to (9) and (10) we now assume

$$\Psi \subseteq \mathcal{M}_c \tag{11}$$

Now $\Phi = \uparrow\Psi$ is κ-open so χ_Φ is κ-continuous. Define:

$$\phi \equiv \lambda(\chi_\Phi)$$

For any $u \in \mathcal{M}$, $\phi u = \mathsf{T}$ if $u \in \Phi$ and $\phi u = \bot$ otherwise; in particular, for any closed $\mathcal{A} \in \Lambda_{\mathcal{M},\mathcal{C}}$,

$$\begin{array}{ll} \mathcal{P} \models \phi\mathcal{A} = \mathsf{T} & \text{iff (the interpretation of) } \mathcal{A} \text{ is in } \Phi \\ \mathcal{P} \models \phi\mathcal{A} = \bot & \text{otherwise} \end{array}$$

Theorem 4.3.1 The first order predicate calculus axioms of map theory are satisfied in any κ-continuous premodel $\mathcal{P}$, provided $\bot$, T, if, ε and ϕ are interpreted as above and provided that (10), (11) and $\mathsf{T} \in \Phi$ hold:

$$\begin{array}{ll} Q_1' : & \dot{\forall} x.\mathcal{A}, \phi\mathcal{B} \to [\mathcal{A}/x{:=}\mathcal{B}] \\ Q_2 : & \varepsilon x.\mathcal{A} = \varepsilon x.(\phi x \dot{\wedge} \mathcal{A}) \\ Q_3 : & \phi(\varepsilon x.\mathcal{A}) = \dot{\forall} x.!\mathcal{A} \\ Q_4' : & \dot{\exists} x.\mathcal{A} \to \dot{\forall} x.!\mathcal{A} \\ Q_5 & \dot{\forall} x.\mathcal{A} = \dot{\forall} x.(\phi x \dot{\wedge} \mathcal{A}) \end{array}$$

The intuition behind each axiom is quite clear: Q_1' says that $\dot{\forall}$ quantifies over no less than Φ and Q_5 that $\dot{\forall}$ quantifies over no more than Φ; in conjunction they say that $\dot{\forall}$ quantifies over Φ. Q_2 says that ε merely depends on $\mathcal{A}$ for $x \in \Phi$ and merely depends on the root of $\mathcal{A}$. Hence, Q_2 both expresses Ackerman's axiom and expresses that ε 'quantifies' over no more than Φ. Q_3 says that $\varepsilon x.\mathcal{A}$ is defined iff $\mathcal{A}$ is defined on any $x \in \phi$, and then belongs to Φ. Among other, this means that ε 'quantifies' over no more than Φ. Q_4' says that if $\exists x.\mathcal{A}$ is true then $\mathcal{A}$ is defined all over Φ.

When writing [18], Q_4 and Q_5 were added late in the development in order to get two important proofs through. This has made the collection of quantification axioms somewhat peculiar and redundant.

Axiom Q_1' above differs from that of [18] in that the premisses of $\rightarrow$ are reversed. Axiom Q_4' above differs from Q_4 in [18] which says $\dot{\exists} x.\mathcal{A} \rightarrow \phi(\varepsilon x.\mathcal{A})$ (if there exists an $x \in \Phi$ that satisfies $\mathcal{A}$, then $\varepsilon x.\mathcal{A}$ is such an x and, in particular, belongs to Φ). Q_4 and Q_4' are equivalent assuming Q_3.

Proof of Theorem 4.3.1. To check that $\mathcal{P}$ satisfies the Q-axioms we work, as usual, with closed terms of $\Lambda_{\mathcal{M},\mathcal{C}}$ instead of working with open terms of $\Lambda_{\mathcal{C}}$. We freely use the fact that $\mathcal{P}$ satisfies the λ-calculus axioms (c.f. Section 3.3) and, in particular, we use the last remark of Section 3.4. We prove Q_1', Q_2 and Q_3, and leave Q_4' and Q_5 as easy exercises.

Q_1': Suppose $\dot{\forall} x.\mathcal{A} = \mathsf{T}$ and $\mathcal{B} \in \Phi$. Then, for all $u \in \Phi$, $[\mathcal{A}/x{:=}u] = \mathsf{T}$ (Lemma 4.2.2). In particular, $[\mathcal{A}/x{:=}\mathcal{B}] = \mathsf{T}$.

Q_2: It is enough to prove that for all $u \in \Phi$ we have: $r((\lambda x.\mathcal{A})u) = r((\lambda x.(\phi x \dot{\wedge} \mathcal{A}))u)$ (c.f. Lemma 4.1.3 (Ackerman's axiom)). But this is equivalent to $r([\mathcal{A}/x{:=}u]) = r([\mathcal{A}/x{:=}u] \dot{\wedge} \phi u)$, which is trivially true since $\phi u = \mathsf{T}$.

Q_3: $!\mathcal{A} \equiv \mathsf{if} \mathcal{A} \mathsf{T} \mathsf{T}$ and, hence, $\dot{\forall} x.!\mathcal{A}$, has value $\perp$ or T in $\mathcal{M}$, and this is also the case for $\phi(\varepsilon x.\mathcal{A})$. Now $\phi(\varepsilon x.\mathcal{A}) = \mathsf{T}$ iff $\varepsilon x.\mathcal{A} \in \Phi$ iff $(\lambda x.\mathcal{A})u \neq \perp$ for all $u \in \Phi$ (here we use $\mathsf{T} \in \Phi$); thus $\phi(\varepsilon x.\mathcal{A}) = \mathsf{T}$ iff, for all $u \in \Phi$, $\mathsf{if}[\mathcal{A}/x{:=}u]\mathsf{T}\mathsf{T} = \mathsf{T}$ iff $\dot{\forall} x.\mathsf{if}\mathcal{A}\mathsf{T}\mathsf{T} = \mathsf{T}$. $\diamondsuit$

Remark. We could recover that $\mathcal{P}$ satisfies all theorems of (usual) predicate calculus, provided the connectives and the quantifiers are replaced by their dotted version and that free variables, if any, are limited to range over Φ. This could in fact be done for predicate calculus over any signature, and is true in fact for any possible modelisation of map theory, since there is a syntactic translation of predicate calculus into map theory ([19], p.8, [18], p.60).

But we have not yet been restrictive enough on Φ so as to ensure that this interpretation of predicate calculus is faithful [at this stage we could have $\Phi = \Psi = \{\mathsf{T}\}$ or $\Phi = \Psi = \{\mathsf{T}, \mathsf{F}\}$] and of course it is not the case:

Example 1. Suppose $|\Psi| < \omega$ (which is necessarily the case if $\kappa = \omega$ since Ψ is κ-small). Then we would have, for any $\mathcal{A} \in \Lambda_{\mathcal{M},\mathcal{C}}$: $\mathcal{P} \models \dot{\exists} x_1 \cdots x_n.[[\mathcal{A}/x{:=}x_1] \dot{\wedge} \cdots \dot{\wedge} [\mathcal{A}/x{:=}x_n] \dot{\Rightarrow} \dot{\forall} x.\mathcal{A}]$, where $n = |\Psi|$.

Example 2. Suppose T is not in Φ. Then $\mathcal{P} \models \dot{\forall} x \dot{\forall} y.(x \dot{\wedge} y \dot{\Leftrightarrow} x \dot{\vee} y)$.

On the contrary, Φ cannot be too rich: Φ is already bounded to be essentially κ-small; on the other hand, requirements like $\lambda x.x \in \Phi$ would contradict the well-foundedness axioms of map theory (syntactic point of view) and the set theoretic properties of the model (semantic point of view). The idea behind Φ is that $(\Phi, \dot{\in})$ is a model of *ZFC* for a particular term $\dot{\in}$ of $\Lambda_{\mathcal{C}}$. For that particular term, $\dot{\forall} y.y \dot{\in} \lambda x.x$ is provable in *MT*, so $\lambda x.x$ represents at least the class of all sets. The requirement $\lambda x.x \in \Phi$ is equivalent to the requirement that the class of all sets is itself a set. Furthermore our intention is not only to model map theory but also to realise all the (semantic) intuitions which were behind it, in particular the strong version of well-foundedness which asserts that if a and $x_1, x_2, \cdots$ are well-founded (i.e. elements of Φ), then there exists an $n \geq 0$ such that $ax_1 \cdots x_n = \mathsf{T}$. Now it is clear that $\lambda x.x \in \Phi$ contradicts this property since if a and $x_1, x_2, \cdots$ are all equal to $\lambda x.x$, then $ax_1 \cdots x_n = \lambda x.x$ which differs from T in any non trivial model of the λ-calculus axioms.

5 Satisfaction of the well-foundedness axioms if $\sigma < \kappa$

The aim of this section is to introduce two semantic conditions on Φ and to show that they are indeed sufficient to ensure the satisfaction of the well-foundedness axioms. The *Strong Induction Principle* (*SIP*) is a strong non-equational way to ensure the satisfaction of the induction rule, via the well-foundedness of Φ w.r.t. a binary relation which will be specified below. All other well-foundedness axioms of map theory may be viewed as simple closure properties of Φ, which will follow essentially from the satisfaction by Φ of a recursive equation that we will call the *Generic Closure Property* (*GCP*). These two principles, together with the *Strong Quartum Non Datur* (*SQND*) are the basic intuitions behind map theory ([18], part I), which can be viewed as a sufficiently powerful equational approximation of them.

As already seen the premodels we are working with are, roughly speaking, those κ-models of λ-calculus which satisfy SQND; the fundamental result of this paper is that, provided there is some inaccessible below κ, they always contain a κ-open set Φ which satisfy the SIP and GCP (in addition to the conditions studied in Section 4, namely that Φ is essentially κ-small and contains T but not $\perp$). Otherwise stated we are providing "strong" models of map theory.

It is worthwhile to mention also the following about the meaning of the Strong Induction Principle; since it is outside the scope of the present paper, we will do it without any justification.

As already mentioned, it is intended in the philosophy of map theory that, given any model M of it (found in a strong enough usual set theoretic universe), the subclass (or subset if M is a set) Φ of all elements u of M such that $\phi u = \mathsf{T}$ (in M), endowed with the interpretation of the term $\dot{\in}$, will be a model of *ZFC*. The Induction rule conveys the fact that this latter *ZFC*-model is well-founded and the Strong Induction Principle says in addition that it is an "ω-model", namely that is has (essentially) the same integers as the universe we started from (two possible definitions for the set of integers in map theory are given in [18], p.21 and p.61, respectively).

5.1 The strong Induction Principle and the Generic Closure Property

The statements of SIP and GCP need the definition of two operators on subsets of M.

$\boxed{G^{(<\omega)}}$ and $\boxed{G^\omega}$ are the sets of finite and infinite sequences, respectively, of elements of G, including the empty one. Such sequences are denoted by $\overline{u}$, $\overline{v}$, $\overline{x}$, etc, even if they are infinite. $\boxed{\overline{x} \leq \overline{y}}$ means that $\ell(\overline{x}) = \ell(\overline{y})$ and $\forall i \leq \ell(\overline{x})$, $x_i \leq y_i$. For $\overline{x} = (x_m)_{m\in\omega} \in G^\omega$, and all $n \geq 0$, we define: $\boxed{\overline{x}_n} \equiv (x_0, \ldots, x_{n-1})$, thus $\overline{x}_0$ is the empty sequence.

Definition. G is *essentially κ-small* (*σ-small*) if there is an H such that $|H| < \kappa$ $(< \sigma)$ and $H \subseteq G \subseteq {\uparrow}H$.

Definition. The *dual* $\boxed{G^\circ}$ of any subset G of M is the set of those elements which are *well-founded w.r.t. G*:

$$G^\circ = \{u{\in}M \mid \forall\overline{x}{\in}G^\omega \exists n{\in}\omega\colon ux_1 \cdots x_n = \mathsf{T}\}$$

(As a special case, $\emptyset^\circ = M \setminus \{\bot\}$). Furthermore, $\boxed{<_G}$ is the binary relation defined on G° by

$$v <_G u \text{ iff } u \neq \mathsf{T} \text{ and } v \in uG \tag{12}$$

In the set-theoretic world $v <_G u$ means that v "belongs to" u and that this fact is witnessed by an element of G. This interpretation gets its full meaning when G is Φ, namely the "class of all sets" (see below and Appendix A).

At this point we only need the most trivial properties of the operator, namely:

- $\mathsf{T} \in G^\circ$, $\bot \notin G^\circ$ and, for all $u \in G^\circ$ and all $\overline{y} \in G^{<\omega}$, $u\overline{y} \in G^\circ$.
- $<_G$ is well-founded on G°.
- $H \subseteq K \Rightarrow K^\circ \subseteq H^\circ$

The name "dual" was originally chosen because of some vague similarity with duality in linear algebra and because Φ was originally desired to satisfy $\Phi = \Phi^{\circ\circ}$ (which did not work out well). Instead of $\Phi = \Phi^{\circ\circ}$, Φ now satisfies $\Phi \subseteq \Phi^{\circ\circ}$ (Lemma A.1.4) and a softening of $\Phi^{\circ\circ} \subseteq \Phi$. The softening of $\Phi^{\circ\circ} \subseteq \Phi$ is formulated in the Generic Closure Property. The property $\Phi \subseteq \Phi^{\circ\circ}$ is syntactically difficult to express, but the Strong Induction Principle below is just as useful.

For any two subsets H and K of M, we define

$$\boxed{H \to K} \equiv \{u{\in}M \mid \forall x{\in}H\colon ux \in K\}$$

The *Strong Induction Principle* (*SIP*) is the (semantic) requirement that

$$\Phi \subseteq \Phi^\circ$$

Because of the well-foundedness of $<_\Phi$ on Φ° and of the SQND it is easy to see that any premodel enriched with an open set Φ satisfying the Strong Induction Principle, will weakly satisfy the Induction rule IND.

The *Generic Closure Property* (*GCP*) is the recursive equation:

$$\Phi = \bigcup\{G^\circ \to \Phi \mid G \in \mathcal{O}_\sigma(\Phi)\}$$

where $\mathcal{O}_\sigma(\Phi)$ is the set of all essentially σ-small open subsets of Φ.

It would be sufficient to satisfy the simpler recursive equation $\Phi = \Phi^\circ \to \Phi$. However, this is inconsistent with the other requirements since it can be proved that $\varepsilon \in \Phi^\circ \to \Phi$ but $\varepsilon \notin \Phi$ (c.f. Appendix A.1).

The GCP admits several useful equivalent formulations which will be given in Section 7.

The existence of an (adequate) solution of the GCP in (all) our premodels will be proved in section 7. We will then need lemmas asserting things like: "If $G \in \mathcal{O}_\sigma(\Phi)$, then $G^\circ \to G$ are in $\mathcal{O}_\sigma(\Phi)$ too". Facts of this kind are not at all obvious; they force us to work with the sets of compact elements which generate the open sets and require a real mathematical work. This motivates the machinery developed in Section 6.

We now work with a fixed, κ-continuous premodel $\mathcal{P} = (\mathcal{M}, \tilde{A}, \tilde{\lambda})$ where $\kappa > \omega$. When needed, σ is an inaccessible cardinal $\leq \kappa$. G, H and K will denote nonempty subsets of $\mathcal{M}$. Assuming the results proved in Section 6 and 7 we suppose that we have available a Φ which satisfies the following requirements.

- Φ is an essentially κ-small open subset of M,
- $\mathsf{T}, \mathsf{F} \in \Phi$ and $\perp \notin \Phi$,
- $\Phi \subseteq \Phi^\circ$ (SIP), and
- $\Phi = \bigcup\{G^\circ \to \Phi \mid G \in \mathcal{O}_\sigma(\Phi)\}$ (GCP).

5.2 Closure properties of Φ and Φ° and satisfaction of the axioms

This subsection proves the semantic versions of the *well-foundedness axioms* (the "set theory" axioms in Appendix C).

Definition. Here, a *κ-continuous model* will be a pair $(\mathcal{P}, j)$ such that there is a $\sigma < \kappa$, σ inaccessible, $\mathcal{P}$ is a κ-continuous premodel, and j interprets the constants of map theory as indicated before; in particular, the interpretation of ϕ and ε is via a κ-open set Φ satisfying the conditions just stated above.

It is easy to see that $\mathsf{T}, \mathsf{F} \in \Phi$, $\mathsf{if}, \phi, \varepsilon \in \Phi^\circ$, and $\perp \notin \Phi^\circ$ (c.f. Appendix A.1).

Notation. $H_1, \ldots, H_n \to K$ means $H_1 \to (H_2 \to \cdots (H_n \to K) \cdots))$ if $n > 1$. $H^n \to K$ means $H, \ldots, H \to K$ where H occurs n times.

Lemma 5.2.1 For all $n \in \mathbf{N}$,

$$\Phi \subseteq (\Phi^\circ)^n \to \Phi \subseteq \Phi^n \to \Phi \subseteq \Phi^n \to \Phi^\circ = \Phi^\circ$$

Proof. By induction on n, using the trivial covariance and contravariance properties of the arrow, and the various properties of Φ. We treat $n = 1$:

If $u \in \Phi$, then $u \in G^\circ \to \Phi$ for some $G \subseteq \Phi$, hence $u \in \Phi^\circ \to \Phi$; $\Phi^\circ \to \Phi \subseteq \Phi \to \Phi \subseteq \Phi \to \Phi^\circ$ since $\Phi \subseteq \Phi^\circ$, and finally $\Phi \to \Phi^\circ = \Phi^\circ$ is obvious. $\diamondsuit$

Corollary 5.2.2 Any κ-continuous model $\mathcal{P}$ satisfies the well-foundedness axioms Well1, Well3, C-A, C-K' and C-P' stated in Appendix C.

