Udvidelse af S-reglerne

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Indhold

1 Introduktion

Dette skriftlige projekt er lavet i forbindelse med Logik-kurset 2006[1].
Vores mål i denne opgave var at bevise en delmængde af udsagnene[2] vedrørende læresætningerne[3] fra S-systemet, som er beskrevet i Mendelson [Men97] i kapitel 3.1. For lethedens skyld er alle nummereringer de samme som i Mendelson.
Mere præcist går opgaven ud på at bevise 3.2(j-o), bevise 3.4, bevise 3.5, tilføje aksiomet $x < y \Leftrightarrow \exists z : z \neq 0 \wedge z + x = y$, bevise 3.7, tilføje aksiomer, der definere $\neg\forall_{obj}\mathsf{z}\colon \neg\bar{\mathsf{y}} = \bar{\mathsf{x}} \cdot \mathsf{z}$, bevise 3.10 og bevise 3.11.
I afsnittene 3 til 7 er beviserne for de påviste lemmaer gennemgået. Alle trivielle hjælpelemmaer er bevist i appendix A.

2 Konklusion

Vi har ikke løst opgaven til fulde, men har dog formået at påvise 3.2j - 3.2o, 3.4, 3.5a -g. På grund af problemer med definitionen af reglen svarende til "existensial rule" side 77 i mendelson, har vi som beskrevet i afsnit 6 ikke været i stand til at kunne bevise lemmaerne fra 3.5h og frem.
Vi har dog som beskrevet i afsnit 6.10 og 6.2.2 gennemgået hvordan beviserne for 3.5h og "existensial rule" ville have set ud, hvis problemet ikke var opstået. Vi har endvidere kort gennemgået de definitioner, som ville være nødvendige for at kunne påvise Lemma 3.7

3 S-reglerne

S-systemet er en første ordens teori, som er udviklet ud fra Peanos postulater og ved hjælp mængdelære. Det skulle være passende at bruge til at bevise

[1] Kursus 061004/202 Logik
[2] Eng.: proposition
[3] Eng.: theorem

basis-resultaterne for tal-teori. Aksiomerne for S er følgende.

$[\mathrm{S} \overset{\text{stmt}}{\to} \mathsf{x}]$

$[\mathrm{MP} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \underline{\mathsf{a}} \vdash \underline{\mathsf{b}}][\mathrm{MP} \overset{\text{proof}}{\to} \text{Rule tactic}]$

$[\mathrm{Gen} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{a}} \colon \underline{\mathsf{a}} \vdash \forall_{\mathrm{obj}} \underline{\mathsf{x}} \colon \underline{\mathsf{a}}][\mathrm{Gen} \overset{\text{proof}}{\to} \text{Rule tactic}]$

$[\mathrm{Ded} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \lambda \mathsf{x}.\mathrm{Ded}_0(\lceil \underline{\mathsf{a}} \rceil, \lceil \underline{\mathsf{b}} \rceil) \Vdash \underline{\mathsf{a}} \vdash \underline{\mathsf{b}}][\mathrm{Ded} \overset{\text{proof}}{\to} \text{Rule tactic}]$

$[\mathrm{S2} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} = \underline{\mathsf{b}} \vdash \underline{\mathsf{a}}' = \underline{\mathsf{b}}'][\mathrm{S2} \overset{\text{proof}}{\to} \text{Rule tactic}]$

$[\mathrm{S3} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \neg 0 = \underline{\mathsf{a}}'][\mathrm{S3} \overset{\text{proof}}{\to} \text{Rule tactic}]$

$[\mathrm{S4} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}}' = \underline{\mathsf{b}}' \vdash \underline{\mathsf{a}} = \underline{\mathsf{b}}][\mathrm{S4} \overset{\text{proof}}{\to} \text{Rule tactic}]$

$[\mathrm{S5} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \underline{\mathsf{a}} + 0 = \underline{\mathsf{a}}][\mathrm{S5} \overset{\text{proof}}{\to} \text{Rule tactic}]$

$[\mathrm{S6} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} + \underline{\mathsf{b}}' = \underline{\mathsf{a}} + \underline{\mathsf{b}}'][\mathrm{S6} \overset{\text{proof}}{\to} \text{Rule tactic}]$

$[\mathrm{S7} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \underline{\mathsf{a}} \cdot 0 = 0][\mathrm{S7} \overset{\text{proof}}{\to} \text{Rule tactic}]$

$[\mathrm{S8} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \cdot \underline{\mathsf{b}}' = \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} + \underline{\mathsf{a}}][\mathrm{S8} \overset{\text{proof}}{\to} \text{Rule tactic}]$

$[\mathrm{S1} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \forall \underline{\mathsf{c}} \colon \underline{\mathsf{a}} = \underline{\mathsf{b}} \vdash \underline{\mathsf{a}} = \underline{\mathsf{c}} \vdash \underline{\mathsf{b}} = \underline{\mathsf{c}}][\mathrm{S1} \overset{\text{proof}}{\to} \text{Rule tactic}]$

Endvidere er reglen *S9*, som er grundlæggende for matematisk induktion, også en del af *S*-teorien. Denne regel kan på almindeligt sprog udtrykkes som:

> *Hvis en egenskab holder for 0 og egenskaben holder for efterfølgeren* x′ *til et naturligt tal* x*, som egenskaben gælder for, så vil egenskaben gælde for alle naturlige tal.*

I pyk er S9 defineret som:

$[\mathrm{S9} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \forall \underline{\mathsf{c}} \colon \langle \lceil \underline{\mathsf{b}} \rceil \equiv^0 \lceil \underline{\mathsf{a}} \rceil | \lceil \underline{\mathsf{x}} \rceil := \lceil 0 \rceil \rangle \Vdash \langle \lceil \underline{\mathsf{c}} \rceil \equiv^0 \lceil \underline{\mathsf{a}} \rceil | \lceil \underline{\mathsf{x}} \rceil := \lceil \underline{\mathsf{x}}' \rceil \rangle \Vdash \underline{\mathsf{b}} \vdash \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{c}} \vdash \underline{\mathsf{a}}][\mathrm{S9} \overset{\text{proof}}{\to} \text{Rule tactic}]$

Reglen for bevis ved modsætninger er den sidste grundlæggende regel, der skal bruges i beviserne. Reglen kan på almindeligt sprog udtrykkes som følgende:

> *Hvis man ud fra en antagelse* A *kan påvise at en egenskab* B *gælder, og at man (når antagelsen stadig gælder) kan vise at den modsatte egenskab* ¬B *også gælder, så må det modsatte af antagelsen gælde, dvs.* ¬A.

I pyk er reglen defineret som:

$[\mathrm{Neg} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \neg \underline{\mathsf{b}} \Rightarrow \neg \underline{\mathsf{a}} \vdash \neg \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}} \vdash \underline{\mathsf{b}}][\mathrm{Neg} \overset{\text{proof}}{\to} \text{Rule tactic}]$

4 Udsagn 3.2

Dette udsagn indeholder de grundlæggende regneregler for tal, som opfører sig som de naturlige tal. Hjælpesætningerne eller lemmaerne er generelt set en direkte konsekvens af aksiomerne.
Alle lemmaerne i dette afsnit kan udledes ud fra de tidligere nævnte og som det er bevist i både check [Gru06] og Mendelson er der for ethvert udtryk $\underline{a}, \underline{b}, \underline{c}$ følgende velformulerede sætninger[4] i systemet S:

[Prop 3.2a $\overset{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a}\colon \underline{a} = \underline{a}$]

[Prop 3.2b $\overset{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} = \underline{b} \vdash \underline{b} = \underline{a}$]

[Prop 3.2c $\overset{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} = \underline{b} \vdash \underline{b} = \underline{c} \vdash \underline{a} = \underline{c}$]

[Prop 3.2d $\overset{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} = \underline{c} \vdash \underline{b} = \underline{c} \vdash \underline{a} = \underline{b}$]

[Prop 3.2e $\overset{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} = \underline{b} \vdash \underline{a} + \underline{c} = \underline{b} + \underline{c}$]

[Prop 3.2f $\overset{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a}\colon \underline{a} = 0 + \underline{a}$]

[Prop 3.2g $\overset{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a}' + \underline{b} = \underline{a} + \underline{b}'$]

[Prop 3.2h $\overset{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} + \underline{b} = \underline{b} + \underline{a}$]

Da 3.2i ikke er bevist i check, (men er i Mendelson), har vi valgt for øvelsens skyld at indskrive beviset:

[Prop 3.2i $\overset{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} = \underline{b} \vdash \underline{c} + \underline{a} = \underline{c} + \underline{b}$]

[Prop 3.2i $\overset{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} = \underline{b} \vdash$ Prop 3.2e $\rhd\ \underline{a} = \underline{b} \gg \underline{a} + \underline{c} = \underline{b} + \underline{c};$ Prop 3.2h $\gg \underline{a} + \underline{c} = \underline{c} + \underline{a};$ Prop 3.2h $\gg \underline{b} + \underline{c} = \underline{c} + \underline{b};$ S1 $\rhd\ \underline{a} + \underline{c} = \underline{b} + \underline{c} \rhd \underline{a} + \underline{c} = \underline{c} + \underline{a} \gg \underline{b} + \underline{c} = \underline{c} + \underline{a};$ Prop 3.2b $\rhd\ \underline{b} + \underline{c} = \underline{c} + \underline{a} \gg \underline{c} + \underline{a} = \underline{b} + \underline{c};$ Prop 3.2c $\rhd\ \underline{c} + \underline{a} = \underline{b} + \underline{c} \rhd \underline{b} + \underline{c} = \underline{c} + \underline{b} \gg \underline{c} + \underline{a} = \underline{c} + \underline{b};$ Ded $\rhd\ \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} = \underline{b} \vdash \underline{c} + \underline{a} = \underline{c} + \underline{b} \gg \underline{a} = \underline{b} \vdash \underline{c} + \underline{a} = \underline{c} + \underline{b}\rceil, p_0, c)$]

De følgende lemmaer er opskrevet, men ikke bevist i Mendelson. Disse er blandt dem som vi ønsker at bevise.

[Prop 3.2j $\overset{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} + \underline{b} + \underline{c} = \underline{a} + \underline{b} + \underline{c}$]

[Prop 3.2k $\overset{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} = \underline{b} \vdash \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c}$]

[Prop 3.2l $\overset{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a}\colon 0 \cdot \underline{a} = 0$]

[Prop 3.2m $\overset{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a}' \cdot \underline{b} = \underline{a} \cdot \underline{b} + \underline{b}$]

[4] Eng.: Wellformed formulas

$[\text{Prop 3.2n} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}]$

$[\text{Prop 3.2o} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} = \underline{b} \vdash \underline{c} \cdot \underline{a} = \underline{c} \cdot \underline{b}]$

Her efter følger beviserne.

