A Step-by-step Description of How to Solve the Problem of Apollonius in 3D

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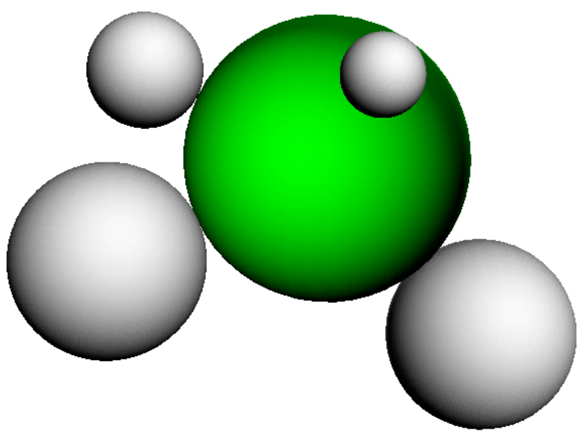
May 14. 2011

[May 27. 2012: There was a small error in the final calculation of r_s which has been fixed.]



The problem of Apollonius is: Given three circles, find a circle that is tangent to all three. A circle can be either internally tangent or externally tangent to one of the given circles (Figure 1). This problem can be generalized to the d-dimensional problem of finding the hypersphere tangent to $d+1$ given hyperspheres.

The wikipedia entry on the problem of Apollonius gives a good description of the 2D problem and some solution methods for finding the tangent circle. This document presents a step-by-step description of how the 'Algebraic solution' on the wikipedia page can be generalized to 3D.

If $(x_i, y_i, z_i, r_i), i = 1 \ldots 4$ are the given spheres, we wish to find (x_s, y_s, z_s, r_z) such that

$$(x_s - x_1)^2 + (y_s - y_1)^2 + (z_s - z_1)^2 = (r_s - s_1 r_1)^2 \quad (1)$$
$$(x_s - x_2)^2 + (y_s - y_2)^2 + (z_s - z_2)^2 = (r_s - s_2 r_2)^2 \quad (2)$$
$$(x_s - x_3)^2 + (y_s - y_3)^2 + (z_s - z_3)^2 = (r_s - s_3 r_3)^2 \quad (3)$$
$$(x_s - x_4)^2 + (y_s - y_4)^2 + (z_s - z_4)^2 = (r_s - s_4 r_4)^2 \quad (4)$$

Where s_i indicates if the solution should be internally ($s_i = 1$) or externally ($s_i = -1$) tangent to the given spheres. The solution to this system of quadratic equations is found in three

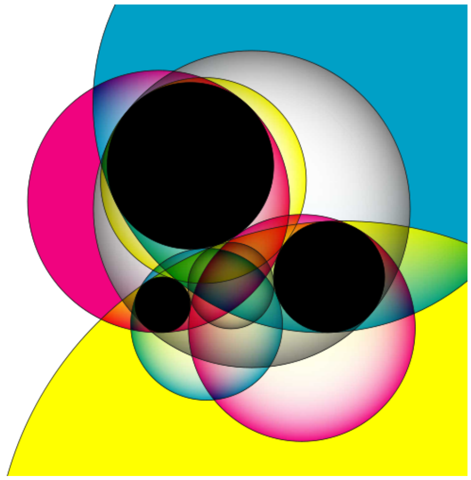


Figure 1: Given the black circles, there are eight spheres that are a combination of internally and externally tangent

steps. First, the four quadratic equations are rewritten to three linear equations each with four unknown variables. Second, the linear system is simplified. Third, by plugging the simplified linear system into one of the original equations r_s can be found. The center of the tangent sphere is easily found using r_s and the simplified linear system.

1 Step 1. Rewrite to linear equations

The four equations above are written out

$$(x_s^2+y_s^2+z_s^2)+(x_1^2+y_1^2+z_1^2)-(2x_1x_s+2y_1y_s+2z_1z_s) = r_s^2+r_1^2-2s_1r_1r_s \quad (5)$$
$$(x_s^2+y_s^2+z_s^2)+(x_2^2+y_2^2+z_2^2)-(2x_2x_s+2y_2y_s+2z_2z_s) = r_s^2+r_2^2-2s_2r_2r_s \quad (6)$$
$$(x_s^2+y_s^2+z_s^2)+(x_3^2+y_3^2+z_3^2)-(2x_3x_s+2y_3y_s+2z_3z_s) = r_s^2+r_3^2-2s_3r_3r_s \quad (7)$$
$$(x_s^2+y_s^2+z_s^2)+(x_4^2+y_4^2+z_4^2)-(2x_4x_s+2y_4y_s+2z_4z_s) = r_s^2+r_4^2-2s_4r_4r_s \quad (8)$$

