Mathematics and Computation
7'th edition
Volume 3

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Contents

Appendix A

Reference Manual

A.1 Error reporting

If (despite what I said in Section 1.1) you find an error in this book, please see if it is already reported at

`http://www.diku.dk/~grue/mob0102/index.html`

If it is not, please notify me at

`grue@diku.dk`

Suggestions for improvement are also welcome.

A.2 Priority

This page is repeated on the last page of this volume.

$$
\begin{array}{lll}
[& \mathsf{Lx} & \breve{>} \\
 & \mathsf{x\,'\,y} & \breve{>} \\
 & \mathsf{x@y} \mathrel{\breve{=}} \mathsf{x@_2y} \mathrel{\breve{=}} \mathsf{x@_+y} \mathrel{\breve{=}} \mathsf{x@_-y} \mathrel{\breve{=}} \mathsf{x@_0y} & \breve{>} \\
 & \#\mathsf{x} & \breve{>} \\
 & \mathsf{x!} \mathrel{\breve{=}} \mathsf{x^y} \mathrel{\breve{=}} \mathsf{x^+} \mathrel{\breve{=}} \mathsf{x^-} \mathrel{\breve{=}} \mathsf{x^*} \mathrel{\breve{=}} & \\
 & \mathsf{x}\text{ head} \mathrel{\breve{=}} \mathsf{x}\text{ tail} \mathrel{\breve{=}} \mathsf{x}\text{ pair} \mathrel{\breve{=}} \mathsf{x}\text{ atom} \mathrel{\breve{=}} \mathsf{x}\text{ Head} \mathrel{\breve{=}} \mathsf{x}\text{ Tail} & \breve{>} \\
 & \mathsf{x} \cdot \mathsf{y} \mathrel{\breve{=}} \mathsf{x} \cdot_+ \mathsf{y} \mathrel{\breve{=}} \mathsf{x} \cdot_- \mathsf{y} \mathrel{\breve{=}} \mathsf{x} \cdot_0 \mathsf{y} \mathrel{\breve{=}} \mathsf{x} \cdot_2 \mathsf{y} \mathrel{\breve{=}} & \\
 & \mathsf{x/y} \mathrel{\breve{=}} \mathsf{x/_+y} \mathrel{\breve{=}} \mathsf{x/_-y} \mathrel{\breve{=}} \mathsf{x/_0y} \mathrel{\breve{=}} \mathsf{x/_2y} \mathrel{\breve{=}} & \\
 & \mathsf{x} \mathbin{//} \mathsf{y} \mathrel{\breve{=}} \mathsf{x} \mathbin{//_+} \mathsf{y} \mathrel{\breve{=}} \mathsf{x} \mathbin{//_-} \mathsf{y} \mathrel{\breve{=}} \mathsf{x} \mathbin{//_0} \mathsf{y} \mathrel{\breve{=}} \mathsf{x} \mathbin{//_2} \mathsf{y} \mathrel{\breve{=}} & \\
 & \mathsf{x} \mathbin{\%} \mathsf{y} \mathrel{\breve{=}} \mathsf{x} \mathbin{\%_+} \mathsf{y} \mathrel{\breve{=}} \mathsf{x} \mathbin{\%_-} \mathsf{y} \mathrel{\breve{=}} \mathsf{x} \mathbin{\%_0} \mathsf{y} \mathrel{\breve{=}} \mathsf{x} \mathbin{\%_2} \mathsf{y} & \breve{>} \\
 & +\mathsf{x} \mathrel{\breve{=}} -\mathsf{x} & \breve{>} \\
 & \mathsf{x}+\mathsf{y} \mathrel{\breve{=}} \mathsf{x}+_+\mathsf{y} \mathrel{\breve{=}} \mathsf{x}+_-\mathsf{y} \mathrel{\breve{=}} \mathsf{x}+_0\mathsf{y} \mathrel{\breve{=}} \mathsf{x}+_2\mathsf{y} \mathrel{\breve{=}} & \\
 & \mathsf{x}-\mathsf{y} \mathrel{\breve{=}} \mathsf{x}-_+\mathsf{y} \mathrel{\breve{=}} \mathsf{x}-_-\mathsf{y} \mathrel{\breve{=}} \mathsf{x}-_0\mathsf{y} \mathrel{\breve{=}} \mathsf{x}-_2\mathsf{y} & \breve{>} \\
 & \mathsf{x} :: \mathsf{y} \mathrel{\breve{=}} \mathsf{x} \therefore \mathsf{y} \mathrel{\breve{=}} \mathsf{x} \times \mathsf{y} & \breve{>} \\
 & \mathsf{x}\text{ append }\mathsf{y} & \breve{>} \\
 & \mathsf{x,y} & \breve{>} \\
 & \mathsf{x}=\mathsf{y} \mathrel{\breve{=}} \mathsf{x}\neq\mathsf{y} \mathrel{\breve{=}} \mathsf{x}<\mathsf{y} \mathrel{\breve{=}} \mathsf{x}\leq\mathsf{y} \mathrel{\breve{=}} \mathsf{x}>\mathsf{y} \mathrel{\breve{=}} \mathsf{x}\geq\mathsf{y} \mathrel{\breve{=}} \mathsf{x}\in\mathsf{y} \mathrel{\breve{=}} \mathsf{x}? & \breve{>} \\
 & \neg\mathsf{x} & \breve{>} \\
 & \mathsf{x}\wedge\mathsf{y} & \breve{>} \\
 & \mathsf{x}\vee\mathsf{y} & \breve{>} \\
 & \forall\mathsf{x}{\in}\mathsf{y}{:}\,\mathsf{z} \mathrel{\breve{=}} \exists\mathsf{x}{\in}\mathsf{y}{:}\,\mathsf{z} \mathrel{\breve{=}} \varepsilon\mathsf{x}{\in}\mathsf{y}{:}\,\mathsf{z} & \breve{>} \\
 & \mathsf{x}\Rightarrow\mathsf{y} \mathrel{\breve{=}} \mathsf{x}\Leftarrow\mathsf{y} \mathrel{\breve{=}} \mathsf{x}\Leftrightarrow\mathsf{y} & \breve{>} \\
 & \mathsf{x}\left\{\begin{array}{l}\mathsf{y}\\ \mathsf{z}\end{array}\right. \mathrel{\breve{=}} \mathsf{x}\left\langle\begin{array}{l}\mathsf{y}\\ \mathsf{z}\end{array}\right. & \breve{>} \\
 & \lambda\mathsf{x.y} & \breve{>} \\
 & \mathsf{x}\equiv\mathsf{y} \mathrel{\breve{=}} \mathsf{x}\overset{+}{\rightarrow}\mathsf{y} \mathrel{\breve{=}} \mathsf{x}\preceq\mathsf{y} & \breve{>} \\
 & \mathsf{x}\rightarrow\mathsf{y} & \breve{>} \\
 & \mathsf{x}\vdash\mathsf{y} & \breve{>} \\
 & \mathsf{x}\rhd\mathsf{y} & \breve{>} \\
 & \mathsf{x}:\mathsf{y} & \breve{>} \\
 & \mathsf{x;y} & \breve{>} \\
 & \textbf{Mac lemma }\mathsf{x{:}y} \mathrel{\breve{=}} \textbf{Mac proof of }\mathsf{x{:}y} \mathrel{\breve{=}} \textbf{Mac rule }\mathsf{x{:}y} &]
\end{array}
$$

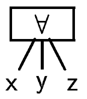
A.3 Associativity and term reduction

This page is repeated on the penultimate page of this volume.

[	$\mathsf{x}\,{}^{\prime}\,\mathsf{y}\,{}^{\prime}\,\mathsf{z}$	$\overset{\smile}{\rightarrow}$	$(\mathsf{x}\,{}^{\prime}\,\mathsf{y})\,{}^{\prime}\,\mathsf{z}$	]
[	$\mathsf{x@y@z}$	$\overset{\smile}{\rightarrow}$	$\mathsf{(x@y)@z}$	]
[	$\mathsf{x}^{\mathsf{y}^{\mathsf{z}}}$	$\overset{\smile}{\rightarrow}$	$\mathsf{x}^{(\mathsf{y}^{\mathsf{z}})}$	]
[	$\mathsf{x} \cdot \mathsf{y} \cdot \mathsf{z}$	$\overset{\smile}{\rightarrow}$	$(\mathsf{x} \cdot \mathsf{y}) \cdot \mathsf{z}$	]
[	$\mathsf{x} + \mathsf{y} + \mathsf{z}$	$\overset{\smile}{\rightarrow}$	$(\mathsf{x} + \mathsf{y}) + \mathsf{z}$	]
[	$\mathsf{x} :: \mathsf{y} :: \mathsf{z}$	$\overset{\smile}{\rightarrow}$	$\mathsf{x} :: (\mathsf{y} :: \mathsf{z})$	]
[	x append y append z	$\overset{\smile}{\rightarrow}$	x append (y append z)	]
[	$\mathsf{x}, \mathsf{y}, \mathsf{z}$	$\overset{\smile}{\rightarrow}$	$\mathsf{x}, (\mathsf{y}, \mathsf{z})$	]
[	$\mathsf{x} \wedge \mathsf{y} \wedge \mathsf{z}$	$\overset{\smile}{\rightarrow}$	$(\mathsf{x} \wedge \mathsf{y}) \wedge \mathsf{z}$	]
[	$\mathsf{x} \vee \mathsf{y} \vee \mathsf{z}$	$\overset{\smile}{\rightarrow}$	$(\mathsf{x} \vee \mathsf{y}) \vee \mathsf{z}$	]
[	$\mathsf{x} \vdash \mathsf{y} \vdash \mathsf{z}$	$\overset{\smile}{\rightarrow}$	$\mathsf{x} \vdash (\mathsf{y} \vdash \mathsf{z})$	]
[	$\mathsf{x} \rhd \mathsf{y} \rhd \mathsf{z}$	$\overset{\smile}{\rightarrow}$	$(\mathsf{x} \rhd \mathsf{y}) \rhd \mathsf{z})$	]
[	$\mathsf{x} : \mathsf{y} : \mathsf{z}$	$\overset{\smile}{\rightarrow}$	$\mathsf{x} : (\mathsf{y} : \mathsf{z})$	]
[	$\mathsf{x}; \mathsf{y}; \mathsf{z}$	$\overset{\smile}{\rightarrow}$	$(\mathsf{x}; \mathsf{y}); \mathsf{z}$	]
[	$\mathsf{x} < \mathsf{y} \leq \mathsf{z}$	$\overset{\smile}{\rightarrow}$	$(\mathsf{x} < \mathsf{y}) \leq \mathsf{z}$	]
[	$\mathsf{x} < \mathsf{y} \leq \mathsf{z}$	$\overset{\circ}{\rightarrow}$	$(\mathsf{x} < \mathsf{y}) \wedge (\mathsf{y} \leq \mathsf{z})\ \overset{\smile}{>}$	
	$\mathsf{x}, \mathsf{y} < \mathsf{z}$	$\overset{\circ}{\rightarrow}$	$\mathsf{x} < \mathsf{z} \wedge \mathsf{y} < \mathsf{z}\ \overset{\smile}{>}$	
	$\mathsf{x} < \mathsf{y}, \mathsf{z}$	$\overset{\circ}{\rightarrow}$	$\mathsf{x} < \mathsf{y} \wedge \mathsf{x} < \mathsf{z}$	]
[	$\forall \mathsf{x}, \mathsf{y} {\in} \mathsf{S} \colon \mathsf{a}$	$\overset{\circ}{\rightarrow}$	$\forall \mathsf{x} {\in} \mathsf{S} \colon \forall \mathsf{y} {\in} \mathsf{S} \colon \mathsf{a}$	]
[	$\mathsf{x} \Rightarrow \mathsf{y} \Leftrightarrow \mathsf{z}$	$\overset{\smile}{\rightarrow}$	$(\mathsf{x} \Rightarrow \mathsf{y}) \Leftrightarrow \mathsf{z}$	]
[	$\mathsf{x} \Rightarrow \mathsf{y} \Leftrightarrow \mathsf{z}$	$\overset{\circ}{\rightarrow}$	$(\mathsf{x} \Rightarrow \mathsf{y}) \wedge (\mathsf{y} \Leftrightarrow \mathsf{z})$	]
[	$\mathsf{x} \rightarrow \mathsf{y} \rightarrow \mathsf{z}$	$\overset{\smile}{\rightarrow}$	$\mathsf{x} \rightarrow (\mathsf{y} \rightarrow \mathsf{z})$	]
[	$\mathsf{x} \rightarrow \mathsf{y} \equiv \mathsf{z}$	$\overset{\circ}{\rightarrow}$	$\mathrm{case}(\mathsf{x}, \mathsf{y}, \mathsf{T}) \equiv \mathrm{case}(\mathsf{x}, \mathsf{z}, \mathsf{T})$	]
[	$\langle \mathsf{x}, \mathsf{y} \rangle$	$\overset{\circ}{\rightarrow}$	$\mathsf{x} :: \langle \mathsf{y} \rangle$	]

A.4 all x in y colon z [∀x∈y: z]˙

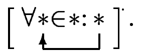
Parse tree



Sort

[∀x∈y: z]˙ is an operator (i.e. it may occur in the syntax tree of a term).

Binding



Description

If [S]˙ is a set then the following holds: [∀x∈S: $\mathcal{A}$ ≡ T]˙ if [$\mathcal{A}$ ∈ **T**]˙ for all [x ∈ S]˙. [∀x∈S: $\mathcal{A}$ ≡ F]˙ if [$\mathcal{A}$ ∈ **B**]˙ for all [x ∈ S]˙ and [$\mathcal{A}$ ∈ **F**]˙ for some [x ∈ S]˙. [∀x∈S: $\mathcal{A}$ ≡ •]˙ if [$\mathcal{A}$]˙ is classical for all [x ∈ S]˙ and [x ∉ **B**]˙ for some [x ∈ S]˙. [∀x∈S: $\mathcal{A}$ ≡ ⊥]˙ if [$\mathcal{A}$]˙ is non-classical for some [x ∈ S]˙.

Mac rules

[**Mac rule Gen** : S ∈ **Set** ⊢ x ∈ S → $\mathcal{A}$ ⊢ ∀x∈S: $\mathcal{A}$]
[**Mac rule ElimAll** : S ∈ **Set** ⊢ ∀x∈S: $\mathcal{A}$ ⊢ $\mathcal{B}$ ∈ S ⊢ ⟨$\mathcal{A}$ | x:=$\mathcal{B}$⟩]
[**Mac rule TypeAll** : x ∈ S → $\mathcal{A}$ ∈ **B** ⊢ (∀x∈S: $\mathcal{A}$) ∈ **B**]
[**Mac rule InverseTypeAll** : (∀x∈S: $\mathcal{A}$) ∈ **B** ⊢ $\mathcal{B}$ ∈ S ⊢ ⟨$\mathcal{A}$ | x:=$\mathcal{B}$⟩ ∈ **B**]
[**Mac rule EqualAll** : x ∈ S → $\mathcal{A}$ ≡ $\mathcal{B}$ ⊢ ∀x∈S: $\mathcal{A}$ ≡ ∀x∈S: $\mathcal{A}$]

Examples

[∀x∈**N**: 2 · x ≥ x	≡	T]˙
[∀x∈**Z**: 2 · x ≥ x	≡	F]˙
[∀x∈**Z**: 1 // x = 1 // x	≡	•]˙
[∀x∈**Z**: x! = x!	≡	⊥]˙

See also

[εx∈y: z]˙	Section A.7
[∃x∈y: z]˙	Section A.12

[∀x∈y: z]˙	all x in y colon z
[Gen]˙	rule Gen
[ElimAll]˙	rule ElimAll
[TypeAll]˙	rule TypeAll
[InverseTypeAll]˙	rule InverseTypeAll
[EqualAll]˙	rule EqualAll

A.5 bottom $[\,\bot\,]^{\cdot}$

Parse tree

$\boxed{\;\bot\;}$

Sort

$[\,\bot\,]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\,\bot\,]^{\cdot}$ denotes total absense of information. If computation of a term proceeds forever and yields no information about the value of the term, then the term equals $[\,\bot\,]^{\cdot}$. Among all mathematical entities, $[\,\bot\,]^{\cdot}$ is the least one with respect to the information comparison relation $[\,\mathsf{x} \preceq \mathsf{y}\,]^{\cdot}$.

Mac rules

$[\,$**Mac rule InfoBottom** $:\ \bot \preceq \mathsf{x}\,]$
$[\,$**Mac rule SubBottom** $:\ \langle \bot \mid \mathsf{x}{:=}\mathcal{S}\rangle \equiv \bot\,]$

Examples

$$[\,(-1)! \quad \equiv \quad \bot\,]^{\cdot}$$
$$[\,(-2)! \quad \equiv \quad \bot\,]^{\cdot}$$

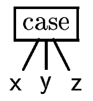
See also

$[\,\bullet\,]^{\cdot}$	Section A.11
$[\,\mathsf{x} \preceq \mathsf{y}\,]^{\cdot}$	Section A.99

$[\,\bot\,]^{\cdot}$	bottom
$[$ SubBottom $]^{\cdot}$	rule SubBottom

A.6 case x comma y comma z end [case(x, y, z)]˙

Parse tree



Sort

[case(x, y, z)]˙ is an operator (i.e. it may occur in the syntax tree of a term).

Description

[case(x, y, z)]˙ equals [y]˙ if [x]˙ equals [N]˙. It equals [z]˙ if [x]˙ is a function, i.e. can be written on form [λu.$\mathcal{A}$]˙. It equals [$\perp$]˙ if [x]˙ equals [$\perp$]˙. [case(x, y, z)]˙ is one of the fundamental constructs of map theory.

Mac rules

[**Mac rule SubCase** : $\langle \mathrm{case}(\mathcal{A}, \mathcal{B}, \mathcal{C}) \mid x{:=}\mathcal{S} \rangle \equiv$ $\mathrm{case}(\langle \mathcal{A} \mid x{:=}\mathcal{S} \rangle, \langle \mathcal{B} \mid x{:=}\mathcal{S} \rangle, \langle \mathcal{C} \mid x{:=}\mathcal{S} \rangle)$]
[**Mac rule CaseNil** : $\mathrm{case}(\mathsf{N}, y, z) \equiv y$]
[**Mac rule CaseT** : $\mathrm{case}(\mathsf{T}, y, z) \equiv y$]
[**Mac rule CaseLambda** : $\mathrm{case}(\lambda u.v, y, z) \equiv z$]
[**Mac rule CaseBottom** : $\mathrm{case}(\perp, y, z) \equiv \perp$]
[**Mac rule CasePair** : $\mathrm{case}(u \mathbin{.\!\!\because} v, y, z) \equiv z$]

Examples

[case(N, 2.0F, 3.00F) $\equiv$ 2.0F]˙

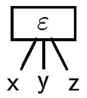
See also

[λx.y]˙	Section A.20
[N]˙	Section A.23
[x ’ y]˙	Section A.47

[case(x, y, z)]˙	case x comma y comma z end
[SubCase]˙	rule SubCase
[CaseNil]˙	rule CaseNil
[CaseT]˙	rule CaseT
[CaseLambda]˙	rule CaseLambda
[CaseBottom]˙	rule CaseBottom
[CasePair]˙	rule CasePair

A.7 choose x in y colon z $[\,\varepsilon \mathsf{x}{\in}\mathsf{y}{:}\,\mathsf{z}\,]^{\cdot}$

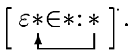
Parse tree



Sort

$[\,\varepsilon \mathsf{x}{\in}\mathsf{y}{:}\,\mathsf{z}\,]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Binding



Description

If $[\,\mathsf{S}\,]^{\cdot}$ is a set then the following holds: If $[\,\mathcal{A} \in \mathbf{B}\,]^{\cdot}$ for all $[\,\mathsf{x} \in \mathsf{S}\,]^{\cdot}$ and if $[\,\mathcal{A}\,]^{\cdot}$ is true for at least one $[\,\mathsf{x} \in \mathsf{S}\,]^{\cdot}$, then $[\,\varepsilon \mathsf{x}{\in}\mathsf{S}{:}\,\mathcal{A}\,]^{\cdot}$ equals some $[\,\mathsf{x} \in \mathsf{S}\,]^{\cdot}$ for which $[\,\mathcal{A}\,]^{\cdot}$ is true. If $[\,\mathcal{A}\,]^{\cdot}$ is true for more than one $[\,\mathsf{x} \in \mathsf{S}\,]^{\cdot}$ then it is impossible to tell which one $[\,\varepsilon\,]^{\circ}$ chooses. If $[\,\mathcal{A}\,]^{\cdot}$ is non-classical for some $[\,\mathsf{x} \in \mathsf{S}\,]^{\cdot}$ then $[\,\varepsilon \mathsf{x}{\in}\mathsf{S}{:}\,\mathcal{A}\,]^{\cdot}$ equals $[\,\bot\,]^{\cdot}$. If $[\,\mathsf{S}\,]^{\cdot}$ is a set and none of the two cases above apply, then $[\,\varepsilon \mathsf{x}{\in}\mathsf{S}{:}\,\mathcal{A}\,]^{\cdot}$ equals $[\,\bullet\,]^{\cdot}$.

Mac rules

[**Mac rule EqualChoice** : $\mathsf{x} \in \mathsf{S} \to \mathcal{A} \equiv \mathcal{B} \vdash \varepsilon \mathsf{x}{\in}\mathsf{S}{:}\,\mathcal{A} \equiv \varepsilon \mathsf{x}{\in}\mathsf{S}{:}\,\mathcal{A}$]

Examples

$[\,\varepsilon \mathsf{x}{\in}\mathbf{Z}{:}\, \mathsf{x} \cdot \mathsf{x} = 9\,]^{\cdot}$ is a classical map that weakly represents $[\,3\,]^{\cdot}$ or $[\,-3\,]^{\cdot}$.

See also

$[\,\forall \mathsf{x}{\in}\mathsf{y}{:}\,\mathsf{z}\,]^{\cdot}$	Section A.4
$[\,\exists \mathsf{x}{\in}\mathsf{y}{:}\,\mathsf{z}\,]^{\cdot}$	Section A.12

$[\,\varepsilon \mathsf{x}{\in}\mathsf{y}{:}\,\mathsf{z}\,]^{\cdot}$	choose x in y colon z
$[$ EqualChoice $]^{\cdot}$	rule EqualChoice

A.8 classical $[\,\ell\,]^{\cdot}$

Parse tree

$\boxed{\ \ell\ }$

Sort

$[\,\ell\,]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\,\ell{}'\mathsf{x}\,]^{\cdot}$ equals $[\,\mathsf{T}\,]^{\cdot}$ if $[\,\mathsf{x}\,]^{\cdot}$ is classical and equals $[\,\bot\,]^{\cdot}$ otherwise.

Mac rules

[**Mac rule ClassicalNil** : $\ell\,\text{'}\,\mathsf{N} \equiv \mathsf{T}$]
[**Mac rule ClassicalBottom** : $\ell\,\text{'}\,\bot \equiv \bot$]
[**Mac rule ClassicalPair** : $\ell\,\text{'}\,(\mathsf{x} \therefore \mathsf{y}) \equiv \ell\,\text{'}\,\mathsf{x} \wedge \ell\,\text{'}\,\mathsf{y}$]
[**Mac rule SubClassical** : $\langle \ell \mid \mathsf{x}{:=}\mathcal{S} \rangle \equiv \ell$]
[**Mac rule ClassicalWellfounded** : $\ell\,\text{'}\,\mathsf{x} \vdash \mathrm{wf}(\mathsf{x})$]

Examples

$[\,\ell\,\text{'}\,(\mathsf{N} \therefore \mathsf{N})$	$\equiv$	$\mathsf{T}\,]^{\cdot}$
$[\,\ell\,\text{'}\,(\mathsf{N} \therefore (\bot \therefore \mathsf{N}))$	$\equiv$	$\bot\,]^{\cdot}$
$[\,\ell\,\text{'}\,\infty\text{-list}(1)$	$\equiv$	$\bot\,]^{\cdot}$

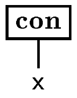
See also

$[\,\mathsf{x} \in \mathsf{y}\,]^{\cdot}$ Section A.50

$[\,\ell\,]^{\cdot}$	classical
[ClassicalNil $]^{\cdot}$	rule ClassicalNil
[ClassicalBottom $]^{\cdot}$	rule ClassicalBottom
[ClassicalPair $]^{\cdot}$	rule ClassicalPair
[SubClassical $]^{\cdot}$	rule SubClassical
[ClassicalWellfounded $]^{\cdot}$	rule ClassicalWellfounded

A.9 construct x $[\,\mathbf{construct}\,\mathsf{x}\,]^{\cdot}$

Parse tree



Sort

$[\,\mathbf{construct}\,\mathsf{x}\,]^{\cdot}$ is a directive (i.e. it cannot occur in the syntax tree of a term).

Description

$[\,\mathbf{construct}\,\mathcal{A}\,]^{\cdot}$ introduces the construct $[\,\mathcal{A}\,]^{\cdot}$ without assigning a meaning to it. $[\,\mathcal{A}\,]^{\cdot}$ gets no meaning and, hence, no value, so it becomes a statement construct rather than an operator.

Examples

$[\,\mathbf{construct}\,\mathsf{x},\mathsf{y}\,]$ introduces the construct $[\,\mathsf{x},\mathsf{y}\,]^{\cdot}$ as a directive with no meaning. $[\,\langle\mathsf{x},\mathsf{y}\rangle\,]^{\cdot}$ is a term even though it containts the directive $[\,\mathsf{x},\mathsf{y}\,]^{\cdot}$. This is so because $[\,\mathsf{x},\mathsf{y}\,]^{\cdot}$ disappears during term reduction so that $[\,\mathsf{x},\mathsf{y}\,]^{\cdot}$ does not occur in the syntax tree of $[\,\langle\mathsf{x},\mathsf{y}\rangle\,]^{\cdot}$.

See also

$[\,\mathsf{x} \doteq \mathsf{y}\,]^{\cdot}$	Section A.58
$[\,\mathbf{variable}\,\mathsf{x}\,]^{\cdot}$	Section A.45

$[\,\mathbf{construct}\,\mathsf{x}\,]^{\cdot}$ construct x

A.10 empty list $[\langle\,\rangle]^{\cdot}$

Parse tree

$\boxed{\langle\,\rangle}$

Sort

$[\langle\,\rangle]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\langle\,\rangle]^{\cdot}$ denotes the empty list, i.e. the unique list that has zero elements. $[\langle\,\rangle]^{\cdot}$ is an atom.

Mac rules

[**Mac rule SubEmptyList** : $\langle\langle\,\rangle \mid \mathsf{x}{:=}\mathcal{S}\rangle \equiv \langle\,\rangle$]
[**Mac rule TypeEmptyListInE** : $\langle\,\rangle \in \mathbf{E}$]

Examples

$[\langle\,\rangle \text{ pair} \equiv \mathsf{F}]^{\cdot}$
$[\langle\,\rangle \text{ atom} \equiv \mathsf{T}]^{\cdot}$
$[\langle\,\rangle \text{ head} \equiv \langle\,\rangle]^{\cdot}$
$[\langle\,\rangle \text{ tail} \equiv \langle\,\rangle]^{\cdot}$

See also

$[\mathbf{E}]^{\cdot}$	Section A.40
$[\mathsf{x} :: \mathsf{y}]^{\cdot}$	Section A.79
$[\langle \mathsf{x}\rangle]^{\cdot}$	Section A.44

$[\langle\,\rangle]^{\cdot}$	empty list
[SubEmptyList]$^{\cdot}$	rule SubEmptyList
[TypeEmptyListInE]$^{\cdot}$	rule TypeEmptyListInE

A.11 exception $[\,\bullet\,]^{\cdot}$

Parse tree

$\boxed{\bullet}$

Sort

$[\,\bullet\,]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\,\bullet\,]^{\cdot}$ denotes nonsense. If a computer can determine within finite time that a term is meaningless, then the value of the term is $[\,\bullet\,]^{\cdot}$. As an example, one divided by zero equals $[\,\bullet\,]^{\cdot}$.

Mac rules

$[\,\textbf{Mac rule SubException} : \langle \bullet \mid \mathsf{x}{:=}\mathcal{S} \rangle \equiv \bullet\,]$
$[\,\textbf{Mac rule TypeXInX} : \bullet \in \mathbf{X}\,]$

Examples

$[\,1/0 \quad \equiv \quad \bullet\,]^{\cdot}$
$[\,1 < \mathsf{F} \quad \equiv \quad \bullet\,]^{\cdot}$

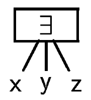
See also

$[\,\perp\,]^{\cdot}$ Section A.5

$[\,\bullet\,]^{\cdot}$	exception
$[$ SubException $]^{\cdot}$	rule SubException
$[$ TypeXInX $]^{\cdot}$	rule TypeXInX

A.12 exists x in y colon z [∃x∈y: z]˙

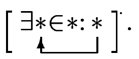
Parse tree



Sort

[∃x∈y: z]˙ is an operator (i.e. it may occur in the syntax tree of a term).

Binding



Description

If [S]˙ is a set then the following holds: [$\exists x \in S{:}\,\mathcal{A} \equiv$ F]˙ if [$\mathcal{A} \in \mathbf{F}$]˙ for all [$x \in S$]˙. [$\exists x \in S{:}\,\mathcal{A} \equiv$ T]˙ if [$\mathcal{A} \in \mathbf{B}$]˙ for all [$x \in S$]˙ and [$\mathcal{A} \in \mathbf{T}$]˙ for some [$x \in S$]˙. [$\exists x \in S{:}\,\mathcal{A} \equiv \bullet$]˙ if [$\mathcal{A}$]˙ is classical for all [$x \in S$]˙ and [$\mathcal{A} \notin \mathbf{B}$]˙ for some [$x \in S$]˙. [$\exists x \in S{:}\,\mathcal{A} \equiv \bot$]˙ if [$\mathcal{A}$]˙ is non-classical for some [$x \in S$]˙.

Mac rules

[**Mac rule IntroExists** : $S \in \mathbf{Set} \vdash x \in S \to \mathcal{A} \in \mathbf{B} \vdash \mathcal{B} \in S \vdash \langle \mathcal{A} \mid x{:=}\mathcal{B} \rangle \vdash \exists x \in S{:}\,\mathcal{A}$]
[**Mac rule ElimExistsB** : $S \in \mathbf{Set} \vdash \exists x \in S{:}\,\mathcal{A} \vdash \mathcal{B} \in S \vdash \langle \mathcal{A} \mid x{:=}\mathcal{B} \rangle \in \mathbf{B}$]
[**Mac rule ElimExistsS** : $S \in \mathbf{Set} \vdash \exists x \in S{:}\,\mathcal{A} \vdash (\varepsilon x \in S{:}\,\mathcal{A}) \in S$]
[**Mac rule ElimExists** : $S \in \mathbf{Set} \vdash \exists x \in S{:}\,\mathcal{A} \vdash \langle \mathcal{A} \mid x{:=}\varepsilon x \in S{:}\,\mathcal{A} \rangle$]
[**Mac rule TypeExists** : $x \in S \to \mathcal{A} \in \mathbf{B} \vdash (\exists x \in S{:}\,\mathcal{A}) \in \mathbf{B}$]
[**Mac rule InverseTypeExists** : $(\exists x \in S{:}\,\mathcal{A}) \in \mathbf{B} \vdash \mathcal{B} \in S \vdash \langle \mathcal{A} \mid x{:=}\mathcal{B} \rangle \in \mathbf{B}$]
[**Mac rule EqualExists** : $x \in S \to \mathcal{A} \equiv \mathcal{B} \vdash \exists x \in S{:}\,\mathcal{A} \equiv \exists x \in S{:}\,\mathcal{A}$]

Examples

[$\exists x \in \mathbf{N}{:}\, x! = 6$	$\equiv$	T]˙
[$\exists x \in \mathbf{N}{:}\, x! = 7$	$\equiv$	F]˙
[$\exists x \in \mathbf{Z}{:}\, x! = 6$	$\equiv$	$\bot$]˙
[$\exists x \in \mathbf{N}{:}\, 10 \,//\, x = 5$	$\equiv$	$\bullet$]˙

[∃x∈y: z]˙	exists x in y colon z
[IntroExists]˙	rule IntroExists
[ElimExistsB]˙	rule ElimExistsB
[ElimExistsS]˙	rule ElimExistsS
[ElimExists]˙	rule ElimExists
[TypeExists]˙	rule TypeExists
[InverseTypeExists]˙	rule InverseTypeExists
[EqualExists]˙	rule EqualExists

See also

[∀x∈y: z]˙	Section A.4
[εx∈y: z]˙	Section A.7

A.13 false [F]˙

Parse tree

$\boxed{\mathsf{F}}$

Sort

[F]˙ is an operator (i.e. it may occur in the syntax tree of a term).

Description

[F]˙ denotes falsehood. If a computer can determine within finite time that a term is false, then the value of the term is [F]˙.

Mac rules

[**Mac rule TypeFInB** : $\mathsf{F} \in \mathbf{B}$]
[**Mac rule TypeFInF** : $\mathsf{F} \in \mathbf{F}$]
[**Mac rule SubFalse** : $\langle \mathsf{F} \mid \mathsf{x}{:=}\mathcal{S} \rangle \equiv \mathsf{F}$]

Examples

[$2 < 1 \quad \equiv \quad \mathsf{F}$]˙
[$1 = \mathsf{T} \quad \equiv \quad \mathsf{F}$]˙

See also

[T]˙	Section A.43
[if(x, y, z)]˙	Section A.18

[F]˙	false
[TypeFInB]˙	rule TypeFInB
[TypeFInF]˙	rule TypeFInF
[SubFalse]˙	rule SubFalse

A.14 f four of x end $[\, f_4(\mathsf{x}) \,]^{\cdot}$

Parse tree

$$\begin{tikzcd}
\boxed{f_4} \arrow[d, dash] \\
\mathsf{x}
\end{tikzcd}$$

Sort

$[\, f_4(\mathsf{x}) \,]^{\cdot}$ is a macro.

Definition

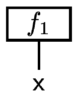
$[\, f_4(\mathsf{x}) \ddot{=} 2 \cdot \mathsf{x} + 4 \,]^{\cdot}$

See also

$[\, f_2 \,]^{\cdot}$	Section A.17
$[\, f_3(\mathsf{x}) \,]^{\cdot}$	Section A.16
$[\, \mathsf{x} \ddot{=} \mathsf{y} \,]^{\cdot}$	Section A.72

A.15 f one of x end $[\, f_1(\mathsf{x}) \,]^{\cdot}$

Parse tree



Sort

$[\, f_1(\mathsf{x}) \,]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Definition

$[\, f_1(\mathsf{x}) \doteq \mathsf{x}^2 - 1 \,]^{\cdot}$

Mac rules

$[\,$ **Mac rule DefFOneOfXEnd** $:\ f_1(\mathsf{x}) \equiv \mathsf{x}^2 - 1 \,]$
$[\,$ **Mac rule SubFOneOfXEnd** $:\ \langle f_1(\mathcal{A}) \mid \mathsf{x}{:=}\mathcal{S} \rangle \equiv f_1(\langle \mathcal{A} \mid \mathsf{x}{:=}\mathcal{S} \rangle) \,]$

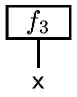
See also

$[\, \mathsf{x} \doteq \mathsf{y} \,]^{\cdot}$ Section A.58

$[\, f_1(\mathsf{x}) \,]^{\cdot}$	f one of x end
$[$ DefFOneOfXEnd $]^{\cdot}$	rule DefFOneOfXEnd
$[$ SubFOneOfXEnd $]^{\cdot}$	rule SubFOneOfXEnd

A.16 f three of x end $[\, f_3(\mathsf{x}) \,]^{\cdot}$

Parse tree



Sort

$[\, f_3(\mathsf{x}) \,]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Definition

$[\, f_3(\mathsf{x}) \doteq 2 \cdot \mathsf{x} + 4 \,]^{\cdot}$

Mac rules

$[\,$ **Mac rule DefFThreeOfXEnd** $: f_3(\mathsf{x}) \equiv 2 \cdot \mathsf{x} + 4 \,]$
$[\,$ **Mac rule SubFThreeOfXEnd** $: \langle f_3(\mathcal{A}) \mid \mathsf{x}{:=}\mathcal{S} \rangle \equiv f_3(\langle \mathcal{A} \mid \mathsf{x}{:=}\mathcal{S} \rangle) \,]$

See also

$[\, f_2 \,]^{\cdot}$	Section A.17
$[\, f_4(\mathsf{x}) \,]^{\cdot}$	Section A.14
$[\, \mathsf{x} \doteq \mathsf{y} \,]^{\cdot}$	Section A.58

$[\, f_3(\mathsf{x}) \,]^{\cdot}$	f three of x end
$[$ DefFThreeOfXEnd $]^{\cdot}$	rule DefFThreeOfXEnd
$[$ SubFThreeOfXEnd $]^{\cdot}$	rule SubFThreeOfXEnd

A.17 f two $[\, f_2 \,]^{\cdot}$

Parse tree

$\boxed{f_2}$

Sort

$[\, f_2 \,]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Definition

$[\, f_2 \doteq \lambda \mathsf{x}.2 \cdot \mathsf{x} + 4 \,]^{\cdot}$

Mac rules

$[\,$**Mac rule DefFTwo** $: f_2 \equiv \lambda \mathsf{x}.2 \cdot \mathsf{x} + 4\,]$
$[\,$**Mac rule ApplyFTwo** $: f_2 \,\text{'}\, \mathsf{x} \equiv 2 \cdot \mathsf{x} + 4\,]$
$[\,$**Mac rule SubFTwo** $: \langle f_2 \mid \mathsf{x}{:=}\mathcal{S} \rangle \equiv f_2\,]$

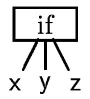
See also

$[\, f_3(\mathsf{x}) \,]^{\cdot}$	Section A.16
$[\, f_4(\mathsf{x}) \,]^{\cdot}$	Section A.14
$[\, \lambda \mathsf{x}.\mathsf{y} \,]^{\cdot}$	Section A.20
$[\, \mathsf{x} \doteq \mathsf{y} \,]^{\cdot}$	Section A.58

$[\, f_2 \,]^{\cdot}$	f two
$[$ DefFTwo $]^{\cdot}$	rule DefFTwo
$[$ ApplyFTwo $]^{\cdot}$	rule ApplyFTwo
$[$ SubFTwo $]^{\cdot}$	rule SubFTwo

A.18 **if x then y else z end** $[\,\mathsf{if}(\mathsf{x},\mathsf{y},\mathsf{z})\,]^{\cdot}$

Parse tree



Sort

$[\,\mathsf{if}(\mathsf{x},\mathsf{y},\mathsf{z})\,]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\,\mathsf{if}(\mathsf{x},\mathsf{y},\mathsf{z})\,]^{\cdot}$ equals $[\,\mathsf{y}\,]^{\cdot}$ and $[\,\mathsf{z}\,]^{\cdot}$ if $[\,\mathsf{x}\,]^{\cdot}$ is true and false, respectively. If $[\,\mathsf{x}\,]^{\cdot}$ is a decimal fraction, an exception, a pair, or the empty list, then $[\,\mathsf{if}(\mathsf{x},\mathsf{y},\mathsf{z}) \equiv \bullet\,]^{\cdot}$. If $[\,\mathsf{x}\,]^{\cdot}$ equals $[\,\perp\,]^{\cdot}$ then $[\,\mathsf{if}(\mathsf{x},\mathsf{y},\mathsf{z}) \equiv \perp\,]^{\cdot}$.