4.1 3.2j

$[\text{Prop 3.2j}_1 \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} + \underline{b} + 0 = \underline{a} + \underline{b} + 0]$

$[\text{Prop 3.2j}_1 \overset{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \mathrm{S5} \gg \underline{a} + \underline{b} + 0 = \underline{a} + \underline{b}; \mathrm{S5} \gg \underline{b} + 0 = \underline{b}; \text{Prop 3.2i} \rhd \underline{b} + 0 = \underline{b} \gg \underline{a} + \underline{b} + 0 = \underline{a} + \underline{b}; \text{Prop 3.2d} \rhd \underline{a} + \underline{b} + 0 = \underline{a} + \underline{b} \rhd \underline{a} + \underline{b} + 0 = \underline{a} + \underline{b} \gg \underline{a} + \underline{b} + 0 = \underline{a} + \underline{b} + 0 \rceil, \mathrm{p}_0, c)]$

$[\text{Prop 3.2j}_2 \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} + \underline{b} + \underline{c} = \underline{a} + \underline{b} + \underline{c} \Rightarrow \underline{a} + \underline{b} + \underline{c}' = \underline{a} + \underline{b} + \underline{c}']$

$[\text{Prop 3.2j}_2 \overset{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} + \underline{b} + \underline{c} = \underline{a} + \underline{b} + \underline{c} \vdash \mathrm{S6} \gg \underline{a} + \underline{b} + \underline{c}' = \underline{a} + \underline{b} + \underline{c}'; \mathrm{S2} \rhd \underline{a} + \underline{b} + \underline{c} = \underline{a} + \underline{b} + \underline{c} \gg \underline{a} + \underline{b} + \underline{c}' = \underline{a} + \underline{b} + \underline{c}'; \text{Prop 3.2c} \rhd \underline{a} + \underline{b} + \underline{c}' = \underline{a} + \underline{b} + \underline{c}' \rhd \underline{a} + \underline{b} + \underline{c}' = \underline{a} + \underline{b} + \underline{c}' \gg \underline{a} + \underline{b} + \underline{c}' = \underline{a} + \underline{b} + \underline{c}'; \mathrm{S6} \gg \underline{b} + \underline{c}' = \underline{b} + \underline{c}'; \text{Prop 3.2i} \rhd \underline{b} + \underline{c}' = \underline{b} + \underline{c}' \gg \underline{a} + \underline{b} + \underline{c}' = \underline{a} + \underline{b} + \underline{c}'; \mathrm{S6} \gg \underline{a} + \underline{b} + \underline{c}' = \underline{a} + \underline{b} + \underline{c}'; \text{Prop 3.2c} \rhd \underline{a} + \underline{b} + \underline{c}' = \underline{a} + \underline{b} + \underline{c}' \rhd \underline{a} + \underline{b} + \underline{c}' = \underline{a} + \underline{b} + \underline{c}' \gg \underline{a} + \underline{b} + \underline{c}' = \underline{a} + \underline{b} + \underline{c}'; \text{Prop 3.2d} \rhd \underline{a} + \underline{b} + \underline{c}' = \underline{a} + \underline{b} + \underline{c}' \rhd \underline{a} + \underline{b} + \underline{c}' = \underline{a} + \underline{b} + \underline{c}' \gg \underline{a} + \underline{b} + \underline{c}' = \underline{a} + \underline{b} + \underline{c}'; \mathrm{Ded} \rhd \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} + \underline{b} + \underline{c} = \underline{a} + \underline{b} + \underline{c} \vdash \underline{a} + \underline{b} + \underline{c}' = \underline{a} + \underline{b} + \underline{c}' \gg \underline{a} + \underline{b} + \underline{c} = \underline{a} + \underline{b} + \underline{c} \Rightarrow \underline{a} + \underline{b} + \underline{c}' = \underline{a} + \underline{b} + \underline{c}' \rceil, \mathrm{p}_0, c)]$

$[\text{Prop 3.2j} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} + \underline{b} + \underline{c} = \underline{a} + \underline{b} + \underline{c}]$

$[\text{Prop 3.2j} \overset{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \text{Prop 3.2j}_1 \gg \bar{x} + \bar{y} + 0 = \bar{x} + \bar{y} + 0; \text{Prop 3.2j}_2 \gg \bar{x} + \bar{y} + \bar{z} = \bar{x} + \bar{y} + \bar{z} \Rightarrow \bar{x} + \bar{y} + \bar{z}' = \bar{x} + \bar{y} + \bar{z}'; \mathrm{S9} \mathbin{@} \bar{z} \rhd \bar{x} + \bar{y} + 0 = \bar{x} + \bar{y} + 0 \rhd \bar{x} + \bar{y} + \bar{z} = \bar{x} + \bar{y} + \bar{z} \Rightarrow \bar{x} + \bar{y} + \bar{z}' = \bar{x} + \bar{y} + \bar{z}' \gg \bar{x} + \bar{y} + \bar{z} = \bar{x} + \bar{y} + \bar{z}; \mathrm{Ded} \rhd \bar{x} + \bar{y} + \bar{z} = \bar{x} + \bar{y} + \bar{z} \gg \underline{a} + \underline{b} + \underline{c} = \underline{a} + \underline{b} + \underline{c} \rceil, \mathrm{p}_0, c)]$

4.2 3.2k

$[\text{Prop 3.2k}_1 \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} = \underline{b} \Rightarrow \underline{a} \cdot 0 = \underline{b} \cdot 0]$

$[\text{Prop 3.2k}_1 \overset{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} = \underline{b} \vdash \mathrm{S7} \gg \underline{a} \cdot 0 = 0; \mathrm{S7} \gg \underline{b} \cdot 0 = 0; \text{Prop 3.2a} \gg 0 = 0; \text{Prop 3.2b} \rhd \underline{b} \cdot 0 = 0 \gg 0 = \underline{b} \cdot 0; \text{Prop 3.2c} \rhd \underline{a} \cdot 0 = 0 \rhd 0 = \underline{b} \cdot 0 \gg \underline{a} \cdot 0 = \underline{b} \cdot 0; \mathrm{Ded} \rhd \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} = \underline{b} \vdash \underline{a} \cdot 0 = \underline{b} \cdot 0 \gg \underline{a} = \underline{b} \Rightarrow \underline{a} \cdot 0 = \underline{b} \cdot 0 \rceil, \mathrm{p}_0, c)]$

$[\text{Prop 3.2k}_2 \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} = \underline{b} \Rightarrow \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c} \Rightarrow \underline{a} = \underline{b} \Rightarrow \underline{a} \cdot \underline{c}' = \underline{b} \cdot \underline{c}']$

$[\text{Prop 3.2k}_2 \overset{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} = \underline{b} \Rightarrow \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c} \vdash \underline{a} = \underline{b} \vdash \mathrm{MP} \rhd \underline{a} = \underline{b} \Rightarrow \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c} \rhd \underline{a} = \underline{b} \gg \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c}; \mathrm{S8} \gg \underline{a} \cdot \underline{c}' = \underline{a} \cdot \underline{c} + \underline{a}; \mathrm{S8} \gg \underline{b} \cdot \underline{c}' = \underline{b} \cdot \underline{c} + \underline{b}; \text{Prop 3.2e} \rhd \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c} \gg \underline{a} \cdot \underline{c} + \underline{a} = \underline{b} \cdot \underline{c} + \underline{a}; \text{Prop 3.2i} \rhd \underline{a} = \underline{b} \gg$

$\underline{b}\cdot\underline{c}+\underline{a} = \underline{b}\cdot\underline{c}+\underline{b}$; Prop 3.2c $\rhd\ \underline{a}\cdot\underline{c}+\underline{a} = \underline{b}\cdot\underline{c}+\underline{a} \rhd \underline{b}\cdot\underline{c}+\underline{a} = \underline{b}\cdot\underline{c}+\underline{b} \gg \underline{a}\cdot\underline{c}+\underline{a} = \underline{b}\cdot\underline{c}+\underline{b}$; Prop 3.2c $\rhd\ \underline{a}\cdot\underline{c}' = \underline{a}\cdot\underline{c}+\underline{a} \rhd \underline{a}\cdot\underline{c}+\underline{a} = \underline{b}\cdot\underline{c}+\underline{b} \gg \underline{a}\cdot\underline{c}' = \underline{b}\cdot\underline{c}+\underline{b}$; Prop 3.2d $\rhd\ \underline{a}\cdot\underline{c}' = \underline{b}\cdot\underline{c}+\underline{b} \rhd \underline{b}\cdot\underline{c}' = \underline{b}\cdot\underline{c}+\underline{b} \gg \underline{a}\cdot\underline{c}' = \underline{b}\cdot\underline{c}'$; Ded $\rhd\ \forall\underline{a}\colon \forall\underline{b}\colon \forall\underline{c}\colon \underline{a} = \underline{b} \Rightarrow \underline{a}\cdot\underline{c} = \underline{b}\cdot\underline{c} \vdash \underline{a} = \underline{b} \vdash \underline{a}\cdot\underline{c}' = \underline{b}\cdot\underline{c}' \gg \underline{a} = \underline{b} \Rightarrow \underline{a}\cdot\underline{c} = \underline{b}\cdot\underline{c} \Rightarrow \underline{a} = \underline{b} \Rightarrow \underline{a}\cdot\underline{c}' = \underline{b}\cdot\underline{c}'\rceil, p_0, c)]$

[Prop 3.2k $\stackrel{\text{stmt}}{\to}$ $S \vdash \forall\underline{a}\colon \forall\underline{b}\colon \forall\underline{c}\colon \underline{a} = \underline{b} \vdash \underline{a}\cdot\underline{c} = \underline{b}\cdot\underline{c}$]

[Prop 3.2k $\stackrel{\text{proof}}{\to}$ $\lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall\underline{a}\colon \forall\underline{b}\colon \forall\underline{c}\colon \underline{a} = \underline{b} \vdash$ Prop $3.2k_1 \gg \bar{x} = \bar{y} \Rightarrow \bar{x}\cdot 0 = \bar{y}\cdot 0$; Prop $3.2k_2 \gg \bar{x} = \bar{y} \Rightarrow \bar{x}\cdot\bar{z} = \bar{y}\cdot\bar{z} \Rightarrow \bar{x} = \bar{y} \Rightarrow \bar{x}\cdot\bar{z}' = \bar{y}\cdot\bar{z}'$; S9 @ $\bar{z} \rhd \bar{x} = \bar{y} \Rightarrow \bar{x}\cdot 0 = \bar{y}\cdot 0 \rhd \bar{x} = \bar{y} \Rightarrow \bar{x}\cdot\bar{z} = \bar{y}\cdot\bar{z} \Rightarrow \bar{x} = \bar{y} \Rightarrow \bar{x}\cdot\bar{z}' = \bar{y}\cdot\bar{z}' \gg \bar{x} = \bar{y} \Rightarrow \bar{x}\cdot\bar{z} = \bar{y}\cdot\bar{z}$; Ded $\rhd\ \bar{x} = \bar{y} \Rightarrow \bar{x}\cdot\bar{z} = \bar{y}\cdot\bar{z} \gg \underline{a} = \underline{b} \Rightarrow \underline{a}\cdot\underline{c} = \underline{b}\cdot\underline{c}$; MP $\rhd\ \underline{a} = \underline{b} \Rightarrow \underline{a}\cdot\underline{c} = \underline{b}\cdot\underline{c} \rhd \underline{a} = \underline{b} \gg \underline{a}\cdot\underline{c} = \underline{b}\cdot\underline{c}\rceil, p_0, c)$]

4.3 3.2l

[Prop $3.2l_2$ $\stackrel{\text{stmt}}{\to}$ $S \vdash \forall\underline{a}\colon 0\cdot\underline{a} = 0 \Rightarrow 0\cdot\underline{a}' = 0$]

[Prop $3.2l_2$ $\stackrel{\text{proof}}{\to}$ $\lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall\underline{a}\colon \forall\underline{a}\colon 0\cdot\underline{a} = 0 \vdash$ S8 $\gg 0\cdot\underline{a}' = 0\cdot\underline{a}+0$; S5 $\gg 0\cdot\underline{a}+0 = 0\cdot\underline{a}$; Prop 3.2c $\rhd\ 0\cdot\underline{a}' = 0\cdot\underline{a}+0 \rhd 0\cdot\underline{a}+0 = 0\cdot\underline{a} \gg 0\cdot\underline{a}' = 0\cdot\underline{a}$; Prop 3.2c $\rhd\ 0\cdot\underline{a}' = 0\cdot\underline{a} \rhd 0\cdot\underline{a} = 0 \gg 0\cdot\underline{a}' = 0$; Ded $\rhd\ \forall\underline{a}\colon 0\cdot\underline{a} = 0 \vdash 0\cdot\underline{a}' = 0 \gg 0\cdot\underline{a} = 0 \Rightarrow 0\cdot\underline{a}' = 0\rceil, p_0, c)$]

