Supplementary Note for the MAC-lecture November 10

November 7, 2003

From Algebraic Proofs to Derivation Proofs

Consider the following algebraic proof:

[**Mac proof of** $3 :: (\lambda x.\ \lambda y.\ \langle\ f_3(x)\ \rangle)\ '\ y \equiv 3 :: (\lambda v.\ 2{\cdot}y + 4 :: \langle\ \rangle)$:

Algebra	▷			$3 :: (\lambda x.\ \lambda y.\ \langle\ f_3(x)\ \rangle)\ '\ y$;
Definition	▷			$3 :: (\lambda x.\ \lambda y.\ \langle\ 2{\cdot}x + 4\ \rangle)\ '\ y$;
Rename	▷			$3 :: (\lambda u.\ \lambda v.\ \langle\ 2{\cdot}u + 4\ \rangle)\ '\ y$;
Replace	▷	ApplyLambda	▷	$3 :: \langle\ \lambda v.\ \langle\ 2{\cdot}u + 4\ \rangle\ \mid\ u := y\ \rangle$;
Reflexivity	▷			$3 :: \langle\ \lambda v.\ 2{\cdot}u + 4 :: \langle\ \rangle\ \mid\ u := y\ \rangle$;
Substitution	▷			$3 :: (\lambda v.\ 2{\cdot}y + 4 :: \langle\ \rangle)$]

The first line in the algebraic proof (i.e. 'Algebra ▷ ...') may be interpreted as a message to the Map-program, saying that

> "Before the first [▷]^o (directive 'concludes') in each of the following lines you have the macro of a Mac rule, and after that [▷]^o you have a special example/**instance** of the rule **<u>if</u>** the term in the *previous* proof line is put equal to the term at the end of this line!"

Consider e.g. the 'axioms':

[**Mac rule Definition**: if definition($\mathcal{A}$, $\mathcal{B}$) then $\underline{\mathcal{A} \equiv \mathcal{B}}$]
[**Mac rule Rename**: if structurally equal($\mathcal{A}$, $\mathcal{B}$) then $\underline{\mathcal{A} \equiv \mathcal{B}}$]
[**Mac rule Reflexivity**: $\underline{\mathcal{A} \equiv \mathcal{A}}$]
[**Mac rule Substitution**: if substitution($\mathcal{A}$, $\mathcal{B}$) then $\underline{\mathcal{A} \equiv \mathcal{B}}$]

(Actually, it is only Reflexivity that is shorthand for an equation that holds unconditionally (i.e. an axiom), but since the conditions of the other rules are written *informally*, it is only the <u>underlined axioms</u> that can be seen in our *formal* proofs).

The macros Definition, Rename, Reflexivity and Substitution are shorthands for the underlined equations, and the corresponding **instances** in our proof are underlined in the following **instantiation statements** (i.e. statements of the form 'Rule concludes Instance of rule'):

[Definition ▷ $\underline{3 :: (\lambda x.\ \lambda y.\ \langle\ f_3(x)\ \rangle)\ '\ y \equiv 3 :: (\lambda x.\ \lambda y.\ \langle\ 2{\cdot}x + 4\ \rangle)\ '\ y}$]$^{\cdot}$
[Rename ▷ $\underline{3 :: (\lambda x.\ \lambda y.\ \langle\ 2{\cdot}x + 4\ \rangle)\ '\ y \equiv 3 :: (\lambda u.\ \lambda v.\ \langle\ 2{\cdot}u + 4\ \rangle)\ '\ y}$]$^{\cdot}$
[Reflexivity ▷ $\underline{3 :: \langle\ \lambda v.\ \langle\ 2{\cdot}u + 4\ \rangle\ \mid\ u := y\ \rangle \equiv 3 :: \langle\ \lambda v.\ 2{\cdot}u + 4 :: \langle\ \rangle\ \mid\ u := y\ \rangle}$]$^{\cdot}$
[Substitution ▷ $\underline{3 :: \langle\ \lambda v.\ 2{\cdot}u + 4 :: \langle\ \rangle\ \mid\ u := y\ \rangle \equiv 3 :: (\lambda v.\ 2{\cdot}y + 4 :: \langle\ \rangle)}$]$^{\cdot}$

How about the macro Replace? Well, that is shorthand for an *inference rule* saying that the equation [u ≡ v]$^{\cdot}$ holds *under the condition*/**premise** [x ≡ y]$^{\cdot}$, if the first equation can be obtained as a replacement of the latter equation:

[**Mac rule Replace**: if replace(x,y,u,v) then $\underline{x \equiv y \vdash u \equiv v}$]

and the corresponding **instance** in our proof is underlined in the following **instantiation statement**:

[Replace ▷ $\underline{(\lambda u.\ \lambda v.\ \langle\ 2{\cdot}u + 4\ \rangle)\ '\ y \equiv \langle\ \lambda v.\ \langle\ 2{\cdot}u + 4\ \rangle\ \mid\ u := y\ \rangle}$
$\underline{\triangleright\ 3 :: (\lambda u.\ \lambda v.\ \langle\ 2{\cdot}u + 4\ \rangle)\ '\ y \equiv 3 :: \langle\ \lambda v.\ \langle\ 2{\cdot}u + 4\ \rangle\ \mid\ u := y\ \rangle}$]$^{\cdot}$

In **Derivation proofs** (Danish: 'Aflednings- eller Udledningsbeviser') all the proof lines are instantiation statements, usually labeled by means of the [label : statement]$^{\cdot}$ directive and the unary L-operator of maximal priority.
Contrary to algebraic proofs the rules in derivation proofs do not have to be applied to 'the term in the previous line', and due to the macro Premise, new *inference rules* (lemmas) may now be proved!