Optimal Reconstruction of Bandlimited Bounded Signals

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Abstract—A formula is presented for restoring bounded, bandlimited signals from a finite number of samples. The restoration is linear in the samples used, and the error bound of the method is the theoretical minimum.

I. Introduction

THIS PAPER is divided into two parts: the main text and an Appendix. The main text contains a minimum of mathematics. It describes the problem that has been solved and practical applications; it also presents quantitative results. The appendix contains all mathematical details and all mathematical proofs. Readers asking "what?" but unconcerned with "how?" should read the main text and omit the Appendix. Those interested in the "how?" as well as the "what?" of this work should read the main text first and then the Appendix.

One of the referees pointed out that similar results are obtained by A. A. Melkman [21]. Melkman concentrates on theoretical results, whereas we aim at practical applications and present as few theoretical results as possible.

Due to the sampling theorem, a finite energy signal g, bandlimited to some frequency f_0, can be determined by measuring $g(p/(2f_0))$ for all integers p. However, this is unsuited for practical applications, as it is impossible to measure the infinitude of values $g(p/(2f_0))$. For this reason, we consider the question of how to restore g from a finite number of samples. This question was considered in [1], and this paper, as well as [2]–[6], follow [1].

In what follows, we assume that the signals to be restored are bounded and bandlimited, and we consider pointwise bounds on the restoration error. The assumption and error definition correspond to those in [1].

The error-bounds for the linear reconstruction formulas in [1]–[6] decrease with the year of publication, but none reaches the theoretical minimum. This article presents a linear reconstruction formula whose error-bound is minimal.

If the assumptions are changed, the formula might become useless. If the error-bound is changed, the formula might become nonoptimal.

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II. Assumptions About Signals

It is not possible to restore signals from any number of samples, if we know nothing about the signals in advance. Hence, we have to assume something about the signals we want to restore. The following assumptions to be used are adopted from [1].

Assumption 1: The signals are bandlimited by some known frequency f_0.

Assumption 2: The signals g are absolutely limited by some known constant C: $|g(t)| \leq C$ for all t.

Many other assumptions have been considered in the literature. A general treatment of assumptions can be found in [7].

One set of assumptions that has attracted much attention is that in which boundedness (Assumption 2) is replaced by a finite-energy requirement: $g(t)$ has finite energy that does not exceed some known constant C_1. The finite-energy requirement is convenient from a mathematical point of view. (The L_2 norm corresponds to an inner product and the surface of a ball in L_2 looks the same at all points.) However, for many applications in connection with speech, music, and other long-duration signals, boundedness is a more reasonable assumption than finite energy. Boundedness is more "messy," though, from a mathematical point of view. (The L_∞ norm does not correspond to any inner product and the surface of a ball in L_∞ has edges.) Optimal estimates of finite energy signals are known [7], [8]. Due to the fundamental differences between L_2 and L_∞, there is no reason to believe that there is any simple relation between optimal estimates in the finite-energy and boundedness cases.

Quite often the choice of assumptions is a trade-off between mathematical simplicity and engineering relevance. The boundedness assumption has been chosen because of its engineering relevance, despite its mathematical properties. On the other hand, bandlimitation (Assumption 1) has been chosen for its mathematical simplicity even though real-world signals do not have spectra that drop off suddenly at some frequency f_0. Finite-sampling approximations of non-bandlimited signals have been considered in [9] and [10], but the estimates presented are nonoptimal. Further work is required to relax Assumption 1 to something more relevant to engineering.

It is tacitly assumed that signals g are deterministic. Of course, this does not exclude signals g originating from

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random processes. The results to be presented are not statistical, and a statistical approach to reconstruction can be found in [11].

III. The Sampling Process

The sampling of a signal g at sampling times $s_1 < \cdots < s_n$ means measurement of the values $v_1, \cdots, v_n$ of the signal at the respective sampling times. The knowledge gained by the sampling is expressed by

$$g(s_1) = v_1, \cdots, g(s_n) = v_n. \tag{1}$$

For the sake of simplicity we consider periodic sampling only. Furthermore, we assume that the sampling rate f_1 is greater than $2f_0$. The methods to be described are, however, applicable to all kinds of sampling.

The assumption that sampling is periodic rules out sampling types like zero sampling. The assumption that $f_1 > 2f_0$ rules out noninteresting cases. These assumptions can be found in [1]–[6] also.

Assumption 2 is insufficient, if one wants to restore signals sampled at rate $f_1 = 2f_0$. References [12] and [13] treat the case $f_1 = 2f_0$ in connection with more restrictive assumptions than Assumption 2.

When a signal g has been sampled, we have the knowledge presented in Assumptions 1 and 2 and (1), and no more. Our knowledge can be represented by the integer n and the $n+3$ real values $f_1, f_0, C, v_1, \cdots, v_n$. Reconstruction formulas should use these known values only.

We assume that the sampling process is accurate, i.e., that the exact values of $n, f_1, v_1, \cdots, v_n$ are known. Reconstruction from inaccurate data is treated in [14], which offers an optimal estimate in the finite energy case.

IV. Approximation-Bound Pairs

Let A and B be fixed functions. If

$$|g(t) - A(n,t,f_1,f_0,C,v_1,\cdots,v_n)| \le B(n,t,f_1,f_0,C,v_1,\cdots,v_n) \tag{2}$$

holds for all signals g satisfying Assumptions 1 and 2 and (1), then (A,B) will be called an approximation-bound pair (an AB pair for short). From now on B will be referred to as a "bound" instead of an "error-bound," because "approximation-error-bound pair" is hardly intelligible.

Consider the following use of AB pairs. A designer designs signal processing equipment that restores sampled signals as part of its processing, so the designer chooses an AB pair (A,B). A is implemented in hardware, whereas the designer calculates bounds B once and for all to see if A is good enough.