Mac rules

[**Mac rule IfTrue** : $\mathsf{if}(\mathsf{T},\mathsf{y},\mathsf{z}) \equiv \mathsf{y}$]
[**Mac rule IfFalse** : $\mathsf{if}(\mathsf{F},\mathsf{y},\mathsf{z}) \equiv \mathsf{z}$]
[**Mac rule IfNil** : $\mathsf{if}(\langle\,\rangle,\mathsf{y},\mathsf{z}) \equiv \bullet$]
[**Mac rule IfPair** : $\mathsf{if}(\mathsf{u} :: \mathsf{v},\mathsf{y},\mathsf{z}) \equiv \bullet$]
[**Mac rule IfException** : $\mathsf{if}(\bullet,\mathsf{y},\mathsf{z}) \equiv \bullet$]
[**Mac rule IfBottom** : $\mathsf{if}(\perp,\mathsf{y},\mathsf{z}) \equiv \perp$]
[**Mac rule SubIf** : $\langle \mathsf{if}(\mathcal{A},\mathcal{B},\mathcal{C}) \mid \mathsf{x}{:=}\mathcal{S}\rangle \equiv \mathsf{if}(\langle \mathcal{A} \mid \mathsf{x}{:=}\mathcal{S}\rangle, \langle \mathcal{B} \mid \mathsf{x}{:=}\mathcal{S}\rangle, \langle \mathcal{C} \mid \mathsf{x}{:=}\mathcal{S}\rangle)$]
[**Mac rule IfT** : $\mathsf{x} \in \mathbf{T} \vdash \mathsf{if}(\mathsf{x},\mathsf{y},\mathsf{z}) \equiv \mathsf{y}$]
[**Mac rule IfF** : $\mathsf{x} \in \mathbf{F} \vdash \mathsf{if}(\mathsf{x},\mathsf{y},\mathsf{z}) \equiv \mathsf{z}$]
[**Mac rule IfD** : $\mathsf{x} \in \mathbf{D} \vdash \mathsf{if}(\mathsf{x},\mathsf{y},\mathsf{z}) \equiv \bullet$]
[**Mac rule IfE** : $\mathsf{x} \in \mathbf{E} \vdash \mathsf{if}(\mathsf{x},\mathsf{y},\mathsf{z}) \equiv \bullet$]
[**Mac rule IfX** : $\mathsf{x} \in \mathbf{X} \vdash \mathsf{if}(\mathsf{x},\mathsf{y},\mathsf{z}) \equiv \bullet$]

$[\,\mathsf{if}(\mathsf{x},\mathsf{y},\mathsf{z})\,]^{\cdot}$	if x then y else z end
$[$ IfTrue $]^{\cdot}$	rule IfTrue
$[$ IfFalse $]^{\cdot}$	rule IfFalse
$[$ IfNil $]^{\cdot}$	rule IfNil
$[$ IfPair $]^{\cdot}$	rule IfPair
$[$ IfException $]^{\cdot}$	rule IfException
$[$ IfBottom $]^{\cdot}$	rule IfBottom
$[$ SubIf $]^{\cdot}$	rule SubIf
$[$ IfT $]^{\cdot}$	rule IfT
$[$ IfF $]^{\cdot}$	rule IfF
$[$ IfD $]^{\cdot}$	rule IfD
$[$ IfE $]^{\cdot}$	rule IfE
$[$ IfX $]^{\cdot}$	rule IfX

Examples

$$\begin{aligned} &[\,\mathrm{if}(\mathsf{T}, 2.0\mathtt{F}, 3.00\mathtt{F}) && \equiv && 2.0\mathtt{F}\,]^{\cdot} \\ &[\,\mathrm{if}(2.0\mathtt{F}, \bot, \bot) && \equiv && \bullet\,]^{\cdot} \end{aligned}$$

See also

$[\,\mathsf{T}\,]^{\cdot}$	Section A.43
$[\,\mathsf{F}\,]^{\cdot}$	Section A.13
$[\,\bullet\,]^{\cdot}$	Section A.11
$[\,\bot\,]^{\cdot}$	Section A.5

A.19 infinity $[\,\infty\,]^{\cdot}$

Parse tree

$\boxed{\infty}$

Sort

$[\,\infty\,]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\,\infty\,]^{\cdot}$ denotes plus infinity. $[\,\infty\,]^{\cdot}$ is a decimal fraction that is weakly greater than any decimal fraction (including itself). $[\,\infty\,]^{\cdot}$ has precision $[\,\infty\,]^{\cdot}$, i.e. it is an exact decimal fraction. All infinite decimal fractions can be constructed from $[\,\infty\,]^{\cdot}$ together with unary minus and rounding. As an example, $[\,-\infty@3\,]^{\cdot}$ denotes minus infinity with three significant digits.

Mac rules

$[\,$**Mac rule SubInfinity** $: \langle\infty \mid \mathsf{x}{:=}\mathcal{S}\rangle \equiv \infty\,]$
$[\,$**Mac rule TypeIiInIi** $: \infty \in \mathbf{D}_{\infty}^{\infty}\,]$
$[\,$**Mac rule TypeMiInMi** $: -\infty \in \mathbf{D}_{\infty}^{-\infty}\,]$
$[\,$**Mac rule TypeImInIm** $: \mathsf{m} \in \mathbf{Z}^{+} \vdash \infty@\mathsf{m} \in \mathbf{D}_{\mathsf{m}}^{\infty}\,]$
$[\,$**Mac rule TypeMmInMm** $: \mathsf{m} \in \mathbf{Z}^{+} \vdash -\infty@\mathsf{m} \in \mathbf{D}_{\mathsf{m}}^{-\infty}\,]$

Examples

$[\,2 < \infty$	$\equiv$	$\mathsf{T}\,]^{\cdot}$
$[\,-\infty < 2$	$\equiv$	$\mathsf{T}\,]^{\cdot}$
$[\,2.3\mathrm{F}@\infty$	$\equiv$	$2.3\,]^{\cdot}$

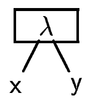
See also

$[\,-\mathsf{x}\,]^{\cdot}$	Section A.22
$[\,\mathsf{x}@\mathsf{y}\,]^{\cdot}$	Section A.85

$[\,\infty\,]^{\cdot}$	infinity
$[$ SubInfinity $]^{\cdot}$	rule SubInfinity
$[$ TypeIiInIi $]^{\cdot}$	rule TypeIiInIi
$[$ TypeMiInMi $]^{\cdot}$	rule TypeMiInMi
$[$ TypeImInIm $]^{\cdot}$	rule TypeImInIm
$[$ TypeMmInMm $]^{\cdot}$	rule TypeMmInMm

A.20 lambda x dot y [λx.y]˙

Parse tree



Sort

[λx.y]˙ is an operator (i.e. it may occur in the syntax tree of a term).

Binding

$\left[\, \lambda \underset{\uparrow\!\_\_\!\lrcorner}{*.*} \,\right]^{\cdot}$.

Description

[λx.y]˙ is the function that maps [x]˙ to [y]˙. It is one of the fundamental constructs of map theory. [x]˙ must be a variable.

Mac rules

[**Mac rule SubLambdaXX** : $\langle \lambda x.\mathcal{A} \mid x{:=}\mathcal{S}\rangle \equiv \lambda x.\mathcal{A}$]
[**Mac rule SubLambdaXYFreeFor** :
if $\mathrm{distinct}(x, y) \wedge \mathrm{freefor}(\lambda x.\mathcal{A}, y, \mathcal{S})$ **then** $\langle \lambda y.\mathcal{A} \mid x{:=}\mathcal{S}\rangle \equiv \lambda y.\ \langle \mathcal{A} \mid x{:=}\mathcal{S}\rangle$]
[**Mac rule SubLambdaXY** :
if $\mathrm{distinct}(x, y)$ **then** $\langle \lambda y.\mathcal{A} \mid x{:=}\langle \mathcal{S} \mid y{:=}\mathsf{T}\rangle\rangle \equiv \lambda y.\ \langle \mathcal{A} \mid x{:=}\langle \mathcal{S} \mid y{:=}\mathsf{T}\rangle\rangle$]
[**Mac rule LambdaSub** : $x \equiv y \vdash \lambda z.x \equiv \lambda z.y$]

Examples

[$(\lambda x.x + 2)\ '\ 3 \quad \equiv \quad 5$]˙
[$(\lambda x.x + y)\ '\ 3 \quad \equiv \quad 3 + y$]˙

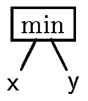
See also

[case(x, y, z)]˙	Section A.6
[N]˙	Section A.23
[$\langle$x $\mid$ y:=z$\rangle$]˙	Section A.28
[**variable** x]˙	Section A.45
[x ' y]˙	Section A.47

[λx.y]˙	lambda x dot y
[SubLambdaXX]˙	rule SubLambdaXX
[SubLambdaXYFreeFor]˙	rule SubLambdaXYFreeFor
[SubLambdaXY]˙	rule SubLambdaXY
[LambdaSub]˙	rule LambdaSub

A.21 minimum x comma y end $[\,\mathrm{min}(\mathsf{x},\mathsf{y})\,]^{\cdot}$

Parse tree



Sort

$[\,\mathrm{min}(\mathsf{x},\mathsf{y})\,]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\,\mathrm{min}(\mathsf{x},\mathsf{y})\,]^{\cdot}$ denotes the least of the two values $[\,\mathsf{x}\,]^{\cdot}$ and $[\,\mathsf{y}\,]^{\cdot}$.

Definition

$$\left[\mathrm{min}(\mathsf{x},\mathsf{y}) \doteq \mathsf{x} < \mathsf{y}\left\{\begin{array}{c}\mathsf{x}\\ \mathsf{y}\end{array}\right.\right].$$

Mac rules

$[$ **Mac rule DefMinimumXCommaYEnd** : $\mathrm{min}(\mathsf{x},\mathsf{y}) \equiv \mathsf{x} < \mathsf{y}\left\{\begin{array}{c}\mathsf{x}\\ \mathsf{y}\end{array}\right.\Big]$

$[$ **Mac rule SubMinimumXCommaYEnd** : $\langle \mathrm{min}(\mathcal{A},\mathcal{B}) \mid \mathsf{x}{:=}\mathcal{S}\rangle \equiv \mathrm{min}(\langle \mathcal{A} \mid \mathsf{x}{:=}\mathcal{S}\rangle, \langle \mathcal{B} \mid \mathsf{x}{:=}\mathcal{S}\rangle)\Big]$

Examples

$[\ \mathrm{min}(2,3)$	$\equiv$	$2\ ]^{\cdot}$
$[\ \mathrm{min}(3,2)$	$\equiv$	$2\ ]^{\cdot}$
$[\ \mathrm{min}(2,2)$	$\equiv$	$2\ ]^{\cdot}$
$[\ \mathrm{min}(2,\infty)$	$\equiv$	$2\ ]^{\cdot}$
$[\ \mathrm{min}(2,-\infty)$	$\equiv$	$-\infty\ ]^{\cdot}$

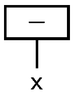
See also

$[\,\mathsf{x} \wedge \mathsf{y}\,]^{\cdot}$ Section A.46

$[\,\mathrm{min}(\mathsf{x},\mathsf{y})\,]^{\cdot}$	minimum x comma y end
$[$ DefMinimumXCommaYEnd $]^{\cdot}$	rule DefMinimumXCommaYEnd
$[$ SubMinimumXCommaYEnd $]^{\cdot}$	rule SubMinimumXCommaYEnd

A.22 minus x $[-\mathsf{x}]^{\cdot}$

Parse tree



Sort

$[-\mathsf{x}]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[-\mathsf{x}]^{\cdot}$ changes the sign of $[\mathsf{x}]^{\cdot}$. In particular, $[-\mathsf{x}]^{\cdot}$ may be used to change the sign of an infinity.

Type table

x	$\mathbf{B}$	$\mathbf{D}$	$\mathbf{X}$	$\mathbf{E}$	$\mathbf{N}$	$\mathbf{Z}$	$\mathbf{Z}^{+}$	$\mathbf{Z}^{-}$
$-\mathsf{x}$	$\mathbf{X}$	$\mathbf{D}$	$\mathbf{X}$	$\mathbf{X}$	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}^{-}$	$\mathbf{Z}^{+}$

x	$\mathbf{D}_{\mathsf{m}}^{\infty}$	$\mathbf{D}_{\mathsf{m}}^{\cdot}$	$\mathbf{D}_{\mathsf{m}}^{-\infty}$	$\mathbf{D}_{\infty}^{\infty}$	$\mathbf{D}_{\infty}^{\cdot}$	$\mathbf{D}_{\infty}^{-\infty}$	$\mathbf{T}$	$\mathbf{F}$
$-\mathsf{x}$	$\mathbf{D}_{\mathsf{m}}^{-\infty}$	$\mathbf{D}_{\mathsf{m}}^{\cdot}$	$\mathbf{D}_{\mathsf{m}}^{\infty}$	$\mathbf{D}_{\infty}^{-\infty}$	$\mathbf{D}_{\infty}^{\cdot}$	$\mathbf{D}_{\infty}^{-\infty}$	$\mathbf{X}$	$\mathbf{X}$

Mac rules

[**Mac rule SubMinusX** : $\langle -\mathcal{A} \mid \mathsf{x}{:=}\mathcal{S}\rangle \equiv -\langle \mathcal{A} \mid \mathsf{x}{:=}\mathcal{S}\rangle$]
[**Mac rule MinusStrict** : $-\bot \equiv \bot$]

Examples

$[--5 \equiv 5]^{\cdot}$
$[--\infty \equiv \infty]^{\cdot}$

See also

$[\infty]^{\cdot}$	Section A.19
$[+\mathsf{x}]^{\cdot}$	Section A.26
$[\mathsf{x}-\mathsf{y}]^{\cdot}$	Section A.73

$[-\mathsf{x}]^{\cdot}$	minus x
[SubMinusX]˙	rule SubMinusX
[MinusStrict]˙	rule MinusStrict

A.23 nil map $[\,\mathsf{N}\,]^{\cdot}$

Parse tree

$\boxed{\;\mathsf{N}\;}$

Sort

$[\,\mathsf{N}\,]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\,\mathsf{N}\,]^{\cdot}$ denotes the nil map. $[\,\mathsf{N}\,]^{\cdot}$ is one of the fundamental constructs of map theory.

Mac rules

[**Mac rule SubNilMap** : $\langle \mathsf{N} \mid \mathsf{x}{:=}\mathcal{S}\rangle \equiv \mathsf{N}$]

Examples

$[\,\mathsf{T} \;\equiv\; \mathsf{N}\,]^{\cdot}$
$[\,\mathsf{F} \;\equiv\; \mathsf{N} \mathrel{.\!\cdot\!.} \mathsf{N}\,]^{\cdot}$

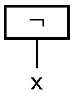
See also

$[\,\mathrm{case}(\mathsf{x},\mathsf{y},\mathsf{z})\,]^{\cdot}$	Section A.6
$[\,\lambda\mathsf{x}.\mathsf{y}\,]^{\cdot}$	Section A.20
$[\,\mathsf{x}\,{}'\,\mathsf{y}\,]^{\cdot}$	Section A.47

$[\,\mathsf{N}\,]^{\cdot}$	nil map
$[\,\mathrm{SubNilMap}\,]^{\cdot}$	rule SubNilMap

A.24 not x [¬x]˙

Parse tree



Sort

[¬x]˙ is an operator (i.e. it may occur in the syntax tree of a term).

Description

[¬x]˙ is true if [x]˙ is false and vice versa. In other words, [¬x]˙ is the logical negation of [x]˙.

Type table

x	**B**	**D**	**X**	**E**	**N**	**Z**	$\mathbf{Z}^+$	$\mathbf{Z}^-$
¬x	**B**	**X**	**X**	**X**	**X**	**X**	**X**	**X**

x	$\mathbf{D}^{\infty}_{\mathsf{m}}$	$\mathbf{D}^{\cdot}_{\mathsf{m}}$	$\mathbf{D}^{-\infty}_{\mathsf{m}}$	$\mathbf{D}^{\infty}_{\infty}$	$\mathbf{D}^{\cdot}_{\infty}$	$\mathbf{D}^{-\infty}_{\infty}$	**T**	**F**
¬x	**X**	**X**	**X**	**X**	**X**	**X**	**F**	**T**

Mac rules

[**Mac rule SubNotX** : $\langle \neg\mathcal{A} \mid \mathsf{x}{:=}\mathcal{S}\rangle \equiv \neg\langle \mathcal{A} \mid \mathsf{x}{:=}\mathcal{S}\rangle$]
[**Mac rule NotStrict** : $\neg\bot \equiv \bot$]

Examples

[¬T ≡ F]˙
[¬F ≡ T]˙
[¬⊥ ≡ ⊥]˙

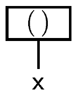
See also

[T]˙	Section A.43
[F]˙	Section A.13

[¬x]˙	not x
[SubNotX]˙	rule SubNotX
[NotStrict]˙	rule NotStrict

A.25 parenthesis x end [(x)]˙

Parse tree



Sort

[(x)]˙ is a macro.

Description

[(x)]˙ means the same as [x]˙. [(x)]˙ is useful for controlling the order of computation like in [$2 \cdot (3+4)$]˙ where the parenthesis forces addition to occur before multiplication.

Definition

[(x) $\ddot{=}$ x].

Examples

[$(2 \cdot 3) + 4$	$\equiv$	10]˙
[$2 \cdot (3+4)$	$\equiv$	14]˙

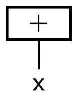
See also

[x $\ddot{=}$ y]˙	Section A.72

[(x)]˙ parenthesis x end

A.26 plus x [+x]˙

Parse tree



Sort

[+x]˙ is an operator (i.e. it may occur in the syntax tree of a term).

Description

[+x]˙ leaves a decimal fraction unchanged. It may be used to present positive quantities in a format similar to that of negative quantities (e.g. [+2]˙ versus [−2]˙ and [+∞]˙ versus [−∞]˙).

Type table

x	**B**	**D**	**X**	**E**	**N**	**Z**	$\mathbf{Z}^{+}$	$\mathbf{Z}^{-}$
+x	**X**	**D**	**X**	**X**	**N**	**Z**	$\mathbf{Z}^{+}$	$\mathbf{Z}^{-}$

x	$\mathbf{D}^{\infty}_{m}$	$\mathbf{D}^{\cdot}_{m}$	$\mathbf{D}^{-\infty}_{m}$	$\mathbf{D}^{\infty}_{\infty}$	$\mathbf{D}^{\cdot}_{\infty}$	$\mathbf{D}^{-\infty}_{\infty}$	**T**	**F**
+x	$\mathbf{D}^{\infty}_{m}$	$\mathbf{D}^{\cdot}_{m}$	$\mathbf{D}^{-\infty}_{m}$	$\mathbf{D}^{\infty}_{\infty}$	$\mathbf{D}^{\cdot}_{\infty}$	$\mathbf{D}^{-\infty}_{\infty}$	**X**	**X**

Mac rules

[**Mac rule SubPlusX** : $\langle +\mathcal{A} \mid x{:=}\mathcal{S}\rangle \equiv +\langle \mathcal{A} \mid x{:=}\mathcal{S}\rangle$]
[**Mac rule PlusStrict** : $+\bot \equiv \bot$]

Examples

[+5 ≡ 5]˙
[+∞ ≡ ∞]˙

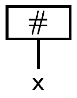
See also

[∞]˙	Section A.19
[−x]˙	Section A.22
[x + y]˙	Section A.80

[+x]˙	plus x
[SubPlusX]˙	rule SubPlusX
[PlusStrict]˙	rule PlusStrict

A.27 precision x [#x]˙

Parse tree



Sort

[#x]˙ is an operator (i.e. it may occur in the syntax tree of a term).

Description

[#x]˙ denotes the precision (number of significant digits) of decimal fractions. If [x]˙ is a decimal fraction then [#x]˙ is a positive integer or [$+\infty$]˙. Exact decimal fractions have precision [$+\infty$]˙.

Type table

x	**B**	**D**	**X**	**E**	**N**	**Z**	$\mathbf{Z}^{+}$	$\mathbf{Z}^{-}$
#x	**X**	**D**	**X**	**X**	$\mathbf{D}_{\infty}^{\infty}$	$\mathbf{D}_{\infty}^{\infty}$	$\mathbf{D}_{\infty}^{\infty}$	$\mathbf{D}_{\infty}^{\infty}$
x	$\mathbf{D}_{\mathsf{m}}^{\infty}$	$\mathbf{D}_{\mathsf{m}}^{\cdot}$	$\mathbf{D}_{\mathsf{m}}^{-\infty}$	$\mathbf{D}_{\infty}^{\infty}$	$\mathbf{D}_{\infty}^{\cdot}$	$\mathbf{D}_{\infty}^{-\infty}$	**T**	**F**
#x	$\mathbf{Z}^{+}$	$\mathbf{Z}^{+}$	$\mathbf{Z}^{+}$	$\mathbf{D}_{\infty}^{\infty}$	$\mathbf{D}_{\infty}^{\infty}$	$\mathbf{D}_{\infty}^{\infty}$	**X**	**X**

Mac rules

[**Mac rule SubPrecisionX** : $\langle \#\mathcal{A} \mid \mathsf{x}{:=}\mathcal{S} \rangle \equiv \#\langle \mathcal{A} \mid \mathsf{x}{:=}\mathcal{S} \rangle$]
[**Mac rule PrecisionStrict** : $\#\bot \equiv \bot$]

Examples

[#1.234F	$\equiv$	4]˙
[#1.234	$\equiv$	∞]˙
[#∞	$\equiv$	∞]˙
[#(∞@3)	$\equiv$	3]˙

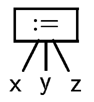
See also

[x@y]˙	Section A.85

[#x]˙	precision x
[SubPrecisionX]˙	rule SubPrecisionX
[PrecisionStrict]˙	rule PrecisionStrict

A.28 substitute x where y is z end [⟨$\mathcal{A}$ | x:=$\mathcal{B}$⟩]˙

Parse tree



Sort

[⟨$\mathcal{A}$ | x:=$\mathcal{B}$⟩]˙ is an operator (i.e. it may occur in the syntax tree of a term).

Binding

$\left[\langle * \mid \overset{\uparrow}{*}{:=}* \rangle\right]^{\cdot}$.

Description

[⟨$\mathcal{A}$ | x:=$\mathcal{B}$⟩]˙ equals [$\mathcal{A}$]˙ in which all free occurrences of [x]˙ have value [$\mathcal{B}$]˙.

Mac rules

[**Mac rule SubXX** : ⟨x | x:=$\mathcal{S}$⟩ ≡ $\mathcal{S}$]
[**Mac rule SubXY** : **if** distinct (x, y) **then** ⟨y | x:=$\mathcal{S}$⟩ ≡ y]
[**Mac rule SubXSubX** : ⟨⟨$\mathcal{A}$ | x:=$\mathcal{B}$⟩ | x:=$\mathcal{S}$⟩ ≡ ⟨$\mathcal{A}$ | x:=⟨$\mathcal{B}$ | x:=$\mathcal{S}$⟩⟩]
[**Mac rule SubXSubY** : **if** distinct(x,y) **then** ⟨⟨$\mathcal{A}$ | x:=$\mathcal{B}$⟩ | y:=⟨$\mathcal{S}$ | x:=T⟩⟩ ≡ ⟨⟨$\mathcal{A}$ | y:=⟨$\mathcal{S}$ | x:=T⟩⟩ | x:=⟨$\mathcal{B}$ | y:=⟨$\mathcal{S}$ | x:=T⟩⟩⟩]
[**Mac rule SubXSubYNotFree** : **if** distinct(x, y) ∧ notfree(x, $\mathcal{S}$) **then** ⟨⟨$\mathcal{A}$ | x:=$\mathcal{B}$⟩ | y:=$\mathcal{S}$⟩ ≡ ⟨⟨$\mathcal{A}$ | y:=$\mathcal{S}$⟩ | x:=⟨$\mathcal{B}$ | y:=$\mathcal{S}$⟩⟩]
[**Mac rule Substitution** : **if** Substitution($\mathcal{A}$, $\mathcal{B}$) **then** $\mathcal{A} \equiv \mathcal{B}$]
[**Mac rule SubNumeral** : **if** numeral(y) **then** ⟨y | x:=$\mathcal{S}$⟩ ≡ y]
([Substitution($\mathcal{A}$, $\mathcal{B}$)]˙ says that [$\mathcal{A}$]˙ and [$\mathcal{B}$]˙ can be made identical by performing substitutions and renaming bound variables).

Examples

[⟨x + 2 | x:=3⟩ ≡ 5]˙
[⟨x + y | x:=3⟩ ≡ 3 + y]˙

[⟨x | y:=z⟩]˙	substitute x where y is z end
[SubXX]˙	rule SubXX
[SubXY]˙	rule SubXY
[SubXSubX]˙	rule SubXSubX
[SubXSubY]˙	rule SubXSubY
[SubXSubYNotFree]˙	rule SubXSubYNotFree
[Substitution]˙	rule Substitution
[SubNumeral]˙	rule SubNumeral

See also

[λx.y]˙	Section A.20
[**variable** x]˙	Section A.45

A.29 the class of sets [Set]˙

Parse tree

$\boxed{\textbf{Set}}$

Sort

[**Set**]˙ is an operator (i.e. it may occur in the syntax tree of a term).

Description

[**Set**]˙ is the class of all sets. [x ∈ **Set**]˙ is true if and only if [x]˙ is a set. For certain, technical reasons, [**Set**]˙ itself is not a set.

Mac rules

[**Mac rule SubSet** : ⟨**Set** | x:=$\mathcal{S}$⟩ ≡ **Set**]

Examples

[**T** ∈ **Set**	≡	T]˙
[**B** ∈ **Set**	≡	T]˙
[**D** ∈ **Set**	≡	T]˙
[**X** ∈ **Set**	≡	T]˙
[117 ∈ **Set**	≡	F]˙
[**Set** ∈ **Set**	≡	F]˙

See also

[ℓ]˙	Section A.8
[x ∈ y]˙	Section A.50

[**Set**]˙	the class of sets
[SubSet]˙	rule SubSet

A.30 the set of decimal fractions $[\,\mathbf{D}\,]^{\cdot}$

Parse tree

$\boxed{\mathbf{D}}$

Sort

$[\,\mathbf{D}\,]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\,\mathbf{D}\,]^{\cdot}$ is the set of decimal fractions. $[\,\mathsf{x} \in \mathbf{D}\,]^{\cdot}$ is true if and only if $[\,\mathsf{x}\,]^{\cdot}$ is a classical map that weakly represents a decimal fraction. Decimal fractions include exact fractions like $[\,1.1\,]^{\cdot}$ and floating fractions like $[\,1.1\mathtt{F}\,]^{\cdot}$. Decimal fractions include infinities like $[\,\infty\,]^{\cdot}$ and $[\,-\infty@4\,]^{\cdot}$.

Mac rules

[**Mac rule SubD** : $\langle \mathbf{D} \mid \mathsf{x}{:=}\mathcal{S}\rangle \equiv \mathbf{D}$]
[**Mac rule SetD** : $\mathbf{D} \in \mathbf{Set}$]
[**Mac rule TypeNumeralInD** : **if** x is a numeral that represents a decimal fraction **then** $\mathsf{x} \in \mathbf{D}$]
[**Mac rule SubtypeDC** : $\mathsf{x} \in \mathbf{D} \vdash \ell'\mathsf{x}$]
[**Mac rule ElimD** : $\mathsf{x} \in \mathbf{D} \vdash \overline{\mathbf{D}}'\mathsf{x}$]
[**Mac rule IntroD** : $\ell'\mathsf{x} \vdash \overline{\mathbf{D}}'\mathsf{x} \vdash \mathsf{x} \in \mathbf{D}$]
[**Mac rule SubtypeNotDC** : $\mathsf{x} \notin \mathbf{D} \vdash \ell'\mathsf{x}$]
[**Mac rule ElimNotD** : $\mathsf{x} \notin \mathbf{D} \vdash \neg\overline{\mathbf{D}}'\mathsf{x}$]
[**Mac rule IntroNotD** : $\ell'\mathsf{x} \vdash \neg\overline{\mathbf{D}}'\mathsf{x} \vdash \mathsf{x} \notin \mathbf{D}$]

$[\,\mathbf{D}\,]^{\cdot}$	the set of decimal fractions
[SubD]˙	rule SubD
[SetD]˙	rule SetD
[TypeNumeralInD]˙	rule TypeNumeralInD
[SubtypeDC]˙	rule SubtypeDC
[ElimD]˙	rule ElimD
[IntroD]˙	rule IntroD
[SubtypeNotDC]˙	rule SubtypeNotDC
[ElimNotD]˙	rule ElimNotD
[IntroNotD]˙	rule IntroNotD

Examples

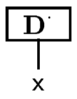
$[\, 0 \in \mathbf{D}$	$\equiv$	$\mathsf{T} \,]^{\cdot}$
$[\, 1 \in \mathbf{D}$	$\equiv$	$\mathsf{T} \,]^{\cdot}$
$[\, 117 \in \mathbf{D}$	$\equiv$	$\mathsf{T} \,]^{\cdot}$
$[\, -1 \in \mathbf{D}$	$\equiv$	$\mathsf{T} \,]^{\cdot}$
$[\, 1.1 \in \mathbf{D}$	$\equiv$	$\mathsf{T} \,]^{\cdot}$
$[\, 117\mathrm{F} \in \mathbf{D}$	$\equiv$	$\mathsf{T} \,]^{\cdot}$
$[\, \mathsf{T} \in \mathbf{D}$	$\equiv$	$\mathsf{F} \,]^{\cdot}$
$[\, \bullet \in \mathbf{D}$	$\equiv$	$\mathsf{F} \,]^{\cdot}$
$[\, 0 :: 1 \in \mathbf{D}$	$\equiv$	$\mathsf{F} \,]^{\cdot}$
$[\, \perp \in \mathbf{D}$	$\equiv$	$\perp \,]^{\cdot}$

See also

$[\, \mathbf{D}^{\cdot}_{\mathsf{x}} \,]^{\cdot}$	Section A.31
$[\, \mathbf{D}^{\infty}_{\mathsf{x}} \,]^{\cdot}$	Section A.34
$[\, \mathbf{D}^{-\infty}_{\mathsf{x}} \,]^{\cdot}$	Section A.36
$[\, \mathbf{Z} \,]^{\cdot}$	Section A.35
$[\, \mathsf{x} \in \mathsf{y} \,]^{\cdot}$	Section A.50

A.31 the set of decimal fractions of precision x end $[\,\mathbf{D}^{\cdot}_{\mathsf{x}}\,]^{\cdot}$

Parse tree



Sort

$[\,\mathbf{D}^{\cdot}_{\mathsf{x}}\,]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\,\mathbf{D}^{\cdot}_{\mathsf{x}}\,]^{\cdot}$ is the set of decimal fractions of precision $[\,\mathsf{x}\,]^{\cdot}$ whose magnitude differs from $[\,\infty\,]^{\cdot}$ and $[\,-\infty\,]^{\cdot}$.

Mac rules

$[$ **Mac rule SubDm** : $\langle \mathbf{D}^{\cdot}_{\mathcal{A}} \mid \mathsf{x}{:=}\mathcal{S}\rangle \equiv \mathbf{D}^{\cdot}_{\langle \mathcal{A} \mid \mathsf{x}:=\mathcal{S}\rangle}$ $]$
$[$ **Mac rule SetDi** : $\mathbf{D}^{\cdot}_{\infty} \in$ **Set** $]$
$[$ **Mac rule TypeNumeralInDm** : **if** x is a numeral that represents a floating fraction of precision m **then** $\mathsf{x} \in \mathbf{D}^{\cdot}_{\mathsf{m}}$ $]$
$[$ **Mac rule TypeNumeralInDi** : **if** x is a numeral that represents an exact fraction **then** $\mathsf{x} \in \mathbf{D}^{\cdot}_{\infty}$ $]$
$[$ **Mac rule TypeNumeralNotInDm** : **if** x is a numeral that does not represent a floating fraction of precision m **then** $\mathsf{x} \notin \mathbf{D}^{\cdot}_{\mathsf{m}}$ $]$
$[$ **Mac rule TypeNumeralNotInDi** : **if** x is a numeral that does not represent an exact fraction **then** $\mathsf{x} \in \mathbf{D}^{\cdot}_{\infty}$ $]$
$[$ **Mac rule SubtypeDiD** : $\mathsf{x} \in \mathbf{D}^{\cdot}_{\infty} \vdash \mathsf{x} \in \mathbf{D}$ $]$
$[$ **Mac rule SubtypeDmD** : $\mathsf{m} \in \mathbf{Z}^{+} \vdash \mathsf{x} \in \mathbf{D}^{\cdot}_{\mathsf{m}} \vdash \mathsf{x} \in \mathbf{D}$ $]$
$[$ **Mac rule SetDm** : $\mathsf{m} \in \mathbf{Z}^{+} \vdash \mathbf{D}^{\cdot}_{\mathsf{m}} \in$ **Set** $]$

$[\,\mathbf{D}^{\cdot}_{\mathsf{x}}\,]^{\cdot}$	the set of decimal fractions of precision x end
[SubDm]˙	rule SubDm
[SetDi]˙	rule SetDi
[TypeNumeralInDm]˙	rule TypeNumeralInDm
[TypeNumeralInDi]˙	rule TypeNumeralInDi
[TypeNumeralNotInDm]˙	rule TypeNumeralNotInDm
[TypeNumeralNotInDi]˙	rule TypeNumeralNotInDi
[SubtypeDiD]˙	rule SubtypeDiD
[SubtypeDmD]˙	rule SubtypeDmD
[SetDm]˙	rule SetDm

Examples

$[\,117 \in \mathbf{D}^{\cdot}_{\infty} \equiv \mathsf{T}\,]^{\cdot}$

$[\,117 \in \mathbf{D}^{\cdot}_{3} \equiv \mathsf{F}\,]^{\cdot}$

$[\,117\mathtt{F} \in \mathbf{D}^{\cdot}_{3} \equiv \mathsf{T}\,]^{\cdot}$

$[\,117\mathtt{F} \in \mathbf{D}^{\cdot}_{\infty} \equiv \mathsf{F}\,]^{\cdot}$

$[\,\infty \in \mathbf{D}^{\cdot}_{\infty} \equiv \mathsf{F}\,]^{\cdot}$

$[\,\mathsf{T} \in \mathbf{D} \equiv \mathsf{F}\,]^{\cdot}$

$[\,\bullet \in \mathbf{D} \equiv \mathsf{F}\,]^{\cdot}$

$[\,0 :: 1 \in \mathbf{D} \equiv \mathsf{F}\,]^{\cdot}$

$[\,\perp \in \mathbf{D} \equiv \perp\,]^{\cdot}$

See also

$[\,\mathbf{D}\,]^{\cdot}$	Section A.30
$[\,\mathsf{x} \in \mathsf{y}\,]^{\cdot}$	Section A.50

A.32 the set of exceptions $[\,\mathbf{X}\,]^{\cdot}$

Parse tree

$\boxed{\mathrm{X}}$

Sort

$[\,\mathbf{X}\,]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\,\mathbf{X}\,]^{\cdot}$ is the set of exceptions. $[\,\mathsf{x} \in \mathbf{X}\,]^{\cdot}$ is true if and only if $[\,\mathsf{x}\,]^{\cdot}$ is a classical map that weakly represents an exception.

Mac rules

$[\,$**Mac rule SubX** $: \langle \mathbf{X} \mid \mathsf{x}{:=}\mathcal{S} \rangle \equiv \mathbf{X}\,]$
$[\,$**Mac rule TypeNumeralNotInX** : **if** x is a numeral **then** $\mathsf{x} \notin \mathbf{X}\,]$
$[\,$**Mac rule SetX** $: \mathbf{X} \in \mathbf{Set}\,]$
$[\,$**Mac rule SubtypeXC** $: \mathsf{x} \in \mathbf{X} \vdash \ell' \mathsf{x}\,]$
$[\,$**Mac rule ElimX** $: \mathsf{x} \in \mathbf{X} \vdash \overline{\mathbf{X}}' \mathsf{x}\,]$
$[\,$**Mac rule IntroX** $: \ell' \mathsf{x} \vdash \overline{\mathbf{X}}' \mathsf{x} \vdash \mathsf{x} \in \mathbf{X}\,]$
$[\,$**Mac rule SubtypeNotXC** $: \mathsf{x} \notin \mathbf{X} \vdash \ell' \mathsf{x}\,]$
$[\,$**Mac rule ElimNotX** $: \mathsf{x} \notin \mathbf{X} \vdash \neg\overline{\mathbf{X}}' \mathsf{x}\,]$
$[\,$**Mac rule IntroNotX** $: \ell' \mathsf{x} \vdash \neg\overline{\mathbf{X}}' \mathsf{x} \vdash \mathsf{x} \notin \mathbf{X}\,]$

Examples

$$
\begin{array}{lcl}
[\,\bullet \in \mathbf{X} & \equiv & \mathsf{T}\,]^{\cdot} \\
[\,\mathsf{T} \in \mathbf{X} & \equiv & \mathsf{F}\,]^{\cdot} \\
[\,\mathsf{F} \in \mathbf{X} & \equiv & \mathsf{F}\,]^{\cdot} \\
[\,117 \in \mathbf{X} & \equiv & \mathsf{F}\,]^{\cdot} \\
[\,0 :: 1 \in \mathbf{X} & \equiv & \mathsf{F}\,]^{\cdot} \\
[\,\bot \in \mathbf{X} & \equiv & \bot\,]^{\cdot}
\end{array}
$$

$[\,\mathbf{X}\,]^{\cdot}$	the set of exceptions
$[\,$SubX$\,]^{\cdot}$	rule SubX
$[\,$TypeNumeralNotInX$\,]^{\cdot}$	rule TypeNumeralNotInX
$[\,$SetX$\,]^{\cdot}$	rule SetX
$[\,$SubtypeXC$\,]^{\cdot}$	rule SubtypeXC
$[\,$ElimX$\,]^{\cdot}$	rule ElimX
$[\,$IntroX$\,]^{\cdot}$	rule IntroX
$[\,$SubtypeNotXC$\,]^{\cdot}$	rule SubtypeNotXC
$[\,$ElimNotX$\,]^{\cdot}$	rule ElimNotX
$[\,$IntroNotX$\,]^{\cdot}$	rule IntroNotX

See also

$[\mathsf{x} \in \mathsf{y}]^{\cdot}$ Section A.50

A.33 the set of false maps $[\mathbf{F}]^{\cdot}$

Parse tree

$\boxed{\mathbf{F}}$

Sort

$[\mathbf{F}]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\mathbf{F}]^{\cdot}$ is the set of classical maps that weakly represent falsehood.

