On the Modularity of Confluence in Infinitary Term Rewriting

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Abstract. We show that, unlike the case in finitary term rewriting, confluence is not a modular property of infinitary term rewriting systems, even when these are non-collapsing. We also give a positive result: two sufficient conditions for the modularity of confluence in the infinitary setting.

1 Introduction

Modularity is the study of properties of rewriting systems that are, or are not, preserved when combining different systems. In finitary term rewriting, a number of properties, e.g., confluence [9, 14], are known to be modular whereas others, e.g. termination [13], are known not to be; see Ch. 8 of [11] for an overview. Modularity has, however, been left completely uninvestigated in the setting of *infinitary term rewriting*, a formalism developed in a series of landmark papers [3, 4, 6, 7]. In this paper, we take the first steps to investigate modularity in the setting of *strongly convergent* infinitary rewriting.

1.1 Contributions

We show that:

- Confluence is *not* a modular property of infinitary term rewriting systems, even for non-collapsing systems.
- Confluence is preserved under disjoint union of a set of *left-linear* iTRSs iff the set has the property of being *essentially non-collapsing*, i.e. at most one system contains collapsing rules.
- Confluence is preserved under disjoint union of a set of arbitrary (i.e. not necessarily left-linear), non-collapsing iTRSs if only terms of *finite rank* are considered.

1.2 Organization of the Paper

Section 2 introduces basic concepts from infinitary rewriting and defines what it means for a property to be modular in this setting. Section 3 contains the counterexample to modularity of confluence. Sections 4 and 5 presents the two sufficient conditions for confluence to be modular, whereas Section 6 briefly discusses the difficulties in extending the results to the setting of *weakly* convergent rewriting.

2 Preliminaries

We assume familiarity with finitary term rewriting (ample introductions are [2, 8, 1] and Chapter 2 of [12]) and basic ordinal theory (see e.g. [10]). The successor of an ordinal α is denoted by $\alpha + 1$, and the least infinite ordinal by ω. If α is a limit ordinal, we indicate this by writing $Lim(\alpha)$. We assume a countable set of variables and a "Hilbert-hotel" style renaming for all terms considered so that fresh variables are always available. Positions in (finite) terms are elements of $\{1, 2, \ldots\}^*$ defined in the usual way. The subterm of term s at position p is denoted $s|_p$. The root symbol of a term is the symbol at position ϵ. If $\mathbf{f}$ is a unary function symbol and $k \in \omega$, we denote by $\mathbf{f}^k(s)$ k successive applications of $\mathbf{f}$ to the term s; we extend the notation to include $\mathbf{f}^\omega$ with the obvious meaning. Let $\Box \notin \Sigma \cup \mathcal{X}$. A term with *holes* is a term over Σ with variable set $\mathcal{X} \cup \{\Box\}$. A term with a hole at position p will be written as $s[]_p$, a term where the holes are at positions $i \in I$ where I is some (possibly infinite) set is written as $s[]_{i \in \mathcal{I}}$ and is called a (many-hole) *context*. Observe that a context may have no holes. Substituting terms from an $\mathcal{I}$-indexed sequence of terms $(s_i)_{i \in \mathcal{I}}$ into a many-hole context is defined in the obvious way. In the following, we recall a number of concepts from infinitary rewriting; our definitions are as in [5], with a few differences in nomenclature.

Definition 1. *Let $Ter(\Sigma)$ be the set of finite terms over the (not necessarily finite) signature Σ with alphabet $\mathcal{X}$. Define the metric $d : Ter(\Sigma) \times Ter(\Sigma) \longrightarrow [0; 1]$ by $d(t, t') \triangleq 0$ if $t = t'$ and $d(t,' t) = 2^{-k}$ otherwise, where k is the length of the shortest position at which t and t' differ. The completion of the metric space $(Ter(\Sigma), d)$, denoted $Ter^\infty(\Sigma)$ is called the* set of finite and infinite terms *(or simply* terms*) over Σ. The* depth *of a position, u, in a term is the length, $|u|$, of u.*

Definition 2. *An* infinitary rewrite rule *is a pair $\mathbf{l} \longrightarrow \mathbf{r}$ where $\mathbf{l} \in Ter(\Sigma)$ and $\mathbf{r} \in Ter^\infty(\Sigma)$ such that $\mathbf{l}$ is not a variable and every variable in $\mathbf{r}$ occurs in $\mathbf{l}$. An* infinitary term rewriting system*, denoted* iTRS*, is a pair $\mathcal{R} \triangleq (\Sigma, R)$, consisting of a signature Σ and a set of infinitary rewrite rules R.*

Definition 3. *A term s is* linear *if every variable of $\mathcal{X}$ occurs at most once in s. A rule $\mathbf{l} \longrightarrow \mathbf{r}$ is* left-linear *if $\mathbf{l}$ is linear. $\mathbf{l} \longrightarrow \mathbf{r}$ is* collapsing *if $\mathbf{r} \in \mathcal{X}$.*

Definition 4. *Let α be any ordinal. A* derivation of length α *is a sequence of rewrite steps $(s_\beta \longrightarrow s_{\beta+1})_{\beta<\alpha}$. In the step $s_\beta \longrightarrow s_{\beta+1}$, assume that the redex contracted is at position u_β of s_β; the* depth*, denoted d_β, of the redex is the depth of u_β. The derivation is called* weakly convergent *(aka.* Cauchy convergent*) if,* *for every limit ordinal $\lambda \leq \alpha$, the distance $d(s_\beta, s_\lambda)$ tends to 0 as β approaches λ from below. It is called* strongly convergent *if it is Cauchy convergent and, in addition, d_β tends to infinity as β approaches λ from below. If $(s_\beta \longrightarrow s_{\beta+1})_{\beta<\alpha}$ is convergent with limit t and $s_0 = s$, we write $s \longrightarrow^\alpha t$, and say that t is a* derivative *of s. When the length of the derivation is bounded above by γ, we shall occasionally write $s \longrightarrow^{\leq\gamma} t$. When the length of a* strongly *convergent derivation is unimportant, we write $s \twoheadrightarrow t$.*

Observe that concatenating any finite number of strongly convergent derivations yields a strongly convergent derivation. The following lemma concerning strongly convergent rewriting is due to Kennaway et al. [5, 6]:

Lemma 1 (Compression). *In every left-linear iTRS, if $s \twoheadrightarrow t$, then $s \longrightarrow^{\leq\omega} t$.*

Definition 5. *A* peak *of an iTRS R is a triple $t \twoheadleftarrow s \twoheadrightarrow t'$ of terms such that $s \twoheadrightarrow t$ and $s \twoheadrightarrow t'$. A* valley *of an iTRS R is a triple $t \twoheadrightarrow s' \twoheadleftarrow t'$ of terms such that $t \twoheadrightarrow s'$ and $t' \twoheadrightarrow s'$. If there exists a valley $t \twoheadrightarrow s' \twoheadleftarrow t'$, then t and t' are said to be* joinable*, written $t \sim t'$, and s' is said to be their* join*. A term s of R is said to be* confluent *(aka.* transfinitely Church-Rosser*) if every peak $t \twoheadleftarrow s \twoheadrightarrow t'$ has a corresponding valley $t \twoheadrightarrow s' \twoheadleftarrow t'$. The iTRS R is said to be confluent if all of its terms are confluent.*

The next auxiliary lemma, the *Dovetailing Lemma*, will prove to be useful in Sections 4 and 5.