The designer controls n, t, and f_1 and assumes values for f_0 and C but does not know the values $v_1, \cdots, v_n$. Hence, the designer is not interested in bounds that depend on $v_1, \cdots, v_n$. For this reason, we shall require B to be independent of $v_1, \cdots, v_n$, which will now be omitted from the parameter list of B. Hence, an AB pair is a pair (A,B) of functions for which

$$|g(t) - A(n,t,f_1,f_0,C,v_1,\cdots,v_n)| \le B(n,t,f_1,f_0,C). \tag{3}$$

V. Optimal AB Pairs

An AB pair (A,B) is said to have a minimal error bound if we have $B(n,t,f_1,f_0,C) \le \tilde{B}(n,t,f_1,f_0,C)$ for all AB pairs $(\tilde{A},\tilde{B})$ and all possible n,t,f_1,f_0,C. It is straightforward to prove the existence of such optimal AB pairs. There are infinitely many of them. The last fact will not be proved because the proof is cumbersome and of little interest.

Let n, f_1, f_0, and C denote fixed values. According to Assumption 2 we have $B(n,t,f_1,f_0,C) \le C$ for optimal (A,B) and all t. As we shall see there are constants t_a, t_b such that $B(n,t,f_1,f_0,C) < C$ for $t \in (t_a,t_b)$ and $B(n,t,f_1,f_0,C) = C$ for $t \notin (t_a,t_b)$. Hence, in general, there is no way possible to obtain information about $g(t)$ for any fixed t outside (t_a,t_b). We shall refer to t_a and t_b as limits of prediction.

VI. The Linear Optimal AB Pair

In the Appendix it is proved that there exists one and only one optimal AB pair (A,B) for which $A(n,t,f_1,f_0,C,v_1,\cdots,v_n)$ is linear in $v_1,\cdots,v_n$. The existence of such a linear optimal AB pair might seem surprising at first. This AB pair is very useful as it is simple to implement in hardware.

In what follows, (A,B) refers to the linear optimal AB pair. As a consequence of the linearity, A can be written in the form

$$A(n,t,f_1,f_0,C,v_1,\cdots,v_n) = \sum_{i=1}^{n} v_i A_i(n,t,f_1,f_0,C),$$

where the A_i are functions.

In the Appendix it is proved that A and A_i are independent of C. From now on, C will be omitted from the parameter list. It will, furthermore, be proved that B is linear in C. We define $B(n,t,f_1,f_0)$ such that $CB(n,t,f_1,f_0) = B(n,t,f_1,f_0,C)$. Formula (3) now reads

$$\left| g(t) - \sum_{i=1}^{n} v_i A_i(n,t,f_1,f_0) \right| \le CB(n,t,f_1,f_0). \tag{4}$$

VII. Computation

In the Appendix a numerical method for calculating A_i and B is outlined. The calculation of a value of A_i and B requires from seconds to days of computation on a PDP11 with floating point processor. The time required mainly depends on n.

The calculation of A_i and B typically requires far too much computational power to be implemented in signal-processing equipment. Instead, tables of A_i as a function of i and t can be calculated once and for all and implemented as coefficients in finite impulse response (FIR) filters (or, equivalently, as coefficients in an interpolation/

extrapolation matrix (cf. [15])). B can be calculated once and for all for accuracy considerations.

Because A_i and B are difficult to calculate, designers of signal-processing equipment should use printed tables for A_i and B if such aids become available.

VIII. Graphs

In Fig. 1, A_1 and A_2 are plotted versus t for $n = 4$ and $f_1/(2f_0) = 1.8$. For convenience, A_3 and A_4 are not plotted. If we choose the symmetry point of the sampling times $s_1, s_2, \cdots$ as the zero of the time axis, then a symmetry argument shows $A_3(t) = A_2(-t)$ and $A_4(t) = A_1(-t)$. We have $A_i(t) = 0$ for $t \notin (t_a, t_b)$, where t_a and t_b are the limits of prediction.

For $t \in (t_a, t_b)$, B is the absolute value of the function b plotted in Fig. 1. For $t \notin (t_a, t_b)$ we have $B(n, t, f_1, f_0) = 1$. The function b is considered in the Appendix.

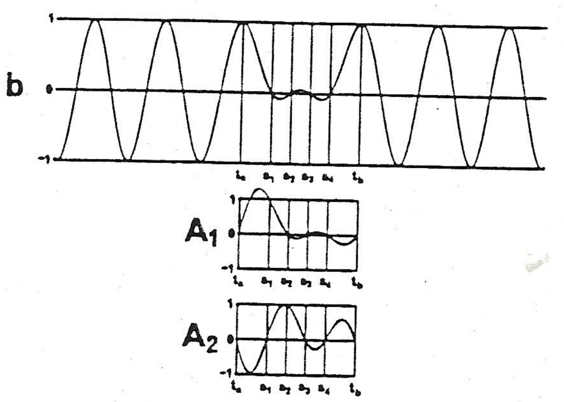

Fig. 1. A_1 and A_2 versus t for $n = 4$ and $f_1/(2f_0) = 1.8$. $A_i(t) = 0$ for $t \notin (t_a, t_b)$, where t_a and t_b are the limits of prediction.

IX. Tables

Consider the following problem. A signal g satisfying Assumptions 1 and 2 is continually sampled at rate f_1. The signal is to be restored for a given time t_0, using an optimal AB pair on the m samples preceding t_0 and the m samples following t_0, where m is some integer. What accuracy will we gain? The bound depends on the distance from t_0 to the nearest sample. It is zero when t_0 coincides with a sample and takes on its maximum value M when t_0 lies right between two samples (cf. Fig. 1). Tables I and II list $-\log_2(M)$ for certain values of $n = 2m$ and ratios $r = f_1/(2f_0)$. All values in the tables are rounded down. The ratio 44.1 kHz/40 kHz is particularly interesting in connection with digital audio disc standards.