Mac rules

[**Mac rule SubF** : $\langle \mathbf{F} \mid \mathsf{x}{:=}\mathcal{S}\rangle \equiv \mathbf{F}$]
[**Mac rule TypeNumeralNotInF** : **if** x is a numeral **then** $\mathsf{x} \notin \mathbf{F}$]
[**Mac rule SetF** : $\mathbf{F} \in \mathbf{Set}$]
[**Mac rule SubtypeFB** : $\mathsf{x} \in \mathbf{F} \vdash \mathsf{x} \in \mathbf{B}$]
[**Mac rule SubtypeFC** : $\mathsf{x} \in \mathbf{F} \vdash \ell' \mathsf{x}$]
[**Mac rule ElimF** : $\mathsf{x} \in \mathbf{F} \vdash \overline{\mathbf{F}}' \mathsf{x}$]
[**Mac rule IntroF** : $\ell' \mathsf{x} \vdash \overline{\mathbf{F}}' \mathsf{x} \vdash \mathsf{x} \in \mathbf{F}$]
[**Mac rule SubtypeNotFC** : $\mathsf{x} \notin \mathbf{F} \vdash \ell' \mathsf{x}$]
[**Mac rule ElimNotF** : $\mathsf{x} \notin \mathbf{F} \vdash \neg\overline{\mathbf{F}}' \mathsf{x}$]
[**Mac rule IntroNotF** : $\ell' \mathsf{x} \vdash \neg\overline{\mathbf{F}}' \mathsf{x} \vdash \mathsf{x} \notin \mathbf{F}$]

Examples

$$\begin{array}{lcl}
[\mathsf{T} \in \mathbf{F} & \equiv & \mathsf{F}]^{\cdot} \\
[\mathsf{F} \in \mathbf{F} & \equiv & \mathsf{T}]^{\cdot} \\
[117 \in \mathbf{F} & \equiv & \mathsf{F}]^{\cdot} \\
[\bullet \in \mathbf{F} & \equiv & \mathsf{F}]^{\cdot} \\
[0 :: 1 \in \mathbf{F} & \equiv & \mathsf{F}]^{\cdot} \\
[\bot \in \mathbf{F} & \equiv & \bot]^{\cdot}
\end{array}$$

$[\mathbf{F}]^{\cdot}$	the set of false maps
$[\text{SubF}]^{\cdot}$	rule SubF
$[\text{TypeNumeralNotInF}]^{\cdot}$	rule TypeNumeralNotInF
$[\text{SetF}]^{\cdot}$	rule SetF
$[\text{SubtypeFB}]^{\cdot}$	rule SubtypeFB
$[\text{SubtypeFC}]^{\cdot}$	rule SubtypeFC
$[\text{ElimF}]^{\cdot}$	rule ElimF
$[\text{IntroF}]^{\cdot}$	rule IntroF
$[\text{SubtypeNotFC}]^{\cdot}$	rule SubtypeNotFC
$[\text{ElimNotF}]^{\cdot}$	rule ElimNotF
$[\text{IntroNotF}]^{\cdot}$	rule IntroNotF

See also

$[\, \mathbf{B} \,]^{\cdot}$	Section A.42
$[\, \mathsf{x} \in \mathsf{y} \,]^{\cdot}$	Section A.50

A.34 the set of infinities of precision x end $[\mathbf{D}^{\infty}_{\mathsf{x}}]^{\cdot}$

Parse tree

$$\begin{tikzcd}
\boxed{\mathbf{D}^{\infty}} \arrow[d, no head] \\
\mathsf{x}
\end{tikzcd}$$

Sort

$[\mathbf{D}^{\infty}_{\mathsf{x}}]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\mathbf{D}^{\infty}_{\mathsf{x}}]^{\cdot}$ is the set of decimal fractions of precision $[\mathsf{x}]^{\cdot}$ whose magnitude equals $[+\infty]^{\cdot}$.

Mac rules

[**Mac rule SubIi** : $\langle \mathbf{D}^{\infty}_{\mathcal{A}} \mid \mathsf{x}{:=}\mathcal{S}\rangle \equiv \mathbf{D}^{\infty}_{\langle \mathcal{A} \mid \mathsf{x}:=\mathcal{S}\rangle}$]
[**Mac rule TypeNumeralNotInIm** : **if** x is a numeral **then** $\mathsf{m} \in \mathbf{Z}^{+} \vdash \mathsf{x} \notin \mathbf{D}^{\infty}_{\mathsf{m}}$]
[**Mac rule TypeNumeralNotInIi** : **if** x is a numeral **then** $\mathsf{x} \notin \mathbf{D}^{\infty}_{\infty}$]
[**Mac rule SetIi** : $\mathbf{D}^{\infty}_{\infty} \in \mathbf{Set}$]
[**Mac rule SetIm** : $\mathsf{m} \in \mathbf{Z}^{+} \vdash \mathbf{D}^{\infty}_{\mathsf{m}} \in \mathbf{Set}$]
[**Mac rule SubtypeIiD** : $\mathsf{x} \in \mathbf{D}^{\infty}_{\infty} \vdash \mathsf{x} \in \mathbf{D}$]
[**Mac rule SubtypeImD** : $\mathsf{m} \in \mathbf{Z}^{+} \vdash \mathsf{x} \in \mathbf{D}^{\infty}_{\mathsf{m}} \vdash \mathsf{x} \in \mathbf{D}$]

Examples

$[\infty \in \mathbf{D}^{\infty}_{\infty}$	$\equiv$	$\mathsf{T}]^{\cdot}$
$[\infty \in \mathbf{D}^{\infty}_{3}$	$\equiv$	$\mathsf{F}]^{\cdot}$
$[\infty@3 \in \mathbf{D}^{\infty}_{\infty}$	$\equiv$	$\mathsf{F}]^{\cdot}$
$[\infty@3 \in \mathbf{D}^{\infty}_{3}$	$\equiv$	$\mathsf{T}]^{\cdot}$
$[-\infty \in \mathbf{D}^{\infty}_{\infty}$	$\equiv$	$\mathsf{F}]^{\cdot}$
$[117 \in \mathbf{D}^{\infty}_{\infty}$	$\equiv$	$\mathsf{F}]^{\cdot}$
$[\mathsf{T} \in \mathbf{D}^{\infty}_{\infty}$	$\equiv$	$\mathsf{F}]^{\cdot}$
$[\bullet \in \mathbf{D}^{\infty}_{\infty}$	$\equiv$	$\mathsf{F}]^{\cdot}$
$[0 :: 1 \in \mathbf{D}^{\infty}_{\infty}$	$\equiv$	$\mathsf{F}]^{\cdot}$
$[\perp \in \mathbf{D}^{\infty}_{\infty}$	$\equiv$	$\perp]^{\cdot}$

$[\mathbf{D}^{\infty}_{\mathsf{x}}]^{\cdot}$	the set of infinities of precision x end
[SubIi]˙	rule SubIi
[TypeNumeralNotInIm]˙	rule TypeNumeralNotInIm
[TypeNumeralNotInIi]˙	rule TypeNumeralNotInIi
[SetIi]˙	rule SetIi
[SetIm]˙	rule SetIm
[SubtypeIiD]˙	rule SubtypeIiD
[SubtypeImD]˙	rule SubtypeImD

See also

$[\,\mathbf{D}\,]^{\cdot}$	Section A.30
$[\,\mathsf{x} \in \mathsf{y}\,]^{\cdot}$	Section A.50

A.35 the set of integers $[\mathbf{Z}]^{\cdot}$

Parse tree

$\boxed{\mathbf{Z}}$

Sort

$[\mathbf{Z}]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\mathbf{Z}]^{\cdot}$ is the set of integers. $[\mathsf{x} \in \mathbf{Z}]^{\cdot}$ is true if and only if $[\mathsf{x}]^{\cdot}$ is a classical map that weakly represents an integer. A number is an *integer* if it is (1) an exact decimal fractions and (2) a whole number.

Mac rules

[**Mac rule SubZ** : $\langle \mathbf{Z} \mid \mathsf{x}{:=}\mathcal{S}\rangle \equiv \mathbf{Z}$]
[**Mac rule TypeNumeralInZ** : **if** x is a numeral that represents an integer **then** $\mathsf{x} \in \mathbf{Z}$]
[**Mac rule TypeNumeralNotInZ** : **if** x is a numeral that does not represent an integer **then** $\mathsf{x} \notin \mathbf{Z}$]
[**Mac rule SetZ** : $\mathbf{Z} \in \mathbf{Set}$]
[**Mac rule SubtypeZDi** : $\mathsf{x} \in \mathbf{Z} \vdash \mathsf{x} \in \mathbf{D}^{\cdot}_{\infty}$]
[**Mac rule SubtypeZD** : $\mathsf{x} \in \mathbf{Z} \vdash \mathsf{x} \in \mathbf{D}$]

Examples

$[0 \in \mathbf{Z}$	$\equiv$	$\mathsf{T}]^{\cdot}$
$[1 \in \mathbf{Z}$	$\equiv$	$\mathsf{T}]^{\cdot}$
$[117 \in \mathbf{Z}$	$\equiv$	$\mathsf{T}]^{\cdot}$
$[-1 \in \mathbf{Z}$	$\equiv$	$\mathsf{T}]^{\cdot}$
$[1.1 \in \mathbf{Z}$	$\equiv$	$\mathsf{F}]^{\cdot}$
$[117\mathrm{F} \in \mathbf{Z}$	$\equiv$	$\mathsf{F}]^{\cdot}$
$[\mathsf{T} \in \mathbf{Z}$	$\equiv$	$\mathsf{F}]^{\cdot}$
$[\bullet \in \mathbf{Z}$	$\equiv$	$\mathsf{F}]^{\cdot}$
$[0 :: 1 \in \mathbf{Z}$	$\equiv$	$\mathsf{F}]^{\cdot}$
$[\perp \in \mathbf{Z}$	$\equiv$	$\perp]^{\cdot}$

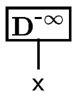
$[\mathbf{Z}]^{\cdot}$	the set of integers
	integer
[SubZ]˙	rule SubZ
[TypeNumeralInZ]˙	rule TypeNumeralInZ
[TypeNumeralNotInZ]˙	rule TypeNumeralNotInZ
[SetZ]˙	rule SetZ
[SubtypeZDi]˙	rule SubtypeZDi
[SubtypeZD]˙	rule SubtypeZD

See also

$[\,\mathbf{D}\,]^{\cdot}$	Section A.30
$[\,\mathbf{N}\,]^{\cdot}$	Section A.37
$[\,\mathbf{Z}^{-}\,]^{\cdot}$	Section A.38
$[\,\mathbf{Z}^{+}\,]^{\cdot}$	Section A.39
$[\,\mathsf{x} \in \mathsf{y}\,]^{\cdot}$	Section A.50

A.36 the set of minus infinities of precision x end $[\,\mathbf{D}^{-\infty}_{\mathsf{x}}\,]^{\cdot}$

Parse tree



Sort

$[\,\mathbf{D}^{-\infty}_{\mathsf{x}}\,]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\,\mathbf{D}^{-\infty}_{\mathsf{x}}\,]^{\cdot}$ is the set of decimal fractions of precision $[\,\mathsf{x}\,]^{\cdot}$ whose magnitude equals $[\,-\infty\,]^{\cdot}$.

Mac rules

$[\,$**Mac rule SubMm** $:\ \langle \mathbf{D}^{-\infty}_{\mathcal{A}} \mid \mathsf{x}{:=}\mathcal{S}\rangle \equiv \mathbf{D}^{-\infty}_{\langle \mathcal{A} \mid \mathsf{x}:=\mathcal{S}\rangle}\,]$
$[\,$**Mac rule TypeNumeralNotInMm** $:$ **if** x is a numeral **then** $\mathsf{m} \in \mathbf{Z}^{+} \vdash \mathsf{x} \notin \mathbf{D}^{-\infty}_{\mathsf{m}}\,]$
$[\,$**Mac rule TypeNumeralNotInMi** $:$ **if** x is a numeral **then** $\mathsf{x} \notin \mathbf{D}^{-\infty}_{\infty}\,]$
$[\,$**Mac rule SetMi** $:\ \mathbf{D}^{-\infty}_{\infty} \in \mathbf{Set}\,]$
$[\,$**Mac rule SetMm** $:\ \mathsf{m} \in \mathbf{Z}^{+} \vdash \mathbf{D}^{-\infty}_{\mathsf{m}} \in \mathbf{Set}\,]$
$[\,$**Mac rule SubtypeMiD** $:\ \mathsf{x} \in \mathbf{D}^{-\infty}_{\infty} \vdash \mathsf{x} \in \mathbf{D}\,]$
$[\,$**Mac rule SubtypeMmD** $:\ \mathsf{m} \in \mathbf{Z}^{+} \vdash \mathsf{x} \in \mathbf{D}^{-\infty}_{\mathsf{m}} \vdash \mathsf{x} \in \mathbf{D}\,]$

$[\,\mathbf{D}^{-\infty}_{\mathsf{x}}\,]^{\cdot}$	the set of minus infinities of precision x end
$[$ SubMm $]^{\cdot}$	rule SubMm
$[$ TypeNumeralNotInMm $]^{\cdot}$	rule TypeNumeralNotInMm
$[$ TypeNumeralNotInMi $]^{\cdot}$	rule TypeNumeralNotInMi
$[$ SetMi $]^{\cdot}$	rule SetMi
$[$ SetMm $]^{\cdot}$	rule SetMm
$[$ SubtypeMiD $]^{\cdot}$	rule SubtypeMiD
$[$ SubtypeMmD $]^{\cdot}$	rule SubtypeMmD

Examples

$[\, -\infty \in \mathbf{D}^{-\infty}_{\infty}$	$\equiv$	$\mathsf{T}\,]^{\cdot}$
$[\, -\infty \in \mathbf{D}^{-\infty}_{3}$	$\equiv$	$\mathsf{F}\,]^{\cdot}$
$[\, -\infty@3 \in \mathbf{D}^{-\infty}_{\infty}$	$\equiv$	$\mathsf{F}\,]^{\cdot}$
$[\, -\infty@3 \in \mathbf{D}^{-\infty}_{3}$	$\equiv$	$\mathsf{T}\,]^{\cdot}$
$[\, \infty \in \mathbf{D}^{-\infty}_{\infty}$	$\equiv$	$\mathsf{F}\,]^{\cdot}$
$[\, 117 \in \mathbf{D}^{-\infty}_{\infty}$	$\equiv$	$\mathsf{F}\,]^{\cdot}$
$[\, \mathsf{T} \in \mathbf{D}^{-\infty}_{\infty}$	$\equiv$	$\mathsf{F}\,]^{\cdot}$
$[\, \bullet \in \mathbf{D}^{-\infty}_{\infty}$	$\equiv$	$\mathsf{F}\,]^{\cdot}$
$[\, 0 :: 1 \in \mathbf{D}^{-\infty}_{\infty}$	$\equiv$	$\mathsf{F}\,]^{\cdot}$
$[\, \perp \in \mathbf{D}^{-\infty}_{\infty}$	$\equiv$	$\perp\,]^{\cdot}$

See also

$[\, \mathbf{D} \,]^{\cdot}$	Section A.30
$[\, \mathsf{x} \in \mathsf{y} \,]^{\cdot}$	Section A.50

A.37 the set of natural numbers [N]˙

Parse tree

$\boxed{\mathrm{N}}$

Sort

[**N**]˙ is an operator (i.e. it may occur in the syntax tree of a term).

Description

[**N**]˙ is the set of natural numbers. [$\mathsf{x} \in \mathbf{N}$]˙ is true if and only if [x]˙ is a classical map that weakly represents a natural number. A number is *natural* if it is (1) an exact decimal fractions and (2) a whole number and (3) non-negative.

Mac rules

[**Mac rule SubN** : $\langle \mathbf{N} \mid \mathsf{x}{:=}\mathcal{S} \rangle \equiv \mathbf{N}$]
[**Mac rule TypeNumeralInN** : **if** x is a numeral that represents a natural number **then** $\mathsf{x} \in \mathbf{N}$]
[**Mac rule TypeNumeralNotInN** : **if** x is a numeral that does not represent a natural number **then** $\mathsf{x} \notin \mathbf{N}$]
[**Mac rule SetN** : $\mathbf{N} \in \mathbf{Set}$]
[**Mac rule SubtypeNZ** : $\mathsf{x} \in \mathbf{N} \vdash \mathsf{x} \in \mathbf{Z}$]
[**Mac rule PeanoA** : $0 \in \mathbf{N}$]
[**Mac rule PeanoB** : $\mathsf{x} \in \mathbf{N} \vdash \mathsf{x}^{+} \in \mathbf{N}$]
[**Mac rule PeanoC** : $\mathsf{x} \in \mathbf{N} \vdash \mathsf{y} \in \mathbf{N} \vdash \mathsf{x}^{+} = \mathsf{y}^{+} \vdash \mathsf{x} = \mathsf{y}$]
[**Mac rule PeanoD** : $\mathsf{x} \in \mathbf{N} \vdash (\mathsf{x} = 0) \in \mathbf{F} \vdash \mathsf{x}^{-} \in \mathbf{N}$]
[**Mac rule PeanoD'** : $\mathsf{x} \in \mathbf{N} \vdash (\mathsf{x} = 0) \in \mathbf{F} \vdash \mathsf{x} - 1 \in \mathbf{N}$]
[**Mac rule PeanoE** : $\mathsf{x} \in \mathbf{N} \vdash (\mathsf{x} = 0) \in \mathbf{F} \vdash \mathsf{x}^{-+} = \mathsf{x}$]
[**Mac rule Induction** :
$\mathsf{n} \in \mathbf{N} \to (\mathsf{n} = 0) \in \mathbf{T} \to \mathcal{A} \vdash$
$\mathsf{n} \in \mathbf{N} \to (\mathsf{n} = 0) \in \mathbf{F} \to \langle \mathcal{A} \mid \mathsf{n}{:=}\mathsf{n} - 1 \rangle \to \mathcal{A} \vdash$
$\mathsf{n} \in \mathbf{N} \to \mathcal{A}$]

[**N**]˙	the set of natural numbers
	natural number
[SubN]˙	rule SubN
[TypeNumeralInN]˙	rule TypeNumeralInN
[TypeNumeralNotInN]˙	rule TypeNumeralNotInN
[SetN]˙	rule SetN
[SubtypeNZ]˙	rule SubtypeNZ
[PeanoA]˙	rule PeanoA
[PeanoB]˙	rule PeanoB
[PeanoC]˙	rule PeanoC
[PeanoD]˙	rule PeanoD
[PeanoD']˙	rule PeanoD'
[PeanoE]˙	rule PeanoE
[Induction]˙	rule Induction

Examples

$[\, 0 \in \mathbf{N}$	$\equiv$	$\mathsf{T}\,]^{\cdot}$
$[\, 1 \in \mathbf{N}$	$\equiv$	$\mathsf{T}\,]^{\cdot}$
$[\, 117 \in \mathbf{N}$	$\equiv$	$\mathsf{T}\,]^{\cdot}$
$[\, -1 \in \mathbf{N}$	$\equiv$	$\mathsf{F}\,]^{\cdot}$
$[\, 1.1 \in \mathbf{Z}$	$\equiv$	$\mathsf{F}\,]^{\cdot}$
$[\, 117\mathrm{F} \in \mathbf{N}$	$\equiv$	$\mathsf{F}\,]^{\cdot}$
$[\, \mathsf{T} \in \mathbf{N}$	$\equiv$	$\mathsf{F}\,]^{\cdot}$
$[\, \bullet \in \mathbf{N}$	$\equiv$	$\mathsf{F}\,]^{\cdot}$
$[\, 0 :: 1 \in \mathbf{N}$	$\equiv$	$\mathsf{F}\,]^{\cdot}$
$[\, \perp \in \mathbf{N}$	$\equiv$	$\perp\,]^{\cdot}$

See also

$[\, \mathbf{Z}\,]^{\cdot}$	Section A.35
$[\, \mathsf{x} \in \mathsf{y}\,]^{\cdot}$	Section A.50

A.38 the set of negative integers $[\mathbf{Z}^-]^{\cdot}$

Parse tree

$\boxed{\mathbf{Z}^-}$

Sort

$[\mathbf{Z}^-]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\mathbf{Z}^-]^{\cdot}$ is the set of negative integers. $[\mathsf{x} \in \mathbf{Z}^-]^{\cdot}$ is true if and only if $[\mathsf{x}]^{\cdot}$ is a classical map that weakly represents a negative integer. A number is a *negative integer* if it is (1) an exact decimal fractions and (2) a whole number and (3) negative (i.e. strongly less than zero).

Mac rules

[**Mac rule SubZm** : $\langle \mathbf{Z}^- \mid \mathsf{x}{:=}\mathcal{S} \rangle \equiv \mathbf{Z}^-$]
[**Mac rule TypeNumeralInZm** : **if** x is a numeral that represents a negative integer **then** $\mathsf{x} \in \mathbf{Z}^-$]
[**Mac rule TypeNumeralNotInZm** : **if** x is a numeral that does not represent a negative integer **then** $\mathsf{x} \notin \mathbf{Z}^-$]
[**Mac rule SetZm** : $\mathbf{Z}^- \in \mathbf{Set}$]
[**Mac rule SubtypeZmZ** : $\mathsf{x} \in \mathbf{Z}^- \vdash \mathsf{x} \in \mathbf{Z}$]

Examples

$[0 \in \mathbf{Z}^-$	$\equiv$	$\mathsf{F}]^{\cdot}$
$[1 \in \mathbf{Z}^-$	$\equiv$	$\mathsf{F}]^{\cdot}$
$[-117 \in \mathbf{Z}^-$	$\equiv$	$\mathsf{T}]^{\cdot}$
$[-1 \in \mathbf{Z}^-$	$\equiv$	$\mathsf{T}]^{\cdot}$
$[-1.1 \in \mathbf{Z}^-$	$\equiv$	$\mathsf{F}]^{\cdot}$
$[-117\mathrm{F} \in \mathbf{Z}^-$	$\equiv$	$\mathsf{F}]^{\cdot}$
$[\mathsf{T} \in \mathbf{Z}^-$	$\equiv$	$\mathsf{F}]^{\cdot}$
$[\bullet \in \mathbf{Z}^-$	$\equiv$	$\mathsf{F}]^{\cdot}$
$[0 :: 1 \in \mathbf{Z}^-$	$\equiv$	$\mathsf{F}]^{\cdot}$
$[\bot \in \mathbf{Z}^-$	$\equiv$	$\bot]^{\cdot}$

$[\mathbf{Z}^-]^{\cdot}$	the set of negative integers
	negative integer
[SubZm]˙	rule SubZm
[TypeNumeralInZm]˙	rule TypeNumeralInZm
[TypeNumeralNotInZm]˙	rule TypeNumeralNotInZm
[SetZm]˙	rule SetZm
[SubtypeZmZ]˙	rule SubtypeZmZ

See also

$[\,\mathbf{Z}\,]^{\cdot}$	Section A.35
$[\,\mathsf{x} \in \mathsf{y}\,]^{\cdot}$	Section A.50

A.39 the set of positive integers $[\,\mathbf{Z}^+\,]^{\cdot}$

Parse tree

$\boxed{\mathbf{Z}^+}$

Sort

$[\,\mathbf{Z}^+\,]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\,\mathbf{Z}^+\,]^{\cdot}$ is the set of positive integers. $[\,\mathsf{x} \in \mathbf{Z}^+\,]^{\cdot}$ is true if and only if $[\,\mathsf{x}\,]^{\cdot}$ is a classical map that weakly represents a positive integer. A number is a *positive integer* if it is (1) an exact decimal fractions and (2) a whole number and (3) positive (i.e. strongly greater than zero).

Mac rules

[**Mac rule SubZp** : $\langle \mathbf{Z}^+ \mid \mathsf{x}{:=}\mathcal{S}\rangle \equiv \mathbf{Z}^+$]
[**Mac rule TypeNumeralInZp** : **if** x is a numeral that represents a positive integer **then** $\mathsf{x} \in \mathbf{Z}^+$]
[**Mac rule TypeNumeralNotInZp** : **if** x is a numeral that does not represent a positive integer **then** $\mathsf{x} \notin \mathbf{Z}^+$]
[**Mac rule SetZp** : $\mathbf{Z}^+ \in \mathbf{Set}$]
[**Mac rule SubtypeZpN** : $\mathsf{x} \in \mathbf{Z}^+ \vdash \mathsf{x} \in \mathbf{N}$]
[**Mac rule SubtypeZpZ** : $\mathsf{x} \in \mathbf{Z}^+ \vdash \mathsf{x} \in \mathbf{Z}$]

Examples

$[\,0 \in \mathbf{Z}^+$	$\equiv$	$\mathsf{F}\,]^{\cdot}$
$[\,1 \in \mathbf{Z}^+$	$\equiv$	$\mathsf{T}\,]^{\cdot}$
$[\,117 \in \mathbf{Z}^+$	$\equiv$	$\mathsf{T}\,]^{\cdot}$
$[\,-1 \in \mathbf{Z}^+$	$\equiv$	$\mathsf{F}\,]^{\cdot}$
$[\,1.1 \in \mathbf{Z}^+$	$\equiv$	$\mathsf{F}\,]^{\cdot}$
$[\,117\mathrm{F} \in \mathbf{Z}^+$	$\equiv$	$\mathsf{F}\,]^{\cdot}$
$[\,\mathsf{T} \in \mathbf{Z}^+$	$\equiv$	$\mathsf{F}\,]^{\cdot}$
$[\,\bullet \in \mathbf{Z}^+$	$\equiv$	$\mathsf{F}\,]^{\cdot}$
$[\,0 :: 1 \in \mathbf{Z}^+$	$\equiv$	$\mathsf{F}\,]^{\cdot}$
$[\,\perp \in \mathbf{Z}^+$	$\equiv$	$\perp\,]^{\cdot}$

$[\,\mathbf{Z}^+\,]^{\cdot}$	the set of positive integers
	positive integer
[SubZp]˙	rule SubZp
[TypeNumeralInZp]˙	rule TypeNumeralInZp
[TypeNumeralNotInZp]˙	rule TypeNumeralNotInZp
[SetZp]˙	rule SetZp
[SubtypeZpN]˙	rule SubtypeZpN
[SubtypeZpZ]˙	rule SubtypeZpZ

See also

$[\,\mathbf{Z}\,]^{\cdot}$	Section A.35
$[\,\mathsf{x} \in \mathsf{y}\,]^{\cdot}$	Section A.50

A.40 the set of the empty list $[\,\mathbf{E}\,]^{\cdot}$

Parse tree

$\boxed{\mathbf{E}}$

Sort

$[\,\mathbf{E}\,]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\,\mathbf{E}\,]^{\cdot}$ is the set of classical maps that weakly represent the empty list.

Mac rules

$[\,$**Mac rule SubE** $:\ \langle \mathbf{E} \mid \mathsf{x}{:=}\mathcal{S}\rangle \equiv \mathbf{E}\,]$
$[\,$**Mac rule TypeNumeralNotInE** $:$ **if** x is a numeral **then** $\mathsf{x} \notin \mathbf{E}\,]$
$[\,$**Mac rule SetE** $:\ \mathbf{E} \in \mathbf{Set}\,]$
$[\,$**Mac rule SubtypeEC** $:\ \mathsf{x} \in \mathbf{E} \vdash \ell'\mathsf{x}\,]$
$[\,$**Mac rule ElimE** $:\ \mathsf{x} \in \mathbf{E} \vdash \overline{\mathbf{E}}'\mathsf{x}\,]$
$[\,$**Mac rule IntroE** $:\ \ell'\mathsf{x} \vdash \overline{\mathbf{E}}'\mathsf{x} \vdash \mathsf{x} \in \mathbf{E}\,]$
$[\,$**Mac rule SubtypeNotEC** $:\ \mathsf{x} \notin \mathbf{E} \vdash \ell'\mathsf{x}\,]$
$[\,$**Mac rule ElimNotE** $:\ \mathsf{x} \notin \mathbf{E} \vdash \neg\overline{\mathbf{E}}'\mathsf{x}\,]$
$[\,$**Mac rule IntroNotE** $:\ \ell'\mathsf{x} \vdash \neg\overline{\mathbf{E}}'\mathsf{x} \vdash \mathsf{x} \notin \mathbf{E}\,]$

Examples

$$\begin{array}{lcl}
[\,\langle\rangle \in \mathbf{E} & \equiv & \mathsf{T}\,]^{\cdot} \\
[\,117 \in \mathbf{E} & \equiv & \mathsf{F}\,]^{\cdot} \\
[\,\bullet \in \mathbf{E} & \equiv & \mathsf{F}\,]^{\cdot} \\
[\,0 :: 1 \in \mathbf{E} & \equiv & \mathsf{F}\,]^{\cdot} \\
[\,\bot \in \mathbf{E} & \equiv & \bot\,]^{\cdot}
\end{array}$$

See also

$[\,\mathsf{x} \in \mathsf{y}\,]^{\cdot}$	Section A.50
$[\,\mathsf{x} \times \mathsf{y}\,]^{\cdot}$	Section A.51

$[\,\mathbf{E}\,]^{\cdot}$	the set of the empty list
[SubE]˙	rule SubE
[TypeNumeralNotInE]˙	rule TypeNumeralNotInE
[SetE]˙	rule SetE
[SubtypeEC]˙	rule SubtypeEC
[ElimE]˙	rule ElimE
[IntroE]˙	rule IntroE
[SubtypeNotEC]˙	rule SubtypeNotEC
[ElimNotE]˙	rule ElimNotE
[IntroNotE]˙	rule IntroNotE

A.41 the set of true maps $[\,\mathbf{T}\,]^{\cdot}$

Parse tree

$\boxed{\mathbf{T}}$

Sort

$[\,\mathbf{T}\,]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\,\mathbf{T}\,]^{\cdot}$ is the set of classical maps that weakly represent truth. $[\,\mathsf{x} \in \mathbf{T}\,]^{\cdot}$ is true if and only if $[\,\mathsf{x} \equiv \mathsf{T}\,]^{\cdot}$.

Mac rules

$[\,$**Mac rule SubT** $: \langle \mathbf{T} \mid \mathsf{x}{:=}\mathcal{S} \rangle \equiv \mathbf{T}\,]$
$[\,$**Mac rule TypeNumeralNotInT** $:$ **if** x is a numeral **then** $\mathsf{x} \notin \mathbf{T}\,]$
$[\,$**Mac rule SetT** $: \mathbf{T} \in \mathbf{Set}\,]$
$[\,$**Mac rule SubtypeTB** $: \mathsf{x} \in \mathbf{T} \vdash \mathsf{x} \in \mathbf{B}\,]$
$[\,$**Mac rule SubtypeTC** $: \mathsf{x} \in \mathbf{T} \vdash \ell'\mathsf{x}\,]$
$[\,$**Mac rule ElimT** $: \mathsf{x} \in \mathbf{T} \vdash \overline{\mathbf{T}}'\mathsf{x}\,]$
$[\,$**Mac rule IntroT** $: \ell'\mathsf{x} \vdash \overline{\mathbf{T}}'\mathsf{x} \vdash \mathsf{x} \in \mathbf{T}\,]$
$[\,$**Mac rule ElimTT** $: \mathsf{x} \in \mathbf{T} \vdash \mathsf{x}\,]$
$[\,$**Mac rule IntroTT** $: \mathsf{x} \vdash \mathsf{x} \in \mathbf{T}\,]$
$[\,$**Mac rule SubtypeNotTC** $: \mathsf{x} \notin \mathbf{T} \vdash \ell'\mathsf{x}\,]$
$[\,$**Mac rule ElimNotT** $: \mathsf{x} \notin \mathbf{T} \vdash \neg\overline{\mathbf{T}}'\mathsf{x}\,]$
$[\,$**Mac rule IntroNotT** $: \ell'\mathsf{x} \vdash \neg\overline{\mathbf{T}}'\mathsf{x} \vdash \mathsf{x} \notin \mathbf{T}\,]$

$[\,\mathbf{T}\,]^{\cdot}$	the set of true maps
$[$ SubT $]^{\cdot}$	rule SubT
$[$ TypeNumeralNotInT $]^{\cdot}$	rule TypeNumeralNotInT
$[$ SetT $]^{\cdot}$	rule SetT
$[$ SubtypeTB $]^{\cdot}$	rule SubtypeTB
$[$ SubtypeTC $]^{\cdot}$	rule SubtypeTC
$[$ ElimT $]^{\cdot}$	rule ElimT
$[$ IntroT $]^{\cdot}$	rule IntroT
$[$ ElimTT $]^{\cdot}$	rule ElimTT
$[$ IntroTT $]^{\cdot}$	rule IntroTT
$[$ SubtypeNotTC $]^{\cdot}$	rule SubtypeNotTC
$[$ ElimNotT $]^{\cdot}$	rule ElimNotT
$[$ IntroNotT $]^{\cdot}$	rule IntroNotT

Examples

$$\begin{array}{lll}
[\,\mathsf{T} \in \mathbf{T} & \equiv & \mathsf{T}\,]^{\cdot} \\
[\,\mathsf{F} \in \mathbf{T} & \equiv & \mathsf{F}\,]^{\cdot} \\
[\,117 \in \mathbf{T} & \equiv & \mathsf{F}\,]^{\cdot} \\
[\,\bullet \in \mathbf{T} & \equiv & \mathsf{F}\,]^{\cdot} \\
[\,0 :: 1 \in \mathbf{T} & \equiv & \mathsf{F}\,]^{\cdot} \\
[\,\perp \in \mathbf{T} & \equiv & \perp\,]^{\cdot}
\end{array}$$

See also

$[\,\mathbf{B}\,]^{\cdot}$	Section A.42
$[\,\mathsf{x} \in \mathsf{y}\,]^{\cdot}$	Section A.50

A.42 the set of truth values $[\mathbf{B}]^{\cdot}$

Parse tree

$\boxed{\mathbf{B}}$

Sort

$[\mathbf{B}]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\mathbf{B}]^{\cdot}$ is the set of truth values. $[\mathsf{x} \in \mathbf{B}]^{\cdot}$ is true if and only if $[\mathsf{x}]^{\cdot}$ is a classical map that weakly represents truth or weakly represents falsehood.

Mac rules

[**Mac rule SubB** : $\langle \mathbf{B} \mid \mathsf{x}{:=}\mathcal{S}\rangle \equiv \mathbf{B}$]
[**Mac rule TypeNumeralNotInB** : **if** x is a numeral **then** $\mathsf{x} \notin \mathbf{B}$]
[**Mac rule SetB** : $\mathbf{B} \in \mathbf{Set}$]
[**Mac rule Cases** : $\mathsf{x} \in \mathbf{T} \to \mathcal{A} \equiv \mathcal{B} \vdash \mathsf{x} \in \mathbf{F} \to \mathcal{A} \equiv \mathcal{B} \vdash \mathsf{x} \in \mathbf{B} \to \mathcal{A} \equiv \mathcal{B}$]
[**Mac rule Tautology** : **if** x is a tautology **then** x]
[**Mac rule SubtypeBC** : $\mathsf{x} \in \mathbf{B} \vdash \ell'\mathsf{x}$]
[**Mac rule ElimB** : $\mathsf{x} \in \mathbf{B} \vdash \overline{\mathbf{B}}'\mathsf{x}$]
[**Mac rule IntroB** : $\ell'\mathsf{x} \vdash \overline{\mathbf{B}}'\mathsf{x} \vdash \mathsf{x} \in \mathbf{B}$]
[**Mac rule SubtypeNotBC** : $\mathsf{x} \notin \mathbf{B} \vdash \ell'\mathsf{x}$]
[**Mac rule ElimNotB** : $\mathsf{x} \notin \mathbf{B} \vdash \neg\overline{\mathbf{B}}'\mathsf{x}$]
[**Mac rule IntroNotB** : $\ell'\mathsf{x} \vdash \neg\overline{\mathbf{B}}'\mathsf{x} \vdash \mathsf{x} \notin \mathbf{B}$]

Examples

$$\begin{array}{lcl}
[\mathsf{T} \in \mathbf{B} & \equiv & \mathsf{T}]^{\cdot} \\
[\mathsf{F} \in \mathbf{B} & \equiv & \mathsf{T}]^{\cdot} \\
[117 \in \mathbf{B} & \equiv & \mathsf{F}]^{\cdot} \\
[\bullet \in \mathbf{B} & \equiv & \mathsf{F}]^{\cdot} \\
[0 :: 1 \in \mathbf{B} & \equiv & \mathsf{F}]^{\cdot} \\
[\bot \in \mathbf{B} & \equiv & \bot]^{\cdot}
\end{array}$$

$[\mathbf{B}]^{\cdot}$	the set of truth values
[SubB]˙	rule SubB
[TypeNumeralNotInB]˙	rule TypeNumeralNotInB
[SetB]˙	rule SetB
[Cases]˙	rule Cases
[Tautology]˙	rule Tautology
[SubtypeBC]˙	rule SubtypeBC
[ElimB]˙	rule ElimB
[IntroB]˙	rule IntroB
[SubtypeNotBC]˙	rule SubtypeNotBC
[ElimNotB]˙	rule ElimNotB
[IntroNotB]˙	rule IntroNotB

See also

$[\,\mathbf{F}\,]^{\cdot}$	Section A.33
$[\,\mathbf{T}\,]^{\cdot}$	Section A.41
$[\,\mathsf{x} \in \mathsf{y}\,]^{\cdot}$	Section A.50

A.43 true $[\,\mathsf{T}\,]^{\cdot}$

Parse tree

$\boxed{\mathsf{T}}$

Sort

$[\,\mathsf{T}\,]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\,\mathsf{T}\,]^{\cdot}$ denotes truth. If a computer can determine within finite time that a term is true, then the value of the term is $[\,\mathsf{T}\,]^{\cdot}$. If Map sees a term in square brackets (such as $[\,2+2=4\,]$) then Map computes the value of the term and protests if the value differs from $[\,\mathsf{T}\,]^{\cdot}$. If a term $[\,\mathcal{A}\,]^{\cdot}$ is stated as a lemma, then it is understood that the lemma says $[\,\mathcal{A} \equiv \mathsf{T}\,]^{\cdot}$.

Mac rules

[**Mac rule SubTrue** : $\langle \mathsf{T} \mid \mathsf{x}{:=}\mathcal{S} \rangle \equiv \mathsf{T}$]
[**Mac rule TypeTInB** : $\mathsf{T} \in \mathbf{B}$]
[**Mac rule TypeTInT** : $\mathsf{T} \in \mathbf{T}$]

Examples

$$[\,2+2=4 \quad \equiv \quad \mathsf{T}\,]^{\cdot}$$
$$[\,2 \neq \mathsf{F} \quad \equiv \quad \mathsf{T}\,]^{\cdot}$$

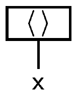
See also

$[\,\mathsf{F}\,]^{\cdot}$	Section A.13
$[\,\mathsf{if}(\mathsf{x},\mathsf{y},\mathsf{z})\,]^{\cdot}$	Section A.18

$[\,\mathsf{T}\,]^{\cdot}$	true
[SubTrue $]^{\cdot}$	rule SubTrue
[TypeTInB $]^{\cdot}$	rule TypeTInB
[TypeTInT $]^{\cdot}$	rule TypeTInT

A.44 tuple x end $[\,\langle \mathsf{x}\rangle\,]^{\cdot}$

Parse tree



Sort

$[\,\langle \mathsf{x}\rangle\,]^{\cdot}$ is a macro.

Description

$[\,\langle \mathsf{x}\rangle\,]^{\cdot}$ denotes a one-element tuple that has $[\,\mathsf{x}\,]^{\cdot}$ as its sole element. The term reduction rule $\left[\,\langle \mathsf{x},\mathsf{y}\rangle \overset{\circ}{\rightarrow} \mathsf{x} :: \langle \mathsf{y}\rangle\,\right]$ allows $[\,\langle \mathsf{x}\rangle\,]^{\cdot}$ to be combined with commas to form tuples of arbitrary length.

Definition

$[\,\langle \mathsf{x}\rangle \ddot{=} \mathsf{x} :: \langle\,\rangle\,]$.

Examples

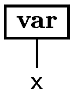
$[\,\langle 1\rangle$	$\equiv$	$1 :: \langle\,\rangle\,]^{\cdot}$
$[\,\langle 1,2\rangle$	$\equiv$	$1 :: 2 :: \langle\,\rangle\,]^{\cdot}$
$[\,\langle 1,2,3\rangle$	$\equiv$	$1 :: 2 :: 3 :: \langle\,\rangle\,]^{\cdot}$
$[\,\langle 1,2,3,4\rangle$	$\equiv$	$1 :: 2 :: 3 :: 4 :: \langle\,\rangle\,]^{\cdot}$
$[\,\langle 1,2,3,4\rangle$ head	$\equiv$	$1\,]^{\cdot}$
$[\,\langle 1,2,3,4\rangle$ tail	$\equiv$	$\langle 2,3,4\rangle\,]^{\cdot}$
$[\,\langle 1,2,3,4\rangle$ pair	$\equiv$	$\mathsf{T}\,]^{\cdot}$
$[\,\langle 1,2,3,4\rangle$ atom	$\equiv$	$\mathsf{F}\,]^{\cdot}$

See also

$[\,\langle\,\rangle\,]^{\cdot}$	Section A.10
$[\,\mathsf{x}$ atom $]^{\cdot}$	Section A.49
$[\,\mathsf{x}$ head $]^{\cdot}$	Section A.64
$[\,\mathsf{x}$ pair $]^{\cdot}$	Section A.78
$[\,\mathsf{x} :: \mathsf{y}\,]^{\cdot}$	Section A.79
$[\,\mathsf{x}$ tail $]^{\cdot}$	Section A.95

A.45 variable x [**variable** x]˙

Parse tree



Sort

[**variable** x]˙ is a directive (i.e. it cannot occur in the syntax tree of a term).