Proof. The axioms state $\mathsf{T} \in \Phi$, $\bot \notin \Phi$, $\Phi \subseteq \Phi \to \Phi$, $\lambda x.\mathsf{T} \in \Phi$, and $\lambda x.(\mathsf{if}\, x\, \mathsf{T}\, \mathsf{T}) \in \Phi$, respectively. We just have to check the last one. But it is clear that $\lambda x.(\mathsf{if} x\, \mathsf{T}\, \mathsf{T}) \in G^\circ \to \mathsf{T} \subseteq \Phi$ for (all) $G \in \mathcal{P}_\sigma(\Phi)$, since $\bot \notin G^\circ$. $\diamondsuit$

Fact 5.2.3 Φ is well-founded w.r.t. $<_\Phi$.

Proof. Follows immediately from $\Phi \subseteq \Phi^\circ$ and the definition of $<_\Phi$ given in (12). $\diamondsuit$

Lemma 5.2.4 Φ is the smallest subset X of $\mathcal{M}$ such that

$$\mathsf{T} \in X \qquad (\Phi \to X) \cap \Phi \subseteq X \tag{13}$$

Proof. That Φ satisfies (13) is obvious; conversely suppose X satisfies (13) and $\Phi \setminus X \neq \emptyset$; take u in $\Phi \setminus X$, minimal w.r.t. $<_\Phi$. Certainly $u \neq \mathsf{T}$; now, by minimality of u, and the fact that $u\Phi \subseteq \Phi$ (since $u \in \Phi \subseteq \Phi \to \Phi$) we get $u\Phi \subseteq X$; since X satisfies (13) we have $u \in X$ which yields a contradiction. $\diamondsuit$

Corollary 5.2.5 Any κ-continuous model $\mathcal{P}$ satisfies the induction rule in Appendix C.

Proof. We assume $\mathcal{A}, \mathcal{B} \in \Lambda_{\mathcal{M},\mathcal{C}}$, $\mathcal{A}$ closed, $\mathrm{FV}(\mathcal{B}) \subseteq \{x\}$. Now, if $\mathcal{P}$ satisfies the premises of the induction rule and $\mathcal{P} \models \mathcal{A} = \mathsf{T}$, then $X = \{u \in \mathcal{M} \mid [\mathcal{B}/x{:=}u] = \mathsf{T}\}$ satisfies (13) of Lemma 5.2.4, so $\Phi \subseteq X$ and $\mathcal{P}$ satisfies the conclusion of the induction rule. $\diamondsuit$

We now turn to the (interpretation of the) combinators that appear explicitly in the well-foundedness axioms of map theory, namely P, *Curry* and *Prim*, as well as some that occur implicitly. The former are treated in Lemma 5.2.6, the latter in Lemma 5.2.8.

Definition. The definitions of P, *Curry* and *Prim* are stated in Appendix C. P is a pairing construct and *Curry* expresses currification; their definitions are repeated here:

$$\begin{array}{lcl} P & = & \lambda a.\lambda b.\lambda x.(\mathsf{if}\, x\, a\, b) \\ \mathit{Curry} & = & \lambda f.\lambda x.\lambda y.(f\,(P\,x\,y)) \end{array}$$

Prim expresses a sort of transfinite primitive recursion, and its definition involves a fixed point operator. Here, we just have to know that for any $f, a, b \in \mathcal{P}$, if $g = \mathit{Prim}\, f\, a\, b$ then for all $x \in \mathcal{P}$:

$$g\,x = \begin{array}{ll} a & \text{if } x = \mathsf{T} \\ \bot & \text{if } x = \bot \\ f\lambda z.g(x(bz)) & \text{otherwise} \end{array}$$

Lemma 5.2.6 In any κ-continuous premodel $\mathcal{P}$ $(\kappa > \sigma)$ we have

(a) $P \in G^\circ, G^\circ \to G^\circ$ for any $G \subseteq \Phi$; (also $P \in \Phi, \Phi \to \Phi$).

(b) $\mathit{Curry} \in \Phi \to \Phi$

(c) $\mathit{Prim} \in (\Phi \to \Phi), \Phi, \Phi \to \Phi$

(In particular they all live in Φ°).

Proof. (a) is clear since for all $u, v \in G^\circ$, $Puv = \lambda x.\mathsf{if}\, x\, u\, v \in G^\circ \to G^\circ \subseteq G \to G^\circ = G^\circ$ (note that $\Phi \subseteq \Phi^\circ$ implies $G \subseteq G^\circ$ for all $G \subseteq \Phi$). If $u, v \in \Phi$, $Puv \in G^\circ \to \Phi$ for any $G \subseteq \Phi$, thus $Puv \in \Phi$.

(b) if $u \in \Phi$, then $u \in G^\circ \to \Phi$ for some $G \in \mathcal{O}_\sigma(\Phi)$, hence $\mathit{Curry}\, u = \lambda x.\lambda y.u(Pxy) \in G^\circ, G^\circ \to \Phi \subseteq \Phi$ (using (a) and Corollary 7.1.4).

(c) suppose $f \in \Phi \to \Phi$ and $a, b \in \Phi$. Let $G \in \mathcal{O}_\sigma(\Phi)$ be such that $a, b \in G^\circ \to G$ (c.f. Lemma 7.1.3 (a)). We prove that $g = \mathit{Prim}\, f\, a\, b$ satisfies $g \in G^\circ \to \Phi$ (thus $g \in \Phi$). Suppose there is an $x \in G^\circ$ such that $gx \notin \Phi$. We choose x minimal for $<_G$ (c.f. (12)); then certainly $x \neq \mathsf{T}$ (otherwise $gx = a \in \Phi$) and $gx = f\lambda u.g(x(bu))$. Now for all $u \in G^\circ$, $bu \in G$, since $x(bu) <_G x$ we have that, for all $u \in G^\circ$, $g(x(bu)) \in \Phi$, hence $\lambda u.g(x(bu)) \in G^\circ \to \Phi \subseteq \Phi$ and $gx \in \Phi$ (since $f \in \Phi \to \Phi$) which yields a contradiction. Hence, $gG^\circ \subseteq \Phi$ and $g \in G^\circ \to \Phi \subseteq \Phi$. $\diamondsuit$

Corollary 5.2.7 Any κ-continuous model $\mathcal{P}$ satisfies C-Curry and C-Prim.

Proof. C-Curry and C-Prim are just the simplest equational ways to express the closure properties of Lemma 5.2.6. $\diamondsuit$

Lemma 5.2.8 (a) $\lambda u.\lambda x.uxx \in \Phi \to \Phi$

(b) $\circ \in (\Phi \to \Phi), \Phi \to \Phi$ where $\circ = \lambda w.\lambda v.\lambda z.w(vz)$

(c) $\lambda z.(w \circ v)zz \in \Phi$ for all $v \in \Phi$ and $w \in \Phi \to \Phi$.

Proof. (c) is a direct consequence of (a) and (b).

(a) let $u \in \Phi$; then $u \in G^\circ, G^\circ \to \Phi$ for some $G \in \mathcal{O}_\sigma(\Phi)$ (c.f. Corollary 7.1.4), hence $\lambda x.uxx \in G^\circ \to \Phi \subseteq \Phi$ as required.

(b) let $w \in \Phi \to \Phi$ and $v \in \Phi$; then $v \in G^\circ \to \Phi$ for some $G \in \mathcal{O}_\sigma(\Phi)$. If $z \in G^\circ$, then $vz \in \Phi$ and $w(vz) \in \Phi$. $\diamondsuit$

Corollary 5.2.9 $\mathcal{P} \models$ C-M1

Proof. We have to show that, for any $\mathcal{A} \in \Lambda_{\mathcal{M},\mathcal{C}}$ such that $\mathrm{FV}(\mathcal{A}) \subseteq \{x, z\}$:

$$\lambda z.\lambda x.\mathcal{A} \in \Phi \to \Phi \Rightarrow \forall v \in \Phi : \lambda x.[\mathcal{A}/z{:=}vx] \in \Phi$$

Let $v \in \Phi$ and $w = \lambda z.\lambda x.\mathcal{A} \in \Phi \to \Phi$. By Lemma 5.2.8 (c), $a = \lambda x.(w \circ v)xx = \lambda x.[\mathcal{A}/z{:=}vx] \in \Phi$ which proves the corollary. $\diamondsuit$

Corollary 5.2.10 $\mathcal{P}$ satisfies Well2 and C-M2.

Proof. (Well2) First we notice that $u \in \Phi$ iff $\phi u = \mathsf{T}$ iff $\phi u \in \Phi$. Second we claim that it suffices to show that for any $\mathcal{A} \in \Lambda_{\mathcal{M},\mathcal{C}}$ such that $\mathrm{FV}(\mathcal{A}) \subseteq \{x\}$, and $G \in \mathcal{O}_\sigma(\Phi)$, $\lambda x.\mathcal{A} \in G^\circ \to \Phi$ iff $\lambda x.\phi\mathcal{A} \in G^\circ \to \Phi$. But this is clear since, for all $w \in G^\circ$, $(\lambda x.\mathcal{A})w \in \Phi$ iff $[\mathcal{A}/x{:=}w] \in \Phi$ iff $[\phi\mathcal{A}/x{:=}w] \in \Phi$ iff $(\lambda x.\phi\mathcal{A})w \in \Phi$.

(C-M2) We have to show that, for any $\mathcal{A} \in \Lambda_{\mathcal{M},\mathcal{C}}$ such that $\mathrm{FV}(\mathcal{A}) \subseteq \{x, z\}$, $\forall v \in \Phi : \lambda x.[\mathcal{A}/z{:=}v] \in \Phi \Rightarrow \forall v \in \Phi : \lambda x.[\mathcal{A}/x := xv/z := v] \in \Phi$. Let us fix $v \in \Phi$; by hypothesis there is a $G \in \mathcal{O}_\sigma(\Phi)$ such that $\lambda x.[\mathcal{A}/z{:=}v] \in G^\circ \to \Phi$. Without loss of generality we assume $v \in G$ (otherwise take $G' = G \cup \{v\}$). Now, for any $w \in G^\circ$, $wv \in G^\circ$, hence $(\lambda x.[\mathcal{A}/z{:=}v])(wv) \in \Phi$; thus $\lambda x.[\mathcal{A}/x := xv, z := v] \in \Phi$. $\diamondsuit$

Remark. $\kappa \geq \sigma$ was enough for Well 1,3, C-A, C-K', C-P' and C-Curry (which do not contain ε, even implicitly).

Remark. As already used in [18], C-M1 and C-M2 are equivalent to the following three axioms:

$$\begin{array}{lcl} \phi a & \to & \lambda x.axx \\ \dot{\forall} x.\phi(ax), \phi b & \to & \phi(a \circ b) \\ \phi a, \phi b & \to & \lambda x.a(xb) \end{array}$$

That C-M1 can be replaced by the two former is evident from the proof of Corollary 5.2.9. In general, the three axioms above are easier to deal with in model-theoretic treatments than C-M1 and C-M2.

6 Duality, types and arrows in a premodel

We still work with a fixed, κ-continuous premodel $\mathcal{P} = (\mathcal{M}, \tilde{A}, \tilde{\lambda})$ where $\kappa > \omega$. When needed, σ is an inaccessible cardinal $\leq \kappa$. G, H and K will denote nonempty subsets of $\mathcal{M}$.

$G_c \equiv \mathcal{M}_c \cap G$ is the set of κ-compact elements of G; recall that G is open iff $G = \uparrow G_c$ (c.f. Section 2.2). $\delta(G)$ is the set of minimal elements of G; if G is open then $\delta(G) \subseteq G_c$ but, even in this case, we may have $\delta(G) = \emptyset$.

For any K, a *choice function w.r.t. K* is a function $q : \uparrow K \to K$ such that $q(x) \in K \cap \downarrow x$. q extends to $(\uparrow K)^\omega$ by $q((x_n)_{n\in\omega}) \equiv (q(x_n))_{n\in\omega}$.

Using a choice function w.r.t. G_c it is easy to prove:

Lemma 6.0.11 If G is open, then G is essentially κ-small (σ-small) iff there is an $H \subseteq G_c$, $|H| < \kappa$ ($< \sigma$) and $G = \uparrow H$.

6.1 Duality and types

We first quote some easy properties of the dual operator G°. For $u \in G^\circ$ and $\overline{x} \in G^\omega$, we let $\boxed{\overline{x}_{[u]}}$ denote the smallest subsequence $\overline{x}_n$ such that $u\overline{x}_n = \mathsf{T}$.

Fact 6.1.1 (a) $\mathsf{T} \in G^\circ$ and $\lambda x_1 \cdots x_n.\mathsf{T} \in G^\circ$ for all n.
(b) $\perp \notin G^\circ$, and for all $u \in G^\circ$ and $\overline{y} \in G^{<\omega}$ we have $u\overline{y} \neq \perp$.

Fact 6.1.2 (a) $G^\circ = \uparrow(G^\circ) = (\uparrow G)^\circ$.
(b) $H \subseteq G \Rightarrow G^\circ \subseteq H^\circ$.
(c) $G \subseteq \uparrow H \Rightarrow H^\circ \subseteq G^\circ$.
(d) $H \subseteq G \subseteq \uparrow H \Rightarrow G^\circ = H^\circ$.
(e) If H is κ-open and $H \neq G$, then $H^\circ \neq G^\circ$
To see (e), note that (the code of) χ_H belongs to $H^\circ \setminus G^\circ$.

The following notion of a *type* was introduced in [18]. As noted in the introduction, it is related to the notion of type in Model Theory.

Definition. The *type of* $u \in \mathcal{M}$ *over* $G \subseteq \mathcal{M}$ $(G \neq \emptyset)$ is the function $\boxed{t(u/G)} : G^{<\omega} \to \{\perp, \mathsf{T}, \lambda x.\mathsf{T}\}$ which associates $r(u\overline{y})$ to any $\overline{y} \in G^{<\omega}$.

We set

$$\boxed{u =_G v} \text{ iff } t(u/G) = t(v/G) \text{ iff } \forall \overline{y} \in G^{<\omega} : r(u\overline{y}) = r(v\overline{y})$$

so

$$u =_G v \text{ iff for all } \overline{y} \in G^{<\omega},\ u\overline{y} = \perp \Leftrightarrow v\overline{y} = \perp \text{ and } u\overline{y} = \mathsf{T} \Leftrightarrow v\overline{y} = \mathsf{T}.$$

This last characterisation does not mention r anymore. Further define:

$$\boxed{u^G} \equiv \{v \in \mathcal{M} \mid v =_G u\}$$

Fact 6.1.3 (a) If $u =_G v$ then $u = \mathsf{T} \Leftrightarrow v = \mathsf{T}$ and $u = \perp \Leftrightarrow v = \perp$.
(b) If $u =_G v$ then $u\overline{y} =_G v\overline{y}$ for all $\overline{y} \in G^{<\omega}$.
(c) $|\mathcal{M}/{=_G}| \leq 2^{\sup(\omega, |G|)}$, thus if G is σ-small then the set of types over G will be σ-small too.
(d) $\perp^G = \{\perp\}$, $\mathsf{T}^G = \{\mathsf{T}\}$, and $u^G \subseteq \mathcal{F}$ if u is proper.
(e) If $u \in G^\circ$, then $u^G \subseteq G^\circ$; thus $G^\circ = \bigcup\{u^G \mid u \in G^\circ\}$.
(f) If $u \in G^\circ$, then $u^G = \uparrow(u^G)$; thus $\delta(G^\circ) = \bigcup\{\delta(u^G) \mid u \in G^\circ\}$.
(g) If $u, v \in G^\circ$ are compatible, then $u^G = v^G$ $(= w^G$ for $w \geq u, v)$.
(h) All $\lambda x_1 \cdots x_n.\mathsf{T}$ $(n \geq 0)$ have different types.

All this is very easy, and so is the exercise below:

Exercise. If $\mathcal{P}$ is a strong premodel then, for all G, u, u^G is closed under sups of bounded subsets and infs of non-empty κ-small subsets; in particular, $|\delta(u^G)| \leq 1$.

For general premodels we will get in fact a much better result than the latter one, but only for $u \in G^\circ$ and G open (or $H \subseteq G \subseteq \uparrow H$ for some $H \subseteq \mathcal{M}_c$): We will show that, in this case, $|\delta(u^G)| = 1$ and $u^G = \uparrow\delta(u^G)$ (c.f. Lemma 6.1.9); in other words, each u^G has a bottom.

Lemma 6.1.4 If $H \subseteq G \subseteq \uparrow H$ and $u \in G^\circ$, then $u^G = u^H$

Proof. $u^G \subseteq u^H$ is clear. Suppose now we have $v =_H u$ (so $u, v \in H^\circ$) and let $\overline{x} \in G^\omega$; let q be a choice function w.r.t. H and $\overline{y} = q(\overline{x}) \in H^\omega$; let also m be such that $\overline{m} = \overline{y}_{[u]}$ ($= \overline{y}_{[v]}$ since $v =_H u$). We have $u\overline{y}_n = \mathsf{T} = v\overline{y}_n$ for each $n \geq m$ and $u\overline{y}_n, v\overline{y}_n \in \mathcal{F}$ for each $n < m$. Thus u and v behave the same way on all $\overline{x} \in G^\omega$, so $u =_G v$. $\diamondsuit$

Lemma 6.1.5 If G is essentially κ-small and $u \in G^\circ$, then u^G is open.

