On Beta-Shifts Having Arithmetical Languages

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Abstract. Let β be a real number with $1 < \beta < 2$. We prove that the language of the β-shift is Δ_n^0 iff β is a Δ_n-real. The special case where n is 1 is the independently interesting result that the language of the β-shift is decidable iff β is a computable real. The "if" part of the proof is non-constructive; we show that for Walters' version of the β-shift, no constructive proof exists.

1 Introduction

Symbolic dynamics is a vast and varied field of research originating with Morse's work in the 1920ies [12], and has a wide variety of applications [6, 1, 11]. A well-known class of symbolic dynamical systems is that of the *β-shifts* introduced by Renyi [16], developed by Parry in the seminal paper [15], and studied intensely [7, 19, 22, 10, 2, 5, 20, 21]. From the vantage point of the computer scientist, the class of β-shifts is also interesting because of the following fact concerning its *topological entropy*, a quantity of major importance in dynamical systems theory also having connections to data compression [6]:

Theorem 1 ([15, 16]). *If β is a non-integral real number > 1, then the topological entropy of the β-shift is* $\log(\beta)$.

The computability of the topological entropy of various dynamical systems has been studied closely [8, 9, 4]. For none of the studied classes of systems, is it known whether, for each *computable* real number α, there exists a system having topological entropy equal to α. As log is a computable function, Theorem 1 thus offers a tantalizing opportunity to have a class of dynamical systems with this property. Ideally, such a correspondence should be effective, ie. we would like to have an *algorithm* that transformed any computable real β, in some suitable representation, to some suitable representation of the β-shift.

As we shall show, β is a computable real iff the socalled "language" of the β-shift is decidable. Therefore, the "suitable representation" of the β-shift is an algorithm for deciding its language. However, we also show that an algorithm as is asked for above does not exist. Our methods are not particular to the setting of decidable sets, but can be recast to fit effective procedures with access to oracles. Consequently, we prove our results for all Δ_n^0 in the Arithmetical Hierarchy. This proof establishes a surprising correspondence with the elegant notion of Δ_n-reals introduced by Weihrauch and Zheng [27].

For ease of notation, we prove our results for reals in the open interval $(1;2)$. The extension of our results to non-integral βs greater than 2 is certainly possible, but requires some awkward encoding.

2 Preliminaries

For ease of notation, we use the computability notions of recursion theory. The reader in need of intuitive understanding may substitute "program" for "partial recursive function" and "program that always halts" for "total recursive function". Good introductions to recursion theory are [18, 14]. Familiarity with computable analysis or any of the varieties of constructive mathematics will be an advantage, but not a prerequisite; Weihrauch's monograph [25] is recommended.

Throughout the paper, $\mathbb{R}$ denotes the usual set of real numbers from classical mathematics, as does any use of the term "real number". As usual, the greatest integer less than or equal to a real number β is denoted by $\lfloor \beta \rfloor$. We denote the set of positive reals by $\mathbb{R}^{>0}$.

We set $\mathbf{2} \triangleq \{0, 1\}$. The set of right-infinite binary sequences is denoted by $\mathbf{2}^{\mathbb{N}}$, the set of bi-infinite such by $\mathbf{2}^{\mathbb{Z}}$; if b is a finite binary string, we denote by b^ω the right-infinite string consisting of an infinite number of concatenations of b. If M is a language of finite binary strings and $k \in \mathbb{N}$, M^k denotes the set of all finite strings obtained by $k-1$ successive concatenations of k elements of M (with $M^1 = M$ as a special case). As usual, we set $M^* \triangleq \{\lambda\} \cup \bigcup_{k=1}^{\infty} M^k$ where λ is the empty string.

The (strict) *lexicographic order* on $\mathbf{2}^{\mathbb{N}}$ (or $\mathbf{2}^k$ for any $k \in \mathbb{N}$) is defined by $\alpha <_{\text{lex}} \gamma$ iff there is an $n \in \mathbb{N}$ such that $\alpha(n) = 0$, $\gamma(n) = 1$, and $\alpha(k) = \gamma(k)$ for all $k < n$. The non-strict lexicographic order is then defined in the obvious way.

We set $\mathbb{N} \triangleq \{1, 2, \ldots\}$, $\mathbb{N}_0 \triangleq \{0\} \cup \mathbb{N}$, and define $\mathbb{Z}$ and $\mathbb{Q}$ as usual. For computability purposes, we assume elements of $\mathbb{N}_0$, $\mathbb{Z}$ and $\mathbb{Q}$ to have suitable representations as elements of $\mathbb{N}$, whence comparison under $<$, $>$ and $=$ are decidable in these sets. Indices k, m, n, s ranges over $\mathbb{N}$.

2.1 The Beta-Shift

For any finite alphabet Σ, the *one-sided shift* map on $\Sigma^{\mathbb{N}}$, denoted σ, is defined by $\sigma(b_1 b_2 \cdots) \triangleq b_2 b_3 \cdots$. The *two-sided shift* on $\Sigma^{\mathbb{Z}}$, also denoted σ, is defined by $\sigma(b)_i = b_{i+1}$ for all $i \in \mathbb{Z}$.

Definition 1. *Let β be a non-integral real number > 1. The (greedy)* expansion of *1* in powers of β^{-1} *is the sequence $a = (a_k)_{k=1}^{\infty}$ where $a_1 = \lfloor \beta \rfloor$, and $a_k = \lfloor \beta^k - \sum_{i=1}^{k-1} a_i \beta^{k-i} \rfloor$ for $k > 1$.*

If there is an $m \in \mathbb{N}$ such that $k \geq m$ implies $a_k = 0$, then the expansion is said to be finite.

