projekt i logik

Maja Hanne Tønnesen, Rune Kildetoft Andresen & Niels Peter Mey

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Indhold

Resumé

1 Introduction

In this paper we will prove the propositions regarding S starting with proposition Prop 3.2i. continuing to 3.4, bevise 3.5, aksiomer, der definerer x—y, bevise 3.10, og bevise 3.11.

We chose to prove chapter 3 from mendelson, starting with Lemma 3.2, since the first ten wfs' are proved in Mendelson, we will start with Lemma 3.2.i.

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A modified version of Mendelsons system S (Peano arithmetic) [Men97] may be formulated thus:

[**Theory** S]

[S **rule** MP: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A}\Rightarrow\mathcal{B}\vdash\mathcal{A}\vdash\mathcal{B}$]

[S **rule** Gen: $\Pi\mathcal{X},\mathcal{A}\colon \mathcal{A}\vdash\forall\mathcal{X}\colon\mathcal{A}$]

[S **rule** Ded: $\Pi\mathcal{A},\mathcal{B}\colon \mathrm{Ded}(\mathcal{A},\mathcal{B})\Vdash\mathcal{A}\vdash\mathcal{B}$]

[S **rule** S2: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A}=\mathcal{B}\vdash\mathcal{A}'=\mathcal{B}'$]

[S **rule** S3: $\Pi\mathcal{A}\colon \neg 0=\mathcal{A}'$]

[S **rule** S4: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A}'=\mathcal{B}'\vdash\mathcal{A}=\mathcal{B}$]

[S **rule** S5: $\Pi\mathcal{A}\colon \mathcal{A}+0=\mathcal{A}$]

[S **rule** S6: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A}+\mathcal{B}'=(\mathcal{A}+\mathcal{B})'$]

[S **rule** S7: $\Pi\mathcal{A}\colon \mathcal{A}\cdot 0=0$]

[S **rule** S8: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A}\cdot(\mathcal{B}')=(\mathcal{A}\cdot\mathcal{B})+\mathcal{A}$]

[S **rule** Neg: $\Pi\mathcal{A}\colon\Pi\mathcal{B}\colon \neg\mathcal{B}\Rightarrow\neg\mathcal{A}\vdash\neg\mathcal{B}\Rightarrow\mathcal{A}\vdash\mathcal{B}$]

[S **rule** S1: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{A} = \mathcal{B} \vdash \mathcal{A} = \mathcal{C} \vdash \mathcal{B} = \mathcal{C}$]

[S **rule** S9: $\Pi\mathcal{X},\mathcal{A},\mathcal{B},\mathcal{C}\colon \langle\mathcal{B}{\equiv}\mathcal{A}|\mathcal{X}{:=}0\rangle \Vdash \langle\mathcal{C}{\equiv}\mathcal{A}|\mathcal{X}{:=}\mathcal{X}'\rangle \Vdash \mathcal{B} \vdash \mathcal{A} \Rightarrow \mathcal{C} \vdash \mathcal{A}$]

[S **lemma** Prop 3.2a: $\Pi\mathcal{A}\colon \mathcal{A} = \mathcal{A}$]

[S **lemma** Prop 3.2b: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A} = \mathcal{B} \vdash \mathcal{B} = \mathcal{A}$]

[S **lemma** Prop 3.2c: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{A} = \mathcal{B} \vdash \mathcal{B} = \mathcal{C} \vdash \mathcal{A} = \mathcal{C}$]

[S **lemma** Prop 3.2d: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{A} = \mathcal{C} \vdash \mathcal{B} = \mathcal{C} \vdash \mathcal{A} = \mathcal{B}$]

[S **lemma** Prop 3.2e: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{A} = \mathcal{B} \vdash \mathcal{A} + \mathcal{C} = \mathcal{B} + \mathcal{C}$]

[S **lemma** Prop 3.2f: $\Pi\mathcal{A}\colon \mathcal{A} = 0 + \mathcal{A}$]

[S **lemma** Prop 3.2g: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A}' + \mathcal{B} = (\mathcal{A} + \mathcal{B})'$]

[S **lemma** Prop 3.2h: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A} + \mathcal{B} = \mathcal{B} + \mathcal{A}$]

[S **lemma** Prop 3.2i: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{A} = \mathcal{B} \vdash \mathcal{C} + \mathcal{A} = \mathcal{C} + \mathcal{B}$]