Proof. Because of Lemma 6.1.4 it is enough to consider the case where $|G| < \kappa$. Now, for all $\overline{y} \in \mathcal{M}^{<\omega}$ define the sets

$$\begin{array}{rcl} \mathcal{T}(\overline{y}) & = & \{v{\in}\mathcal{M} \mid v\overline{y} = \mathsf{T}\} \\ \mathcal{F}(\overline{y}) & = & \{v{\in}\mathcal{M} \mid v\overline{y} \in \mathcal{F}\} \end{array}$$

Using Fact 3.1.1 these sets are easily seen to be κ-open. Now,

$$\begin{array}{rcl} u^G & = & \bigcap\{\mathcal{T}(\overline{y}) \mid \overline{y} \in G^{<\omega} \wedge u\overline{y} = \mathsf{T}\} \\ & \cap & \bigcap\{\mathcal{F}(\overline{y}) \mid \overline{y} \in G^{<\omega} \wedge u\overline{y} \in \mathcal{F}\} \end{array}$$

thus u^G is an intersection of $|G^{<\omega}|$ open sets. If G is κ-small, then $G^{<\omega}$ is κ-small too (since $\kappa > \omega$) and u^G is open (c.f. fact 2.1.1). $\diamondsuit$

Corollary 6.1.6 If G is essentially κ-small then G° is open.

Proof. Use Lemma 6.1.5 and Fact 6.1.3 (e). $\diamondsuit$

Lemma 6.1.7 If G is open, then, for all $u \in \mathcal{M}$, $u^G = u^{G_c}$.

Proof. Obviously $u =_G v$ implies $u =_{G_c} v$. Now, each element of G is the directed sup of elements in G_c; using that iterated application is κ-continuous we have that

$$\forall \overline{y}{\in}G^{<\omega} : u\overline{y} = \sup\{u\overline{z} \mid \overline{z} \leq \overline{y} \wedge \overline{z} \in G_c^{<\omega}\}$$

Thus $t(u/G)$ is completely determined by $t(u/G_c)$, i.e. $u =_{G_c} v \Rightarrow u =_G v$. $\diamondsuit$

To continue the study of u^G and G° we need an intermediate definition.

Definition. For G open, $\boxed{\Downarrow_G}$ is the element of $\mathcal{M}$ defined by:

$$\Downarrow_G \equiv \mathsf{Y}\lambda k.\lambda u.\mathsf{if}\, u\, \mathsf{T}\, \lambda y.[\mathsf{if}\,(\chi_G y)\,(k(uy))\,\bot]$$

where Y is some fixed point operator in $\mathcal{M}$ and χ_G is the characteristic function of G. Thus, for all $u \in \mathcal{M}$ we have:

$$\Downarrow_G u = \mathsf{if}\, u\, \mathsf{T}\, \lambda y.[\mathsf{if}\,(\chi_G y)\,(\Downarrow_G(uy))\,\bot]$$

Hence:

$$\begin{array}{l} r(\Downarrow_G u) = r(u) \\ y \in G \Rightarrow \Downarrow_G uy = \Downarrow_G(uy) \end{array}$$

From these properties we easily get:

Lemma 6.1.8 For all open G and $u \in \mathcal{M}$, we have $\Downarrow_G u \in u^G$.

Proof. We have, for all $\overline{x} \in G^\omega$ and $n \in \mathbf{N}$: $\Downarrow_G u\overline{x}_n = \Downarrow_G (u\overline{x}_n)$. Now, $\Downarrow_G(u\overline{x}_n) = \bot$ (or T) iff $u\overline{x}_n = \bot$ (or T). Thus $(\Downarrow_G u)\overline{x}_n \equiv \Downarrow_G u\overline{x}_n = \bot$ (or T) iff $u\overline{x}_n = \bot$ (or T), so $u =_G \Downarrow_G u$ which proves $\Downarrow_G u \in u^G$. $\diamondsuit$

Lemma 6.1.9 For all open G and all $u \in G^\circ$:
(a) $\Downarrow_G u$ is the minimum of u^G, and $u^G = \uparrow\{\Downarrow_G u\}$.
(b) If G is essentially κ-small then u^G is open and $\Downarrow_G u \in \mathcal{M}_c$ (or G°_c).

Proof. (a) We first show that, if $v =_G u =_G \Downarrow_G u$ and $\Downarrow_G u \not\leq v$ then there is an infinite sequence $\overline{x} \in G^\omega$ such that, for all $n \in \omega$,

$$\Downarrow_G u\overline{x}_n \not\leq v\overline{x}_n \tag{14}$$

Since (14) implies

$$u\overline{x}_n \neq \bot, \mathsf{T} \tag{15}$$

(because u, $\Downarrow_G u$ and v have the same type over G) the existence of such an $\overline{x}$ contradicts $u \in G^\circ$.

We build $\overline{x}_n$ by induction in n. As usual, $\overline{x}_0$ is the empty sequence. Suppose we know $\overline{x}_n$ satisfying (14). Then both sides of the inequality are proper maps (because of (15), and there is an $x_{n+1} \in \mathcal{M}$ such that $\Downarrow_G u\overline{x}_n x_{n+1} \not\leq v\overline{x}_n x_{n+1}$ (c.f. Lemma 3.1.6). This forces x_{n+1} to be in G, because of the obvious:

Fact 6.1.10 If $\Downarrow_G u x_1 \cdots x_n \neq \mathsf{T}$ and $x_{n+1} \notin G$ then $\Downarrow_G u x_1 \cdots x_{n+1} = \bot$.

The second claim follows, since $u \in G^\circ$ implies $u^G = \uparrow u^G$ (c.f. Fact 6.1.3).

(b) follows from Lemma 6.1.5 and (a). $\diamondsuit$

Theorem 6.1.11 For all open G, $G \neq \emptyset$,
(a) $G^\circ = \uparrow\delta(G^\circ)$ and $\delta(G^\circ) = \{\Downarrow_G u \mid u \in G^\circ\}$ is an infinite set of incompatible elements.
(b) if G is essentially κ-small, then G° is open and $\delta(G^\circ) \subseteq \mathcal{M}_c$.
(c) if G is essentially σ-small, then G° is essentially σ-small (and, hence, κ-small).

We cannot deduce that G° is essentially κ-small in case (b) above, unless κ is inaccessible.
Proof. (a) use Lemma 6.1.9 and Fact 6.1.3 (e–h).
(b) use Corollary 6.1.6.
(c) follows from $|\delta(G^\circ)| = |G^\circ/=_G|$ and Fact 6.1.3 (c). $\diamondsuit$

Since $G^\circ = H^\circ$ for those H such that $H \subseteq G \subseteq \uparrow H$ (c.f. Fact 6.1.2 (d)) we have:

Corollary 6.1.12 The same conclusions hold for those $G \subseteq \mathcal{M}$ such that there exist $H \subseteq \mathcal{M}_c$ such that $H \subseteq G \subseteq \uparrow H$ (and $|H| < \kappa$ or σ if needed).

6.2 Arrows

Definition. To each $u \in \mathcal{M}$ is associated the open set

$$\boxed{\operatorname{dom} u} \equiv \{x{\in}\mathcal{M} \mid ux \neq \bot\}$$

In order to be able to control the nature and size of

$$H \to K \equiv \{u{\in}\mathcal{M} \mid \forall x{\in}H : ux \in K\}$$

from that of H and K, we define, for $G \neq \emptyset$:

$$\begin{array}{lcl} \boxed{H \to_G K} & \equiv & \{u \in H \to K \mid \\ & & \operatorname{dom} u \subseteq H \wedge \forall x, y{\in}\mathcal{M} : (x =_G y \Rightarrow ux = uy)\} \end{array}$$

Fact 6.2.1 (a) $|H \to_G K| \leq 2 + |K|^{|\mathcal{M}/=_G|}$. In particular, if G and K are σ-small, then $H \to_G K$ is σ-small.

Note that $|(H \to_G K) \setminus \{\mathsf{T}, \bot\}| \leq |K|^{|\mathcal{M}/=_G|}$; this explains where "2" comes in.
(b) $G^\circ \to K$ is obviously increasing (i.e. "covariant") both w.r.t. G and K (for $\subseteq$). This is still true w.r.t. K for $G^\circ \to_G K$, but false w.r.t. G (unless $\bot \in K$, a case we are not interested in).
(c) If $\bot \notin K$, then $\bot, \mathsf{T} \notin G^\circ \to_G K$.
(d) For all $u \in G^\circ \to_G K$ we have $u\bot = \bot$, since $\bot \notin G^\circ$.
(e) $G \to G^\circ = G^\circ$ for any G.
(f) $G^\circ \to_G \delta(K) \subseteq \delta(G^\circ \to K)$ for any G and K.

In fact the constraints in the definition of $H \to_G K$ are so strong that $H \to_G K$ is empty, for example, if H is not enough upwards closed, or if $\bot \notin K$ and some v^G meets both H and $\mathcal{M} \setminus H$. However, these constraints are completely coherent in the cases we are interested in, namely $H = G^\circ$ and G κ-small (where H is open by Corollary 6.1.6).

Lemma 6.2.2 Let G open, $G \neq \emptyset$, $K \subseteq \mathcal{M}$, and $s : \delta(G^\circ) \to K$. Then $g \equiv s_{G^\circ}$ defined by

$$g(x) = \begin{cases} s(\Downarrow_G x) & \text{if } x \in G^\circ \\ \bot & \text{otherwise} \end{cases}$$

is κ-continuous and $\lambda(g) \in G^\circ \to_G K$.

Proof. First we show

$$x \leq y \wedge g(x) \neq \bot \Rightarrow g(x) = g(y) \tag{16}$$

Indeed $g(x) \neq \bot$ implies $x \in G^\circ$, so $y \in G^\circ$ and $x^G = y^G$ (c.f. Fact 6.1.3 (g)). Hence, $\Downarrow_G x = \Downarrow_G y$ (Lemma 6.1.9), and $g(x) = g(y)$.

Suppose now $a = \sup B$ for some κ-directed B. We have to show $g(a) \leq \sup g(B)$. The non-trivial case is $a \in G^\circ$; then, since G° is open (Corollary 6.1.6), there exists a $c \in B \cap G^\circ$. Now, $c \leq a$; hence $g(a) = g(c) \leq \sup g(B)$ thus g is κ-continuous. It is now clear that $\lambda(g) \in G^\circ \to_G K$. $\diamondsuit$

Corollary 6.2.3 If G is open and essentially κ-small, then $G^\circ \neq G$.

Proof. It is sufficient to consider $G \neq \emptyset$. For any permutation s of $\delta(G^\circ)$ let s_{G° be defined as in Lemma 6.2.2. We have $s_{G^\circ} \in G^\circ \to_G \delta(G^\circ)$, by Lemma 6.2.2, hence $s_{G^\circ} \in \delta(G^\circ \to G^\circ)$ (c.f. Fact 6.2.1 (f)); moreover $s \neq s'$ implies $s_{G^\circ} \neq s'_{G^\circ}$. Thus $|\delta(G^\circ \to G^\circ)| \geq 2^{|\delta(G^\circ)|}$. Now, if $G^\circ = G$, then $G^\circ = G \to G^\circ = G^\circ \to G^\circ$, so $\delta(G^\circ \to G^\circ) = \delta(G^\circ)$. This is a contradiction. $\diamondsuit$

Corollary 6.2.4 If $G \subseteq \mathcal{M}_c$, $|G| < \kappa$, then

$$G^\circ \to \uparrow K = \uparrow(G^\circ \to K) = \begin{cases} \uparrow(G^\circ \to_G K) & \text{if } \mathsf{T} \notin K \\ \uparrow(G^\circ \to_G K) \cup \{\mathsf{T}\} & \text{if } \mathsf{T} \in K \end{cases}$$

Proof. $\supseteq$ is obvious. Now let q be a choice function w.r.t. K, $u \in G^\circ \to \uparrow K$, $f \equiv A(u)$ and $g \equiv (g \circ f)_{G^\circ}$. Then $v \equiv \lambda(g) \in G^\circ \to_G K$ and $v \leq u$ since $vx = q(u(\Downarrow_G x))$ if $x \in G^\circ$ and $\perp$ otherwise. $\diamondsuit$

Lemma 6.2.5 If $G, K \subseteq \mathcal{M}_c$ and $|G| < \sigma$, then $G^\circ \to_G K \subseteq \mathcal{M}_c$.

Proof. We consider the functions $g_{u,v}$ defined, for $u \in G^\circ$ and $v \in K$, by

$$g_{u,v} = \begin{cases} v & \text{if } x \in u^G \\ \perp & \text{otherwise} \end{cases}$$

$g_{u,v}$ is κ-continuous because $g_{u,v} = s_{G^\circ}$ for $s : \delta G^\circ \to K \cup \{\perp\}$ defined by $s(\Downarrow_G u) = v$, $s(\Downarrow_G u') = \perp$ if ${u'}^G \neq u^G$ (c.f. Lemma 6.2.2. That $g_{u,v}$ is κ-compact follows from the fact that v is compact and that, for any κ-continuous f, we have:

$$g_{u,v} \leq f \text{ iff } g_{u,v}(\Downarrow_G u) \leq f(\Downarrow_G u)$$

Now it is enough to notice that each element $w \in G^\circ \to_G K$ is either $\perp$ (if $\perp \in K$) or (the code of) the sup of at most $|G^\circ / =_G|$ compatible functions $g_{u,v}$. If G is σ-small then w is the sup of a σ-small, hence κ-small, set of κ-compact elements, so it is κ-compact too. $\diamondsuit$

Corollary 6.2.6 Suppose $G \subseteq \mathcal{M}_c$ and $|G| < \sigma$; then $G^\circ \to_G G \subseteq \mathcal{M}_c$ and $|G^\circ \to_G G| < \sigma$.

Proof. The first assertion follows from Lemma 6.2.5. Now $|G^\circ \to_G G| \leq |G|^{|\mathcal{M}/=_G|}$ (Fact 6.2.1) and G and $\mathcal{M}/=_G$ are σ-small (Fact 6.1.3; since σ is inaccessible we get that $|G^\circ \to_G G|$ is σ-small too. $\diamondsuit$

Corollary 6.2.7 Suppose G and H are open and G is essentially κ-small, then $G^\circ \to H$ is open.

Proof. $G = \uparrow K$ for some $K \subseteq G_c$, $|K| < \kappa$, and $G^\circ \to H = K^\circ \to H = \uparrow(K^\circ \to_K H)$ (c.f. Corollary 6.2.4) and $K^\circ \to_K H \subseteq \mathcal{M}_c$ (c.f. Lemma 6.2.5). $\diamondsuit$

Lemma 6.2.8 If G and H are open, and G is essentially σ-small, then

$$G^\circ \to H = \bigcup\{G^\circ \to H' \mid \emptyset \neq H' \subseteq H,\ H' \text{ open and essentially } \sigma\text{-small}\}$$

Proof. Let q be a choice function w.r.t. H_c. By Theorem 6.1.11, $G^\circ = \uparrow\delta(G^\circ)$ and $\delta(G^\circ)$ is σ-small. For any $u \in G^\circ \to H$, $uG^\circ \subseteq \uparrow u\delta(G^\circ) \subseteq \uparrow q(u\delta(G^\circ))$. Now $q(u\delta(G^\circ))$ is a σ-small subset of H_c, hence $H' = \uparrow q(u\delta(G^\circ))$ is open and essentially σ-small; and of course $u \in G^\circ \to H'$. $\diamondsuit$

7 The existence of a well behaved Φ

ϕ and ε will be defined, as in Section 4, from an open set Φ of $\mathcal{M}$, but we now put further constraints on Φ in order to model the well-foundedness axioms. This amounts more or less to ensure that $(\Phi/\doteq, \dot\in)$ is a model of *ZFC* where $\dot\in$ and $\doteq$ are the interpretations of two $\Lambda_\mathcal{C}$-terms defined in [18], p.20 and recalled in Appendix C. See Appendix A.3 for a closer examination of $(\Phi/\doteq, \dot\in)$.

We prove in Section 7.1 the existence of a well-behaved Φ (in the sense of Theorem 7.1.1 below).

Recall that for any $E \subseteq \mathcal{M}$, $\mathcal{P}_\sigma(E)$ is the set of σ-small subsets of E and $\mathcal{O}_\sigma(E)$ is the set of essentially σ-small open subsets of E. This notation will merely be used for E open. Note that $\mathcal{O}_\sigma(E)$ is closed under σ-small unions since σ is regular.

7.1 Solving the Generic Closure Property

This section is devoted to the proof of the following theorem:

Theorem 7.1.1 Suppose $\kappa \geq \sigma$, σ inaccessible. Then in any κ-continuous premodel $\mathcal{P}$ there is an open set Φ such that:

(1) $\mathsf{T} \in \Phi$, $\perp \notin \Phi$, $\mathsf{F} \in \Phi$.

(2) $\Phi \subseteq \Phi^\circ$

(3) $\Phi = \bigcup\{G^\circ \to \Phi \mid G \in \mathcal{O}_\sigma(\Phi)\}$

(4) $\Phi = \uparrow\Psi$ for some $\Psi \subseteq \mathcal{M}_c$ such that $|\Psi| \leq \sigma$; in particular, if $\sigma < \kappa$, then Φ will be essentially κ-small.

This will be sufficient to prove the consistency of the well-foundedness axioms. However, we will use two refinements of (3) which motivate the definition below:

Definition. For any open set Φ of $\mathcal{M}$ we define:

$$\begin{array}{rcl} \mathcal{F}_1(\Phi) & = & \bigcup\{G^\circ \to \Phi \mid G \in \mathcal{O}_\sigma(\Phi)\} \\ \mathcal{F}_2(\Phi) & = & \bigcup\{G^\circ \to G \mid G \in \mathcal{O}_\sigma(\Phi)\} \\ \mathcal{F}_3(\Phi) & = & \bigcup\{G^\circ, G^\circ \to G \mid G \in \mathcal{O}_\sigma(\Phi)\} \end{array}$$

where $A, B \to C$ means $A \to (B \to C)$. All the $\mathcal{F}_i(\Phi)$ are open sets (c.f. Corollary 6.2.7).