TABLE I
NUMBER OF SIGNIFICANT BITS GAINED RESTORING SIGNALS SAMPLED AT RATE $2f_0r$ FROM n SAMPLES[a]

n \ r	1.2	1.4	1.6	1.8	2.0
2	0.6	1.1	1.5	1.9	2.2
4	1.3	2.4	3.3	4.1	4.8
6	2.1	3.8	5.2	6.3	7.3
8	2.9	5.1	7.0	8.5	9.9
10	3.6	6.5	8.8	10.7	12.4
12	4.4	7.8	10.6	13.0	15.0
14	5.2	9.2	12.4	15.2	17.5
16	5.9	10.5	14.2	17.4	20.1
18	6.7	11.9	16.0	19.6	22.6
20	7.5	13.2	17.9	21.8	25.2

[a] f_0 denotes an upper bandlimit of the signal.

TABLE II
NUMBER OF SIGNIFICANT BITS GAINED RESTORING SIGNALS BANDLIMITED BY 20 kHz AND SAMPLED AT 44.1 kHz FROM $n + N$ SAMPLES

n \ N	0	20	40	60
2	0.3	4.5	8.8	13.0
4	0.7	4.9	9.2	13.4
6	1.1	5.4	9.6	13.8
8	1.6	5.8	10.0	14.3
10	2.0	6.2	10.5	14.7
12	2.4	6.6	10.9	15.1
14	2.8	7.1	11.3	15.5
16	3.3	7.5	11.7	16.0
18	3.7	7.9	12.1	16.4
20	4.1	8.3	12.6	16.8

The following conclusion can be drawn from Table II. Suppose that a signal bandlimited by 20 kHz is sampled at 44.1 kHz and recorded on a digital audio disc. If a record player for the disc raises the sampling rate by FIR filters, and if an accuracy of 16 b is required, the FIR filters must have an impulse response of length at least 76.

Equation (4) is an extrapolation formula when $t \in (s_n, t_b)$. Within the setting considered, prediction of $g(t)$ beyond t_b is impossible. Table III lists $(t_b - s_n)f_0$ for various values of n and $r = f_1/(2f_0)$. The value $(t_b - s_n)f_0$ is the length of (s_n, t_b) measured in cycles of the limit frequency f_0.

TABLE III
RANGE OF PREDICTION MEASURED IN CYCLES OF THE LIMIT FREQUENCY f_0[a]

n \ r	1.0	1.2	2.0	20.0	50.0
1	0.25	0.25	0.25	0.25	0.25
2	0.25	0.27	0.32	0.42	0.42
4	0.25	0.30	0.45	0.74	0.76
10	0.25	0.38	0.81	NC	NC
50	0.25	0.75	3.00	NC	NC

[a] NC: not calculated.

X. Comparison of Accuracies

Knab [6], considers an AB pair $(\tilde{A}, \tilde{B})$ whose bound is approximately $C\tilde{B}(n, t, f_1, f_0) = C/\sinh(\pi(1 - 2f_0/f_1)(n-1)/2)$ for $n \geq 7$, n odd. This seems to be the smallest error bound published for the problem considered.

The error bound of optimal AB pairs (equal for all optimal AB pairs) has been calculated for n even only, so a direct comparison between Knab's AB pair and optimal AB pairs is not possible on the basis of Tables I and II. Tables IV and V, however, give some idea of the distance from Knabs AB pair to optimum. The tables compare $-\log_2(M)$ values; all values are rounded down.

TABLE IV
SAMPLING RATE 44.1 kHz, UPPER BANDLIMIT 20 kHz

n	Knab	Optimal
79	15.4	
80		16.8
81	15.8	
83	16.2	
85	16.6	
87	17.1	

TABLE V
SAMPLING RATE $4f_0$, UPPER BANDLIMIT f_0

n	Knab	Optimal
6		7.3
7	6.9	
8		9.9
9	8.0	
10		12.4
11	10.3	
12		15.0
13	12.5	
14		17.5
15	14.8	
16		20.1
17	17.1	

XI. COMPARISON WITH ASYMPTOTICAL RESULTS

The asymptotical behavior of optimal error bounds has not been found. This behavior is of little importance for practical applications but is interesting in itself.

Values of $-\log_2(M)$ for Knab's error bound asymptotically equal $2.2662(1 - 2f_0/f_1)(n - 1) - 1$. It is interesting to compare this result with Tables I and II. The table entries are almost linear in n but they are nonlinear in $(1 - 2f_0/f_1)$. Knab's AB pair seems to be close to optimum for a small degree of oversampling ($f_1/(2f_0) \simeq 1$) and seems to become less close for an increasing degree of oversampling (increasing $f_1/(2f_0)$).

The values of Tables II and I equal approximately $0.2117n - 0.1$ and $1.275n - 0.3$ for values of 44.1/40 and 2, respectively, for $f_1/(2f_0)$. Knab's accuracy asymptotically equals $0.2107n - 1.2$ and $1.1331n - 2.1$, respectively.

XII. COMPARISON OF STRUCTURES

In [1]–[6] AB pairs are presented whose structure equals that of (4). All these AB pairs have the property that A_i for different i are equal except for a translation along the t-axis. The linear optimal AB pair happens not to have this property (cf. Fig. 1).

Knab's AB pair is constructed by using prolate spheroidal wave functions. These functions are the solutions to optimization problems that differ fundamentally from the optimization problem solved in this paper. There does not seem to by any relation between prolate spheroidal wave functions and optimal AB pairs.

