Supplementary Note for the MAC-lecture November 17

November 14, 2003

Equations expressing that $[\mathcal{A} \equiv \mathcal{B}]^{\cdot}$ holds under some hypothesis

In the Mac derivation system there is a basic rule saying that:

If
    the equation $[\mathcal{A} \equiv \mathcal{B}]^{\cdot}$ holds when $\mathrm{x} \in \mathbf{T} \equiv \mathrm{T}$
and
    the equation $[\mathcal{A} \equiv \mathcal{B}]^{\cdot}$ holds when $\mathrm{x} \in \mathbf{F} \equiv \mathrm{T}$
then
    the equation $[\mathcal{A} \equiv \mathcal{B}]^{\cdot}$ holds when $\mathrm{x} \in \mathbf{B} \equiv \mathrm{T}$

Until now we have not seen any axiom expressing an 'if-then-statement', so we may expect this rule to be an inference rule of the form:

[**Mac rule MacroName**: $([\mathrm{x} \in \mathbf{T} \equiv \mathrm{T}]^{\cdot}$ implies that $[\mathcal{A} \equiv \mathcal{B}]^{\cdot}$ holds)
$\vdash$ $([\mathrm{x} \in \mathbf{F} \equiv \mathrm{T}]^{\cdot}$ implies that $[\mathcal{A} \equiv \mathcal{B}]^{\cdot}$ holds)
$\vdash$ $([\mathrm{x} \in \mathbf{B} \equiv \mathrm{T}]^{\cdot}$ implies that $[\mathcal{A} \equiv \mathcal{B}]^{\cdot}$ holds)]^{o}

We know that **premises** and **conclusions** are something that may *hold* or not hold, i.e. they **must be equations** (or terms interpreted as equations) and thus they cannot contain the infer-directive! But how can we then write premises and a conclusion expressing that $[\mathcal{A} \equiv \mathcal{B}]^{\cdot}$ holds under some hypothesis?

Well, consider the equation

$$\mathrm{case}(\mathrm{x} \in \mathbf{B}, \mathcal{A}, \mathrm{T}) \equiv \mathrm{case}(\mathrm{x} \in \mathbf{B}, \mathcal{B}, \mathrm{T})$$

Since **hypotheses** are *true* or not, they **must be terms**, and thus they may appear as the first argument of the case-operator.
If the hypothesis $[\mathrm{x} \in \mathbf{B}]^{\cdot}$ is true then this equation tells us (or rather Map) that the equation $[\mathcal{A} \equiv \mathcal{B}]^{\cdot}$ holds, *whereas* the equation does not say more than the rule Reflexivity, when $[\mathrm{x} \in \mathbf{B}]^{\cdot}$ is $[\perp]^{\cdot}$ or a λ-function like $[\mathrm{F}]^{\cdot}$.
Hence, we may write the rule in question as:

[**Mac rule Cases**: $\mathrm{case}(\mathrm{x} \in \mathbf{T}, \mathcal{A}, \mathrm{T}) \equiv \mathrm{case}(\mathrm{x} \in \mathbf{T}, \mathcal{B}, \mathrm{T})$
$\vdash$ $\mathrm{case}(\mathrm{x} \in \mathbf{F}, \mathcal{A}, \mathrm{T}) \equiv \mathrm{case}(\mathrm{x} \in \mathbf{F}, \mathcal{B}, \mathrm{T})$
$\vdash$ $\mathrm{case}(\mathrm{x} \in \mathbf{B}, \mathcal{A}, \mathrm{T}) \equiv \mathrm{case}(\mathrm{x} \in \mathbf{B}, \mathcal{B}, \mathrm{T})]^{\cdot}$

In Chapter 12 the new 'deduce-directive' $[\rightarrow]^{o}$ is introduced by the *term reduction rule*:

$$[\mathrm{x} \rightarrow \mathrm{y} \equiv \mathrm{z} \overset{o}{\rightarrow} \mathrm{case}(\mathrm{x}, \mathrm{y}, \mathrm{T}) \equiv \mathrm{case}(\mathrm{x}, \mathrm{z}, \mathrm{T})]^{\cdot}$$

and due to the priorities $[\mathrm{x} \equiv \mathrm{y} \breve{>} \mathrm{x} \rightarrow \mathrm{y} \breve{>} \mathrm{x} \vdash \mathrm{y}]^{\cdot}$ the inference rule Cases may thus be written in a more compact way:

[**Mac rule Cases**: $\mathrm{x} \in \mathbf{T} \rightarrow \mathcal{A} \equiv \mathcal{B} \ \vdash\ \mathrm{x} \in \mathbf{F} \rightarrow \mathcal{A} \equiv \mathcal{B} \ \vdash\ \mathrm{x} \in \mathbf{B} \rightarrow \mathcal{A} \equiv \mathcal{B}]^{\cdot}$

In Sections 12.4 to 12.11 it is shown how the so-called deduction proof technique may be used for proving lemmas containing the deduce-directive (*without introducing the 'case-equations'* !).

Since all the lemmas in Chapter 12 can be proved without using *blocks*, you may have a glance at the proof of Lemma L14.1.1, where blocks are really necessary!

Exercise 1

Consider the following lemma (proved in Chapter 12)

$[$**Mac lemma L12.5.1**: $\mathrm{x} \in \mathbf{Z} \rightarrow \mathrm{y} \in \mathbf{Z} \rightarrow \mathrm{x} + \mathrm{y} \cdot \mathrm{x} \in \mathbf{Z}]$

L12.5.1 is a macro for the *implication* $[\mathrm{x} \in \mathbf{Z} \rightarrow \mathrm{y} \in \mathbf{Z} \rightarrow \mathrm{x} + \mathrm{y} \cdot \mathrm{x} \in \mathbf{Z}]^{\cdot}$, which consists of two *hypothesis* and a *consequence* (the term $[\mathrm{x} + \mathrm{y} \cdot \mathrm{x} \in \mathbf{Z}]^{\cdot}$ interpreted as the equation $[\mathrm{x} + \mathrm{y} \cdot \mathrm{x} \in \mathbf{Z} \equiv \mathrm{T}]^{\cdot}$). The new directive $[\rightarrow]^{o}$ is right-associative (like $[\vdash]^{o}$), but is L12.5.1 an axiom or an inference rule?

Exercise 2

Assume that x, y, z, u and v are terms. Which of the following expressions are then syntactically correct?

$[\mathrm{x} \rightarrow \mathrm{y} \rightarrow \mathrm{z}]^{o}$ (the *implication* may be read as: x *implies* that y *implies* that $\mathrm{z} \equiv \mathrm{T}$)
$[\mathrm{x} \vdash \mathrm{y} \vdash \mathrm{z}]^{o}$ (the *inference* may be read as: $\mathrm{x} \equiv \mathrm{T}$ *infers* that $\mathrm{y} \equiv \mathrm{T}$ *infers* that $\mathrm{z} \equiv \mathrm{T}$)
$[\mathrm{x} \rightarrow \mathrm{y} \rightarrow \mathrm{z} \equiv \mathrm{u}]^{o}$
$[\mathrm{x} \rightarrow \mathrm{y} \equiv \mathrm{u} \rightarrow \mathrm{z} \equiv \mathrm{v}\ ]^{o}$
$[(\mathrm{x} \rightarrow \mathrm{y} \equiv \mathrm{u}) \rightarrow \mathrm{z} \equiv \mathrm{v}\ ]^{o}$
$[(\mathrm{x} \rightarrow \mathrm{y} \equiv \mathrm{u}) \vdash \mathrm{z} \equiv \mathrm{v}\ ]^{o}$
$[\mathrm{x} \equiv \mathrm{u} \rightarrow \mathrm{y} \rightarrow \mathrm{z}]^{o}$
$[\mathrm{x} \equiv \mathrm{u} \vdash \mathrm{y} \vdash \mathrm{z}]^{o}$

Exercise 3

In Section E.2, Volume 3, you have a Mac-statement expressing the priorities of most of the operators and directives in the Mac System (e.g. *not* the substitution operator). Until now you have learned about all these constructs except the following four:

The directive $[\preceq]^{o}$ (i.e. 'weakly less information than') and
the three ternary operators $[\forall \mathrm{x} {\in} \mathrm{y}{:}\mathrm{z}]^{\cdot}$, $[\exists \mathrm{x} {\in} \mathrm{y}{:}\mathrm{z}]^{\cdot}$, $[\varepsilon \mathrm{x} {\in} \mathrm{y}{:}\mathrm{z}]^{\cdot}$.

Try to recall what all the other constructs stand for!