Lemma 2 (Dovetailing). *If $\{s_i\}_{i\in\mathcal{I}}$ is a set of parallel subterms of some term $s = C[s_i]_{i\in\mathcal{I}}$ such that there are terms t_i with $s_i \twoheadrightarrow t_i$ for all $i \in \mathcal{I}$, then $s \twoheadrightarrow C[t_i]_{i\in\mathcal{I}}$.*

Proof. Since function symbols have finite arity, there are finitely many of s_i with root symbol at any given depth k. Since concatenation of a finite number of strongly convergent derivations yields a strongly convergent derivation, there exists, for every $k \in \omega$, a strongly convergent derivation S_k turning all s_i with root symbol at depth k into the respective t_i. Clearly, the derivation $S_0 \cdot S_1 \cdot S_2 \cdots$ is strongly convergent with limit $C[t_i]_{i\in\mathcal{I}}$. □

Definition 6. *The* direct sum *of a set $\mathcal{R} = \{R_k\}_{k\in\mathcal{K}}$ of iTRSs over signatures $\{\Sigma_k\}_{k\in\mathcal{K}}$, denoted by $\oplus\mathcal{R}$, is the iTRS $\bigcup_{k\in\mathcal{K}} R_k$ over signature $\biguplus_{k\in\mathcal{K}} \Sigma_k$, where $\biguplus$ is disjoint union of sets. If $\{R_k\}_{k\in\mathcal{K}} = \{R_0, R_1\}$, we write $R_0 \oplus R_1$. A term over $\biguplus_{k\in\mathcal{K}} \Sigma_k$ is called* monochrome *if all of its function symbols are elements of a single Σ_k.*

When the involved signatures are disjoint, we refer to the iTRSs of $\mathcal{R}$ as being disjoint. Observe that, unlike the finitary case, we allow sums of any (finite or infinite) number of iTRSs. As we shall see, this has no impact on the modularity of confluence.

Definition 7. *The* rank *of a term s over $\{\Sigma_k\}_{k \in \mathcal{K}}$, denoted by* $\mathbf{rank}(s)$*, is the maximal number of signature changes in maximal paths starting from the root, if such a number exists, and ∞ otherwise. If* $\mathbf{rank}(s) \neq \infty$*, we say that s* is of finite rank.

Note that $\mathbf{rank}(s) = \infty$ does not imply the existence of a maximal path encountering infinitely many signature changes. All maximal paths could encounter only finitely many signature changes, but an upper bound on the number of such changes may not exist.

Definition 8. *A predicate P on the class of iTRSs is said to be* modular *if, given an arbitrary set $\{R_k\}_{k \in \mathcal{K}}$, the direct sum $\oplus\mathcal{R}$ has property P iff all elements of $\{R_k\}_{k \in \mathcal{K}}$ have property P.*

In (finitary) TRSs, the number of elements of $\{\Sigma_k\}_{k \in \mathcal{K}}$ contributing function symbols to any term s is finite. Hence, it is sufficient to consider finite sets $\mathcal{K}$ in this setting, showing that our definition reduces to the usual one in the finitary case.

Definition 9. *Let the root symbol of the term s over $\biguplus_{k \in \mathcal{K}} \Sigma_k$ belong to the signature Σ_r. The* cap *of s, denoted* $\mathbf{cap}(s)$*, is the maximal monochrome, linear term $C[x_i]_{i \in \mathcal{I}}$ containing the root symbol of s such that there is a sequence $(s_i)_{i \in \mathcal{I}}$ of terms with root symbols in $\biguplus_{k \in \mathcal{K} \setminus \{r\}} \Sigma_k$ and $s = C[s_i]_{i \in \mathcal{I}}$, in which case we write $C[\![s_i]\!]_{i \in \mathcal{I}}$ for clarity. The s_i are called the* principal subterms *of s.*

Observe that a term s may have an infinite number of principal subterms.

Definition 10. *The set of* blocks *of a term s, denoted $Bl(s)$ over $\biguplus_{k \in \mathcal{K}} \Sigma_k$ is defined by the following coinduction:*

1. $\mathbf{cap}(s) \in Bl(s)$.
2. *If $s = C[\![s_i]\!]_{i \in \mathcal{I}}$, then, for all $i \in \mathcal{I}$, $Bl(s_i) \subseteq Bl(s)$.*

A block, b, is collapsing *if $b \twoheadrightarrow x$ for some $x \in \mathcal{X}$, and we say that b* collapses to *x. If the underlying iTRS is confluent, each collapsing block b can collapse to at most one x and we call that x the* collapsing variable *of that block.*

Definition 11. *A rewrite step $s \longrightarrow t$ is* outer *if the redex contracted is in the cap of s, otherwise it is* inner*. An outer step is indicated by $\stackrel{o}{\longrightarrow}$, an inner step by $\stackrel{i}{\longrightarrow}$.*

In the remainder of the paper, we make essential use of the *descendant relation* well-known from strongly convergent rewriting [5, Def. 12.5.1], that tracks positions across derivations. Observe that since we do not in general track *residuals* (i.e. what "happens" to redexes), we do not require the iTRS involved to be left-linear (certain residuals *will* be tracked in Section 4 where all systems are assumed to be left-linear).