It is easy to see that $1 = \sum_{n=1}^{\infty} a_n \beta^{-n}$, and that β is the unique positive solution to $1 = \sum_{k=1}^{\infty} a_k x^{-k}$. Observe that if $k = \lfloor \beta \rfloor + 1$, then $0 \leq a_n \leq k - 1$

for all $n \in \mathbb{N}$, and thus $a = (a_n)_{n=1}^{\infty}$ is an element of the full shift on k letters (i.e. the set of all right-infinite sequences of words from a k-letter alphabet—this set is unique up to injective renaming of the letters). As we restrict our attention to the open interval $(1; 2)$, we may take $\Sigma = \mathbf{2}$ in the remainder of the paper.

Note that $\sigma^n(a) \leq_{\text{lex}} a$ for all $n \in \mathbb{N}$; this gives rise to the standard definition of the β-shift:

Definition 2. *Let β be a real number with $1 < \beta < 2$, and let $a = (a_n)_{n=1}^{\infty}$ be the expansion of 1 in powers of β^{-1}. The* one-sided W-β-shift *is the subset $\tilde{X}_\beta$ of $\mathbf{2}^{\mathbb{N}}$ containing exactly those b such that, for all $n \in \mathbb{N}_0$, we have $\sigma^n(b) \leq_{\text{lex}} a$.*

The one-sided β-shift*, denoted X_β, is defined to be $\tilde{X}_\beta$ if a is not finite. If a is finite, i.e. $a = a_1 a_2, \ldots a_k 0^\omega$ such that $a_k = 1$, define $a' \triangleq (a_1 a_2 \cdots a_{k-1} 0)^\omega$. Then, X_β is defined to be the subset of $\mathbf{2}^{\mathbb{N}}$ such that, for all $n \in \mathbb{N}_0$, we have $\sigma^n(b) \leq_{\text{lex}} a'$*

The two-sided *W-β-shift is the subset of $\mathbf{2}^{\mathbb{Z}}$ containing exactly those b such that, for all $i \in \mathbb{Z}$, we have we have $b_i b_{i+1} b_{i+2} \cdots \in \tilde{X}_\beta$. The two-sided β-shift is defined analogously, using X_β.*

It is easy to see that both the one- and two-sided (W-)β-shifts are shift-invariant subsets of $\{0, \ldots, \lfloor \beta \rfloor\}^{\mathbb{N}}$ and $\{0, \ldots, \lfloor \beta \rfloor\}^{\mathbb{Z}}$, ie., $\sigma(\tilde{X}_\beta) = \tilde{X}_\beta$ and $\sigma(X_\beta) = X_\beta$.

The term "W-β-shift" is short for "Walters-β-shift", since $\tilde{X}_\beta$ is studied in Walters' book [24] (a point of confusion is that the W-β-shift is occasionally called the β-shift in the literature). The special case where the definition of the W-β-shift differs from the β-shift (i.e. with finite a) stems from the original research of the β-shift [15] where it was necessary to consider the special case to study aspects of number theory. Both the W-β-shift and the β-shift satisfy Theorem 1.

A fundamental concept in the study of shift spaces is that of *language*:

Definition 3. *Let β be a real number with $1 < \beta < 2$. The* language *of the W-β-shift, denoted $\mathcal{L}(\tilde{X}_\beta)$, is the set of all finite binary strings occurring in elements of $\tilde{X}_\beta$, ie. $\mathcal{L}(\tilde{X}_\beta) \triangleq \left\{ b_i b_{i+1} \cdots b_j \mid b \in \tilde{X}_\beta \wedge i, j \in \mathbb{N} \wedge i \leq j \right\}$. $\mathcal{L}(X_\beta)$ is defined analogously.*

Define the shift map σ_{fin} on *finite* strings by $\sigma_{\text{fin}}(b_1 b_2 \cdots b_k) \triangleq b_2 \cdots b_k$ and note that $|\sigma_{\text{fin}}(a)| + 1 = |a|$. Extend the map to sets of finite strings by letting σ_{fin} act on each string in the set. We have:

Proposition 1. *For all $j, k \in \mathbb{N}$, we have $\mathcal{L}(\tilde{X}_\beta) \cap \mathbf{2}^{jk} \subseteq (\mathcal{L}(\tilde{X}_\beta) \cap \mathbf{2}^k)^j$, and $\mathcal{L}(X_\beta) \cap \mathbf{2}^{jk} \subseteq (\mathcal{L}(X_\beta) \cap \mathbf{2}^k)^j$*

Proof. As $\sigma(\tilde{X}_\beta) = \tilde{X}_\beta$, we see that $\sigma_{\text{fin}}(\mathcal{L}(\tilde{X}_\beta)) = \mathcal{L}(\tilde{X}_\beta)$. From the above, we see that $\sigma^k(\mathcal{L}(\tilde{X}_\beta) \cap \mathbf{2}^{jk}) = \mathcal{L}(\tilde{X}_\beta) \cap \mathbf{2}^{(j-1)k}$ (where $\mathbf{2}^0 = \{\lambda\}$). Hence, $\mathcal{L}(\tilde{X}_\beta) \cap \mathbf{2}^{jk} \subseteq (\mathcal{L}(\tilde{X}_\beta) \cap \mathbf{2}^k) \cdot (\mathcal{L}(\tilde{X}_\beta) \cap \mathbf{2}^{(j-1)k})$, and the result follows by a simple induction on j. The proof for X_β is completely analogous. □

2.2 Computable Reals

There are several definitions of "computable reals" in the literature, but these are all equivalent [23, 13, 17, 25, 26]. The definition that will be easiest to work with in this paper is, essentially, that of [25]:

Definition 4. *A sequence* $(I_s)_{s\in\mathbb{N}} = ([p_s; q_s])_{s\in\mathbb{N}}$ *of closed intervals with endpoints in* $\mathbb{Q}$ *is said to be computable if there is a total recursive function* $\phi : \mathbb{N} \longrightarrow \mathbb{Q}$ *where, for all* $s \in \mathbb{N}$*, we have* $\phi(2s) = p_s$ *and* $\phi(2s+1) = q_s$*. A* computable name *is a computable sequence* $(I_s)_{s\in\mathbb{N}}$ *of closed intervals with endpoints in* $\mathbb{Q}$ *such that, for all* $s \in \mathbb{N}$*, we have* $I_{s+1} \subseteq I_s$ *such that* $\bigcap_{s\in\mathbb{N}}$ *is a singleton.*

A real number α *is said to be* computable *if there is a computable name* $(I_s)_{s\in\mathbb{N}}$ *with* $\{\alpha\} = \bigcap_{s\in\mathbb{N}} I_s$.