[Prop 3.2l $\stackrel{\text{stmt}}{\to}$ $S \vdash \forall\underline{a}\colon 0\cdot\underline{a} = 0$]

[Prop 3.2l $\stackrel{\text{proof}}{\to}$ $\lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall\underline{a}\colon$ S7 $\gg 0\cdot 0 = 0$; Prop $3.2l_2 \gg 0\cdot\bar{x} = 0 \Rightarrow 0\cdot\bar{x}' = 0$; S9 @ $\bar{x} \rhd 0\cdot 0 = 0 \rhd 0\cdot\bar{x} = 0 \Rightarrow 0\cdot\bar{x}' = 0 \gg 0\cdot\bar{x} = 0$; Ded $\rhd\ 0\cdot\bar{x} = 0 \gg 0\cdot\underline{a} = 0\rceil, p_0, c)$]

4.4 3.2m

[Prop $3.2m_1$ $\stackrel{\text{stmt}}{\to}$ $S \vdash \forall\underline{a}\colon \underline{a}'\cdot 0 = \underline{a}\cdot 0+0$]

[Prop $3.2m_1$ $\stackrel{\text{proof}}{\to}$ $\lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall\underline{a}\colon$ S7 $\gg \underline{a}'\cdot 0 = 0$; Prop 3.2f $\gg 0 = 0+0$; S7 $\gg \underline{a}\cdot 0 = 0$; Prop 3.2b $\rhd\ \underline{a}\cdot 0 = 0 \gg 0 = \underline{a}\cdot 0$; Prop 3.2e $\rhd\ 0 = \underline{a}\cdot 0 \gg 0+0 = \underline{a}\cdot 0+0$; Prop 3.2c $\rhd\ 0 = 0+0 \rhd 0+0 = \underline{a}\cdot 0+0 \gg 0 = \underline{a}\cdot 0+0$; Prop 3.2c $\rhd\ \underline{a}'\cdot 0 = 0 \rhd 0 = \underline{a}\cdot 0+0 \gg \underline{a}'\cdot 0 = \underline{a}\cdot 0+0\rceil, p_0, c)$]

[Prop $3.2m_2$ $\stackrel{\text{stmt}}{\to}$ $S \vdash \forall\underline{a}\colon \forall\underline{b}\colon \underline{a}'\cdot\underline{b} = \underline{a}\cdot\underline{b}+\underline{b} \Rightarrow \underline{a}'\cdot\underline{b}' = \underline{a}\cdot\underline{b}'+\underline{b}'$]

[Prop $3.2m_2$ $\stackrel{\text{proof}}{\to}$ $\lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall\underline{a}\colon \forall\underline{b}\colon \forall\underline{a}\colon \forall\underline{b}\colon \underline{a}'\cdot\underline{b} = \underline{a}\cdot\underline{b}+\underline{b} \vdash$ S8 $\gg \underline{a}'\cdot\underline{b}' = \underline{a}'\cdot\underline{b}+\underline{a}'$; Prop 3.2e $\rhd\ \underline{a}'\cdot\underline{b} = \underline{a}\cdot\underline{b}+\underline{b} \gg \underline{a}'\cdot\underline{b}+\underline{a}' = \underline{a}\cdot\underline{b}+\underline{b}+\underline{a}'$; S6 $\gg \underline{b}+\underline{a}' = \underline{b}+\underline{a}'$; Prop 3.2g $\gg \underline{b}'+\underline{a} = \underline{b}+\underline{a}'$; Prop 3.2d $\rhd\ \underline{b}+\underline{a}' = \underline{b}+\underline{a}' \rhd \underline{b}'+\underline{a} = \underline{b}+\underline{a}' \gg \underline{b}+\underline{a}' = \underline{b}'+\underline{a}$; Prop 3.2h $\gg \underline{b}'+\underline{a} = \underline{a}+\underline{b}'$; Prop 3.2c $\rhd\ \underline{b}+\underline{a}' = \underline{b}'+\underline{a} \rhd \underline{b}'+\underline{a} = \underline{a}+\underline{b}' \gg \underline{b}+\underline{a}' = \underline{a}+\underline{b}'$; Prop 3.2i $\rhd\ \underline{b}+\underline{a}' = \underline{a}+\underline{b}' \gg \underline{a}\cdot\underline{b}+\underline{b}+\underline{a}' = \underline{a}\cdot\underline{b}+\underline{a}+\underline{b}'$; Prop 3.2j $\gg \underline{a}\cdot\underline{b}+\underline{a}+\underline{b}' = \underline{a}\cdot\underline{b}+\underline{a}+\underline{b}'$; Prop 3.2d $\rhd\ \underline{a}\cdot\underline{b}+\underline{b}+\underline{a}' = \underline{a}\cdot\underline{b}+\underline{a}+\underline{b}' \rhd \underline{a}\cdot\underline{b}+\underline{a}+\underline{b}' = \underline{a}\cdot\underline{b}+\underline{a}+\underline{b}' \gg \underline{a}\cdot\underline{b}+\underline{b}+\underline{a}' = \underline{a}\cdot\underline{b}+\underline{a}+\underline{b}'$; S8 $\gg \underline{a}\cdot\underline{b}' =$

$\underline{a} \cdot \underline{b} + \underline{a}; \text{Prop 3.2e} \rhd \underline{a} \cdot \underline{b}' = \underline{a} \cdot \underline{b} + \underline{a} \gg \underline{a} \cdot \underline{b}' + \underline{b}' = \underline{a} \cdot \underline{b} + \underline{a} + \underline{b}'; \text{Prop 3.2d} \rhd \underline{a} \cdot \underline{b} + \underline{b} + \underline{a}' = \underline{a} \cdot \underline{b} + \underline{a} + \underline{b}' \rhd \underline{a} \cdot \underline{b}' + \underline{b}' = \underline{a} \cdot \underline{b} + \underline{a} + \underline{b}' \gg \underline{a} \cdot \underline{b} + \underline{b} + \underline{a}' = \underline{a} \cdot \underline{b}' + \underline{b}'; \text{Prop 3.2j} \gg \underline{a} \cdot \underline{b} + \underline{b} + \underline{a}' = \underline{a} \cdot \underline{b} + \underline{b} + \underline{a}'; \text{Prop 3.2c} \rhd \underline{a}' \cdot \underline{b} + \underline{a}' = \underline{a} \cdot \underline{b} + \underline{b} + \underline{a}' \rhd \underline{a} \cdot \underline{b} + \underline{b} + \underline{a}' = \underline{a} \cdot \underline{b} + \underline{b} + \underline{a}' \gg \underline{a}' \cdot \underline{b} + \underline{a}' = \underline{a} \cdot \underline{b} + \underline{b} + \underline{a}'; \text{Prop 3.2c} \rhd \underline{a}' \cdot \underline{b} + \underline{a}' = \underline{a} \cdot \underline{b} + \underline{b} + \underline{a}' \rhd \underline{a} \cdot \underline{b} + \underline{b} + \underline{a}' = \underline{a} \cdot \underline{b}' + \underline{b}' \gg \underline{a}' \cdot \underline{b} + \underline{a}' = \underline{a} \cdot \underline{b}' + \underline{b}'; \text{Prop 3.2c} \rhd \underline{a}' \cdot \underline{b}' = \underline{a}' \cdot \underline{b} + \underline{a}' \rhd \underline{a}' \cdot \underline{b} + \underline{a}' = \underline{a} \cdot \underline{b}' + \underline{b}' \gg \underline{a}' \cdot \underline{b}' = \underline{a} \cdot \underline{b}' + \underline{b}'; \text{Ded} \rhd \forall \underline{a} \colon \forall \underline{b} \colon \underline{a}' \cdot \underline{b} = \underline{a} \cdot \underline{b} + \underline{b} \vdash \underline{a}' \cdot \underline{b}' = \underline{a} \cdot \underline{b}' + \underline{b}' \gg \underline{a}' \cdot \underline{b} = \underline{a} \cdot \underline{b} + \underline{b} \Rightarrow \underline{a}' \cdot \underline{b}' = \underline{a} \cdot \underline{b}' + \underline{b}' \rceil, p_0, c)]$

$[\text{Prop 3.2m} \overset{\text{stmt}}{\to} S \vdash \forall \underline{a} \colon \forall \underline{b} \colon \underline{a}' \cdot \underline{b} = \underline{a} \cdot \underline{b} + \underline{b}]$

$[\text{Prop 3.2m} \overset{\text{proof}}{\to} \lambda c. \lambda x. \mathcal{P}(\lceil S \vdash \forall \underline{a} \colon \forall \underline{b} \colon \text{Prop 3.2m}_1 \gg \overline{x}' \cdot 0 = \overline{x} \cdot 0 + 0; \text{Prop 3.2m}_2 \gg \overline{x}' \cdot \overline{y} = \overline{x} \cdot \overline{y} + \overline{y} \Rightarrow \overline{x}' \cdot \overline{y}' = \overline{x} \cdot \overline{y}' + \overline{y}'; \text{S9} \mathbin{@} \overline{y} \rhd \overline{x}' \cdot 0 = \overline{x} \cdot 0 + 0 \rhd \overline{x}' \cdot \overline{y} = \overline{x} \cdot \overline{y} + \overline{y} \Rightarrow \overline{x}' \cdot \overline{y}' = \overline{x} \cdot \overline{y}' + \overline{y}' \gg \overline{x}' \cdot \overline{y} = \overline{x} \cdot \overline{y} + \overline{y}; \text{Ded} \rhd \overline{x}' \cdot \overline{y} = \overline{x} \cdot \overline{y} + \overline{y} \gg \underline{a}' \cdot \underline{b} = \underline{a} \cdot \underline{b} + \underline{b} \rceil, p_0, c)]$

4.5 3.2n

$[\text{Prop 3.2n}_1 \overset{\text{stmt}}{\to} S \vdash \forall \underline{a} \colon \underline{a} \cdot 0 = 0 \cdot \underline{a}]$

$[\text{Prop 3.2n}_1 \overset{\text{proof}}{\to} \lambda c. \lambda x. \mathcal{P}(\lceil S \vdash \forall \underline{a} \colon \text{S7} \gg \underline{a} \cdot 0 = 0; \text{Prop 3.2l} \gg 0 \cdot \underline{a} = 0; \text{Prop 3.2d} \rhd \underline{a} \cdot 0 = 0 \rhd 0 \cdot \underline{a} = 0 \gg \underline{a} \cdot 0 = 0 \cdot \underline{a} \rceil, p_0, c)]$

$[\text{Prop 3.2n}_2 \overset{\text{stmt}}{\to} S \vdash \forall \underline{a} \colon \forall \underline{b} \colon \underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a} \Rightarrow \underline{a} \cdot \underline{b}' = \underline{b}' \cdot \underline{a}]$