Definition 12 (The Descendant Relation). *Let R be an iTRS, let s be a term of R, and let $s \twoheadrightarrow t$. The set of* descendants *of any position $u \in Pos(s)$ across $s \twoheadrightarrow t$, denoted $u/(s \twoheadrightarrow t)$, is defined by induction on the length α of $s \twoheadrightarrow t$:*

- *$\alpha = 0$. Then, $u/(s \twoheadrightarrow t) = \{u\}$.*
- *$\alpha = \beta+1$. Let q be any position of s_β and assume that the redex r contracted in $s_\beta \longrightarrow s_{\beta+1}$ is of the rule $\mathbf{l} \longrightarrow \mathbf{r}$ and situated at position u. If $q \preceq u$, then $q/(s_\beta \longrightarrow s_{\beta+1}) = \{q\}$. If $u \prec q$, then there is exactly one variable occurrence x in $\mathbf{l}$ at position p_x such that $u \cdot p_x \cdot p' = q$ for some position p'. Let $\{p_x^k\}_{k \in \mathcal{K}}$ be the set of positions of occurrences of x in $\mathbf{r}$. Then, $q/(s \twoheadrightarrow s_{\beta+1}) = \{u \cdot p_x^k \cdot p'\}$ We then define $u/(s \longrightarrow s_{\beta+1})$ to be $\bigcup_{q \in u/(s \longrightarrow s_\beta)} (u/(s_\beta \longrightarrow s_{\beta+1}))$.*
- *$Lim(\alpha)$. Here, a position q of s_α is a descendant of a position u of s iff q is a descendant of u in s_β for all sufficiently large $\beta < \alpha$.*

We shall speak of descendants of variable occurrences and principal subterms, meaning "the position of a variable occurrence" and "position of the root symbol of a principal subterm". Note that the definition of descendant entails that if $C[\![s_i]\!]_{i \in \mathcal{I}} \twoheadrightarrow C'[\![t_j]\!]_{j \in \mathcal{J}}$ and some t_j is a descendant of s_i, then $s_i \twoheadrightarrow t_j$. Strong convergence is crucial in this respect (cf. Section 6).

3 A General Counterexample to Modularity of Confluence

We now turn to the modularity of confluence. As in the case of orthogonal systems, there is a trivial counterexample based on the presence of two collapsing rules: If $R_0 = \{\mathbf{f}(x) \longrightarrow x\}$ and $R_1 = \{\mathbf{g}(x) \longrightarrow x\}$, then both R_0 and R_1 are confluent, but in $R_0 \oplus R_1$ there is a peak $\mathbf{f}^\omega \twoheadleftarrow \mathbf{f}(\mathbf{g}(\mathbf{f}(\mathbf{g}(\cdots)))) \twoheadrightarrow \mathbf{g}^\omega$, and $\mathbf{f}^\omega$ and $\mathbf{g}^\omega$ are obviously not joinable. We will therefore restrict our attention to non-collapsing systems.

In infinitary rewriting, we may need "balancing" rules to make non-left-linear rules applicable if we desire confluence. To appreciate this, consider $S \triangleq \{\mathbf{f}(x,x) \longrightarrow \mathbf{a}\}$ which is (finitarily) confluent by Newman's Lemma; but when considering S as an iTRS, we lose (infinitary) confluence:

Example 1. Consider S. From the term $h \triangleq \mathbf{f}(h,h)$ we get the following two derivatives $k \triangleq \mathbf{f}(\mathbf{a},k)$ and $p \triangleq \mathbf{f}(p,\mathbf{a})$, both of which are normal forms of S, i.e. S cannot be confluent. □

Suitably extending S yields a confluent iTRS; consider the following right-ground system:

$$R \triangleq \begin{cases} \mathbf{f}(x,x) \longrightarrow \mathbf{a} \\ \mathbf{f}(\mathbf{a},x) \longrightarrow \mathbf{a} \\ \mathbf{f}(x,\mathbf{a}) \longrightarrow \mathbf{a} \\ \mathbf{f}(\mathbf{f}(x,y),z) \longrightarrow \mathbf{a} \\ \mathbf{f}(x,\mathbf{f}(y,z)) \longrightarrow \mathbf{a} \end{cases}$$

We have the following:

Proposition 1. *R is confluent.*

Proof. We claim that if $\mathbf{f}(s, s')$ is a term and if $\mathbf{f}(s, s') \twoheadrightarrow t$ is a strongly convergent derivation of length at least 1, then $t \longrightarrow \mathbf{a}$ (observe that $t \notin \mathcal{X}$). We reason as follows: If $t = \mathbf{a}$, we are done. Otherwise, write $t = \mathbf{f}(w, w')$ and split on cases according to w and w':

1. $w = \mathbf{a}$ or $w' = \mathbf{a}$. Here, $t \longrightarrow \mathbf{a}$ by an application of either the rule $\mathbf{f}(\mathbf{a}, x) \longrightarrow \mathbf{a}$ or $\mathbf{f}(x, \mathbf{a}) \longrightarrow \mathbf{a}$.
2. $w = \mathbf{f}(r, r')$ or $w' = \mathbf{f}(r, r')$. In this case, $t \longrightarrow \mathbf{a}$ by an application of either the rule $\mathbf{f}(\mathbf{f}(x, y), z) \longrightarrow \mathbf{a}$ or the rule $\mathbf{f}(x, \mathbf{f}(y, z)) \longrightarrow \mathbf{a}$.
3. $w = x$ and $w' = y$ for $x, y \in \mathcal{X}$. Since there are no collapsing rules, this is only possible if $s = x$ and $s' = y$. If $x \neq y$, $\mathbf{f}(x, y)$ is a normal form, which contradicts the assumption that $\mathbf{f}(s, s') \twoheadrightarrow t$ has length at least 1. Thus, we must have $x = y$, i.e. $w = w'$ and the rule $\mathbf{f}(x, x) \longrightarrow \mathbf{a}$ yields $\mathbf{f}(w, w') \longrightarrow \mathbf{a}$. □

Make a "copy", R', of R, renaming $\mathbf{f}$ to $\mathbf{g}$ and $\mathbf{a}$ to $\mathbf{b}$, and copying the rules *mutatis mutandis*. The resulting system is clearly confluent, but $R \oplus R'$ is not confluent:

Proposition 2. *The term $s \triangleq \mathbf{f}(\mathbf{g}(s, s), \mathbf{g}(s, s))$ is not confluent (in $R \oplus R'$).*

Proof. It is clear that $s \longrightarrow \mathbf{a}$ and that $\mathbf{g}(s, s) \longrightarrow \mathbf{b}$. There is a strongly convergent derivation of the "right" subterm $\mathbf{g}(s, s)$ with limit $s'' \triangleq \mathbf{g}(\mathbf{a}, \mathbf{f}(\mathbf{b}, s''))$. Since the "left" subterm $\mathbf{g}(s, s)$ rewrites in one step to $\mathbf{b}$, s can in ω steps be rewritten to $s' \triangleq \mathbf{f}(\mathbf{b}, \mathbf{g}(\mathbf{a}, s'))$, which is a normal form. Thus, there is a peak $\mathbf{a} \twoheadleftarrow s \twoheadrightarrow s'$ for which no corresponding valley exists. □

Corollary 1. *$R \oplus R'$ is not confluent.*

Corollary 2. *Confluence is not a modular property of iTRSs.*

The counterexample to confluence crucially employs two facts:

1. One of the considered systems has a rule that is not left-linear.
2. The specific term considered does not have finite rank.

