The Minimum Spanning Tree Problem

The Road to Linear Complexity

Marius-Florin Cristian

Thesis supervisor: Jyrki Katajainen

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Content

Overview

- Problem definition
- Goals of the thesis
- General overview of the road
- Greedy overview of the road
- Non-greedy overview of the road

Problem Definition and Goals

- Problem Definition

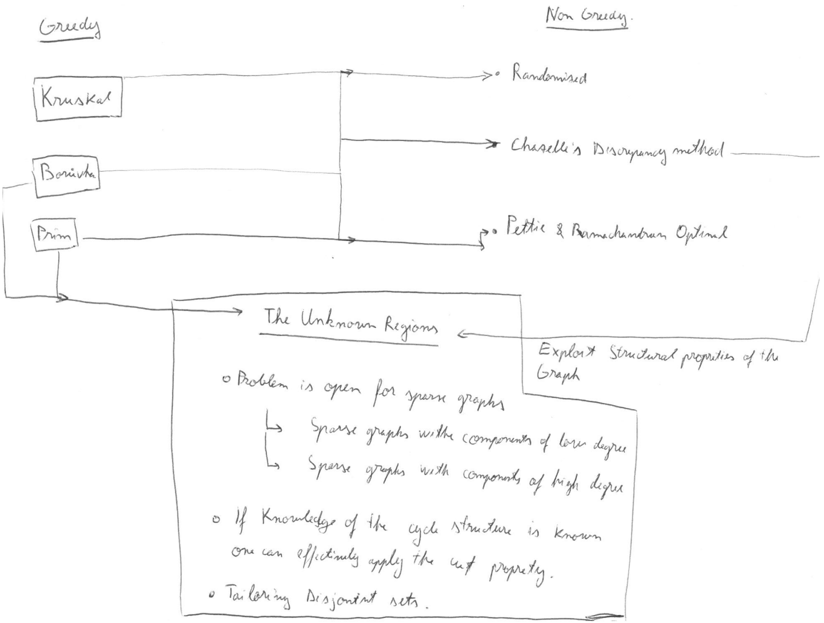
Given an undirected, connected graph $G = (V, E, C)$, where $V = \{v_1, v_2, ...v_n\}$ is the vertex set, $E = \{e_1, e_2, ..., e_m\}$ the edge set, and each edge e_i has a cost (or weight) associated $c_i \in C$, where $C = \{c_1, c_2, ..., c_m\}$. Let $|V| = n$ and $|E| = m$. A *spanning tree* $T = (V, E')$ of G connects its vertices and the edge set $E' \subseteq E$ has $n - 1$ edges. The *weight* of that spanning tree is the sum of the cost of its edges, i.e. $\sum_{e_{i,j} \in E'} c_{i,j}$. In the *minimum-spanning-tree problem* the task is to find a spanning tree of G for which the cost is minimum.
We do not allow duplicate edges.

Problem Definition and Goals

- Goals of the thesis.

In order to solve the open problem, one must compute the minimum spanning tree of a given graph in a deterministic fashion in linear time with respect to the number of edges.
This thesis explores the progress done until the current day and tries to solve the problem.

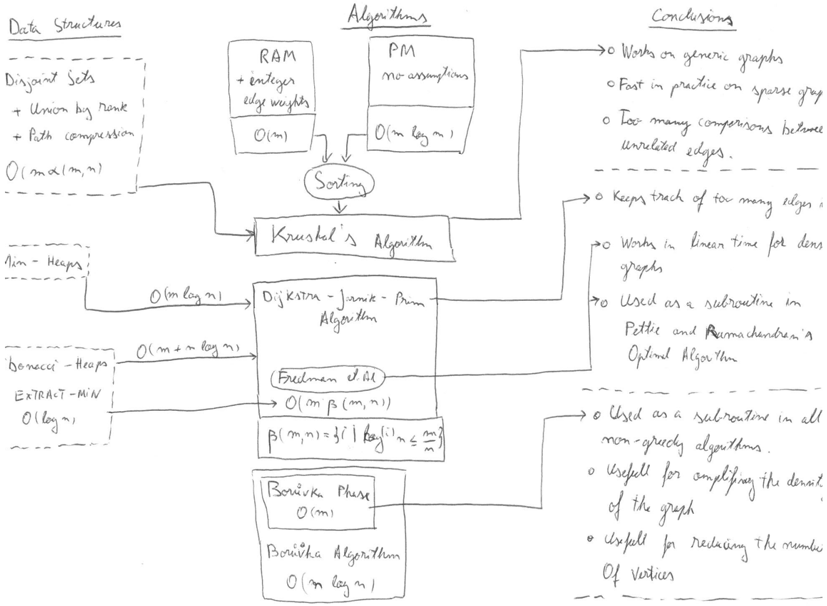
General Overview


Greedy
Non Greedy.
Kruskal
Borůvka
Prim
Randomised
Chazelle's Discrepancy method
Pettie & Ramachandran Optimal
Exploit Structural properties of the Graph