$[\text{Prop 3.2n}_2 \overset{\text{proof}}{\to} \lambda c. \lambda x. \mathcal{P}(\lceil S \vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{a} \colon \forall \underline{b} \colon \underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a} \vdash \text{S8} \gg \underline{a} \cdot \underline{b}' = \underline{a} \cdot \underline{b} + \underline{a}; \text{Prop 3.2e} \rhd \underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a} \gg \underline{a} \cdot \underline{b} + \underline{a} = \underline{b} \cdot \underline{a} + \underline{a}; \text{Prop 3.2m} \gg \underline{b}' \cdot \underline{a} = \underline{b} \cdot \underline{a} + \underline{a}; \text{Prop 3.2b} \rhd \underline{b}' \cdot \underline{a} = \underline{b} \cdot \underline{a} + \underline{a} \gg \underline{b} \cdot \underline{a} + \underline{a} = \underline{b}' \cdot \underline{a}; \text{Prop 3.2c} \rhd \underline{a} \cdot \underline{b} + \underline{a} = \underline{b} \cdot \underline{a} + \underline{a} \rhd \underline{b} \cdot \underline{a} + \underline{a} = \underline{b}' \cdot \underline{a} \gg \underline{a} \cdot \underline{b} + \underline{a} = \underline{b}' \cdot \underline{a}; \text{Prop 3.2c} \rhd \underline{a} \cdot \underline{b}' = \underline{a} \cdot \underline{b} + \underline{a} \rhd \underline{a} \cdot \underline{b} + \underline{a} = \underline{b}' \cdot \underline{a} \gg \underline{a} \cdot \underline{b}' = \underline{b}' \cdot \underline{a}; \text{Ded} \rhd \forall \underline{a} \colon \forall \underline{b} \colon \underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a} \vdash \underline{a} \cdot \underline{b}' = \underline{b}' \cdot \underline{a} \gg \underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a} \Rightarrow \underline{a} \cdot \underline{b}' = \underline{b}' \cdot \underline{a} \rceil, p_0, c)]$

$[\text{Prop 3.2n} \overset{\text{stmt}}{\to} S \vdash \forall \underline{a} \colon \forall \underline{b} \colon \underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}]$

$[\text{Prop 3.2n} \overset{\text{proof}}{\to} \lambda c. \lambda x. \mathcal{P}(\lceil S \vdash \forall \underline{a} \colon \forall \underline{b} \colon \text{Prop 3.2n}_1 \gg \overline{x} \cdot 0 = 0 \cdot \overline{x}; \text{Prop 3.2n}_2 \gg \overline{x} \cdot \overline{y} = \overline{y} \cdot \overline{x} \Rightarrow \overline{x} \cdot \overline{y}' = \overline{y}' \cdot \overline{x}; \text{S9} \mathbin{@} \overline{y} \rhd \overline{x} \cdot 0 = 0 \cdot \overline{x} \rhd \overline{x} \cdot \overline{y} = \overline{y} \cdot \overline{x} \Rightarrow \overline{x} \cdot \overline{y}' = \overline{y}' \cdot \overline{x} \gg \overline{x} \cdot \overline{y} = \overline{y} \cdot \overline{x}; \text{Ded} \rhd \overline{x} \cdot \overline{y} = \overline{y} \cdot \overline{x} \gg \underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a} \rceil, p_0, c)]$

4.6 3.2o

$[\text{Prop 3.2o} \overset{\text{stmt}}{\to} S \vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \underline{a} = \underline{b} \vdash \underline{c} \cdot \underline{a} = \underline{c} \cdot \underline{b}]$

$[\text{Prop 3.2o} \overset{\text{proof}}{\to} \lambda c. \lambda x. \mathcal{P}(\lceil S \vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \underline{a} = \underline{b} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \underline{a} = \underline{b} \vdash \text{Prop 3.2k} \rhd \underline{a} = \underline{b} \gg \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c}; \text{Prop 3.2n} \gg \underline{a} \cdot \underline{c} = \underline{c} \cdot \underline{a}; \text{Prop 3.2n} \gg \underline{b} \cdot \underline{c} = \underline{c} \cdot \underline{b}; \text{Prop 3.2c} \rhd \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c} \rhd \underline{b} \cdot \underline{c} = \underline{c} \cdot \underline{b} \gg \underline{a} \cdot \underline{c} = \underline{c} \cdot \underline{b}; \text{S1} \rhd \underline{a} \cdot \underline{c} = \underline{c} \cdot \underline{a} \rhd \underline{a} \cdot \underline{c} = \underline{c} \cdot \underline{b} \gg \underline{c} \cdot \underline{a} = \underline{c} \cdot \underline{b}; \text{Ded} \rhd \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \underline{a} = \underline{b} \vdash \underline{c} \cdot \underline{a} = \underline{c} \cdot \underline{b} \gg \underline{a} = \underline{b} \Rightarrow \underline{c} \cdot \underline{a} = \underline{c} \cdot \underline{b}; \text{MP} \rhd \underline{a} = \underline{b} \Rightarrow \underline{c} \cdot \underline{a} = \underline{c} \cdot \underline{b} \rhd \underline{a} = \underline{b} \gg \underline{c} \cdot \underline{a} = \underline{c} \cdot \underline{b} \rceil, p_0, c)]$

5 Udsagn 3.4

De følgende udsagn er en udvidelse af egenskaberne ved addition og multiplikation.

[Prop 3.4a $\overset{\text{stmt}}{\rightarrow} \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a}\cdot\underline{b}+\underline{c} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}$]

[Prop 3.4b $\overset{\text{stmt}}{\rightarrow} \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{b}+\underline{c}\cdot\underline{a} = \underline{b}\cdot\underline{a}+\underline{c}\cdot\underline{a}$]

[Prop 3.4c $\overset{\text{stmt}}{\rightarrow} \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a}\cdot\underline{b}\cdot\underline{c} = \underline{a}\cdot\underline{b}\cdot\underline{c}$]

[Prop 3.4d $\overset{\text{stmt}}{\rightarrow} \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a}+\underline{c} = \underline{b}+\underline{c} \Rightarrow \underline{a} = \underline{b}$]

5.1 3.4a

[Prop $3.4\mathrm{a}_1 \overset{\text{stmt}}{\rightarrow} \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a}\cdot\underline{b}+0 = \underline{a}\cdot\underline{b}+\underline{a}\cdot 0$]

[Prop $3.4\mathrm{a}_1 \overset{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \mathrm{S5} \gg \underline{b}+0 = \underline{b}; \text{Prop 3.2o} \rhd \underline{b}+0 = \underline{b} \gg \underline{a}\cdot\underline{b}+0 = \underline{a}\cdot\underline{b}; \mathrm{S5} \gg \underline{a}\cdot\underline{b}+0 = \underline{a}\cdot\underline{b}; \text{Prop 3.2d} \rhd \underline{a}\cdot\underline{b}+0 = \underline{a}\cdot\underline{b} \rhd \underline{a}\cdot\underline{b}+0 = \underline{a}\cdot\underline{b} \gg \underline{a}\cdot\underline{b}+0 = \underline{a}\cdot\underline{b}+0; \mathrm{S7} \gg \underline{a}\cdot 0 = 0; \text{Prop 3.2i} \rhd \underline{a}\cdot 0 = 0 \gg \underline{a}\cdot\underline{b}+\underline{a}\cdot 0 = \underline{a}\cdot\underline{b}+0; \text{Prop 3.2d} \rhd \underline{a}\cdot\underline{b}+0 = \underline{a}\cdot\underline{b}+0 \rhd \underline{a}\cdot\underline{b}+\underline{a}\cdot 0 = \underline{a}\cdot\underline{b}+0 \gg \underline{a}\cdot\underline{b}+0 = \underline{a}\cdot\underline{b}+\underline{a}\cdot 0\rceil, \mathrm{p}_0, c)$]

[Prop $3.4\mathrm{a}_2 \overset{\text{stmt}}{\rightarrow} \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a}\cdot\underline{b}+\underline{c} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c} \Rightarrow \underline{a}\cdot\underline{b}+\underline{c}' = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}'$]

[Prop $3.4\mathrm{a}_2 \overset{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a}\cdot\underline{b}+\underline{c} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c} \vdash \mathrm{S6} \gg \underline{b}+\underline{c}' = \underline{b}+\underline{c}'; \text{Prop 3.2o} \rhd \underline{b}+\underline{c}' = \underline{b}+\underline{c}' \gg \underline{a}\cdot\underline{b}+\underline{c}' = \underline{a}\cdot\underline{b}+\underline{c}'; \mathrm{S8} \gg \underline{a}\cdot\underline{b}+\underline{c}' = \underline{a}\cdot\underline{b}+\underline{c}+\underline{a}; \text{Prop 3.2e} \rhd \underline{a}\cdot\underline{b}+\underline{c} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c} \gg \underline{a}\cdot\underline{b}+\underline{c}+\underline{a} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}+\underline{a}; \text{Prop 3.2j} \gg \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}+\underline{a} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}+\underline{a}; \text{Prop 3.2c} \rhd \underline{a}\cdot\underline{b}+\underline{c}+\underline{a} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}+\underline{a} \rhd \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}+\underline{a} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}+\underline{a} \gg \underline{a}\cdot\underline{b}+\underline{c}+\underline{a} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}+\underline{a}; \mathrm{S8} \gg \underline{a}\cdot\underline{c}' = \underline{a}\cdot\underline{c}+\underline{a}; \text{Prop 3.2i} \rhd \underline{a}\cdot\underline{c}' = \underline{a}\cdot\underline{c}+\underline{a} \gg \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}' = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}+\underline{a}; \text{Prop 3.2d} \rhd \underline{a}\cdot\underline{b}+\underline{c}+\underline{a} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}+\underline{a} \rhd \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}' = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}+\underline{a} \gg \underline{a}\cdot\underline{b}+\underline{c}+\underline{a} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}'; \text{Prop 3.2c} \rhd \underline{a}\cdot\underline{b}+\underline{c}' = \underline{a}\cdot\underline{b}+\underline{c}+\underline{a} \rhd \underline{a}\cdot\underline{b}+\underline{c}+\underline{a} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}' \gg \underline{a}\cdot\underline{b}+\underline{c}' = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}'; \text{Prop 3.2c} \rhd \underline{a}\cdot\underline{b}+\underline{c}' = \underline{a}\cdot\underline{b}+\underline{c}' \rhd \underline{a}\cdot\underline{b}+\underline{c}' = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}' \gg \underline{a}\cdot\underline{b}+\underline{c}' = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}'; \text{Ded} \rhd \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a}\cdot\underline{b}+\underline{c} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c} \vdash \underline{a}\cdot\underline{b}+\underline{c}' = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}' \gg \underline{a}\cdot\underline{b}+\underline{c} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c} \Rightarrow \underline{a}\cdot\underline{b}+\underline{c}' = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}'\rceil, \mathrm{p}_0, c)$]

[Prop 3.4a $\overset{\text{stmt}}{\rightarrow} \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a}\cdot\underline{b}+\underline{c} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}$]

[Prop 3.4a $\overset{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \text{Prop } 3.4\mathrm{a}_1 \gg \overline{x}\cdot\overline{y}+0 = \overline{x}\cdot\overline{y}+\overline{x}\cdot 0; \text{Prop } 3.4\mathrm{a}_2 \gg \overline{x}\cdot\overline{y}+\overline{z} = \overline{x}\cdot\overline{y}+\overline{x}\cdot\overline{z} \Rightarrow \overline{x}\cdot\overline{y}+\overline{z}' = \overline{x}\cdot\overline{y}+\overline{x}\cdot\overline{z}'; \mathrm{S9}\ @\ \overline{z} \rhd \overline{x}\cdot\overline{y}+0 = \overline{x}\cdot\overline{y}+\overline{x}\cdot 0 \rhd \overline{x}\cdot\overline{y}+\overline{z} = \overline{x}\cdot\overline{y}+\overline{x}\cdot\overline{z} \Rightarrow \overline{x}\cdot\overline{y}+\overline{z}' = \overline{x}\cdot\overline{y}+\overline{x}\cdot\overline{z}' \gg \overline{x}\cdot\overline{y}+\overline{z} = \overline{x}\cdot\overline{y}+\overline{x}\cdot\overline{z}; \text{Ded} \rhd \overline{x}\cdot\overline{y}+\overline{z} = \overline{x}\cdot\overline{y}+\overline{x}\cdot\overline{z} \gg \underline{a}\cdot\underline{b}+\underline{c} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}\rceil, \mathrm{p}_0, c)$]