[S **lemma** Prop 3.2j: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon (\mathcal{A} + \mathcal{B}) + \mathcal{C} = \mathcal{A} + (\mathcal{B} + \mathcal{C})$]

[S **lemma** Prop 3.2k: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{A} = \mathcal{B} \vdash \mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{C}$]

[S **lemma** Prop 3.2l: $\Pi\mathcal{A}\colon 0 \cdot \mathcal{A} = 0$]

[S **lemma** Prop 3.2m: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A}' \cdot \mathcal{B} = \mathcal{A} \cdot \mathcal{B} + \mathcal{B}$]

[S **lemma** Prop 3.2n: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A} \cdot \mathcal{B} = \mathcal{B} \cdot \mathcal{A}$]

[S **lemma** Prop 3.2o: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{A} = \mathcal{B} \vdash \mathcal{C} \cdot \mathcal{A} = \mathcal{C} \cdot \mathcal{B}$]

we will start by doing Prop 3.2i

[S **lemma** Prop 3.2i: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{A} = \mathcal{B} \vdash \mathcal{C} + \mathcal{A} = \mathcal{C} + \mathcal{B}$]

S **proof of** Prop 3.2i:

L01:	Arbitrary $\gg$	$\mathcal{A},\mathcal{B},\mathcal{C}$	;
L02:	Block $\gg$	Begin	;
L03:	Arbitrary $\gg$	$\mathcal{A},\mathcal{B},\mathcal{C}$	;
L04:	Prop 3.2e $\gg$	$\mathcal{A} = \mathcal{B} \Rightarrow \mathcal{A} + \mathcal{C} = \mathcal{B} + \mathcal{C}$	;
L05:	Prop 3.2h $\gg$	$\mathcal{A} + \mathcal{C} = \mathcal{C} + \mathcal{A}$	;
L06:	Prop 3.2h $\gg$	$\mathcal{B} + \mathcal{C} = \mathcal{C} + \mathcal{B}$	;
L07:	Premise $\gg$	$\mathcal{A} = \mathcal{B}$	;
L08:	MP $\triangleright$ L04 $\triangleright$ L07 $\gg$	$\mathcal{A} + \mathcal{C} = \mathcal{B} + \mathcal{C}$	;
L09:	S1 $\gg$	$\mathcal{A} + \mathcal{C} = \mathcal{B} + \mathcal{C} \Rightarrow (\mathcal{A} + \mathcal{C} = \mathcal{C} + \mathcal{A} \Rightarrow \mathcal{B} + \mathcal{C} = \mathcal{C} + \mathcal{A})$	;
L10:	MP $\triangleright$ L09 $\triangleright$ L08 $\gg$	$\mathcal{A} + \mathcal{C} = \mathcal{C} + \mathcal{A} \Rightarrow \mathcal{B} + \mathcal{C} = \mathcal{C} + \mathcal{A}$	;
L08:	MP $\triangleright$ L10 $\triangleright$ L05 $\gg$	$\mathcal{B} + \mathcal{C} = \mathcal{C} + \mathcal{A}$	;
L11:	Prop 3.2b $\triangleright$ L08 $\gg$	$\mathcal{C} + \mathcal{A} = \mathcal{B} + \mathcal{C}$	;

L12:	Prop 3.2e $\gg$	$\mathcal{C}+\mathcal{A}=\mathcal{B}+\mathcal{C} \Rightarrow (\mathcal{B}+\mathcal{C}=\mathcal{C}+\mathcal{B} \Rightarrow \mathcal{C}+\mathcal{A}=\mathcal{C}+\mathcal{B})$	;
L13:	MP $\rhd$ L12 $\rhd$ L11 $\gg$	$\mathcal{B}+\mathcal{C}=\mathcal{C}+\mathcal{B} \Rightarrow \mathcal{C}+\mathcal{A}=\mathcal{C}+\mathcal{B}$	;
L14:	MP $\rhd$ L13 $\rhd$ L06 $\gg$	$\mathcal{C}+\mathcal{A}=\mathcal{C}+\mathcal{B}$	;
L15:	Block $\gg$	End	;
L16:	Ded $\rhd$ L15 $\gg$	$\mathcal{A}=\mathcal{B} \Rightarrow \mathcal{C}+\mathcal{A}=\mathcal{C}+\mathcal{B}$	$\square$

[S **lemma** Prop 3.2j$_1$: $\Pi\mathcal{A},\mathcal{B}\colon (\mathcal{A}+\mathcal{B})+0=\mathcal{A}+(\mathcal{B}+0)$]