The further main results of this paper are that if restrictions are imposed on one of the two facts above, modularity of confluence may be recovered.

4 Modularity of Confluence for Left-Linear Systems

In this section we consider combinations of confluent, left-linear, pairwise disjoint systems, and subsequently derive necessary and sufficient conditions for modularity of confluence. We begin by proving our results for non-collapsing iTRSs and later extend them to sets of iTRSs $\mathcal{R}$ such that $\oplus\mathcal{R}$ is essentially non-collapsing.

Definition 13. *The term s is said to be* insulated *if it contains no collapsing blocks.*

Proposition 3. *If R is a left-linear iTRS, s is insulated and $s \twoheadrightarrow t$, then t is insulated.*

Proof. By left-linearity and insulation of s. □

Thus, for *left-linear* terms, insulation of s corresponds to the notion of *preservation* well known from the study of modularity in finitary rewriting [9].

Proposition 4 (Outer and Inner Derivations Commute). *Let $\mathcal{R}$ be a set of left-linear, pairwise disjoint iTRSs and let s be an insulated term with a peak $t \overset{i}{\twoheadleftarrow} s \overset{o}{\twoheadrightarrow} t'$. Then, there exists a term s' and a valley $t' \overset{i}{\twoheadrightarrow} s' \overset{o}{\twoheadleftarrow} t$.*

Proof. Straightforward induction on the length of the longest of the two derivations in the peak (in case of equal length, pick any of them). □

Proposition 5 (Postponement of Inner Derivation). *Let $\mathcal{R}$ be a set of left-linear, pairwise disjoint iTRSs and let s be an insulated term with $s \twoheadrightarrow t$ (in $\oplus\mathcal{R}$). Then, there is a term t' such that $s \overset{o}{\twoheadrightarrow} t' \overset{i}{\twoheadrightarrow} t$.*

Proof. By left-linearity and insulation, inner rewrite steps can neither destroy nor create outer redexes, and there is thus a term t' and a strongly convergent *outer* derivation $s \twoheadrightarrow t'$ such that $\mathbf{cap}(t') = \mathbf{cap}(t)$, and such that the set of descendants of any position of a variable occurrence in $\mathbf{cap}(s)$ is identical under $s \twoheadrightarrow t$ and $s \twoheadrightarrow t'$. Every principal subterm t_j of t is a descendant of some subterm s_i of s, and by disjointness of the iTRSs and strong convergence, we have $s_i \twoheadrightarrow t_j$. By strong convergence of $s \twoheadrightarrow t$ and the definition of descendant, t_j is eventually "fixed" at a single position p_j. The principal subterm of t' at p_j is a descendant of s_i, and since there were no inner steps in $s \twoheadrightarrow t'$ must be identical to s_i. Hence, $s_i \twoheadrightarrow t_j$. Since t_j was arbitrary, the same argument holds for all principal subterms of t, and an application of the Dovetailing Lemma concludes the proof. □

Proposition 6. *Let $\mathcal{R}$ be a set of left-linear, pairwise disjoint, confluent iTRSs, and let s be an insulated term. Then the following diagram commutes for any peak $t \twoheadleftarrow s \twoheadrightarrow t'$:*

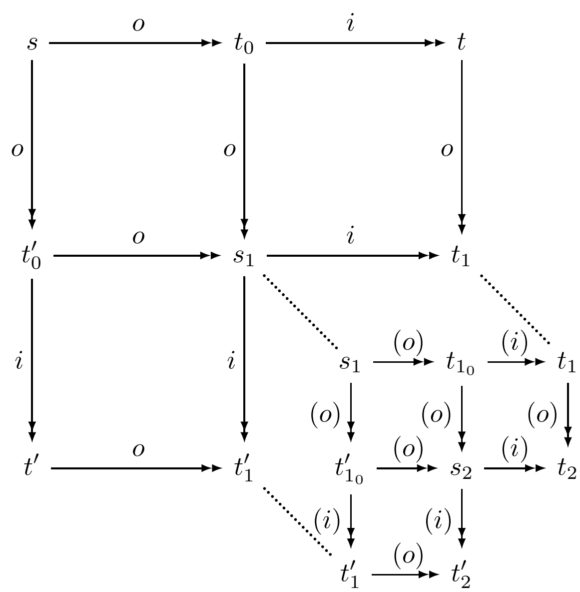
$$\begin{array}{ccccc}
s & \overset{o}{\twoheadrightarrow} & t_0 & \overset{i}{\twoheadrightarrow} & t \\
{\scriptstyle o}\big\downarrow & & {\scriptstyle o}\big\downarrow & & {\scriptstyle o}\big\downarrow \\
t'_0 & \overset{o}{\twoheadrightarrow} & s_1 & \overset{i}{\twoheadrightarrow} & t_1 \\
{\scriptstyle i}\big\downarrow & & {\scriptstyle i}\big\downarrow & & \\
t' & \overset{o}{\twoheadrightarrow} & t'_1 & &
\end{array}$$

where all rewrite steps in the peak $t_1 \overset{i}{\twoheadleftarrow} s_1 \overset{i}{\twoheadrightarrow} t'_1$ take place at depth ≥ 1.