5.2 3.4b

[Prop 3.4b $\overset{\text{stmt}}{\rightarrow} \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{b}+\underline{c}\cdot\underline{a} = \underline{b}\cdot\underline{a}+\underline{c}\cdot\underline{a}$]

[Prop 3.4b $\stackrel{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \text{Prop 3.4a} \gg \underline{a}\cdot\underline{b}+\underline{c} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}; \text{Prop 3.2n} \gg \underline{a}\cdot\underline{b}+\underline{c} = \underline{b}+\underline{c}\cdot\underline{a}; \text{Prop 3.2n} \gg \underline{a}\cdot\underline{b} = \underline{b}\cdot\underline{a}; \text{Prop 3.2n} \gg \underline{a}\cdot\underline{c} = \underline{c}\cdot\underline{a}; \text{S1} \rhd \underline{a}\cdot\underline{b}+\underline{c} = \underline{b}+\underline{c}\cdot\underline{a} \rhd \underline{a}\cdot\underline{b}+\underline{c} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c} \gg \underline{b}+\underline{c}\cdot\underline{a} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}; \text{Prop 3.2e} \rhd \underline{a}\cdot\underline{b} = \underline{b}\cdot\underline{a} \gg \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c} = \underline{b}\cdot\underline{a}+\underline{a}\cdot\underline{c}; \text{Prop 3.2i} \rhd \underline{a}\cdot\underline{c} = \underline{c}\cdot\underline{a} \gg \underline{b}\cdot\underline{a}+\underline{a}\cdot\underline{c} = \underline{b}\cdot\underline{a}+\underline{c}\cdot\underline{a}; \text{Prop 3.2c} \rhd \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c} = \underline{b}\cdot\underline{a}+\underline{a}\cdot\underline{c} \rhd \underline{b}\cdot\underline{a}+\underline{a}\cdot\underline{c} = \underline{b}\cdot\underline{a}+\underline{c}\cdot\underline{a} \gg \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c} = \underline{b}\cdot\underline{a}+\underline{c}\cdot\underline{a}; \text{Prop 3.2c} \rhd \underline{b}+\underline{c}\cdot\underline{a} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c} \rhd \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c} = \underline{b}\cdot\underline{a}+\underline{c}\cdot\underline{a} \gg \underline{b}+\underline{c}\cdot\underline{a} = \underline{b}\cdot\underline{a}+\underline{c}\cdot\underline{a}\rceil, p_0, c)$]

5.3 3.4c

[Prop 3.4c$_1 \stackrel{\text{stmt}}{\to} S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a}\cdot\underline{b}\cdot 0 = \underline{a}\cdot\underline{b}\cdot 0$]

[Prop 3.4c$_1 \stackrel{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \text{S7} \gg \underline{a}\cdot\underline{b}\cdot 0 = 0; \text{S7} \gg \underline{b}\cdot 0 = 0; \text{Prop 3.2o} \rhd \underline{b}\cdot 0 = 0 \gg \underline{a}\cdot\underline{b}\cdot 0 = \underline{a}\cdot 0; \text{S7} \gg \underline{a}\cdot 0 = 0; \text{Prop 3.2c} \rhd \underline{a}\cdot\underline{b}\cdot 0 = \underline{a}\cdot 0 \rhd \underline{a}\cdot 0 = 0 \gg \underline{a}\cdot\underline{b}\cdot 0 = 0; \text{Prop 3.2d} \rhd \underline{a}\cdot\underline{b}\cdot 0 = 0 \rhd \underline{a}\cdot\underline{b}\cdot 0 = 0 \gg \underline{a}\cdot\underline{b}\cdot 0 = \underline{a}\cdot\underline{b}\cdot 0\rceil, p_0, c)$]

[Prop 3.4c$_2 \stackrel{\text{stmt}}{\to} S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a}\cdot\underline{b}\cdot\underline{c} = \underline{a}\cdot\underline{b}\cdot\underline{c} \Rightarrow \underline{a}\cdot\underline{b}\cdot\underline{c}' = \underline{a}\cdot\underline{b}\cdot\underline{c}'$]

[Prop 3.4c$_2 \stackrel{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a}\cdot\underline{b}\cdot\underline{c} = \underline{a}\cdot\underline{b}\cdot\underline{c} \vdash \text{S8} \gg \underline{a}\cdot\underline{b}\cdot\underline{c}' = \underline{a}\cdot\underline{b}\cdot\underline{c}+\underline{a}\cdot\underline{b}; \text{Prop 3.2e} \rhd \underline{a}\cdot\underline{b}\cdot\underline{c} = \underline{a}\cdot\underline{b}\cdot\underline{c} \gg \underline{a}\cdot\underline{b}\cdot\underline{c}+\underline{a}\cdot\underline{b} = \underline{a}\cdot\underline{b}\cdot\underline{c}+\underline{a}\cdot\underline{b}; \text{Prop 3.2c} \rhd \underline{a}\cdot\underline{b}\cdot\underline{c}' = \underline{a}\cdot\underline{b}\cdot\underline{c}+\underline{a}\cdot\underline{b} \rhd \underline{a}\cdot\underline{b}\cdot\underline{c}+\underline{a}\cdot\underline{b} = \underline{a}\cdot\underline{b}\cdot\underline{c}+\underline{a}\cdot\underline{b} \gg \underline{a}\cdot\underline{b}\cdot\underline{c}' = \underline{a}\cdot\underline{b}\cdot\underline{c}+\underline{a}\cdot\underline{b}; \text{Prop 3.4a} \gg \underline{a}\cdot\underline{b}\cdot\underline{c}+\underline{b} = \underline{a}\cdot\underline{b}\cdot\underline{c}+\underline{a}\cdot\underline{b}; \text{Prop 3.2d} \rhd \underline{a}\cdot\underline{b}\cdot\underline{c}' = \underline{a}\cdot\underline{b}\cdot\underline{c}+\underline{a}\cdot\underline{b} \rhd \underline{a}\cdot\underline{b}\cdot\underline{c}+\underline{b} = \underline{a}\cdot\underline{b}\cdot\underline{c}+\underline{a}\cdot\underline{b} \gg \underline{a}\cdot\underline{b}\cdot\underline{c}' = \underline{a}\cdot\underline{b}\cdot\underline{c}+\underline{b}; \text{S8} \gg \underline{b}\cdot\underline{c}' = \underline{b}\cdot\underline{c}+\underline{b}; \text{Prop 3.2o} \rhd \underline{b}\cdot\underline{c}' = \underline{b}\cdot\underline{c}+\underline{b} \gg \underline{a}\cdot\underline{b}\cdot\underline{c}' = \underline{a}\cdot\underline{b}\cdot\underline{c}+\underline{b}; \text{Prop 3.2d} \rhd \underline{a}\cdot\underline{b}\cdot\underline{c}' = \underline{a}\cdot\underline{b}\cdot\underline{c}+\underline{b} \rhd \underline{a}\cdot\underline{b}\cdot\underline{c}' = \underline{a}\cdot\underline{b}\cdot\underline{c}+\underline{b} \gg \underline{a}\cdot\underline{b}\cdot\underline{c}' = \underline{a}\cdot\underline{b}\cdot\underline{c}'; \text{Ded} \rhd \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a}\cdot\underline{b}\cdot\underline{c} = \underline{a}\cdot\underline{b}\cdot\underline{c} \vdash \underline{a}\cdot\underline{b}\cdot\underline{c}' = \underline{a}\cdot\underline{b}\cdot\underline{c}' \gg \underline{a}\cdot\underline{b}\cdot\underline{c} = \underline{a}\cdot\underline{b}\cdot\underline{c} \Rightarrow \underline{a}\cdot\underline{b}\cdot\underline{c}' = \underline{a}\cdot\underline{b}\cdot\underline{c}'\rceil, p_0, c)$]

[Prop 3.4c $\stackrel{\text{stmt}}{\to} S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a}\cdot\underline{b}\cdot\underline{c} = \underline{a}\cdot\underline{b}\cdot\underline{c}$]

[Prop 3.4c $\stackrel{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \text{Prop 3.4c}_1 \gg \bar{x}\cdot\bar{y}\cdot 0 = \bar{x}\cdot\bar{y}\cdot 0; \text{Prop 3.4c}_2 \gg \bar{x}\cdot\bar{y}\cdot\bar{z} = \bar{x}\cdot\bar{y}\cdot\bar{z} \Rightarrow \bar{x}\cdot\bar{y}\cdot\bar{z}' = \bar{x}\cdot\bar{y}\cdot\bar{z}'; \text{S9} @ \bar{z} \rhd \bar{x}\cdot\bar{y}\cdot 0 = \bar{x}\cdot\bar{y}\cdot 0 \rhd \bar{x}\cdot\bar{y}\cdot\bar{z} = \bar{x}\cdot\bar{y}\cdot\bar{z} \Rightarrow \bar{x}\cdot\bar{y}\cdot\bar{z}' = \bar{x}\cdot\bar{y}\cdot\bar{z}' \gg \bar{x}\cdot\bar{y}\cdot\bar{z} = \bar{x}\cdot\bar{y}\cdot\bar{z}; \text{Ded} \rhd \bar{x}\cdot\bar{y}\cdot\bar{z} = \bar{x}\cdot\bar{y}\cdot\bar{z} \gg \underline{a}\cdot\underline{b}\cdot\underline{c} = \underline{a}\cdot\underline{b}\cdot\underline{c}\rceil, p_0, c)$]

5.4 3.4d

[Prop 3.4d$_1 \stackrel{\text{stmt}}{\to} S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a}+0 = \underline{b}+0 \Rightarrow \underline{a} = \underline{b}$]

[Prop 3.4d$_1 \stackrel{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{a}\colon \forall \underline{b}\colon \underline{a}+0 = \underline{b}+0 \vdash \text{S5} \gg \underline{a}+0 = \underline{a}; \text{S5} \gg \underline{b}+0 = \underline{b}; \text{S1} \rhd \underline{a}+0 = \underline{a} \rhd \underline{a}+0 = \underline{b}+0 \gg \underline{a} = \underline{b}+0; \text{Prop 3.2c} \rhd \underline{a} = \underline{b}+0 \rhd \underline{b}+0 = \underline{b} \gg \underline{a} = \underline{b}; \text{Ded} \rhd \forall \underline{a}\colon \forall \underline{b}\colon \underline{a}+0 = \underline{b}+0 \vdash \underline{a} = \underline{b} \gg \underline{a}+0 = \underline{b}+0 \Rightarrow \underline{a} = \underline{b}\rceil, p_0, c)$]

[Prop 3.4d$_2 \stackrel{\text{stmt}}{\to} S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a}+\underline{c} = \underline{b}+\underline{c} \Rightarrow \underline{a} = \underline{b} \Rightarrow \underline{a}+\underline{c}' = \underline{b}+\underline{c}' \Rightarrow \underline{a} = \underline{b}$]

[Prop 3.4d$_2 \stackrel{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a}+\underline{c} = \underline{b}+\underline{c} \Rightarrow \underline{a} = \underline{b} \vdash \underline{a}+\underline{c}' = \underline{b}+\underline{c}' \vdash \text{S6} \gg \underline{a}+\underline{c}' = \underline{a}+\underline{c}'; \text{S6} \gg \underline{b}+\underline{c}' = \underline{b}+\underline{c}'; \text{S1} \rhd \underline{a}+\underline{c}' =$