Description

[**variable** $\mathcal{A}$]˙ declares the construct [$\mathcal{A}$]˙ to be a variable. Any construct introduced by [**variable** $\mathcal{A}$]˙ becomes a null-ary outfix operator. A statement like [**variable** [L[n]]] introduces infinitely many variables in one go (c.f. Section 11.6).

Examples

[**variable** a]	[**variable** n]	[**variable** $\mathcal{A}$]	[**variable** $\mathcal{N}$]
[**variable** b]	[**variable** o]	[**variable** $\mathcal{B}$]	[**variable** $\mathcal{O}$]
[**variable** c]	[**variable** p]	[**variable** $\mathcal{C}$]	[**variable** $\mathcal{P}$]
[**variable** d]	[**variable** q]	[**variable** $\mathcal{D}$]	[**variable** $\mathcal{Q}$]
[**variable** e]	[**variable** r]	[**variable** $\mathcal{E}$]	[**variable** $\mathcal{R}$]
[**variable** f]	[**variable** s]	[**variable** $\mathcal{F}$]	[**variable** $\mathcal{S}$]
[**variable** g]	[**variable** t]	[**variable** $\mathcal{G}$]	[**variable** $\mathcal{T}$]
[**variable** h]	[**variable** u]	[**variable** $\mathcal{H}$]	[**variable** $\mathcal{U}$]
[**variable** i]	[**variable** v]	[**variable** $\mathcal{I}$]	[**variable** $\mathcal{V}$]
[**variable** j]	[**variable** w]	[**variable** $\mathcal{J}$]	[**variable** $\mathcal{W}$]
[**variable** k]	[**variable** x]	[**variable** $\mathcal{K}$]	[**variable** $\mathcal{X}$]
[**variable** l]	[**variable** y]	[**variable** $\mathcal{L}$]	[**variable** $\mathcal{Y}$]
[**variable** m]	[**variable** z]	[**variable** $\mathcal{M}$]	[**variable** $\mathcal{Z}$]
[**variable** $*$]			

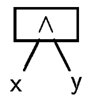
See also

[**construct** x]˙	Section A.9
[x $\doteq$ y]˙	Section A.58

[**variable** x]˙ variable x

A.46 x and y $[\,\mathsf{x} \wedge \mathsf{y}\,]^{\cdot}$

Parse tree



Sort

$[\,\mathsf{x} \wedge \mathsf{y}\,]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\,\mathsf{x} \wedge \mathsf{y}\,]^{\cdot}$ denotes the minimum of $[\,\mathsf{x}\,]^{\cdot}$ and $[\,\mathsf{y}\,]^{\cdot}$. In particular, for truth values $[\,\mathsf{x}\,]^{\cdot}$ and $[\,\mathsf{y}\,]^{\cdot}$, $[\,\mathsf{x} \wedge \mathsf{y}\,]^{\cdot}$ denotes "$[\,\mathsf{x}\,]^{\cdot}$ and $[\,\mathsf{y}\,]^{\cdot}$". For truth values $[\,\mathsf{x}\,]^{\cdot}$ and $[\,\mathsf{y}\,]^{\cdot}$, $[\,\mathsf{x} \wedge \mathsf{y}\,]^{\cdot}$ coincides with $[\,\mathsf{x} \cdot \mathsf{y}\,]^{\cdot}$. $[\,\mathsf{x} \wedge \mathsf{y} \equiv \bullet\,]^{\cdot}$ if $[\,\mathsf{x}\,]^{\cdot}$ and $[\,\mathsf{y}\,]^{\cdot}$ are decimal fractions of different precision.

Type table

$\wedge$	**B**	**D**	**X**	**E**	$\wedge$	**N**	**Z**	$\mathbf{Z}^+$	$\mathbf{Z}^-$
B	**B**	**X**	**X**	**X**	**N**	**N**	**Z**	**N**	$\mathbf{Z}^-$
D	**X**	–	**X**	**X**	**Z**	**Z**	**Z**	**Z**	$\mathbf{Z}^-$
X	**X**	**X**	**X**	**X**	$\mathbf{Z}^+$	**N**	**Z**	$\mathbf{Z}^+$	$\mathbf{Z}^-$
E	**X**	**X**	**X**	**X**	$\mathbf{Z}^-$	$\mathbf{Z}^-$	$\mathbf{Z}^-$	$\mathbf{Z}^-$	$\mathbf{Z}^-$
$\wedge$	$\mathbf{D}_m^{\infty}$	$\mathbf{D}_m^{\cdot}$	$\mathbf{D}_m^{-\infty}$	$\mathbf{D}_{\infty}^{\infty}$	$\mathbf{D}_{\infty}^{\cdot}$	$\mathbf{D}_{\infty}^{-\infty}$	$\mathbf{D}_n^{\infty}$	$\mathbf{D}_n^{\cdot}$	$\mathbf{D}_n^{-\infty}$
$\mathbf{D}_m^{\infty}$	$\mathbf{D}_m^{\infty}$	$\mathbf{D}_m^{\cdot}$	$\mathbf{D}_m^{-\infty}$	**X**	**X**	**X**	**X**	**X**	**X**
$\mathbf{D}_m^{\cdot}$	$\mathbf{D}_m^{\cdot}$	$\mathbf{D}_m^{\cdot}$	$\mathbf{D}_m^{-\infty}$	**X**	**X**	**X**	**X**	**X**	**X**
$\mathbf{D}_m^{-\infty}$	$\mathbf{D}_m^{-\infty}$	$\mathbf{D}_m^{-\infty}$	$\mathbf{D}_m^{-\infty}$	**X**	**X**	**X**	**X**	**X**	**X**
$\mathbf{D}_{\infty}^{\infty}$	**X**	**X**	**X**	$\mathbf{D}_{\infty}^{\infty}$	$\mathbf{D}_{\infty}^{\cdot}$	$\mathbf{D}_{\infty}^{-\infty}$	$\wedge$	**T**	**F**
$\mathbf{D}_{\infty}^{\cdot}$	**X**	**X**	**X**	$\mathbf{D}_{\infty}^{\cdot}$	$\mathbf{D}_{\infty}^{\cdot}$	$\mathbf{D}_{\infty}^{-\infty}$	**T**	**T**	**F**
$\mathbf{D}_{\infty}^{-\infty}$	**X**	**X**	**X**	$\mathbf{D}_{\infty}^{-\infty}$	$\mathbf{D}_{\infty}^{-\infty}$	$\mathbf{D}_{\infty}^{-\infty}$	**F**	**F**	**F**

Mac rules

[**Mac rule StrictAndX** : $\perp \wedge \mathsf{x} \equiv \perp$]
[**Mac rule XAndStrict** : $\mathsf{x} \wedge \perp \equiv \perp$]
[**Mac rule SubXAndY** : $\langle \mathcal{A} \wedge \mathcal{B} \mid \mathsf{x} {:=} \mathcal{S} \rangle \equiv \langle \mathcal{A} \mid \mathsf{x} {:=} \mathcal{S} \rangle \wedge \langle \mathcal{B} \mid \mathsf{x} {:=} \mathcal{S} \rangle$]

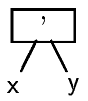
$[\,\mathsf{x} \wedge \mathsf{y}\,]^{\cdot}$	x and y
[StrictAndX $]^{\cdot}$	rule StrictAndX
[XAndStrict $]^{\cdot}$	rule XAndStrict
[SubXAndY $]^{\cdot}$	rule SubXAndY

See also

$[\,\mathrm{min}(\mathsf{x},\mathsf{y})\,]^{\cdot}$	Section A.21
$[\,\mathsf{x} \vee \mathsf{y}\,]^{\cdot}$	Section A.77
$[\,\mathsf{x} \cdot \mathsf{y}\,]^{\cdot}$	Section A.97

A.47 x apply y [x ’ y]˙

Parse tree



Sort

[x ’ y]˙ is an operator (i.e. it may occur in the syntax tree of a term).

Description

[x ’ y]˙ denotes the function [x]˙ applied to the argument [y]˙. [x ’ y]˙ is one of the fundamental constructs of map theory.

Mac rules

[**Mac rule SubXApplyY** : $\langle \mathcal{A}\text{ ’ }\mathcal{B} \mid \mathsf{x}{:=}\mathcal{S}\rangle \equiv \langle \mathcal{A} \mid \mathsf{x}{:=}\mathcal{S}\rangle\text{ ’ }\langle \mathcal{B} \mid \mathsf{x}{:=}\mathcal{S}\rangle$]
[**Mac rule ApplyNil** : $\mathsf{N}\text{ ’ }\mathsf{y} \equiv \mathsf{N}$]
[**Mac rule ApplyT** : $\mathsf{T}\text{ ’ }\mathsf{y} \equiv \mathsf{T}$]
[**Mac rule ApplyLambda** : $(\lambda \mathsf{u}.\mathsf{v})\text{ ’ }\mathsf{y} \equiv \langle \mathsf{v} \mid \mathsf{u}{:=}\mathsf{y}\rangle$]
[**Mac rule ApplyBottom** : $\perp\text{ ’ }\mathsf{y} \equiv \perp$]

Examples

[$(\lambda \mathsf{x}.\mathsf{x}+2)$ ’ 3	$\equiv$	5]˙
[$(\lambda \mathsf{x}.\mathsf{x}+\mathsf{y})$ ’ 3	$\equiv$	$3+\mathsf{y}$]˙

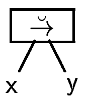
See also

[case(x, y, z)]˙	Section A.6
[N]˙	Section A.23
[λx.y]˙	Section A.20

[x ’ y]˙	x apply y
[SubXApplyY]˙	rule SubXApplyY
[ApplyNil]˙	rule ApplyNil
[ApplyT]˙	rule ApplyT
[ApplyLambda]˙	rule ApplyLambda
[ApplyBottom]˙	rule ApplyBottom

A.48 x associates as y $[\mathsf{x} \overset{\smile}{\dashrightarrow} \mathsf{y}]^{\cdot}$

Parse tree



Sort

$[\mathsf{x} \overset{\smile}{\dashrightarrow} \mathsf{y}]^{\cdot}$ is a directive (i.e. it cannot occur in the syntax tree of a term).

Description

$[\mathcal{A} \overset{\smile}{\dashrightarrow} \mathcal{B}]^{\cdot}$ states that the term $[\mathcal{A}]^{\cdot}$ should be interpretted like term $[\mathcal{B}]^{\cdot}$. $[\mathcal{A}]^{\cdot}$ must consist of two operators of equal priority that are applied to three variables. $[\mathcal{B}]^{\cdot}$ must consist of the same two operators applied to the same three variables plus a parenthesis which indicates how the operator associates.

Examples

- $[\mathsf{x} + \mathsf{y} + \mathsf{z} \overset{\smile}{\dashrightarrow} (\mathsf{x} + \mathsf{y}) + \mathsf{z}]$ makes $[+]^{\circ}$ left associative.
- $[\mathsf{x} + \mathsf{y} - \mathsf{z} \overset{\smile}{\dashrightarrow} (\mathsf{x} + \mathsf{y}) - \mathsf{z}]$ is identical to the associativity rule above since $[+]^{\circ}$ and $[-]^{\circ}$ have equal priority.
- $[\mathsf{x} :: \mathsf{y} :: \mathsf{z} \overset{\smile}{\dashrightarrow} \mathsf{x} :: (\mathsf{y} :: \mathsf{z})]$ makes $[::]^{\circ}$ right associative.
- $\left[\mathsf{x}^{\mathsf{y}^{\mathsf{z}}} \overset{\smile}{\dashrightarrow} \mathsf{x}^{(\mathsf{y}^{\mathsf{z}})}\right]$ makes $[\mathsf{x}^{\mathsf{y}}]^{\cdot}$ associate in a way that is neither left nor right associativity.

See also

$[\mathsf{x} \overset{\smile}{=} \mathsf{y}]^{\cdot}$	Section A.87
$[\mathsf{x} \overset{\circ}{\dashrightarrow} \mathsf{y}]^{\cdot}$	Section A.96

$[\mathsf{x} \overset{\smile}{\dashrightarrow} \mathsf{y}]^{\cdot}$ x associates as y

A.49 x atom [x atom]$^{\cdot}$

Parse tree

$$\begin{tikzcd}\boxed{\text{atom}} \arrow[d, no head] \\ \text{x}\end{tikzcd}$$

Sort

[x atom]$^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

[x atom]$^{\cdot}$ is true if [x]$^{\cdot}$ is an atom, i.e. not a pair. Truth values, exceptions, decimal fractions, and the empty list are atoms. [$\perp$]$^{\cdot}$ is neither an atom nor a pair.

Type table

x	**B**	**D**	**X**	**E**	**N**	**Z**	$\mathbf{Z}^{+}$	$\mathbf{Z}^{-}$
x atom	**T**	**T**	**T**	**T**	**T**	**T**	**T**	**T**

x	$\mathbf{D}_{m}^{\infty}$	$\mathbf{D}_{m}^{\cdot}$	$\mathbf{D}_{m}^{-\infty}$	$\mathbf{D}_{\infty}^{\infty}$	$\mathbf{D}_{\infty}^{\cdot}$	$\mathbf{D}_{\infty}^{-\infty}$	**T**	**F**
x atom	**T**	**T**	**T**	**T**	**T**	**T**	**T**	**T**

Mac rules

[**Mac rule PairIsNotAtom** : (x : : y) atom $\equiv$ F]
[**Mac rule SubXAtom** : $\langle\mathcal{A}$ atom $\mid$ x:=$\mathcal{S}\rangle \equiv \langle\mathcal{A} \mid$ x:=$\mathcal{S}\rangle$ atom]
[**Mac rule StrictAtom** : $\perp$ atom $\equiv \perp$]

Examples

[2.0F atom	$\equiv$	T]$^{\cdot}$
[$\langle$x, y, z$\rangle$ atom	$\equiv$	F]$^{\cdot}$

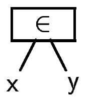
See also

[x pair]$^{\cdot}$	Section A.78
[x : : y]$^{\cdot}$	Section A.79

[x atom]$^{\cdot}$	x atom
[PairIsNotAtom]$^{\cdot}$	rule PairIsNotAtom
[SubXAtom]$^{\cdot}$	rule SubXAtom
[StrictAtom]$^{\cdot}$	rule StrictAtom

A.50 x belongs to y $[\mathsf{x} \in \mathsf{y}]^{\cdot}$

Parse tree



Sort

$[\mathsf{x} \in \mathsf{y}]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

If $[\mathsf{y}]^{\cdot}$ is a set then the following hold: $[\mathsf{x} \in \mathsf{y}]^{\cdot}$ equals $[\mathsf{T}]^{\cdot}$ if $[\mathsf{x}]^{\cdot}$ belongs to $[\mathsf{y}]^{\cdot}$. $[\mathsf{x} \in \mathsf{y}]^{\cdot}$ equals $[\mathsf{F}]^{\cdot}$ if $[\mathsf{x}]^{\cdot}$ is classical and $[\mathsf{x}]^{\cdot}$ does not belong to $[\mathsf{y}]^{\cdot}$. $[\mathsf{x} \in \mathsf{y}]^{\cdot}$ equals $[\perp]^{\cdot}$ if $[\mathsf{x}]^{\cdot}$ is non-classical. The definition

$$\left[\boxed{\mathsf{x} \notin \mathsf{y}} \ddot{=} \neg x \in y\right]$$

makes it easy to express non-membership.

Mac rules

[**Mac rule SubtypeSetC** : $\mathsf{y} \in \mathbf{Set} \vdash \mathsf{x} \in \mathsf{y} \vdash \ell' \mathsf{x}$]
[**Mac rule SubtypeNotSetC** : $\mathsf{y} \in \mathbf{Set} \vdash \mathsf{x} \notin \mathsf{y} \vdash \ell' \mathsf{x}$]
[**Mac rule TypeCInSet** : $\mathsf{y} \in \mathbf{Set} \vdash \ell' \mathsf{x} \vdash (\mathsf{x} \in \mathsf{y}) \in \mathbf{B}$]
[**Mac rule XBelongsToStrict** : $\mathsf{x} \in \perp \equiv \perp$]
[**Mac rule StrictBelongsToY** : $\mathsf{y} \in \mathbf{Set} \vdash \perp \in \mathsf{y} \equiv \perp$]
[**Mac rule SubXBelongsToY** : $\langle \mathcal{A} \in \mathcal{B} \mid \mathsf{x}{:=}\mathcal{S} \rangle \equiv \langle \mathcal{A} \mid \mathsf{x}{:=}\mathcal{S} \rangle \in \langle \mathcal{B} \mid \mathsf{x}{:=}\mathcal{S} \rangle$]

Examples

$$\begin{array}{lcr} [2 \in \mathsf{N} & \equiv & \mathsf{T}]^{\cdot} \\ [-2 \in \mathsf{N} & \equiv & \mathsf{F}]^{\cdot} \\ [\perp \in \mathsf{N} & \equiv & \perp]^{\cdot} \\ [2 \in \perp & \equiv & \perp]^{\cdot} \end{array}$$

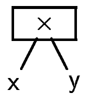
$[\mathsf{x} \in \mathsf{y}]^{\cdot}$	x belongs to y
$[\mathsf{x} \notin \mathsf{y}]^{\cdot}$	x not in y
[SubtypeSetC]˙	rule SubtypeSetC
[SubtypeNotSetC]˙	rule SubtypeNotSetC
[TypeCInSet]˙	rule TypeCInSet
[XBelongsToStrict]˙	rule XBelongsToStrict
[StrictBelongsToY]˙	rule StrictBelongsToY
[SubXBelongsToY]˙	rule SubXBelongsToY

See also

$[\ \ell\ ]^{\cdot}$	Section A.8
$[\ \mathbf{Set}\ ]^{\cdot}$	Section A.29
$[\ \mathbf{D}\ ]^{\cdot}$	Section A.30
$[\ \mathbf{X}\ ]^{\cdot}$	Section A.32
$[\ \mathbf{E}\ ]^{\cdot}$	Section A.40
$[\ \mathbf{B}\ ]^{\cdot}$	Section A.42
$[\ \mathsf{x} \times \mathsf{y}\ ]^{\cdot}$	Section A.51

A.51 x cartesian y $[\mathsf{x} \times \mathsf{y}]^{\cdot}$

Parse tree



Sort

$[\mathsf{x} \times \mathsf{y}]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\mathsf{x} \times \mathsf{y}]^{\cdot}$ is the set of pairs whose first component belongs to $[\mathsf{x}]^{\cdot}$ and whose second component belongs to $[\mathsf{y}]^{\cdot}$.

Mac rules

[**Mac rule SubXCartesianY** : $\langle \mathcal{A} \times \mathcal{B} \mid \mathsf{x}{:=}\mathcal{S}\rangle \equiv \langle \mathcal{A} \mid \mathsf{x}{:=}\mathcal{S}\rangle \times \langle \mathcal{B} \mid \mathsf{x}{:=}\mathcal{S}\rangle$]
[**Mac rule SetXCartesianY** : $\mathsf{A} \in \mathbf{Set} \vdash \mathsf{B} \in \mathbf{Set} \vdash \mathsf{A} \times \mathsf{B} \in \mathbf{Set}$]
[**Mac rule ElimCartesianPair** : $\mathsf{A} \in \mathbf{Set} \vdash \mathsf{B} \in \mathbf{Set} \vdash \mathsf{x} \in \mathsf{A} \times \mathsf{B} \vdash \overline{\mathbf{P}}'\mathsf{x}$]
[**Mac rule ElimCartesianHead** : $\mathsf{A} \in \mathbf{Set} \vdash \mathsf{B} \in \mathbf{Set} \vdash \mathsf{x} \in \mathsf{A} \times \mathsf{B} \vdash \mathsf{x}$ head $\in \mathsf{A}$]
[**Mac rule ElimCartesianTail** : $\mathsf{A} \in \mathbf{Set} \vdash \mathsf{B} \in \mathbf{Set} \vdash \mathsf{x} \in \mathsf{A} \times \mathsf{B} \vdash \mathsf{x}$ tail $\in \mathsf{B}$]
[**Mac rule IntroCartesian** : $\mathsf{A} \in \mathbf{Set} \vdash \mathsf{B} \in \mathbf{Set} \vdash \ell'\mathsf{x} \vdash \overline{\mathbf{P}}'\mathsf{x} \vdash \mathsf{x}$ head $\in \mathsf{A} \vdash \mathsf{x}$ tail $\in \mathsf{B} \vdash \mathsf{x} \in \mathsf{A} \times \mathsf{B}$]
[**Mac rule IntroCartesianPair** : $\mathsf{A} \in \mathbf{Set} \vdash \mathsf{B} \in \mathbf{Set} \vdash \mathsf{a} \in \mathsf{A} \vdash \mathsf{b} \in \mathsf{B} \vdash \mathsf{a} :: \mathsf{b} \in \mathsf{A} \times \mathsf{B}$]

Examples

$[1 :: \mathsf{T} \in \mathbf{N} \times \mathbf{B} \quad \equiv \quad \mathsf{T}]^{\cdot}$
$[\mathsf{T} :: 1 \in \mathbf{N} \times \mathbf{B} \quad \equiv \quad \mathsf{F}]^{\cdot}$

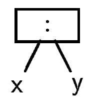
See also

$[\mathbf{E}]^{\cdot}$	Section A.40
$[\mathsf{x} \in \mathsf{y}]^{\cdot}$	Section A.50
$[\mathsf{x} :: \mathsf{y}]^{\cdot}$	Section A.79

$[\mathsf{x} \times \mathsf{y}]^{\cdot}$	x cartesian y
[SubXCartesianY]˙	rule SubXCartesianY
[SetXCartesianY]˙	rule SetXCartesianY
[ElimCartesianPair]˙	rule ElimCartesianPair
[ElimCartesianHead]˙	rule ElimCartesianHead
[ElimCartesianTail]˙	rule ElimCartesianTail
[IntroCartesian]˙	rule IntroCartesian
[IntroCartesianPair]˙	rule IntroCartesianPair

A.52 x colon y [x : y]˙

Parse tree



Sort

[x : y]˙ is a directive (i.e. it cannot occur in the syntax tree of a term).

Description

[x : y]˙ means the same as [y]˙, and also introduces [x]˙ as shorthand for the meaning of [y]˙. In particular, if [y]˙ is an instantiation statement, then [x]˙ is introduced as shorthand for the conclusion of [y]˙.

Examples

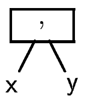
[x: TailPair ⊳ (1 :: 2) tail ≡ 2]˙ means the same as [TailPair ⊳ (1 :: 2) tail ≡ 2]˙ and introduces [x]˙ as shorthand for [(1 :: 2) tail ≡ 2]˙.

See also

[x **proof of** y:z]˙	Section A.83
[x ⊳ y]˙	Section A.56

A.53 x comma y $[\mathsf{x},\mathsf{y}]^{\cdot}$

Parse tree



Sort

$[\mathsf{x},\mathsf{y}]^{\cdot}$ is a directive (i.e. it cannot occur in the syntax tree of a term).

Description

$[\mathsf{x},\mathsf{y}]^{\cdot}$ has no meaning of its own. It has a meaning by virtue of term reduction rules (See Appendix A.3). If Map is asked to compute an expression whose syntax tree contains $[\mathsf{x},\mathsf{y}]^{\cdot}$ then Map protests because the expression is not a term.

Definition

$[$ **construct** x,y $]$.

Examples

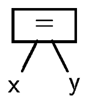
$$\begin{array}{lcl}
[\,\langle\mathsf{x},\mathsf{y},\mathsf{z}\rangle & \equiv & \mathsf{x} :: \mathsf{y} :: \mathsf{z} :: \langle\,\rangle\,]^{\cdot} \\
[\,\mathsf{x},\mathsf{y} < \mathsf{z} & \equiv & \mathsf{x} < \mathsf{z} \wedge \mathsf{y} < \mathsf{z}\,]^{\cdot} \\
[\,\mathsf{x} < \mathsf{y},\mathsf{z} & \equiv & \mathsf{x} < \mathsf{y} \wedge \mathsf{x} < \mathsf{z}\,]^{\cdot}
\end{array}$$

See also

$[$ **construct** $\mathsf{x}]^{\cdot}$	Section A.9
$[\langle\mathsf{x}\rangle]^{\cdot}$	Section A.44

A.54 x computational equal y $[\mathsf{x} = \mathsf{y}]^{\cdot}$

Parse tree



Sort

$[\mathsf{x} = \mathsf{y}]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\mathsf{x} = \mathsf{y}]^{\cdot}$ compares for equality. If $[\mathsf{x}]^{\cdot}$ and $[\mathsf{y}]^{\cdot}$ are decimal fractions, then $[\mathsf{x} = \mathsf{y}]^{\cdot}$ disregards precision. If $[\mathsf{x}]^{\cdot}$ and $[\mathsf{y}]^{\cdot}$ are truth values then $[\mathsf{x} = \mathsf{y}]^{\cdot}$ coincides with $[\mathsf{x} \Leftrightarrow \mathsf{y}]^{\cdot}$.

Type table

=	**B**	**D**	**X**	**E**	=	**N**	**Z**	$\mathbf{Z}^+$	$\mathbf{Z}^-$
B	**B**	**F**	**X**	**F**	**N**	**B**	**B**	**B**	**F**
D	**F**	**B**	**X**	**F**	**Z**	**B**	**B**	**B**	**B**
X	**X**	**X**	**X**	**X**	$\mathbf{Z}^+$	**B**	**B**	**B**	**F**
E	**F**	**F**	**X**	**T**	$\mathbf{Z}^-$	**F**	**B**	**F**	**B**
=	$\mathbf{D}_{\mathsf{m}}^{\infty}$	$\mathbf{D}_{\mathsf{m}}^{\cdot}$	$\mathbf{D}_{\mathsf{m}}^{-\infty}$	$\mathbf{D}_{\infty}^{\infty}$	$\mathbf{D}_{\infty}^{\cdot}$	$\mathbf{D}_{\infty}^{-\infty}$	$\mathbf{D}_{\mathsf{n}}^{\infty}$	$\mathbf{D}_{\mathsf{n}}^{\cdot}$	$\mathbf{D}_{\mathsf{n}}^{-\infty}$
$\mathbf{D}_{\mathsf{m}}^{\infty}$	**T**	**F**	**F**	**T**	**F**	**F**	**T**	**F**	**F**
$\mathbf{D}_{\mathsf{m}}^{\cdot}$	**F**	**B**	**F**	**F**	**B**	**F**	**F**	**B**	**F**
$\mathbf{D}_{\mathsf{m}}^{-\infty}$	**F**	**F**	**T**	**F**	**F**	**T**	**F**	**F**	**T**
$\mathbf{D}_{\infty}^{\infty}$	**T**	**F**	**F**	**T**	**F**	**F**	=	**T**	**F**
$\mathbf{D}_{\infty}^{\cdot}$	**F**	**B**	**F**	**F**	**B**	**F**	**T**	**T**	**F**
$\mathbf{D}_{\infty}^{-\infty}$	**F**	**F**	**T**	**F**	**F**	**T**	**F**	**F**	**T**

Mac rules

[**Mac rule StrictComputationalEqualX** : $\perp = \mathsf{x} \equiv \perp$]
[**Mac rule XComputationalEqualStrict** : $\mathsf{x} = \perp \equiv \perp$]
[**Mac rule EqualPair** : $\mathsf{x} :: \mathsf{y} = \mathsf{u} :: \mathsf{v} \equiv \mathsf{x} = \mathsf{u} \wedge \mathsf{y} = \mathsf{v}$]
[**Mac rule SubXComputationalEqualY** : $\langle \mathcal{A} = \mathcal{B} \mid \mathsf{x}{:=}\mathcal{S} \rangle \equiv \langle \mathcal{A} \mid \mathsf{x}{:=}\mathcal{S} \rangle = \langle \mathcal{B} \mid \mathsf{x}{:=}\mathcal{S} \rangle$]

$[\mathsf{x} = \mathsf{y}]^{\cdot}$	x computational equal y
[StrictComputationalEqualX $]^{\cdot}$	rule StrictComputationalEqualX
[XComputationalEqualStrict $]^{\cdot}$	rule XComputationalEqualStrict
[EqualPair $]^{\cdot}$	rule EqualPair
[SubXComputationalEqualY $]^{\cdot}$	rule SubXComputationalEqualY

Examples

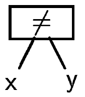
$$[\,\langle 1,2\rangle = \langle 1,2\rangle \equiv \mathsf{T}\,]^{\cdot}$$
$$[\,\langle 1,2\rangle = \langle 1,3\rangle \equiv \mathsf{F}\,]^{\cdot}$$
$$[\,\langle 1,2\rangle = \langle 1,2,3\rangle \equiv \mathsf{F}\,]^{\cdot}$$
$$[\,\langle 1,2\rangle = \langle 1,\bullet\rangle \equiv \bullet\,]^{\cdot}$$
$$[\,\langle 1,2\rangle = \langle 1,\bot\rangle \equiv \bot\,]^{\cdot}$$

See also

$[\,\mathsf{x} \neq \mathsf{y}\,]^{\cdot}$	Section A.55
$[\,\mathsf{x} < \mathsf{y}\,]^{\cdot}$	Section A.93

A.55 x computational unequal y [x ≠ y]˙

Parse tree



Sort

[x ≠ y]˙ is an operator (i.e. it may occur in the syntax tree of a term).

Description

[x ≠ y]˙ compares for inequality. If [x]˙ and [y]˙ are decimal fractions, then [x ≠ y]˙ disregards precision. If [x]˙ and [y]˙ are truth values then [x ≠ y]˙ coincides with exclusive "or".

Type table

≠	**B**	**D**	**X**	**E**	≠	**N**	**Z**	$\mathbf{Z}^{+}$	$\mathbf{Z}^{-}$
B	**B**	**T**	**X**	**T**	**N**	**B**	**B**	**B**	**T**
D	**T**	**B**	**X**	**T**	**Z**	**B**	**B**	**B**	**B**
X	**X**	**X**	**X**	**X**	$\mathbf{Z}^{+}$	**B**	**B**	**B**	**T**
E	**T**	**T**	**X**	**F**	$\mathbf{Z}^{-}$	**T**	**B**	**T**	**B**
≠	$\mathbf{D}^{\infty}_{m}$	$\mathbf{D}^{\cdot}_{m}$	$\mathbf{D}^{-\infty}_{m}$	$\mathbf{D}^{\infty}_{\infty}$	$\mathbf{D}^{\cdot}_{\infty}$	$\mathbf{D}^{-\infty}_{\infty}$	$\mathbf{D}^{\infty}_{n}$	$\mathbf{D}^{\cdot}_{n}$	$\mathbf{D}^{-\infty}_{n}$
$\mathbf{D}^{\infty}_{m}$	**F**	**T**	**T**	**F**	**T**	**T**	**F**	**T**	**T**
$\mathbf{D}^{\cdot}_{m}$	**T**	**B**	**T**	**T**	**B**	**T**	**T**	**B**	**T**
$\mathbf{D}^{-\infty}_{m}$	**T**	**T**	**F**	**T**	**T**	**F**	**T**	**T**	**F**
$\mathbf{D}^{\infty}_{\infty}$	**F**	**T**	**T**	**F**	**T**	**T**	≠	**T**	**F**
$\mathbf{D}^{\cdot}_{\infty}$	**T**	**B**	**T**	**T**	**B**	**T**	**T**	**F**	**T**
$\mathbf{D}^{-\infty}_{\infty}$	**T**	**T**	**F**	**T**	**T**	**F**	**F**	**T**	**F**

Mac rules

[**Mac rule StrictComputationalUnequalX** : ⊥ ≠ x ≡ ⊥]
[**Mac rule XComputationalUnequalStrict** : x ≠ ⊥ ≡ ⊥]
[**Mac rule UnequalPair** : x :: y ≠ u :: v ≡ x ≠ u ∨ y ≠ v]
[**Mac rule SubXComputationalUnequalY** : ⟨$\mathcal{A}$ ≠ $\mathcal{B}$ | x:=$\mathcal{S}$⟩ ≡ ⟨$\mathcal{A}$ | x:=$\mathcal{S}$⟩ ≠ ⟨$\mathcal{B}$ | x:=$\mathcal{S}$⟩]

[x ≠ y]˙	x computational unequal y
[StrictComputationalUnequalX]˙	rule StrictComputationalUnequalX
[XComputationalUnequalStrict]˙	rule XComputationalUnequalStrict
[UnequalPair]˙	rule UnequalPair
[SubXComputationalUnequalY]˙	rule SubXComputationalUnequalY

Examples

$[\, \langle 1,2\rangle \neq \langle 1,2\rangle$	$\equiv$	$\mathsf{F} \,]^{\cdot}$
$[\, \langle 1,2\rangle \neq \langle 1,3\rangle$	$\equiv$	$\mathsf{T} \,]^{\cdot}$
$[\, \langle 1,2\rangle \neq \langle 1,2,3\rangle$	$\equiv$	$\mathsf{T} \,]^{\cdot}$
$[\, \langle 1,2\rangle \neq \langle 1,\bullet\rangle$	$\equiv$	$\bullet \,]^{\cdot}$
$[\, \langle 1,2\rangle \neq \langle 1,\perp\rangle$	$\equiv$	$\perp \,]^{\cdot}$

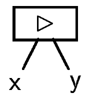
See also

$[\, \mathsf{x} = \mathsf{y} \,]^{\cdot}$	Section A.54
$[\, \mathsf{x} < \mathsf{y} \,]^{\cdot}$	Section A.93

A.56 x concludes y $[\, \mathsf{x} \rhd \mathsf{y} \,]^{\cdot}$

$[\, \boxed{\mathsf{x} \rhd \mathsf{y}} \,]^{\cdot}$

Parse tree



Sort

$[\, \mathsf{x} \rhd \mathsf{y} \,]^{\cdot}$ is a directive (i.e. it cannot occur in the syntax tree of a term).

Description

$[\, \mathsf{x} \rhd \mathsf{y} \,]^{\cdot}$ is used to form argumentations from inference rules and equations.

Examples

$[\, \mathsf{x} \equiv \mathsf{y} \vdash \mathsf{y} \equiv \mathsf{z} \vdash \mathsf{x} \equiv \mathsf{z} \rhd \langle\rangle \text{ head} \equiv \langle\rangle \rhd \langle\rangle \equiv \langle\rangle \text{ tail} \rhd \langle\rangle \text{ head} \equiv \langle\rangle \text{ tail} \,]^{\cdot}$ states that $[\, \langle\rangle \text{ head} \equiv \langle\rangle \text{ tail} \,]^{\cdot}$ follows from $[\, \langle\rangle \text{ head} \equiv \langle\rangle \,]^{\cdot}$ and $[\, \langle\rangle \equiv \langle\rangle \text{ tail} \,]^{\cdot}$ according to the inference rules $[\, \mathsf{x} \equiv \mathsf{y} \vdash \mathsf{y} \equiv \mathsf{z} \vdash \mathsf{x} \equiv \mathsf{z} \,]^{\cdot}$. Another way to state this is

$$[\, \text{Transitivity} \rhd \langle\rangle \text{ head} \equiv \langle\rangle \rhd \langle\rangle \equiv \langle\rangle \text{ tail} \rhd \langle\rangle \text{ head} \equiv \langle\rangle \text{ tail} \,]^{\cdot}$$

since Transitivity is shorthand for $[\, \mathsf{x} \equiv \mathsf{y} \vdash \mathsf{y} \equiv \mathsf{z} \vdash \mathsf{x} \equiv \mathsf{z} \,]^{\cdot}$.

See also

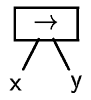
$[\, \mathsf{x}$ **proof of** $\mathsf{y}{:}\mathsf{z} \,]^{\cdot}$	Section A.83
$[\, \mathsf{x} \unrhd \mathsf{y} \,]^{\cdot}$	Section A.75
$[\, \mathsf{x} : \mathsf{y} \,]^{\cdot}$	Section A.52
$[\, \mathsf{x} \vdash \mathsf{y} \,]^{\cdot}$	Section A.68

$[\, \mathsf{x} \rhd \mathsf{y} \,]^{\cdot}$ x concludes y

A.57 x deduces y $[\,x \rightarrow y\,]^{\cdot}$

$[\,\boxed{x \rightarrow y}\,]^{\cdot}$

Parse tree



Sort

$[\,x \rightarrow y\,]^{\cdot}$ is a directive (i.e. it cannot occur in the syntax tree of a term).

Description

$[\,x \rightarrow y\,]^{\cdot}$ is used to state that $[\,x\,]^{\cdot}$ implies $[\,y\,]^{\cdot}$ when $[\,x\,]^{\cdot}$ is a term and $[\,y\,]^{\cdot}$ is an equation. $[\,x \rightarrow y\,]^{\cdot}$ is shorthand for an equation.

Mac rules

$[\,$**Mac rule ModusPonens** $: x \equiv \mathsf{T} \vdash x \rightarrow y \equiv z \vdash y \equiv z\,]$

Examples

$[\,x \in \mathbf{Z} \rightarrow y \in \mathbf{Z} \rightarrow x + y \equiv y + x\,]^{\cdot}$ states that if $[\,x\,]^{\cdot}$ and $[\,y\,]^{\cdot}$ are integers then $[\,x + y\,]^{\cdot}$ equals $[\,y + x\,]^{\cdot}$.

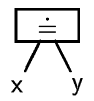
See also

$[\,x \vdash y\,]^{\cdot}$	Section A.68
$[\,x \Rightarrow y\,]^{\cdot}$	Section A.67
$[\,x \trianglerighteq y\,]^{\cdot}$	Section A.75

$[\,x \rightarrow y\,]^{\cdot}$	x deduces y
[ModusPonens]˙	rule ModusPonens

A.58 x defined equal y $[\mathsf{x} \doteq \mathsf{y}]^{\cdot}$

Parse tree



Sort

$[\mathsf{x} \doteq \mathsf{y}]^{\cdot}$ is a directive (i.e. it cannot occur in the syntax tree of a term).

Description

Among other, $[\mathcal{A} \doteq \mathcal{B}]^{\cdot}$ introduces a new operator $[\mathcal{A}]^{\cdot}$, the definition axiom $[\mathcal{A} \equiv \mathcal{B}]^{\cdot}$, and the reduction rule $\left[\mathcal{A} \overset{+}{\dashrightarrow} \mathcal{B}\right]^{\cdot}$. Contrary to constructs introduced by $[\mathcal{A} \stackrel{..}{=} \mathcal{B}]^{\cdot}$, constructs introduced by $[\mathcal{A} \doteq \mathcal{B}]^{\cdot}$ can occur in syntax trees.

Examples

$[\,\infty\text{-list}(\mathsf{n}) \doteq \mathsf{n} :: \infty\text{-list}(\mathsf{n}+1)\,]$

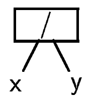
See also

$[$ **construct** $\mathsf{x}]^{\cdot}$	Section A.9
$[$ **variable** $\mathsf{x}]^{\cdot}$	Section A.45
$[\mathsf{x} \equiv \mathsf{y}]^{\cdot}$	Section A.60
$[\mathsf{x} \stackrel{..}{=} \mathsf{y}]^{\cdot}$	Section A.72
$[\mathsf{x} \acute{=} \mathsf{y}]^{\cdot}$	Section A.76

$[\mathsf{x} \doteq \mathsf{y}]^{\cdot}$ x defined equal y

A.59 x divide y [x/y]·

Parse tree



Sort

[x/y]· is an operator (i.e. it may occur in the syntax tree of a term).