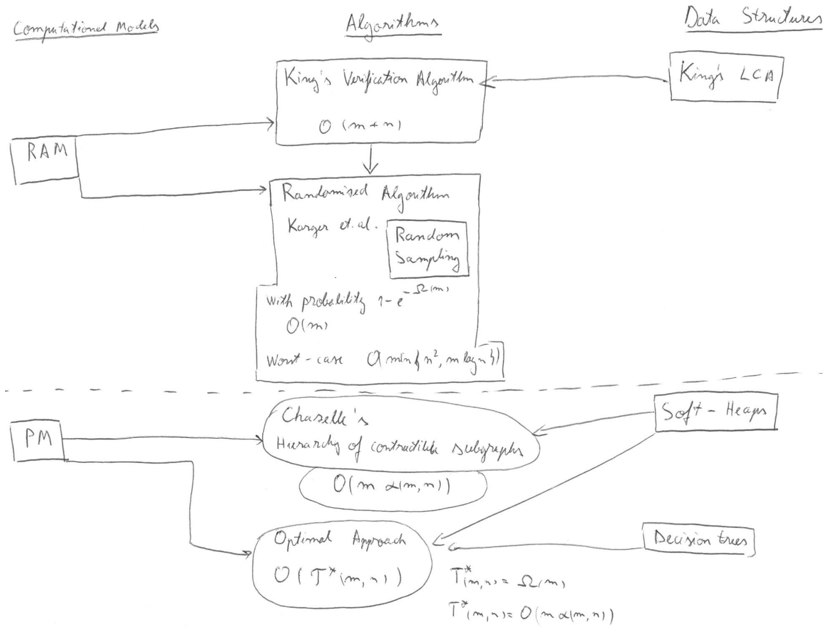
The Unknown Regions

- Problem is open for sparse graphs
  - Sparse graphs with components of low degree
  - Sparse graphs with components of high degree
- If Knowledge of the cycle structure is known one can effectively apply the cut property.
- Tailoring Disjoint sets.

Greedy Overview


Data Structures
Algorithms
Conclusions
Disjoint Sets
+ Union by rank
+ Path compression
$O(m\alpha(m,n))$
RAM
+ integer edge weights
$O(m)$
PM
no assumptions
$O(m \log m)$
Sorting
Kruskal's Algorithm
Works on generic graphs
Fast in practice on sparse grap
Too many comparisons betwee unrelated edges.
Min - Heaps
$O(m \log n)$
Dijkstra - Jarnik - Prim Algorithm
Keeps track of too many edges
Works in linear time for dens graphs
Used as a subroutine in Pettie and Ramachandran's Optimal Algorithm
bonacci - Heaps
EXTRACT-MIN
$O(\log n)$
$O(m + n \log n)$
Fredman et Al
$O(m\beta(m,n))$
$\beta(m,n) = \{i \mid \log^{(i)} n \le \frac{m}{n}\}$
Boruvka Phase
$O(m)$
Boruvka Algorithm
$O(m \log n)$
Used as a subroutine in all non-greedy algorithms.
Usefull for simplifying the densit of the graph
Usefull for reducing the numbe Of vertices

Non-greedy Overview



Developing ideas

- A look at special cases of graphs
- Similarities between planar graphs and sparse graphs
- A classification of sparse graphs
- Trimming the subtrees of a graph
- A hierarchy of cycles

Graph structural properties

- The problem is not solved for general case graphs.
- The problem is solved for dense enough graphs.
- The problem is solved for planar graphs.
- The problem is NOT solved for sparse graphs with no apparent structural properties.
  How can we force structure to our graph?

Planar Graph properties

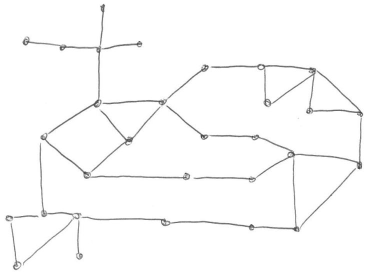
One quickly notices that when dealing with a planar graph, the triangle inequality can be exploited in order to bound the neighbourhood of each point.
A linear time algorithm follows just from the Delaunay triangulation of such a graph.
The main property that a planar graph has in common with a general sparse graph is that the degree of each vertex under inspection is low.
For a general sparse graph (number of edges $< 4n$) with vertices of low degree (< 4), the problem should be solvable in linear time.