Lemma 7.1.2 $\mathcal{F}_1 = \mathcal{F}_2$

Proof. Let Φ by any open set; then $G^\circ \to G \subseteq G^\circ \to \Phi$ for any $G \in \mathcal{O}_\sigma(\Phi)$. Conversely, $G^\circ \to \Phi = \bigcup\{G^\circ \to K \mid G, K \in \mathcal{O}_\sigma(\Phi)\}$ (Lemma 6.2.8) $\subseteq \bigcup\{L^\circ \to L \mid L \in \mathcal{O}_\sigma(\Phi)\}$ (just take $L = G \cup K$). So $\mathcal{F}_1(\Phi) = \mathcal{F}_2(\Phi)$. $\diamondsuit$

Lemma 7.1.3 If $\Phi = \mathcal{F}_1(\Phi)$, then

(a) $\forall K{\in}\mathcal{P}_\sigma(\Phi)\exists H{\in}\mathcal{O}_\sigma(\Phi) : K \subseteq H^\circ \to H \quad (\subseteq H^\circ \to \Phi)$

(b) $\forall G{\in}\mathcal{O}_\sigma(\Phi)\exists H{\in}\mathcal{O}_\sigma(\Phi) : G \subseteq H^\circ \to H \quad (\subseteq H^\circ \to \Phi)$

(c) $\Phi = \mathcal{F}_3(\Phi)$

Proof. We use $\mathcal{F}_1 = \mathcal{F}_2$ freely.

(b) follows from (a): indeed if $G \in \mathcal{O}_\sigma(\Phi)$, then $G = \uparrow K$ for some $K \in \mathcal{P}_\sigma(\Phi)$, and $K \subseteq H^\circ \to H$ implies $G \subseteq H^\circ \to H$ for some $K \in \mathcal{P}_\sigma(\Phi)$.

(c) follows from (b), indeed $G^\circ \to G \subseteq G^\circ \to (H^\circ \to H)$ for some $H \in \mathcal{O}_\sigma(\Phi)$; hence $G^\circ \to G \subseteq L^\circ \to (L^\circ \to L)$ with $L = G \cup H$.

(a) Let $K \in \mathcal{P}_\sigma(\Phi)$; for any $v \in K$ there is $H_v \in \mathcal{O}_\sigma(\Phi)$ such that $v \in H_v^\circ \to H_v$ (since $\Phi \subseteq \mathcal{F}_2(\Phi)$). Now $H = \bigcup\{H_v \mid v \in K\}$ is still in $\mathcal{O}_\sigma(\Phi)$ (since K is σ-small) and $v \in H^\circ \to H$ for all $v \in K$. Thus $K \subseteq H^\circ \to H$. $\diamondsuit$

Corollary 7.1.4 Any Φ as in Theorem 7.1.1 satisfies:

$$\begin{array}{lcl} \Phi & = & \bigcup\{G^\circ \to G \mid G \in \mathcal{O}_\sigma(\Phi)\} \\ \Phi & = & \bigcup\{G^\circ, G^\circ \to G \mid G \in \mathcal{O}_\sigma(\Phi)\} \end{array}$$

There are many other decompositions of Φ that would be sufficient for our purpose (all the reasonable ones work). A way of proving Theorem 7.1.1 is to view it as a corollary of:

Theorem 7.1.5 Suppose $\kappa \geq \sigma$. Then, in any κ-continuous premodel $\mathcal{P}$ there is a subset Ψ of $\mathcal{M}_c$ such that:

(1) $\Psi = \{\top\} \cup \bigcup\{H^\circ \to_H H \mid H \in \mathcal{P}_\sigma(\Psi)\}$

(2) $\Psi \subseteq \Psi^\circ$

(3) $|\Psi| \leq \sigma$ (this is the only place we need that σ is inaccessible).

Proof of Theorem 7.1.1 from Theorem 7.1.5. Let $\Phi = \uparrow\Psi$. Since $\top \in \Psi$ and $\bot \notin \Psi$, we have $\top \in \Phi$ and $\bot \notin \Phi$. Also $\Phi = \uparrow\Psi \subseteq \Psi^\circ = \Phi^\circ$ (Fact 6.1.2). We now show that $\Phi = \mathcal{F}_2(\Phi)$. Indeed:

$$\begin{array}{rcll} \Phi = \uparrow\Psi & = & \bigcup\{\uparrow(H^\circ \to_H H) \mid H \in \mathcal{P}_\sigma(\Psi)\} \cup \{\top\} & \\ & = & \bigcup\{H^\circ \to \uparrow H \mid H \in \mathcal{P}_\sigma(\Psi)\} & \text{(Corollary 6.2.4)} \\ & \subseteq & \bigcup\{G^\circ \to G \mid G \in \mathcal{O}_\sigma(\Phi)\} & \text{(just take } G = \uparrow H) \end{array}$$

Conversely, if $G \in \mathcal{O}_\sigma(\Phi)$, then there is a $K \subseteq G$ such that $G = \uparrow K$ and $|K| < \sigma$; if q is a choice function w.r.t. Ψ as defined at the beginning of Section 6 then $H = q(K)$ satisfies $H \in \mathcal{P}_\sigma(\Phi)$, $G \subseteq \uparrow H$, and $H^\circ \subseteq G^\circ$ (c.f. fact 6.1.2) thus $G^\circ \to G \subseteq H^\circ \to \uparrow H$, and the inclusion $\Phi \subseteq \mathcal{F}_2(\Phi)$ above is in fact an equality.

Finally, $\mathsf{F} = \lambda x.\mathsf{T} \in \{\mathsf{T}\}^\circ \to \{\mathsf{T}\}$, hence $\mathsf{F} \in \Phi$. $\diamondsuit$

The proof of Theorem 7.1.5 occupies the rest of the subsection.

Definition. Ψ is the least subset X of $\mathcal{M}$ such that:

$$(1)\ \mathsf{T} \in X \qquad (2)\ G \in \mathcal{P}_\sigma(X) \Rightarrow G^\circ \to_G G \subseteq X \tag{17}$$

Since property (17) is closed under intersection, Ψ is the intersection of all $X \subseteq \mathcal{M}$ which satisfy (17).

The direct analogous definition for Φ as "the least open X of $\mathcal{M}$" such that

$$(1)\ \mathsf{T} \in X \qquad (2)\ G \in \mathcal{O}_\sigma(X) \Rightarrow G^\circ \to G \subseteq X \tag{18}$$

would not have worked since the family of open sets X satisfying (18) is not of limited size and the intersection needs not be open.

Lemma 7.1.6 $\Psi \subseteq \mathcal{M}_c$

Proof. $\mathcal{M}_c$ satisfies (17) (c.f. Lemma 6.2.5). $\diamondsuit$

Lemma 7.1.7 $\perp \notin \Psi$

Proof. $\Psi \setminus \{\perp\}$ satisfies (17) because Ψ satisfies (17) and because $\perp \notin G$ implies $\perp \notin G^\circ \to_G G$; thus, by minimality of Ψ, we have $\Psi = \Psi \setminus \{\perp\}$. Hence, $\perp \notin \Psi$. $\diamondsuit$

Lemma 7.1.8 $\Psi = \Psi'$ where $\Psi' = \bigcup\{G^\circ \to_G G \mid G \in \mathcal{P}_\sigma(\Psi)\} \cup \{\mathsf{T}\}$

Proof. $\Psi' \subseteq \Psi$ is obvious; conversely, if $x \in \mathcal{M} \setminus \Psi'$, then $\Psi \setminus \{x\}$ satisfies (17), hence $x \notin \Psi$. $\diamondsuit$

To prove $\Psi \subseteq \Psi^\circ$ we need another characterisation of Ψ:

Lemma 7.1.9 Ψ is the least subset X of $\mathcal{M}$ such that

$$(1)\ T \in X \qquad (2)\ G \in \mathcal{P}_\sigma(X \cap \Psi) \Rightarrow G^\circ \to_G G \subseteq X \tag{19}$$

(and is the intersection of all these subsets X).

Proof. Let Ψ'' be the intersection of all X satisfying (19). Obviously, Ψ satisfies (19), therefore $\Psi'' \subseteq \Psi$. But now it is clear that Ψ'' satisfies (17), hence $\Psi \subseteq \Psi''$ which ends the proof. $\diamondsuit$

Lemma 7.1.10 $\Psi \subseteq \Psi^\circ$

Proof. It is enough to prove that Ψ° satisfies (19) (c.f. Lemma 7.1.9); and we already know that $\mathsf{T} \in \Psi^\circ$.

Let $G \in \mathcal{P}_\sigma(\Psi^\circ \cap \Psi)$; we have to prove that $G^\circ \to_G G \subseteq \Psi^\circ$. Since $G^\circ \to_G G \subseteq G^\circ \to G \subseteq G^\circ \to \Psi^\circ$ and since $\Psi^\circ = \Psi \to \Psi^\circ$ (c.f. Fact 6.2.1 (e)) it is enough to prove that $\Psi \subseteq G^\circ$ and, for this, to prove that G° satisfies (19). We already know that $\mathsf{T} \in G^\circ$. Let $H \in \mathcal{P}_\sigma(G^\circ \cap \Psi)$; we have to prove that $H^\circ \to_H H \subseteq G^\circ$. Now $H \subseteq \Psi$, hence $\Psi^\circ \subseteq H^\circ$, hence $G \subseteq H^\circ$. Thus $H^\circ \to_H H \subseteq H^\circ \to H \subseteq G \to H \subseteq G \to G^\circ = G^\circ$ as required. $\diamondsuit$

For dealing with the size of Ψ we need a last characterisation: The induction one.

Definition. $(\Psi_\alpha)_{\alpha \leq \sigma}$ is the increasing sequence of subsets of $\mathcal{M}$ defined by

$$\begin{array}{lcll} \Psi_0 & = & \{\mathsf{T}\} & \\ \Psi_{\alpha+1} & = & \{\mathsf{T}\} \cup \bigcup \{G^\circ \to_G G \mid G \subseteq \Psi_\alpha\} & \\ \Psi_\alpha & = & \bigcup_{\beta<\alpha} \Psi_\beta & \text{for limit ordinals } \alpha \end{array}$$

Lemma 7.1.11 $\forall \alpha < \sigma : |\Psi_\alpha| < \sigma$

This is the only point where we use that σ is inaccessible.
Proof. By induction on α. For limit ordinals α we use that σ is regular. For successor ordinals α we use Corollary 6.2.6 (if G is σ-small then $G^\circ \to_G G$ is σ-small too), the fact that $|\Psi_\alpha| < \sigma \Rightarrow 2^{|\Psi_\alpha|} < \sigma$ (σ inaccessible), and that the union of less than σ sets of cardinality less that σ is less than σ (regularity of σ). $\diamondsuit$

Lemma 7.1.12 $\Psi = \Psi_\sigma$ and $|\Psi| \leq \sigma$

Proof. It is obvious that for any $\alpha \leq \sigma$, $\Psi_\alpha \subseteq \Psi$ (induction in α), hence $\Psi_\sigma \subseteq \Psi$. Conversely any σ-small subset G of Ψ_σ is already in one of the Ψ_α, $\alpha < \sigma$; hence Ψ_σ satisfies (17) and $\Psi \subseteq \Psi_\sigma$. Now $|\Psi| \leq \sigma$ since Ψ is the increasing union of σ subsets of $\mathcal{P}_\sigma(\Psi)$. $\diamondsuit$

It has now been verified that Ψ as defined by (17) satisfies Theorem 7.1.5, which ends the proof of that theorem.

7.2 Conclusion

Theorem 7.2.1 If κ is $>$ some inaccessible cardinal σ, then any κ-continuous premodel can be expanded to a model of map theory.

Proof. It is sufficient to interpret ϕ by (the code of) the characteristic function of some open set $\Phi = \uparrow\Psi$ satisfying the constraints in Theorem 7.1.1, and to interpret ε by a choice function w.r.t. Φ, as in Section 4.1. $\diamondsuit$

Remark. The preceding construction provides of course as many solutions ϕ as there are inaccessible cardinals σ below κ. For a given σ it yields the smallest possible Φ (this is indeed clear from the proof of Lemma A.1.1).

8 Elementary construction of a premodel

A solution $\mathcal{D}$ to (1) will be obtained from a structured web C (a p.o with coherence relation) satisfying a more simple recursive equation. As mentioned in Section 2.4, $\mathcal{D}$ will in fact be a prime algebraic domain and the web C will be isomorphic to $\mathcal{D}_p$. This solution $\mathcal{D}$ will be referred to as a *canonical (κ-continuous) premodel of MT*; the model obtained by interpreting furthermore ϕ and ε as in Section 7 will be referred to as a *canonical model of MT*. All canonical models and premodels are obtained by letting σ, κ, $\sigma \leq \kappa$ range over all regular cardinals. A canonical model is only a model of *MT* if σ is inaccessible and $\sigma < \kappa$].

8.1 Preordered coherent spaces

A preordered coherent space (pcs) is a triple $C = (D, \sim, \preceq)$ such that

(a) $\sim$ is a reflexive and symmetric relation on D.

(b) $\preceq$ is a reflexive and transitive relation on D.

(c) $x \sim y \wedge x' \preceq x \wedge y' \preceq y \Rightarrow x' \sim y'$

C is an ocs (ordered coherent space) if furthermore $\preceq$ is a p.o (i.e. is antisymmetric).

Example. Starting from any κ-Scott domain $\mathcal{D}$, any subset E of $\mathcal{D}$, equipped with the induced partial order and compatibility relation ($x \sim y$ if $\{x, y\}$ has an upper bound in $\mathcal{D}$), is an ocs, but the relevant example for our purpose is $E = \mathcal{D}_p$.

$(D, \sim, \preceq) \sqsubseteq (D', \sim', \preceq')$ will mean $D \subseteq D' \wedge \sim = \sim' \cap D^2 \wedge \preceq = \preceq' \cap D^2$. The following are easy:

(d) The 'empty pcs' $(\emptyset, \emptyset, \emptyset)$ is the smallest pcs (w.r.t. $\sqsubseteq$).

(e) If $(C_\beta)_{\beta<\alpha}$ is an increasing sequence of pcs's, then $C = \cup_{\beta<\alpha} C_\beta$ (with the obvious meaning: union of domains, union of relations), is a pcs; it is the smallest pcs such that $C_\beta \sqsubseteq C$ for all $\beta < \alpha$.

Elements of C (i.e. elements of D) will be denoted p, q, x, y. Subsets of C will be denoted u, v. $u \subseteq C$ is coherent if $\forall x, y {\in} u : x \sim y$. $u \subseteq C$ is an initial segment if $\forall x, y : (x \in u \wedge y \preceq x \Rightarrow y \in u)$. $u \sim v$ if $u \cup v$. For any $u \subseteq C$ let $\overline{u}$ denote the initial segment generated by u. The following is clearly equivalent to (c):

(c') $\forall u : u$ coherent $\Rightarrow \overline{u}$ coherent

Let $C \cong C'$ denote that the pcs's C and C' are isomorphic.

8.2 Transferring the problem to pcs's

To any pcs $C = (D, \sim, \preceq)$, we associate the p.o $\mathcal{D} = S(C)$ of all coherent, initial segments of D ordered by inclusion. $\mathcal{D}$ will be called "the domain of web C".

It is easy to check that $S(C)$ is a prime algebraic domain where sup, when defined, is union, and that $u \in S(C)$ is prime iff there is a $p \in C$ such that $u = \overline{\{p\}}$.

Thus $\mathcal{D}$ is a κ-prime algebraic domain (for any regular κ). If C is an ocs then $\mathcal{D}_p$ is isomorphic to C (as ocs) (in the general case we would have to quotient C by the equivalence relation induced by the preorder) and $\mathcal{D}$ appears as the completion of C which add sups to all subsets of compatible elements of C.

We now fix κ, and let $C^\star$ be the set of coherent κ-small subsets of C. It is easily seen (or c.f. remarks in Section 2.4) that $u \in \mathcal{D}$ is κ-compact iff $u = \overline{a}$ for some $a \in C^\star$. These remarks link proposition 8.2.1 below to Section 2.5 and motivate the following definition:

Definition. F is the function which associates to any pcs $C = (D, \sim, \preceq)$ the triple $F(C) = (D_F, \sim_F, \preceq_F)$ where $D_F = C^\star \times D$, $(u, x) \sim_F (v, y) \Leftrightarrow (u \sim v \Rightarrow x \sim y)$ and $(u, x) \preceq_F (v, y) \Leftrightarrow x \preceq y \wedge \overline{v} \subseteq \overline{u}$.

It is straightforward to check that $F(C)$ is a pcs, and that F is monotone (w.r.t. $\sqsubseteq$). Moreover, since κ is regular, F commutes with increasing unions indexed by κ: If $C = \cup_{\alpha<\kappa} C_\alpha$ is an increasing union of pcs, then $F(C) = \cup_{\alpha<\kappa} F(C_\alpha)$. We denote also $F(C)$ by $C^\star \times C$.

Lemma 8.2.1 If $\mathcal{D}$ is the domain of web C, then $[\mathcal{D} \to \mathcal{D}]_\kappa$ is order isomorphic, and hence κ-isomorphic, to the domain of web $F(C) \equiv C^\star \times C$.

Proof. Let $C = (D, \sim, \preceq)$. Define Tr and A' on $[\mathcal{D} \to \mathcal{D}]_\kappa$ and $S(C^\star \times C)$, respectively, by:

$$\begin{array}{lcll} \mathrm{Tr}(g) & = & \{(a, p) \in D^\star \times D \mid p \in g(\overline{a})\} & \text{for } g \in [\mathcal{D} \to \mathcal{D}]_\kappa \\ A'(u)(v) & = & \{x \in D \mid \exists a \subseteq v : (a, x) \in u\} & \text{for } u, v \in S(C^\star \times C) \end{array}$$

It is straightforward to prove that Tr and A' are inverse order isomorphisms, and hence κ-isomorphisms between the two domains. (Hint: 7 points to check).