S **proof of** Prop 3.2j$_1$:

L01:	Arbitrary $\gg$	$\mathcal{A},\mathcal{B}$	;
L02:	S5 $\gg$	$(\mathcal{A}+\mathcal{B})+0=\mathcal{A}+\mathcal{B}$	;
L03:	S5 $\gg$	$\mathcal{B}+0=\mathcal{B}$	;
L04:	Prop 3.2i $\rhd$ L03 $\gg$	$\mathcal{A}+(\mathcal{B}+0)=\mathcal{A}+\mathcal{B}$	;
L05:	Prop 3.2d $\rhd$ L02 $\rhd$ L04 $\gg$	$(\mathcal{A}+\mathcal{B})+0=\mathcal{A}+(\mathcal{B}+0)$	$\square$

[S **lemma** Prop 3.2j$_2$: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon (\mathcal{A}+\mathcal{B})+\mathcal{C}=\mathcal{A}+(\mathcal{B}+\mathcal{C}) \Rightarrow (\mathcal{A}+\mathcal{B})+\mathcal{C}'=\mathcal{A}+(\mathcal{B}+\mathcal{C}')$]

S **proof of** Prop 3.2j$_2$:

L01:	Arbitrary $\gg$	$\mathcal{A},\mathcal{B},\mathcal{C}$	;
L02:	Block $\gg$	Begin	;
L03:	Arbitrary $\gg$	$\mathcal{A},\mathcal{B},\mathcal{C}$	;
L04:	Premise $\gg$	$(\mathcal{A}+\mathcal{B})+\mathcal{C}=\mathcal{A}+(\mathcal{B}+\mathcal{C})$	;
L05:	S6 $\gg$	$(\mathcal{A}+\mathcal{B})+\mathcal{C}'=\mathcal{A}+(\mathcal{B}+\mathcal{C})'$	;
L06:	S2 $\rhd$ L04 $\gg$	$((\mathcal{A}+\mathcal{B})+\mathcal{C})'=(\mathcal{A}+(\mathcal{B}+\mathcal{C}))'$	;
L07:	Prop 3.2c $\rhd$ L05 $\rhd$ L06 $\gg$	$(\mathcal{A}+\mathcal{B})+\mathcal{C}'=(\mathcal{A}+(\mathcal{B}+\mathcal{C}))'$	;
L08:	S6 $\gg$	$\mathcal{B}+\mathcal{C}'=(\mathcal{B}+\mathcal{C})'$	;
L09:	Prop 3.2i $\rhd$ L08 $\gg$	$\mathcal{A}+(\mathcal{B}+\mathcal{C}')=\mathcal{A}+(\mathcal{B}+\mathcal{C})'$	;
L10:	S6 $\gg$	$\mathcal{A}+(\mathcal{B}+\mathcal{C})'=(\mathcal{A}+(\mathcal{B}+\mathcal{C}))'$	;
L11:	Prop 3.2c $\rhd$ L09 $\rhd$ L10 $\gg$	$\mathcal{A}+(\mathcal{B}+\mathcal{C}')=(\mathcal{A}+(\mathcal{B}+\mathcal{C}))'$	;
L12:	Prop 3.2d $\rhd$ L07 $\rhd$ L11 $\gg$	$(\mathcal{A}+\mathcal{B})+\mathcal{C}'=\mathcal{A}+(\mathcal{B}+\mathcal{C}')$	;
L13:	Block $\gg$	End	;
L14:	Ded $\rhd$ L13 $\gg$	$(\mathcal{A}+\mathcal{B})+\mathcal{C}=\mathcal{A}+(\mathcal{B}+\mathcal{C}) \Rightarrow (\mathcal{A}+\mathcal{B})+\mathcal{C}'=\mathcal{A}+(\mathcal{B}+\mathcal{C}')$	$\square$

S **proof of** Prop 3.2j:

L01:	Arbitrary $\gg$	$\mathcal{A},\mathcal{B},\mathcal{C}$	;
L02:	Block $\gg$	Begin	;
L03:	Prop 3.2j$_1$ $\gg$	$(x+y)+0=x+(y+0)$	;
L04:	Prop 3.2j$_2$ $\gg$	$(x+y)+z=x+(y+z) \Rightarrow (x+y)+z'=x+(y+z')$	;
L05:	S9 @ z $\rhd$ L03 $\rhd$ L04 $\gg$	$(x+y)+z=x+(y+z)$	;
L06:	Block $\gg$	End	;
L07:	Ded $\rhd$ L06 $\gg$	$(\mathcal{A}+\mathcal{B})+\mathcal{C}=\mathcal{A}+(\mathcal{B}+\mathcal{C})$	$\square$