Description

If [x]· and [y]· are floating fractions of the same precision, then [x/y]· denotes [x]· divided by [y]· rounded to the precision of [x]· and [y]· using round to even. [$x/_{+}y$]·, [$x/_{-}y$]·, [$x/_{0}y$]·, and [$x/_{2}y$]· are versions of [x/y]· that use round to plus infinity, minus infinity, zero, and even, respectively. [$x/_{2}y$]· is identical to [x/y]·. If [x]· and [y]· are exact fractions or if they have different precision then [x/y ≡ •]·.

Type table

/	**B**	**D**	**X**	**E**	/	**N**	**Z**	$\mathbf{Z}^+$	$\mathbf{Z}^-$
B	**X**	**X**	**X**	**X**	**N**	**X**	**X**	**X**	**X**
D	**X**	–	**X**	**X**	**Z**	**X**	**X**	**X**	**X**
X	**X**	**X**	**X**	**X**	$\mathbf{Z}^+$	**X**	**X**	**X**	**X**
E	**X**	**X**	**X**	**X**	$\mathbf{Z}^-$	**X**	**X**	**X**	**X**
/	$\mathbf{D}_m^{\infty}$	$\mathbf{D}_m^{\cdot}$	$\mathbf{D}_m^{-\infty}$	$\mathbf{D}_\infty^{\infty}$	$\mathbf{D}_\infty^{\cdot}$	$\mathbf{D}_\infty^{-\infty}$	$\mathbf{D}_n^{\infty}$	$\mathbf{D}_n^{\cdot}$	$\mathbf{D}_n^{-\infty}$
$\mathbf{D}_m^{\infty}$	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**
$\mathbf{D}_m^{\cdot}$	**X**	–	**X**	**X**	**X**	**X**	**X**	**X**	**X**
$\mathbf{D}_m^{-\infty}$	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**
$\mathbf{D}_\infty^{\infty}$	**X**	**X**	**X**	**X**	**X**	**X**	/	**T**	**F**
$\mathbf{D}_\infty^{\cdot}$	**X**	**X**	**X**	**X**	**X**	**X**	**T**	**X**	**X**
$\mathbf{D}_\infty^{-\infty}$	**X**	**X**	**X**	**X**	**X**	**X**	**F**	**X**	**X**

Mac rules

[**Mac rule StrictDivideX** : $\bot/x \equiv \bot$]
[**Mac rule XDivideStrict** : $x/\bot \equiv \bot$]
[**Mac rule SubXDivideY** : $\langle \mathcal{A}/\mathcal{B} \mid x{:=}\mathcal{S}\rangle \equiv \langle \mathcal{A} \mid x{:=}\mathcal{S}\rangle / \langle \mathcal{B} \mid x{:=}\mathcal{S}\rangle$]

[x/y]·	x divide y
[StrictDivideX]·	rule StrictDivideX
[XDivideStrict]·	rule XDivideStrict
[SubXDivideY]·	rule SubXDivideY

[**Mac rule DivideNonzero** : $\mathsf{m} \in \mathbf{Z}^{+} \vdash \mathsf{x} \in \mathbf{D}^{\cdot}_{\mathsf{m}} \vdash \mathsf{y} \in \mathbf{D}^{\cdot}_{\mathsf{m}} \vdash (\mathsf{y} = 0) \in \mathbf{F} \vdash \mathsf{x}/\mathsf{y} \in \mathbf{D}^{\cdot}_{\mathsf{m}}$]
[**Mac rule DivideZero** : $\mathsf{x} \in \mathbf{D} \vdash \mathsf{y} \in \mathbf{D} \vdash (\mathsf{y} = 0) \in \mathbf{T} \vdash \mathsf{x}/\mathsf{y} \in \mathbf{X}$]

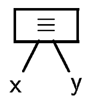
See also

[x // y]˙	Section A.69
[x@y]˙	Section A.85

[DivideNonzero]˙	rule DivideNonzero
[DivideZero]˙	rule DivideZero

A.60 x equal y $[\mathsf{x} \equiv \mathsf{y}]^{\cdot}$

Parse tree



Sort

$[\mathsf{x} \equiv \mathsf{y}]^{\cdot}$ is a directive (i.e. it cannot occur in the syntax tree of a term).

Description

$[\mathcal{A} \equiv \mathcal{B}]^{\cdot}$ states that $[\mathcal{A}]^{\cdot}$ equals $[\mathcal{B}]^{\cdot}$. $[\mathcal{A} \equiv \mathcal{B}]^{\cdot}$ has no value since it is a statement construct. Instead, $[\mathcal{A} \equiv \mathcal{B}]^{\cdot}$ may "hold" or "fail". $[\mathcal{A} \equiv \mathcal{B}]^{\cdot}$ holds if $[\mathcal{A}]^{\cdot}$ equals $[\mathcal{B}]^{\cdot}$ for all values of free variables in $[\mathcal{A}]^{\cdot}$ and $[\mathcal{B}]^{\cdot}$. $[\mathcal{A} \equiv \mathcal{B}]^{\cdot}$ fails if $[\mathcal{A}]^{\cdot}$ is unequal to $[\mathcal{B}]^{\cdot}$ for some values of the free variables in $[\mathcal{A}]^{\cdot}$ and $[\mathcal{B}]^{\cdot}$.

Mac rules

[**Mac antirule Contradiction** : $\bot \equiv \mathsf{T}$]
[**Mac rule Repetition** : $\mathsf{x} \equiv \mathsf{y} \vdash \mathsf{x} \equiv \mathsf{y}$]
[**Mac rule Reflexivity** : $\mathsf{x} \equiv \mathsf{x}$]
If $[\mathsf{u} \doteq \mathsf{v}]^{\cdot}$ or $[\mathsf{u} \overset{\prime}{=} \mathsf{v}]^{\cdot}$ and if $[\mathsf{x} \equiv \mathsf{y}]^{\cdot}$ is a replacement or reverse replacement of $[\mathsf{u} \equiv \mathsf{v}]^{\cdot}$, then $[\mathsf{x} \equiv \mathsf{y}]^{\cdot}$:
[**Mac rule Definition** : **if** definition(x, y) **then** $\mathsf{x} \equiv \mathsf{y}$]
If $[\mathsf{x}]^{\cdot}$ and $[\mathsf{y}]^{\cdot}$ are closed and compute to the same value, then $[\mathsf{x} \equiv \mathsf{y}]^{\cdot}$:
[**Mac rule Computation** : **if** closed(x) ∧ closed(y) ∧ equal(x, y) **then** $\mathsf{x} \equiv \mathsf{y}$]
[**Mac rule Commutativity** : $\mathsf{x} \equiv \mathsf{y} \vdash \mathsf{y} \equiv \mathsf{x}$]
[**Mac rule Transitivity** : $\mathsf{x} \equiv \mathsf{y} \vdash \mathsf{y} \equiv \mathsf{z} \vdash \mathsf{x} \equiv \mathsf{z}$]
[**Mac rule Rename** : **if** structurally equal($\mathcal{A}, \mathcal{B}$) **then** $\mathcal{A} \equiv \mathcal{B}$]
[**Mac rule Replace** : **if** replace(x, y, u, v) **then** $\mathsf{x} \equiv \mathsf{y} \vdash \mathsf{u} \equiv \mathsf{v}$]
[**Mac rule Reverse** : **if** replace(x, y, u, v) **then** $\mathsf{x} \equiv \mathsf{y} \vdash \mathsf{v} \equiv \mathsf{u}$]

$[\mathsf{x} \equiv \mathsf{y}]^{\cdot}$	x equal y
[Contradiction]ˑ	rule Contradiction
[Repetition]ˑ	rule Repetition
[Reflexivity]ˑ	rule Reflexivity
[Definition]ˑ	rule Definition
[Computation]ˑ	rule Computation
[Commutativity]ˑ	rule Commutativity
[Transitivity]ˑ	rule Transitivity
[Rename]ˑ	rule Rename
[Replace]ˑ	rule Replace
[Reverse]ˑ	rule Reverse

[**Mac rule Replace'** : **if** replace$(\mathsf{x},\mathsf{y},\mathsf{u},\mathsf{a}) \wedge$ replace$(\mathsf{x},\mathsf{y},\mathsf{v},\mathsf{b})$ **then** $\mathsf{x} \equiv \mathsf{y} \vdash \mathsf{u} \equiv \mathsf{v} \vdash \mathsf{a} \equiv \mathsf{b}$]
[**Mac rule Reverse'** : **if** replace$(\mathsf{x},\mathsf{y},\mathsf{u},\mathsf{a}) \wedge$ replace$(\mathsf{x},\mathsf{y},\mathsf{v},\mathsf{b})$ **then** $\mathsf{x} \equiv \mathsf{y} \vdash \mathsf{a} \equiv \mathsf{b} \vdash \mathsf{u} \equiv \mathsf{v}$]

Examples

[$(\mathsf{x} :: \mathsf{y})$ head	$\equiv$	x]˙	holds
[$(\mathsf{x} :: \mathsf{y})$ tail	$\equiv$	x]˙	fails
[$\perp$	$\equiv$	$\perp$]˙	holds
[1	$\equiv$	$\perp$]˙	fails
[1.0F	$\equiv$	1.00F]˙	fails

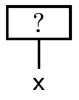
See also

[$\mathsf{x} \doteq \mathsf{y}$]˙	Section A.58
[$\mathsf{x} = \mathsf{y}$]˙	Section A.54

[Replace']˙	rule Replace'
[Reverse']˙	rule Reverse'

A.61 x exceptional $[\, x? \,]^{\cdot}$

Parse tree



Sort

$[\, x? \,]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\, x? \,]^{\cdot}$ is true if $[\, x \,]^{\cdot}$ is exceptional.

Type table

x	**B**	**D**	**X**	**E**	**N**	**Z**	$\mathbf{Z}^{+}$	$\mathbf{Z}^{-}$
x!	**F**	**F**	**T**	**F**	**F**	**F**	**F**	**F**

x	$\mathbf{D}^{\infty}_{\mathsf{m}}$	$\mathbf{D}^{\cdot}_{\mathsf{m}}$	$\mathbf{D}^{-\infty}_{\mathsf{m}}$	$\mathbf{D}^{\infty}_{\infty}$	$\mathbf{D}^{\cdot}_{\infty}$	$\mathbf{D}^{-\infty}_{\infty}$	**T**	**F**
x!	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**F**

Mac rules

[**Mac rule SubXExceptional** : $\langle \mathcal{A}? \mid \mathsf{x}{:=}\mathcal{S} \rangle \equiv \langle \mathcal{A} \mid \mathsf{x}{:=}\mathcal{S} \rangle ?$]
[**Mac rule StrictExceptional** : $\bot? \equiv \bot$]

Examples

$[\, \infty?$	$\equiv$	$\mathsf{F} \,]^{\cdot}$
$[\, (1/0)?$	$\equiv$	$\mathsf{T} \,]^{\cdot}$

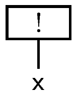
See also

$[\, \bullet \,]^{\cdot}$ Section A.11

$[\, x? \,]^{\cdot}$	x exceptional
$[$ SubXExceptional $]^{\cdot}$	rule SubXExceptional
$[$ StrictExceptional $]^{\cdot}$	rule StrictExceptional

A.62 x faculty [x!]˙

Parse tree



Sort

[x!]˙ is an operator (i.e. it may occur in the syntax tree of a term).

Description

For positive integers [x]˙, [x!]˙ denotes the product of all numbers from [1]˙ to [x]˙, inclusive. [$0! \equiv 1$]˙.

Definition

[$\mathsf{n}! \doteq \mathrm{if}(\mathsf{n} = 0, 1, \mathsf{n} \cdot (\mathsf{n} - 1)!)$].

Type table

x	**B**	**D**	**X**	**E**	**N**	**Z**	$\mathbf{Z}^{+}$	$\mathbf{Z}^{-}$
x!	**X**	–	**X**	**X**	$\mathbf{Z}^{+}$	–	$\mathbf{Z}^{+}$	–

x	$\mathbf{D}^{\infty}_{m}$	$\mathbf{D}^{\cdot}_{m}$	$\mathbf{D}^{-\infty}_{m}$	$\mathbf{D}^{\infty}_{\infty}$	$\mathbf{D}^{\cdot}_{\infty}$	$\mathbf{D}^{-\infty}_{\infty}$	**T**	**F**
x!	**X**	–	**X**	**X**	–	**X**	**X**	**X**

Mac rules

[**Mac rule DefXFaculty** : $\mathsf{n}! \equiv \mathrm{if}(\mathsf{n} = 0, 1, \mathsf{n} \cdot (\mathsf{n} - 1)!)$]
[**Mac rule SubXFaculty** : $\langle \mathcal{A}! \mid \mathsf{x}{:=}\mathcal{S} \rangle \equiv \langle \mathcal{A} \mid \mathsf{x}{:=}\mathcal{S} \rangle!$]
[**Mac rule StrictFaculty** : $\bot! \equiv \bot$]

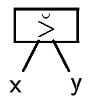
See also

[$\mathsf{x} \doteq \mathsf{y}$]˙ Section A.62

[x!]˙	x faculty
[DefXFaculty]˙	rule DefXFaculty
[SubXFaculty]˙	rule SubXFaculty
[StrictFaculty]˙	rule StrictFaculty

A.63 x greater priority y $[\mathsf{x} \mathrel{\breve{>}} \mathsf{y}]^{\cdot}$

Parse tree



Sort

$[\mathsf{x} \mathrel{\breve{>}} \mathsf{y}]^{\cdot}$ is a directive (i.e. it cannot occur in the syntax tree of a term).

Description

$[\mathcal{A} \mathrel{\breve{>}} \mathcal{B}]^{\cdot}$ states that the construct $[\mathcal{A}]^{\cdot}$ has greater priority than the construct $[\mathcal{B}]^{\cdot}$ when computing expressions. Statements of form $[\mathcal{A} \mathrel{\breve{>}} \mathcal{B}]^{\cdot}$ affect the process of forming parse trees. They affect syntax trees indirectly in that they affect the parse trees from which the syntax trees are constructed. In $[\mathcal{A} \mathrel{\breve{>}} \mathcal{B}]^{\cdot}$, $[\mathcal{A}]^{\cdot}$ and $[\mathcal{B}]^{\cdot}$ must have the form of an operator applied to variables. $[\mathcal{A} \mathrel{\breve{>}} \mathcal{B}]^{\cdot}$ has no effect on outfix operators.

Examples

$[\mathsf{x} \cdot \mathsf{y} \mathrel{\breve{>}} \mathsf{x} + \mathsf{y}]^{\cdot}$ makes Map interpret $[2 \cdot 3 + 4]^{\cdot}$ as $[(2 \cdot 3) + 4]^{\cdot}$.

See also

$[\mathsf{x} \mathrel{\breve{\dashrightarrow}} \mathsf{y}]^{\cdot}$	Section A.48
$[\mathsf{x} \mathrel{\breve{\doteq}} \mathsf{y}]^{\cdot}$	Section A.87

$[\mathsf{x} \mathrel{\breve{>}} \mathsf{y}]^{\cdot}$ x greater priority y

A.64 x head [x head]˙

Parse tree

$$\begin{tikzcd}\boxed{\text{head}} \arrow[d, dash] \\ \mathsf{x}\end{tikzcd}$$

Sort

[x head]˙ is an operator (i.e. it may occur in the syntax tree of a term).

Description

[x head]˙ denotes the first element of a pair [x]˙ or (which is the same) the first element of a list. If [x]˙ is a truth value, a decimal fraction, an exception or the empty list, then [x head]˙ equals [x]˙.

Type table

x	**B**	**D**	**X**	**E**	**N**	**Z**	$\mathbf{Z}^+$	$\mathbf{Z}^-$
x head	**B**	**D**	**X**	**E**	**N**	**Z**	$\mathbf{Z}^+$	$\mathbf{Z}^-$

x	$\mathbf{D}^{\infty}_{\mathsf{m}}$	$\mathbf{D}^{\cdot}_{\mathsf{m}}$	$\mathbf{D}^{-\infty}_{\mathsf{m}}$	$\mathbf{D}^{\infty}_{\infty}$	$\mathbf{D}^{\cdot}_{\infty}$	$\mathbf{D}^{-\infty}_{\infty}$	**T**	**F**
x head	$\mathbf{D}^{\infty}_{\mathsf{m}}$	$\mathbf{D}^{\cdot}_{\mathsf{m}}$	$\mathbf{D}^{-\infty}_{\mathsf{m}}$	$\mathbf{D}^{\infty}_{\infty}$	$\mathbf{D}^{\cdot}_{\infty}$	$\mathbf{D}^{-\infty}_{\infty}$	**T**	**F**

Mac rules

[**Mac rule HeadEmpty** : $\langle\,\rangle$ head $\equiv \langle\,\rangle$]
[**Mac rule HeadPair** : (x : : y) head $\equiv$ x]
[**Mac rule HeadBottom** : $\perp$ head $\equiv \perp$]
[**Mac rule SubXHead** : $\langle\mathcal{A}$ head | x:=$\mathcal{S}\rangle \equiv \langle\mathcal{A}$ | x:=$\mathcal{S}\rangle$ head]

Examples

[$\langle 1,2,3,4,5\rangle$ head $\equiv$ 1]˙

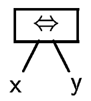
See also

[x : : y]˙	Section A.79
[x tail]˙	Section A.95

[x head]˙	x head
[HeadEmpty]˙	rule HeadEmpty
[HeadPair]˙	rule HeadPair
[HeadBottom]˙	rule HeadBottom
[SubXHead]˙	rule SubXHead

A.65 x if and only if y $[\,x \Leftrightarrow y\,]^{\cdot}$

Parse tree



Sort

$[\,x \Leftrightarrow y\,]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\,x \Leftrightarrow y\,]^{\cdot}$ expresses "$[\,x\,]^{\cdot}$ if and only if $[\,y\,]^{\cdot}$". $[\,x \Leftrightarrow y\,]^{\cdot}$ coincides with $[\,x = y\,]^{\cdot}$ for truth values $[\,x\,]^{\cdot}$ and $[\,y\,]^{\cdot}$.

Type table

⇔	**B**	**D**	**X**	**E**	⇔	**N**	**Z**	$\mathbf{Z}^{+}$	$\mathbf{Z}^{-}$
B	**B**	**X**	**X**	**X**	**N**	**X**	**X**	**X**	**X**
D	**X**	**X**	**X**	**X**	**Z**	**X**	**X**	**X**	**X**
X	**X**	**X**	**X**	**X**	$\mathbf{Z}^{+}$	**X**	**X**	**X**	**X**
E	**X**	**X**	**X**	**X**	$\mathbf{Z}^{-}$	**X**	**X**	**X**	**X**

⇔	D^{∞}_{m}	$D^{\cdot}_{m}$	$D^{-\infty}_{m}$	D^{∞}_{∞}	$D^{\cdot}_{\infty}$	$D^{-\infty}_{\infty}$	D^{∞}_{n}	$D^{\cdot}_{n}$	$D^{-\infty}_{n}$
D^{∞}_{m}	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**
$D^{\cdot}_{m}$	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**
$D^{-\infty}_{m}$	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**
D^{∞}_{∞}	**X**	**X**	**X**	**X**	**X**	**X**	⇔	**T**	**F**
$D^{\cdot}_{\infty}$	**X**	**X**	**X**	**X**	**X**	**X**	**T**	**T**	**F**
$D^{-\infty}_{\infty}$	**X**	**X**	**X**	**X**	**X**	**X**	**F**	**F**	**T**

Mac rules

[**Mac rule StrictIfAndOnlyIfX** : $\bot \Leftrightarrow x \equiv \bot$]
[**Mac rule XIfAndOnlyIfStrict** : $x \Leftrightarrow \bot \equiv \bot$]
[**Mac rule SubXIfAndOnlyIfY** : $\langle \mathcal{A} \Leftrightarrow \mathcal{B} \mid x{:=}\mathcal{S}\rangle \equiv \langle \mathcal{A} \mid x{:=}\mathcal{S}\rangle \Leftrightarrow \langle \mathcal{B} \mid x{:=}\mathcal{S}\rangle$]

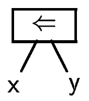
$[\,x \Leftrightarrow y\,]^{\cdot}$	x if and only if y
[StrictIfAndOnlyIfX $]^{\cdot}$	rule StrictIfAndOnlyIfX
[XIfAndOnlyIfStrict $]^{\cdot}$	rule XIfAndOnlyIfStrict
[SubXIfAndOnlyIfY $]^{\cdot}$	rule SubXIfAndOnlyIfY

See also

$[\,\mathsf{x} = \mathsf{y}\,]^{\cdot}$	Section A.54
$[\,\mathsf{x} \Rightarrow \mathsf{y}\,]^{\cdot}$	Section A.67
$[\,\mathsf{x} \Leftarrow \mathsf{y}\,]^{\cdot}$	Section A.66

A.66 x implied by y $[\mathsf{x} \Leftarrow \mathsf{y}]^{\cdot}$

Parse tree



Sort

$[\mathsf{x} \Leftarrow \mathsf{y}]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\mathsf{x} \Leftarrow \mathsf{y}]^{\cdot}$ expresses "$[\mathsf{x}]^{\cdot}$ follows from $[\mathsf{y}]^{\cdot}$". $[\mathsf{x} \Leftarrow \mathsf{y}]^{\cdot}$ coincides with $[\mathsf{x} \geq \mathsf{y}]^{\cdot}$ for truth values $[\mathsf{x}]^{\cdot}$ and $[\mathsf{y}]^{\cdot}$.

Type table

⇐	**B**	**D**	**X**	**E**	⇐	**N**	**Z**	$\mathbf{Z}^{+}$	$\mathbf{Z}^{-}$
B	**B**	**X**	**X**	**X**	**N**	**X**	**X**	**X**	**X**
D	**X**	**X**	**X**	**X**	**Z**	**X**	**X**	**X**	**X**
X	**X**	**X**	**X**	**X**	$\mathbf{Z}^{+}$	**X**	**X**	**X**	**X**
E	**X**	**X**	**X**	**X**	$\mathbf{Z}^{-}$	**X**	**X**	**X**	**X**
⇐	$\mathbf{D}_{\mathsf{m}}^{\infty}$	$\mathbf{D}_{\mathsf{m}}^{\cdot}$	$\mathbf{D}_{\mathsf{m}}^{-\infty}$	$\mathbf{D}_{\infty}^{\infty}$	$\mathbf{D}_{\infty}^{\cdot}$	$\mathbf{D}_{\infty}^{-\infty}$	$\mathbf{D}_{\mathsf{n}}^{\infty}$	$\mathbf{D}_{\mathsf{n}}^{\cdot}$	$\mathbf{D}_{\mathsf{n}}^{-\infty}$
$\mathbf{D}_{\mathsf{m}}^{\infty}$	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**
$\mathbf{D}_{\mathsf{m}}^{\cdot}$	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**
$\mathbf{D}_{\mathsf{m}}^{-\infty}$	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**
$\mathbf{D}_{\infty}^{\infty}$	**X**	**X**	**X**	**X**	**X**	**X**	⇐	**T**	**F**
$\mathbf{D}_{\infty}^{\cdot}$	**X**	**X**	**X**	**X**	**X**	**X**	**T**	**T**	**T**
$\mathbf{D}_{\infty}^{-\infty}$	**X**	**X**	**X**	**X**	**X**	**X**	**F**	**F**	**T**

Mac rules

[**Mac rule StrictImpliedByX** : $\bot \Leftarrow \mathsf{x} \equiv \bot$]
[**Mac rule XImpliedByStrict** : $\mathsf{x} \Leftarrow \bot \equiv \bot$]
[**Mac rule SubXImpliedByY** : $\langle \mathcal{A} \Leftarrow \mathcal{B} \mid \mathsf{x}{:=}\mathcal{S} \rangle \equiv \langle \mathcal{A} \mid \mathsf{x}{:=}\mathcal{S} \rangle \Leftarrow \langle \mathcal{B} \mid \mathsf{x}{:=}\mathcal{S} \rangle$]

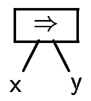
$[\mathsf{x} \Leftarrow \mathsf{y}]^{\cdot}$	x implied by y
$[\text{StrictImpliedByX}]^{\cdot}$	rule StrictImpliedByX
$[\text{XImpliedByStrict}]^{\cdot}$	rule XImpliedByStrict
$[\text{SubXImpliedByY}]^{\cdot}$	rule SubXImpliedByY

See also

$[\,\mathsf{x} \Leftrightarrow \mathsf{y}\,]^{\cdot}$	Section A.65
$[\,\mathsf{x} \Rightarrow \mathsf{y}\,]^{\cdot}$	Section A.67
$[\,\mathsf{x} \geq \mathsf{y}\,]^{\cdot}$	Section A.98

A.67 x implies y $[\mathsf{x} \Rightarrow \mathsf{y}]^{\cdot}$

Parse tree



Sort

$[\mathsf{x} \Rightarrow \mathsf{y}]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\mathsf{x} \Rightarrow \mathsf{y}]^{\cdot}$ expresses "$[\mathsf{x}]^{\cdot}$ implies $[\mathsf{y}]^{\cdot}$". $[\mathsf{x} \Rightarrow \mathsf{y}]^{\cdot}$ coincides with $[\mathsf{x} \leq \mathsf{y}]^{\cdot}$ for truth values $[\mathsf{x}]^{\cdot}$ and $[\mathsf{y}]^{\cdot}$.

Type table

⇒	**B**	**D**	**X**	**E**	⇒	**N**	**Z**	$\mathbf{Z}^+$	$\mathbf{Z}^-$
B	**B**	**X**	**X**	**X**	**N**	**X**	**X**	**X**	**X**
D	**X**	**X**	**X**	**X**	**Z**	**X**	**X**	**X**	**X**
X	**X**	**X**	**X**	**X**	$\mathbf{Z}^+$	**X**	**X**	**X**	**X**
E	**X**	**X**	**X**	**X**	$\mathbf{Z}^-$	**X**	**X**	**X**	**X**
⇒	$\mathbf{D}^{\infty}_{\mathsf{m}}$	$\mathbf{D}^{\cdot}_{\mathsf{m}}$	$\mathbf{D}^{-\infty}_{\mathsf{m}}$	$\mathbf{D}^{\infty}_{\infty}$	$\mathbf{D}^{\cdot}_{\infty}$	$\mathbf{D}^{-\infty}_{\infty}$	$\mathbf{D}^{\infty}_{\mathsf{n}}$	$\mathbf{D}^{\cdot}_{\mathsf{n}}$	$\mathbf{D}^{-\infty}_{\mathsf{n}}$
$\mathbf{D}^{\infty}_{\mathsf{m}}$	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**
$\mathbf{D}^{\cdot}_{\mathsf{m}}$	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**
$\mathbf{D}^{-\infty}_{\mathsf{m}}$	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**
$\mathbf{D}^{\infty}_{\infty}$	**X**	**X**	**X**	**X**	**X**	**X**	⇒	**T**	**F**
$\mathbf{D}^{\cdot}_{\infty}$	**X**	**X**	**X**	**X**	**X**	**X**	**T**	**T**	**F**
$\mathbf{D}^{-\infty}_{\infty}$	**X**	**X**	**X**	**X**	**X**	**X**	**F**	**T**	**T**

Mac rules

[**Mac rule StrictImpliesX** : $\bot \Rightarrow \mathsf{x} \equiv \bot$]
[**Mac rule XImpliesStrict** : $\mathsf{x} \Rightarrow \bot \equiv \bot$]
[**Mac rule SubXImpliesY** : $\langle \mathcal{A} \Rightarrow \mathcal{B} \mid \mathsf{x}{:=}\mathcal{S} \rangle \equiv \langle \mathcal{A} \mid \mathsf{x}{:=}\mathcal{S} \rangle \Rightarrow \langle \mathcal{B} \mid \mathsf{x}{:=}\mathcal{S} \rangle$]

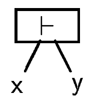
See also

$[\mathsf{x} \to \mathsf{y}]^{\cdot}$	Section A.57
$[\mathsf{x} \Leftrightarrow \mathsf{y}]^{\cdot}$	Section A.65
$[\mathsf{x} \Leftarrow \mathsf{y}]^{\cdot}$	Section A.66
$[\mathsf{x} \leq \mathsf{y}]^{\cdot}$	Section A.98

$[\mathsf{x} \Rightarrow \mathsf{y}]^{\cdot}$	x implies y
$[$ StrictImpliesX $]^{\cdot}$	rule StrictImpliesX
$[$ XImpliesStrict $]^{\cdot}$	rule XImpliesStrict
$[$ SubXImpliesY $]^{\cdot}$	rule SubXImpliesY

A.68 x infers y $[\,x \vdash y\,]^{\cdot}$

Parse tree



Sort

$[\,x \vdash y\,]^{\cdot}$ is a directive (i.e. it cannot occur in the syntax tree of a term).

Description

$[\,x \vdash y\,]^{\cdot}$ is used to form inference rules from equations.

Examples

$[\,x \equiv y \vdash y \equiv z \vdash x \equiv z\,]^{\cdot}$ states that if $[\,x\,]^{\cdot}$ equals $[\,y\,]^{\cdot}$ and $[\,y\,]^{\cdot}$ equals $[\,z\,]^{\cdot}$ then $[\,x\,]^{\cdot}$ equals $[\,z\,]^{\cdot}$.

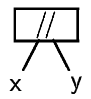
See also

$[\,x \rhd y\,]^{\cdot}$	Section A.56
$[\,x \rightarrow y\,]^{\cdot}$	Section A.57

$[\,x \vdash y\,]^{\cdot}$ x infers y

A.69 x integer divide y [x // y]˙

Parse tree



Sort

[x // y]˙ is an operator (i.e. it may occur in the syntax tree of a term).

Description

[x // y]˙ denotes [x]˙ divided by [y]˙, rounded to the nearest integer in the direction of [$-\infty$]˙. [x $//_+$ y]˙, [x $//_-$ y]˙, [x $//_0$ y]˙, and [x $//_2$ y]˙ are versions of [x // y]˙ that use round to plus infinity, minus infinity, zero, and even, respectively. [x $//_-$ y]˙ is identical to [x // y]˙. [x // y]˙ disregards the precision of [x]˙ and [y]˙. When [x]˙ is a finite decimal fraction and [y]˙ is a positive finite decimal fraction, [x // y]˙ produces an integer result, i.e. an exact fraction that is a whole number. [x // y]˙ equals [•]˙ if [y]˙ is zero or negative.

Type table

//	**B**	**D**	**X**	**E**	//	**N**	**Z**	**Z^+**	**Z^-**
B	**X**	**X**	**X**	**X**	**N**	–	–	**N**	**X**
D	**X**	–	**X**	**X**	**Z**	–	–	**Z**	**X**
X	**X**	**X**	**X**	**X**	**Z^+**	–	–	**N**	**X**
E	**X**	**X**	**X**	**X**	**Z^-**	–	–	**Z^-**	**X**
//	D_m^∞	$D_m^\cdot$	$D_m^{-\infty}$	D_∞^∞	$D_\infty^\cdot$	$D_\infty^{-\infty}$	D_n^∞	$D_n^\cdot$	$D_n^{-\infty}$
D_m^∞	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**
$D_m^\cdot$	**X**	–	**X**	**X**	–	**X**	**X**	–	**X**
$D_m^{-\infty}$	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**
D_∞^∞	**X**	**X**	**X**	**X**	**X**	**X**	//	**T**	**F**
$D_\infty^\cdot$	**X**	–	**X**	**X**	–	**X**	**T**	**X**	**X**
$D_\infty^{-\infty}$	**X**	**X**	**X**	**X**	**X**	**X**	**F**	**X**	**X**

Mac rules

[**Mac rule StrictIntegerDivideX** : $\perp$ // y $\equiv$ $\perp$]
[**Mac rule XIntegerDivideStrict** : x // $\perp$ $\equiv$ $\perp$]
[**Mac rule SubXIntegerDivideY** : $\langle\mathcal{A}$ // $\mathcal{B}$ | x:=$\mathcal{S}\rangle$ $\equiv$ $\langle\mathcal{A}$ | x:=$\mathcal{S}\rangle$ // $\langle\mathcal{B}$ | x:=$\mathcal{S}\rangle$]

[x // y]˙	x integer divide y
[StrictIntegerDivideX]˙	rule StrictIntegerDivideX
[XIntegerDivideStrict]˙	rule XIntegerDivideStrict
[SubXIntegerDivideY]˙	rule SubXIntegerDivideY

[**Mac rule IntegerDivideZero** : $x \in \mathbf{D} \vdash y \in \mathbf{D} \vdash y \leq 0 \vdash x \mathbin{//} y \in \mathbf{X}$]
[**Mac rule TypeDmIntegerDivideDm** : $m \in \mathbf{Z}^+ \vdash x \in \mathbf{D}^{\cdot}_{m} \vdash y \in \mathbf{D}^{\cdot}_{m} \vdash y > 0 \vdash x \mathbin{//} y \in \mathbf{D}^{\cdot}_{\infty}$]
[**Mac rule TypeDmIntegerDivideDi** : $m \in \mathbf{Z}^+ \vdash x \in \mathbf{D}^{\cdot}_{m} \vdash y \in \mathbf{D}^{\cdot}_{\infty} \vdash y > 0 \vdash x \mathbin{//} y \in \mathbf{D}^{\cdot}_{\infty}$]
[**Mac rule TypeDiIntegerDivideDm** : $m \in \mathbf{Z}^+ \vdash x \in \mathbf{D}^{\cdot}_{\infty} \vdash y \in \mathbf{D}^{\cdot}_{m} \vdash y > 0 \vdash x \mathbin{//} y \in \mathbf{D}^{\cdot}_{\infty}$]
[**Mac rule TypeDiIntegerDivideDi** : $x \in \mathbf{D}^{\cdot}_{\infty} \vdash y \in \mathbf{D}^{\cdot}_{\infty} \vdash y > 0 \vdash x \mathbin{//} y \in \mathbf{D}^{\cdot}_{\infty}$]
[**Mac rule TypeDmIntegerDivideDn** : $m \in \mathbf{Z}^+ \vdash n \in \mathbf{Z}^+ \vdash x \in \mathbf{D}^{\cdot}_{m} \vdash y \in \mathbf{D}^{\cdot}_{n} \vdash y > 0 \vdash x \mathbin{//} y \in \mathbf{D}^{\cdot}_{\infty}$]

See also

[x % y]˙	Section A.74
[x/y]˙	Section A.59

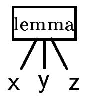
[IntegerDivideZero]˙	rule IntegerDivideZero
[TypeDmIntegerDivideDm]˙	rule TypeDmIntegerDivideDm
[TypeDmIntegerDivideDi]˙	rule TypeDmIntegerDivideDi
[TypeDiIntegerDivideDm]˙	rule TypeDiIntegerDivideDm
[TypeDiIntegerDivideDi]˙	rule TypeDiIntegerDivideDi
[TypeDmIntegerDivideDn]˙	rule TypeDmIntegerDivideDn

A.70 x lemma y colon z

Syntax

$\left[\,\boxed{\text{x } \textbf{lemma} \text{ y:z}}\,\right]^{\cdot}$.

Parse tree



Sort

[x **lemma** y:z]˙ is a directive (i.e. it cannot occur in the syntax tree of a term).

Description

Lemmas have the form [x **lemma** y:z]˙ where [y]˙ is the name of the lemma to be proved and [z]˙ is the statement that the lemma claims to hold. [x]˙ is the system in which the proof has to be stated. [x **lemma** y:z]˙ defines [y]˙ as shorthand for [z]˙. For each lemma there must be a proof of the lemma.

Examples

The following statement introduces L11.4.1 as shorthand for [(1 : : 2) head ≡ (2 : : 1) tail]˙.

[**Mac lemma L11.4.1** : (1 : : 2) head ≡ (2 : : 1) tail]

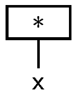
See also

[x **proof of** y:z]˙	Section A.83
[x **rule** y:z]˙	Section A.86

[x **lemma y:z**]˙ x lemma y colon z

A.71 x listset $[\, \mathsf{x}^* \,]^{\cdot}$

Parse tree



Sort

$[\, \mathsf{x}^* \,]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\, \mathsf{x}^* \,]^{\cdot}$ is the set of lists whose components belongs to $[\, \mathsf{x} \,]^{\cdot}$.

Mac rules

$[\, \textbf{Mac rule SubXListSet} : \langle \mathcal{A}^* \mid \mathsf{x}{:=}\mathcal{S} \rangle \equiv \langle \mathcal{A} \mid \mathsf{x}{:=}\mathcal{S} \rangle^* \,]$
$[\, \textbf{Mac rule SetXListSet} : \mathsf{A} \in \textbf{Set} \vdash \mathsf{A}^* \in \textbf{Set} \,]$
$[\, \textbf{Mac rule ElimListSet} : \mathsf{A} \in \textbf{Set} \vdash \mathsf{x} \in \mathsf{A}^* \vdash \mathsf{x} \in \mathbf{E} \vee \mathsf{x} \in \mathsf{A} \times \mathsf{A}^* \,]$
$[\, \textbf{Mac rule IntroListSet} : \mathsf{A} \in \textbf{Set} \vdash \mathsf{x} \in \mathbf{E} \vee \mathsf{x} \in \mathsf{A} \times \mathsf{A}^* \vdash \mathsf{x} \in \mathsf{A}^* \,]$
$[\, \textbf{Mac rule IntroListSetEmpty} : \mathsf{A} \in \textbf{Set} \vdash \langle\, \rangle \in \mathsf{A}^* \,]$
$[\, \textbf{Mac rule IntroListSetPair} : \mathsf{A} \in \textbf{Set} \vdash \mathsf{x} \in \mathsf{A} \vdash \mathsf{y} \in \mathsf{A}^* \vdash \mathsf{x} :: \mathsf{y} \in \mathsf{A}^* \,]$

Examples

$[[\, \langle 1,2,3 \rangle \in \mathbf{N}^*$	$\equiv$	$\mathsf{T} \,]^{\cdot}\,]^{\cdot}$
$[[\, \langle 0,1,2 \rangle \in (\mathbf{Z}^+)^*$	$\equiv$	$\mathsf{F} \,]^{\cdot}\,]^{\cdot}$
$[[\, \infty\text{-list}(1) \in \mathbf{N}^*$	$\equiv$	$\perp \,]^{\cdot}\,]^{\cdot}$

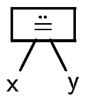
See also

$[\, \langle\, \rangle \,]^{\cdot}$	Section A.10
$[\, \mathbf{E} \,]^{\cdot}$	Section A.40
$[\, \langle \mathsf{x} \rangle \,]^{\cdot}$	Section A.44
$[\, \mathsf{x} \in \mathsf{y} \,]^{\cdot}$	Section A.50
$[\, \mathsf{x} \times \mathsf{y} \,]^{\cdot}$	Section A.51
$[\, \mathsf{x} :: \mathsf{y} \,]^{\cdot}$	Section A.79

$[\, \mathsf{x}^* \,]^{\cdot}$	x listset
$[$ SubXListSet $]^{\cdot}$	rule SubXListSet
$[$ SetXListSet $]^{\cdot}$	rule SetXListSet
$[$ ElimListSet $]^{\cdot}$	rule ElimListSet
$[$ IntroListSet $]^{\cdot}$	rule IntroListSet
$[$ IntroListSetEmpty $]^{\cdot}$	rule IntroListSetEmpty
$[$ IntroListSetPair $]^{\cdot}$	rule IntroListSetPair

A.72 x macro equal y $[\, \mathsf{x} \ddot{=} \mathsf{y} \,]^{\cdot}$

Parse tree



Sort

$[\, \mathsf{x} \ddot{=} \mathsf{y} \,]^{\cdot}$ is a directive (i.e. it cannot occur in the syntax tree of a term).