XIII. ANOTHER OPTIMALITY CRITERION

As stated in Section V, bounds are defined as functions of n, t, f_1, f_0, and C. If the bounds are allowed to depend also on the sample values $v_1, \cdots, v_n$, then the definition of "optimal" becomes stricter. The linear optimal AB pair is not optimal under this stricter definition. It is simple to prove that only one AB pair is optimal under the stricter definition. The bound of this AB pair is, of course, the same as the bound of the linear AB pair when bounds are defined in the less strict way.

The less strict definition has been chosen for the following reasons: 1) it makes analysis possible; 2) it permits a linear AB pair to be optimal, which simplifies hardware implementation; 3) last, but not least, it is unreasonable to use the stricter definition, as discussed in Section IV.

The AB pair, which is optimal under the stricter definition, can be analyzed numerically.

XIV. CONCLUSION

An optimal approximation-bound pair has been found. It is better, but not much better than known results. It is suited to hardware implementation. It is particularly interesting in that it gives us the exact limits on the accuracy that can be gained. To make optimal reconstruction popular, it is necessary that tables of the functions involved be published.

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APPENDIX

In this Appendix, which consists of five sections, the postulates of the main text are proved.

Section B states an optimization problem that turns out to be the key to finding optimal AB pairs. Solutions to the problem are called *optimal functions*. The function b of Fig. 1 is an optimal function.

It is time-consuming but manageable work to calculate optimal functions. Section E outlines the method used by the author.

It is simple to find the functions A and B when the proper optimal functions b have been found. Section D describes the method.

Sections A and C supply auxiliary definitions and results to Sections B, D, and E. Section A states general definitions, and Section C makes a general investigation of optimal functions.

To make optimal reconstruction, proceed as follows. 1) Choose the values of n, f_1, and f_0. 2) Normalize f_0 to 1/2 Hz (see Section A-1)). 3) Calculate the sampling times $s_1, \cdots, s_n$ (see Section A-2)). 4) Find the zeros of the real polynomials M and Z to be introduced. The zeros of M are called $m_1, \cdots, m_{n-1}$ and are real. The zeros of z are called $z_1, \cdots, z_{n-1}, \bar{z}_1, \cdots, \bar{z}_{n-1}$ and occur as conjugate pairs of nonreal zeros. Use (8) and (9) in Section E-4) to find the zeros numerically. 5) Tabulate the optimal function b, using (7) in Section E-3). 6) Tabulate the function b defined in Section C-5). 7) Calculate $b'(s_1), \cdots, b'(s_n)$.

8) Tabulate A_i and B by using (5), (6), and the definition of b_i in Section D-1). Formula (5) defines A and not A_i, but this causes no problems (see Section VI). 9) Use the A_i to do the reconstruction, and B to find the accuracy of the reconstruction.

If tables of b, b, and $b'(s_1), \cdots, b'(s_n)$ are available, then only the trivial steps 8) and 9) have to be carried out. Step 4) requires the most work.

The optimization problem in Section B is related to the problem considered in [19]. Unfortunately, it is not possible to use the results in [19] in this article, as the problems considered are too different. It is easy to devise a general problem of which the problem of [19] and the problem of Section B are both special cases. There are many analogies between the solutions of the two problems. One such can be found in the differential equation in Section E-3).

A. Definitions

1) Normalization: According to Assumptions 1 and 2 we deal with signals bandlimited by f_0 and absolutely limited by C. By scaling the time axis and signal value axis, we can make f_0 equal to 1/2 Hz and C equal to 1. In what follows, $f_0 = 1/2$ Hz and $C = 1$ are assumed.

2) Sampling Times: Signals are to be restored from n samples. The cases n even and n odd differ slightly and it is inconvenient to treat the two together. However, the cases are so much alike that only one of them will be treated in what follows. Considering the problem stated in Section IX, the case n even is more interesting than n odd. Hence we assume that n is even.

Signals are sampled periodically at the rate $f_1 > 2f_0 = 1$ Hz. Choose the zero of the time axis as the symmetry center of the sampling times. The sampling times s_i now satisfy $s_i = (-n-1)/(2f_1) + i/f_1, 1 \le i \le n$.

3) Bandlimitation: A real function g is said to be bandlimited by 1/2 Hz, if it is the inverse Fourier transform of some distribution that is zero outside [−1/2 Hz, 1/2 Hz]. This definition permits some infinite power signals, such as polynomials, to be bandlimited by 1/2 Hz. The set of real functions that are bandlimited by 1/2 Hz is denoted P (for Paley–Wiener [16]).

For signals g satisfying $|g(t)| \le D$ for some D and all t, the definition above is equivalent to ordinary definitions. Hence, the definition does not influence any statement stated until now.

The following lemmas are simple consequences of the theorem of Paley–Wiener [16, p. 183], Hadamard's factorization theorem [17, p. 22], Bernsteins theorem [17, p. 206], and [17, theorem 5.1.12(b)].

Lemma 1: Any g in P has an entire analytical extension, i.e., a complex function $\tilde{g}$ that is defined and analytical in the entire complex plane for which $\tilde{g}(t) = g(t)$ for all real t.

Lemma 2: If the nonzero functions g and h are in P, and their (entire) analytical extensions $\tilde{g}$ and $\tilde{h}$ have the same zeros (counted with multiplicity), then $g = ah$ for some real nonzero constant a.

Lemma 3: If g and h are in P and a is real, then $g + h$, ag, $tg(t)$, and $g'(t)$ are in P.

Lemma 4: If g is in P, a is real, and $g(a) = 0$, then $g(t)(t-a)^{-1}$ is in P.