Proof. Use Proposition 5 twice to erect the leftmost and uppermost sides of the diagram. Since the systems were assumed to be confluent and left-linear, outer derivation is confluent, whence we get commutativity of the upper-left square. Two applications of Proposition 4 furnish commutativity of the two remaining squares. All rewrite steps in the peak $t'_1 \twoheadleftarrow s_1 \twoheadrightarrow t_1$ are inner, and so by insulation take place at depth ≥ 1. □

Lemma 3. *Let $\mathcal{R}$ be a set of left-linear, confluent, pairwise disjoint iTRSs. Then every insulated term is confluent (in $\oplus\mathcal{R}$).*

Proof. Let $t \twoheadleftarrow s \twoheadrightarrow t'$ be a peak of $\oplus\mathcal{R}$ with s insulated. By Proposition 6, we can erect a diagram as in that proposition. Consider the peak $t_1 \twoheadleftarrow s_1 \twoheadrightarrow t'_1$ and observe that $\mathbf{cap}(s_1) = \mathbf{cap}(t_1) = \mathbf{cap}(t'_1)$, since s was insulated. Write $s_1 = C[\![s'_i]\!]_{i \in \mathcal{I}}$. Then the inner derivations in $t_1 \twoheadleftarrow s_1 \twoheadrightarrow t'_1$ occur in the s_i. Applying the proposition coinductively (*viz.* the below diagram) to the inner derivations in the s_i – using the Dovetailing Lemma to order arrange derivations in parallel subterms – yields strongly convergent derivations $t \twoheadrightarrow s' \twoheadleftarrow t'$ for some term s', as all redex contractions at the "kth application" of Proposition 6 take place at depth $\geq k$.



□

Corollary 3. *Modularity is confluent for left-linear, non-collapsing, iTRSs.*

4.1 Essentially Non-Collapsing Sets of iTRSs

We now give a simple condition on sets of iTRSs that will turn out to be a necessary and sufficient condition for the modularity of confluence.

Definition 14. *A set, $\mathcal{R}$, of pairwise disjoint iTRSs is said to be* essentially non-collapsing *if at most one iTRS of $\mathcal{R}$ contains a collapsing rule. If there exists an R such that R is the unique iTRS among $\mathcal{R}$ that contains a collapsing rule, we call R the* collapsing colour *of $\mathcal{R}$.*

The definition is similar to the notion of *almost-non-collapsing* iTRS well-known from the study of *orthogonal* iTRSs [5]; observe however, that the term is used here as a property of a *set* of iTRSs, not the individual iTRSs.

Proposition 7. *Let R be a left-linear iTRS. If there is a strongly convergent derivation $s \longrightarrow^{\alpha} x$ for some $x \in \mathcal{X}$, then $s \longrightarrow^{k} x$ for some $k \in \omega$.*

Proof. By the Compression Lemma, we may assume that $\alpha \leq \omega$. The fact that $s \longrightarrow^{\leq \omega} x$ is *convergent* now furnishes the desideratum. ◻

Lemma 4. *If $\mathcal{R}$ is a set of left-linear, pairwise disjoint, confluent iTRSs such that $\oplus\mathcal{R}$ is confluent, then $\mathcal{R}$ is essentially non-collapsing.*

Proof. By contraposition. If $\mathcal{R}$ were *not* essentially non-collapsing, there would be at least two iTRSs, (Σ_1, R_1) and (Σ_2, R_2), each containing a collapsing rule. We may write as $C_1[x] \longrightarrow x$, resp. $C_2[x] \longrightarrow x$ where the first rule is from R_1, and the second from R_2. The term $C_1[C_2[C_1[C_2[\cdots]]]]$ has the two derivatives $C_1[C_1[\cdots]]$ and $C_2[C_2[\cdots]]$ which are terms over disjoint alphabets and are hence joinable only if both terms can be rewritten to a variable, y. By Proposition 7, this can only happen if $C_i[C_i[\cdots]] \longrightarrow^{*} y$. Since this derivation is finite, there exists an n such that a stack, $C_i[\cdots[C_i[x]]]$ of n copies of $C_i[x]$ rewrites to y. But, clearly, $C_i[\cdots[C_i[x]]] \longrightarrow^{*} x$, whence confluence of the underlying iTRS yields $x = y$. By left-linearity, we may assume that there are no copies of x in $C_i[C_i[\cdots]]$. Thus, $C_1[C_1[\cdots]]$ and $C_2[C_2[\cdots]]$ have no common join, contradicting confluence of $\oplus\mathcal{R}$. ◻

Thus, essential non-collapsingness is a necessary condition for modularity of confluence. To see that it is also sufficient, we proceed as follows:

Definition 15. *Let $\mathcal{R}$ be a set of left-linear, confluent, pairwise disjoint iTRSs. Let s be a term and write $s = C[\![s_i]\!]_{i \in \mathcal{I}}$ (observe that if s is monochrome, we have $\mathcal{I} = \emptyset$). We define the term $\tilde{s}$ as follows:*

$$\begin{array}{ll} C[\tilde{s}_i]_{i \in \mathcal{I}} & \text{if } C[x_i]_{i \in \mathcal{I}} \text{ is not collapsing} \\ \tilde{s}_m & \text{if } C[x_i]_{i \in \mathcal{I}} \twoheadrightarrow x_m \end{array}$$

That is, $\tilde{s}$ is the term obtained from s by collapsing all collapsing blocks in a top-down fashion; by essential non-collapsingness, only blocks of a single colour will be collapsed. Observe that by confluence of the elements of $\mathcal{R}$, each block can collapse in at most one way, whence $\tilde{s}$ is well-defined. Note also that $s \twoheadrightarrow \tilde{s}$.

Proposition 8. *Let $\mathcal{R}$ be an essentially non-collapsing set of left-linear, pairwise disjoint, confluent iTRSs. Then $\tilde{s}$ is insulated, and the set of descendants of any $u \in Pos(s)$ is the same for any strongly convergent derivation $s \twoheadrightarrow \tilde{s}$ and satisfies $|u/(s \twoheadrightarrow \tilde{s})| \leq 1$.*

Proof. By left-linearity, contraction of redexes in one block can only create redexes in another if the block collapses. Since the considered systems are left-linear and there is at most one collapsing colour, $\tilde{s}$ must be insulated. By confluence of the systems, each collapsing block $C[x_1, \ldots, x_m]$ has a unique collapsing variable x_i, and we hence have $|u/(s \twoheadrightarrow \tilde{s})| \leq 1$ for any such derivation. $u/(s \twoheadrightarrow \tilde{s})$ is clearly independent of the choice of derivation. □

Definition 16. *Let $\mathcal{R}$ be an essentially non-collapsing set of left-linear, confluent, pairwise disjoint iTRSs, For any term s, we define $P_s(u)$ as the predicate on $Pos(s)$ that is true iff u has a descendant across $s \twoheadrightarrow \tilde{s}$. Furthermore, we set $U_s \triangleq \{u \in Pos(s) : P_s(u)\}$.*

By the previous proposition, we see that $P_s(u)$, and hence also U_s, is well-defined.