Formally the proof of 8.2.1 does not differ from the ω-case. For the ω-semantics this is worked out in [25] in a slightly different formalism, in [24] for pcs's with trivial coherence; [16] is not really relevant for pcs's with trivial preorders since it deals with the more accurate class of "stable" functions. $\diamondsuit$

Remark. A corollary of Lemma 8.2.1 is that the problem of solving $\mathcal{D} \cong [\mathcal{D} \to \mathcal{D}]$ in the κ-ccc reduces to finding a pcs C such that $C \cong F(C)$ (it is quite obvious that isomorphic pcs's generate κ-isomorphic domains), or even $C = F(C)$. The second equation has no solution in a well-founded universe.

In order to take $\tilde{\top}$ and $\tilde{\bot}$ into account we define a function G on pcs's such that the problem of solving (1) reduces to finding a pcs C such that $C = G(F(C))$, where G will be chosen such that $C = G(F(C))$ has a solution in any well-founded universe.

First we fix two elements f and t of the universe which are not pairs. If $C = (D, \sim, \preceq)$ and $f, t \notin D$ we define $G(C) = (D_G, \sim_G, \preceq_G)$ where $D_G =$

$D \cup \{t, f\}$, $x \sim_G y \Leftrightarrow x \sim y \vee x = y = t \vee x = f \wedge y \neq t \vee y = f \wedge x \neq t$ and $x \preceq_G y \Leftrightarrow x \preceq y \vee x = y = t \vee x = f \wedge y \neq t$.

It is easy to see that $G(C)$, when defined, is a pcs (an ocs if C is), that G is monotone and that $G(F(C))$ is always defined (since f and t are not pairs). Moreover, G commutes with all increasing unions of pcs's not containing f or t. Hence, $G \circ F$ commutes with all increasing unions of sequences, indexed by κ.

That solving 1 amounts to finding a fixed point to $H = G \circ F$ follows from Lemma 8.2.1 and:

Lemma 8.2.2 If $\mathcal{D}$ is a domain of web C and $f, t \notin \mathcal{D}$, then $\mathcal{D} \oplus_{\tilde{\perp}} \{\tilde{\mathsf{T}}\}$ is order isomorphic to the domain of web $G(C)$.

Proof. Indeed $u \in S(G(C))$ iff $u = \emptyset$ or $u = \{t\}$ or $u = \{f\} \cup u'$ with $u' \in S(C)$. $\diamondsuit$

8.3 Solving $C = H(C)$

We know that $H(C)$ is defined and monotonic for all pcs's C. Hence, the ordinal sequence C_α defined by $C_0 = \emptyset$ and $C_\alpha = \cup_{\beta<\alpha} H(C_\beta)$ is increasing. Since κ is a limit ordinal we have $C_\kappa = \cup_{\beta<\kappa} C_\beta$; since H commutes with such increasing unions we get $H(C_\kappa) = \cup_{\beta<\kappa} H(C_\beta) = C_\kappa$. Hence, $C = C_\kappa$ is a fixed point of H and $\mathcal{D} = S(C)$ is a solution of (1).

Remark 1. As announced at the end of Section 2.2, $|\mathcal{D}_c| = \kappa$ ($= |\mathcal{D}_p|$). Indeed $|C| = \kappa$ (we started from $\emptyset$, added elements at each step and obviously $|C'| \leq \kappa \Rightarrow |H(C')| \leq \kappa$). Obviously $|\mathcal{D}_p| \leq |\mathcal{D}_c| \leq |C^\star| = \kappa$. It remains to prove $\kappa \leq |\mathcal{D}_p|$. Since $\mathcal{D}_p$ and $C/=_{\mathcal{D}}$ are order isomorphic, where $=_{\mathcal{D}}$ is the equivalence relation induced by the preorder $\preceq$, and since κ is regular, it is enough to show that the equivalence class of any $x \in C$ is κ-small. This can be proved easily by induction on the smallest β such that $x \in C_\beta$, using once more that κ is regular.

Remark 2. The following makes explicit the applicative behaviour of $\mathcal{D}$ and the way $\mathcal{D}$ encodes its continuous functions; it is the starting point of any concrete use of the canonical premodel (and model) of *MT*.

In $\mathcal{D} = S(C)$ we have, for all $u, v \in \mathcal{D}$ and any function $h \in [\mathcal{D} \to \mathcal{D}]$:

1. $\perp = \emptyset$, $\mathsf{T} = \{t\}$, $\lambda x.\mathsf{T} = \{f\}$, and if $u \neq \emptyset, \{t\}$ then $f \in u$.
2. $\mathsf{T}v = \mathsf{T}$ and $uv = \{p \in \mathcal{D} \mid \exists a \subseteq v\colon (a, p) \in u\}$ if $u \neq \mathsf{T}$.
3. $\lambda(h) = \{f\} \cup \{(a, p) \in D^\star \times D \mid p \in h(\bar{a})\}$ and the interpretation of any abstraction of $\Lambda^0_{M,C}$ is given by $|\lambda x.\mathcal{A}| = \{f\} \cup \{(a, p) \in D^\star \times D \mid p \in |\mathcal{A}[x := \bar{a}]|$.

Remark 3. Working with $\kappa = \omega$ gives a solution of (1) in the usual ccc of ω-ccpos and ω-continuous functions. This "premodel" is indeed sufficient when one is interested only in the concrete computational features of *MT*. In this case $D^\star$ is the set of coherent finite subsets of D.

In Appendix B we will show that all canonical premodels, and in particular the ω-one, are adequate for concrete computation.

It is worth noticing that C_ω consists of the elements of C_κ which are hereditarily finite sets but that $\mathcal{D}_\omega$ is no substructure of $\mathcal{D}_\kappa$.]

9 Conclusion

The success of *ZF* as a system for founding mathematics is due to both that *ZF* is syntactically simple and that the class of its models is mathematically rich and well structured; in particular each model of *ZF* has a lot of meaningful 'sub' and 'sur'-models (internal models and generic extensions), some of them being 'canonical' (from different points of view). The conviction of the authors is that *MT* shares this wealth with *ZFC*. On one hand models of *ZFC* and *MT* are closely linked, as shown by the syntactic translation in [18] and the constructions in the present paper (c.f. Appendix A); in particular our construction shows that there are at least as many models of *MT* as there are models of *ZFC* + *SI* (and probably *SI* can be weakened).

On the other hand it seems that we will have a lot of freedom in building models of *MT*: indeed we think that most of the techniques or frameworks available for building models of untyped λ-calculus can most probably be adapted to yield models of *MT*: the essence of the work done in the present paper seems to be in fact that all big solutions of (1) in any suitable framework can be expanded to a model of *MT*. For example the constructions made in the paper seem to carry over without problems, along the same lines, to the κ-stable semantics (on κ-dI-domains).

Now, that the model should satisfy (1) is not a necessary condition. In particular we conjecture that the forcing techniques initiated in [2] to show that $\Omega = \delta\delta$ is an easy term (namely that any equation $\Omega = t$, t any closed λ-term, is consistent with untyped λ-calculus) and which are used also in [21, 22, 37] to build models of extended λ-calculi, would apply (for example one could use them to show that Ω is an easy term also w.r.t. *MT*).

It is already interesting to have an explicit and (rather) simple model of *MT* like ours, since first it gives a comprehensible proof of the consistency of *MT* and second provides a concrete support for a simplification of the axiomatisation which would not change its spirit much. But the existence of a great variety of models, realising possibly different equational extensions of *MT*, allows much more freedom for the subsequent axiomatisations of *MT*, and may give hope to justify intuitively-correct other proposals.

Such new axiomatisations are proposed in [17]. Contrary to the present one they incorporate order and monotonicity (and part of stability) at the level of syntax, and hence rule out from the beginning the possibility to carry over a modelisation like in [29]: as a matter of fact such modelisation is incompatible with order and monotonicity since any two elements of the model can be exchanged via a function which is representable in the model.

10 Acknowledgement

The first author is grateful to René Cori, Claude Sureson, Klaus Keimel and Michel Parigot for their support when she was working on [18], including many helpful and stimulating discussions. The second author is grateful to the group at Equipe de Logique Mathématique for discussions that have been essential for putting Map Theory into context, not least witnessed by the present paper. Thanks are also due to Thierry Vallée for comments on the manuscript and for pointing out an error in a proof.

The present paper is issued from a bare version of the consistency proof, called "A simple semantic consistency proof for Map Theory based on ξ-denotational semantics". Following the request of one of the referees a presentation of Map Theory has been added, as well as many motivations; moreover the proof, which formerly followed a bottom-up approach, has been completely restructured. We have furthermore added a more comprehensive treatment of the set theoretic properties of the models, and the proof of the computational adequation of the canonical model. We are grateful to the referees for having given us this opportunity of providing a more mature and comprehensive paper and to correct a lot of small bugs in the meanwhile. We thank also the referees for drawing our attention to several interesting papers, including [13, 15, 29]. In particular, the consistency proof of Flagg-Myhill's system, which we were not aware of, appealed for a comparison with our own work. Such a comparison has been included in the present version. A further step would be, as already mentioned in Section 1.6, to look whether the alternative arguments of [29] could be carried out for Map Theory.

A Further properties of κ-continuous models

A.1 Structural properties of Φ and Φ°

Lemma A.1.1 Ψ and Φ are not essentially σ-small, and $|\Psi| = \sigma$.

Proof. We have not yet proved (neither claimed) that the sequence Ψ_α was strictly increasing. We show here something stronger, namely that $\Phi_\alpha = \uparrow\Psi_\alpha$ is strictly increasing (limit: $\Phi_\sigma = \Phi$). First we note that $\Phi_{\alpha+1} = \Phi^\circ_\alpha \to \Phi_\alpha$. Indeed, $\Phi_{\alpha+1} = \uparrow\Psi_{\alpha+1} = \bigcup\{\uparrow(G^\circ \to_G G) \mid G \subseteq \Psi_\alpha, G \neq \emptyset\} = \bigcup\{G^\circ \to \uparrow G \mid \emptyset \neq G \subseteq \Psi_\alpha\}$(corollary 6.2.4) $= \Psi^\circ_\alpha \to \uparrow\Psi_\alpha = \Phi^\circ_\alpha \to \Phi_\alpha$. And we have, for all $\alpha \leq \sigma$:

$$\Phi_\alpha = \bigcup_{\beta<\alpha} (\Phi^\circ_\beta \to \Phi_\beta)$$

and also $\Phi^\circ_\beta \supseteq \Phi^\circ_\alpha$ (since $\Phi_\beta \subseteq \Phi_\alpha$). Suppose now that α is such that

$$\Phi_\alpha = \Phi^\circ_\alpha \to \Phi_\alpha \tag{20}$$

(where, clearly, $\alpha \neq 0$). Since $\chi_{\Phi^\circ_\alpha} \in \Phi^\circ_\alpha \to \{\mathsf{T}\} \subseteq \Phi^\circ_\alpha \to \Phi_\alpha$ we have $\chi_{\Phi^\circ_\alpha} \in \Phi^\circ_\beta \to \Phi_\beta$ for some $\beta < \alpha$. But this forces $\Phi^\circ_\beta \subseteq \Phi^\circ_\alpha$, hence $\Phi^\circ_\beta = \Phi^\circ_\alpha$ and

$\Phi_\beta = \Phi_\alpha$ (Fact 6.1.2 (f)). Hence, $\Phi_\beta = \Phi_{\beta+1} = \Phi_\beta^\circ \to \Phi_\beta$. Thus there is no first α such that (20) is true and Φ_α is strictly increasing. $\diamondsuit$

Lemma A.1.2 If $\sigma < \kappa$, then all inclusions of Lemma 5.2.1 are strict.

Proof. If $\sigma < \kappa$ then Φ is essentially κ-small (by Lemma 7.1.12 or A.1.1). Hence $\Phi \subseteq \Phi^\circ$ (by Corollary 6.2.3).

Now, if $u \in \Phi^\circ \setminus \Phi$, then $\lambda x.u \in \Phi \to \Phi^\circ$ but $\lambda x.u \notin \Phi \to \Phi$; ϕ and $\lambda x.x$ belong to $\Phi \to \Phi$ but not to $\Phi^\circ \to \Phi$ (similarly: if belongs to $\Phi^3 \to \Phi$ but not $(\Phi^\circ)^3 \to \Phi$). Finally $\varepsilon \in \Phi^\circ \to \Phi$, since elements of Φ° never take the value $\perp$ when applied to elements of Φ, but $\varepsilon \notin \Phi$ since, for all $G \in \mathcal{O}_\sigma(\Phi)$, $\varepsilon \notin G^\circ \to \Phi$: indeed $G \in \mathcal{O}_\sigma(\Phi)$ implies $G \neq \Phi$ since Φ is not essentially σ-small, so $\varepsilon(\chi_G) = \perp$, but obviously $\chi_G \in G^\circ$. $\diamondsuit$

Lemma A.1.3 $\delta(\Phi) = \delta(\Psi) = \{\mathsf{T}\}$

Proof. Let $u \in \Phi$, $u \neq \mathsf{T}$. Then $u \in \Phi_\beta^\circ \to \Phi_\beta$ for some $\beta < \sigma$; since Φ_α is strictly increasing (c.f. the proof of Lemma A.1.1), then $\Phi_\beta \subset \Phi_{\beta+1}$ and $\Phi_{\beta+1}^\circ \subset \Phi_\beta^\circ$ (Fact 6.1.2 (e)). Now the restriction v of u to $\Phi_{\beta+1}^\circ$ is clearly in $\Phi_{\beta+1}^\circ \to \Phi_\beta \subseteq \Phi_{\beta+1}^\circ \to \Phi_{\beta+1} \subseteq \Phi$; moreover $v < u$. $\diamondsuit$

Lemma A.1.4 $\Phi \subseteq \Phi^{\circ\circ} \subseteq \Phi^\circ$. If $\sigma < \kappa$ then both inclusions are strict.

Proof. From $\Phi \subseteq \Phi^\circ$ we immediately get $\Phi^{\circ\circ} \subseteq \Phi^\circ$. For proving $\Phi \subseteq \Phi^{\circ\circ}$ it is enough to prove $\Psi \subseteq \Psi^{\circ\circ}$ and for this to check that $\Psi^{\circ\circ}$ satisfies (19) in Lemma 7.1.9. Now $T \in \Psi^{\circ\circ}$ is clear. Suppose $G \subseteq \Psi \cap \Psi^{\circ\circ}$ and G is σ-small. We have: $G^\circ \to_G G \subseteq G^\circ \to \Psi^{\circ\circ} \subseteq \Psi^\circ \to \Psi^{\circ\circ} = \Psi^{\circ\circ}$. Thus $\Psi^{\circ\circ}$ satisfies (19) as required. Now $\Phi^{\circ\circ} = \Phi^\circ$ implies $\Phi^\circ = \Phi$ (Fact 6.1.2 (e)) but this contradicts $\sigma < \kappa$ (c.f. Corollary 6.2.3. Also $\Phi^{\circ\circ} = \Phi$ implies $\Phi = \uparrow\delta(\Phi)$ (c.f. Theorem 6.1.11) and, hence, $\Phi = \{\mathsf{T}\}$, a contradiction. $\diamondsuit$

Exercise. Define inductively: $\Phi^{\circ(0)} = \Phi$ and $\Phi^{\circ(k+1)} = (\Phi^{\circ(k)})^\circ$. Deduce from Lemma A.1.4 that

(a) $(\Phi^{\circ(2k)})_k$ is an increasing sequence (w.r.t. $\subseteq$) and $(\Phi^{\circ(2k+1)})_k$ is decreasing.

(b) for all $k, m \geq 1$, $\Phi \subseteq \Phi^{\circ(2k)} \subseteq \Phi^{\circ(2m+1)} \subseteq \Phi^\circ$.

(c) all inclusions are strict if $\sigma < \kappa$.

The following lemma expresses where (the interpretations of) the solvable terms of pure lambda-calculus live within κ-continuous models.

Lemma A.1.5 Let $\mathcal{A}$ be a pure, closed λ-term ($\mathcal{A} \in \Lambda$).

(a) if $\mathcal{A}$ is normalisable and of order n then $\mathcal{A} \in \Phi^n \to \Phi \subseteq \Phi^\circ$.

(b) if $\mathcal{A}$ is solvable and $\sigma < \kappa$ then $\mathcal{A} \notin \Phi$.

Recall that $\mathcal{A}$ is (β-)normal of order n iff $\mathcal{A} = \lambda\overline{x}.z\mathcal{A}_1 \cdots \mathcal{A}_m$ where $\ell(\overline{z}) = n \geq 0$, $m \geq 0$, z is a variable and the $\mathcal{A}_i$'s are (β-)normal. $\mathcal{A}$ is normalisable (of order n) iff $\mathcal{A}$ is β-equivalent to a normal term (of order n). $\mathcal{A}$ closed is solvable iff $\exists m \geq 0 \exists \mathcal{B}_1, \ldots, \mathcal{B}_m : \mathcal{A}\mathcal{B}_1 \cdots \mathcal{B}_m \simeq_\beta \lambda x.x$ and the $\mathcal{B}_i$'s are closed normal terms. Any normalisable term is solvable. (See [4] for a treatment of these concepts).