[S **lemma** Prop 3.2k$_1$: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A}=\mathcal{B} \vdash \mathcal{A}\cdot 0=\mathcal{B}\cdot 0$]

S **proof of** Prop $3.2\mathrm{k}_1$:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}$	;
L02:	S7 $\gg$	$\mathcal{A} \cdot 0 = 0$	;
L03:	S7 $\gg$	$\mathcal{B} \cdot 0 = 0$	;
L04:	Prop 3.2a $\gg$	$0 = 0$	;
L05:	Prop 3.2b $\rhd$ L03 $\gg$	$0 = \mathcal{B} \cdot 0$	;
L06:	Prop 3.2c $\gg$	$\mathcal{A} \cdot 0 = 0 \Rightarrow 0 = \mathcal{B} \cdot 0 \Rightarrow \mathcal{A} \cdot 0 = \mathcal{B} \cdot 0$	;
L07:	L06 $\rhd$ L02 $\rhd$ L05 $\gg$	$\mathcal{A} \cdot 0 = \mathcal{B} \cdot 0$	$\square$

[S **lemma** Prop $3.2\mathrm{k}_2$: $\Pi\mathcal{A}, \mathcal{B}, \mathcal{C}$: $\mathcal{A} = \mathcal{B} \Rightarrow \mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{C} \vdash \mathcal{A} = \mathcal{B} \Rightarrow \mathcal{A} \cdot \mathcal{C}' = \mathcal{B} \cdot \mathcal{C}'$]

S **proof of** Prop $3.2\mathrm{k}_2$:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}, \mathcal{C}$	;
L02:	Block $\gg$	Begin	;
L03:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}, \mathcal{C}$	;
L04:	Premise $\gg$	$\mathcal{A} = \mathcal{B} \Rightarrow \mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{C}$	;
L05:	Premise $\gg$	$\mathcal{A} = \mathcal{B}$	;
L06:	L04 $\rhd$ L05 $\gg$	$\mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{C}$	;
L07:	S8 $\gg$	$\mathcal{A} \cdot \mathcal{C}' = \mathcal{A} \cdot \mathcal{C} + \mathcal{A}$	;
L08:	S8 $\gg$	$\mathcal{B} \cdot \mathcal{C}' = \mathcal{B} \cdot \mathcal{C} + \mathcal{B}$	;
L09:	Prop 3.2e $\gg$	$\mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{C} \Rightarrow \mathcal{A} \cdot \mathcal{C} + \mathcal{A} = \mathcal{B} \cdot \mathcal{C} + \mathcal{A}$	;
L10:	L09 $\rhd$ L06 $\gg$	$\mathcal{A} \cdot \mathcal{C} + \mathcal{A} = \mathcal{B} \cdot \mathcal{C} + \mathcal{A}$	;
L11:	Prop 3.2i $\gg$	$\mathcal{A} = \mathcal{B} \Rightarrow \mathcal{B} \cdot \mathcal{C} + \mathcal{A} = \mathcal{B} \cdot \mathcal{C} + \mathcal{B}$	;
L12:	L11 $\rhd$ L06 $\gg$	$\mathcal{B} \cdot \mathcal{C} + \mathcal{A} = \mathcal{B} \cdot \mathcal{C} + \mathcal{B}$	;
L13:	Prop 3.2c $\gg$	$\mathcal{A} \cdot \mathcal{C} + \mathcal{A} = \mathcal{B} \cdot \mathcal{C} + \mathcal{A} \Rightarrow \mathcal{B} \cdot \mathcal{C} + \mathcal{A} = \mathcal{B} \cdot \mathcal{C} + \mathcal{B} \Rightarrow \mathcal{A} \cdot \mathcal{C} + \mathcal{A} = \mathcal{B} \cdot \mathcal{C} + \mathcal{B}$	;
L14:	L13 $\rhd$ L10 $\rhd$ L12 $\gg$	$\mathcal{A} \cdot \mathcal{C} + \mathcal{A} = \mathcal{B} \cdot \mathcal{C} + \mathcal{B}$	;
L15:	L14 $\rhd$ L07 $\rhd$ L08 $\gg$	$\mathcal{A} \cdot \mathcal{C}' = \mathcal{B} \cdot \mathcal{C}'$	;
L12:	Block $\gg$	End	;
L16:	Ded $\rhd$ L12 $\gg$	$\mathcal{A} = \mathcal{B} \Rightarrow \mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{C} \vdash \mathcal{A} = \mathcal{B} \Rightarrow \mathcal{A} \cdot \mathcal{C}' = \mathcal{B} \cdot \mathcal{C}'$	$\square$