To get rid of the quadratic terms of x_s, y_s, z_s and r_s, equations 6, 7 and 8 are subtracted from equation 5 giving the following three equations

$$(x_1^2+y_1^2+z_1^2)-(x_2^2+y_2^2+z_2^2)-(2x_1x_s+2y_1y_s+2z_1z_s)+(2x_2x_s+2y_2y_s+2z_2z_s) = r_1^2-r_2^2-2s_1r_1r_s+2s_2r_2r_s$$
$$(x_1^2+y_1^2+z_1^2)-(x_3^2+y_3^2+z_3^2)-(2x_1x_s+2y_1y_s+2z_1z_s)+(2x_3x_s+2y_3y_s+2z_3z_s) = r_1^2-r_3^2-2s_1r_1r_s+2s_3r_3r_s$$
$$(x_1^2+y_1^2+z_1^2)-(x_4^2+y_4^2+z_4^2)-(2x_1x_s+2y_1y_s+2z_1z_s)+(2x_4x_s+2y_4y_s+2z_4z_s) = r_1^2-r_4^2-2s_1r_1r_s+2s_4r_4r_s$$

This yields a linear system of equations

$$a_{11}x_s + a_{12}y_s + a_{13}z_s + a_{14}r_s = b_1 \tag{9}$$
$$a_{21}x_s + a_{22}y_s + a_{23}z_s + a_{24}r_s = b_2 \tag{10}$$
$$a_{31}x_s + a_{32}y_s + a_{33}z_s + a_{34}r_s = b_3 \tag{11}$$

where

$$a_{i1} = 2x_{i+1} - 2x_1 \tag{12}$$
$$a_{i2} = 2y_{i+1} - 2y_1 \tag{13}$$
$$a_{i3} = 2z_{i+1} - 2z_1 \tag{14}$$
$$a_{i4} = 2s_1r_1 - 2s_{i+1}r_{i+1} \tag{15}$$
$$b_i = (x_{i+1}^2 + y_{i+1}^2 + z_{i+1}^2) - (x_1^2 + y_1^2 + z_1^2) + r_1^2 - r_{i+1}^2 \tag{16}$$

for $i = 1, 2, 3$.

2 Step 2. Simplifying the linear system

Using gaussian elimination the system above can be rewritten to

$$\begin{array}{cccccc} x_s & & & +a'_{14}r_s & = & b'_1 \\ & y_s & & +a'_{24}r_s & = & b'_2 \\ & & z_s & +a'_{34}r_s & = & b'_3 \end{array}$$

And further to

$$x_s = M + Nr_s \tag{17}$$
$$y_s = P + Qr_s \tag{18}$$
$$z_s = R + Sr_s \tag{19}$$

where

$$M = b'_1 \tag{20}$$
$$N = -a'_{14} \tag{21}$$
$$P = b'_2 \tag{22}$$
$$Q = -a'_{24} \tag{23}$$
$$R = b'_3 \tag{24}$$
$$S = -a'_{34} \tag{25}$$

(The above equations should be rewritten to matrix-notation so they generalize better to other dimensions).

3 Step 3. Finding the tangent sphere

By plugging equations 17, 18 and 19 into equation 1 we get the following expression for r_s.

$$
\begin{aligned}
(x_s - x_1)^2 + (y_s - y_1)^2 + (z_s - z_1)^2 &= (r_s - s_1 r_1)^2 && (26)\\
(M + N r_s - x_1)^2 + (P + Q r_s - y_1)^2 + (R + S r_s - z_1)^2 &= (r_s - s_1 r_1)^2 && (27)\\
(N^2 + Q^2 + S^2 - 1) r_s^2 + & \\
(2(M - x_1)N + 2(P - y_1)Q + 2(R - z_1)S + 2 s_1 r_1)\, r_s + & \\
\left((M - x_1)^2 + (P - y_1)^2 + (R - z_1)^2 - r_1^2\right) &= 0 && (28)\\
a r_s^2 + b r_s + c &= 0 && (29)\\
\max\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}, \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right) &= r_s && (30)
\end{aligned}
$$

There will often exist a negative solution to the quadratic equation but it can be ignored. Instead of figuring out which solution is larger, we simply force the denominator to be positive and use addition in the denominator. By plugging r_s into the simplified linear system (equations 17 to 19) the center of the tangent sphere is found.