From any computable name $(I_s)_{s\in\mathbb{N}}$ of some real α, we may effectively obtain a computable name $(I'_s)_{s\in\mathbb{N}}$ of α such that $|I'_s| \leq 2^{-s}$ for all $s \in \mathbb{N}$: Since we know that $|I_s| \to 0$ for $s \to \infty$ and we can, in finite time, check the length of an interval I_s, we may simply wait for $(I_s)_{s\in\mathbb{N}}$ to produce sufficiently small intervals.

Definition 5. *Let* α *be a real number. Then,* α *is said to be* left-computable *(resp.* right-computable*) if there is a total recursive function* $\phi : \mathbb{N} \longrightarrow \mathbb{Q}$ *such that* $\sup_s \phi(s) = \alpha$ *(resp.* $\inf_s \phi(s) = \alpha$*).*

It is well-known that a real number is computable iff it is both left- and right-computable. Also:

Proposition 2. *For each fixed computable name of some real* α*, the following problem is undecidable:*

Given: *A computable name* $(I_n)_{n\in\mathbb{N}}$ *of some computable real* β*.*

To decide*: Is* $\beta < \alpha$*?*

Proof. Standard. See e.g. [3, 25]. □

We use the above proposition in Section 6, specialized to the case where α is the Golden Mean $(1+\sqrt{5})/2$.

We shall need an effective way of finding the unique positive root of equations of the form $1 = \sum_{j=1}^{k} c_j x^{-j}$ where all $c_j \in \mathbf{2}$ and at least one of the c_j equals 1.

Lemma 1. *There is a total recursive function* $\psi : \mathbb{N} \longrightarrow \mathbb{N}$ *such that, for each* $k \in \mathbb{N}$*,* $\phi_{\psi(k)} : \mathbf{2}^k \longrightarrow \mathbb{N}$ *is a partial recursive function such that, if* $c_1, \ldots, c_k \in \mathbf{2}$ *with at least one* $c_j = 1$*, then* $\phi_{\psi(k)}(c_1, \ldots, c_k)$ *is defined and* $\phi_{\phi_{\psi(k)}(c_1,\ldots,c_k)} : \mathbb{N} \longrightarrow \mathbb{Q}$ *is a computable name of the unique positive solution to* $1 = \sum_{j=1}^{k} c_j x^{-j}$*.*

Proof. The positive solution of $1 = \sum_{j=1}^{k} c_j x^{-j}$ is an isolated zero of $f(x) \triangleq \sum_{j=1} c_j x^{-j} - 1$, which is a computable function in the sense of Weihrauch [25]. The result now follows from standard root-finding algorithms, indeed from the fact that every isolated zero of a computable function is a computable real, and that there is an effective way of finding a computable name for it [25, Ch. 6]. □

In the above lemma, ψ is merely a way of getting the right arity, and $\phi_{\psi(k)}$ an "algorithm" for converting the relevant "coefficients" to a computable name of the solution.

2.3 The Arithmetical Hierarchy of Reals

We briefly summarize a few notions from recursion theory:

Definition 6. *Let $A \subseteq \mathbb{N}$. We let $(\phi_i^A)_{i\in\mathbb{N}}$ be an effective enumeration of all partial functions from $\mathbb{N} \longrightarrow \mathbb{N}$ that are recursive-in-A (ie., computable by Turing Machines with access to an oracle for A). Observe that $A = \emptyset$ gives the usual partial recursive functions, and we write ϕ_i in place of $\phi_i^\emptyset$. We will usually suppress the index i if it is not necessary for the exposition.*

We overload the ϕ_i^A to denote partial recursive-in-A functions with domain or codomain any of the sets $\mathbf{2}, \mathbb{N}, \mathbb{Z}, \mathbb{Q}$ (using suitable representations). If B is any of these sets, observe that $C \subseteq B$ is decidable iff there exists a total recursive function $\phi_i : B \longrightarrow \mathbf{2}$ such that $\phi_i(x) = 1$ iff $x \in C$.

Let, for each $n \in \mathbb{N}$, $\langle \cdot, \ldots, \cdot \rangle : \mathbb{N}^n \longrightarrow \mathbb{N}$ be a total recursive pairing function, e.g. the one obtained by repeated use of the Cantor pairing function $\langle i, j \rangle \triangleq (i + j)(i + j + 1)/2 + j$ and its accompanying projections.

Using the pairing function, we may extend the concepts introduced above to finite Cartesian products of any of these sets. If $\phi : \mathbb{N} \longrightarrow \mathbb{N}$ is a *total* function, we say that $\psi : \mathbb{N} \longrightarrow \mathbb{N}$ is recursive-in-ϕ if it is recursive-in-$\{\langle n, \phi(n)\rangle \mid n \in \mathbb{N}\}$.