S **proof of** Prop 3.2k:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}, \mathcal{C}$	;
L02:	Block $\gg$	Begin	;
L03:	Prop $3.2\mathrm{k}_1$ $\gg$	$x = y \Rightarrow x \cdot 0 = y \cdot 0$	;
L04:	Prop $3.2\mathrm{k}_2$ $\gg$	$x = y \Rightarrow x \cdot z = y \cdot z \Rightarrow x = y \Rightarrow x \cdot z' = y \cdot z'$	;
L05:	S9 @ z $\rhd$ L03 $\rhd$ L04 $\gg$	$x = y \Rightarrow x \cdot z = y \cdot z$	;
L06:	Block $\gg$	End	;
L07:	Ded $\rhd$ L06 $\gg$	$\mathcal{A} = \mathcal{B} \vdash \mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{C}$	$\square$

[S **lemma** Prop $3.2\mathrm{l}_1$: $\Pi\mathcal{A}$: $0 \cdot 0 = 0$]

S **proof of** Prop $3.2\mathrm{l}_1$:

L01:	S7 $\gg$	$0 \cdot 0 = 0$	$\square$

[S **lemma** Prop 3.2l$_2$: $\Pi\mathcal{A}$: $0 \cdot \mathcal{A} = 0 \Rightarrow 0 \cdot \mathcal{A}' = 0$]

S **proof of** Prop 3.2l$_2$:

L01:	Arbitrary $\gg$	$\mathcal{A}$	;
L02:	Block $\gg$	Begin	;
L03:	Arbitrary $\gg$	$\mathcal{A}$	;
L04:	Premise $\gg$	$0 \cdot \mathcal{A} = 0$	;
L05:	S8 $\gg$	$0 \cdot \mathcal{A}' = 0 \cdot \mathcal{A} + 0$	;
L06:	S5 $\gg$	$0 \cdot \mathcal{A} + 0 = 0 \cdot \mathcal{A}$	;
L07:	Prop 3.2c $\rhd$ L06 $\rhd$ L05 $\gg$	$0 \cdot \mathcal{A}' = 0 \cdot \mathcal{A}$	;
L08:	Prop 3.2c $\rhd$ L07 $\rhd$ L04 $\gg$	$0 \cdot \mathcal{A}' = 0$	;
L09:	Block $\gg$	End	;
L10:	Ded $\rhd$ L09 $\gg$	$0 \cdot \mathcal{A} = 0 \Rightarrow 0 \cdot \mathcal{A}' = 0$	$\Box$

S **proof of** Prop 3.2l:

L01:	Arbitrary $\gg$	$\mathcal{A}$	;
L02:	Block $\gg$	Begin	;
L03:	Prop 3.2l$_1$ $\gg$	$0 \cdot 0 = 0$	;
L04:	Prop 3.2l$_2$ $\gg$	$0 \cdot x = 0 \Rightarrow 0 \cdot x' = 0$	;
L05:	S9 @ x $\rhd$ L03 $\rhd$ L04 $\gg$	$0 \cdot x = 0$	;
L06:	Block $\gg$	End	;
L07:	Ded $\rhd$ L06 $\gg$	$0 \cdot \mathcal{A} = 0$	$\Box$

[S **lemma** Prop 3.2m$_1$: $\Pi\mathcal{A}$: $\mathcal{A}' \cdot 0 = \mathcal{T} \cdot 0 + 0$]

S **proof of** Prop 3.2m$_1$:

L01:	Arbitrary $\gg$	$\mathcal{A}$	;
L02:	S7 $\gg$	$\mathcal{A}' \cdot 0 = 0$	;
L03:	Prop 3.2f $\gg$	$0 = 0 + 0$	;
L04:	S7 $\gg$	$0 = \mathcal{A} \cdot 0$	;
L05:	Prop 3.2e $\gg$	$0 = \mathcal{A} \cdot 0 \Rightarrow 0 + 0 = \mathcal{A} \cdot 0 + 0$	;
L06:	L05 $\rhd$ L04 $\gg$	$0 + 0 = \mathcal{A} \cdot 0 + 0$	;
L07:	Prop 3.2c $\gg$	$0 = 0 + 0 \Rightarrow 0 + 0 = \mathcal{A} \cdot 0 + 0 \Rightarrow 0 = \mathcal{A} \cdot 0 + 0$	;
L08:	L07 $\rhd$ L03 $\rhd$ L06 $\gg$	$0 = \mathcal{A} \cdot 0 + 0$	;
L09:	Prop 3.2c $\gg$	$\mathcal{A}' \cdot 0 = 0 \Rightarrow 0 = \mathcal{A} \cdot 0 + 0 \Rightarrow \mathcal{A}' \cdot 0 = \mathcal{A} \cdot 0 + 0$	;
L10:	L09 $\rhd$ L02 $\rhd$ L09 $\gg$	$\mathcal{A}' \cdot 0 = \mathcal{A} \cdot 0 + 0$	$\Box$

[S **lemma** Prop 3.2m$_2$: $\Pi\mathcal{A}, \mathcal{B}$: $\mathcal{A}' \cdot \mathcal{B} = \mathcal{A} \cdot \mathcal{B} + \mathcal{B} \Rightarrow \mathcal{A}' \cdot \mathcal{B}' = \mathcal{A} \cdot \mathcal{B}' + \mathcal{B}'$]

S **proof of** Prop 3.2m$_2$:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}$	;
L02:	Block $\gg$	Begin	;
L03:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}$	;
L04:	Premise $\gg$	$\mathcal{A}' \cdot \mathcal{B} = \mathcal{A} \cdot \mathcal{B} + \mathcal{B}$	;
L05:	S8 $\gg$	$\mathcal{A}' \cdot \mathcal{B}' = \mathcal{A}' \cdot \mathcal{B}' + \mathcal{A}'$	;
L06:	Prop 3.2e $\gg$	$\mathcal{A}' \cdot \mathcal{B} = \mathcal{A} \cdot \mathcal{B} + \mathcal{B} \Rightarrow \mathcal{A}' \cdot \mathcal{B} + \mathcal{A}' = \mathcal{A} \cdot \mathcal{B} + \mathcal{B} + \mathcal{A}'$	;

L07:	L06 $\rhd$ L04 $\gg$	$\mathcal{A}' \cdot \mathcal{B} + \mathcal{A}' = \mathcal{A} \cdot \mathcal{B} + \mathcal{B} + \mathcal{A}'$	;
L08:	S6 $\gg$	$\mathcal{B} + \mathcal{A}' = (\mathcal{B} + \mathcal{A})'$	;
L09:	Prop 3.2g $\gg$	$\mathcal{B}' + \mathcal{A} = (\mathcal{B} + \mathcal{A})'$	;
L10:	Prop 3.2d $\rhd$ L08 $\rhd$ L09 $\gg$	$\mathcal{B} + \mathcal{A}' = \mathcal{B}' + \mathcal{A}$	;
L11:	Prop 3.2h $\gg$	$\mathcal{B}' + \mathcal{A} = \mathcal{A} + \mathcal{B}'$	;
L12:	Prop 3.2c $\rhd$ L10 $\rhd$ L11 $\gg$	$\mathcal{B} + \mathcal{A}' = \mathcal{A} + \mathcal{B}'$	;
L13:	Prop 3.2i $\rhd$ L12 $\gg$	$\mathcal{A} \cdot \mathcal{B} + \mathcal{B} + \mathcal{A}' = \mathcal{A} \cdot \mathcal{B} + \mathcal{A} + \mathcal{B}'$	;
L14:	S8 $\gg$	$\mathcal{A} \cdot \mathcal{B}' = \mathcal{A} \cdot \mathcal{B} + \mathcal{A}$	;
L15:	Prop 3.2e $\rhd$ L14 $\gg$	$\mathcal{A} \cdot \mathcal{B}' + \mathcal{B}' = \mathcal{A} \cdot \mathcal{B} + \mathcal{A} + \mathcal{B}'$	;
L16:	Prop 3.2d $\rhd$ L15 $\rhd$ L13 $\gg$	$\mathcal{A} \cdot \mathcal{B} + \mathcal{B} + \mathcal{A}' = \mathcal{A} \cdot \mathcal{B}' + \mathcal{B}'$	;
L17:	Prop 3.2c $\rhd$ L07 $\rhd$ L16 $\gg$	$\mathcal{A}' \cdot \mathcal{B} + \mathcal{A}' = \mathcal{A} \cdot \mathcal{B}' + \mathcal{B}'$	;
L18:	Prop 3.2c $\rhd$ L05 $\rhd$ L17 $\gg$	$\mathcal{A}' \cdot \mathcal{B}' = \mathcal{A} \cdot \mathcal{B}' + \mathcal{B}'$	;
L19:	Block $\gg$	End	;
L20:	Ded $\rhd$ L19 $\gg$	$\mathcal{A}' \cdot \mathcal{B} = \mathcal{A} \cdot \mathcal{B} + \mathcal{B} \Rightarrow \mathcal{A}' \cdot \mathcal{B}' = \mathcal{A} \cdot \mathcal{B}' + \mathcal{B}'$	□