Lemma 5: If g is in P with analytical extension $\tilde{g}$, and if $\tilde{g}(a) = 0$ for some complex a with $\mathrm{Im}(a) \ne 0$, then $g(t)(t-a)^{-1}(t-\bar{a})^{-1}$ is in P, where $\bar{a}$ is the complex conjugate of a.

Lemma 6: If g is in P and $|g(t)| \le 1$ for all t, then $|g'(t)| \le \pi$, $|g''(t)| \le \pi^2$, $|g'''(t)| \le \pi^3$, etc.

B. Optimal Functions

Section B-1) states an optimization problem whose solutions are called optimal functions, and B-2) proves that solutions to the problem exist.

1) An Optimization Problem: The following optimization problem turns out to be the key to finding optimal AB pairs.

Let $b \in P$ satisfy $|b(t)| \le 1$ for all t and $b(s_1) = \cdots = b(s_n) = 0$. How large can $b(0)$ be? Which functions, if any, maximize $b(0)$? For the sake of this problem we define K to be the set of functions $g \in P$ for which $|g(t)| \le 1$ for all t and $g(s_1) = \cdots = g(s_n) = 0$. A function $b \in K$ is said to be optimal if $b(0) \ge g(0)$ for all g in K. In what follows we shall prove that there is one and only one optimal function b; we shall investigate this function and construct the linear optimal AB pair from it. The graph of b is shown in Fig. 1 for $f_1 = 1.8$ Hz, $n = 4$. b is closely related to the error bound B of optimal AB pairs.

2) The Existence of Optimal Functions: The existence proof below is inspired by a similar proof in [19]. According to [17, theorem 6.2.6] and [20, definition 7.1], K is locally uniformly bounded. Choose (which is possible) $g_1, g_2, \cdots \in K$ such that $g_1(0), g_2(0), \cdots$ converges towards the supremum of $\{g(0) | g \in K\}$ (which exists). According to [20, corollary 7.6], $g_1, g_2, \cdots$ has a locally, uniformly convergent subsequence. Let b be the limit of such a sequence. It is easily shown that b is an optimal function.

C. Properties of Optimal Functions

Section C-1) introduces the optimal function b. Section C-2) proves $b(0) > 0$. Section C-3) proves $|b(t)| < 1$ for $t \in [s_1, s_n]$. Section C-4) proves that the least upper bandlimit of b is 1/2 Hz. Sections C-5) and C-6) investigate the zeros of $b'(t)$, leading to results that are crucial for Sections D and E.

1) The Definition of b: We have established the existence, but not the uniqueness, of the optimal function. The uniqueness proof is given in Section D-3) so, in what follows, we cannot assume uniqueness.

In what follows, b will denote one among the optimal functions (the only, it will turn out). According to the definition of optimal functions, we have $b \in P$, $|b(t)| \le 1$, $b(s_1) = \cdots = b(s_n) = 0$ and $b(0) \ge g(0)$ for all $g \in K$.

2) $b(0) > 0$: Define g as the product

$$g(t) = (-1)^{n/2} \sin(\pi(t - s_1)/n) \times \cdots \times \sin(\pi(t - s_n)/n).$$

We have $g \in K$ and $b(0) \ge g(0) > 0$. A consequence of $b(0) > 0$ is that b is not a constant function and, in particular, b is not the zero function. This will be used implicitly in what follows.

3) $|b(t)| < 1$ for $t \in [s_1, s_n]$: We first prove Lemma 7.

Lemma 7: If $g \in P$, $|g(t)| \le 1$ for all t, and $g(0) = 1$, then $g(t) > 0$ for $t \in (-1/2, 1/2)$.

Proof: Define $\phi_i(t) = (-1)^i t \sin(\pi t)/(\pi i(t-i))$ for $i \ne 0$, $\phi_0(t) = \sin(\pi t)/(\pi t)$ and $\bar{\phi}(t) = \sin(\pi t)/\pi$. According to [17, Theorem 11.5.10], we have the following sampling theorem:

$$g(t) = g'(0)\bar{\phi}(t) + \sum_{i=-\infty}^{+\infty} g(i)\phi_i(t).$$

(Note that [17, equation 11.5.11] contains a printing error.) A special case is

$$\cos(\pi t) = \sum_{i=-\infty}^{+\infty} (-1)^i \phi_i(t).$$

For $t \in (-1/2, 1/2)$, $i \neq 0$ we have $(-1)^i \phi_i(t) \leq 0$. From $g(0) = 1$, $g(t) \leq 1$ we deduce $g'(0) = 0$. Using all this we have

$$g(t) = \phi_0(t) + \sum_{i=1}^{+\infty} \left(g(i)\phi_i(t) + g(-i)\phi_{-i}(t) \right)$$

$$\geq \phi_0(t) - \sum_{i=1}^{+\infty} \left(|\phi_i(t)| + |\phi_{-i}(t)| \right) = \cos(\pi t) > 0$$

for $t \in (-1/2, 1/2)$. This completes the proof.

A consequence of the Lemma 7 is that if $g \in P$, $|g(t)| \leq 1$ for all t, and if g has two zeros with distance less than 1, then $|g(t)| < 1$ between the zeros.

From $b(s_1) = \cdots = b(s_n) = 0$ and $s_n - s_{n-1} = \cdots = s_2 - s_1 = 1/f_1 < 1$ we deduce $|b(t)| < 1$ for $t \in [s_1, s_n]$.