$\underline{a} + \underline{c}' \triangleright \underline{a} + \underline{c}' = \underline{b} + \underline{c}' \gg \underline{a} + \underline{c}' = \underline{b} + \underline{c}'$; Prop 3.2c $\triangleright \underline{a} + \underline{c}' = \underline{b} + \underline{c}' \triangleright \underline{b} + \underline{c}' = \underline{b} + \underline{c}' \gg \underline{a} + \underline{c}' = \underline{b} + \underline{c}'$; S4 $\triangleright \underline{a} + \underline{c}' = \underline{b} + \underline{c}' \gg \underline{a} + \underline{c} = \underline{b} + \underline{c}$; MP $\triangleright \underline{a} + \underline{c} = \underline{b} + \underline{c} \Rightarrow \underline{a} = \underline{b} \triangleright \underline{a} + \underline{c} = \underline{b} + \underline{c} \gg \underline{a} = \underline{b}$; Ded $\triangleright \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \underline{a} + \underline{c} = \underline{b} + \underline{c} \Rightarrow \underline{a} = \underline{b} \vdash \underline{a} + \underline{c}' = \underline{b} + \underline{c}' \vdash \underline{a} = \underline{b} \gg \underline{a} + \underline{c} = \underline{b} + \underline{c} \Rightarrow \underline{a} = \underline{b} \Rightarrow \underline{a} + \underline{c}' = \underline{b} + \underline{c}' \Rightarrow \underline{a} = \underline{b}\rceil, p_0, c)]$

[Prop 3.4d $\stackrel{\text{stmt}}{\to}$ $S \vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \underline{a} + \underline{c} = \underline{b} + \underline{c} \Rightarrow \underline{a} = \underline{b}$]

[Prop 3.4d $\stackrel{\text{proof}}{\to}$ $\lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon$ Prop 3.4d$_1 \gg \bar{x} + 0 = \bar{y} + 0 \Rightarrow \bar{x} = \bar{y}$; Prop 3.4d$_2 \gg \bar{x} + \bar{z} = \bar{y} + \bar{z} \Rightarrow \bar{x} = \bar{y} \Rightarrow \bar{x} + \bar{z}' = \bar{y} + \bar{z}' \Rightarrow \bar{x} = \bar{y}$; S9 @ $\bar{z} \triangleright \bar{x} + 0 = \bar{y} + 0 \Rightarrow \bar{x} = \bar{y} \triangleright \bar{x} + \bar{z} = \bar{y} + \bar{z} \Rightarrow \bar{x} = \bar{y} \Rightarrow \bar{x} + \bar{z}' = \bar{y} + \bar{z}' \Rightarrow \bar{x} = \bar{y} \gg \bar{x} + \bar{z} = \bar{y} + \bar{z} \Rightarrow \bar{x} = \bar{y}$; Ded $\triangleright \bar{x} + \bar{z} = \bar{y} + \bar{z} \Rightarrow \bar{x} = \bar{y} \gg \underline{a} + \underline{c} = \underline{b} + \underline{c} \Rightarrow \underline{a} = \underline{b}\rceil, p_0, c)]$

6 Udsagn 3.5

I denne del har vi kun kunne løse indtil h, da denne og de efterfølgende kræver reglen “existensial rule”. Dog er der i afsnit 6.2.2 genemgået hvilket problem der forhindrer os i at bevise ” existencial rule”, og hvordan vi ville have bevist reglen, hvis problemet ikke var opstået. Endvidere er der i 6.2.2 genemgået hvordan vi ville have løst 3.5h.
I de følgende udsagn indgår numeraler opfører sig som de naturlige tal og er defineret i forhold til 0 på følgende måde:

$$\begin{array}{lcl} 0 & = & \overline{0} \\ 0' & = & \overline{1} \\ 0'' & = & \overline{2} \\ \vdots & & \vdots \\ 0^{n*'} & = & \overline{n} \end{array}$$

Dvs. hvis $\overline{0}$ er et numeral, og hvis $\overline{n}$ er et numeral, så er $\overline{n}'$ også.

[Prop 3.5a $\stackrel{\text{stmt}}{\to}$ $S \vdash \forall \underline{a} \colon \underline{a} + 0' = \underline{a}'$]

[Prop 3.5b $\stackrel{\text{stmt}}{\to}$ $S \vdash \forall \underline{a} \colon \underline{a} \cdot 0' = \underline{a}$]

[Prop 3.5c $\stackrel{\text{stmt}}{\to}$ $S \vdash \forall \underline{a} \colon \underline{a} \cdot 0'' = \underline{a} + \underline{a}$]

[Prop 3.5d $\stackrel{\text{stmt}}{\to}$ $S \vdash \forall \underline{a} \colon \forall \underline{b} \colon \underline{a} + \underline{b} = 0 \Rightarrow \neg \underline{a} = 0 \Rightarrow \neg \underline{b} = 0$]

[Prop 3.5e $\stackrel{\text{stmt}}{\to}$ $S \vdash \forall \underline{a} \colon \forall \underline{b} \colon \neg \underline{b} = 0 \Rightarrow \underline{a} \cdot \underline{b} = 0 \Rightarrow \underline{a} = 0$]

[Prop 3.5f $\stackrel{\text{stmt}}{\to}$ $S \vdash \forall \underline{a} \colon \forall \underline{b} \colon \underline{a} + \underline{b} = 0' \Rightarrow \neg\neg \underline{a} = 0 \Rightarrow \neg \underline{b} = 0' \Rightarrow \neg \underline{a} = 0' \Rightarrow \neg \underline{b} = 0$]

[Prop 3.5g $\stackrel{\text{stmt}}{\to}$ $S \vdash \forall \underline{a} \colon \forall \underline{b} \colon \underline{a} \cdot \underline{b} = 0' \Rightarrow \neg \underline{a} = 0' \Rightarrow \neg \underline{b} = 0'$]

[Prop 3.5h $\overset{\text{stmt}}{\rightarrow}$ S $\vdash \forall \underline{a} \colon \neg \underline{a} = 0 \Rightarrow \neg\forall_{\text{obj}}\underline{b} \colon \neg \underline{a} = \underline{b}'$]

[Prop 3.5i $\overset{\text{stmt}}{\rightarrow}$ S $\vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \neg \underline{c} = 0 \Rightarrow \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c} \Rightarrow \underline{a} = \underline{b}$]

[Prop 3.5j $\overset{\text{stmt}}{\rightarrow}$ S $\vdash \forall \underline{a} \colon \neg \underline{a} = 0 \Rightarrow \neg \underline{a} = 0' \Rightarrow \neg\forall_{\text{obj}}\underline{b} \colon \neg \underline{a} = \underline{b}''$]

Efterfø[]lgende vil vi gennemgåde nø[]dvendige hjælpesætninger og definitioner for at kunne bevise 3.5a-3.5g.

6.1 Definitioner i forbindelse med ∧ og ∨

Fra 3.5d bruges ∧ og ∨, som begge er makrodefinerede udtryk.

$[\mathsf{x} \wedge \mathsf{y} \overset{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil [\mathsf{x} \wedge \mathsf{y} \doteq \neg(\mathsf{x} \Rightarrow \neg \mathsf{y})] \rceil)]$

$[\mathsf{x} \vee \mathsf{y} \overset{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil [\mathsf{x} \vee \mathsf{y} \doteq (\neg \mathsf{x}) \Rightarrow \mathsf{y}] \rceil)]$

Endvidere er det klart at reglerne for introduktion og eliminering af ∧ og ∨ dermed også skal bruges. Alle disse er bevist i de følgende afsnit.

6.1.1 Introduktion af ∧

[Con $\overset{\text{stmt}}{\rightarrow}$ S $\vdash \forall \underline{a} \colon \forall \underline{b} \colon \underline{a} \vdash \underline{b} \vdash \neg \underline{a} \Rightarrow \neg \underline{b}$]

[Con $\overset{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil$S $\vdash \forall \underline{a} \colon \forall \underline{b} \colon \underline{a} \vdash \underline{b} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \underline{a} \Rightarrow \neg \underline{b} \vdash$ Repetition $\rhd \underline{a} \gg \underline{a}$; MP $\rhd \underline{a} \Rightarrow \neg \underline{b} \rhd \underline{a} \gg \neg \underline{b}$; Ded $\rhd \forall \underline{a} \colon \forall \underline{b} \colon \underline{a} \Rightarrow \neg \underline{b} \vdash \neg \underline{b} \gg \underline{a} \Rightarrow \neg \underline{b} \Rightarrow \neg \underline{b}$; Lem1.11b $\gg \underline{b} \Rightarrow \neg\neg \underline{b}$; MP $\rhd \underline{b} \Rightarrow \neg\neg \underline{b} \rhd \underline{b} \gg \neg\neg \underline{b}$; MT $\rhd \underline{a} \Rightarrow \neg \underline{b} \Rightarrow \neg \underline{b} \rhd \neg\neg \underline{b} \gg \neg \underline{a} \Rightarrow \neg \underline{b} \rceil, \mathrm{p}_0, \mathsf{c})$]

6.1.2 Elimination af ∧ 1

[Con1 $\overset{\text{stmt}}{\rightarrow}$ S $\vdash \forall \underline{a} \colon \forall \underline{b} \colon \neg \underline{a} \Rightarrow \neg \underline{b} \vdash \underline{a}$]

[Con1 $\overset{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil$S $\vdash \forall \underline{a} \colon \forall \underline{b} \colon \neg \underline{a} \Rightarrow \neg \underline{b} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \neg \underline{a} \vdash$ A1′ $\gg \neg \underline{a} \Rightarrow \neg\neg \underline{b} \Rightarrow \neg \underline{a}$; MP $\rhd \neg \underline{a} \Rightarrow \neg\neg \underline{b} \Rightarrow \neg \underline{a} \rhd \neg \underline{a} \gg \neg\neg \underline{b} \Rightarrow \neg \underline{a}$; Lem1.11d $\gg \neg\neg \underline{b} \Rightarrow \neg \underline{a} \Rightarrow \underline{a} \Rightarrow \neg \underline{b}$; MP $\rhd \neg\neg \underline{b} \Rightarrow \neg \underline{a} \Rightarrow \underline{a} \Rightarrow \neg \underline{b} \rhd \neg\neg \underline{b} \Rightarrow \neg \underline{a} \gg \underline{a} \Rightarrow \neg \underline{b}$; Ded $\rhd \forall \underline{a} \colon \forall \underline{b} \colon \neg \underline{a} \vdash \underline{a} \Rightarrow \neg \underline{b} \gg \neg \underline{a} \Rightarrow \underline{a} \Rightarrow \neg \underline{b}$; MT $\rhd \neg \underline{a} \Rightarrow \underline{a} \Rightarrow \neg \underline{b} \rhd \neg \underline{a} \Rightarrow \neg \underline{b} \gg \neg\neg \underline{a}$; Lem1.11a $\gg \neg\neg \underline{a} \Rightarrow \underline{a}$; MP $\rhd \neg\neg \underline{a} \Rightarrow \underline{a} \rhd \neg\neg \underline{a} \gg \underline{a} \rceil, \mathrm{p}_0, \mathsf{c})$]

6.1.3 Elimination af ∧ 2

[Con2 $\overset{\text{stmt}}{\rightarrow}$ S $\vdash \forall \underline{a} \colon \forall \underline{b} \colon \neg \underline{a} \Rightarrow \neg \underline{b} \vdash \underline{b}$]