Definition 7. *For any $A \subseteq \mathbb{N}$, the* jump*, A' is defined by $A' \triangleq \{i \in \mathbb{N} \mid \phi_i^A(i)$ is defined$\}$. For $n \in \mathbb{N}$, the nth jump $A^{(n)}$ is defined by $A^{(1)} \triangleq A'$, and $A^{(n+1)} \triangleq \left(A^{(n-1)}\right)'$. For convenience, we set $A^{(0)} \triangleq A$.*

Define $\Sigma_0^0 = \Pi_0^0 = \Delta_0^0$ to be the set of decidable subsets of $\mathbb{N}$. For any $n \in \mathbb{N}$, define the sets of subsets of $\mathbb{N}$ called Σ_n^0, Π_n^0, and Δ_n^0 as follows: a set $A \subseteq \mathbb{N}$ satisfies $A \in \Sigma_n^0$ iff there is a decidable set $R \subseteq \mathbb{N}$ such that, for any $i \in \mathbb{N}$: $i \in A$ iff $(\exists m_1)(\forall m_2)(\exists m_3)\cdots(Qm_n).(\langle i, m_1, \ldots, m_n\rangle \in R)$ where Q is '$\exists$' if n is odd and '$\forall$' otherwise. $A \subseteq \Pi_n^0$, iff the complement $\overline{A} \in \Sigma_n^0$, and we define $\Delta_n^0 \triangleq \Sigma_n^0 \cap \Pi_n^0$.

It is easy to see that Σ_1^0 contains precisely the recursively enumerable (henceforth "r.e.") subsets of $\mathbb{N}$, and Π_1^0 precisely the co-r.e. sets. It is a standard result that, for $n \in \mathbb{N}$, $A \in \Delta_n^0$ iff there is a total recursive-in-$\emptyset^{(n-1)}$ function $\phi : \mathbb{N} \longrightarrow \mathbf{2}$ such that $\phi(j) = 1$ iff $j \in A$.

Recognizing the similarity between alternating quantifiers in the usual notion of arithmetical hierarchy for $\mathbb{N}$ and the alternating uses of inf and sup in certain generalizations of the computable reals, Weihrauch and Zheng introduced the *arithmetical hierarchy of reals* [27]. Each class in the hierarchy constitutes a closed subfield of $\mathbb{R}$ corresponding to a degree of unsolvability.

A full introduction to the arithmetical hierarchy of reals is beyond the scope of this paper; we shall only need to recapitulate a few facts. The lemma below may be taken as a definition of the classes.

Lemma 2 (Lemma 7.2 of [27]). *With the convention $\emptyset^{(0)} = \emptyset$, the following hold for all $n \in \mathbb{N}_0$ and all $x \in \mathbb{R}$:*

1. *$x \in \Sigma_{n+1}$ iff there is a recursive-in-$\emptyset^{(n)}$ total function $\phi_i : \mathbb{N} \longrightarrow \mathbb{Q}$ with $x = \sup_s \phi_i(s)$.*
2. *$x \in \Pi_{n+1}$ if there is a recursive-in-$\emptyset^{(n)}$ total function $\phi_i : \mathbb{N} \longrightarrow \mathbb{Q}$ with $x = \inf_s \phi_i(s)$.*
3. *$x \in \Delta_{n+1}$ if there is a total function as above such that $x = \lim_{s\to\infty} \phi_i(s)$ that converges effectively, ie. there is a recursive-in-$\emptyset^{(n)}$ total function $\xi : \mathbb{N} \longrightarrow \mathbb{N}$ such that for all $s, j \in \mathbb{N}$, we have $s \geq \xi(j) \Rightarrow |x - \phi_i(s)| \leq 2^{-j}$.*
4. *$x \in \Delta_{n+2}$ if there is a total function as above such that $x = \lim_{s\to\infty} \phi_i(s)$.*

In [27], the lemma is stated only for $n \geq 1$, but the case $n = 0$ is proved elsewhere *loc. cit.*

From the above lemma, it is not hard to see that $\Delta_n = \Sigma_n \cap \Pi_n$ for all $n \in \mathbb{N}$, that Δ_1 coincides with the set of computable reals, and Σ_1 (resp. Π_1) coincides with the set of left-computable (resp. right-computable) reals.

Proposition 3 (First part of Prop. 7.6 of [27]). *For any $n \in \mathbb{N}$, Δ_n is an algebraic field, ie. is closed under the arithmetical operations of addition, subtraction, multiplication and division.*

Examination of the proof in [27] and the standard proof of algebraic closure of the computable reals [25] yields that the closure under algebraic operations is effective. For example, if $\phi_i, \phi_j : \mathbb{N} \longrightarrow \mathbb{Q}$ are total recursive-in-$\emptyset^{(n-1)}$ functions with $\lim_{s\to\infty} \phi_i(s) = \alpha$ and $\lim_{s\to\infty} \phi_j(s) = \beta$ (where the convergence is effective in both cases), then there is a total recursive-in-$\emptyset^{(n-1)}$ function $\psi : \mathbb{N} \longrightarrow \mathbb{Q}$ such that $\lim_{s\to\infty} \psi(s) = \alpha + \beta$, effectively.

We now prove a series of ancillary propositions and lemmas.

Proposition 4. *For any $n \in \mathbb{N}$, if α is a Π_n-real, then so is 2^α.*

Proof. As α is Π_n, there is, by Lemma 2, a total recursive-in-$\emptyset^{(n-1)}$ function $\phi : \mathbb{N} \longrightarrow \mathbb{Q}$ such that $\alpha = \inf_k f(k)$. Using standard methods from computable analysis, it is easy to show that there is a total recursive function $\xi : \mathbb{N} \times \mathbb{Q} \longrightarrow \mathbb{Q}$ such that, for each $k \in \mathbb{N}$ and $p/q \in \mathbb{Q}$, we have $0 \leq \xi(k, p/q) - 2^{p/q} < 2^{-k}$. Hence, $0 \leq \xi(k, f(k)) - 2^{f(k)} < 2^{-k}$ for all $k \in \mathbb{N}$. The function $\zeta : \mathbb{N} \longrightarrow \mathbb{Q}$ defined by $\zeta(k) \triangleq \xi(k, f(k))$ is thus recursive-in-$\emptyset^{(n-1)}$ and, since $x \mapsto 2^x$ is an increasing map, satisfies $\inf_k \zeta(k) = 2^\alpha$. Thus, $2^\alpha \in \Pi_n$. ☐