4) The Bandlimit of b: We have that 1/2 Hz is among the upper bandlimits of b. We now prove that 1/2 Hz is the least among these bandlimits. Suppose b had a bandlimit less than 1/2 Hz. Then $b((1-\epsilon)^{-1}t)$ would be bandlimited by 1/2 Hz for small positive ϵ. Now define

$$b_1(t) = b\left((1-\epsilon)^{-1}t\right)\left(t - s_1(1-\epsilon)\right)^{-1} \cdots$$
$$\left(t - s_n(1-\epsilon)\right)^{-1}(t - s_1) \cdots (t - s_n).$$

One easily proves $b_1 \in K$, $b_1(0) > b(0)$ for small positive ϵ, contradicting that b is an optimal function.

5) Some Definitions: From $b(s_1) = \cdots = b(s_n) = 0$ we deduce that b' must have at least one zero in each of the intervals (s_i, s_{i+1}). Now define $m_i \in (s_i, s_{i+1})$ such that $b'(m_i) = 0$,

$$M(t) = (t - m_1)(t - m_2) \cdots (t - m_{n-1}),$$
$$S(t) = (t - s_1)(t - s_2) \cdots (t - s_n),$$
$$b(t) = b'(t) S(t)/M(t).$$

We have $b(s_1) = \cdots = b(s_n) = 0$. According to Lemmas 3 and 4 we have $b \in P$. S has degree n, M degree $n-1$ and $|b'(t)| \leq \pi$ according to Lemma 6. Hence $|b(t)| \leq \alpha(|t| + 1)$ for some α and all t.

6) Zeros of $b'(t)$: In this section we investigate the zeros of $b'(t)$:. With limited effort, one can deduce Lemma 8 from Lemma 6 and Lemma 9 from Lemma 8.

Lemma 8: $|b(t)| \leq 1 - b'(t)^2/(2\pi^2)$.

Lemma 9: $|b(t) + \epsilon b'(t)| \leq 1 + \pi^2\epsilon^2/2$.

Lemma 9 enables us to prove five lemmas regarding the zeros of $b'(t)$ and $b(t)$.

Lemma 10: $b(0) \neq 0$.

Lemma 11: The analytical extension of $b'(t)$ has real zeros only.

Lemma 12: The only zeros of $b'(t)$ in $[s_1, s_n]$ are $m_1, \cdots, m_{n-1}$, and these zeros each have order one.

Lemma 13: If $b'(t) = 0$ and $t \notin [s_1, s_n]$, then $|b(t)| = 1$.

Lemma 14: All zeros of $b'(t)$ have order one.

We first prove lemma 10.

Proof: Suppose $b(0) = 0$. Let p be the order of the zero of b at 0 (as we can prove, b is not the zero function). Let $b_1(t) = b(t)t^{-p}$. We have $b_1(0) \neq 0$. Choose b_2 among b_1 and $-b_1$ such that $b_2(0) > 0$. We have $b_2 \in P$, $b_2(0) > 0$, $b_2(s_1) = \cdots = b_2(s_n) = 0$ and $|b_2(t)| \leq \beta$ for some β and all t.

Now consider the function $b(t) + \delta b_2(t)$ for some small positive δ. We have $|b(t)| < 1$ for $t \in [s_1, s_n]$, so for sufficiently small δ we have $|b(t) + \delta b_2(t)| < 1$ for $t \in [s_1, s_n]$.

We have $b_2(t) = \pm b'(t)t^{-p}S(t)/M(t)$. $t^{-p}S(t)/M(t)$ is bounded outside $[s_1, s_n]$. Let γ be an upper bound on $|t^{-p}S(t)/M(t)|$ outside $[s_1, s_n]$. For $t \notin [s_1, s_n]$ we have, according to Lemma 9, $|b(t) + \delta b_2(t)| \leq |b(t) \pm \delta\gamma b'(t)| \leq 1 + \pi^2\delta^2\gamma^2/2$.

We now have that $b_3(t) = (b(t) + \delta b_2(t))/(1 + \pi^2\delta^2\gamma^2/2)$ is in K. For sufficiently small positive δ we have $b_3(0) > b(0)$ contradicting that b is an optimal function. This proves Lemma 10.

The proofs of Lemmas 11–14 are similar. The main difference is that the definition of b_1 above differs from proof to proof, as follows.

For Lemma 11: Let $\tilde{b}'$ be the analytical extension of b' and let a be nonreal. Suppose $\tilde{b}'(a) = 0$. Define $b_1(t) = b(t)(t - a)^{-1}(t - \bar{a})^{-1}$ to get a contradiction.

For Lemma 12: Suppose $b'(a) = 0$ for some $a \in [s_1, s_n]$, $a \notin \{m_1, \cdots, m_{n-1}\}$: define $b_1(t) = b(t)(t - a)$. Suppose b' has a zero of order higher than 1 at m_i: define $b_1(t) = b(t)(t - m_i)$.

For Lemma 13: Suppose $b'(a) = 0$, $|b(a)| \neq 1$ and $a \notin [s_1, s_n]$: define $b_1(t) = b(t)(t - a)$. To carry through the proof one has to prove $|b(t) + \delta b_2(t)| < 1$ not only for $t \in [s_1, s_n]$ but also for some neighborhood of a.

For Lemma 14: Lemma 12 established the lemma for zeros in $[s_1, s_n]$. Now suppose that b' has a multiorder zero at $a \notin [s_1, s_n]$. From Lemma 13 we have $|b(a)| = 1$. From $|b(t)| \leq 1$ we deduce that the order of the zero of b' at a must be odd, i.e., at least 3. Define $b_1(t) = b(t)(t - a)$. To carry through the proof one has to prove $|b(t) + \delta b_2(t)| \leq 1 + \pi^2\delta^2\gamma^2/2$ for some small neighborhood of a separately.

D. The Linear Optimal AB Pair

Section D-1) presents a linear AB pair, and D-2) proves that it is optimal. Section D-3) proves the uniqueness of b.