Description

$[\, \mathcal{A} \ddot{=} \mathcal{B} \,]^{\cdot}$ introduces $[\, \mathcal{A} \,]^{\cdot}$ as a macro construct which is shorthand for $[\, \mathcal{B} \,]^{\cdot}$. Contrary to constructs introduced by $[\, \mathcal{A} \doteq \mathcal{B} \,]^{\cdot}$, constructs introduced by $[\, \mathcal{A} \ddot{=} \mathcal{B} \,]^{\cdot}$ cannot occur in syntax trees. Constructs introduced by $[\, \mathcal{A} \ddot{=} \mathcal{B} \,]^{\cdot}$ can occur in parse trees but disappear when constructing the syntax tree. Constructs introduced by $[\, \mathcal{A} \ddot{=} \mathcal{B} \,]^{\cdot}$ may expand to terms or statement constructs.

Examples

$$\begin{array}{lcr} [\, \langle \mathsf{x} \rangle & \ddot{=} & \mathsf{x} :: \langle\, \rangle \,]^{\cdot} \\ [\, (\mathsf{x}) & \ddot{=} & \mathsf{x} \,]^{\cdot} \end{array}$$

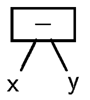
See also

$[\, \mathsf{x} \doteq \mathsf{y} \,]^{\cdot}$	Section A.58

$[\, \mathsf{x} \ddot{=} \mathsf{y} \,]^{\cdot}$ x macro equal y

A.73 x minus y $[\mathsf{x}-\mathsf{y}]^{\cdot}$

Parse tree



Sort

$[\mathsf{x}-\mathsf{y}]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

If $[\mathsf{x}]^{\cdot}$ and $[\mathsf{y}]^{\cdot}$ are exact fractions then $[\mathsf{x}-\mathsf{y}]^{\cdot}$ denotes the exact difference of $[\mathsf{x}]^{\cdot}$ and $[\mathsf{y}]^{\cdot}$. If $[\mathsf{x}]^{\cdot}$ and $[\mathsf{y}]^{\cdot}$ are floating fractions of equal precision, then $[\mathsf{x}-\mathsf{y}]^{\cdot}$ denotes the exact difference of $[\mathsf{x}]^{\cdot}$ and $[\mathsf{y}]^{\cdot}$ rounded to the common precision of $[\mathsf{x}]^{\cdot}$ and $[\mathsf{y}]^{\cdot}$ using round to even. If $[\mathsf{x}]^{\cdot}$ and $[\mathsf{y}]^{\cdot}$ are floating fractions of different precisions, then $[\mathsf{x}-\mathsf{y}]^{\cdot}$ equals $[\bullet]^{\cdot}$. If $[\mathsf{x}]^{\cdot}$ is an exact fraction and $[\mathsf{y}]^{\cdot}$ is a floating fraction or vice versa, then $[\mathsf{x}-\mathsf{y}]^{\cdot}$ denotes the exact difference of $[\mathsf{x}]^{\cdot}$ and $[\mathsf{y}]^{\cdot}$ rounded to the precision of the floating fraction using round to even. $[\mathsf{x} -_{+} \mathsf{y}]^{\cdot}$, $[\mathsf{x} -_{-} \mathsf{y}]^{\cdot}$, $[\mathsf{x} -_{0} \mathsf{y}]^{\cdot}$, and $[\mathsf{x} -_{2} \mathsf{y}]^{\cdot}$ are versions of $[\mathsf{x}-\mathsf{y}]^{\cdot}$ that use round to plus infinity, minus infinity, zero, and even, respectively. $[\mathsf{x} -_{2} \mathsf{y}]^{\cdot}$ is identical to $[\mathsf{x}-\mathsf{y}]^{\cdot}$.

Type table

−	**B**	**D**	**X**	**E**	−	**N**	**Z**	$\mathbf{Z}^{+}$	$\mathbf{Z}^{-}$
B	**X**	**X**	**X**	**X**	**N**	**Z**	**Z**	**Z**	$\mathbf{Z}^{+}$
D	**X**	−	**X**	**X**	**Z**	**Z**	**Z**	**Z**	**Z**
X	**X**	**X**	**X**	**X**	$\mathbf{Z}^{+}$	**Z**	**Z**	**Z**	$\mathbf{Z}^{+}$
E	**X**	**X**	**X**	**X**	$\mathbf{Z}^{-}$	$\mathbf{Z}^{-}$	**Z**	$\mathbf{Z}^{-}$	**Z**
−	$\mathbf{D}^{\infty}_{\mathsf{m}}$	$\mathbf{D}^{\cdot}_{\mathsf{m}}$	$\mathbf{D}^{-\infty}_{\mathsf{m}}$	$\mathbf{D}^{\infty}_{\infty}$	$\mathbf{D}^{\cdot}_{\infty}$	$\mathbf{D}^{-\infty}_{\infty}$	$\mathbf{D}^{\infty}_{\mathsf{n}}$	$\mathbf{D}^{\cdot}_{\mathsf{n}}$	$\mathbf{D}^{-\infty}_{\mathsf{n}}$
$\mathbf{D}^{\infty}_{\mathsf{m}}$	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**
$\mathbf{D}^{\cdot}_{\mathsf{m}}$	**X**	$\mathbf{D}^{\cdot}_{\mathsf{m}}$	**X**	**X**	$\mathbf{D}^{\cdot}_{\mathsf{m}}$	**X**	**X**	**X**	**X**
$\mathbf{D}^{-\infty}_{\mathsf{m}}$	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**
$\mathbf{D}^{\infty}_{\infty}$	**X**	**X**	**X**	**X**	**X**	**X**	−	**T**	**F**
$\mathbf{D}^{\cdot}_{\infty}$	**X**	$\mathbf{D}^{\cdot}_{\mathsf{m}}$	**X**	**X**	$\mathbf{D}^{\cdot}_{\infty}$	**X**	**T**	**X**	**X**
$\mathbf{D}^{-\infty}_{\infty}$	**X**	**X**	**X**	**X**	**X**	**X**	**F**	**X**	**X**

Mac rules

[**Mac rule StrictMinusX** : $\perp - \mathsf{x} \equiv \perp$]

$[\mathsf{x}-\mathsf{y}]^{\cdot}$	x minus y
$[\text{StrictMinusX}]^{\cdot}$	rule StrictMinusX

[**Mac rule XMinusStrict** : $\mathsf{x} - \bot \equiv \bot$]
[**Mac rule SubXMinusY** : $\langle \mathcal{A} - \mathcal{B} \mid \mathsf{x}{:=}\mathcal{S} \rangle \equiv \langle \mathcal{A} \mid \mathsf{x}{:=}\mathcal{S} \rangle - \langle \mathcal{B} \mid \mathsf{x}{:=}\mathcal{S} \rangle$]

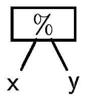
See also

$[\, \mathsf{x} + \mathsf{y} \,]^{\cdot}$	Section A.80
$[\, \mathsf{x@y} \,]^{\cdot}$	Section A.85
$[\, \mathsf{x} \cdot \mathsf{y} \,]^{\cdot}$	Section A.97

$[$ XMinusStrict $]^{\cdot}$	rule XMinusStrict
$[$ SubXMinusY $]^{\cdot}$	rule SubXMinusY

A.74 x modulo y [x % y]˙

Parse tree



Sort

[x % y]˙ is an operator (i.e. it may occur in the syntax tree of a term).

Description

[x % y]˙ denotes the remainder when integer dividing [x]˙ by [y]˙. [x % y]˙ disregards the precision of [x]˙ and [y]˙. When [x]˙ is a finite decimal fraction and [y]˙ is a positive finite decimal fraction, [x % y]˙ produces an exact fraction. [x % y]˙ equals [•]˙ if [y]˙ is zero or negative.

[x $\%_+$ y]˙, [x $\%_-$ y]˙, [x $\%_0$ y]˙, and [x $\%_2$ y]˙ are versions of [x % y]˙ that express the remainders of [x $//_+$ y]˙, [x $//_-$ y]˙, [x $//_0$ y]˙, and [x $//_2$ y]˙, respectively. [x $\%_-$ y]˙ is identical to [x % y]˙.

For exact decimal fractions [x]˙ and positive exact decimal fractions [y]˙ you have:

$$
\begin{array}{lcl}
[\,x \% y & \equiv & x - (x // y) \cdot y\,]^{\cdot} \\
[\,x \%_+ y & \equiv & x - (x //_+ y) \cdot y\,]^{\cdot} \\
[\,x \%_- y & \equiv & x - (x //_- y) \cdot y\,]^{\cdot} \\
[\,x \%_0 y & \equiv & x - (x //_0 y) \cdot y\,]^{\cdot} \\
[\,x \%_2 y & \equiv & x - (x //_2 y) \cdot y\,]^{\cdot}
\end{array}
$$

Type table

%	**B**	**D**	**X**	**E**	%	**N**	**Z**	$\mathbf{Z}^+$	$\mathbf{Z}^-$
B	**X**	**X**	**X**	**X**	**N**	–	–	**N**	**X**
D	**X**	–	**X**	**X**	**Z**	–	–	**N**	**X**
X	**X**	**X**	**X**	**X**	$\mathbf{Z}^+$	–	–	**N**	**X**
E	**X**	**X**	**X**	**X**	$\mathbf{Z}^-$	–	–	**N**	**X**
%	$\mathbf{D}_m^\infty$	$\mathbf{D}_m^\cdot$	$\mathbf{D}_m^{-\infty}$	$\mathbf{D}_\infty^\infty$	$\mathbf{D}_\infty^\cdot$	$\mathbf{D}_\infty^{-\infty}$	$\mathbf{D}_n^\infty$	$\mathbf{D}_n^\cdot$	$\mathbf{D}_n^{-\infty}$
$\mathbf{D}_m^\infty$	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**
$\mathbf{D}_m^\cdot$	**X**	–	**X**	**X**	–	**X**	**X**	–	**X**
$\mathbf{D}_m^{-\infty}$	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**
$\mathbf{D}_\infty^\infty$	**X**	**X**	**X**	**X**	**X**	**X**	%	**T**	**F**
$\mathbf{D}_\infty^\cdot$	**X**	–	**X**	**X**	–	**X**	**T**	**X**	**X**
$\mathbf{D}_\infty^{-\infty}$	**X**	**X**	**X**	**X**	**X**	**X**	**F**	**X**	**X**

[x % y]˙ x modulo y

Mac rules

[**Mac rule StrictModuloX** : $\perp \,\%\, \mathsf{x} \equiv \perp$]
[**Mac rule XModuloStrict** : $\mathsf{x} \,\%\, \perp \equiv \perp$]
[**Mac rule SubXModuloY** : $\langle \mathcal{A} \,\%\, \mathcal{B} \mid \mathsf{x}{:=}\mathcal{S}\rangle \equiv \langle \mathcal{A} \mid \mathsf{x}{:=}\mathcal{S}\rangle \,\%\, \langle \mathcal{B} \mid \mathsf{x}{:=}\mathcal{S}\rangle$]
[**Mac rule ModuloZero** : $\mathsf{x} \in \mathbf{D} \vdash \mathsf{y} \in \mathbf{D} \vdash \mathsf{y} \le 0 \vdash \mathsf{x} \,\%\, \mathsf{y} \in \mathbf{X}$]
[**Mac rule TypeDmModuloDm** : $\mathsf{m} \in \mathbf{Z}^{+} \vdash \mathsf{x} \in \mathbf{D}^{\cdot}_{\mathsf{m}} \vdash \mathsf{y} \in \mathbf{D}^{\cdot}_{\mathsf{m}} \vdash \mathsf{y} > 0 \vdash \mathsf{x} \,\%\, \mathsf{y} \in \mathbf{D}^{\cdot}_{\infty}$]
[**Mac rule TypeDmModuloDi** : $\mathsf{m} \in \mathbf{Z}^{+} \vdash \mathsf{x} \in \mathbf{D}^{\cdot}_{\mathsf{m}} \vdash \mathsf{y} \in \mathbf{D}^{\cdot}_{\infty} \vdash \mathsf{y} > 0 \vdash \mathsf{x} \,\%\, \mathsf{y} \in \mathbf{D}^{\cdot}_{\infty}$]
[**Mac rule TypeDiModuloDm** : $\mathsf{m} \in \mathbf{Z}^{+} \vdash \mathsf{x} \in \mathbf{D}^{\cdot}_{\infty} \vdash \mathsf{y} \in \mathbf{D}^{\cdot}_{\mathsf{m}} \vdash \mathsf{y} > 0 \vdash \mathsf{x} \,\%\, \mathsf{y} \in \mathbf{D}^{\cdot}_{\infty}$]
[**Mac rule TypeDiModuloDi** : $\mathsf{x} \in \mathbf{D}^{\cdot}_{\infty} \vdash \mathsf{y} \in \mathbf{D}^{\cdot}_{\infty} \vdash \mathsf{y} > 0 \vdash \mathsf{x} \,\%\, \mathsf{y} \in \mathbf{D}^{\cdot}_{\infty}$]
[**Mac rule TypeDmModuloDn** : $\mathsf{m} \in \mathbf{Z}^{+} \vdash \mathsf{n} \in \mathbf{Z}^{+} \vdash \mathsf{x} \in \mathbf{D}^{\cdot}_{\mathsf{m}} \vdash \mathsf{y} \in \mathbf{D}^{\cdot}_{\mathsf{n}} \vdash \mathsf{y} > 0 \vdash \mathsf{x} \,\%\, \mathsf{y} \in \mathbf{D}^{\cdot}_{\infty}$]

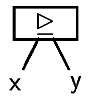
See also

$[\,\mathsf{x} \mathbin{//} \mathsf{y}\,]^{\cdot}$ Section A.69

$[$ StrictModuloX $]^{\cdot}$	rule StrictModuloX
$[$ XModuloStrict $]^{\cdot}$	rule XModuloStrict
$[$ SubXModuloY $]^{\cdot}$	rule SubXModuloY
$[$ ModuloZero $]^{\cdot}$	rule ModuloZero
$[$ TypeDmModuloDm $]^{\cdot}$	rule TypeDmModuloDm
$[$ TypeDmModuloDi $]^{\cdot}$	rule TypeDmModuloDi
$[$ TypeDiModuloDm $]^{\cdot}$	rule TypeDiModuloDm
$[$ TypeDiModuloDi $]^{\cdot}$	rule TypeDiModuloDi
$[$ TypeDmModuloDn $]^{\cdot}$	rule TypeDmModuloDn

A.75 x modus ponens y $[\,\mathsf{x} \trianglerighteq \mathsf{y}\,]^{\cdot}$

Parse tree



Sort

$[\,\mathsf{x} \trianglerighteq \mathsf{y}\,]^{\cdot}$ is a directive (i.e. it cannot occur in the syntax tree of a term).

Description

$[\,\mathsf{x} \trianglerighteq \mathsf{y}\,]^{\cdot}$ is used to form argumentations from equations using the infererence rules of Modus Ponens.

Examples

$$[\,\mathsf{x} \in \mathbf{Z} \to \mathsf{x} + 2 \in \mathbf{Z} \trianglerighteq \mathsf{x} \in \mathbf{Z} \triangleright \mathsf{x} + 2 \in \mathbf{Z}\,]^{\cdot}$$

states that $[\,\mathsf{x} + 2 \in \mathbf{Z}\,]^{\cdot}$ follows from $[\,\mathsf{x} \in \mathbf{Z} \to \mathsf{x} + 2 \in \mathbf{Z}\,]^{\cdot}$ and $[\,\mathsf{x} \in \mathbf{Z}\,]^{\cdot}$ according to the inference rule of ModusPonens. The statement is shorthand for

$$[\,\text{ModusPonens} \triangleright \mathsf{x} \in \mathbf{Z} \triangleright \mathsf{x} \in \mathbf{Z} \to \mathsf{x} + 2 \in \mathbf{Z} \triangleright \mathsf{x} + 2 \in \mathbf{Z}\,]^{\cdot}.$$

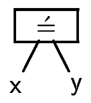
See also

$[\,\mathsf{x} \triangleright \mathsf{y}\,]^{\cdot}$	Section A.56
$[\,\mathsf{x} \to \mathsf{y}\,]^{\cdot}$	Section A.57

$[\,\mathsf{x} \trianglerighteq \mathsf{y}\,]^{\cdot}$ x modus ponens y

A.76 x optimised equal y $[\mathsf{x} \acute{=} \mathsf{y}]^{\cdot}$

Parse tree



Sort

$[\mathsf{x} \acute{=} \mathsf{y}]^{\cdot}$ is a directive (i.e. it cannot occur in the syntax tree of a term).

Description

$[\mathcal{A} \acute{=} \mathcal{B}]^{\cdot}$ means the same as $[\mathcal{A} \doteq \mathcal{B}]^{\cdot}$ but may affect the time Map spends on computing $[\mathcal{A}]^{\cdot}$.

Examples

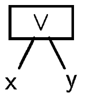
$[\mathsf{n}! \acute{=} \mathrm{if}(\mathsf{n} = 0, 1, \mathsf{n} \cdot (\mathsf{n} - 1)!)]$.

See also

$[\mathsf{x} \doteq \mathsf{y}]^{\cdot}$ Section A.58

A.77 x or y $[\,\mathsf{x} \vee \mathsf{y}\,]^{\cdot}$

Parse tree



Sort

$[\,\mathsf{x} \vee \mathsf{y}\,]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\,\mathsf{x} \vee \mathsf{y}\,]^{\cdot}$ denotes the maximum of $[\,\mathsf{x}\,]^{\cdot}$ and $[\,\mathsf{y}\,]^{\cdot}$. In particular, for truth values $[\,\mathsf{x}\,]^{\cdot}$ and $[\,\mathsf{y}\,]^{\cdot}$, $[\,\mathsf{x} \vee \mathsf{y}\,]^{\cdot}$ denotes "$[\,\mathsf{x}\,]^{\cdot}$ or $[\,\mathsf{y}\,]^{\cdot}$". For truth values $[\,\mathsf{x}\,]^{\cdot}$ and $[\,\mathsf{y}\,]^{\cdot}$, $[\,\mathsf{x}{\vee}\mathsf{y}\,]^{\cdot}$ coincides with $[\,\mathsf{x}{+}\mathsf{y}\,]^{\cdot}$. $[\,\mathsf{x}{\vee}\mathsf{y} \equiv \bullet\,]^{\cdot}$ if $[\,\mathsf{x}\,]^{\cdot}$ and $[\,\mathsf{y}\,]^{\cdot}$ are decimal fractions of different precision.

Type table

$\vee$	**B**	**D**	**X**	**E**	$\vee$	**N**	**Z**	$\mathbf{Z}^{+}$	$\mathbf{Z}^{-}$
B	**B**	**X**	**X**	**X**	**N**	**N**	**N**	$\mathbf{Z}^{+}$	**N**
D	**X**	–	**X**	**X**	**Z**	**N**	**Z**	$\mathbf{Z}^{+}$	**Z**
X	**X**	**X**	**X**	**X**	$\mathbf{Z}^{+}$	$\mathbf{Z}^{+}$	$\mathbf{Z}^{+}$	$\mathbf{Z}^{+}$	$\mathbf{Z}^{+}$
E	**X**	**X**	**X**	**X**	$\mathbf{Z}^{-}$	**N**	**Z**	$\mathbf{Z}^{+}$	$\mathbf{Z}^{-}$
$\vee$	$\mathbf{D}^{\infty}_{m}$	$\mathbf{D}^{\cdot}_{m}$	$\mathbf{D}^{-\infty}_{m}$	$\mathbf{D}^{\infty}_{\infty}$	$\mathbf{D}^{\cdot}_{\infty}$	$\mathbf{D}^{-\infty}_{\infty}$	$\mathbf{D}^{\infty}_{n}$	$\mathbf{D}^{\cdot}_{n}$	$\mathbf{D}^{-\infty}_{n}$
$\mathbf{D}^{\infty}_{m}$	$\mathbf{D}^{\infty}_{m}$	$\mathbf{D}^{\infty}_{m}$	$\mathbf{D}^{\infty}_{m}$	**X**	**X**	**X**	**X**	**X**	**X**
$\mathbf{D}^{\cdot}_{m}$	$\mathbf{D}^{\infty}_{m}$	$\mathbf{D}^{\cdot}_{m}$	$\mathbf{D}^{\cdot}_{m}$	**X**	**X**	**X**	**X**	**X**	**X**
$\mathbf{D}^{-\infty}_{m}$	$\mathbf{D}^{\infty}_{m}$	$\mathbf{D}^{\cdot}_{m}$	$\mathbf{D}^{-\infty}_{m}$	**X**	**X**	**X**	**X**	**X**	**X**
$\mathbf{D}^{\infty}_{\infty}$	**X**	**X**	**X**	$\mathbf{D}^{\infty}_{\infty}$	$\mathbf{D}^{\infty}_{\infty}$	$\mathbf{D}^{\infty}_{\infty}$	$\vee$	**T**	**F**
$\mathbf{D}^{\cdot}_{\infty}$	**X**	**X**	**X**	$\mathbf{D}^{\infty}_{\infty}$	$\mathbf{D}^{\cdot}_{\infty}$	$\mathbf{D}^{\cdot}_{\infty}$	**T**	**T**	**T**
$\mathbf{D}^{-\infty}_{\infty}$	**X**	**X**	**X**	$\mathbf{D}^{\infty}_{\infty}$	$\mathbf{D}^{\cdot}_{\infty}$	$\mathbf{D}^{-\infty}_{\infty}$	**F**	**T**	**F**

Mac rules

[**Mac rule StrictOrX** : $\perp \vee \mathsf{x} \equiv \perp$]
[**Mac rule XOrStrict** : $\mathsf{x} \vee \perp \equiv \perp$]
[**Mac rule SubXOrY** : $\langle \mathcal{A} \vee \mathcal{B} \mid \mathsf{x}{:=}\mathcal{S} \rangle \equiv \langle \mathcal{A} \mid \mathsf{x}{:=}\mathcal{S} \rangle \vee \langle \mathcal{B} \mid \mathsf{x}{:=}\mathcal{S} \rangle$]

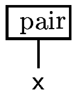
$[\,\mathsf{x} \vee \mathsf{y}\,]^{\cdot}$	x or y
$[$ StrictOrX $]^{\cdot}$	rule StrictOrX
$[$ XOrStrict $]^{\cdot}$	rule XOrStrict
$[$ SubXOrY $]^{\cdot}$	rule SubXOrY

See also

$[\,\mathsf{x} \wedge \mathsf{y}\,]^{\cdot}$	Section A.46
$[\,\mathsf{x} + \mathsf{y}\,]^{\cdot}$	Section A.80

A.78 **x pair** [x pair]˙

Parse tree



Sort

[x pair]˙ is an operator (i.e. it may occur in the syntax tree of a term).

Description

[x pair]˙ is true if [x]˙ is a pair. Truth values, exceptions, decimal fractions, and the empty list are atoms rather than pairs. [⊥]˙ is neither an atom nor a pair.

Type table

x	**B**	**D**	**X**	**E**	**N**	**Z**	$\mathbf{Z}^+$	$\mathbf{Z}^-$
x pair	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**F**

x	$\mathbf{D}_m^{\infty}$	$\mathbf{D}_m^{\cdot}$	$\mathbf{D}_m^{-\infty}$	$\mathbf{D}_\infty^{\infty}$	$\mathbf{D}_\infty^{\cdot}$	$\mathbf{D}_\infty^{-\infty}$	**T**	**F**
x pair	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**F**

Mac rules

[**Mac rule PairIsPair** : (x :: y) pair ≡ T]
[**Mac rule SubXPair** : $\langle \mathcal{A}$ pair | x:=$\mathcal{S}\rangle \equiv \langle \mathcal{A}$ | x:=$\mathcal{S}\rangle$ pair]
[**Mac rule StrictPair** : ⊥ pair ≡ ⊥]

Examples

[2.0F pair	≡	F]˙
[⟨x, y⟩ pair	≡	T]˙

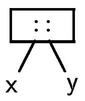
See also

[x atom]˙	Section A.49
[x :: y]˙	Section A.79

[x pair]˙	x pair
[PairIsPair]˙	rule PairIsPair
[SubXPair]˙	rule SubXPair
[StrictPair]˙	rule StrictPair

A.79 x pair y [x :: y]˙

Parse tree



Sort

[x :: y]˙ is an operator (i.e. it may occur in the syntax tree of a term).

Description

[x :: y]˙ denotes the pair whose first element is [x]˙ and whose second element is [y]˙. [x :: y]˙ aggregates two mathematical values [x]˙ and [y]˙ into a single mathematical value such that they can be taken appart using [z head]˙ and [z tail]˙ and in such a way that [z pair]˙ can recognise the aggregate as a pair. [x :: y]˙ contains the information contained in [x]˙ plus the information contained in [y]˙ plus the information that it is a pair. [x :: y]˙ is non-strict in both [x]˙ and [y]˙. [x :: y]˙ is known as “lazy cons” in the litterature.

[x :: y $\equiv$ u :: v]˙ holds if [x $\equiv$ u]˙ and [y $\equiv$ v]˙ both hold. [x :: y $\equiv$ u :: v]˙ fails if [x $\equiv$ u]˙ fails or [y $\equiv$ v]˙ fails or both fail. [x :: y $\equiv$ z]˙ fails for all [x]˙ and [y]˙ if [z]˙ is a truth value, a decimal fraction, an exception, the empty list, or [$\perp$]˙. In particular, [$\perp$ :: $\perp$ $\equiv$ $\perp$]˙ fails.

Mac rules

[**Mac rule SubXPairY** : $\langle \mathcal{A} :: \mathcal{B} \mid \mathsf{x}{:=}\mathcal{S}\rangle \equiv \langle \mathcal{A} \mid \mathsf{x}{:=}\mathcal{S}\rangle :: \langle \mathcal{B} \mid \mathsf{x}{:=}\mathcal{S}\rangle$]

Examples

[$\langle 1,2,3\rangle \quad \equiv \quad 1 :: 2 :: 3 :: \langle\,\rangle$]˙

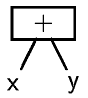
See also

[x × y]˙	Section A.51
[x head]˙	Section A.64
[x tail]˙	Section A.95
[x pair]˙	Section A.78
[x atom]˙	Section A.49

[x :: y]˙	x pair y
[SubXPairY]˙	rule SubXPairY

A.80 x plus y $[\mathsf{x} + \mathsf{y}]^{\cdot}$

Parse tree



Sort

$[\mathsf{x} + \mathsf{y}]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

If $[\mathsf{x}]^{\cdot}$ and $[\mathsf{y}]^{\cdot}$ are truth values then $[\mathsf{x} + \mathsf{y}]^{\cdot}$ coincides with $[\mathsf{x} \vee \mathsf{y}]^{\cdot}$. If $[\mathsf{x}]^{\cdot}$ and $[\mathsf{y}]^{\cdot}$ are exact fractions then $[\mathsf{x} + \mathsf{y}]^{\cdot}$ denotes the exact sum of $[\mathsf{x}]^{\cdot}$ and $[\mathsf{y}]^{\cdot}$. If $[\mathsf{x}]^{\cdot}$ and $[\mathsf{y}]^{\cdot}$ are floating fractions of equal precision, then $[\mathsf{x} + \mathsf{y}]^{\cdot}$ denotes the exact sum of $[\mathsf{x}]^{\cdot}$ and $[\mathsf{y}]^{\cdot}$ rounded to the common precision of $[\mathsf{x}]^{\cdot}$ and $[\mathsf{y}]^{\cdot}$ using round to even. If $[\mathsf{x}]^{\cdot}$ and $[\mathsf{y}]^{\cdot}$ are floating fractions of different precisions, then $[\mathsf{x} + \mathsf{y}]^{\cdot}$ equals $[\bullet]^{\cdot}$. If $[\mathsf{x}]^{\cdot}$ is an exact fraction and $[\mathsf{y}]^{\cdot}$ is a floating fraction or vice versa, then $[\mathsf{x} + \mathsf{y}]^{\cdot}$ denotes the exact sum of $[\mathsf{x}]^{\cdot}$ and $[\mathsf{y}]^{\cdot}$ rounded to the precision of the floating fraction using round to even. $[\mathsf{x} +_{+} \mathsf{y}]^{\cdot}$, $[\mathsf{x} +_{-} \mathsf{y}]^{\cdot}$, $[\mathsf{x} +_{0} \mathsf{y}]^{\cdot}$, and $[\mathsf{x} +_{2} \mathsf{y}]^{\cdot}$ are versions of $[\mathsf{x} + \mathsf{y}]^{\cdot}$ that use round to plus infinity, minus infinity, zero, and even, respectively. $[\mathsf{x} +_{2} \mathsf{y}]^{\cdot}$ is identical to $[\mathsf{x} + \mathsf{y}]^{\cdot}$.

Type table

+	**B**	**D**	**X**	**E**	+	**N**	**Z**	$\mathbf{Z}^{+}$	$\mathbf{Z}^{-}$
B	**B**	**X**	**X**	**X**	**N**	**N**	**Z**	$\mathbf{Z}^{+}$	**Z**
D	**X**	–	**X**	**X**	**Z**	**Z**	**Z**	**Z**	**Z**
X	**X**	**X**	**X**	**X**	$\mathbf{Z}^{+}$	$\mathbf{Z}^{+}$	**Z**	$\mathbf{Z}^{+}$	**Z**
E	**X**	**X**	**X**	**X**	$\mathbf{Z}^{-}$	**Z**	**Z**	**Z**	$\mathbf{Z}^{-}$
+	$\mathbf{D}^{\infty}_{\mathsf{m}}$	$\mathbf{D}^{\cdot}_{\mathsf{m}}$	$\mathbf{D}^{-\infty}_{\mathsf{m}}$	$\mathbf{D}^{\infty}_{\infty}$	$\mathbf{D}^{\cdot}_{\infty}$	$\mathbf{D}^{-\infty}_{\infty}$	$\mathbf{D}^{\infty}_{\mathsf{n}}$	$\mathbf{D}^{\cdot}_{\mathsf{n}}$	$\mathbf{D}^{-\infty}_{\mathsf{n}}$
$\mathbf{D}^{\infty}_{\mathsf{m}}$	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**
$\mathbf{D}^{\cdot}_{\mathsf{m}}$	**X**	$\mathbf{D}^{\cdot}_{\mathsf{m}}$	**X**	**X**	$\mathbf{D}^{\cdot}_{\mathsf{m}}$	**X**	**X**	**X**	**X**
$\mathbf{D}^{-\infty}_{\mathsf{m}}$	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**
$\mathbf{D}^{\infty}_{\infty}$	**X**	**X**	**X**	**X**	**X**	**X**	+	**T**	**F**
$\mathbf{D}^{\cdot}_{\infty}$	**X**	$\mathbf{D}^{\cdot}_{\mathsf{m}}$	**X**	**X**	$\mathbf{D}^{\cdot}_{\infty}$	**X**	**T**	**T**	**T**
$\mathbf{D}^{-\infty}_{\infty}$	**X**	**X**	**X**	**X**	**X**	**X**	**F**	**T**	**F**

Mac rules

[**Mac rule StrictPlusX** : $\perp + \mathsf{x} \equiv \perp$]

$[\mathsf{x} + \mathsf{y}]^{\cdot}$	x plus y
$[\,\text{StrictPlusX}\,]^{\cdot}$	rule StrictPlusX

[**Mac rule XPlusStrict** : $\mathsf{x} + \bot \equiv \bot$]
[**Mac rule SubXPlusY** : $\langle \mathcal{A} + \mathcal{B} \mid \mathsf{x}{:=}\mathcal{S} \rangle \equiv \langle \mathcal{A} \mid \mathsf{x}{:=}\mathcal{S} \rangle + \langle \mathcal{B} \mid \mathsf{x}{:=}\mathcal{S} \rangle$]

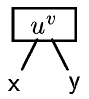
See also

[$\mathsf{x} - \mathsf{y}$]˙	Section A.73
[$\mathsf{x} \vee \mathsf{y}$]˙	Section A.77
[x@y]˙	Section A.85
[$\mathsf{x} \cdot \mathsf{y}$]˙	Section A.97

[XPlusStrict]˙	rule XPlusStrict
[SubXPlusY]˙	rule SubXPlusY

A.81 x power y end $[\, \mathsf{x}^{\mathsf{y}} \,]^{\cdot}$

Parse tree



Sort

$[\, \mathsf{x}^{\mathsf{y}} \,]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\, \mathsf{x}^{\mathsf{y}} \,]^{\cdot}$ denotes $[\, \mathsf{x} \,]^{\cdot}$ raised to the power of $[\, \mathsf{y} \,]^{\cdot}$. For decimal fractions $[\, \mathsf{x} \,]^{\cdot}$ and $[\, \mathsf{y} \,]^{\cdot}$, $[\, \mathsf{x}^{\mathsf{y}} \,]^{\cdot}$ differs from $[\, \bullet \,]^{\cdot}$ in the following cases:

(a) $[\, \mathsf{x} \,]^{\cdot}$ and $[\, \mathsf{y} \,]^{\cdot}$ are floating fractions with equal precision and $[\, \mathsf{x} \,]^{\cdot}$ is positive. In this case $[\, \mathsf{x}^{\mathsf{y}} \,]^{\cdot}$ has the same precision as $[\, \mathsf{x} \,]^{\cdot}$ and $[\, \mathsf{y} \,]^{\cdot}$.

(b) $[\, \mathsf{x} \,]^{\cdot}$ is a floating fraction and $[\, \mathsf{y} \,]^{\cdot}$ is an exact fraction or vice versa, and $[\, \mathsf{x} \,]^{\cdot}$ is positive. In this case $[\, \mathsf{x}^{\mathsf{y}} \,]^{\cdot}$ has the same precision as the floating fraction.

(c) $[\, \mathsf{x} \,]^{\cdot}$ is a decimal fraction and $[\, \mathsf{y} \,]^{\cdot}$ is a natural number. In this case $[\, \mathsf{x}^{\mathsf{y}} \,]^{\cdot}$ has the same precision as $[\, \mathsf{x} \,]^{\cdot}$. In particular, $[\, \mathsf{x}^{0} \equiv 1@\#\mathsf{x} \,]^{\cdot}$.

(d) $[\, \mathsf{x} = 10 \,]^{\cdot}$ and $[\, \mathsf{y} \,]^{\cdot}$ is an integer. In this case $[\, \mathsf{x}^{\mathsf{y}} \,]^{\cdot}$ has precision $[\, \infty \,]^{\cdot}$.

Type table

u^v	**B**	**D**	**X**	**E**
B	**X**	**X**	**X**	**X**
D	**X**	–	**X**	**X**
X	**X**	**X**	**X**	**X**
E	**X**	**X**	**X**	**X**

u^v	**N**	**Z**	$\mathbf{Z}^{+}$	$\mathbf{Z}^{-}$
N	**N**	–	**N**	–
Z	**Z**	–	**Z**	–
$\mathbf{Z}^{+}$	$\mathbf{Z}^{+}$	–	$\mathbf{Z}^{+}$	–
$\mathbf{Z}^{-}$	**Z**	–	**Z**	–

u^v	$\mathbf{D}^{\infty}_{m}$	$\mathbf{D}^{\cdot}_{m}$	$\mathbf{D}^{-\infty}_{m}$	$\mathbf{D}^{\infty}_{\infty}$	$\mathbf{D}^{\cdot}_{\infty}$	$\mathbf{D}^{-\infty}_{\infty}$	$\mathbf{D}^{\infty}_{n}$	$\mathbf{D}^{\cdot}_{n}$	$\mathbf{D}^{-\infty}_{n}$
$\mathbf{D}^{\infty}_{m}$	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**
$\mathbf{D}^{\cdot}_{m}$	**X**	–	**X**	**X**	–	**X**	**X**	–	**X**
$\mathbf{D}^{-\infty}_{m}$	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**
$\mathbf{D}^{\infty}_{\infty}$	**X**	**X**	**X**	**X**	**X**	**X**			
$\mathbf{D}^{\cdot}_{\infty}$	**X**	–	**X**	**X**	–	**X**			
$\mathbf{D}^{-\infty}_{\infty}$	**X**	**X**	**X**	**X**	**X**	**X**			

u^v	**T**	**F**
T	**X**	**X**
F	**X**	**X**

$[\, \mathsf{x}^{\mathsf{y}} \,]^{\cdot}$ x power y end

Mac rules

[**Mac rule StrictPowerXEnd** : $\perp^{\mathsf{x}} \equiv \perp$]
[**Mac rule XPowerStrictEnd** : $\mathsf{x}^{\perp} \equiv \perp$]
[**Mac rule SubXPowerYEnd** : $\langle \mathcal{A}^{\mathcal{B}} \mid \mathsf{x}{:=}\mathcal{S} \rangle \equiv \langle \mathcal{A} \mid \mathsf{x}{:=}\mathcal{S} \rangle^{\langle \mathcal{B} \mid \mathsf{x}:=\mathcal{S} \rangle}$]

[StrictPowerXEnd]$^\cdot$	rule StrictPowerXEnd
[XPowerStrictEnd]$^\cdot$	rule XPowerStrictEnd
[SubXPowerYEnd]$^\cdot$	rule SubXPowerYEnd

A.82 x predecessor $[\,\mathsf{x}^-\,]^{\cdot}$

Parse tree

$$\begin{tikzcd} \boxed{\mathsf{u}^-} \arrow[d, no head] \\ \mathsf{x} \end{tikzcd}$$

Sort

$[\,\mathsf{x}^-\,]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\,\mathsf{x}^-\,]^{\cdot}$ denotes $[\,\mathsf{x}-1\,]^{\cdot}$. All integers can be expressed using $[\,0\,]^{\cdot}$, $[\,\mathsf{x}^+\,]^{\cdot}$ and $[\,\mathsf{x}^-\,]^{\cdot}$.

Definition

$[\,\mathsf{x}^- \doteq \mathsf{x}-1\,]$.

Type table

x	**B**	**D**	**X**	**E**	**N**	**Z**	$\mathbf{Z}^+$	$\mathbf{Z}^-$
x tail	**X**	–	**X**	**X**	**Z**	**Z**	**N**	$\mathbf{Z}^-$

x	$\mathbf{D}^{\infty}_{\mathsf{m}}$	$\mathbf{D}^{\cdot}_{\mathsf{m}}$	$\mathbf{D}^{-\infty}_{\mathsf{m}}$	$\mathbf{D}^{\infty}_{\infty}$	$\mathbf{D}^{\cdot}_{\infty}$	$\mathbf{D}^{-\infty}_{\infty}$	**T**	**F**
x tail	**X**	$\mathbf{D}^{\cdot}_{\mathsf{m}}$	**X**	**X**	$\mathbf{D}^{\cdot}_{\infty}$	**X**	**X**	**X**

Mac rules

[**Mac rule SubXPredecessor** : $\langle \mathcal{A}^- \mid \mathsf{x}{:=}\mathcal{S}\rangle \equiv \langle \mathcal{A} \mid \mathsf{x}{:=}\mathcal{S}\rangle^-$]

Examples

$[\,5^{---} \equiv 2\,]^{\cdot}$
$[\,0^{-----} \equiv -5\,]^{\cdot}$

See also

$[\,\mathsf{x}^+\,]^{\cdot}$ Section A.94

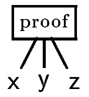
$[\,\mathsf{x}^-\,]^{\cdot}$	x predecessor
[SubXPredecessor]$^{\cdot}$	rule SubXPredecessor

A.83 x proof of y colon z

Syntax

$\left[\,\boxed{\mathsf{x}\ \textbf{proof of}\ \mathsf{y{:}z}}\,\right]^{\cdot}$.