Classification of sparse graphs

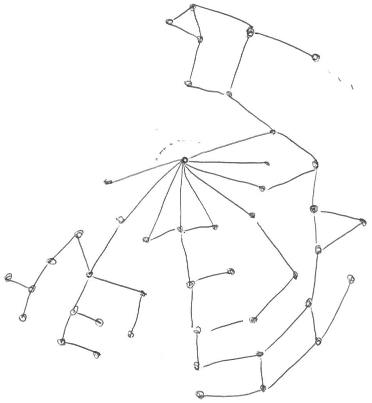
- Sparse graphs with only low degree components
  We superficially argue that it should be possible to solve this case in a greedy fashion, either by applying one of the classical algorithms, or by detecting the cycles and trimming the trees. (more on this technique next)
- Sparse graphs with some high degree components
  High degree components are part of more cycles.
  If we use a greedy approach to keep track of the edges of such components we will end up with n edges, and the running time will be in a factor of n for just one such component.

Two types of sparse graphs

Low degree sparse graphs



High degree sparse graphs



High degree sparse graph

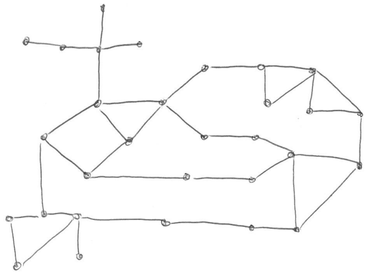
- How shall we trim the vertices of degree 1?
- It can cascade in a succession of trims that could cover the whole graph
- What happens if there are more components of high degree?

High degree sparse graph

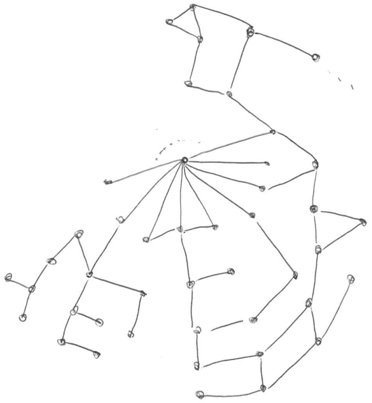
- How shall we trim the vertices of degree 1?
  An idea would be to keep track of the cycles from the shortest to the longest, and given a vertex of degree 1, to answer in constant time to which tree it belongs such that the root of the tree is a cycle.
- It can cascade in a succession of trims that could cover the whole graph
  We can avoid any such cascades by classifying such a hierarchy of cycles on levels, and then start the trimming process from the "longest" cycle.
  First we remove the heaviest edge from each disjoint cycle on each level.
- What happens if there are more components of high degree?

Detecting cycles

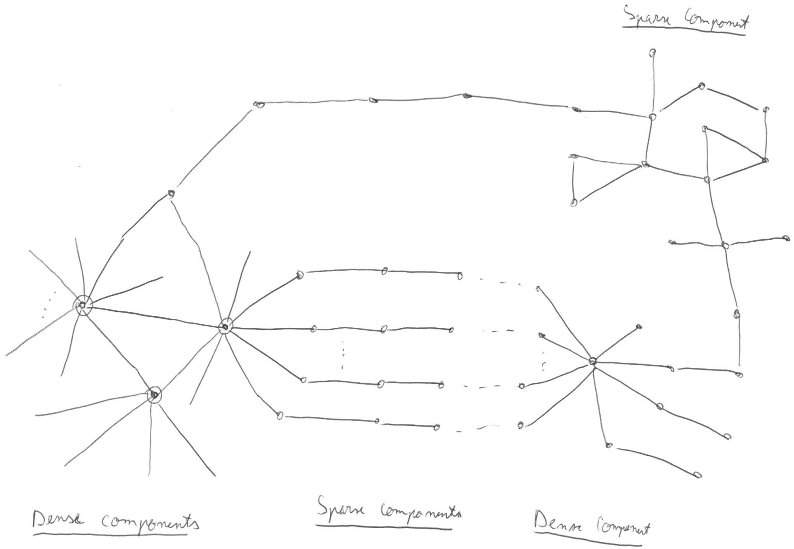
Low degree sparse graphs



High degree sparse graphs



Sparse graphs with mixed structure



A General Algorithm

- Reduce the density of the components such that cycle detection is easy
- The bottleneck will be the data structure used to detect cycles
- Disjoint Sets with table look-up
- Fighting the error produced by the disjoint sets

Density Reduction

- Reduce the density of the components such that cycle detection is easy
  Here we can artificially group components of high degree to cluster by applying Borůvka Phases, then apply the dense case algorithm to reduce the number of edges.
  The problem is more subtle, as we inspect only local components and we might discard good edges by doing such things. Thus we must consider the border edges as well.