1) A Linear AB Pair: In Section C-5) we defined b. As we can easily deduce, b satisfies $b \in P$. $b(s_1) = \cdots = b(s_n) = 0$. All zeros of b have order one. All zeros of the analytical extension of b are real. $|b(t)| \leq \alpha(|t| + 1)$ for some α and all t. There is no $\beta < \infty$ such that $|b(t)| \leq \beta$ for all t. The least upper bandlimit of b is 1/2 Hz.

Now, define $s_0, s_{-1}, s_{-2}, \cdots$ and $s_{n+1}, s_{n+2}, \cdots$ such that $\cdots < s_{-1} < s_0 < s_1 < \cdots$ and such that $\{\cdots, s_{-1}, s_0, s_1, \cdots\}$ is exactly the set of zeros of b. Define $b_i(t) = b(t)(t - s_i)^{-1}b'(s_i)^{-1}$. With some effort, we can prove the sampling theorem below for all $g \in P$ for which $|g(t)| \leq 1$:

$$g(t) = \sum_{i=-\infty}^{+\infty} g(s_i) b_i(t).$$

A special case is

$$b(t) = \sum_{i=-\infty}^{0} (-1)^{i+n/2} b_i(t) + \sum_{i=n+1}^{+\infty} (-1)^{i+1+n/2} b_i(t).$$

Define $t_a = s_0$ and $t_b = s_{n+1}$ (cf. Fig. 1). For $t \in [t_a, t_b]$ we have

$$\left| g(t) - \sum_{i=1}^{n} g(s_i) b_i(t) \right|$$

$$= \left| \sum_{i=-\infty}^{0} g(s_i) b_i(t) + \sum_{i=n+1}^{+\infty} g(s_i) b_i(t) \right|$$

$$\leq \sum_{i=-\infty}^{0} |b_i(t)| + \sum_{i=n+1}^{+\infty} |b_i(t)| = |b(t)|.$$

(The last equality requires a small investigation of signs.) Now, define

$$A(n,t,f_1,f_0,C,v_1,\cdots,v_n) = \sum_{i=1}^{n} v_i b_i(2f_0 t) \tag{5}$$

for $t \in [t_a, t_b]$ and equals 0 elsewhere;

$$B(n,t,f_1,f_0,C) = C|b(2f_0 t)| \tag{6}$$

for $t \in [t_a, t_b]$ and equals C elsewhere, where the right sides implicitly depend on n and f_1 through the definitions of b and b_i. (A,B) is a linear AB pair.

2) (A,B) is Optimal: We now prove that (A,B) is optimal. Suppose we have sampled a signal g satisfying Assumptions 1 and 2, and suppose we got $g(s_1) = \cdots = g(s_n) = 0$. Now, g *could* equal $g_1(t) = Cb(2f_0 t)$, and it also *could* equal $g_2(t) = -Cb(2f_0 t)$. For all AB pairs $(\tilde{A}, \tilde{B})$ we have

$$|g(t) - \tilde{A}(n,t,f_1,f_0,C,0,\cdots,0)| \le \tilde{B}(n,t,f_1,f_0,C)$$

for $g = g_1$ as well as $g = g_2$. Hence,

$$2\tilde{B}(n,t,f_1,f_0,C) \ge |g_1(t) - g_2(t)| = 2B(n,t,f_1,f_0,C)$$

proving that (A,B) is optimal.

3) b is Unique: We have $C|b(2f_0 t)| = B(n,t,f_1,f_0,C)$ for $t \in [t_a, t_b]$. B is uniquely defined as the bound of optimal AB pairs. B is not zero in $I = [s_{n/2}, s_{n/2+1}]$, so we have either $b(2f_0 t) = B(n,t,f_1,f_0,C)/C$ or $b(2f_0 t) = -B(n,t,f_1,f_0,C)/C$ for $t \in I$. Now, $0 \in I$ and $b(0) > 0$, so we have $b(2f_0 t) = B(n,t,f_1,f_0,C)/C$ for $t \in I$. $b(2f_0 t)$ is analytical and coincides with $B(n,t,f_1,f_0,C)/C$ on an interval, so b is unique. Hence, the optimization problem of Section B-1) has a unique solution.

E. Numerical Calculations

Sections E-1)–E-3) establish for the function Z introduced in Section E-2 the differential equation $Z(t)b'(t)^2 = \pi^2 M(t)^2(1 - b(t)^2)$. Its solution (7) provides a numerical method for calculating b in Sections E-4)–E-6).

1) Root Structure of $\mathrm{Im}(\tilde{b}(z)) = 0$: Let $\tilde{b}$ denote the analytical extension of b. In order to solve $1 - \tilde{b}(z)^2 = 0$, we investigate the equation $\mathrm{Im}(\tilde{b}(z)) = 0$. The root structure of the equation $\mathrm{Im}(\tilde{b}(z)) = 0$ consists of curves except where $\tilde{b}'(z) = 0$. At the points where $\tilde{b}'(z) = 0$, four curves end in a common point (other numbers of curves would be possible if $\tilde{b}'$ had had multiple zeros). For all real z, $\mathrm{Im}(\tilde{b}(z)) = 0$, so the real line is part of the root structure. Locally around the real axis, the root structure must have the form shown in Fig. 2. The root structure is symmetric about the real axis. Fig. 2 displays the upper half-plane only. The arrows indicate the direction of increasing $\mathrm{Re}(\tilde{b}(z))$.

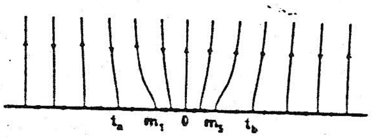

Fig. 2. Root structure of $\mathrm{Im}(\tilde{b}(z)) = 0$ for $n = 6$ and $f_1 = 4f_0$.

