Supplementary Note for the MAC-lecture November 6

November 5, 2003

The Classical Sets of Values

In Chapter 9 we saw how all the **values** in the sets

$\mathbf{B}$, $\mathbf{X}$, $\mathbf{D}$ and $\mathbf{E}$

could be represented by **maps** (i.e. N, $\perp$ or a function).
Likewise, we saw how all **pairs** $[\ x :: y\ ]^{\cdot}$ **of values** from these sets could be represented.
Using a cross $[\ \times\ ]^{o}$ to denote **the cartesian product** (Section 10.7), we thus also know how to represent all the values in sets like

$\mathbf{B} \times \mathbf{B},\ \mathbf{B} \times \mathbf{X},\ \mathbf{B} \times \mathbf{D},\ \mathbf{B} \times \mathbf{E},$
$\mathbf{X} \times \mathbf{B},\ \mathbf{X} \times \mathbf{X},\ \mathbf{X} \times \mathbf{D}$,..., or
$\mathbf{B} \times \mathbf{X} \times \mathbf{D}$ (i.e. pairs with head in $\mathbf{B}$ and tail in $\mathbf{X} \times \mathbf{D}$)

as well as values in sets like

$\mathbf{B}^{*}$ (i.e. lists with a *finite* number of boolean elements, e.g. the empty list)

As an example we (or rather *the Map-program*!) will represent the value of the term $[\ \langle\ \mathrm{F}\ \rangle\ ]^{\cdot}$ in $\mathbf{B} \times \mathbf{E}$ by

$$\begin{aligned} \mathrm{S}\ \text{'}\ \langle\ \mathrm{F}\ \rangle \equiv \mathrm{S}\ \text{'}\ (\mathrm{F} :: \langle\ \rangle) &\equiv (\mathrm{N} \therefore \mathrm{N}) \therefore (\mathrm{S}\ \text{'}\ \mathrm{F}) \therefore (\mathrm{S}\ \text{'}\ \langle\ \rangle) \therefore \mathrm{N} \\ &\equiv (\mathrm{N} \therefore \mathrm{N}) \therefore (\mathrm{N} \therefore \mathrm{N}) \therefore ((\mathrm{N} \therefore \mathrm{N}) \therefore \mathrm{N}) \therefore \mathrm{N} \end{aligned}$$

It is worth noting that Klaus in fact does *not* use the map $[\ \perp\ ]^{\cdot}$ in the strong representation of these values! Why not?

Well, all the values T, F, decimal fractions, etc. and *finite* lists of such values are **well-founded classical** mathematical objects, and thus it is not necessary to include any lack of knowledge (e.g. $[\ \perp\ ]^{\cdot}$) in their (strong) representations.
However, during the computing process these values may be appear in form of their **weak** representations, and the weak representations may e.g. be *infinite* lists (for which we will never know the whole contents!) or contain $[\ \perp\ ]^{\cdot}$.
These map-representations may still be **interpreted** (by the W-function) as values from the classical mathematics, but the **representations** themselves(!) should be classified as *ill-founded* or just *non-classical*, due to some lack of knowledge.

As an example we may note that

$$\mathrm{W}\ \text{'}\ ((\mathrm{N} \therefore \mathrm{N}) \therefore (\mathrm{N} \therefore \perp) \therefore ((\mathrm{N} \therefore \perp) \therefore \mathrm{N}) \therefore \mathrm{N}\ ) \equiv \langle\ \mathrm{F}\ \rangle,$$

since it represents a pair, and $\bar{\mathbf{F}}\ \text{'}\ (\mathrm{N} \therefore \perp) \equiv \mathrm{T}$ and $\bar{\mathbf{E}}\ \text{'}\ ((\mathrm{N} \therefore \perp) \therefore \mathrm{N}) \equiv \mathrm{T}$,
but the representation itself is non-classical, since

$\ell\ \text{'}\ ((\mathrm{N} \therefore \mathrm{N}) \therefore (\mathrm{N} \therefore \perp) \therefore ((\mathrm{N} \therefore \perp) \therefore \mathrm{N}) \therefore \mathrm{N}\ ) \equiv \perp$ (cf. the Mac rules page 341)

Since $\mathbf{B} \times \mathbf{E}$, like all the other sets, only should contain the *classical* mathematical values, and Map represents the terms $[\langle\ \mathrm{F}\ \rangle]^{\cdot}$, $[\langle\ 2\ \rangle]^{\cdot}$ by their strong (classical) representations, we thus find that