S **proof of** Prop 3.2m:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}$	;
L02:	Block $\gg$	Begin	;
L03:	Prop $3.2\mathrm{m}_1$ $\gg$	$x' \cdot 0 = x \cdot 0 + 0$	;
L04:	Prop $3.2\mathrm{m}_2$ $\gg$	$x' \cdot y = x \cdot y + y \Rightarrow x' \cdot y' = x \cdot y' + y$	;
L05:	S9 @ y $\rhd$ L03 $\rhd$ L04 $\gg$	$x' \cdot y = x \cdot y + y$	;
L06:	Block $\gg$	End	;
L07:	Ded $\rhd$ L06 $\gg$	$\mathcal{A}' \cdot \mathcal{B} = \mathcal{A} \cdot \mathcal{B} + \mathcal{B}$	□

[S **lemma** Prop $3.2\mathrm{n}_1$: $\Pi\mathcal{A}$: $\mathcal{A} \cdot 0 = 0 \cdot \mathcal{A}$]

S **proof of** Prop $3.2\mathrm{n}_1$:

L01:	Arbitrary $\gg$	$\mathcal{A}$	;
L02:	S7 $\gg$	$\mathcal{A} \cdot 0 = 0$	;
L03:	Prop 3.2l $\gg$	$0 \cdot \mathcal{A} = 0$	;
L04:	Prop 3.2c $\rhd$ L02 $\rhd$ L03 $\gg$	$\mathcal{A} \cdot 0 = 0 \cdot \mathcal{A}$	□

[S **lemma** Prop $3.2\mathrm{n}_2$: $\Pi\mathcal{A}, \mathcal{B}$: $\mathcal{A} \cdot \mathcal{B} = \mathcal{B} \cdot \mathcal{A} \Rightarrow \mathcal{A} \cdot \mathcal{B}' = \mathcal{B}' \cdot \mathcal{A}$]

S **proof of** Prop $3.2\mathrm{n}_2$:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}$	;
L02:	Block $\gg$	Begin	;
L03:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}$	;
L04:	Premise $\gg$	$\mathcal{A} \cdot \mathcal{B} = \mathcal{B} \cdot \mathcal{A}$	;
L05:	S8 $\gg$	$\mathcal{A} \cdot \mathcal{B}' = \mathcal{A} \cdot \mathcal{B} + \mathcal{A}$	;
L06:	Prop 3.2e $\rhd$ L04 $\gg$	$\mathcal{A} \cdot \mathcal{B} + \mathcal{A} = \mathcal{B} \cdot \mathcal{A} + \mathcal{A}$	;
L07:	Prop 3.2b $\rhd$ Prop 3.2m $\gg$	$\mathcal{B} \cdot \mathcal{A} + \mathcal{A} = \mathcal{B}' \cdot \mathcal{A}$	;
L08:	Prop 3.2c $\rhd$ L08 $\rhd$ L07 $\gg$	$\mathcal{A} \cdot \mathcal{B} + \mathcal{A} = \mathcal{B}' \cdot \mathcal{A}$	;
L09:	Prop 3.2c $\rhd$ L05 $\rhd$ L08 $\gg$	$\mathcal{A} \cdot \mathcal{B}' = \mathcal{B}' \cdot \mathcal{A}$	;
L10:	Block $\gg$	End	;
L11:	Ded $\rhd$ L10 $\gg$	$\mathcal{A} \cdot \mathcal{B} = \mathcal{B} \cdot \mathcal{A} \Rightarrow \mathcal{A} \cdot \mathcal{B}' = \mathcal{B}' \cdot \mathcal{S}$	□