Parse tree



Sort

[x **proof of** y:z]˙ is a directive (i.e. it cannot occur in the syntax tree of a term).

Description

Proofs have the form [x **proof of** y:z]˙ where [y]˙ is the lemma to be proved, [z]˙ is a list of instantiation statements separated by semicolons, and [x]˙ is the axiomatic system in use.

Examples

[**Mac proof of** L11.4.1**:**

L1 :	HeadPair ▷	(1 : : 2) head ≡ 1	;
L2 :	TailPair ▷	(2 : : 1) tail ≡ 1	;
L3 :	Commutativity ▷ L2 ▷	1 ≡ (2 : : 1) tail	;
L4 :	Transitivity ▷ L1 ▷ L3 ▷	(1 : : 2) head ≡ (2 : : 1) tail	]

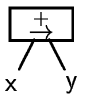
See also

[x **lemma** y:z]˙	Section A.70
[x **rule** y:z]˙	Section A.86
[x: y]˙	Section A.52
[x ▷ y]˙	Section A.56
[x; y]˙	Section A.88

[x **proof of** y:z]˙ x proof of y colon z

A.84 x reduces to y $\left[\mathsf{x} \overset{+}{\to} \mathsf{y} \right]^{\cdot}$

Parse tree



Sort

$\left[\mathsf{x} \overset{+}{\to} \mathsf{y} \right]^{\cdot}$ is a directive (i.e. it cannot occur in the syntax tree of a term).

Description

$\left[\mathcal{A} \overset{+}{\to} \mathcal{B} \right]^{\cdot}$ states that $[\, \mathsf{x} \,]^{\cdot}$ reduces to $[\, \mathsf{y} \,]^{\cdot}$ in finite, non-zero time. If $\left[\mathcal{A} \overset{+}{\to} \mathcal{B} \right]^{\cdot}$ then $[\, \mathcal{A} \equiv \mathcal{B} \,]^{\cdot}$.

Examples

$\left[2 \cdot 3 + 4 \overset{+}{\to} 6 + 4 \overset{+}{\to} 10 \right]^{\cdot}$

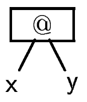
See also

$[\, \mathsf{x} \equiv \mathsf{y} \,]^{\cdot}$	Section A.60
$\left[\mathsf{x} \overset{\circ}{\to} \mathsf{y} \right]^{\cdot}$	Section A.96

$\left[\mathsf{x} \overset{+}{\to} \mathsf{y} \right]^{\cdot}$ x reduces to y

A.85 x round y [x@y]˙

Parse tree



Sort

[x@y]˙ is an operator (i.e. it may occur in the syntax tree of a term).

Description

If [x]˙ is a decimal fraction and [y]˙ is a positive integer or [$+\infty$]˙, then [x@y]˙ rounds [x]˙ to precision [y]˙. If [y]˙ is weakly greater than the precision of [x]˙, then [x@y]˙ extends the precision of [x]˙ without affecting the value of [x]˙. If [y]˙ is strongly less than the precision of [x]˙, then [x@y]˙ rounds [x]˙ to the nearest value of precision [y]˙. If two values of precision [y]˙ are equally distant from [x]˙, then [x@y]˙ chooses the one of the two values whose least significant digit is even. [$x@_{+}y$]˙, [$x@_{-}y$]˙, [$x@_{0}y$]˙, and [$x@_{2}y$]˙ are versions of [x@y]˙ that round towards plus infinity, minus infinity, zero, and even, respectively. [$x@_{2}y$]˙ is identical to [x@y]˙.

Type table

@	**B**	**D**	**X**	**E**
B	**X**	**X**	**X**	**X**
D	**X**	–	**X**	**X**
X	**X**	**X**	**X**	**X**
E	**X**	**X**	**X**	**X**

@	**N**	**Z**	$\mathbf{Z}^{+}$	$\mathbf{Z}^{-}$
N	–	–	**D**	–
Z	–	–	**D**	–
$\mathbf{Z}^{+}$	–	–	**D**	–
$\mathbf{Z}^{-}$	–	–	**D**	–

@	$\mathbf{D}^{\infty}_{m}$	$\mathbf{D}^{\cdot}_{m}$	$\mathbf{D}^{-\infty}_{m}$	$\mathbf{D}^{\infty}_{\infty}$	$\mathbf{D}^{\cdot}_{\infty}$	$\mathbf{D}^{-\infty}_{\infty}$	$\mathbf{D}^{\infty}_{n}$	$\mathbf{D}^{\cdot}_{n}$	$\mathbf{D}^{-\infty}_{n}$
$\mathbf{D}^{\infty}_{m}$	**X**	**X**	**X**	$\mathbf{D}^{\infty}_{\infty}$	–	**X**	**X**	**X**	**X**
$\mathbf{D}^{\cdot}_{m}$	**X**	**X**	**X**	$\mathbf{D}^{\cdot}_{\infty}$	–	**X**	**X**	**X**	**X**
$\mathbf{D}^{-\infty}_{m}$	**X**	**X**	**X**	$\mathbf{D}^{-\infty}_{\infty}$	–	**X**	**X**	**X**	**X**
$\mathbf{D}^{\infty}_{\infty}$	**X**	**X**	**X**	$\mathbf{D}^{\infty}_{\infty}$	–	**X**			
$\mathbf{D}^{\cdot}_{\infty}$	**X**	**X**	**X**	$\mathbf{D}^{\cdot}_{\infty}$	–	**X**			
$\mathbf{D}^{-\infty}_{\infty}$	**X**	**X**	**X**	$\mathbf{D}^{-\infty}_{\infty}$	–	**X**			

@	**T**	**F**
T	**X**	**X**
F	**X**	**X**

Mac rules

[**Mac rule StrictRoundX** : $\bot$@x $\equiv$ $\bot$]
[**Mac rule XRoundStrict** : x@$\bot$ $\equiv$ $\bot$]

[x@y]˙	x round y
[StrictRoundX]˙	rule StrictRoundX
[XRoundStrict]˙	rule XRoundStrict

[**Mac rule SubRound** : $\langle \mathcal{A}@\mathcal{B} \mid \mathsf{x}{:=}\mathcal{S}\rangle \equiv \langle \mathcal{A} \mid \mathsf{x}{:=}\mathcal{S}\rangle@\langle \mathcal{B} \mid \mathsf{x}{:=}\mathcal{S}\rangle$]
[**Mac rule RoundException** : $\mathsf{x} \in \mathbf{D} \vdash \mathsf{y} \in \mathbf{D} \vdash \mathsf{y} \notin \mathbf{Z}^{+} \vdash \mathsf{y} \notin \mathbf{D}^{\infty}_{\infty} \vdash \mathsf{x}@\mathsf{y} \in \mathbf{X}$]
[**Mac rule RoundIiIn** : $\mathsf{x} \in \mathbf{D}^{\infty}_{\infty} \vdash \mathsf{n} \in \mathbf{Z}^{+} \vdash \mathsf{x}@\mathsf{y} \in \mathbf{D}^{\infty}_{\mathsf{n}}$]
[**Mac rule RoundDiDn** : $\mathsf{x} \in \mathbf{D}^{\cdot}_{\infty} \vdash \mathsf{n} \in \mathbf{Z}^{+} \vdash \mathsf{x}@\mathsf{y} \in \mathbf{D}^{\cdot}_{\mathsf{n}}$]
[**Mac rule RoundMiMn** : $\mathsf{x} \in \mathbf{D}^{-\infty}_{\infty} \vdash \mathsf{n} \in \mathbf{Z}^{+} \vdash \mathsf{x}@\mathsf{y} \in \mathbf{D}^{-\infty}_{\mathsf{n}}$]
[**Mac rule RoundImIn** : $\mathsf{m} \in \mathbf{Z}^{+} \vdash \mathsf{x} \in \mathbf{D}^{\infty}_{\mathsf{m}} \vdash \mathsf{n} \in \mathbf{Z}^{+} \vdash \mathsf{x}@\mathsf{y} \in \mathbf{D}^{\infty}_{\mathsf{n}}$]
[**Mac rule RoundDmDn** : $\mathsf{m} \in \mathbf{Z}^{+} \vdash \mathsf{x} \in \mathbf{D}^{\cdot}_{\mathsf{m}} \vdash \mathsf{n} \in \mathbf{Z}^{+} \vdash \mathsf{x}@\mathsf{y} \in \mathbf{D}^{\cdot}_{\mathsf{n}}$]
[**Mac rule RoundMmMn** : $\mathsf{m} \in \mathbf{Z}^{+} \vdash \mathsf{x} \in \mathbf{D}^{-\infty}_{\mathsf{m}} \vdash \mathsf{n} \in \mathbf{Z}^{+} \vdash \mathsf{x}@\mathsf{y} \in \mathbf{D}^{-\infty}_{\mathsf{n}}$]

Examples

x	x@2	$x@_{+}2$	$x@_{-}2$	$x@_{0}2$
0F	0.0F	0.0F	0.0F	0.0F
0.0F	0.0F	0.0F	0.0F	0.0F
0.00F	0.0F	0.0F	0.0F	0.0F
0	0.0F	0.0F	0.0F	0.0F
1.14F	1.1F	1.2F	1.1F	1.1F
1.16F	1.2F	1.2F	1.1F	1.1F
1.15F	1.2F	1.2F	1.1F	1.1F
1.25F	1.2F	1.3F	1.2F	1.2F
1.35F	1.4F	1.4F	1.3F	1.3F
−1.14F	−1.1F	−1.1F	−1.2F	−1.1F
−1.16F	−1.2F	−1.1F	−1.2F	−1.1F
−1.15F	−1.2F	−1.1F	−1.2F	−1.1F
−1.25F	−1.2F	−1.2F	−1.3F	−1.2F
−1.35F	−1.4F	−1.3F	−1.4F	−1.3F

See also

[#x]˙	Section A.27
[x + y]˙	Section A.80

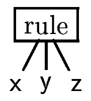
[SubRound]˙	rule SubRound
[RoundException]˙	rule RoundException
[RoundIiIn]˙	rule RoundIiIn
[RoundDiDn]˙	rule RoundDiDn
[RoundMiMn]˙	rule RoundMiMn
[RoundImIn]˙	rule RoundImIn
[RoundDmDn]˙	rule RoundDmDn
[RoundMmMn]˙	rule RoundMmMn

A.86 x rule y colon z

Syntax

$\left[\boxed{\text{x } \textbf{rule} \text{ y:z}}\right]^{\cdot}$.

Parse tree



Sort

[**Mac rule** x:y]˙ is a directive (i.e. it cannot occur in the syntax tree of a term).

Description

Rules have the form [x **rule** y:z]˙ where [y]˙ is the name of the rule, [z]˙ is the rule itself, i.e. a statement that claimed to hold without proof, and [x]˙ is the axiomatic system in which the rule holds without proof. [x **rule** y:z]˙ defines [y]˙ as shorthand for [z]˙. Rules may occur in argumentations in proofs.

Examples

The following statement introduces Transitivity as shorthand for [x ≡ y ⊢ y ≡ z ⊢ x ≡ z]˙ and states that Transitivity holds without proof in the [Mac]˙ system.

[**Mac rule Transitivity** : x ≡ y ⊢ y ≡ z ⊢ x ≡ z]

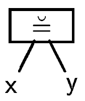
See also

[x **lemma** y:z]˙	Section A.70
[x **proof of** y:z]˙	Section A.83

[x **rule** y:z]˙ x rule y colon z

A.87 x same priority y $[\,\mathsf{x} \breve{=} \mathsf{y}\,]^{\cdot}$

Parse tree



Sort

$[\,\mathsf{x} \breve{=} \mathsf{y}\,]^{\cdot}$ is a directive (i.e. it cannot occur in the syntax tree of a term).

Description

$[\,\mathcal{A} \breve{=} \mathcal{B}\,]^{\cdot}$ states that the constructs $[\,\mathcal{A}\,]^{\cdot}$ and $[\,\mathcal{B}\,]^{\cdot}$ have the same priority and the same associativity when forming parse trees. Furthermore, $[\,\mathcal{A}\,]^{\cdot}$ and $[\,\mathcal{B}\,]^{\cdot}$ are subject to the same term reduction rules when reducing parse trees to syntax trees.

Examples

$[\,\mathsf{x} - \mathsf{y} \breve{=} \mathsf{x} + \mathsf{y}\,]$ and $[\,\mathsf{x} + \mathsf{y} + \mathsf{z} \overset{\smile}{\rightharpoonup} (\mathsf{x} + \mathsf{y}) + \mathsf{z}\,]^{\cdot}$ together make Map interpret $[\,2 + 3 - 4\,]^{\cdot}$ as $[\,(2 + 3) - 4\,]^{\cdot}$ and $[\,2 - 3 + 4\,]^{\cdot}$ as $[\,(2 - 3) + 4\,]^{\cdot}$.

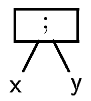
See also

$[\,\mathsf{x} \overset{\smile}{\rightharpoonup} \mathsf{y}\,]^{\cdot}$	Section A.48
$[\,\mathsf{x} \breve{>} \mathsf{y}\,]^{\cdot}$	Section A.63
$[\,\mathsf{x} \overset{\circ}{\rightharpoonup} \mathsf{y}\,]^{\cdot}$	Section A.96

$[\,\mathsf{x} \breve{=} \mathsf{y}\,]^{\cdot}$ x same priority y

A.88 x semicolon y [x; y]˙

Parse tree



Sort

[x; y]˙ is a directive (i.e. it cannot occur in the syntax tree of a term).

Description

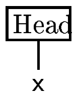
[x; y]˙ is used to form lists of statements. It is used between lines of derivation proofs, between equations in counterexamples, and between rules and lemmas in statements that proclaim more than one rule or lemma.

See also

[x **proof of** y:z]˙	Section A.83

A.89 x simple head [x Head]˙

Parse tree



Sort

[x Head]˙ is an operator (i.e. it may occur in the syntax tree of a term).

Description

[x Head]˙ denotes the first element of a simple pair [x]˙.

Mac rules

[**Mac rule SimpleHeadNil** : N Head $\equiv$ N]
[**Mac rule SimpleHeadPair** : (x $\therefore$ y) Head $\equiv$ x]
[**Mac rule SimpleHeadBottom** : $\perp$ Head $\equiv$ $\perp$]
[**Mac rule SubXSimpleHead** : $\langle \mathcal{A}\ \mathrm{Head} \mid \mathsf{x}{:=}\mathcal{S} \rangle \equiv \langle \mathcal{A} \mid \mathsf{x}{:=}\mathcal{S} \rangle\ \mathrm{Head}$]

Examples

[F Head $\equiv$ T]˙

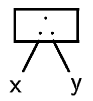
See also

[x $\therefore$ y]˙	Section A.90
[x Tail]˙	Section A.91

[x Head]˙	x simple head
[SimpleHeadNil]˙	rule SimpleHeadNil
[SimpleHeadPair]˙	rule SimpleHeadPair
[SimpleHeadBottom]˙	rule SimpleHeadBottom
[SubXSimpleHead]˙	rule SubXSimpleHead

A.90 x simple pair y [x ∴ y]˙

Parse tree



Sort

[x ∴ y]˙ is an operator (i.e. it may occur in the syntax tree of a term).

Description

[x ∴ y]˙ denotes the simple pair whose first element is [x]˙ and whose second element is [y]˙.

Mac rules

[**Mac rule SubXSimplePairY** : $\langle \mathcal{A} \therefore \mathcal{B} \mid$ x:=$\mathcal{S} \rangle \equiv \langle \mathcal{A} \mid$ x:=$\mathcal{S} \rangle \therefore \langle \mathcal{B} \mid$ x:=$\mathcal{S} \rangle$]

Examples

[F ≡ N ∴ N]˙

See also

[x Head]˙	Section A.89
[x Tail]˙	Section A.91

[x ∴ y]˙	x simple pair y
[SubXSimplePairY]˙	rule SubXSimplePairY

A.91 x simple tail $[\,\mathsf{x}\ \text{Tail}\,]^{\cdot}$

Parse tree

$$\begin{tikzcd}\boxed{\text{Tail}} \arrow[d, no head] \\ \mathsf{x}\end{tikzcd}$$

Sort

$[\,\mathsf{x}\ \text{Tail}\,]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\,\mathsf{x}\ \text{Tail}\,]^{\cdot}$ denotes the first element of a simple pair $[\,\mathsf{x}\,]^{\cdot}$.

Mac rules

[**Mac rule SimpleTailNil** : N Tail $\equiv$ N]
[**Mac rule SimpleTailPair** : $(\mathsf{x} \therefore \mathsf{y})$ Tail $\equiv$ y]
[**Mac rule SimpleTailBottom** : $\perp$ Tail $\equiv$ $\perp$]
[**Mac rule SubXSimpleTail** : $\langle \mathcal{A}$ Tail $\mid \mathsf{x}{:=}\mathcal{S}\rangle \equiv \langle \mathcal{A} \mid \mathsf{x}{:=}\mathcal{S}\rangle$ Tail]

Examples

$[\,\mathsf{F}\ \text{Tail} \quad \equiv \quad \mathsf{T}\,]^{\cdot}$

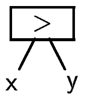
See also

$[\,\mathsf{x}\ \text{Head}\,]^{\cdot}$	Section A.89
$[\,\mathsf{x} \therefore \mathsf{y}\,]^{\cdot}$	Section A.90

$[\,\mathsf{x}\ \text{Tail}\,]^{\cdot}$	x simple tail
[SimpleTailNil]˙	rule SimpleTailNil
[SimpleTailPair]˙	rule SimpleTailPair
[SimpleTailBottom]˙	rule SimpleTailBottom
[SubXSimpleTail]˙	rule SubXSimpleTail

A.92 x strongly greater than y [x > y]˙

Parse tree



Sort

[x > y]˙ is an operator (i.e. it may occur in the syntax tree of a term).

Description

[x > y]˙ is true if [x]˙ is strongly greater than [y]˙ (i.e. greater than and, in particular, not equal to). [x > y]˙ uses the convention that truth is greater than falsehood when comparing truth values. When comparing decimal fractions, [x > y]˙ disregards precision.

Type table

>	**B**	**D**	**X**	**E**	>	**N**	**Z**	$\mathbf{Z}^+$	$\mathbf{Z}^-$
B	**B**	**X**	**X**	**X**	**N**	**B**	**B**	**B**	**T**
D	**X**	**B**	**X**	**X**	**Z**	**B**	**B**	**B**	**B**
X	**X**	**X**	**X**	**X**	$\mathbf{Z}^+$	**B**	**B**	**B**	**T**
E	**X**	**X**	**X**	**X**	$\mathbf{Z}^-$	**F**	**B**	**F**	**B**
>	$\mathbf{D}^{\infty}_{m}$	$\mathbf{D}^{\cdot}_{m}$	$\mathbf{D}^{-\infty}_{m}$	$\mathbf{D}^{\infty}_{\infty}$	$\mathbf{D}^{\cdot}_{\infty}$	$\mathbf{D}^{-\infty}_{\infty}$	$\mathbf{D}^{\infty}_{n}$	$\mathbf{D}^{\cdot}_{n}$	$\mathbf{D}^{-\infty}_{n}$
$\mathbf{D}^{\infty}_{m}$	**F**	**T**	**T**	**F**	**T**	**T**	**F**	**T**	**T**
$\mathbf{D}^{\cdot}_{m}$	**F**	**B**	**T**	**F**	**B**	**T**	**F**	**B**	**T**
$\mathbf{D}^{-\infty}_{m}$	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**F**
$\mathbf{D}^{\infty}_{\infty}$	**F**	**T**	**T**	**F**	**T**	**T**	>	**T**	**F**
$\mathbf{D}^{\cdot}_{\infty}$	**F**	**B**	**T**	**F**	**B**	**T**	**T**	**F**	**T**
$\mathbf{D}^{-\infty}_{\infty}$	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**F**

Mac rules

[**Mac rule StrictStronglyGreaterThanX** : $\perp > \mathsf{x} \equiv \perp$]
[**Mac rule XStronglyGreaterThanStrict** : $\mathsf{x} > \perp \equiv \perp$]
[**Mac rule SubXStronglyGreaterThanY** : $\langle \mathcal{A} > \mathcal{B} \mid \mathsf{x}{:=}\mathcal{S} \rangle \equiv \langle \mathcal{A} \mid \mathsf{x}{:=}\mathcal{S} \rangle > \langle \mathcal{B} \mid \mathsf{x}{:=}\mathcal{S} \rangle$]

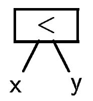
[x > y]˙	x strongly greater than y
[StrictStronglyGreaterThanX]˙	rule StrictStronglyGreaterThanX
[XStronglyGreaterThanStrict]˙	rule XStronglyGreaterThanStrict
[SubXStronglyGreaterThanY]˙	rule SubXStronglyGreaterThanY

See also

$[\,\mathsf{x} = \mathsf{y}\,]^{\cdot}$	Section A.54
$[\,\mathsf{x} < \mathsf{y}\,]^{\cdot}$	Section A.93
$[\,\mathsf{x} \geq \mathsf{y}\,]^{\cdot}$	Section A.98
$[\,\mathsf{x} \leq \mathsf{y}\,]^{\cdot}$	Section A.100

A.93 x strongly less than y $[\,x < y\,]^{\cdot}$

Parse tree



Sort

$[\,x < y\,]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\,x < y\,]^{\cdot}$ is true if $[\,x\,]^{\cdot}$ is strongly less than $[\,y\,]^{\cdot}$ (i.e. less than and, in particular, not equal to). $[\,x < y\,]^{\cdot}$ uses the convention that truth is greater than falsehood when comparing truth values. When comparing decimal fractions, $[\,x < y\,]^{\cdot}$ disregards precision.

Type table

$<$	**B**	**D**	**X**	**E**	$<$	**N**	**Z**	$\mathbf{Z}^+$	$\mathbf{Z}^-$
B	**B**	**X**	**X**	**X**	**N**	**B**	**B**	**B**	**F**
D	**X**	**B**	**X**	**X**	**Z**	**B**	**B**	**B**	**B**
X	**X**	**X**	**X**	**X**	$\mathbf{Z}^+$	**B**	**B**	**B**	**F**
E	**X**	**X**	**X**	**X**	$\mathbf{Z}^-$	**T**	**B**	**T**	**B**
$<$	$\mathbf{D}_m^\infty$	$\mathbf{D}_m^\cdot$	$\mathbf{D}_m^{-\infty}$	$\mathbf{D}_\infty^\infty$	$\mathbf{D}_\infty^\cdot$	$\mathbf{D}_\infty^{-\infty}$	$\mathbf{D}_n^\infty$	$\mathbf{D}_n^\cdot$	$\mathbf{D}_n^{-\infty}$
$\mathbf{D}_m^\infty$	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**F**
$\mathbf{D}_m^\cdot$	**T**	**B**	**F**	**T**	**B**	**F**	**T**	**B**	**F**
$\mathbf{D}_m^{-\infty}$	**T**	**T**	**F**	**T**	**T**	**F**	**T**	**T**	**F**
$\mathbf{D}_\infty^\infty$	**F**	**F**	**F**	**F**	**F**	**F**	$<$	**T**	**F**
$\mathbf{D}_\infty^\cdot$	**T**	**B**	**F**	**T**	**B**	**F**	**T**	**F**	**F**
$\mathbf{D}_\infty^{-\infty}$	**T**	**T**	**F**	**T**	**T**	**F**	**F**	**T**	**F**

Mac rules

[**Mac rule StrictStronglyLessThanX** : $\perp < x \equiv \perp$]
[**Mac rule XStronglyLessThanStrict** : $x < \perp \equiv \perp$]
[**Mac rule SubXStronglyLessThanY** : $\langle \mathcal{A} < \mathcal{B} \mid x{:=}\mathcal{S} \rangle \equiv \langle \mathcal{A} \mid x{:=}\mathcal{S} \rangle < \langle \mathcal{B} \mid x{:=}\mathcal{S} \rangle$]

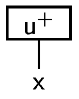
$[\,x < y\,]^{\cdot}$	x strongly less than y
[StrictStronglyLessThanX $]^{\cdot}$	rule StrictStronglyLessThanX
[XStronglyLessThanStrict $]^{\cdot}$	rule XStronglyLessThanStrict
[SubXStronglyLessThanY $]^{\cdot}$	rule SubXStronglyLessThanY

See also

$[\,\mathsf{x} = \mathsf{y}\,]^{\cdot}$	Section A.54
$[\,\mathsf{x} > \mathsf{y}\,]^{\cdot}$	Section A.92
$[\,\mathsf{x} \geq \mathsf{y}\,]^{\cdot}$	Section A.98
$[\,\mathsf{x} \leq \mathsf{y}\,]^{\cdot}$	Section A.100

A.94 x successor $[\mathsf{x}^{+}]^{\cdot}$

Parse tree



Sort

$[\mathsf{x}^{+}]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\mathsf{x}^{+}]^{\cdot}$ denotes $[\mathsf{x}+1]^{\cdot}$. All natural numbers can be expressed using $[0]^{\cdot}$ and $[\mathsf{x}^{+}]^{\cdot}$.

Definition

$[\mathsf{x}^{+} \doteq \mathsf{x}+1]$.

Type table

x	**B**	**D**	**X**	**E**	**N**	**Z**	$\mathbf{Z}^{+}$	$\mathbf{Z}^{-}$
x tail	**X**	–	**X**	**X**	**N**	**Z**	$\mathbf{Z}^{+}$	**Z**

x	$\mathbf{D}_{\mathsf{m}}^{\infty}$	$\mathbf{D}_{\mathsf{m}}^{\cdot}$	$\mathbf{D}_{\mathsf{m}}^{-\infty}$	$\mathbf{D}_{\infty}^{\infty}$	$\mathbf{D}_{\infty}^{\cdot}$	$\mathbf{D}_{\infty}^{-\infty}$	**T**	**F**
x tail	**X**	$\mathbf{D}_{\mathsf{m}}^{\cdot}$	**X**	**X**	$\mathbf{D}_{\infty}^{\cdot}$	**X**	**X**	**X**

Mac rules

[**Mac rule SubXSuccessor** : $\langle \mathcal{A}^{+} \mid \mathsf{x}{:=}\mathcal{S} \rangle \equiv \langle \mathcal{A} \mid \mathsf{x}{:=}\mathcal{S} \rangle^{+}$]

Examples

$[0^{+++++} \quad \equiv \quad 5]^{\cdot}$

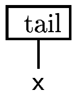
See also

$[\mathsf{x}^{-}]^{\cdot}$ Section A.82

$[\mathsf{x}^{+}]^{\cdot}$	x successor
$[\text{SubXSuccessor}]^{\cdot}$	rule SubXSuccessor

A.95 x tail [x tail]˙

Parse tree



Sort

[x tail]˙ is an operator (i.e. it may occur in the syntax tree of a term).

Description

[x tail]˙ denotes the second element of a pair [x]˙ or (which is the same) all of a list except the first element. If [x]˙ is a truth value, a decimal fraction, an exception or the empty list, then [x tail]˙ equals [x]˙.

Type table

x	$\mathbf{B}$	$\mathbf{D}$	$\mathbf{X}$	$\mathbf{E}$	$\mathbf{N}$	$\mathbf{Z}$	$\mathbf{Z}^+$	$\mathbf{Z}^-$
x tail	$\mathbf{B}$	$\mathbf{D}$	$\mathbf{X}$	$\mathbf{E}$	$\mathbf{N}$	$\mathbf{Z}$	$\mathbf{Z}^+$	$\mathbf{Z}^-$

x	$\mathbf{D}_{m}^{\infty}$	$\mathbf{D}_{m}^{\cdot}$	$\mathbf{D}_{m}^{-\infty}$	$\mathbf{D}_{\infty}^{\infty}$	$\mathbf{D}_{\infty}^{\cdot}$	$\mathbf{D}_{\infty}^{-\infty}$	$\mathbf{T}$	$\mathbf{F}$
x tail	$\mathbf{D}_{m}^{\infty}$	$\mathbf{D}_{m}^{\cdot}$	$\mathbf{D}_{m}^{-\infty}$	$\mathbf{D}_{\infty}^{\infty}$	$\mathbf{D}_{\infty}^{\cdot}$	$\mathbf{D}_{\infty}^{-\infty}$	$\mathbf{T}$	$\mathbf{F}$

Mac rules

[**Mac rule TailEmpty** : $\langle\,\rangle$ tail $\equiv$ $\langle\,\rangle$]
[**Mac rule TailPair** : (x :: y) tail $\equiv$ y]
[**Mac rule TailBottom** : $\perp$ tail $\equiv$ $\perp$]
[**Mac rule SubXTail** : $\langle\mathcal{A}$ tail | x:=$\mathcal{S}\rangle$ $\equiv$ $\langle\mathcal{A}$ | x:=$\mathcal{S}\rangle$ tail]

Examples

[$\langle 1,2,3,4,5\rangle$ tail $\equiv$ $\langle 2,3,4,5\rangle$]˙

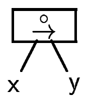
See also

[x head]˙	Section A.64
[x :: y]˙	Section A.79

[x tail]˙	x tail
[TailEmpty]˙	rule TailEmpty
[TailPair]˙	rule TailPair
[TailBottom]˙	rule TailBottom
[SubXTail]˙	rule SubXTail

A.96 x term reduces to y $[\mathsf{x} \overset{\circ}{\rightarrow} \mathsf{y}]^{\cdot}$

Parse tree



Sort

$[\mathsf{x} \overset{\circ}{\rightarrow} \mathsf{y}]^{\cdot}$ is a directive (i.e. it cannot occur in the syntax tree of a term).

Description

$[\mathcal{A} \overset{\circ}{\rightarrow} \mathcal{B}]^{\cdot}$ states that $[\mathcal{A}]^{\cdot}$ reduces to $[\mathcal{B}]^{\cdot}$. $[\mathcal{A} \doteq \mathcal{B}]^{\cdot}$ also states that $[\mathcal{A}]^{\cdot}$ reduces to $[\mathcal{B}]^{\cdot}$. The former rule is a *term reduction rule* whereas the latter is a *graph reduction rule*. The term *reduction rule* stands for “graph reduction rule”.

The difference between term and graph reduction rules is that term reduction rules are applied before graph reduction rules when computing the value of an expression.

To compute the value of an expression, do as follows: Convert the expression into a parse tree using priority and associativity rules. Then convert the parse tree into a syntax tree using term reduction rules and *macro reduction rules*. Finally convert the syntax tree into a value using graph reduction rules.

To check the correctness of a proof, do as follows: Convert all expressions in the proof into parse trees using priority and associativity rules. Then convert all the parse trees into syntax trees using term reduction rules and macro reduction rules. Finally check each step of the proof on basis of the syntax trees.

Hence, another way to state the difference between term reduction rules and graph reduction rules is: Term reduction rules have to be applied both before computing the value of a term and before checking the correctness of a proof. Graph reduction rules are not applied when checking the correctness of a proof.

For each graph reduction rule there is a corresponding axiom (but not vice versa: there are axioms that do not correspond to graph redution rules). Graph reduction rules may enter proofs through the axioms they correspond to, but when they do, they do so explicitly. Term reduction rules do not correspond to axioms.

For an introduction to term reduction, see Section 4.8. For an overview of the process of forming parse and syntax trees, see Section 4.10. For details on the order of performing term reduction, see Section 4.12. For a list of term

$[\mathsf{x} \overset{\circ}{\rightarrow} \mathsf{y}]^{\cdot}$	x term reduces to y
	term reduction rule
	graph reduction rule
	reduction rule
	macro reduction rule

reduction rules, see Appendix A.3. For ways of making proofs more readable when notation is heavy due to term reduction, see Section 6.13. For the construct $[\, \mathsf{x}, \mathsf{y} \,]^{\cdot}$ that has meaning only because of term reduction, see Section 4.11 and Section A.53.

Examples

$\left[\, \langle \mathsf{x}, \mathsf{y} \rangle \overset{\circ}{\rightarrow} \mathsf{x} :: \langle \mathsf{y} \rangle \,\right]$ introduces a term reduction rule and $[\, \langle \mathsf{x} \rangle \ddot{=} \mathsf{x} :: \langle\, \rangle \,]$ introduces a macro reduction rule. Due to these term and macro reduction rules, $[\, \langle \mathsf{x}, \mathsf{y} \rangle \,]^{\cdot}$ and $[\, \mathsf{x} :: \mathsf{y} :: \langle\, \rangle \,]^{\cdot}$ have the same syntax trees.

In contrast, $[\, \infty\text{-list}(\mathsf{n}) \doteq \mathsf{n} :: \infty\text{-list}(\mathsf{n}+1) \,]$ introduces the graph reduction rule $\left[\, \infty\text{-list}(\mathsf{n}) \overset{+}{\rightarrow} \mathsf{n} :: \infty\text{-list}(\mathsf{n}+1) \,\right]^{\cdot}$ and the corresponding axiom $[\, \infty\text{-list}(\mathsf{n}) \equiv \mathsf{n} :: \infty\text{-list}(\mathsf{n}+1) \,]^{\cdot}$.

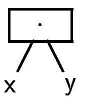
$[\, \perp \text{ head} \equiv \perp \,]^{\cdot}$ is an example of an axiom that does not correspond to any kind of reduction rule.

See also

$[\, \mathsf{x} \doteq \mathsf{y} \,]^{\cdot}$	Section A.58
$[\, \mathsf{x} \ddot{=} \mathsf{y} \,]^{\cdot}$	Section A.72

A.97 x times y $[x \cdot y]^{\cdot}$

Parse tree



Sort

$[x \cdot y]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

If $[x]^{\cdot}$ and $[y]^{\cdot}$ are truth values then $[x \cdot y]^{\cdot}$ coincides with $[x \wedge y]^{\cdot}$. If $[x]^{\cdot}$ and $[y]^{\cdot}$ are exact fractions then $[x \cdot y]^{\cdot}$ denotes the exact product of $[x]^{\cdot}$ and $[y]^{\cdot}$. If $[x]^{\cdot}$ and $[y]^{\cdot}$ are floating fractions of equal precision, then $[x \cdot y]^{\cdot}$ denotes the exact product of $[x]^{\cdot}$ and $[y]^{\cdot}$ rounded to the common precision of $[x]^{\cdot}$ and $[y]^{\cdot}$ using round to even. If $[x]^{\cdot}$ and $[y]^{\cdot}$ are floating fractions of different precisions, then $[x \cdot y]^{\cdot}$ equals $[\bullet]^{\cdot}$. If $[x]^{\cdot}$ is an exact fraction and $[y]^{\cdot}$ is a floating fraction or vice versa, then $[x \cdot y]^{\cdot}$ denotes the exact product of $[x]^{\cdot}$ and $[y]^{\cdot}$ rounded to the precision of the floating fraction using round to even. $[x \cdot_{+} y]^{\cdot}$, $[x \cdot_{-} y]^{\cdot}$, $[x \cdot_{0} y]^{\cdot}$, and $[x \cdot_{2} y]^{\cdot}$ are versions of $[x \cdot y]^{\cdot}$ that use round to plus infinity, minus infinity, zero, and even, respectively. $[x \cdot_{2} y]^{\cdot}$ is identical to $[x \cdot y]^{\cdot}$.

Type table

·	**B**	**D**	**X**	**E**	·	**N**	**Z**	$\mathbf{Z}^{+}$	$\mathbf{Z}^{-}$
B	**B**	**X**	**X**	**X**	**N**	**N**	**Z**	**N**	**Z**
D	**X**	–	**X**	**X**	**Z**	**Z**	**Z**	**Z**	**Z**
X	**X**	**X**	**X**	**X**	$\mathbf{Z}^{+}$	**N**	**Z**	$\mathbf{Z}^{+}$	$\mathbf{Z}^{-}$
E	**X**	**X**	**X**	**X**	$\mathbf{Z}^{-}$	**Z**	**Z**	$\mathbf{Z}^{-}$	$\mathbf{Z}^{+}$
·	$\mathbf{D}_{m}^{\infty}$	$\mathbf{D}_{m}^{\cdot}$	$\mathbf{D}_{m}^{-\infty}$	$\mathbf{D}_{\infty}^{\infty}$	$\mathbf{D}_{\infty}^{\cdot}$	$\mathbf{D}_{\infty}^{-\infty}$	$\mathbf{D}_{n}^{\infty}$	$\mathbf{D}_{n}^{\cdot}$	$\mathbf{D}_{n}^{-\infty}$
$\mathbf{D}_{m}^{\infty}$	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**
$\mathbf{D}_{m}^{\cdot}$	**X**	$\mathbf{D}_{m}^{\cdot}$	**X**	**X**	$\mathbf{D}_{m}^{\cdot}$	**X**	**X**	**X**	**X**
$\mathbf{D}_{m}^{-\infty}$	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**	**X**
$\mathbf{D}_{\infty}^{\infty}$	**X**	**X**	**X**	**X**	**X**	**X**	·	**T**	**F**
$\mathbf{D}_{\infty}^{\cdot}$	**X**	$\mathbf{D}_{m}^{\cdot}$	**X**	**X**	$\mathbf{D}_{\infty}^{\cdot}$	**X**	**T**	**T**	**F**
$\mathbf{D}_{\infty}^{-\infty}$	**X**	**X**	**X**	**X**	**X**	**X**	**F**	**F**	**F**

Mac rules

[**Mac rule StrictTimesX** : $\perp \cdot y \equiv \perp$]

$[x \cdot y]^{\cdot}$	x times y
$[\text{StrictTimesX}]^{\cdot}$	rule StrictTimesX

[**Mac rule XTimesStrict** : $\mathsf{x} \cdot \perp \equiv \perp$]
[**Mac rule SubXTimesY** : $\langle \mathcal{A} \cdot \mathcal{B} \mid \mathsf{x}{:=}\mathcal{S} \rangle \equiv \langle \mathcal{A} \mid \mathsf{x}{:=}\mathcal{S} \rangle \cdot \langle \mathcal{B} \mid \mathsf{x}{:=}\mathcal{S} \rangle$]

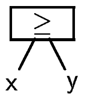
See also

[$\mathsf{x} \wedge \mathsf{y}$]$^\cdot$	Section A.46
[$\mathsf{x} - \mathsf{y}$]$^\cdot$	Section A.73
[$\mathsf{x} + \mathsf{y}$]$^\cdot$	Section A.80
[x@y]$^\cdot$	Section A.85

[XTimesStrict]$^\cdot$	rule XTimesStrict
[SubXTimesY]$^\cdot$	rule SubXTimesY

A.98 x weakly greater than y $[x \geq y]^{\cdot}$

Parse tree



Sort

$[x \geq y]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[x \geq y]^{\cdot}$ is true if $[x]^{\cdot}$ is weakly greater than $[y]^{\cdot}$ (i.e. greater than or equal to). $[x \geq y]^{\cdot}$ uses the convention that truth is greater than falsehood when comparing truth values. For truth values $[x]^{\cdot}$ and $[y]^{\cdot}$, $[x \geq y]^{\cdot}$ coincides with $[x \Leftarrow y]^{\cdot}$. When comparing decimal fractions, $[x \geq y]^{\cdot}$ disregards precision.