We need the concept of Δ^0_n-good sequences to make some of the subsequent proofs more readable:

Definition 8. *Let $n \in \mathbb{N}$. A sequence $(x_s)_{s\in\mathbb{N}}$ of computable reals is called Δ^0_n-*good *if there is a $\emptyset^{(n-1)}$-computable total function $\psi : \mathbb{N} \longrightarrow \mathbb{N}$ such that, for each $s \in \mathbb{N}$, $\phi_{\psi(s)} : \mathbb{N} \longrightarrow \mathbb{Q}$ is a computable name of x_s.*

Taking the sup or inf of such sequences does not force us into a higher level of the arithmetical hierarchy:

Proposition 5. *Let $n \in \mathbb{N}$, and let $(x_s)_{s\in\mathbb{N}}$ be a Δ_n^0-good, convergent sequence of computable reals. Then:*

1. *If $\forall s \in \mathbb{N}.x_s \leq \lim_s x_s$, then $\lim_{s\longrightarrow\infty} x_s = \sup_s x_s \in \Sigma_n$.*
2. *If $\forall s \in \mathbb{N}.x_s \geq \lim_s x_s$, then $\lim_{s\longrightarrow\infty} x_s = \inf x_s \in \Pi_n$.*

Proof. We prove (1); the proof of (2) is similar.

As we have $\forall s \in \mathbb{N}.x_s \leq \lim_{s\rightarrow\infty} x_s$, we immediately get $\lim_{s\rightarrow\infty} x_s = \sup_s x_s$. As $(x_s)_{s\in\mathbb{N}}$ is Δ_n^0-good, there is a total recursive-in-$\emptyset^{(n-1)}$ function ψ with the properties of Definition 8. For each s, $\phi_{\psi(s)}(2s)$ is a left endpoint of an interval a name of x_s; there is clearly a total recursive-in-$\emptyset^{(n-1)}$ function $\xi : \mathbb{N} \longrightarrow \mathbb{Q}$ such that $\xi(s) = \phi_{\psi(s)}(2s)$, for all $s \in \mathbb{N}$.

By the comments after Definition 4, we may assume wlog. that for each $s \in \mathbb{N}$, we have $|x_s - \phi_{\psi(s)}(2s)| \leq 2^{-s}$, Furthermore, for each $s \in \mathbb{N}$, $\phi_{\psi(s)}(2s)$ is a *left* endpoint of a name of x_s, and we thus have $x_s \geq \phi_{\psi(s)}(2s)$ for all $s \in \mathbb{N}$, and thus $\lim_s \phi_{\psi(s)}(2s) = \sup_s \phi_{\psi(s)}(2s) = \sup_s \xi(s) \in \Sigma_n$, as desired. □

3 Beta-Shifts Having Arithmetical Languages

In this and the remaining sections, we assume a $\beta \in \mathbb{R}$ with $1 < \beta < 2$. Furthermore, we freely refer to $(a_k)_{k\in\mathbb{N}}$ as the expansion of 1 in powers of β^{-1}.

Let log be the logarithm to base 2; we now establish a sufficient condition for $\log(\beta)$ to be in Π_n:

Proposition 6. *Let $\mathcal{L}(\tilde{X}_\beta)$ be Δ_n^0. Then, the quantity*

$$\log(\beta) = h_{top}\left(\tilde{X}_\beta\right) = \lim_{k\rightarrow\infty}\left(\frac{\log(|\mathcal{L}(\tilde{X}_\beta) \cap \mathbf{2}^k|)}{k}\right)$$

is a Π_n-real. The result holds with $\mathcal{L}(\tilde{X}_\beta)$ replaced by $\mathcal{L}(X_\beta)$.

Proof. The limit always exists and equals $\log(\beta)$ by the standard theory of the β-shift [24]. We want to use Proposition 5 and proceed as follows:

- If $\mathcal{L}(\tilde{X}_\beta)$ is Δ_n^0, then there is a total recursive-in-$\emptyset^{(n-1)}$ function $\zeta : \mathbf{2}^* \longrightarrow \mathbf{2}$ such that $\zeta(a) = 1$ iff $a \in \mathcal{L}(\tilde{X}_\beta)$; hence, there is a total recursive-in-$\emptyset^{(n-1)}$ function $\xi : \mathbb{N} \longrightarrow \mathbb{N}$ such that $\xi(k) = |\mathcal{L}(\tilde{X}_\beta) \cap \mathbf{2}^k|$ for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, $\log(|\mathcal{L}(\tilde{X}_\beta) \cap \mathbf{2}^k|)/k$ is a computable real, and we can effectively find a computable name for it given the natural number $|\mathcal{L}(\tilde{X}_\beta) \cap \mathbf{2}^k|$ as input. Thus, there is a total recursive-in-$\emptyset^{(n-1)}$ function $\psi : \mathbb{N} \longrightarrow \mathbb{N}$ such that $\phi_{\psi(k)} : \mathbb{N} \longrightarrow \mathbb{Q}$ is a computable name of $\log(\mathcal{L}(\tilde{X}_\beta) \cap \mathbf{2}^k)/k$ for all $k \in \mathbb{N}$, proving that $(\log(\mathcal{L}(\tilde{X}_\beta) \cap \mathbf{2}^k)/k)_{k\in\mathbb{N}}$ is a Δ_n^0-good sequence.
- For all $j, k \in \mathbb{N}$, Proposition 1 entails that $\mathcal{L}(\tilde{X}_\beta) \cap \mathbf{2}^{kj} \subseteq (\mathcal{L}(\tilde{X}_\beta) \cap \mathbf{2}^k)^j$, hence that $|\mathcal{L}(\tilde{X}_\beta) \cap \mathbf{2}^{kj}| \leq |(\mathcal{L}(\tilde{X}_\beta) \cap \mathbf{2}^k)^j| = |\mathcal{L}(\tilde{X}_\beta) \cap \mathbf{2}^k|^j$.
  Thus:

$$\frac{\log(|\mathcal{L}(\tilde{X}_\beta) \cap \mathbf{2}^{kj}|)}{kj} \leq \frac{\log(|\mathcal{L}(\tilde{X}_\beta) \cap \mathbf{2}^{k}|^{j})}{kj} = \frac{\log(|\mathcal{L}(\tilde{X}_\beta) \cap \mathbf{2}^{k}|)}{k}.$$