Proof of Lemma A.1.5. (a) Since two β-equivalent terms have the same interpretation it is enough to prove (a) for closed normal terms, and we do it by induction on the length $\ell\mathcal{A}$ of $\mathcal{A}$. So suppose $\mathcal{A} = \lambda\overline{x}.z\mathcal{A}_1 \cdots \mathcal{A}_m$ is normal of order n, and closed. Then $z \in \overline{x}$ and all $\lambda\overline{x}.\mathcal{A}_i$ are closed, normal of order $n_i = n + m_i$. Moreover $\ell(\lambda\overline{x}.\mathcal{A}_i) < \ell\mathcal{A}$. Let now $\overline{u} \in \Phi^n$. $\mathcal{A}\overline{u} = [z/\overline{x}{:=}\overline{u}][\mathcal{A}_1/\overline{x}{:=}\overline{u}] \cdots [\mathcal{A}_m/\overline{x}{:=}\overline{u}]$. Now, $[z/\overline{x}{:=}\overline{u}] \in \Phi$ (since $z \in \overline{x}$). Hence $[z/\overline{x}{:=}\overline{u}] \in \Phi^{\circ m} \to \Phi$ (c.f. Lemma 5.2.1). Also $[\mathcal{A}/\overline{x}{:=}\overline{u}] = (\lambda\overline{x}.\mathcal{A}_i)\overline{u}$; by induction hypothesis $\lambda\overline{x}.\mathcal{A}_i \in \Phi^{n_i} \to \Phi = \Phi^n \to (\Phi^m \to \Phi) \subseteq \Phi^n \to \Phi^\circ$ (c.f. Lemma 5.2.1). Hence $[\mathcal{A}_i/\overline{x}{:=}\overline{u}] \in \Phi^\circ$ for each i. Thus $\mathcal{A}\overline{u} \in \Phi$ as claimed.

(b) Suppose we have $\mathcal{A}\mathcal{B}_1 \cdots \mathcal{B}_m \simeq_\beta \lambda x.x$ for some $m \geq 0$, $\mathcal{B}_i$ closed normal terms. Then $\mathcal{B}_i \in \Phi^\circ$ for all i (by (a)). If we had $\mathcal{A} \in \Phi \subseteq \Phi^{\circ m} \to \Phi$ (c.f. Lemma 5.2.1), we would have $\lambda x.x \in \Phi$. But this contradicts Fact A.1.2). $\diamondsuit$

As an example of a solvable term which does not belong to Φ° we may take any fixed point operator Z. To see this, we first notice that $\dot{\neg} = \lambda x.\mathsf{if}\, x\, \mathsf{F}\, \mathsf{T}$ is in Φ since it is in $\{\mathsf{T},\mathsf{F}\}^\circ \to \{\mathsf{T},\mathsf{F}\}$. Suppose now that Z is in Φ°. Sunce $\dot{\neg}$ is in Φ we have $Z\dot{\neg} \in \Phi^\circ$. This contradicts the fact (left to the reader; follows from QND$'$) that any fixed point of $\dot{\neg}$ is provably equal to $\perp$.

For unsolvable terms the situation is less definite. Let us concentrate on unsolvables of order zero, like Ω. On one hand none is provably in ϕ, since they are all equated to $\perp$ in the canonical models (c.f. Theorem B.0.2 in Appendix B); but on the other hand some might as well be equated in some model to some term which is in Φ (even provably), for example to T itself); this would occur for example with $\Omega \equiv \delta\delta$ if, as we conjectured, it is an easy term for MT.

Unsolvable terms of infinite order, like any fixed point of $K \equiv \lambda x.\lambda y.x$, can never be in Φ°.

A.2 The size of $\Phi/{=_\Phi}$

In the following we prove $|\Phi/{=_\Phi}| = \sigma$ and that for all $\alpha < \sigma$ there exists an essentially σ-small $G \subseteq \Phi$ such that $|\delta G^\circ| \geq |\alpha|$. The latter result will be used for finding a model of V_σ in Section A.4. To establish these results we will prove that there exist "self-extensional sets" $G \subseteq \Phi$ such that $G/{=_\Phi}$ can have any size less that σ.

Definition. G is a *self-extensional set* if

(a) $G \subseteq \Phi$.

(b) $G \neq \emptyset$.

(c) G is essentially σ-small.

(d) $G \subseteq G^{\circ\circ}$.

(e) $x =_G y \Rightarrow x =_\Phi y$ for all $x, y \in G$.

The name "self-extensional" refers to property (e) above.

Definition. $\mathcal{H}$ is a *self-extensional chain* if

- $\mathcal{H}$ is a non-empty set of self-extensional sets.
- $\forall G, G' \in \mathcal{H} \colon G \subseteq G' \vee G' \subseteq G$.

Theorem A.2.1 (A) If G is a self-extensional set then so is $G^{\circ\circ}$ and $|G/{=_\Phi}| \leq |\delta G^\circ| < |G^{\circ\circ}/{=_\Phi}|$.

(B) If $\mathcal{H}$ is a self-extensional chain then $\bigcup \mathcal{H}$ is a self-extensional set and $|G/{=_\Phi}| \leq |\bigcup \mathcal{H}/{=_\Phi}|$ for all $G \in \mathcal{H}$.

(C) $|\Phi/{=_\Phi}| = \sigma$.

(D) For all $\alpha < \sigma$ there exists an essentially σ-small $G \subseteq \Phi$ such that $|\delta G^\circ| \geq |\alpha|$.

Proof. (A) and (B) are proved as lemmas below. (C) and (D) follow from (A) and (B) as follows: Define $G_0 = \{\mathsf{T}\}$, $G_{\alpha'} = G_\alpha^{\circ\circ}$ and $G_\delta = \bigcup_{\alpha \in \delta} G_\alpha$ for limit ordinals δ. It follows from (A) and (B) by transfinite induction that $|G_\alpha/{=_\Phi}| \geq |\alpha|$ so $|G_\sigma/{=_\Phi}| \geq \sigma$ which together with $G_\sigma \subseteq \Phi$ proves $|\Phi/{=_\Phi}| \geq \sigma$. $|\Phi/{=_\Phi}| \leq \sigma$ follows from Lemma A.1.1. Furthermore, G_α satisfies (D). $\diamondsuit$

We now state and prove (B).

Lemma A.2.2 If $\mathcal{H}$ is a self-extensional chain then $U = \bigcup \mathcal{H}$ is a self-extensional set and $|G/{=_\Phi}| \leq |U/{=_\Phi}|$ for all $G \in \mathcal{H}$.

Proof. The latter claim is trivial. That U is a self-extensional set requires verification of (a) to (e). Point (a) to (c) are trivial. To see $U \subseteq U^{\circ\circ}$ assume $g \in U$ and choose G such that $g \in G \in \mathcal{H}$. Now $G \in \mathcal{H} \Rightarrow G \subseteq U \Rightarrow G^\circ \supseteq U^\circ \Rightarrow G^{\circ\circ} \subseteq U^{\circ\circ}$. To see (e) assume $x, y \in U$, $x =_U y$. Choose $G \in \mathcal{H}$ such that $x, y \in G$. Now $x =_U y \Rightarrow x =_G y$ because $G \subseteq U$ and $x =_G y \Rightarrow x {=_\Phi} y$ because G is self-extensional. $\diamondsuit$

Before we state and prove (A) as a lemma, we need some auxiliary concepts and lemmas.

In this section, for all $f \in \mathcal{M}$ and $\bar{y} = (y_1, \ldots, y_n) \in \mathcal{M}^{<\omega}$ let $f[\bar{y}]$ denote $fy_1 \cdots y_n \in \mathcal{M}$ and let $f\bar{y}$ denote $(fy_1, \ldots, fy_n) \in \mathcal{M}^{<\omega}$. As an example, $f[g\bar{y}] = f(gy_1) \cdots (gy_n)$.

For relations R on $\mathcal{M}$, for $\bar{y} = (y_1, \ldots, y_p) \in \mathcal{M}^{<\omega}$ and $\bar{z} = (z_1, \ldots, z_q) \in \mathcal{M}^{<\omega}$ let $\bar{y}\mathsf{R}\bar{z}$ stand for $p = q \wedge y_1 \mathsf{R} z_1 \wedge \cdots \wedge y_p \mathsf{R} z_p$. Recall that for $\bar{y} = (y_1, \ldots) \in \mathcal{M}^\omega$ and $n \in \omega$, $\bar{y}_n$ denotes $(y_1, \ldots, y_n)$.

For all $f \in \mathcal{M}$ define $f^* \in \mathcal{M}$ by

$$f^* = \mathsf{Y}\lambda k.\lambda x.(\mathsf{if}\, x\, \mathsf{T}\, \lambda y.k(x(fy)))$$

For all $x \in \mathcal{M}$ and $\bar{y} \in \mathcal{M}^{<\omega}$ we have

$$\begin{array}{rcl} r((f^*x)[y]) & = & r(f^*xy_1\cdots y_n) \\ & = & r(x(fy_1)\cdots(fy_n)) \\ & = & r(x[f\bar{y}]) \end{array}$$

Lemma A.2.3 If G is self-extensional then $\Downarrow_G$ is injective from $G/=_\Phi$ to δG° in the sense that $\Downarrow_G x \in \delta G^\circ$ and $\Downarrow_G x = \Downarrow_G y \Rightarrow x =_\Phi y$ for all $x, y \in G$.

Proof. If $x \in G$ then $G \subseteq \Phi \subseteq G^\circ$ and Theorem 6.1.11 gives $\Downarrow_G x \in \delta G^\circ$. If $x, y \in G$ then $\Downarrow_G x = \Downarrow_G y$ implies $x =_G y$ which, since $G \subseteq \Phi$, implies $x =_\Phi y$. $\diamondsuit$

Definition of $\Uparrow_G$ and $\Uparrow_G^*$. Now assume G is self-extensional. δG° is a σ-small set of incompatible, compact elements according to Theorem 6.1.11. Hence, there exist "left inverses" $\Uparrow_G \in \mathcal{M}$ of $\Downarrow_G$ which satisfy $\Uparrow_G x \in G$ for all $x \in \delta G^\circ$ ($G \neq \emptyset$ is needed here) and $\Uparrow_G(\Downarrow_G z) =_G z$ for all $z \in G$. Let $\Uparrow_G$ be one such left inverse. Let $\Uparrow_G^*$ stand for $(\Uparrow_G)^*$.

Lemma A.2.4 If G is self-extensional and $x \in \delta G^\circ$ then $\Uparrow_G^* x \in G^{\circ\circ}$ and $x \le \Uparrow_G^* x$.

Proof. Let $y = \Uparrow_G^* x$, let $\bar{z} \in (G^\circ)^\omega$, and let $\bar{z}' = \Uparrow_G \bar{z} \in G^\omega$. Choose $n \in \omega$ such that $r(x[\bar{z}'_n]) = \mathsf{T}$ (this is possible since $x \in G^\circ$ and $\bar{z}' \in G^\omega$). Now $r(y[\bar{z}_n]) = r((\Uparrow_G^* x)[\bar{z}_n]) = r(x[\Uparrow_G \bar{z}_n]) = r(x[\bar{z}'_n]) = \mathsf{T}$ which proves $y \in G^{\circ\circ}$.

Now let $\bar{u} \in G^{<\omega}$. We have $(\Downarrow_G \bar{u}) \le \bar{u}$ so $\{\mathsf{T}, \mathsf{F}\} \ni r(y[\Downarrow_G \bar{u}]) \le r(y[\bar{u}])$ which proves $r(y[\Downarrow_G \bar{u}]) = r(y[\bar{u}])$. This in combination with $r(y[\Downarrow_G \bar{u}]) = r(x[\Uparrow_G(\Downarrow_G \bar{u})]) = r(x[\bar{u}])$ gives $r(y[\bar{u}]) = r(x[\bar{u}])$. Hence, $y =_G x$ which combined with $x \in \delta G^\circ$ gives $x \le y$. $\diamondsuit$

Lemma A.2.5 If G is a self-extensional set then so is $G^{\circ\circ}$.

Proof. We have to check (a) to (e). (a) follows from GCP by transfinite induction in $<_{G^\circ}$: Let $x \in G^{\circ\circ}$. We shall prove $x \in \Phi$. As inductive hypothesis assume $y \in \Phi$ for all $y <_{G^\circ} x$. If $x = \mathsf{T}$ then $x \in \Phi$ holds. If $x \neq \mathsf{T}$ then the inductive hypothesis states that $xz \in \Phi$ for all $z \in G^\circ$ so $x \in G^\circ \to \Phi \subseteq \Phi$. (b) follows from $\mathsf{T} \in G^{\circ\circ}$. (c) follows from Theorem 6.1.11(c). (d) can be seen as follows: $G \subseteq G^{\circ\circ} \Rightarrow G^\circ \supseteq G^{\circ\circ\circ} \Rightarrow G^{\circ\circ} \subseteq G^{\circ\circ\circ\circ}$. (e) goes as follows: let $x, y \in G^{\circ\circ}$ satisfy $x =_{G^{\circ\circ}} y$ and let $\bar{z} \in \Phi^{<\omega}$. We shall prove $r(x[\bar{z}]) = r(y[\bar{z}])$. Let $\bar{z}' = \Downarrow_G \bar{z} \in (\delta G^\circ)^{<\omega}$ and let $\bar{z}'' = \Uparrow_G^* \bar{z}' \in (G^{\circ\circ})^{<\omega}$. We have $\bar{z} \ge \bar{z}' \le \bar{z}''$ so $r(x[\bar{z}]) \ge r(x[\bar{z}']) \le r(x[\bar{z}''])$. Furthermore, since $x \in G^{\circ\circ}$ and $\bar{z} \in (G^\circ)^{<\omega}$ we have $r(x[\bar{z}']) \neq \bot$ so $r(x[\bar{z}]) = r(x[\bar{z}']) = r(x[\bar{z}''])$. Likewise, $r(y[\bar{z}]) = r(y[\bar{z}''])$. Furthermore, $x =_{G^{\circ\circ}} y$ gives $r(x[\bar{z}'']) = r(y[\bar{z}''])$ so $r(x[\bar{z}]) = r(y[\bar{z}])$ as required. $\diamondsuit$

Lemma A.2.6 If G is a self-extensional set then $|G/{=_\Phi}| \le |\delta G^\circ| < |G^{\circ\circ}/{=_\Phi}|$.

Proof. $|G/{=_\Phi}| \leq |\delta G^\circ|$ follows from Lemma A.2.3. $|\delta G^\circ| < |G^{\circ\circ}/{=_\Phi}|$ follows from $|\mathcal{P}(\delta G^\circ)| \leq |G^{\circ\circ}/{=_\Phi}|$ which can be seen as follows: For all $S \subseteq \delta G^\circ$, define $\hat{\chi}_S \in \mathcal{M}$ by

$$\hat{\chi}_S x = \begin{cases} \mathsf{T} & \text{if } x \in \uparrow S \\ \mathsf{F} & \text{if } x \in \uparrow(\delta G^\circ \setminus S) \\ \perp & \text{if } x \notin G^\circ \end{cases}$$

Now $\hat{\chi}_S \in G^{\circ\circ}$. It remains to prove $S \neq T \Rightarrow \hat{\chi}_S \neq_\Phi \hat{\chi}_T$ for $S, T \in \delta G^\circ$. However, if $S \neq T$ then choose $y \in \delta G^\circ$ so that y is in one of S and T but not in the other, and let $z = \Uparrow^*_G y$. Then $y \leq z \in G^{\circ\circ} \subseteq \Phi$ by Lemma A.2.5 and $r(\hat{\chi}_S z) \neq r(\hat{\chi}_T z)$. $\diamondsuit$

A.3 Models of *ZFC* within models of *MT*

There are two ways of finding a model of *ZFC* within the κ-continuous models of *MT*. The first is to deduce it from the syntactical translation of *ZFC* into *MT*, which is a difficult theorem of [18], plus the fact that the models we consider satisfy the SQND. This translation is recalled later and uses the constructs $\dot{\in}$ and $\doteq$ which are defined in Appendix C. The second is to prove directly that, for our models, $\mathcal{N} \equiv (\Phi/{\doteq}, \dot{\in})$ is isomorphic to $(V_\sigma, \in)$ (within our big universe). This last way, though providing a stronger result, is in fact much more easy; as a matter of fact it requires no more than what we have already proved (c.f. Section A.4).

In this section we comment on the syntactical argument and show where the SQND comes in. First, consider a model of a theory $\mathcal{A}$ inside a theory $\mathcal{B}$. If both $\mathcal{A}$ and $\mathcal{B}$ are based on predicate calculus, then it is customary to model e.g. implication in $\mathcal{A}$ by implication in $\mathcal{B}$. In general, it is customary to model the predicate calculus part of $\mathcal{A}$ by the predicate calculus part of $\mathcal{B}$. Hence, to give a model of $\mathcal{A}$ in $\mathcal{B}$ it is sufficient to define the functions and relations of $\mathcal{A}$ inside $\mathcal{B}$. Furthermore, $\mathcal{A}$ and $\mathcal{B}$ are sure to use the "same" predicate calculus.

Matters are more complicated if $\mathcal{A}$ is based on predicate calculus and $\mathcal{B}$ is not. In this case, not only the functions and relations, but also the logical connectives and quantifiers of $\mathcal{A}$ have to be defined in $\mathcal{B}$. This is exactly the case when $\mathcal{A}$ is *ZFC* and $\mathcal{B}$ is *MT*, and it opens the possibility that the modeling of predicate calculus may be non-standard and may have pathological properties. This, however, may be ruled out by assuming SQND.

The syntactic result which is proved in [18] has the following shape:

Theorem A.3.1 If $A[x_1, \cdots, x_n]$ is a theorem of *ZFC* (including predicate calculus), then the equation $\phi x_1, \ldots, \phi x_n \to \dot{A}$ is a theorem of *MT*.