Proposition 9. *Let $\mathcal{R}$ be an essentially non-collapsing set of left-linear, confluent, pairwise disjoint iTRSs, and s be a term. Then U_s is partially ordered by $\prec$ and the graph of $(U_s, \prec)$ is a (possibly infinite) directed tree $\mathcal{T}_s$. The number of children of $\mathcal{T}_s$ at any vertex u is the arity of the function symbol at position u in s.*

Proof. U_s is partially ordered by $\prec$ since $Pos(s)$ is. The graph of $(Pos(s), \prec)$ is a directed tree, and clearly U_s is connected, hence also a directed tree. If a block collapses, at least one position below it has a descendant across $s \twoheadrightarrow \tilde{s}$, whence a position $u \in Pos(s)$ has exactly as many children in $(U_s, \prec)$ as it has in $(Pos(s), \prec)$. □

Proposition 10. *Let $\mathcal{R}$ be an essentially non-collapsing set of left-linear, confluent, pairwise disjoint iTRSs, and let $s \twoheadrightarrow s'$ have length at most ω (by the Compression Lemma, if need be). Let d be any non-negative integer. There is a non-negative integer d' such that, for all $u \in U_{s'}$ with $|u| \geq d'$, the depth, d'', of the single element in $u/(s' \twoheadrightarrow \tilde{s}')$ satisfies $d'' \geq d$. Furthermore, there is a $k \in \omega$ such that for $k' > k$, if the redex, r, contracted in $s_{k'} \xrightarrow{r} s_{k'+1}$ is at position u, then the unique residual of r by $s_{k'} \twoheadrightarrow \tilde{s}_{k'}$ is at depth $\geq d$.*

Proof. By Proposition 9 and the pigeon hole principle, the number of vertices at each depth in $\mathcal{T}_{s'}$ is finite. Write $u = p_1^{nc} \cdot p_1^{c} \cdot p_2^{nc} \cdot p_2^{c} \cdots p_m^{nc}$ where the p_i^{nc} and p_i^{c} are the positions of variables in non-collapsing and collapsing blocks of s', respectively, and where p_1^{nc} is possibly the empty position. Clearly, the depth of the single element in $u/(s' \twoheadrightarrow \tilde{s}')$ is $cl(u) \triangleq |p_1^{nc} \cdot p_2^{nc} \cdots p_m^{nc}|$, and $cl(u)$ is thus the depth of u in $\mathcal{T}_{s'}$. Let $\{u_1, \ldots, u_m\}$ be the set of vertices in $\mathcal{T}_{s'}$ at depth d, and set $d' \triangleq \max\{|u_1|, \ldots, |u_m|\}$; then $P_{s'}(u)$ and $|u| \geq d'$ implies $cl(u) \geq d$. Strong convergence of $s \twoheadrightarrow s'$ and the fact that the length is at most ω yield existence of a $k \in \omega$ such that all redexes contracted in $s_k \twoheadrightarrow s'$ are at depths $\geq d'$. Hence, for any $k' > k$, if $s_{k'} \xrightarrow{r} s_{k'+1}$, the unique residual of r by $s_{k'} \twoheadrightarrow \tilde{s}_{k'}$ will be at depth $\geq d$ in $\tilde{s}_{k'}$. □

Proposition 11. *Let $\mathcal{R}$ be an essentially non-collapsing set of left-linear, confluent, pairwise disjoint iTRSs, and let $s \twoheadrightarrow t$. Then $\tilde{s} \twoheadrightarrow \tilde{t}$.*

Proof. By the Compression Lemma, we may assume that $s \longrightarrow^{\leq\omega} t$, and proceed by induction on the length, α, of the derivation:

- $\alpha = 0$. Trivial.
- $\alpha = j + 1$.
  Consider $s_j \xrightarrow{r} s_{j+1}$; if the redex r is at position u and $u \notin U_{s_j}$, we have $\tilde{s_j} = \tilde{s}_{j+1}$, and we are done. If $u \in U_{s_j}$, contracting $r/(s_j \twoheadrightarrow \tilde{s}_j)$ clearly yields $\tilde{s}_{j+1}$ in one step.
- $\alpha = \omega$. By Proposition 10, for each depth $d \in \omega$, there is a $k \in \omega$ such that all steps in $\tilde{s}_{k'} \longrightarrow \tilde{s}_{k'+1}$ are below depth d for $k' > k$, showing that the resulting derivation is strongly convergent with limit $\tilde{t}$. □

Proposition 12. *Let R be the collapsing colour of an essentially non-collapsing set $\mathcal{R}$ of left-linear, confluent iTRSs. If $\tilde{s} \twoheadrightarrow t$, then $s \twoheadrightarrow t$.*

Proof. $s \twoheadrightarrow \tilde{s} \twoheadrightarrow t$. □

We can now prove the first positive result of the paper:

Theorem 1. *Let $\mathcal{R}$ be a set of confluent, left-linear, pairwise disjoint iTRS. Then, $\oplus\mathcal{R}$ is confluent iff $\mathcal{R}$ is essentially non-collapsing.*

Proof. If $\oplus\mathcal{R}$ is confluent, it follows from Lemma 4 that $\mathcal{R}$ must be essentially non-collapsing. Conversely, if $\mathcal{R}$ is essentially non-collapsing, let $t \twoheadleftarrow s \twoheadrightarrow t'$ be a peak of $\oplus\mathcal{R}$. By Proposition 11, there exists a peak $\tilde{t} \twoheadleftarrow \tilde{s} \twoheadrightarrow \tilde{t}'$. Lemma 3 now yields existence of a term s' and strongly convergent sequences $\tilde{t} \longrightarrow s'$ and $\tilde{t}' \longrightarrow s'$. An application of Proposition 12 concludes the proof. □

4.2 Mutually Orthogonal Systems

Confluence of left-linear systems in finitary rewriting can be ensured by less strict demands than that of disjointness. In both first- and higher-order finitary rewriting, *mutual orthogonality* (and the more lax *mutual weak orthogonality*) is sufficient for confluent systems to be confluent under direct sum [15]. The techniques of [6, 5] for proving confluence results in orthogonal (strongly convergent) transfinite rewriting use reasoning about residuals and the depths of redexes contracted in valleys as their linchpin; this does not generalize to arbitrary confluent iTRSs, hence not to the setting of modularity, since we cannot necessarily track residuals in non-orthogonal systems. Unlike the case with disjoint systems, contraction of a redex in one system can create redexes in others without being the application of a collapsing rule; as we cannot properly gauge the effect of such creations without tracking residuals, there appears to be no easy way of extending our results for left-linear systems to the setting of mutual orthogonality.