[Con2 $\overset{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil$S $\vdash \forall \underline{a} \colon \forall \underline{b} \colon \neg \underline{a} \Rightarrow \neg \underline{b} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \neg \underline{b} \vdash$ A1′ $\gg \neg \underline{b} \Rightarrow \underline{a} \Rightarrow \neg \underline{b}$; MP $\rhd \neg \underline{b} \Rightarrow \underline{a} \Rightarrow \neg \underline{b} \rhd \neg \underline{b} \gg \underline{a} \Rightarrow \neg \underline{b}$; Ded $\rhd \forall \underline{a} \colon \forall \underline{b} \colon \neg \underline{b} \vdash \underline{a} \Rightarrow \neg \underline{b} \gg \neg \underline{b} \Rightarrow \underline{a} \Rightarrow \neg \underline{b}$; MT $\rhd \neg \underline{b} \Rightarrow \underline{a} \Rightarrow \neg \underline{b} \rhd \neg \underline{a} \Rightarrow \neg \underline{b} \gg \neg\neg \underline{b}$; Lem1.11a $\gg \neg\neg \underline{b} \Rightarrow \underline{b}$; MP $\rhd \neg\neg \underline{b} \Rightarrow \underline{b} \rhd \neg\neg \underline{b} \gg \underline{b} \rceil, \mathrm{p}_0, \mathsf{c})$]

6.1.4 Introduktion af $\vee$ 1

$[\mathrm{Dis1} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \vdash \neg \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}]$

$[\mathrm{Dis1} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \vdash \mathrm{Lem1.11c} \gg \neg\neg \underline{\mathsf{a}} \Rightarrow \neg \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}; \mathrm{Lem1.11b} \gg \underline{\mathsf{a}} \Rightarrow \neg\neg \underline{\mathsf{a}}; \mathrm{MP} \rhd \underline{\mathsf{a}} \Rightarrow \neg\neg \underline{\mathsf{a}} \rhd \underline{\mathsf{a}} \gg \neg\neg \underline{\mathsf{a}}; \mathrm{MP} \rhd \neg\neg \underline{\mathsf{a}} \Rightarrow \neg \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \rhd \neg\neg \underline{\mathsf{a}} \gg \neg \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}\rceil, \mathrm{p}_0, \mathsf{c})]$

6.1.5 Introduktion af $\vee$ 2

$[\mathrm{Dis2} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{b}} \vdash \neg \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}]$

$[\mathrm{Dis2} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{b}} \vdash \mathrm{A1'} \gg \underline{\mathsf{b}} \Rightarrow \neg \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}; \mathrm{MP} \rhd \underline{\mathsf{b}} \Rightarrow \neg \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \rhd \underline{\mathsf{b}} \gg \neg \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}\rceil, \mathrm{p}_0, \mathsf{c})]$

6.1.6 Elimination af $\vee$

Da vi ikke har haft brug for at fjerne $\vee$, har vi ikke bevist denne regel.

6.2 Andre hjælpesætninger

Vi har til beviset af 3.5a-3.5g haft brug for nogle af reglerne fra [Gru06], dog med `imply` istedet for `infer`. Sådanne regler er navngivet med det <oprindelige navn>' og beviserne for disse kan ses i Appendix A.

6.2.1 $\mathcal{A} \neq \mathcal{B}$

En ny hjælperegel er **regel** H3 , som skal bruges i forbindelse med $\mathcal{A} \neq \mathcal{B}$ for at kunne konkludere sammenhænge i mellem $(\mathrm{x} = \mathrm{y}) \wedge (\mathrm{x} \neq 3) \Rightarrow (\mathrm{y} \neq 3)$.

$[\mathrm{H3} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \forall \underline{\mathsf{c}} \colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \vdash \underline{\mathsf{a}} \vdash \neg \underline{\mathsf{c}} \vdash \neg \underline{\mathsf{b}}]$

$[\mathrm{H3} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \forall \underline{\mathsf{c}} \colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \vdash \underline{\mathsf{a}} \vdash \neg \underline{\mathsf{c}} \vdash \mathrm{MP} \rhd \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \rhd \underline{\mathsf{a}} \gg \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}}; \mathrm{MT} \rhd \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \rhd \neg \underline{\mathsf{c}} \gg \neg \underline{\mathsf{b}}\rceil, \mathrm{p}_0, \mathsf{c})]$

6.2.2 $\exists$ x

Selv om vi ikke har nået at implementere fra opgave 3.5h, hvor der gøres brug af $\exists$, har vi alligevel makrodefineret kvantoren som følger.
$[\exists \mathsf{x} \colon \mathsf{y} \stackrel{\mathrm{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil[\exists \mathsf{x} \colon \mathsf{y} \ddot{=} \neg(\forall \mathsf{x} \colon \neg \mathsf{y})]\rceil)]$
Vores problem har været, at vi ikke har kunne udtrykke *"$\mathcal{B}(\mathrm{x}, \mathrm{t})$, hvor* t *kan indsættes i stedet for* x*'erne uden at blive bundet til en alkvantor"*, i pyk. Beviset for hjælpesætningen, der indfører eksistenskvantoren, ville dog have været opbygget nogenlunde som følger.
Følgende tautologi,

$$(\mathcal{A} \Rightarrow \neg \mathcal{B}) \Rightarrow (\mathcal{B} \Rightarrow \neg \mathcal{A})$$

antages vist nogenlunde som Modus Tollens, dog med variation.
Ved axiom A4′ kan vi få instansen:

$$(\forall x)\neg\mathcal{A}(x,t) \Rightarrow \neg A(t,t)$$

Idet man kan få følgende instans af tautologien:

$$\big((\forall x)\neg\mathcal{A}(x,t) \Rightarrow \neg A(t,t)\big) \Rightarrow \big(A(t,t) \Rightarrow \neg(\forall x)\neg\mathcal{A}(x,t)\big)$$

kan man vha. MP på instansen og axiomet få:

$$\big(A(t,t) \Rightarrow \neg(\forall x)\neg\mathcal{A}(x,t)\big)$$

og pga. af makrodefinitionen er denne nu vist.
Herefter vises Lemmaerne 3.5a-3.5g

6.3 3.5a

[Prop 3.5a $\overset{\text{stmt}}{\rightarrow}$ $\mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \underline{\mathsf{a}} + 0' = \underline{\mathsf{a}}'$]

[Prop 3.5a $\overset{\text{proof}}{\rightarrow}$ $\lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \mathrm{S6} \gg \underline{\mathsf{a}} + 0' = \underline{\mathsf{a}} + 0'; \mathrm{S5} \gg \underline{\mathsf{a}} + 0 = \underline{\mathsf{a}}; \mathrm{S2} \rhd \underline{\mathsf{a}} + 0 = \underline{\mathsf{a}} \gg \underline{\mathsf{a}} + 0' = \underline{\mathsf{a}}'; \text{Prop 3.2c} \rhd \underline{\mathsf{a}} + 0' = \underline{\mathsf{a}} + 0' \rhd \underline{\mathsf{a}} + 0' = \underline{\mathsf{a}}' \gg \underline{\mathsf{a}} + 0' = \underline{\mathsf{a}}'; \text{Prop 3.2a} \gg \underline{\mathsf{a}} + 0' = \underline{\mathsf{a}} + 0'; \mathrm{S1} \rhd \underline{\mathsf{a}} + 0' = \underline{\mathsf{a}} + 0' \rhd \underline{\mathsf{a}} + 0' = \underline{\mathsf{a}}' \gg \underline{\mathsf{a}} + 0' = \underline{\mathsf{a}}' \rceil, \mathrm{p}_0, \mathsf{c})$]

6.4 3.5b

[Prop 3.5b $\overset{\text{stmt}}{\rightarrow}$ $\mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \underline{\mathsf{a}} \cdot 0' = \underline{\mathsf{a}}$]

[Prop 3.5b $\overset{\text{proof}}{\rightarrow}$ $\lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \mathrm{S8} \gg \underline{\mathsf{a}} \cdot 0' = \underline{\mathsf{a}} \cdot 0 + \underline{\mathsf{a}}; \mathrm{S7} \gg \underline{\mathsf{a}} \cdot 0 = 0; \text{Prop 3.2e} \rhd \underline{\mathsf{a}} \cdot 0 = 0 \gg \underline{\mathsf{a}} \cdot 0 + \underline{\mathsf{a}} = 0 + \underline{\mathsf{a}}; \text{Prop 3.2c} \rhd \underline{\mathsf{a}} \cdot 0' = \underline{\mathsf{a}} \cdot 0 + \underline{\mathsf{a}} \rhd \underline{\mathsf{a}} \cdot 0 + \underline{\mathsf{a}} = 0 + \underline{\mathsf{a}} \gg \underline{\mathsf{a}} \cdot 0' = 0 + \underline{\mathsf{a}}; \text{Prop 3.2f} \gg \underline{\mathsf{a}} = 0 + \underline{\mathsf{a}}; \text{Prop 3.2b} \rhd \underline{\mathsf{a}} = 0 + \underline{\mathsf{a}} \gg 0 + \underline{\mathsf{a}} = \underline{\mathsf{a}}; \text{Prop 3.2c} \rhd \underline{\mathsf{a}} \cdot 0' = 0 + \underline{\mathsf{a}} \rhd 0 + \underline{\mathsf{a}} = \underline{\mathsf{a}} \gg \underline{\mathsf{a}} \cdot 0' = \underline{\mathsf{a}}; \text{Prop 3.2a} \gg \underline{\mathsf{a}} \cdot 0' = \underline{\mathsf{a}} \cdot 0'; \mathrm{S1} \rhd \underline{\mathsf{a}} \cdot 0' = \underline{\mathsf{a}} \cdot 0' \rhd \underline{\mathsf{a}} \cdot 0' = \underline{\mathsf{a}} \gg \underline{\mathsf{a}} \cdot 0' = \underline{\mathsf{a}} \rceil, \mathrm{p}_0, \mathsf{c})$]

6.5 3.5c

[Prop 3.5c $\overset{\text{stmt}}{\rightarrow}$ $\mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \underline{\mathsf{a}} \cdot 0'' = \underline{\mathsf{a}} + \underline{\mathsf{a}}$]

[Prop 3.5c $\overset{\text{proof}}{\rightarrow}$ $\lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \mathrm{S8} \gg \underline{\mathsf{a}} \cdot 0'' = \underline{\mathsf{a}} \cdot 0' + \underline{\mathsf{a}}; \text{Prop 3.5b} \gg \underline{\mathsf{a}} \cdot 0' = \underline{\mathsf{a}}; \text{Prop 3.2e} \rhd \underline{\mathsf{a}} \cdot 0' = \underline{\mathsf{a}} \gg \underline{\mathsf{a}} \cdot 0' + \underline{\mathsf{a}} = \underline{\mathsf{a}} + \underline{\mathsf{a}}; \text{Prop 3.2c} \rhd \underline{\mathsf{a}} \cdot 0'' = \underline{\mathsf{a}} \cdot 0' + \underline{\mathsf{a}} \rhd \underline{\mathsf{a}} \cdot 0' + \underline{\mathsf{a}} = \underline{\mathsf{a}} + \underline{\mathsf{a}} \gg \underline{\mathsf{a}} \cdot 0'' = \underline{\mathsf{a}} + \underline{\mathsf{a}}; \text{Prop 3.2a} \gg \underline{\mathsf{a}} \cdot 0'' = \underline{\mathsf{a}} \cdot 0''; \mathrm{S1} \rhd \underline{\mathsf{a}} \cdot 0'' = \underline{\mathsf{a}} \cdot 0'' \rhd \underline{\mathsf{a}} \cdot 0'' = \underline{\mathsf{a}} + \underline{\mathsf{a}} \gg \underline{\mathsf{a}} \cdot 0'' = \underline{\mathsf{a}} + \underline{\mathsf{a}} \rceil, \mathrm{p}_0, \mathsf{c})$]

6.6 3.5d

[Prop $3.5\mathrm{d}_1$ $\overset{\text{stmt}}{\rightarrow}$ $\mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \underline{\mathsf{a}} + 0 = 0 \Rightarrow \neg \underline{\mathsf{a}} = 0 \Rightarrow \neg 0 = 0$]