Here $\dot{A}$ is the term of *MT* obtained by replacing $\in$, $=$, and each connective and quantifier in A by its doted version as a term of *MT*. The definition of $\mathcal{A}_1, \ldots, \mathcal{A}_n \to \mathcal{B}$ may be found in Section 3.2 and Appendix C.

Thus, for any theorem $A[\bar{x}]$ of *ZFC* (including predicate calculus; that is why free variables may occur), for any model $\mathcal{M}$ of *MT*, and for any $\bar{u} \subseteq \Phi$ the model

$\mathcal{M}$ will satisfy the equation $\dot{A}[\bar{u}] = \mathsf{T}$. However we need something slightly different, namely that $\mathcal{N}$ satisfies the *ZFC*-formula $A[\bar{u}]$. This last assertion must be justified by a lemma which links the two notions of satisfaction in the two different settings, like the following:

Lemma A.3.2 For any *ZFC*-formula $A[\bar{x}]$ and for any model $\mathcal{M}$ of *MT*, we have:

$$\text{For any } \bar{u} \subseteq \Phi, \quad \mathcal{M} \models \dot{A}[\bar{u}] = \mathsf{T} \Rightarrow \mathcal{N} \models A[\bar{u}]$$

where $\mathcal{N} = (\Phi/\doteq, \dot{\in})$.

Proof. The proof is of course by induction in the structure of $A[\bar{x}]$. However to achieve it we need to know (because of the negation case) that a closed term of the shape $\dot{A}[\bar{x}]$ can take only the two values T and F. It is provable from *MT* that $\dot{\forall}\bar{x}(\mathsf{if}\,\dot{A}[\bar{x}]\,\mathsf{T}\,\mathsf{T}) = \mathsf{T}$, which rules out the value $\perp$, and that $\dot{A}[\bar{x}] = (\mathsf{if}\,\dot{A}\,\mathsf{T}\,\mathsf{F})$; the SQND is needed to conclude from this last equation that a term of the shape $\dot{A}[\bar{u}]$ is always equal to T or F. $\diamondsuit$

If SQND holds, then the above is a model of *ZFC* in the traditional sense, and if SQND does not hold, then the above may have some pathological properties. It should be noted that it is still open whether or not *MT* is strictly stronger than *ZFC* even though the above gives a model of *ZFC* in some sense.

SIP implies that $\mathcal{N}$ is an ω-model in the following sense: Let ω be the set of integers in the universe where our model is built and let $\dot{\omega}$ be a map term such that $\phi\dot{\omega} = \mathsf{T}$ and which satisfies

$$\dot{\omega}x = \mathsf{if}\,x\,\mathsf{T}\,\lambda y.\dot{\omega}(x\mathsf{T})$$

(Take $\dot{\omega} = \mathit{Prim}(\lambda x.x)\,\mathsf{T}\,\mathsf{T}$, where *Prim* is defined in Appendix C). The intuition behind is that $\dot{\omega}$ is the set of Zermelo's integers $\{\cdots\{\{\emptyset\}\}\cdots\}$; we choose Zermelo's integers here because they are more simple than Von Neumann integers, and are equivalent for our discussion. The n'th Zermelo integer can be represented by $A_n \equiv \lambda x_1 \dots x_n.\mathsf{T}$, $n \in \omega$; it is easy to check that $\phi A_n = \mathsf{T}$ and $\dot{\omega}A_n = A_n$, so all the Zermelo integers are indeed in the range of $\dot{\omega}$ and, hence, are "elements of $\dot{\omega}$" in the sense of $\dot{\in}$. Now it is easy to show that in any model satisfying SIP and SQND, $\dot{\omega}$ will have no other elements, while non satisfaction of SIP introduces "non standard" integers and non satisfaction of SQND introduces still stranger integers. Note that ϕ and ε does not occur in the definition of $\dot{\omega}$, so $\dot{\omega}$ is a computable function. In general, if SIP and SQND hold then sets of integers can be represented by computable functions if and only if the sets are recursively enumerable. We finally end up this example by noticing that computation of $\dot{\omega}x$ merely requires knowledge of $r(x\mathsf{T}\mathsf{T}\cdots\mathsf{T})$, so the *inner range* of $\dot{\omega}$ is nothing else than $\{\mathsf{T}\}$.

A.4 Finding V_σ in a κ-continuous model

Theorem A.4.1 If $\mathcal{M}$ is a κ-continuous model, then $\mathcal{N} \equiv (\Phi/\doteq, \dot{\in})$ is isomorphic to $(V_\sigma, \in)$.

Proof. In what follows the metavariables u, v range over elements of Φ. The rank of v is the smallest ordinal $\alpha < \sigma$ such that $v \in \Phi_\alpha$, and "induction in v" means induction in the rank of v.

We define a function s on Φ by recursion in the rank of v by:

$$s(v) \equiv \begin{cases} \emptyset & \text{if } v = \mathsf{T} \\ \{s(vx) \mid x \in \Phi\} & \text{otherwise} \end{cases}$$

The aim is to show that the function s' induced by s on $\Phi/\dot{=}$ is the required isomorphism. It is easy to prove successively that:

(1) $s(v) \in V_\sigma$, by induction in v.

(2) $u \geq v \Rightarrow s(w) = s(v)$, by induction in v.

(3) $u \dot{=} v = \mathsf{T} \Leftrightarrow s(u) = s(v)$, by induction in v and using SQND.

(4) $u \dot{\in} v = \mathsf{T} \Leftrightarrow s(u) \in s(v)$, by induction in v and using SQND.

It remains to prove $\exists u \in \Phi\colon s(u) = x$ for all $x \in V_\sigma$. This is trivial for $x = \emptyset$ and the rest is by induction in the rank of x: Assume $x \in V_\sigma$, $x \neq \emptyset$, and assume $s(u_y) = y$ for all $y \in x$. From $x \in V_\sigma$ we have that x is σ-small. Choose $G \subseteq \Phi$ such that $|\delta(G^\circ)| \geq |x|$ (using Theorem A.2.1 (D)), and choose $u' \in G^\circ \to \Phi$ such that the range of u' equals $\{u_y \mid y \in x\}$ (by Lemma 6.2.2). Using GCP, let u be the element of Φ corresponding to u'. Now $s(u) = x$. $\diamondsuit$

B Computational adequacy of the canonical model

In this section we restrict our attention to the set $\boxed{\Lambda^c}$ (c for computable) of *MT*-terms in which ϕ, ε and $\perp$ do not occur. For convenience, and without loss of generality, we require all occurrences of if to be followed by three arguments. Λ^c can be seen as the syntax class defined thus:

$$\begin{array}{lcl} \mathsf{variable} & ::= & x \mid y \mid z \mid \ldots \\ \Lambda^c & ::= & \mathsf{variable} \mid \lambda\mathsf{variable}.\Lambda^c \mid (\Lambda^c\,\Lambda^c) \mid \mathsf{T} \mid \mathsf{if}\,\Lambda^c\,\Lambda^c\,\Lambda^c \end{array}$$

Let Λ^c_c be the set of terms in Λ^c that are closed.

Elements of Λ^c will be considered to be equal if they differ only in naming of bound variables.

Now let $\to_L$ be the least relation on Λ^c that satisfies the following statements for all $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{A}' \in \Lambda^c$.

$$\begin{array}{lcll} \mathsf{T}\mathcal{B} & \to_L & \mathsf{T} & \\ (\lambda x.\mathcal{A})\mathcal{B} & \to_L & [\mathcal{A}/x{:=}\mathcal{B}] & \\ \mathsf{if}\,\mathsf{T}\,\mathcal{B}\,\mathcal{C} & \to_L & \mathcal{B} & \\ \mathsf{if}\,(\lambda x.\mathcal{A})\,\mathcal{B}\,\mathcal{C} & \to_L & \mathcal{C} & \\ \mathcal{A}\mathcal{B} & \to_L & \mathcal{A}'\mathcal{B} & \text{if } \mathcal{A} \to_L \mathcal{A}' \\ \mathsf{if}\,\mathcal{A}\,\mathcal{B}\,\mathcal{C} & \to_L & \mathsf{if}\,\mathcal{A}'\,\mathcal{B}\,\mathcal{C} & \text{if } \mathcal{A} \to_L \mathcal{A}' \end{array}$$

Above, $[\mathcal{A}/x{:=}\mathcal{B}]$ denotes substitution with suitable renaming of bound variables.

Note that if $\mathcal{S} \in \Lambda^c$ then $\mathcal{S} \to_L \mathcal{T}$ holds for at most one $\mathcal{T} \in \Lambda^c$ (up to renaming of bound variables), so $\to_L$ may be seen as a partial function from Λ^c to Λ^c. Terms $\mathcal{S} \in \Lambda^c$ for which $\mathcal{S} \to_L \mathcal{T}$ holds for no $\mathcal{T} \in \Lambda^c$ will be said to be in *root normal form*.

Let $\twoheadrightarrow_L$ be the reflexive, transitive closure of $\to_L$. This relation represents the leftmost reduction strategy [4] applied to Axioms Apply 1, Apply 2, Select 1, and Select 2 in Appendix C when these axioms are read as reduction rules.

Now define $\mathcal{N}_t$, $\mathcal{N}_f$ and $\mathcal{N}_\perp$ thus:

$$\begin{array}{lcl} \mathcal{N}_t & = & \{\mathcal{A} \in \Lambda^c \mid \mathcal{A} \twoheadrightarrow_L \mathsf{T}\} \\ \mathcal{N}_f & = & \{\mathcal{A} \in \Lambda^c \mid \exists \mathcal{B} \in \Lambda^c \colon \mathcal{A} \twoheadrightarrow_L \lambda x.\mathcal{B}\} \\ \mathcal{N}_\perp & = & \Lambda^c \setminus (\mathcal{N}_t \cup \mathcal{N}_f) \end{array}$$

(where x may be free in $\mathcal{B}$ in the second equation above).

It is straightforward to write a computer program which performs the leftmost reduction strategy, so it is straightforward e.g. to write a computer program which, given a term $\mathcal{T} \in \Lambda^c_c$, prints "T" after a while if $\mathcal{T} \in \mathcal{N}_t$, prints "F" after a while if $\mathcal{T} \in \mathcal{N}_f$, and proceeds forever without printing anything if $\mathcal{T} \in \mathcal{N}_\perp$. The purpose of the present section is to prove that canonical models are faithful to this computational aspect in the sense that such a program prints "T" iff $\mathcal{T} = \mathsf{T}$ in the model and similarly for the other two possibilities. This is formally expressed in the computational adequacy theorem below.

For the time being let $\mathcal{M}$ be any model of *MT*, let $\mathcal{F} = \mathcal{M} \setminus \{\mathsf{T}, \perp\}$, and let $|\mathcal{T}|$ denote the interpretation of $\mathcal{T}$ in $\mathcal{M}$. For $\mathcal{T} \in \Lambda^c_c$ we obviously have

$$\begin{array}{lcl} \mathcal{T} \in \mathcal{N}_t & \Rightarrow & |\mathcal{T}| = \mathsf{T} \\ \mathcal{T} \in \mathcal{N}_f & \Rightarrow & |\mathcal{T}| \in \mathcal{F} \\ \mathcal{T} \in \mathcal{N}_\perp & \Leftarrow & |\mathcal{T}| = \perp \end{array}$$

The main result of the present section is that the converse is also true if $\mathcal{M}$ is canonical:

Theorem B.0.2 (Computational adequacy) If $\mathcal{M}$ is canonical and $\mathcal{T} \in \Lambda^c_c$ then

$$\begin{array}{lcl} \mathcal{T} \in \mathcal{N}_t & \Leftrightarrow & |\mathcal{T}| = \mathsf{T} \\ \mathcal{T} \in \mathcal{N}_f & \Leftrightarrow & |\mathcal{T}| \in \mathcal{F} \\ \mathcal{T} \in \mathcal{N}_\perp & \Leftrightarrow & |\mathcal{T}| = \perp \end{array}$$

Readers familiar with Tait's reducibility (also called computability) technique and to intersection type systems will notice that the following proof is based on these ideas, and could be written within a type assignment setting. Doing so is not necessary here but the correspondence is sketched in a remark later on.

Now let $\mathcal{M}$ be a canonical model. Recall that elements of $\mathcal{M}$ are subsets of the set C where the elements of C represent prime elements of $\mathcal{M}$ and recall that $\mathcal{M}$ is a p.o. ordered by a relation $\leq$ which is simply the subset relation. There is a canonical injection from C to $\mathcal{M}$ (which takes prime elements to their initial segments) and a canonical injection from Λ^c_c to $\mathcal{M}$ (which takes terms to their interpretation). Using these injections it makes sense to write e.g. $p \leq \mathcal{T}$ for prime elements p and closed terms $\mathcal{T}$. For open terms $\mathcal{T}$, $p \leq \mathcal{T}$ will be taken to mean that $p \leq \mathcal{T}$ holds for all values of free variables.

For any closed term $\mathcal{T}$ we have $|\mathcal{T}| \subseteq C$. We trivially have $p \in |\mathcal{T}| \Leftrightarrow p \leq \mathcal{T}$. Hence, for any term $\mathcal{T}$, $|\mathcal{T}|$ may be seen as the set of prime elements smaller than $\mathcal{T}$.

Now we introduce the converse. For all prime elements $p \in C$ we introduce the set $\boxed{I(p)}$ of terms $\mathcal{T} \in \Lambda^c$ for which, intuitively, $p \leq \mathcal{T}$. $I(p)$ will be referred to as the *interpretation* of p. The formal definition of $I(p)$ will not be based on $\leq$ but will be a syntactic definition defined by recursion in the rank of p. Since $\mathcal{T} \in I(p)$ and $p \in |\mathcal{T}|$ intuitively both express $p \leq \mathcal{T}$ we would expect to have $\mathcal{T} \in I(p) \Leftrightarrow p \in |\mathcal{T}|$. We shall prove a lemma similar to this from which the computational adequacy theorem follows.

Now recall that $C = (C^\star \times C) \cup \{t, f\}$. t represents T (in the sense that their canonical injections into $\mathcal{M}$ are equal), so we would like to have

$$I(t) = \mathcal{N}_t$$

Likewise, f represents $\lambda x.\bot$ so we would like to have

$$I(f) = \mathcal{N}_f$$

Elements of $C^\star$ represent compact elements. If $c \in C^\star$ then c is a set of prime elements and c represents the least upper bound of these elements. Hence, the set $I(c)$ of terms $\mathcal{A} \in \Lambda^c$ greater than c can be defined from $I(p)$ thus:

$$I(c) = \{\mathcal{A} \in \Lambda^c \mid \forall p{\in}c\colon \mathcal{A} \in I(p)\}$$

Elements of $C^\star \times C$ represent prime elements. If $(c, p) \in C^\star \times C$ then (c, p) represents the least map f for which $f c = p$. Hence, we would like to have

$$I(c, p) = I(c) \to I(p)$$

where

$$X \to Y = \{\mathcal{A} \in \mathcal{N}_f \mid \forall \mathcal{B} \in X\colon \mathcal{A}\mathcal{B} \in Y\}$$

The above equations define $I(p)$ uniquely for all $p \in C$ by recursion in the rank of p.

A set $X \subseteq \Lambda^c$ of terms will be said to be *saturated* if $\mathcal{T} \in X$ and $\mathcal{S} \twoheadrightarrow_L \mathcal{T}$ implies $\mathcal{S} \in X$. Hence, $X \subseteq \Lambda^c$ is saturated if:

$$\begin{array}{lcl}
\mathsf{T}\mathcal{E}_1 \cdots \mathcal{E}_n \in X & \Rightarrow & \mathsf{T}\mathcal{B}\mathcal{E}_1 \cdots \mathcal{E}_n \in X \\
[\mathcal{A}/x{:=}\mathcal{B}]\mathcal{E}_1 \cdots \mathcal{E}_n \in X & \Rightarrow & (\lambda x.\mathcal{A})\mathcal{B}\mathcal{E}_1 \cdots \mathcal{E}_n \in X \\
\mathcal{A} \in \mathcal{N}_t \wedge \mathcal{B}\mathcal{E}_1 \cdots \mathcal{E}_n \in X & \Rightarrow & (\mathsf{if}\,\mathcal{A}\,\mathcal{B}\,\mathcal{C})\mathcal{E}_1 \cdots \mathcal{E}_n \in X \\
\mathcal{A} \in \mathcal{N}_f \wedge \mathcal{C}\mathcal{E}_1 \cdots \mathcal{E}_n \in X & \Rightarrow & (\mathsf{if}\,\mathcal{A}\,\mathcal{B}\,\mathcal{C})\mathcal{E}_1 \cdots \mathcal{E}_n \in X
\end{array}$$

Note that $\mathcal{N}_t$ and $\mathcal{N}_f$ are saturated and that $X \to Y$ is saturated whenever Y is saturated. The proof of the adequacy theorem has two key steps, the first of which is the following lemma.

Lemma B.0.3 $I(p)$ is saturated for all $p \in C$.

Proof. The proof is by induction in the rank of p. $I(t) = \mathcal{N}_t$ and $I(f) = \mathcal{N}_f$ are saturated as noted above. If $p = (c, p')$ then $I(p) = I(c, p') = I(c) \to I(p')$ which is saturated because $I(p')$ is saturated by the inductive hypothesis. $\diamondsuit$

The second key step is proved by structural induction over Λ^c where the first was by induction over C. The second key step establishes the previously mentioned link between $p \in |\mathcal{T}|$ and $\mathcal{T} \in I(p)$ which both intuitively express $p \leq \mathcal{T}$.