Density Reduction

When dealing with density reduction we must consider the following components:

- Components of high degree and high degree neighbours
- Components of high degree and low degree neighbours
- Components of high degree and mixed degree neighbours
- Components of low degree and high degree neighbours
- Components of low degree and low degree neighbours
- Components of low degree and mixed degree neighbours

Disjoint sets for cycle detection

- The disjoint sets data structure is good at detecting cycles
- We pick a vertex of high degree and find all of it's closest cycles; Union-Find each of it's neighbours, continue until a cycle is found. (example on blackboard)
- We mark the visited vertices and edges
- We pick an unmarked vertex of high degree and repeat the procedure
  The vertex will either form cycles with already marked ones, or new cycles.
- Such a pass will take $O(m+n)$ time. Bounding the number of passes (blackboard).
- Each formed cycle is considered to be 1 vertex in future iterations
  We must also consider if during the density reduction a component had several expansions towards border edges.

Disjoint sets with table lookup

- The upper bound for disjoint set with union by rank and path compression is $O(m\alpha(m, n))$
- If we know the structure of unions in advance Gabow and Tarjan show that look-up tables can be computed for components of size $\Omega(\log \log n)$ such that the running time of the disjoint sets will be $O(m + n)$
- We can approximate such a union structure, by expanding the dense components and computing the look-up tables
- Dense components are more likely to be part of more cycles

Disjoint sets with table lookup

However two side effects might occur from using such a table look-up with the approximation of the union structure:

- The components are connected but the answer to a Find is false. A Union is usually performed, then the components will get connected again which will take O(1) time since there is nothing to do but try to connect. We can correct the value of the table in O(1) to properly reflect the component, and mark the Find as faulty; thus only after a Union we can check if the edge under inspection is part of a cycle or not. This should not worry us as there is no harm done.

Disjoint sets with table lookup

- The components are not connected and the answer to a Find is true, then the components will not get connected at this point in time. We might consider an edge as part of a cycle even if it isnt. This can create phantom cycles in the running of the algorithm. The phantom cycles can be addressed after the hierarchy of cycles has been established. The obvious side-effect is that some edges that are part of the MST might be omitted
  This should worry us as there is a lot of harm done.
  To counter this, we must either try to predict the structure of the MST pretty good, or aim in creating a pseudoforest, where we allow some cycles, and some disconnected components. (blackboard explanation)

Fighting the errors

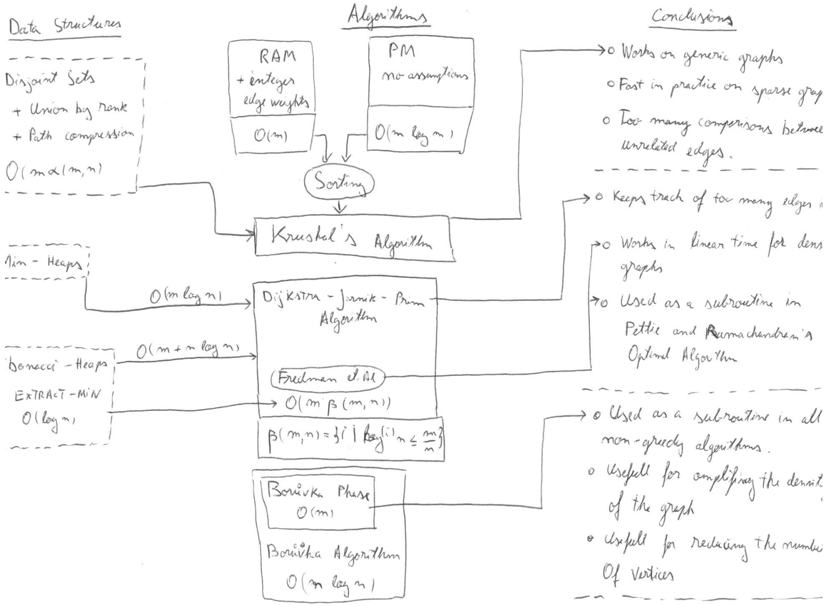
In order to fight the badness produced by the hardcoded tables we need to:

- Craft the tables carefully
- After the algorithm terminates expand any isolated component with it's lowest degree edge (make use of the cut property)
- Unfortunately we don't know how to properly bound the produced segmentation but we can make the following remarks:
  - The errors might occur only at components of low degree that connect components of high degree
  - Some of such errors get remediated in time as the components of high degree tend to become one
  - If the components of high degree don't colapse into eachother in the near future, then a gap is formed
  - Such a gap is bad only if it is part of a tree that gets trimmed!