S **proof of** Prop 3.2n:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}$	;
L02:	Block $\gg$	Begin	;
L03:	Prop 3.2n$_1$ $\gg$	$x \cdot 0 = 0 \cdot x$	;
L04:	Prop 3.2n$_2$ $\gg$	$x \cdot y = y \cdot x \Rightarrow x \cdot y' = y' \cdot x$	;
L05:	S9 @ $y \rhd$ L03 $\rhd$ L04 $\gg$	$x \cdot y = y = x$	;
L06:	Block $\gg$	End	;
L07:	Ded $\rhd$ L06 $\gg$	$\mathcal{A} \cdot \mathcal{B} = \mathcal{B} \cdot \mathcal{A}$	□

S **proof of** Prop 3.2o:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}, \mathcal{C}$	;
L02:	Block $\gg$	Begin	;
L03:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}, \mathcal{C}$	;
L04:	Premise $\gg$	$\mathcal{A} = \mathcal{B}$	;
L05:	Prop 3.2k $\gg$	$\mathcal{A} = \mathcal{B} \Rightarrow \mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{C}$	;
L06:	L05 $\rhd$ L04 $\gg$	$\mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{C}$	;
L07:	Prop 3.2n $\gg$	$\mathcal{A} \cdot \mathcal{C} = \mathcal{C} \cdot \mathcal{A}$	;
L08:	Prop 3.2n $\gg$	$\mathcal{B} \cdot \mathcal{C} = \mathcal{C} \cdot \mathcal{B}$	;
L09:	Prop 3.2c $\gg$	$\mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{C} \Rightarrow \mathcal{B} \cdot \mathcal{C} = \mathcal{C} \cdot \mathcal{B} \Rightarrow \mathcal{A} \cdot \mathcal{C} = \mathcal{C} \cdot \mathcal{B}$	;
L10:	L09 $\rhd$ L06 $\rhd$ L08 $\gg$	$\mathcal{A} \cdot \mathcal{C} = \mathcal{C} \cdot \mathcal{B}$	;
L11:	S1 $\gg$	$\mathcal{A} \cdot \mathcal{C} = \mathcal{C} \cdot \mathcal{B} \Rightarrow \mathcal{A} \cdot \mathcal{C} = \mathcal{C} \cdot \mathcal{A} \Rightarrow \mathcal{C} \cdot \mathcal{A} = \mathcal{C} \cdot \mathcal{B}$	;
L12:	L11 $\rhd$ L10 $\rhd$ L07 $\gg$	$\mathcal{C} \cdot \mathcal{A} = \mathcal{C} \cdot \mathcal{B}$	;
L13:	Block $\gg$	End	;
L14:	Ded $\rhd$ L13 $\gg$	$\mathcal{A} = \mathcal{B} \Rightarrow \mathcal{C} \cdot \mathcal{A} = \mathcal{C} \cdot \mathcal{B}$	□

[S **lemma** Prop 3.4a: $\Pi \mathcal{A}, \mathcal{B}, \mathcal{C} \colon \mathcal{A} \cdot (\mathcal{B} + \mathcal{C}) = (\mathcal{A} \cdot \mathcal{B}) + (\mathcal{A} \cdot \mathcal{C})$]

[S **lemma** Prop 3.4b: $\Pi \mathcal{A}, \mathcal{B}, \mathcal{C} \colon (\mathcal{B} + \mathcal{C}) \cdot \mathcal{A} = (\mathcal{B} \cdot \mathcal{A}) + (\mathcal{C} \cdot \mathcal{B})$]

[S **lemma** Prop 3.4c: $\Pi \mathcal{A}, \mathcal{B}, \mathcal{C} \colon (\mathcal{A} \cdot \mathcal{B}) \cdot \mathcal{C} = \mathcal{A} \cdot (\mathcal{B} \cdot \mathcal{C})$]

[S **lemma** Prop 3.4d: $\Pi \mathcal{A}, \mathcal{B}, \mathcal{C} \colon \mathcal{A} + \mathcal{C} = \mathcal{B} + \mathcal{C} \Rightarrow \mathcal{A} = \mathcal{B}$]

Litteratur

[Men97] E. Mendelson. *Introduction to Mathematical Logic*. Chapmann & Hall, 4. edition, 1997.