Along curves in the root structure, the direction of increasing $\mathrm{Re}(\tilde{b}(z))$ can change only where $\tilde{b}'(z) = 0$. It is easily proved that for all entire nonconstant functions $\tilde{g}$, the root structure of $\mathrm{Im}(\tilde{g}(z)) = 0$ has no loops. Hence, the curves perpendicular to the real axis in Fig. 2 extend to infinity without meeting one another.

2) The Zeros of $1 - \tilde{b}(z)^2$: We now search the zeros of $1 - \tilde{b}(z)^2$. $1 - \tilde{b}(z)^2$ has zeros where $\tilde{b}(z) \in \{-1, 1\}$, so all zeros of $1 - \tilde{b}(z)^2$ are on the root structure of $\mathrm{Im}(\tilde{b}(z))$. Using $|b(t)| < 1$ for $t \in [s_1, s_n]$, we deduce that $1 - \tilde{b}(z)^2$ has exactly one zero at each of the $2n - 2$ curves starting at some m_i perpendicular to the real axis. Call these zeros $z_1, \bar{z}_1, \cdots, z_{n-1}, \bar{z}_{n-1}$. All these zeros have order one. Using $|b(t)| = 1$ for $b'(t) = 0$, $t \notin [s_1, s_n]$, we deduce that $1 - b'(z)^2$ has order-two zeros where $b'(t) = 0$, $t \notin [s_1, s_n]$. As we see, $1 - \tilde{b}(z)^2$ has no other zeros. Define

$$Z(t) = (t - z_1)(t - \bar{z}_1) \cdots (t - z_{n-1})(t - \bar{z}_{n-1}).$$

3) A Differential Equation: A comparison of the zeros of $Z(z)\tilde{b}'(z)^2$ and $M(z)^2(1 - \tilde{b}(z)^2)$ reveals that these two functions have the same zeros counted with multiplicity. Hence, according to Lemma 2, $Z(z)\tilde{b}'(z)^2 = \pm a^2 M(z)^2(1 - \tilde{b}(z)^2)$ for some real nonzero constant a. As all factors on both sides of the equality are nonnegative for real z, the $\pm$ sign must be a plus. The solution to this equation (using $\tilde{b}(s_1) = 0$) is

$$\tilde{b}(z) = \sin\left(\int_{s_1}^{z} aM(u)Z(u)^{-1/2}\,du\right).$$

The integral depends on the choice of integration path, but the dependence is removed by the many-to-one sine function making $\tilde{b}$ independent of the choice of path.

From the Paley–Wiener theorem [16, p. 183], we deduce that the least upper bandlimit of the right side of the equation above is $|a|$ rad/s. The least upper bandlimit of b is 1/2 Hz, proving $|a| = \pi$. Using $b(0) > 0$ we deduce $a = (-1)^{n/2}\pi$. Thus

$$b(t) = (-1)^{n/2}\sin\left(\pi\int_{s_1}^{t} M(x)Z(x)^{-1/2}\,dx\right). \tag{7}$$

4) Determining M and Z: We can use (7) to calculate b provided we know M and Z. We now derive equations for determining $M = (t - m_1)\cdots(t - m_{n-1})$ and $Z = (t - z_1)(t - \bar{z}_1)\cdots(t - z_{n-1})(t - \bar{z}_{n-1})$, i.e., the $n - 1$ real numbers $m_1, \cdots, m_{n-1}$, and $n - 1$ complex numbers $z_1, \cdots, z_{n-1}$.

From $b(s_2) = \cdots = b(s_n) = 0$, $|b(t)| < 1$ for $t \in [s_1, s_n]$ and (7) we obtain

$$\int_{s_1}^{s_i} M(x)Z(x)^{-1/2}\,dx = 0, \qquad \text{for } i = 2, \cdots, n, \tag{8}$$

where the integration path should not leave the real axis.

$\tilde{b}$, the analytical expression of b, is entire. In particular, $\tilde{b}$ is differentiable at each of the points $z_1, \cdots, z_{n-1}$. Considering the differentiability of the right side of (7) in these points, we obtain

$$\cos\left(\pi\int_{s_1}^{z_i} M(u)Z(u)^{-1/2}\,du\right) = 0, \qquad \text{for } i = 1, \cdots, n-1. \tag{9}$$

Equation (9) is independent of the integration path.

We now have $n - 1$ real equations (8) and $n - 1$ complex equations (9) for determining $n - 1$ real values m_i and $n - 1$ complex values z_i. As we shall prove, (8) and (9) have no other solution than those desired.

5) Uniqueness of Solution: Suppose M_1 and Z_1 satisfy (8) and (9). Define

$$b_1(t) = (-1)^{n/2}\sin\left(\pi\int_{s_1}^{t} M_1(x)Z_1(x)^{-1/2}\,dx\right).$$

We now prove $b_1 = b$ (cf. (7)). In Sections C-1)–C-6) certain properties of b were stated. One can prove that b_1 shares all these properties. In Sections D-1) and D-2) it was proved that $A|b(2f_0 t)|$ equals the error bound of optimal AB pairs. The proof used properties of b that were stated in Sections C-1)–C-6) and nothing else. Hence, the proof holds for b_1 also. $b_1 = b$ is now easily established. Hence, (8) and (9) have the desired solutions only.

6) Numerical Methods: Equations (8) and (9) can be solved numerically. The results, i.e., the polynomials M and Z, and (5), (6), and (7) can be used for numerical calculation of A and B.

Much computation is required to solve (8) and (9), and it is necessary to be careful when choosing numerical algorithms. The entry $n + N = 80$ in Table II was calculated within 19 hours using a Pascal program on a PDP11 with floating point processor. The processor time strongly depends on n.

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