5 Confluence of Terms of Finite Rank

In this section, we show that when only terms of *finite rank* are considered, confluence is modular for non-collapsing, not necessarily left-linear, systems. The methods employed are akin to Toyama's original proof of (finitary) confluence of TRSs [14] and the initial part of the later, more elegant proof [9]. The parts of these papers dealing with collapsing rules do not appear to be applicable when working with *strongly* convergent derivations.

Proposition 13. *If the confluent terms s and s' satisfy $s \sim s'$ (i.e. s and t are joinable), then any derivative, t, of s is joinable with any derivative, t', of s'.*

Proof. Straightforward. □

Definition 17. *For sequences of terms $(s_k)_{k\in\mathcal{K}}$ and $(t_k)_{k\in\mathcal{K}}$, we write $(s_k)_{k\in\mathcal{K}} \propto (t_k)_{k\in\mathcal{K}}$ when it is the case that $t_{k'} = t_{k''}$ if $s_{k'} \sim s_{k''}$ for all $k', k'' \in \mathcal{K}$.*

Proposition 14. *Let $\mathcal{R}$ be a set of non-collapsing, pairwise disjoint iTRSs, let $s = C[\![s_i]\!]_{i\in\mathcal{I}}$, and assume that $s \twoheadrightarrow t$ with $t = C'[\![t_j]\!]_{j\in\mathcal{J}}$. Choose variables $(x_i)_{i\in\mathcal{I}}$ such that $(s_i)_{i\in\mathcal{I}} \propto (x_i)_{i\in\mathcal{I}}$. Then $C[x_i]_{i\in\mathcal{I}} \twoheadrightarrow C'[y_j]_{j\in\mathcal{J}}$ such that y_j is a descendant of x_i across $C[x_i]_{i\in\mathcal{I}} \twoheadrightarrow C'[y_j]_{j\in\mathcal{J}}$ iff t_j is a descendant of s_i across $s \twoheadrightarrow t$ for all $i \in \mathcal{I}$, $j \in \mathcal{J}$.*

Proof. By induction on the length, α, of $s \twoheadrightarrow t$.

- $\alpha = 0$. Straightforward.
- $\alpha = \beta+1$. Write $s_\beta = D[\![t'_k]\!]_{k\in\mathcal{K}}$; by the induction hypothesis we may assume that there exists a strongly convergent derivation $C[x_i]_{i\in\mathcal{I}} \twoheadrightarrow D[z_k]_{k\in\mathcal{K}}$ such that z_k is a descendant of x_i iff t'_k is a descendant of s_i, for all $k \in \mathcal{K}$, $i \in \mathcal{I}$ Consider the single rewrite step $s_\beta \longrightarrow s_{\beta+1}$. Assume that the redex contracted is of the rule $\mathbf{l} \longrightarrow \mathbf{r}$ in s_β and at position u. If the redex is not outer, or the rule is left-linear, the desideratum follows immediately. Assume, then that the redex is outer and that the rule is not left-linear. Since the induction hypothesis furnishes that z_k is a descendant of x_i iff t'_k is a descendant of s_i, for all $k \in \mathcal{K}$, $i \in \mathcal{I}$, whence $(t'_k)_{k\in\mathcal{K}} \propto (z_k)_{k\in\mathcal{K}}$, and the rule $\mathbf{l} \longrightarrow \mathbf{r}$ is applicable at position u in $D[z_k]_{k\in\mathcal{K}}$. The demand on the descendants is clearly fulfilled.
- $Lim(\alpha)$. Observe that the rewrite steps of $C[x_i]_{i\in\mathcal{I}} \twoheadrightarrow C'[y_j]_{j\in\mathcal{J}}$ correspond exactly to the outer steps of $s \twoheadrightarrow t$. If $C[x_i]_{i\in\mathcal{I}} \twoheadrightarrow C'[y_j]_{j\in\mathcal{J}}$ were not strongly convergent, neither would $s \twoheadrightarrow t$ be. It is clear by the definition of the descendant relation that the demand on the descendants is fulfilled. □

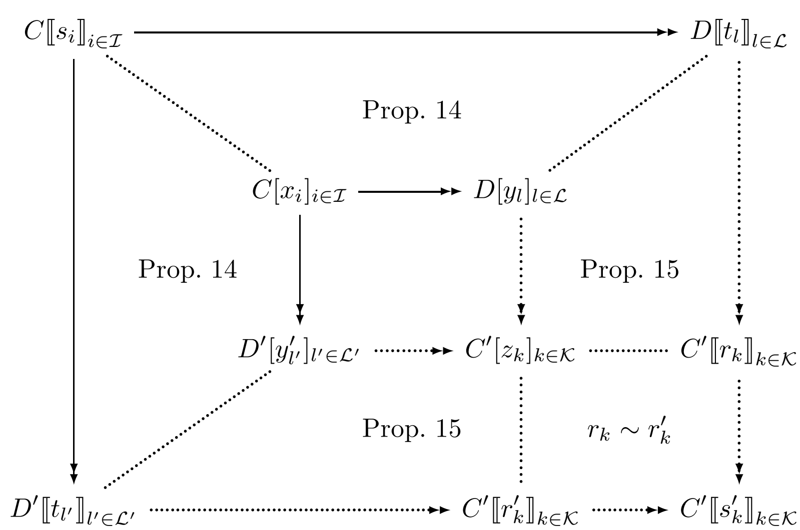
Proposition 15. *Let R be a non-collapsing iTRS, let $s = C[\![s_i]\!]_{i\in\mathcal{I}}$ such that the s_i are all confluent, and choose variables $(x_i)_{i\in\mathcal{I}}$ such that $(s_i)_{i\in\mathcal{I}} \propto (x_i)_{i\in\mathcal{I}}$. If $C[x_i]_{i\in\mathcal{I}} \twoheadrightarrow C'[z_k]_{k\in\mathcal{K}}$, then $C[\![s_i]\!]_{i\in\mathcal{I}} \twoheadrightarrow C'[\![t_k]\!]_{k\in\mathcal{K}}$ such that t_k is a descendant of s_i across $C[\![s_i]\!]_{i\in\mathcal{I}} \twoheadrightarrow C'[\![t_k]\!]_{k\in\mathcal{K}}$ iff z_k is a descendant of x_i across $C[x_i]_{i\in\mathcal{I}} \twoheadrightarrow C'[z_k]_{k\in\mathcal{K}}$.*

Proof. By induction on the length, α, of $C[x_i]_{i\in\mathcal{I}} \twoheadrightarrow C'[z_k]_{k\in\mathcal{K}}$.