$[\ \langle\ \mathrm{F}\ \rangle \in \mathbf{B} \times \mathbf{E} \equiv \mathrm{T}\ ]\quad [\ \langle\ 2\ \rangle \in \mathbf{B} \times \mathbf{E} \equiv \mathrm{F}\ ]$
$[\ (\mathrm{N} \therefore \mathrm{N}) \therefore (\mathrm{N} \therefore \mathrm{N}) \therefore ((\mathrm{N} \therefore \mathrm{N}) \therefore \mathrm{N}) \therefore \mathrm{N} \in \mathbf{B} \times \mathbf{E} \equiv \mathrm{T}\ ]$
$[\ (\mathrm{N} \therefore \mathrm{N}) \therefore (\mathrm{N} \therefore \perp) \therefore ((\mathrm{N} \therefore \perp) \therefore \mathrm{N}) \therefore \mathrm{N} \in \mathbf{B} \times \mathbf{E} \equiv \perp\ ]$

In other words: For classical values/map-representations we **know** whether they belong to a set or not, but for non-classical maps we (and Map) lack this knowledge.

How to decide whether a map-representation is classical

For most maps (i.e. N, $\perp$ or functions) we may be able to decide whether it is classical by using the three rules on page 341. Since classical maps also are well-founded:

[**Mac rule ClassicalWellFounded** : $(\ell$ ' x $\equiv$ T$) \vdash ($wf(x) $\equiv$ T$)$] (Volume 3, page 508)

we may also show that a map is *non-classical* by showing that it is *not* well-founded, i.e. that [wf(x) $\equiv \perp$]$^{\cdot}$ holds ($\perp$ is not equal to T, due to the *antirule* on page 580).

A third possibility is to examine the **information contents** of the representation. If [x]$^{\cdot}$ contains at least as much information as a *classical* map [y]$^{\cdot}$, it is sufficient, i.e. then [x]$^{\cdot}$ is also a classical map!

Unfortunately, we have not yet read about 'information contents' (Chapter 17), so the following is just an example of what may be useful when we have studied Chapter 17:

> Consider e.g. the map [λx. N]$^{\cdot}$. This is neither N, $\perp$ nor a simple pair, so the three rules on page 341 are not applicable, and the unary wf-operator just tells us that the map is well-founded, not that it is classical.
> However, if a term [y]$^{\cdot}$ contains at least as much information as a term [z]$^{\cdot}$ (denoted [z $\preceq$ y]$^{\cdot}$) then [ℓ ' z $\equiv$ T]$^{\cdot}$ implies that [ℓ ' y $\equiv$ T]$^{\cdot}$ due to:
>
> [**Mac rule InfoImply** : (z $\preceq$ y) $\vdash$ (f ' z $\to$ f ' y)] (for e.g f $\equiv \ell$, cf. page 634)
>
> Hence, if [λx. N]$^{\cdot}$ contains at least as much information as a classical map, it must be classical.
>
> Depending on the free variable [x]$^{\cdot}$ in [case(x, N, N)]$^{\cdot}$ the latter term will have the value [$\perp$]$^{\cdot}$ or [N]$^{\cdot}$, and according to
>
> [**Mac lemma InfoReflexive** : x $\preceq$ x] and
> [**Mac rule InfoBottom** : $\perp \preceq$ x] (Volume 3, page 634)
>
> [case(x, N, N) $\preceq$ N] thus holds for *all* x. From
>
> [**Mac rule InfoLambda** : $(\mathcal{A} \preceq \mathcal{B}) \vdash (\lambda$x. $\mathcal{A} \preceq \lambda$x. $\mathcal{B})$] (Volume 3, page 634)
>
> we thus conclude that our function [λx. N]$^{\cdot}$ is classical, since it contains at least as much information as the classical strong representation of F :
>
> [λx. case(x, N, N) $\preceq \lambda$x. N]
>
> However, you are of course not *yet* expected to be able to use 'information contents' for deciding whether a map-representation is classical or not! If you want to try anyway, you could e.g. try to show that [(N $\therefore$ N) $\therefore$ (N $\therefore$ N) $\preceq \lambda$x. λx. N] and thus [λx. λx. N]$^{\cdot}$ is classical, too.