The rightmost expression above does not depend on j, whence we have, for each $k \in \mathbb{N}$:

$$\lim_{j\to\infty}\left(\frac{\log(|\mathcal{L}(\tilde{X}_\beta) \cap \mathbf{2}^{j}|)}{j}\right) = \lim_{j\to\infty}\left(\frac{\log(|\mathcal{L}(\tilde{X}_\beta) \cap \mathbf{2}^{kj}|)}{kj}\right) \leq \frac{\log(|\mathcal{L}(\tilde{X}_\beta) \cap \mathbf{2}^{k}|)}{k}.$$

Thus, for each $k \in \mathbb{N}$, $\log(|\mathcal{L}(\tilde{X}_\beta) \cap \mathbf{2}^{k}|)/k$ is an upper bound on $h_{\text{top}}\left(\tilde{X}_\beta\right)$.

Finally, Proposition 5 yields $h_{\text{top}}\left(\tilde{X}_\beta\right) \in \Pi_n$. The proof for $\mathcal{L}(X_\beta)$ can be carried out by copying the arguments for $\mathcal{L}(\tilde{X}_\beta)$ verbatim. □

The following lemma establishes a useful correspondence between $\mathcal{L}(\tilde{X}_\beta)$ and $\{k \mid a_k = 1\}$.

Lemma 3. *$\mathcal{L}(\tilde{X}_\beta)$ is Δ^0_n iff $\{k \in \mathbb{N} \mid a_k = 1\}$ is Δ^0_n.*

Proof. Let, for each $k \in \mathbb{N}$, $D_k \triangleq \{d \in \mathbf{2}^k \mid \forall j \in \{0,\ldots,k-1\}.\sigma^j(d) \leq_{\text{lex}} a_1 \cdots a_{k-j}\}$. Observe that if $d \in D_k$, then $d\cdot 0^\omega \in \tilde{X}_\beta$, and thus $D_k \subseteq \mathcal{L}(\tilde{X}_\beta) \cap \mathbf{2}^k$. Conversely, if $d \in \mathcal{L}(\tilde{X}_\beta) \cap \mathbf{2}^k$, then $\sigma^j_{\text{fin}}(d) \leq_{\text{lex}} a_1 \cdots a_{k-j}$ for $j \in \{0,\ldots,k-1\}$, ie. $d \in D_k$. Hence, $D_k = \mathcal{L}(\tilde{X}_\beta) \cap \mathbf{2}^k$.

If $\mathcal{L}(\tilde{X}_\beta)$ is Δ^0_n, then we can obviously establish a total recursive-in-$\emptyset^{(n-1)}$ function $\phi : \mathbb{N} \longrightarrow \mathbf{2}$ such that $\phi(k) = b_k$ where $b_1 \cdots b_k$ is the lexicographically greatest element of D_k. By definition of the β-shift, the lexicographically greatest element of D_k is the prefix of length k of $a_1 a_2 \cdots$. But then $\phi(k) = 1$ iff $a_k = 1$, ie. $\{k \in \mathbb{N} \mid a_k = 1\}$.

Conversely, if $\{k \in \mathbb{N} \mid a_k = 1\}$ is Δ^0_n, we can recursively-in-$\emptyset^{(n-1)}$ establish $a_1 \cdots a_k$ for each $k \in \mathbb{N}$. With $a_1 \cdots a_k$ in hand, we can effectively establish D_k. For a given $d \in \mathbf{2}^*$, to decide whether $d \in \mathcal{L}(\tilde{X}_\beta)$, we need only examine whether $d \in D_{|d|}$, which is thus recursive-in-$\emptyset^{(n-1)}$, ie. there is a total recursive-in-$\emptyset^{(n-1)}$ function $\psi : \mathbf{2}^* \longrightarrow \mathbf{2}$ such that $\psi(d) = 1$ iff $d \in \mathcal{L}(\tilde{X}_\beta)$. □

Observe that the proof is constructive, ie. we have an effective way of producing decision procedures for $\{k \mid a_k = 1\}$ given decision procedures for $\mathcal{L}(\tilde{X}_\beta)$ as input, and vice versa.

Proposition 7. *$\mathcal{L}(X_\beta)$ is Δ^0_n iff $\mathcal{L}(\tilde{X}_\beta)$ is Δ^0_n.*

Proof. If a is not finite, we have $\mathcal{L}(X_\beta) = \mathcal{L}(\tilde{X}_\beta)$, and the result follows. If a is finite, then $\{k \mid a_k = 1\}$ is Δ^0_1 (there are only a finite number of 1s), whence Lemma 3 furnishes that $\mathcal{L}(\tilde{X}_\beta)$ is Δ^0_1. Also, we have that $\mathcal{L}(X_\beta)$ is Δ^0_1, since we can use the same construction as in the second part of the proof of Lemma 3 applied to the sequence $a' = (a_1 a_2 \cdots a_{k-1} 0)^\omega$ where k is the largest integer with $a_k = 1$. □

