Firing Games*

Mikkel Thorup†
University of Copenhagen

September 11, 1996

Abstract

This is a general theorem, with a simple proof, on the uniqueness of termination of firing games. It implies theorems of Björner, Lovász, and Shor on chip firing games on graphs. Such games can be seen as models for perfectly asynchronous parallel computations, and then the uniqueness refers to the result of these computations.

We consider *firing games* which are single-player games played on finitely many distinct cannons. At any stage of a firing game, a cannon is either *ready* to fire, or not. A move consists of the player firing a ready cannon of his own choice. The game *terminates* at some stage if and only if no cannons are ready. We call a firing game *monotone* if it satisfies the following rule:

> Suppose X and Y are playing the game (on identical boards), and that they have fired a certain cannon c equally many times. Suppose, moreover, that player Y has fired all other cannons at least as many times as player X. Then, if cannon c is ready for player X, it is also ready for player Y.

Theorem 1 *If a monotone firing game terminates for a given player A, it will terminate for any player B. Moreover, A and B will end up firing each cannon exactly the same number of times.*

As an application of the theorem, consider the following game studied in [2]: each node of a graph contains a pile of chips, and a move consists of selecting a node with at least as many chips on it as its degree, and letting it send one chip to each of its neighbors. The game terminates if there is no such node. Clearly, the graph together with an initial distribution of chips constitutes a firing game with the nodes as the cannons. At some specific stage of the game, the number of chips on a node v is

$$c_v = s_v + (\sum_{w \in N(v)} f_w) - f_v |N(v)| \tag{1}$$

where for any node x, s_x is the initial number of chips on x, $N(x)$ is the set of neighbors of x, and f_x is the number of times that that x has fired. A node v is ready to fire if and only if $c_v \geq |N(v)|$, so the game is monotone. Thus, by Theorem 1, it is independent of the player whether the game terminates and, if it terminates, how many times each node fires in total. By (1) the latter determines uniquely the number of chips on each node in the terminal stage. This uniqueness

*Technical Report 94/15, Department of Computer Science, University of Copenhagen, 1994. This result was found while the author was at Oxford University supported by the Danish Technical Research Council.

†University of Copenhagen, Department of Computer Science, Universitetsparken 1, 2100 København Ø, Denmark; e-mail: `mthorup@diku.dk`

of the termination of the game is a main result from [2]. Similarly, Theorem 1 implies uniqueness of termination for a corresponding game on directed graphs studied in [1].

Proof of Theorem 1: By symmetry, we may assume that the total number of firings of player A is no bigger than that of player B. We want to show that for each cannon, they do exactly the same number of firings.

First, we ask player A to redo his game putting an A on every cannon he fires (some cannons could get several As). Similarly, we let player B redo his game putting a B on every cannon he fires. If no cannon receives more Bs than As, we conclude that the players fired each cannon exactly the same number of times, as desired. Thus assume, for a contradiction, that some cannon receives more Bs and As. Consider the very first time player B is about to put a B on a cannon c that already has as many Bs as As. Compare this stage of player B's game with the terminal stage of player A's game. Now, cannon c has been fired exactly the same number of times by the two players, whilst all other cannons have been fired at least as many times by player A as they have been fired by player B. Moreover, since player B is about to put a B on cannon c, cannon c is ready for player B. Thus, by monotonicity, cannon c is ready for player A. This contradicts that player A has terminated. ■

Note A proof similar to the one presented here has been found independently by Kimmo Eriksson.

References

[1] BJÖRNER, A., AND LOVÁSZ, L. Chip firing games on directed graphs. *Journal of Algebraic Combinatorics 1* (1992) 305–328.

[2] BJÖRNER, A., LOVÁSZ, L., AND SHOR, P.W. Chip-firing games on graphs. *European Journal of Combinatorics 12* (1991) 283–291.