Lemma B.0.4 Let $\mathcal{T} \in \Lambda^c$. Let $x_1, \ldots, x_n$ be the free variables of $\mathcal{T}$. Let $e_1, \ldots, e_n \in C^\star$. Let $\mathcal{E}_1 \in I(e_1), \ldots, \mathcal{E}_n \in I(e_n)$. For all terms $\mathcal{G}$ let $\mathcal{G}_e$ stand for $[\mathcal{G}/x_1{:=}e_1, \cdots, x_n{:=}e_n]$ and let $\mathcal{G}_\mathcal{E}$ stand for $[\mathcal{G}/x_1{:=}\mathcal{E}_1, \cdots, x_n{:=}\mathcal{E}_n]$. For all $p \in C$ we have:

$$p \in |\mathcal{T}_e| \Rightarrow \mathcal{T}_\mathcal{E} \in I(p)$$

Note that for $\mathcal{T} \in \Lambda^c_c$ the lemma gives

$$\begin{array}{lllll} |\mathcal{T}| = \mathsf{T} & \Leftrightarrow & t \in |\mathcal{T}| & \Rightarrow & \mathcal{T} \in I(t) = \mathcal{N}_t \\ |\mathcal{T}| \in \mathcal{F} & \Leftrightarrow & f \in |\mathcal{T}| & \Rightarrow & \mathcal{T} \in I(f) = \mathcal{N}_f \end{array}$$

from which the computational adequacy theorem follows.

Before proving Lemma B.0.4, we state and prove an auxiliary lemma:

Lemma B.0.5 If $p, q \in C$ and $p \preceq q$ then $I(q) \subseteq I(p)$.

Proof. The proof is by induction in the rank of p and q. $p, q \in C$ and $p \preceq q$ gives four cases to consider: (1) $p = q = t$, (2) $p = q = f$, (3) $p = f$ and $q = (q_1, q_2) \in C^\star \times C$, and (4) $p = (p_1, p_2) \in C^\star \times C$ and $q = (q_1, q_2) \in C^\star \times C$. In case (1) and (2) the lemma is trivial. In case (3), $I(p) = \mathcal{N}_f$ and $I(q) = I(q_1) \to I(q_2) = \{\mathcal{A} \in \mathcal{N}_f \mid \cdots\}$. In case (4) we have $p_2 \preceq q_2$ and $\forall q_3{\in}q_1 \exists p_3{\in}p_1{:}\, q_3 \preceq p_3$. Hence, by inductive hypothesis, $I(q_2) \subseteq I(p_2)$ and $\forall q_3{\in}q_1 \exists p_3{\in}p_1{:}\, I(p_3) \subseteq I(q_3)$. The latter implies $I(p_1) \subseteq I(q_1)$ which, together with the former, yields $I(q_1, q_2) \subseteq I(p_1, p_2)$. $\diamondsuit$

Proof of Lemma B.0.4. As mentioned, the proof is by structural induction in $\mathcal{T}$. This gives rise to five cases:

Case 1. Assume $\mathcal{T} \equiv x_i$. If $p \in |\mathcal{T}_e| = |e_i|$ then p is in the initial segment generated by e_i so $p \preceq p' \in e_i$ for some p'. $p' \in e_i$ gives $I(e_i) \subseteq I(p')$ by the definition of I and $p \preceq p'$ gives $I(p') \subseteq I(p)$ by lemma B.0.5 below. Hence, $\mathcal{T}_\mathcal{E} \equiv \mathcal{E}_i \in I(e_i) \subseteq I(p') \subseteq I(p)$.

Case 2. Assume $\mathcal{T} \equiv \mathsf{T}$. We have $p \in |\mathcal{T}_e| = |\mathsf{T}| = \{t\} \Rightarrow p = t \Rightarrow \mathcal{T}_\mathcal{E} \equiv \mathsf{T} \in I(p)$.

Case 3. Assume $\mathcal{T} \equiv \mathcal{A}\mathcal{B}$ and $p \in |\mathcal{T}_e| = |\mathcal{A}_e\mathcal{B}_e|$. If $t \in |\mathcal{A}_e|$ then $p = t$ and $t \in |\mathcal{A}_e| \Rightarrow \mathcal{A}_\mathcal{E} \in I(t) = \mathcal{N}_t \Rightarrow \mathcal{A}_\mathcal{E}\mathcal{B}_\mathcal{E} \in \mathcal{N}_t = I(t) = I(p)$. If $t \notin |\mathcal{A}_e|$ then choose c such that $(c,p) \in |\mathcal{A}_e|$ and $c \subseteq |\mathcal{B}_e|$. We have $c \subseteq |\mathcal{B}_e| \Rightarrow \forall p' {\in} c{:}\, p' \in |\mathcal{B}_e| \Rightarrow \forall p' {\in} c{:}\, \mathcal{B}_\mathcal{E} \in I(p') \Rightarrow \mathcal{B}_\mathcal{E} \in I(c)$. Furthermore, $(c,p) \in |\mathcal{A}_e| \Rightarrow \mathcal{A}_\mathcal{E} \in I(c,p) = I(c) \to I(p)$. Finally, $\mathcal{A}_\mathcal{E} \in I(c) \to I(p)$ and $\mathcal{B}_\mathcal{E} \in I(c)$ gives $\mathcal{T}_\mathcal{E} \equiv \mathcal{A}_\mathcal{E}\mathcal{B}_\mathcal{E} \in I(p)$.

Case 4. Assume $\mathcal{T} \equiv \mathsf{if}\,\mathcal{A}\,\mathcal{B}\,\mathcal{C}$. If $p \in |\mathcal{T}_e| = |\mathsf{if}\,\mathcal{A}\,\mathcal{B}\,\mathcal{C}|$ then $t \in |\mathcal{A}_e| \wedge p \in |\mathcal{B}_e| \vee f \in |\mathcal{A}_e| \wedge p \in |\mathcal{C}_e|$. Hence, by the inductive hypothesis, $\mathcal{A}_\mathcal{E} \in I(t) \wedge \mathcal{B}_\mathcal{E} \in I(p) \vee \mathcal{A}_\mathcal{E} \in I(f) \wedge \mathcal{C}_\mathcal{E} \in I(p)$. Using the definition of $I(t)$ and $I(f)$ this gives $\mathcal{A}_\mathcal{E} \in \mathcal{N}_t \wedge \mathcal{B}_\mathcal{E} \in I(p) \vee \mathcal{A}_\mathcal{E} \in \mathcal{N}_f \wedge \mathcal{C}_\mathcal{E} \in I(p)$. Finally, using the saturation of $I(p)$ this gives $\mathcal{T}_\mathcal{E} \equiv \mathsf{if}\,\mathcal{A}_\mathcal{E}\,\mathcal{B}_\mathcal{E}\,\mathcal{C}_\mathcal{E} \in I(p)$.

Case 5. Assume $\mathcal{T} \equiv \lambda y.\mathcal{A}$ where y is not among $x_1, \ldots, x_n$ and does not occur free in $\mathcal{E}_1, \ldots, \mathcal{E}_n$. Further assume $p \in |\mathcal{T}_e| = |\lambda y.\mathcal{A}_e|$. If $p = f$ then $\mathcal{T}_\mathcal{E} = \lambda y.\mathcal{A}_\mathcal{E} \in \mathcal{N}_f = I(f) = I(p)$. If $p \neq f$ then p has the form (c,p') and $p' \in |(\lambda y.\mathcal{A}_e)c| = |[\mathcal{A}/x_1{:=}e_1, \cdots, x_n{:=}e_n, y{:=}c]|$. Now assume $\mathcal{B} \in I(c)$. From the inductive hypothesis we have $[\mathcal{A}_\mathcal{E}/y{:=}\mathcal{B}] \equiv [\mathcal{A}/x_1{:=}\mathcal{E}_1, \cdots, x_n{:=}\mathcal{E}_n, y{:=}\mathcal{B}] \in I(p')$ so, by the saturation of $I(p')$ we have $\mathcal{T}_\mathcal{E}\mathcal{B} \equiv (\lambda y.\mathcal{A}_\mathcal{E})\mathcal{B} \in I(p')$ so $\mathcal{T}_\mathcal{E} \in I(c) \to I(p') = I(c,p') = I(p)$. $\diamondsuit$

Remark Elements of D may be viewed as formulas of an extended but strict "intersection type system". For this it is enough to change the notation (c,p) of pairs into $c \to p$, to read a as the "conjunction" of its elements, and $\to$ as implication. We use only "strict formulas" in the sense that no conjunction is allowed on the right side of the arrow (moreover "external" conjunction is not needed); they are "extended" in the sense that we use a global and unordered conjunction of $<\omega$ or $<\kappa$ elements, instead of usual binary conjunction. Note that $\to$ is no more, here, than the inclusion of $D^\star \times D$ into D. Then one can easily produce rules typing each term of Λ^c in such a way that it is equivalent for a formula p to belong to the interpretation of a (closed) term $\mathcal{A}$ or to be a type for $\mathcal{A}$ in the system (for a systematic treatment of such a view, c.f. [5]; strict intersection type systems have also been studied in [3].

C Syntax and axioms of map theory

The grammar of map theory

variable	::=	$x \mid y \mid z \mid \ldots$
term	::=	variable $\mid$ λvariable.term $\mid$ (term term) $\mid$ T $\mid$ $\perp$ $\mid$ if $\mid$ ε $\mid$ ϕ
wff	::=	term = term

Various definitions in map theory

$$
\begin{array}{lcl}
\mathsf{F} & = & \lambda x.\mathsf{T} \\
\dot{\neg} x & = & \mathsf{if}\, x\, \mathsf{F}\, \mathsf{T} \\
\approx x & = & \mathsf{if}\, x\, \mathsf{T}\, \mathsf{F} \\
!x & = & \mathsf{if}\, x\, \mathsf{T}\, \mathsf{T} \\
¡x & = & \mathsf{if}\, x\, \mathsf{F}\, \mathsf{F} \\
x \dot{\wedge} y & = & \mathsf{if}\, x\, (\mathsf{if}\, y\, \mathsf{F}\, \mathsf{T})\, (\mathsf{if}\, y\, \mathsf{F}\, \mathsf{F}) \\
x \dot{\vee} y & = & \mathsf{if}\, x\, (\mathsf{if}\, y\, \mathsf{T}\, \mathsf{T})\, (\mathsf{if}\, y\, \mathsf{T}\, \mathsf{F}) \\
x \dot{\Rightarrow} y & = & \mathsf{if}\, x\, (\mathsf{if}\, y\, \mathsf{T}\, \mathsf{F})\, (\mathsf{if}\, y\, \mathsf{T}\, \mathsf{T}) \\
x \dot{\Leftrightarrow} y & = & \mathsf{if}\, x\, (\mathsf{if}\, y\, \mathsf{T}\, \mathsf{F})\, (\mathsf{if}\, y\, \mathsf{F}\, \mathsf{T}) \\
\dot{\exists} \mathcal{A} & = & \approx (\mathcal{A}\, \varepsilon \mathcal{A}) \\
\dot{\exists} x.\mathcal{A} & = & \dot{\exists}(\lambda x.\mathcal{A}) \\
\dot{\forall} x.\mathcal{A} & = & \dot{\neg}\dot{\exists} x.\dot{\neg}\mathcal{A} \\
x \doteq y & = & (\mathsf{if}\, x\, (\mathsf{if}\, y\, \mathsf{T}\, \mathsf{F})\, (\mathsf{if}\, y\, \mathsf{F} \\
 & & \quad (\dot{\forall} u \dot{\exists} v.(x\, u) \doteq (y\, v)) \dot{\wedge} (\dot{\forall} v \dot{\exists} u.(x\, u) \doteq (y\, v)))) \\
x \dot{\in} y & = & \mathsf{if}\, y\, \mathsf{F}\, \dot{\exists} v.x \doteq (y\, v) \\
\mathsf{Y} & = & \lambda f.((\lambda x.(f\, (x\, x)))\, (\lambda x.(f\, (x\, x)))) \\
\mathsf{Y} f.\mathcal{A} & = & (\mathsf{Y}\, \lambda f.\mathcal{A}) \\
P & = & \lambda a.\lambda b.\lambda x.\mathsf{if}\, x\, a\, b \\
\mathit{Curry} & = & \lambda f.\lambda x.\lambda y.(f\, (P\, x\, y)) \\
\mathit{Prim} & = & \lambda f.\lambda a.\lambda b.\mathsf{Y} g.\lambda x.\mathsf{if}\, x\, a\, (f\, \lambda u.(g\, (x\, (b\, u)))) \\
\mathsf{F}' & = & \lambda f.\lambda x.(f\, x) \\
\phi x.\mathcal{A} & = & \phi \lambda x.\mathcal{A} \\
x{:}y & = & \mathsf{if}\, x\, y\, \mathsf{T}
\end{array}
$$

$\mathcal{A}_1, \ldots, \mathcal{A}_n \rightarrow (\mathcal{C} = \mathcal{D})$ is shorthand for the equation $\mathcal{A}_1{:} \ldots {:}\mathcal{A}_n{:}\mathcal{C} = \mathcal{A}_1{:} \ldots {:}\mathcal{A}_n{:}\mathcal{D}$
$\mathcal{A}_1, \ldots, \mathcal{A}_n \rightarrow \mathcal{C}$ is shorthand for the equation $\mathcal{A}_1{:} \ldots {:}\mathcal{A}_n{:}\mathcal{C} = \mathcal{A}_1{:} \ldots {:}\mathcal{A}_n{:}\mathsf{T}$

Axioms/inference rules

Axioms of Section 4 in [18] (λ-calculus)

Trans	$\mathcal{A} = \mathcal{B}; \mathcal{A} = \mathcal{C} \vdash \mathcal{B} = \mathcal{C}$
Sub1	$\mathcal{A} = \mathcal{B}; \mathcal{C} = \mathcal{D} \vdash (\mathcal{A}\,\mathcal{C}) = (\mathcal{B}\,\mathcal{D})$
Sub2	$\mathcal{A} = \mathcal{B} \vdash \lambda x.\mathcal{A} = \lambda x.\mathcal{B}$
Apply 1	$(\mathsf{T}\,\mathcal{B}) = \mathsf{T}$
Apply 2	$((\lambda x.\mathcal{A})\,\mathcal{B}) = [\mathcal{A}/x{:=}\mathcal{B}]$ if $\mathcal{B}$ is free for x in $\mathcal{A}$
Apply 3	$(\perp \mathcal{B}) = \perp$
Select 1	$\mathsf{if}\,\mathsf{T}\,\mathcal{B}\,\mathcal{C} = \mathcal{B}$
Select 2	$\mathsf{if}\,(\lambda x.\mathcal{A})\,\mathcal{B}\,\mathcal{C} = \mathcal{C}$
Select 3	$\mathsf{if}\perp\mathcal{B}\,\mathcal{C} = \perp$
Rename	$\lambda x.[\mathcal{A}/y{:=}x] = \lambda y.[\mathcal{A}/x{:=}y]$ if x is free for y in $\mathcal{A}$ and vice versa

Axioms of Section 5 in [18] (propositional calculus)

QND'	$[\mathcal{A}/x{:=}\mathsf{T}] = [\mathcal{B}/x{:=}\mathsf{T}]$; $[\mathcal{A}/x{:=}(\mathsf{F}'\,x)] = [\mathcal{B}/x{:=}(\mathsf{F}'\,x)]$; $[\mathcal{A}/x{:=}\perp] = [\mathcal{B}/x{:=}\perp]$ $\vdash \mathcal{A} = \mathcal{B}$

Axioms of Section 6 in [18] (first order predicate calculus)

Quantify 1	$\phi\mathcal{A}, \dot\forall x.\mathcal{B} \rightarrow ((\lambda x.\mathcal{B})\,\mathcal{A})$
Quantify 2	$\varepsilon x.\mathcal{A} = \varepsilon x.(\phi x \dot\wedge \mathcal{A})$
Quantify 3	$\phi\varepsilon x.\mathcal{A} = \dot\forall x.!\mathcal{A}$
Quantify 4	$\dot\exists x.\mathcal{A} \rightarrow \phi\varepsilon x.\mathcal{A}$
Quantify 5	$\dot\forall x.\mathcal{A} = \dot\forall x.(\phi x \dot\wedge \mathcal{A})$

Axioms of Section 7 in [18] (set theory)

Well 1	$\phi\mathsf{T} = \mathsf{T}$
Well 2	$\phi x.\mathcal{A} = \phi x.\phi\mathcal{A}$
Well 3	$\phi\perp = \perp$
C-A	$\phi a, \phi b \rightarrow \phi(a\,b)$
C-K'	$\phi x.\mathsf{T} = \mathsf{T}$
C-P'	$\phi x.\mathsf{if}\,x\,\mathsf{T}\,\mathsf{T} = \mathsf{T}$
C-Curry	$\phi a \rightarrow \phi(\mathit{Curry}\,a)$
C-Prim	$\dot\forall x.\phi(f\,x), \phi a, \phi b \rightarrow \phi(\mathit{Prim}\,f\,a\,b)$
C-M1	$\dot\forall z.\phi x.\mathcal{A} \rightarrow \dot\forall z.\phi x.((\lambda z.\mathcal{A})\,(z\,x))$
C-M2	$\dot\forall z.\phi x.\mathcal{A} \rightarrow \dot\forall z.\phi x.((\lambda x.\mathcal{A})\,(x\,z))$
Induction	If x does not occur free in $\mathcal{A}$ and y does not occur (free or bound) in $\mathcal{B}$, then $\mathcal{A}, x \rightarrow \mathcal{B}; \mathcal{A}, \dot\neg x, \phi x, \dot\forall y.[\mathcal{B}/x{:=}(x\,y)] \rightarrow \mathcal{B} \vdash \mathcal{A}, \phi x \rightarrow \mathcal{B}$

The axioms of Section 7 in [18] are referred to as the *well-foundedness axioms*.

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