Let $s \in \mathbb{N}$, $a_1 = 1$ and $a_j \in \mathbf{2}$ for $j \in \{2, \ldots, s\}$. Consider the map $f_s : \mathbb{R}^{>0} \longrightarrow \mathbb{R}^{>0}$ defined by $f_s(x) = \sum_{j=1}^{s} a_j x^{-j}$. Now, $f_s(x)$ is strictly decreasing, continuous and onto, whence $1 = f_s(x)$ has a unique positive real solution for all s. We now show that this solution is a computable real, and that there is an effective way to find it given $a_1, \ldots, a_s$ as input:

Proposition 8. *If $\{k \in \mathbb{N} \mid a_k = 1\}$ is Δ_n^0, then the sequence $(\alpha_s)_{s \in \mathbb{N}}$ of positive solutions to $1 = \sum_{j=1}^{s} a_j x^{-j}$ is a Δ_n^0-good sequence of computable reals, convergent with limit β, and satisfying $\forall s \in \mathbb{N}.\alpha_s \leq \beta$.*

Proof. Observe that we always have $a_1 = 1$. By Lemma 1, there is an effective procedure yielding a computable name of the unique positive real solution to $1 = \sum_{j=1}^{s} a_j x^{-j}$, when given $(a_1, \ldots, a_s)$ as input. Let the notation and names of recursive functions be as in Lemma 1; Then $\phi_{\phi_{\psi(s)}(a_1,\ldots,a_s)} : \mathbb{N} \longrightarrow \mathbb{Q}$ is a computable name of the unique positive solution, and the function $\psi : \mathbb{N} \longrightarrow \mathbb{N}$ is total recursive. As $\{k \in \mathbb{N} \mid a_k = 1\}$ is Δ_n^0, there is a total recursive-in-$\emptyset^{(n-1)}$ function $\xi : \mathbb{N} \longrightarrow \mathbf{2}$ with $\xi(k) = 1$ iff $a_k = 1$, and hence a total recursive-in-$\emptyset^{(n-1)}$ function $\zeta : \mathbb{N} \longrightarrow \mathbf{2}$ such that $\zeta(k) = a_k$ for all $k \in \mathbb{N}$. Hence, there is a total recursive-in-$\emptyset^{(n-1)}$ function mapping $s \in \mathbb{N}$ to an index of $\phi_{\psi(s)}(\zeta(s), \ldots, \zeta(1))$, whence $(\alpha_s)_{s \in \mathbb{N}}$ is a Δ_n^0-good sequence of computable reals. The sequence is non-decreasing, since $\alpha_{s+1} = \alpha_s$ if $a_{s+1} = 0$ and $\alpha_{s+1} > \alpha_s$ if $a_{s+1} = 1$. Now, β is the unique positive solution to $1 = \sum_{j=1}^{\infty} a_j x^{-j}$, and clearly all of the α_s are less than or equal to this solution. Hence, $\forall s \in \mathbb{N}.\alpha_s \leq \beta$. Proving that $\lim_{s \to \infty} \alpha_s = \beta$ is a standard exercise in undergraduate (classical) mathematics. □

We now have the following key lemma:

Lemma 4. *If $\mathcal{L}(\tilde{X}_\beta)$ is Δ_n^0, then β is a Δ_n-real.*

Proof. Propositions 6 and 4 furnish that $\beta \in \Pi_n$. Furthermore, Lemma 3, and Propositions 8 and 5 furnish that $\beta \in \Sigma_n$, whence the result. □

4 Arithmetical Betas

In the first lemma of this section, we give a sufficient condition for $\{k \mid a_k = 1\}$ to be Δ_n^0.

Lemma 5. *Let $n \in \mathbb{N}$, and assume that, for all $k \in \mathbb{N}$, we have $1 \neq \beta^k - \sum_{j=1}^{k-1} a_j \beta^{k-j}$. Then there is a total recursive-in-$\emptyset^{(n-1)}$ function $\xi : \mathbb{N} \longrightarrow \mathbf{2}$ such that $\xi(n) = 1$ iff $\beta^k - \sum_{j=1}^{k-1} a_j \beta^{k-j} \geq 1$, ie. $\{k \mid a_k = 1\}$ is a Δ_n^0 subset of $\mathbb{N}$.*

Proof. By Lemma 2, there is a recursive-in-$\emptyset^{(n-1)}$ total function $f : \mathbb{N} \longrightarrow \mathbb{Q}$ such that $\beta = \lim_{i \to \infty} f(i)$ effectively (that is, there is an $\emptyset^{(n-1)}$-computable total function $\psi : \mathbb{N} \longrightarrow \mathbb{N}$ such that, for all $m \in \mathbb{N}$, $|\beta - f(i)| < 2^{-m}$ for all $i \geq \psi(m)$). By Proposition 3, Δ_n is an algebraic field, and we thus have

$\beta^k - \sum_{j=1}^{k-1} a_j \beta^{k-j} \in \Delta_n$ for all $k \in \mathbb{N}$. By the comments after the proposition, the algebraic operations are recursive, and there is thus a total recursive-in-$\emptyset^{(n-1)}$ function $\xi : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{Q}$ such that, for all $k, m \in \mathbb{N}$, $|\beta^k - \sum_{j=1}^{k-1} a_j \beta^{k-j} - \xi(k,m)| < 2^{-m}$.

Consider the recursive-in-ξ procedure that does the following: For each $k \in \mathbb{N}$, run $\xi(k,i)$ on successively greater i until an i is found for which $|1 - \xi(k,i)| > 2^{-(i-1)}$ (the assumption $1 \neq \beta^k - \sum_{j=1}^{k-1} a_j \beta^{k-j}$ implies existence of such an i). As $|\beta^k - \sum_{j=1}^{k-1} a_j \beta^{k-j} - \xi(k,i)| < 2^{-i}$, we have $\xi(k,i) > 1$ iff $\beta^k - \sum_{j=1}^{n-1} a_j \beta^{k-j} > 1$.