- $\alpha = 0$. Straightforward
- $\alpha = \beta + 1$. Write $s_\beta = D[r_l]_{l\in\mathcal{L}}$. We have $C[x_i]_{i\in\mathcal{I}} \longrightarrow^\beta D[z_l]_{l\in\mathcal{L}}$. By the induction hypothesis, we have $C[s_i]_{i\in\mathcal{I}} \twoheadrightarrow D[r_l]_{l\in\mathcal{L}}$ for suitable $(r_l)_{l\in\mathcal{L}}$ such that the demand on the descendant relation is satisfied. Assume that the redex contracted is of the rule $\mathbf{l} \longrightarrow \mathbf{r}$ in s_β and at position u in $D[z_l]_{l\in\mathcal{L}}$. If the rule is left-linear, the desideratum follows immediately. If the rule is not left-linear, applicability of the rule in $D[z_l]_{l\in\mathcal{L}}$, $(s_i)_{i\in\mathcal{I}} \propto (x_i)_{i\in\mathcal{I}}$ and the descendants part of the induction hypothesis furnishes that if $z_j = z_{j'}$, r_j and $r_{j'}$ are joinable. Since $\mathbf{l}$ is a finite term, only a finite number of principal subterms need to be reduced to a common term in order for the rule to be applicable in $D[r_l]_{l\in\mathcal{L}}$. Thus, by Proposition 13, there exists a strongly convergent derivation $D[r_l]_{l\in\mathcal{L}} \twoheadrightarrow D[r'_l]_{l\in\mathcal{L}}$ (with all steps performed at depth $\geq |u|$) such that $\mathbf{l} \longrightarrow \mathbf{r}$ is applicable at position u in $D[r'_l]_{l\in\mathcal{L}}$ and the demand on the descendant relation is satisfied.
- $Lim(\alpha)$. There are two kinds of rewrite steps performed in $C[\![s_i]\!]_{i\in\mathcal{I}} \twoheadrightarrow C'[\![t_k]\!]_{k\in\mathcal{K}}$: "outer" steps corresponding to (and of the same depth as) the steps in $s \twoheadrightarrow C'[z_k]_{k\in\mathcal{K}}$, and "inner" steps performed to make non-left-linear rules applicable in the successor case above. The inner steps are all performed at a depth greater than that of the non-left-linear outer step that prompted them. Hence, the resulting derivation is strongly convergent; the demand on the descendant relation is clearly satisfied. □

Lemma 5. *Let $\mathcal{R}$ be a set of non-collapsing iTRSs and let $s = C[\![s_i]\!]_{i\in\mathcal{I}}$. Assume that outer derivation and the s_i are confluent for all $i \in \mathcal{I}$, and let $D[\![t_l]\!]_{l\in\mathcal{L}} \twoheadleftarrow s \twoheadrightarrow D'[\![t_{l'}]\!]_{l'\in\mathcal{L}'}$ be a peak. Then there exists a valley $t \twoheadrightarrow s' \twoheadleftarrow t'$.*

Proof. Apply Propositions 14 and 15 twice:



For the lower right rectangle, observe that the demand on the descendant relations in Propositions 14 and 15 ensure that r_k and r'_k are descendants of the same s_i for all $k \in \mathcal{K}$. Since the s_i were confluent, we get $r_k \sim r'_k$ for all $k \in \mathcal{K}$. An application of the Dovetailing Lemma yields that performing the $|\mathcal{K}|$ derivations needed to obtain $C'[\![s'_k]\!]$ from $C'[r_k]$ (resp. $C'[r'_k]$) can be done in a strongly convergent fashion. $\square$

We now have the second positive result of this paper:

Theorem 2. *Let $\mathcal{R}$ be a set of non-collapsing, confluent iTRSs. Then, every term s over $\biguplus_{i \in \mathcal{I}} \Sigma_i$ with finite rank is confluent.*

Proof. By induction on $\mathbf{rank}(s)$. If $\mathbf{rank}(s) = 0$, the result follows immediately, since monochrome terms were assumed to be confluent. If $\mathbf{rank}(s) > 0$, note that outer derivation is confluent, as are all principal subterms of s, since they have rank strictly less than $\mathbf{rank}(s)$. The result follows by an application of Lemma 5. $\square$

Systems containing collapsing rules exhibit severe technical complications that appear to be solvable neither with the techniques presented herein, nor with the standard techniques from finitary rewriting [9, 14].

6 Weakly Convergent Rewriting

In the previous sections, we have considered *strongly* convergent derivations. The more general setting of *weak* convergence is not very well understood, and sports far fewer auxiliary results. A major hurdle in this setting is that it is not clear how to define a suitable descendant relation. To appreciate the impact of this on the study of modularity, observe that the techniques of finitary rewriting, as well as those in this paper, depend crucially on the property of non-collapsing rewriting that, in a derivation $s \twoheadrightarrow t$, we can identify principal subterms of t with descendants of principal subterms of s. This is lost in weakly convergent rewriting, to wit the following example:

Example 2. We give an example of a weakly convergent derivation $s \longrightarrow^{\alpha} t$ such that, for a principal subterm t_j of t, there is *no* principal subterm s_i of s satisfying $s_i \twoheadrightarrow t_j$. Let $R_0 \triangleq \{\mathbf{a}(x) \longrightarrow \mathbf{b}(x)\}$ and let R_1 by the system consisting of the following infinite set of rules:

$$\mathbf{f}(x, \mathbf{g}^k(\mathbf{c}), \mathbf{d}(y, z)) \longrightarrow \mathbf{f}(y, \mathbf{g}^{k+1}(\mathbf{c}), z) \text{ for } k \in \omega$$

Clearly, the two systems are disjoint, and both are orthogonal. Let $s \triangleq \mathbf{d}(\mathbf{a}^{\omega}, s)$ and ponder the term:

$$\mathbf{f}(\mathbf{a}(\mathbf{c}), \mathbf{g}(\mathbf{c}), \mathbf{d}(\mathbf{a}(\mathbf{c}), \mathbf{d}(\mathbf{a}(\mathbf{a}(\mathbf{c})), \mathbf{d}(\mathbf{a}(\mathbf{a}(\mathbf{a}(\mathbf{c}))), \mathbf{d}(\cdots)))))$$

from which there is a weakly convergent derivation having limit $\mathbf{f}(\mathbf{a}^{\omega}, \mathbf{g}^{\omega}, s)$ (contract redexes at position ϵ repeatedly). But there is no principal subterm, s_i of the starting term such that a weakly convergent derivation $s_i \longrightarrow^{\beta} \mathbf{a}^{\omega}$ exists for any ordinal β. $\square$

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