Conclusions

- The Road
  - Greedy
    - Kruskal
    - Prim
    - Borůvka
  - Non-greedy
    - Randomized (Karger et. al.)
    - Chazelle
    - Pettie and Ramachandran's Optimal
- The Proposed solution

Greedy Overview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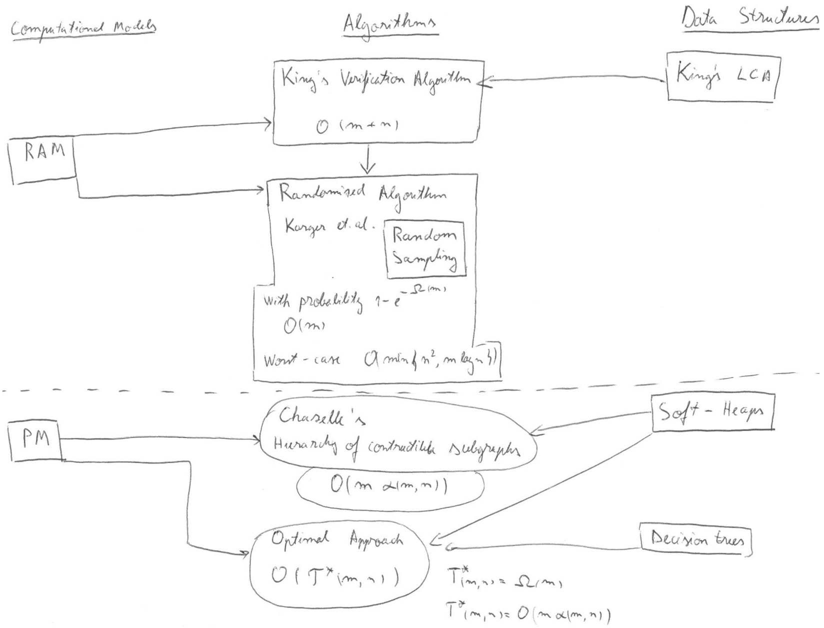
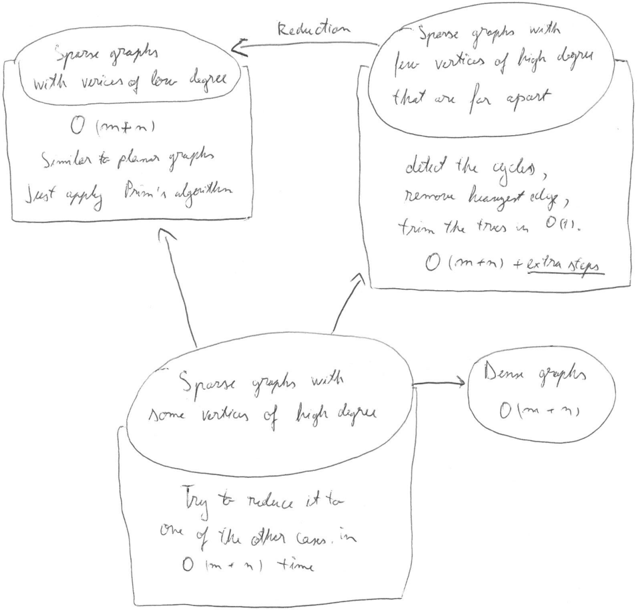
β(m, n) = {i | log^(i) n ≤ m/n}
Boruvka Phase
O(m)
Boruvka Algorithm
O(m log n)
o Used as a subroutine in all non-greedy algorithms.
o Usefull for amplifying the densit of the graph
o Usefull for reducing the numbe Of Vertices

Non-greedy Overview



The proposed solution



Reduction
Sparse graphs with vertices of low degree
O (m+n)
Similar to planar graphs
Just apply Prim's algorithm
Sparse graphs with few vertices of high degree that are far apart
detect the cycles, remove heaviest edge, trim the trees in O(1).
O (m+n) + extra steps
Sparse graphs with some vertices of high degree
Try to reduce it to one of the other cases. in O (m+n) time
Dense graphs
O (m+n)

Future work

- Solve the problem !!!