This procedure can clearly be made into a total recursive-in-$\emptyset^{(n-1)}$-function $h : \mathbb{N} \longrightarrow \mathbf{2}$ such that $h(n) = 1$ iff $\beta^k - \sum_{j=1}^{n-1} a_j \beta^{k-j} > 1$. □

The next lemma is a counterpart to Lemma 4.

Lemma 6. *Let $\beta \in \Delta_n$. Then, $\{k \mid a_k = 1\}$ is Δ_n^0.*

Proof. Consider $(a_k)_{k\in\mathbb{N}}$. Either there is a $k \in \mathbb{N}$ such that $1 = \beta^k - \sum_{i=1}^{k-1} a_i \beta^{k-i}$, or there is not[1]. If there is no such k, then Lemma 5 furnishes the result. If there is no such k, the a_i, for $1 \leq i \leq k-1$, are the initial coefficients of the expansion of 1 in negative powers of β. Hence, $a_k = \lfloor \beta^k - \sum_{i=1}^{k-1} a_i \beta^{k-i} \rfloor = \beta^k - \sum i = 1^{k-1} a_i \beta^{k-i} = 1$, showing that $1 = \sum_{i=1}^{k} a_i \beta^{-i}$ is the β-expansion of 1, all further coefficients therefore being 0. Thus, there is a total *recursive* function $\phi : \mathbb{N} \longrightarrow \mathbf{2}$ such that $\phi(k) = 1$ iff $a_k = 1$. □

5 The Correspondence Theorem

We now prove our main result:

Theorem 2. *Let β be a real number with $1 < \beta < 2$, and let $n \in \mathbb{N}$. The following are equivalent:*

1. *β is a Δ_n-real.*
2. *$\{k \lfloor a_k = 1\}$ is a Δ_n^0 subset of $\mathbb{N}$.*
3. *$\mathcal{L}(\tilde{X}_\beta)$ is a Δ_n^0 subset of $\mathbf{2}^*$.*
4. *$\mathcal{L}(X_\beta)$ is a Δ_n^0 subset of $\mathbf{2}^*$.*

Proof. (1) ⇒ (2) is Lemma 6, (2) ⇒ (3) is one-half of Lemma 3, and (3) ⇒ (1) is Lemma 4. Finally, Proposition 7 furnishes equivalence of (3) and (4). □

The case where n is 1 is of particular interest:

Corollary 1. *Let β be a real number with $1 < \beta < 2$. The following are equivalent:*

1. *β is a computable real.*
2. *The set $\{k \mid a_k = 1\}$ is a decidable subset of $\mathbb{N}$.*
3. *$\mathcal{L}(\tilde{X}_\beta)$ is a decidable subset of $\mathbf{2}^*$.*
4. *$\mathcal{L}(X_\beta)$ is a decidable subset of $\mathbf{2}^*$.*

[1] This use of the Law of the Excluded Middle is the essential non-constructive part of the proof: We are asking for an answer to the undecidable problem of whether such a k exists.

6 Absence of a Constructive Proof

Inspection of the proof of Lemma 4 reveals that it is constructive and thus yields an effective procedure for converting a decision procedure for $\mathcal{L}(\tilde{X}_\beta)$ to a computable name of β. Hence, (3) $\Rightarrow$ (1) of Theorem 2 is effective in the case where n equals 1.

Unfortunately, that fact is not very interesting; what we *really* want is for (1) $\Rightarrow$ (3) to be constructive, ie. we desire a *program* to generate a decision procedure for $\mathcal{L}(\tilde{X}_\beta)$ when given a computable name of a computable real β as input. Alas, this is impossible:

Theorem 3. *There is no partial recursive function* $\psi : \mathbb{N} \longrightarrow \mathbb{N}$ *such that if* $\phi_i : \mathbb{N} \longrightarrow \mathbb{Q}$ *is a computable name of a computable real* $\beta \in (1;2)$*, then* $i \in dom(\psi)$ *and* $\phi_{\psi(i)} : \mathbf{2}^* \longrightarrow \mathbf{2}$ *is a total recursive function such that* $\phi_{\psi(i)}(c) = 1$ *iff* $c \in \mathcal{L}(\tilde{X}_\beta)$ *for all* $c \in \mathbf{2}^*$.

Proof. Observe that for any $\beta \in (1,2)$, we have $a_1 = 1$. Also, $a_2 = 0$ iff $\lfloor \beta^2 - \beta \rfloor = 0$ iff $\beta^2 - \beta < 1$ iff $\beta < (1+\sqrt{5})/2$. If ψ existed, we could, by Lemma 3 and the comments thereafter, effectively establish the sequence $(a_n)_{n\in\mathbb{N}}$. Thus, we could decide whether $a_2 = 0$ or $a_2 = 1$, and hence decide whether $\beta < (1+\sqrt{5})/2$, which is impossible by Proposition 2. □

In other words, the proof of the theorem shows that there is no program converting computable names to decision procedures for the associated shifts. Note also that the proof can immediately be adapted to show that (1) $\Rightarrow$ (2) in Theorem 2 cannot be made effective. As $x \mapsto 2^x$ is a computable function on the computable reals, another adaptation of the proof yields:

Corollary 2. *There is no partial recursive function* $\psi : \mathbb{N} \longrightarrow \mathbb{N}$ *such that if* $\phi_i : \mathbb{N} \longrightarrow \mathbb{Q}$ *is a computable name of a computable real* $\beta \in (0;1)$*, then* $i \in dom(\psi)$ *and* $\phi_{\psi(i)} : \mathbb{N} \longrightarrow \mathbf{2}$ *is a total recursive function with* $\phi_{\psi(i)}(c) = 1$ *iff* $c \in \mathcal{L}(\tilde{X}_\beta)$.

Thus, there is no effective way to find decision procedures for the W-β-shift given its topological entropy $\log(\beta)$.

Whether the corresponding result holds for X_β is still open; we strongly conjecture that it does.

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