Type table

≥	**B**	**D**	**X**	**E**	≥	**N**	**Z**	$\mathbf{Z}^+$	$\mathbf{Z}^-$
B	**B**	**X**	**X**	**X**	**N**	**B**	**B**	**B**	**T**
D	**X**	**B**	**X**	**X**	**Z**	**B**	**B**	**B**	**B**
X	**X**	**X**	**X**	**X**	$\mathbf{Z}^+$	**B**	**B**	**B**	**T**
E	**X**	**X**	**X**	**X**	$\mathbf{Z}^-$	**F**	**B**	**F**	**B**
≥	$\mathbf{D}_m^\infty$	$\mathbf{D}_m^\cdot$	$\mathbf{D}_m^{-\infty}$	$\mathbf{D}_\infty^\infty$	$\mathbf{D}_\infty^\cdot$	$\mathbf{D}_\infty^{-\infty}$	$\mathbf{D}_n^\infty$	$\mathbf{D}_n^\cdot$	$\mathbf{D}_n^{-\infty}$
$\mathbf{D}_m^\infty$	**T**	**T**	**T**	**T**	**T**	**T**	**T**	**T**	**T**
$\mathbf{D}_m^\cdot$	**F**	**B**	**T**	**F**	**B**	**T**	**F**	**B**	**T**
$\mathbf{D}_m^{-\infty}$	**F**	**F**	**T**	**F**	**F**	**T**	**F**	**F**	**T**
$\mathbf{D}_\infty^\infty$	**T**	**T**	**T**	**T**	**T**	**T**	≥	**T**	**F**
$\mathbf{D}_\infty^\cdot$	**F**	**B**	**T**	**F**	**B**	**T**	**T**	**T**	**T**
$\mathbf{D}_\infty^{-\infty}$	**F**	**F**	**T**	**F**	**F**	**T**	**F**	**F**	**T**

Mac rules

[**Mac rule StrictWeaklyGreaterThanX** : $\bot \geq y \equiv \bot$]
[**Mac rule XWeaklyGreaterThanStrict** : $x \geq \bot \equiv \bot$]
[**Mac rule SubXWeaklyGreaterThanY** : $\langle \mathcal{A} \geq \mathcal{B} \mid x{:=}\mathcal{S} \rangle \equiv \langle \mathcal{A} \mid x{:=}\mathcal{S} \rangle \geq \langle \mathcal{B} \mid x{:=}\mathcal{S} \rangle$]

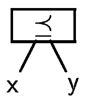
$[x \geq y]^{\cdot}$	x weakly greater than y
[StrictWeaklyGreaterThanX]$^{\cdot}$	rule StrictWeaklyGreaterThanX
[XWeaklyGreaterThanStrict]$^{\cdot}$	rule XWeaklyGreaterThanStrict
[SubXWeaklyGreaterThanY]$^{\cdot}$	rule SubXWeaklyGreaterThanY

See also

$[\,\mathsf{x} = \mathsf{y}\,]^{\cdot}$	Section A.54
$[\,\mathsf{x} > \mathsf{y}\,]^{\cdot}$	Section A.92
$[\,\mathsf{x} < \mathsf{y}\,]^{\cdot}$	Section A.93
$[\,\mathsf{x} \leq \mathsf{y}\,]^{\cdot}$	Section A.100
$[\,\mathsf{x} \Leftarrow \mathsf{y}\,]^{\cdot}$	Section A.66

A.99 x weakly less information y $[\mathsf{x} \preceq \mathsf{y}]^{\cdot}$

Parse tree



Sort

$[\mathsf{x} \preceq \mathsf{y}]^{\cdot}$ is a directive (i.e. it cannot occur in the syntax tree of a term).

Description

$[\mathsf{x} \preceq \mathsf{y}]^{\cdot}$ holds if any property of $[\mathsf{x}]^{\cdot}$ is also a property of $[\mathsf{y}]^{\cdot}$. The following fact explains why $[\mathsf{x} \preceq \mathsf{y}]^{\cdot}$ is relevant to computer science:

Fact A.99.1 If $[\mathsf{f}, \mathsf{a}_1, \mathsf{a}_2, \mathsf{a}_3, \ldots]^{\circ}$ are computable by machine and if

$$[\mathsf{a}_1, \mathsf{a}_2, \mathsf{a}_3, \ldots]^{\circ}$$

is an increasing sequence with supremum $[\mathsf{a}]^{\cdot}$, then

$$[f\,\text{'}\,\mathsf{a}_1, f\,\text{'}\,\mathsf{a}_2, f\,\text{'}\,\mathsf{a}_3, \ldots]^{\circ}$$

is an increasing sequence with supremum $[f\,\text{'}\,\mathsf{a}]^{\cdot}$.

Mac rules

[**Mac lemma InfoReflexive** : $\mathsf{x} \preceq \mathsf{x}$]
[**Mac rule InfoBottom** : $\bot \preceq \mathsf{x}$]
[**Mac rule InfoImply** : $\mathsf{x} \preceq \mathsf{y} \vdash \mathsf{f}\,\text{'}\,\mathsf{x} \rightarrow \mathsf{f}\,\text{'}\,\mathsf{y}$]
[**Mac rule ImplyInfo** : **if** $\text{notfree}(\mathsf{f},\mathsf{x}) \wedge \text{notfree}(\mathsf{f},\mathsf{y})$ **then** $\mathsf{f}\,\text{'}\mathsf{x} \rightarrow \mathsf{f}\,\text{'}\mathsf{y} \vdash \mathsf{x} \preceq \mathsf{y}$]
[**Mac rule InfoAntiSymmetry** : $\mathsf{x} \preceq \mathsf{y} \vdash \mathsf{y} \preceq \mathsf{x} \vdash \mathsf{x} \equiv \mathsf{y}$]
[**Mac lemma InfoTransitivity** : $\mathsf{x} \preceq \mathsf{y} \vdash \mathsf{y} \preceq \mathsf{z} \vdash \mathsf{x} \preceq \mathsf{z}$]
[**Mac lemma Monotonicity** : $\mathsf{x} \preceq \mathsf{y} \vdash \mathsf{g}\,\text{'}\,\mathsf{x} \preceq \mathsf{g}\,\text{'}\,\mathsf{y}$]
[**Mac lemma Monotonicity'** : $\mathsf{x} \preceq \mathsf{y} \vdash \langle \mathcal{A} \mid \mathsf{u}{:=}\mathsf{x} \rangle \preceq \langle \mathcal{A} \mid \mathsf{u}{:=}\mathsf{y} \rangle$]
[**Mac rule InfoLambda** : $\mathcal{A} \preceq \mathcal{B} \vdash \lambda \mathsf{x}.\mathcal{A} \preceq \lambda \mathsf{x}.\mathcal{B}$]
[**Mac lemma MonotonicityHead** : $\mathsf{x} \preceq \mathsf{y} \vdash \mathsf{x}$ head $\preceq \mathsf{y}$ head]
[**Mac lemma MonotonicityTail** : $\mathsf{x} \preceq \mathsf{y} \vdash \mathsf{x}$ tail $\preceq \mathsf{y}$ tail]
[**Mac lemma MonotonicityHead'** : $\mathsf{x} :: \mathsf{y} \preceq \mathsf{u} :: \mathsf{v} \vdash \mathsf{x} \preceq \mathsf{u}$]
[**Mac lemma MonotonicityTail'** : $\mathsf{x} :: \mathsf{y} \preceq \mathsf{u} :: \mathsf{v} \vdash \mathsf{y} \preceq \mathsf{v}$]

$[\mathsf{x} \preceq \mathsf{y}]^{\cdot}$	x weakly less information y
[InfoBottom]˙	rule InfoBottom
[InfoImply]˙	rule InfoImply
[ImplyInfo]˙	rule ImplyInfo
[InfoAntiSymmetry]˙	rule InfoAntiSymmetry
[InfoLambda]˙	rule InfoLambda

[**Mac lemma MonotonicityHead"** : $\mathsf{x} \preceq \mathsf{y} \vdash \mathsf{x :: z} \preceq \mathsf{y :: z}$]
[**Mac lemma MonotonicityTail"** : $\mathsf{y} \preceq \mathsf{z} \vdash \mathsf{x :: y} \preceq \mathsf{x :: z}$]
[**Mac lemma MonotonicityPair** : $\mathsf{x} \preceq \mathsf{u} \vdash \mathsf{y} \preceq \mathsf{v} \vdash \mathsf{x :: y} \preceq \mathsf{u :: v}$]
[**Mac rule MinimalY** : $\mathsf{f\,'\,x} \equiv \mathsf{x} \vdash \mathsf{Y\,'\,f} \preceq \mathsf{x}$]
[**Mac rule Minimal** : $\mathsf{f\,'\,x} \preceq \mathsf{x} \vdash \mathsf{Y\,'\,f} \preceq \mathsf{x}$]
[**Mac lemma InfoLessBottom** : $\mathsf{x} \preceq \perp \vdash \mathsf{x} \equiv \perp$]

Examples

[$\perp :: \perp$	$\preceq$	$\perp :: \perp$]$^{\cdot}$	holds
[$\perp :: \perp$	$\preceq$	$\perp :: 2$]$^{\cdot}$	holds
[$\perp :: \perp$	$\preceq$	$\mathsf{T} :: \perp$]$^{\cdot}$	holds
[$\perp :: 2$	$\preceq$	$\mathsf{T} :: 2$]$^{\cdot}$	holds
[$\mathsf{T} :: \perp$	$\preceq$	$\mathsf{T} :: 2$]$^{\cdot}$	holds
[$\perp :: \perp$	$\preceq$	$\mathsf{T} :: 2$]$^{\cdot}$	holds
[$\mathsf{T} :: \perp$	$\preceq$	$\mathsf{F} :: 2$]$^{\cdot}$	fails

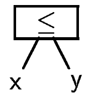
See also

[$\perp$]$^{\cdot}$	Section A.5
[$\mathsf{x} \equiv \mathsf{y}$]$^{\cdot}$	Section A.60

[MinimalY]$^{\cdot}$	rule MinimalY
[Minimal]$^{\cdot}$	rule Minimal

A.100 x weakly less than y $[\mathsf{x} \leq \mathsf{y}]^{\cdot}$

Parse tree



Sort

$[\mathsf{x} \leq \mathsf{y}]^{\cdot}$ is an operator (i.e. it may occur in the syntax tree of a term).

Description

$[\mathsf{x} \leq \mathsf{y}]^{\cdot}$ is true if $[\mathsf{x}]^{\cdot}$ is weakly less than $[\mathsf{y}]^{\cdot}$ (i.e. less than or equal to). $[\mathsf{x} \leq \mathsf{y}]^{\cdot}$ uses the convention that truth is greater than falsehood when comparing truth values. For truth values $[\mathsf{x}]^{\cdot}$ and $[\mathsf{y}]^{\cdot}$, $[\mathsf{x} \leq \mathsf{y}]^{\cdot}$ coincides with $[\mathsf{x} \Rightarrow \mathsf{y}]^{\cdot}$. When comparing decimal fractions, $[\mathsf{x} \leq \mathsf{y}]^{\cdot}$ disregards precision.

Type table

$\leq$	**B**	**D**	**X**	**E**	$\leq$	**N**	**Z**	$\mathbf{Z}^{+}$	$\mathbf{Z}^{-}$
B	**B**	**X**	**X**	**X**	**N**	**B**	**B**	**B**	**F**
D	**X**	**B**	**X**	**X**	**Z**	**B**	**B**	**B**	**B**
X	**X**	**X**	**X**	**X**	$\mathbf{Z}^{+}$	**B**	**B**	**B**	**F**
E	**X**	**X**	**X**	**X**	$\mathbf{Z}^{-}$	**T**	**B**	**T**	**B**
$\leq$	$\mathbf{D}^{\infty}_{m}$	$\mathbf{D}^{\cdot}_{m}$	$\mathbf{D}^{-\infty}_{m}$	$\mathbf{D}^{\infty}_{\infty}$	$\mathbf{D}^{\cdot}_{\infty}$	$\mathbf{D}^{-\infty}_{\infty}$	$\mathbf{D}^{\infty}_{n}$	$\mathbf{D}^{\cdot}_{n}$	$\mathbf{D}^{-\infty}_{n}$
$\mathbf{D}^{\infty}_{m}$	**T**	**F**	**F**	**T**	**F**	**F**	**T**	**F**	**F**
$\mathbf{D}^{\cdot}_{m}$	**T**	**B**	**F**	**T**	**B**	**F**	**T**	**B**	**F**
$\mathbf{D}^{-\infty}_{m}$	**T**	**T**	**T**	**T**	**T**	**T**	**T**	**T**	**T**
$\mathbf{D}^{\infty}_{\infty}$	**T**	**F**	**F**	**T**	**F**	**F**	$\leq$	**T**	**F**
$\mathbf{D}^{\cdot}_{\infty}$	**T**	**B**	**F**	**T**	**B**	**F**	**T**	**T**	**F**
$\mathbf{D}^{-\infty}_{\infty}$	**T**	**T**	**T**	**T**	**T**	**T**	**F**	**T**	**T**

Mac rules

[**Mac rule StrictWeaklyLessThanX** : $\perp \leq \mathsf{x} \equiv \perp$]
[**Mac rule XWeaklyLessThanStrict** : $\mathsf{x} \leq \perp \equiv \perp$]
[**Mac rule XWeaklyLessThanY** : $\langle \mathcal{A} \leq \mathcal{B} \mid \mathsf{x}{:=}\mathcal{S} \rangle \equiv \langle \mathcal{A} \mid \mathsf{x}{:=}\mathcal{S} \rangle \leq \langle \mathcal{B} \mid \mathsf{x}{:=}\mathcal{S} \rangle$]

$[\mathsf{x} \leq \mathsf{y}]^{\cdot}$	x weakly less than y
$[$ StrictWeaklyLessThanX $]^{\cdot}$	rule StrictWeaklyLessThanX
$[$ XWeaklyLessThanStrict $]^{\cdot}$	rule XWeaklyLessThanStrict
$[$ XWeaklyLessThanY $]^{\cdot}$	rule XWeaklyLessThanY

See also

$[\,\mathsf{x} = \mathsf{y}\,]^{\cdot}$	Section A.54
$[\,\mathsf{x} > \mathsf{y}\,]^{\cdot}$	Section A.92
$[\,\mathsf{x} < \mathsf{y}\,]^{\cdot}$	Section A.93
$[\,\mathsf{x} \geq \mathsf{y}\,]^{\cdot}$	Section A.98
$[\,\mathsf{x} \Rightarrow \mathsf{y}\,]^{\cdot}$	Section A.67

Appendix B

Map theory

B.1 Introduction

This appendix presents a formal definition of *map theory*. All operators in this book are definable from the fundamental constructs of map theory. All rules and lemmas in this book are provable in map theory.

B.2 Priority

$[\, \mathsf{x}\,'\,\mathsf{y} \;\breve{>}$
$\mathsf{x} \downarrow \mathsf{y} \;\breve{>}$
$\mathsf{x} \mathrel{\hat{=}} \mathsf{y} \mathrel{\breve{=}} \mathsf{x} \mathrel{\hat{=}}_{\mathsf{z}} \mathsf{y} \;\breve{>}$
$\hat{\approx}\mathsf{x} \mathrel{\breve{=}} \hat{\neg}\mathsf{x} \mathrel{\breve{=}} \hat{!}\mathsf{x} \mathrel{\breve{=}} \hat{?}\mathsf{x} \;\breve{>}$
$\mathsf{x} \mathbin{\bar{\wedge}} \mathsf{y} \mathrel{\breve{=}} \mathsf{x} \mathbin{\hat{\wedge}} \mathsf{y} \;\breve{>}$
$\mathsf{x} \mathrel{\hat{\Rightarrow}} \mathsf{y} \;\breve{>}$
$\mathsf{x} \left\langle \begin{matrix} \mathsf{y} \\ \mathsf{z} \end{matrix} \right. \;\breve{>}$
$\lambda \mathsf{x}.\mathsf{y} \;\breve{>}$
$\mathsf{x} \equiv \mathsf{y} \mathrel{\breve{=}} \mathsf{x} \preceq \mathsf{y} \;\breve{>}$
$\mathsf{x} \rightarrow \mathsf{y} \;\breve{>}$
$\mathsf{x} \vdash \mathsf{y} \,]$

B.3 Associativity

$[\, \mathsf{x}\,'\,\mathsf{y}\,'\,\mathsf{z} \mathrel{\breve{\rightarrow}} (\mathsf{x}\,'\,\mathsf{y})\,'\,\mathsf{z} \,].$
$[\, \mathsf{x} \downarrow \mathsf{y} \downarrow \mathsf{z} \mathrel{\breve{\rightarrow}} (\mathsf{x} \downarrow \mathsf{y}) \downarrow \mathsf{z} \,].$
$[\, \mathsf{x} \mathbin{\bar{\wedge}} \mathsf{y} \mathbin{\bar{\wedge}} \mathsf{z} \mathrel{\breve{\rightarrow}} (\mathsf{x} \mathbin{\bar{\wedge}} \mathsf{y}) \mathbin{\bar{\wedge}} \mathsf{z} \,].$
$[\, \mathsf{x} \rightarrow \mathsf{y} \rightarrow \mathsf{z} \mathrel{\breve{\rightarrow}} \mathsf{x} \rightarrow (\mathsf{y} \rightarrow \mathsf{z}) \,].$
$[\, \mathsf{x} \vdash \mathsf{y} \vdash \mathsf{z} \mathrel{\breve{\rightarrow}} \mathsf{x} \vdash (\mathsf{y} \vdash \mathsf{z}) \,].$

map theory

B.4 Fundamental constructs

$[\,\mathsf{N}\,]^{\cdot}$	The uhr-element.
$[\,\lambda\mathsf{x}.\mathcal{A}\,]^{\cdot}$	The function that maps $[\,\mathsf{x}\,]^{\cdot}$ to $[\,\mathcal{A}\,]^{\cdot}$
$[\,\mathsf{x}\,'\,\mathsf{y}\,]^{\cdot}$	$[\,\mathsf{x}\,]^{\cdot}$ applied to $[\,\mathsf{y}\,]^{\cdot}$
$[\,\mathrm{case}(\mathsf{x},\mathsf{y},\mathsf{z})\,]^{\cdot}$	$\begin{cases}[\,\mathsf{y}\,]^{\cdot} \text{ if } [\,\mathsf{x}\,]^{\cdot} \text{ is the uhr-element}\\ [\,\mathsf{z}\,]^{\cdot} \text{ if } [\,\mathsf{x}\,]^{\cdot} \text{ is a function}\\ [\,\perp\,]^{\cdot} \text{ if } [\,\mathsf{x}\,]^{\cdot} \text{ equals } [\,\perp\,]^{\cdot}\end{cases}$
$[\,\bar{\varepsilon}\mathsf{x}\,]^{\cdot}$	The choice operator
$[\,\bar{\exists}\mathsf{x}\,]^{\cdot}$	Fundamental existence

B.5 Definitions

$$[\,\perp \qquad \doteq (\lambda \mathsf{x}.\mathsf{x}\,'\,\mathsf{x})\,'(\lambda \mathsf{x}.\mathsf{x}\,'\,\mathsf{x})\,]$$

$$[\,\tilde{F} \qquad \doteq \lambda \mathsf{x}.\lambda \mathsf{y}.\mathsf{x}\,'\,\mathsf{y}\,]$$

$$[\,\mathsf{Y} \qquad \doteq \lambda f.(\lambda \mathsf{x}.f\,'(\mathsf{x}\,'\,\mathsf{x}))\,'(\lambda \mathsf{x}.f\,'(\mathsf{x}\,'\,\mathsf{x}))\,]$$

$$[\,\mathsf{T} \qquad \doteq \mathsf{N}\,]$$

$$[\,\hat{\mathsf{F}} \qquad \doteq \lambda \mathsf{x}.\mathsf{N}\,]$$

$$[\,\hat{\approx}\mathsf{x} \qquad \doteq \mathrm{case}(\mathsf{x},\mathsf{T},\hat{\mathsf{F}})\,]$$

$$\left[\mathsf{x}\left\langle\begin{array}{l}\mathsf{y}\\ \mathsf{z}\end{array}\right. \qquad \ddot{=}\ \mathrm{case}(\mathsf{x},\mathsf{y},\mathsf{z})\right]$$

$$\left[\mathsf{x}\ \bar{\wedge}\ \mathsf{y} \qquad \doteq \mathsf{x}\left\langle\begin{array}{l}\mathrm{case}(\mathsf{y},\mathsf{T},\hat{\mathsf{F}})\\ \mathrm{case}(\mathsf{y},\hat{\mathsf{F}},\hat{\mathsf{F}})\end{array}\right.\right]$$

$$[\,\mathsf{x}\ \hat{\wedge}\ \mathsf{y} \qquad \doteq \mathrm{case}(\mathsf{x},\mathsf{y},\hat{\mathsf{F}})\,]$$

$$[\,\mathsf{x}\ \hat{\Rightarrow}\ \mathsf{y} \qquad \doteq \mathrm{case}(\mathsf{x},\mathsf{y},\mathsf{T})\,]$$

$$[\,\hat{\neg}\mathsf{x} \qquad \doteq \mathrm{case}(\mathsf{x},\hat{\mathsf{F}},\mathsf{T})\,]$$

$$[\,\hat{!}\mathsf{x} \qquad \doteq \mathrm{case}(\mathsf{x},\mathsf{T},\mathsf{T})\,]$$

$$\left[\mathsf{x}\downarrow\mathsf{y} \qquad \doteq \mathsf{x}\left\langle\begin{array}{l}\mathrm{case}(\mathsf{y},\mathsf{N},\perp)\\ \mathrm{case}(\mathsf{y},\perp,\lambda \mathsf{z}.\mathsf{x}\,'\,\mathsf{z}\downarrow\mathsf{y}\,'\,\mathsf{z})\end{array}\right.\right]$$

$$[\,\hat{?}\mathsf{x} \qquad \doteq \mathrm{case}(\mathsf{x},\mathsf{T},\perp)\,]$$

$$[\,\varepsilon\mathsf{x}{:}\,\mathsf{y} \qquad \doteq \bar{\varepsilon}\lambda \mathsf{x}.\mathsf{y}\,]$$

$$[\,\bar{\exists}\mathsf{x}{:}\,\mathsf{y} \qquad \doteq \bar{\exists}\lambda \mathsf{x}.\mathsf{y}\,]$$

$$[\,\exists\mathsf{x}{:}\,\mathsf{y} \qquad \doteq \hat{\approx}(\lambda \mathsf{x}.\mathsf{y})\ '\ \varepsilon\mathsf{x}{:}\,\mathsf{y}\,]$$

$$[\,\forall\mathsf{x}{:}\,\mathsf{y} \qquad \doteq \hat{\neg}\exists\mathsf{x}{:}\,\hat{\neg}\mathsf{y}\,]$$

$$\left[\mathsf{x}\ \hat{=}\ \mathsf{y} \qquad \doteq \mathsf{x}\left\langle\begin{array}{l}\mathrm{case}(\mathsf{y},\mathsf{T},\hat{\mathsf{F}})\\ \mathrm{case}(\mathsf{y},\hat{\mathsf{F}},\forall\mathsf{z}{:}\,\mathsf{x}\ '\ \mathsf{z}\ \hat{=}\ \mathsf{y}\ '\ \mathsf{z})\end{array}\right.\right]$$

$$\left[\mathsf{x}\ \hat{=}_{\mathsf{u}}\ \mathsf{y} \qquad \doteq \mathsf{x}\left\langle\begin{array}{l}\mathrm{case}(\mathsf{y},\mathsf{T},\hat{\mathsf{F}})\\ \mathrm{case}(\mathsf{y},\hat{\mathsf{F}},\forall\mathsf{z}{:}\,\mathsf{x}\ '\ (\mathsf{u}\ '\ \mathsf{z})\ \hat{=}_{\mathsf{u}}\ \mathsf{y}\ '\ (\mathsf{u}\ '\ \mathsf{z}))\end{array}\right.\right]$$

$$\left[\ell \qquad \doteq \mathsf{Y}'\lambda\mathsf{f}.\lambda\mathsf{x}.\mathsf{x}\left\langle\begin{array}{l}\mathsf{T}\\ (\forall\mathsf{y}{:}\,\mathsf{f}'(\mathsf{x}'\mathsf{y}))\ \bar{\wedge}\ \bar{\exists}\mathsf{u}{:}\,\mathsf{f}'\mathsf{u}\ \bar{\wedge}\ \forall\mathsf{v}{:}\,\forall\mathsf{w}{:}\,(\mathsf{v}\ \hat{=}_{\mathsf{u}}\ \mathsf{w}\Rightarrow\mathsf{x}'\mathsf{v}\ \hat{=}\ \mathsf{x}'\mathsf{w})\end{array}\right.\right]$$

Whenever a term $[\,\mathcal{A}\,]^{\cdot}$ occurs in a position where a formula is expected, $[\,\mathcal{A}\,]^{\cdot}$ is shorthand for $[\,\mathcal{A} \equiv \mathsf{T}\,]^{\cdot}$.

$[\,\mathcal{A} \rightarrow (\mathcal{B} \equiv \mathcal{C})\,]^{\cdot}$ is shorthand for $[\,\mathcal{A} \mathbin{\hat{\wedge}} \mathcal{B} \equiv \mathcal{A} \mathbin{\hat{\wedge}} \mathcal{C}\,]^{\cdot}$.

$[\,\mathsf{x} \preceq \mathsf{y}\,]^{\cdot}$ is shorthand for $[\,\mathsf{x} \equiv \mathsf{x} \downarrow \mathsf{y}\,]^{\cdot}$.

B.6 Axioms and inferences

[**Map rule MapTrans** : $\mathcal{A} \equiv \mathcal{B} \vdash \mathcal{A} \equiv \mathcal{C} \vdash \mathcal{B} \equiv \mathcal{C}$]
[**Map rule MapSubLambda** : $\mathcal{A} \equiv \mathcal{B} \vdash \lambda\mathsf{x}.\mathcal{A} \equiv \lambda\mathsf{x}.\mathcal{B}$]
[**Map rule MapSubApply** : $\mathcal{A} \equiv \mathcal{B} \vdash \mathcal{C}\,'\,\mathcal{A} \equiv \mathcal{C}\,'\,\mathcal{B}$]
[**Map rule MapApplyNil** : $\mathsf{N}\,'\,\mathcal{B} \equiv \mathsf{N}$]
[**Map rule MapApplyLambda** : $(\lambda\mathsf{x}.\mathcal{A})\,'\,\mathcal{B} \equiv \langle \mathcal{A} \mid \mathsf{x}{:=}\mathcal{B} \rangle$]
[**Map rule MapApplyBottom** : $\bot\,'\,\mathcal{B} \equiv \bot$]
[**Map rule MapIfNil** : $\mathrm{case}(\mathsf{N}, \mathcal{B}, \mathcal{C}) \equiv \mathcal{B}$]
[**Map rule MapIfLambda** : $\mathrm{case}(\lambda\mathsf{x}.\mathcal{A}, \mathcal{B}, \mathcal{C}) \equiv \mathcal{C}$]
[**Map rule MapIfBottom** : $\mathrm{case}(\bot, \mathcal{B}, \mathcal{C}) \equiv \bot$]
[**Map rule MapMonotonicity** : $\mathcal{A} \preceq \mathcal{B} \vdash \mathcal{C}\,'\,\mathcal{A} \preceq \mathcal{C}\,'\,\mathcal{B}$]
[**Map rule MapMinimality** : $\mathcal{A}'\mathcal{B} \preceq \mathcal{B} \vdash \mathsf{Y}\,'\,\mathcal{A} \preceq \mathcal{B}$]
[**Map rule MapQND** : $\mathcal{A}\,'\,\mathsf{N} \equiv \mathcal{B}\,'\,\mathsf{N} \vdash \mathcal{A}'(\tilde{F}\,'\,\mathcal{C}) \equiv \mathcal{A}'(\tilde{F}\,'\,\mathcal{C}) \vdash$
$\mathcal{A}\,'\,\bot \equiv \mathcal{B}\,'\,\bot \vdash \mathcal{A}'\mathcal{C} \equiv \mathcal{B}'\mathcal{C}$]
[**Map rule MapExtensionality** : **if** $\mathrm{distinct}(\mathsf{u}, \mathsf{v}) \wedge$
$\mathrm{notfree}(\mathsf{u}, \mathcal{A}) \wedge \mathrm{notfree}(\mathsf{v}, \mathcal{A}) \wedge \mathrm{notfree}(\mathsf{u}, \mathcal{B}) \wedge \mathrm{notfree}(\mathsf{v}, \mathcal{B})$ **then**
$\hat{\approx}(\mathcal{A}\,'\,\mathsf{u}) \equiv \hat{\approx}(\mathcal{B}\,'\,\mathsf{u}) \vdash \mathcal{A}\,'\,\mathsf{u}\,'\,\mathsf{v} \equiv \mathcal{A}'\mathcal{C} \vdash \mathcal{B}\,'\,\mathsf{u}\,'\,\mathsf{v} \equiv \mathcal{B}'\mathcal{C} \vdash \mathcal{A}\,'\,\mathsf{u} \equiv \mathcal{B}\,'\,\mathsf{u}$]
[**Map rule MapQuantifyA** : $\ell\,\text{'}\,\mathsf{a} \rightarrow \forall\mathsf{x}\colon \mathcal{A} \rightarrow \langle \mathcal{A} \mid \mathsf{x}{:=}\mathsf{a} \rangle$]
[**Map rule MapQuantifyB** : $\forall\mathsf{x}\colon \hat{!}\mathcal{A} \equiv \hat{!}\forall\mathsf{x}\colon \mathcal{A}$]
[**Map rule MapQuantifyC** : $\forall\mathsf{x}\colon \hat{!}\mathcal{A} \equiv \ell\,\text{'}\,\varepsilon\mathsf{x}\colon \mathcal{A}$]
[**Map rule MapQuantifyD** : $\varepsilon\mathsf{x}\colon \mathcal{A} \equiv \varepsilon\mathsf{x}\colon \ell\,\text{'}\,\mathsf{x} \mathbin{\bar{\wedge}} \mathcal{A}$]
[**Map rule MapExistsA** : $\mathsf{x}\,\text{'}\,\mathsf{y} \rightarrow \bar{\exists}\mathsf{x}$]
[**Map rule MapExistsB** : **if** $\mathrm{notfree}(\mathsf{y}, \mathsf{u})$ **then** $\mathsf{x}\,\text{'}\,\mathsf{y} \rightarrow \mathsf{u}\,\text{'}\,\mathsf{v} \vdash \bar{\exists}\mathsf{x} \rightarrow \bar{\exists}\mathsf{u}$]

$[\,\text{MapTrans}\,]^{\cdot}$	rule MapTrans
$[\,\text{MapSubLambda}\,]^{\cdot}$	rule MapSubLambda
$[\,\text{MapSubApply}\,]^{\cdot}$	rule MapSubApply
$[\,\text{MapApplyNil}\,]^{\cdot}$	rule MapApplyNil
$[\,\text{MapApplyLambda}\,]^{\cdot}$	rule MapApplyLambda
$[\,\text{MapApplyBottom}\,]^{\cdot}$	rule MapApplyBottom
$[\,\text{MapIfNil}\,]^{\cdot}$	rule MapIfNil
$[\,\text{MapIfLambda}\,]^{\cdot}$	rule MapIfLambda
$[\,\text{MapIfBottom}\,]^{\cdot}$	rule MapIfBottom
$[\,\text{MapMonotonicity}\,]^{\cdot}$	rule MapMonotonicity
$[\,\text{MapMinimality}\,]^{\cdot}$	rule MapMinimality
$[\,\text{MapQND}\,]^{\cdot}$	rule MapQND
$[\,\text{MapExtensionality}\,]^{\cdot}$	rule MapExtensionality
$[\,\text{MapQuantifyA}\,]^{\cdot}$	rule MapQuantifyA
$[\,\text{MapQuantifyB}\,]^{\cdot}$	rule MapQuantifyB
$[\,\text{MapQuantifyC}\,]^{\cdot}$	rule MapQuantifyC
$[\,\text{MapQuantifyD}\,]^{\cdot}$	rule MapQuantifyD
$[\,\text{MapExistsA}\,]^{\cdot}$	rule MapExistsA
$[\,\text{MapExistsB}\,]^{\cdot}$	rule MapExistsB

[**Map rule MapExistsC** : $\bar{\exists}\mathsf{x} \equiv \hat{?}\bar{\exists}\mathsf{x}$]

[MapExistsC]$^{\cdot}$ rule MapExistsC

Appendix C

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$[x \notin y]^{\cdot}$	x not in y, 566
$[x \mathrel{\acute{=}} y]^{\cdot}$	x optimised equal y, 113, 602
$[x \vee y]^{\cdot}$	x or y, 156, 603
$[x :: y]^{\cdot}$	x pair y, 165, 606
$[x + y]^{\cdot}$	x plus y, 102, 144, 607
$[x^{y}]^{\cdot}$	x power y end, 103, 148, 609
$[x^{-}]^{\cdot}$	x predecessor, 113, 398, 611
$[x \overset{+}{\rightarrow} y]^{\cdot}$	x reduces to y, 120, 613
$[x@_{-}y]^{\cdot}$	x round down y, 143
$[x@_{2}y]^{\cdot}$	x round even y, 143
$[x@_{+}y]^{\cdot}$	x round up y, 143
$[x@y]^{\cdot}$	x round y, 142, 614
$[x@_{0}y]^{\cdot}$	x round zero y, 143
$[x \mathrel{\breve{=}} y]^{\cdot}$	x same priority y, 112, 617
$\left[x\left\{\begin{array}{c}y\\z\end{array}\right.\right]^{\cdot}$	x select y else z end, 122

Appendix D

Bibliography

[1] K. Grue. Map theory. *Theoretical Computer Science*, 102(1):1–133, July 1992.

[2] Donald E. Knuth. *Fundamental Algorithms*, volume 1 of *The Art of Computer Programming*. Addison-Wesley, Reading, Massachusetts, second edition, 1981.

[3] J. McCarthy. Recursive functions of symbolic expressions and their computation by machine. *Communications of the ACM*, pages 184–195, 1960.

[4] E. Mendelson. *Introduction to Mathematical Logic*. Wadsworth and Brooks, Reading, Massachusetts, third edition, 1987.

Appendix E

Summary of syntax

E.1 Associativity and term reduction

[	$x \,'\, y \,'\, z$	$\overset{\smile}{\rightarrow}$	$(x \,'\, y) \,'\, z$	]
[	$x@y@z$	$\overset{\smile}{\rightarrow}$	$(x@y)@z$	]
[	$x^{y^{z}}$	$\overset{\smile}{\rightarrow}$	$x^{(y^{z})}$	]
[	$x \cdot y \cdot z$	$\overset{\smile}{\rightarrow}$	$(x \cdot y) \cdot z$	]
[	$x + y + z$	$\overset{\smile}{\rightarrow}$	$(x + y) + z$	]
[	$x :: y :: z$	$\overset{\smile}{\rightarrow}$	$x :: (y :: z)$	]
[	x append y append z	$\overset{\smile}{\rightarrow}$	x append (y append z)	]
[	x, y, z	$\overset{\smile}{\rightarrow}$	$x, (y, z)$	]
[	$x \wedge y \wedge z$	$\overset{\smile}{\rightarrow}$	$(x \wedge y) \wedge z$	]
[	$x \vee y \vee z$	$\overset{\smile}{\rightarrow}$	$(x \vee y) \vee z$	]
[	$x \vdash y \vdash z$	$\overset{\smile}{\rightarrow}$	$x \vdash (y \vdash z)$	]
[	$x \rhd y \rhd z$	$\overset{\smile}{\rightarrow}$	$(x \rhd y) \rhd z)$	]
[	$x : y : z$	$\overset{\smile}{\rightarrow}$	$x : (y : z)$	]
[	$x; y; z$	$\overset{\smile}{\rightarrow}$	$(x; y); z$	]
[	$x < y \leq z$	$\overset{\smile}{\rightarrow}$	$(x < y) \leq z$	]
[	$x < y \leq z$	$\overset{\circ}{\rightarrow}$	$(x < y) \wedge (y \leq z)$ $\overset{\smile}{>}$	
	$x, y < z$	$\overset{\circ}{\rightarrow}$	$x < z \wedge y < z$ $\overset{\smile}{>}$	
	$x < y, z$	$\overset{\circ}{\rightarrow}$	$x < y \wedge x < z$	]
[	$\forall x, y{\in}S\colon a$	$\overset{\circ}{\rightarrow}$	$\forall x{\in}S\colon \forall y{\in}S\colon a$	]
[	$x \Rightarrow y \Leftrightarrow z$	$\overset{\smile}{\rightarrow}$	$(x \Rightarrow y) \Leftrightarrow z$	]
[	$x \Rightarrow y \Leftrightarrow z$	$\overset{\circ}{\rightarrow}$	$(x \Rightarrow y) \wedge (y \Leftrightarrow z)$	]
[	$x \rightarrow y \rightarrow z$	$\overset{\smile}{\rightarrow}$	$x \rightarrow (y \rightarrow z)$	]
[	$x \rightarrow y \equiv z$	$\overset{\circ}{\rightarrow}$	$\mathrm{case}(x, y, T) \equiv \mathrm{case}(x, z, T)$	]
[	$\langle x, y \rangle$	$\overset{\circ}{\rightarrow}$	$x :: \langle y \rangle$	]

E.2 Priority

[$\mathsf{L}x$ $\breve{>}$

$x\,'\,y$ $\breve{>}$

$x@y \breve{=} x@_2y \breve{=} x@_+y \breve{=} x@_-y \breve{=} x@_0y$ $\breve{>}$

$\#x$ $\breve{>}$

$x! \breve{=} x^y \breve{=} x^+ \breve{=} x^- \breve{=} x^* \breve{=}$

x head $\breve{=}$ x tail $\breve{=}$ x pair $\breve{=}$ x atom $\breve{=}$ x Head $\breve{=}$ x Tail $\breve{>}$

$x \cdot y \breve{=} x \cdot_+ y \breve{=} x \cdot_- y \breve{=} x \cdot_0 y \breve{=} x \cdot_2 y \breve{=}$

$x/y \breve{=} x/_+y \breve{=} x/_-y \breve{=} x/_0y \breve{=} x/_2y \breve{=}$

$x\, //\, y \breve{=} x\, //_+\, y \breve{=} x\, //_-\, y \breve{=} x\, //_0\, y \breve{=} x\, //_2\, y \breve{=}$

$x \,\%\, y \breve{=} x \,\%_+\, y \breve{=} x \,\%_-\, y \breve{=} x \,\%_0\, y \breve{=} x \,\%_2\, y$ $\breve{>}$

$+x \breve{=} -x$ $\breve{>}$

$x + y \breve{=} x +_+ y \breve{=} x +_- y \breve{=} x +_0 y \breve{=} x +_2 y \breve{=}$

$x - y \breve{=} x -_+ y \breve{=} x -_- y \breve{=} x -_0 y \breve{=} x -_2 y$ $\breve{>}$

$x :: y \breve{=} x \therefore y \breve{=} x \times y$ $\breve{>}$

x append y $\breve{>}$

x, y $\breve{>}$

$x = y \breve{=} x \neq y \breve{=} x < y \breve{=} x \leq y \breve{=} x > y \breve{=} x \geq y \breve{=} x \in y \breve{=} x?$ $\breve{>}$

$\neg x$ $\breve{>}$

$x \wedge y$ $\breve{>}$

$x \vee y$ $\breve{>}$

$\forall x{\in}y\colon z \breve{=} \exists x{\in}y\colon z \breve{=} \varepsilon x{\in}y\colon z$ $\breve{>}$

$x \Rightarrow y \breve{=} x \Leftarrow y \breve{=} x \Leftrightarrow y$ $\breve{>}$

$x\left\{\begin{array}{l} y \\ z \end{array}\right. \breve{=} x\left\langle\begin{array}{l} y \\ z \end{array}\right.$ $\breve{>}$

$\lambda x.y$ $\breve{>}$

$x \equiv y \breve{=} x \overset{+}{\to} y \breve{=} x \preceq y$ $\breve{>}$

$x \to y$ $\breve{>}$

$x \vdash y$ $\breve{>}$

$x \rhd y$ $\breve{>}$

$x : y$ $\breve{>}$

$x; y$ $\breve{>}$

Mac lemma $x{:}y$ $\breve{=}$ **Mac proof of** $x{:}y$ $\breve{=}$ **Mac rule** $x{:}y$]