$[\text{Prop } 3.5\text{d}_1 \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{a}} \colon \underline{\mathsf{a}} + 0 = 0 \vdash \text{S5} \gg \underline{\mathsf{a}} + 0 = \underline{\mathsf{a}}; \text{S1} \rhd \underline{\mathsf{a}} + 0 = \underline{\mathsf{a}} \rhd \underline{\mathsf{a}} + 0 = 0 \gg \underline{\mathsf{a}} = 0; \text{Prop } 3.2\text{a} \gg 0 = 0; \text{Con} \rhd \underline{\mathsf{a}} = 0 \rhd 0 = 0 \gg \neg \underline{\mathsf{a}} = 0 \Rightarrow \neg 0 = 0; \text{Ded} \rhd \forall \underline{\mathsf{a}} \colon \underline{\mathsf{a}} + 0 = 0 \vdash \neg \underline{\mathsf{a}} = 0 \Rightarrow \neg 0 = 0 \gg \underline{\mathsf{a}} + 0 = 0 \Rightarrow \neg \underline{\mathsf{a}} = 0 \Rightarrow \neg 0 = 0 \rceil, \text{p}_0, \mathsf{c})]$

$[\text{Prop } 3.5\text{d}_2 \stackrel{\text{stmt}}{\rightarrow} \text{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} + \underline{\mathsf{b}} = 0 \Rightarrow \neg \underline{\mathsf{a}} = 0 \Rightarrow \neg \underline{\mathsf{b}} = 0 \Rightarrow \underline{\mathsf{a}} + \underline{\mathsf{b}}' = 0 \Rightarrow \neg \underline{\mathsf{a}} = 0 \Rightarrow \neg \underline{\mathsf{b}}' = 0]$

$[\text{Prop } 3.5\text{d}_2 \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} + \underline{\mathsf{b}} = 0 \Rightarrow \neg \underline{\mathsf{a}} = 0 \Rightarrow \neg \underline{\mathsf{b}} = 0 \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} + \underline{\mathsf{b}}' = 0 \vdash \text{S3} \gg \neg 0 = \underline{\mathsf{a}} + \underline{\mathsf{b}}'; \text{Prop3.2b}' \gg \underline{\mathsf{a}} + \underline{\mathsf{b}}' = 0 \Rightarrow 0 = \underline{\mathsf{a}} + \underline{\mathsf{b}}'; \text{Lem1.11e} \gg \underline{\mathsf{a}} + \underline{\mathsf{b}}' = 0 \Rightarrow 0 = \underline{\mathsf{a}} + \underline{\mathsf{b}}' \Rightarrow \neg 0 = \underline{\mathsf{a}} + \underline{\mathsf{b}}' \Rightarrow \neg \underline{\mathsf{a}} + \underline{\mathsf{b}}' = 0; \text{MP} \rhd \underline{\mathsf{a}} + \underline{\mathsf{b}}' = 0 \Rightarrow 0 = \underline{\mathsf{a}} + \underline{\mathsf{b}}' \Rightarrow \neg 0 = \underline{\mathsf{a}} + \underline{\mathsf{b}}' \Rightarrow \neg \underline{\mathsf{a}} + \underline{\mathsf{b}}' = 0 \rhd \underline{\mathsf{a}} + \underline{\mathsf{b}}' = 0 \Rightarrow 0 = \underline{\mathsf{a}} + \underline{\mathsf{b}}' \gg \neg 0 = \underline{\mathsf{a}} + \underline{\mathsf{b}}' \Rightarrow \neg \underline{\mathsf{a}} + \underline{\mathsf{b}}' = 0; \text{MP} \rhd \neg 0 = \underline{\mathsf{a}} + \underline{\mathsf{b}}' \Rightarrow \neg \underline{\mathsf{a}} + \underline{\mathsf{b}}' = 0 \rhd \neg 0 = \underline{\mathsf{a}} + \underline{\mathsf{b}}' \gg \neg \underline{\mathsf{a}} + \underline{\mathsf{b}}' = 0; \text{S6} \gg \underline{\mathsf{a}} + \underline{\mathsf{b}}' = \underline{\mathsf{a}} + \underline{\mathsf{b}}'; \text{S1}'' \gg \underline{\mathsf{a}} + \underline{\mathsf{b}}' = \underline{\mathsf{a}} + \underline{\mathsf{b}}' \Rightarrow \underline{\mathsf{a}} + \underline{\mathsf{b}}' = 0 \Rightarrow \underline{\mathsf{a}} + \underline{\mathsf{b}}' = 0; \text{H3} \rhd \underline{\mathsf{a}} + \underline{\mathsf{b}}' = \underline{\mathsf{a}} + \underline{\mathsf{b}}' \Rightarrow \underline{\mathsf{a}} + \underline{\mathsf{b}}' = 0 \Rightarrow \underline{\mathsf{a}} + \underline{\mathsf{b}}' = 0 \rhd \underline{\mathsf{a}} + \underline{\mathsf{b}}' = \underline{\mathsf{a}} + \underline{\mathsf{b}}' \rhd \neg \underline{\mathsf{a}} + \underline{\mathsf{b}}' = 0 \gg \neg \underline{\mathsf{a}} + \underline{\mathsf{b}}' = 0; \text{Lem1.11c} \gg \neg \underline{\mathsf{a}} + \underline{\mathsf{b}}' = 0 \Rightarrow \underline{\mathsf{a}} + \underline{\mathsf{b}}' = 0 \Rightarrow \neg \underline{\mathsf{a}} = 0 \Rightarrow \neg \underline{\mathsf{b}}' = 0; \text{MP} \rhd \neg \underline{\mathsf{a}} + \underline{\mathsf{b}}' = 0 \Rightarrow \underline{\mathsf{a}} + \underline{\mathsf{b}}' = 0 \Rightarrow \neg \underline{\mathsf{a}} = 0 \Rightarrow \neg \underline{\mathsf{b}}' = 0 \rhd \neg \underline{\mathsf{a}} + \underline{\mathsf{b}}' = 0 \gg \underline{\mathsf{a}} + \underline{\mathsf{b}}' = 0 \Rightarrow \neg \underline{\mathsf{a}} = 0 \Rightarrow \neg \underline{\mathsf{b}}' = 0; \text{MP} \rhd \underline{\mathsf{a}} + \underline{\mathsf{b}}' = 0 \Rightarrow \neg \underline{\mathsf{a}} = 0 \Rightarrow \neg \underline{\mathsf{b}}' = 0 \rhd \underline{\mathsf{a}} + \underline{\mathsf{b}}' = 0 \gg \neg \underline{\mathsf{a}} = 0 \Rightarrow \neg \underline{\mathsf{b}}' = 0; \text{Ded} \rhd \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} + \underline{\mathsf{b}}' = 0 \vdash \neg \underline{\mathsf{a}} = 0 \Rightarrow \neg \underline{\mathsf{b}}' = 0 \gg \underline{\mathsf{a}} + \underline{\mathsf{b}}' = 0 \Rightarrow \neg \underline{\mathsf{a}} = 0 \Rightarrow \neg \underline{\mathsf{b}}' = 0; \text{Ded} \rhd \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} + \underline{\mathsf{b}} = 0 \Rightarrow \neg \underline{\mathsf{a}} = 0 \Rightarrow \neg \underline{\mathsf{b}} = 0 \vdash \underline{\mathsf{a}} + \underline{\mathsf{b}}' = 0 \Rightarrow \neg \underline{\mathsf{a}} = 0 \Rightarrow \neg \underline{\mathsf{b}}' = 0 \gg \underline{\mathsf{a}} + \underline{\mathsf{b}} = 0 \Rightarrow \neg \underline{\mathsf{a}} = 0 \Rightarrow \neg \underline{\mathsf{b}} = 0 \Rightarrow \underline{\mathsf{a}} + \underline{\mathsf{b}}' = 0 \Rightarrow \neg \underline{\mathsf{a}} = 0 \Rightarrow \neg \underline{\mathsf{b}}' = 0 \rceil, \text{p}_0, \mathsf{c})]$

$[\text{Prop } 3.5\text{d} \stackrel{\text{stmt}}{\rightarrow} \text{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} + \underline{\mathsf{b}} = 0 \Rightarrow \neg \underline{\mathsf{a}} = 0 \Rightarrow \neg \underline{\mathsf{b}} = 0]$

$[\text{Prop } 3.5\text{d} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \text{Prop } 3.5\text{d}_1 \gg \bar{\mathsf{x}} + 0 = 0 \Rightarrow \neg \bar{\mathsf{x}} = 0 \Rightarrow \neg 0 = 0; \text{Prop } 3.5\text{d}_2 \gg \bar{\mathsf{x}} + \bar{\mathsf{y}} = 0 \Rightarrow \neg \bar{\mathsf{x}} = 0 \Rightarrow \neg \bar{\mathsf{y}} = 0 \Rightarrow \bar{\mathsf{x}} + \bar{\mathsf{y}}' = 0 \Rightarrow \neg \bar{\mathsf{x}} = 0 \Rightarrow \neg \bar{\mathsf{y}}' = 0; \text{S9} \,@\, \bar{\mathsf{y}} \rhd \bar{\mathsf{x}} + 0 = 0 \Rightarrow \neg \bar{\mathsf{x}} = 0 \Rightarrow \neg 0 = 0 \rhd \bar{\mathsf{x}} + \bar{\mathsf{y}} = 0 \Rightarrow \neg \bar{\mathsf{x}} = 0 \Rightarrow \neg \bar{\mathsf{y}} = 0 \Rightarrow \bar{\mathsf{x}} + \bar{\mathsf{y}}' = 0 \Rightarrow \neg \bar{\mathsf{x}} = 0 \Rightarrow \neg \bar{\mathsf{y}}' = 0 \gg \bar{\mathsf{x}} + \bar{\mathsf{y}} = 0 \Rightarrow \neg \bar{\mathsf{x}} = 0 \Rightarrow \neg \bar{\mathsf{y}} = 0; \text{Ded} \rhd \bar{\mathsf{x}} + \bar{\mathsf{y}} = 0 \Rightarrow \neg \bar{\mathsf{x}} = 0 \Rightarrow \neg \bar{\mathsf{y}} = 0 \gg \underline{\mathsf{a}} + \underline{\mathsf{b}} = 0 \Rightarrow \neg \underline{\mathsf{a}} = 0 \Rightarrow \neg \underline{\mathsf{b}} = 0 \rceil, \text{p}_0, \mathsf{c})]$

6.7 3.5e

$[\text{Prop } 3.5\text{e}_1 \stackrel{\text{stmt}}{\rightarrow} \text{S} \vdash \forall \underline{\mathsf{a}} \colon \neg 0 = 0 \Rightarrow \underline{\mathsf{a}} \cdot 0 = 0 \Rightarrow \underline{\mathsf{a}} = 0]$

$[\text{Prop } 3.5\text{e}_1 \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{a}} \colon \neg 0 = 0 \vdash \text{Prop } 3.2\text{a} \gg 0 = 0; \text{Lem1.11c} \gg \neg 0 = 0 \Rightarrow 0 = 0 \Rightarrow \underline{\mathsf{a}} \cdot 0 = 0 \Rightarrow \underline{\mathsf{a}} = 0; \text{MP} \rhd \neg 0 = 0 \Rightarrow 0 = 0 \Rightarrow \underline{\mathsf{a}} \cdot 0 = 0 \Rightarrow \underline{\mathsf{a}} = 0 \rhd \neg 0 = 0 \gg 0 = 0 \Rightarrow \underline{\mathsf{a}} \cdot 0 = 0 \Rightarrow \underline{\mathsf{a}} = 0; \text{MP} \rhd 0 = 0 \Rightarrow \underline{\mathsf{a}} \cdot 0 = 0 \Rightarrow \underline{\mathsf{a}} = 0 \rhd 0 = 0 \gg \underline{\mathsf{a}} \cdot 0 = 0 \Rightarrow \underline{\mathsf{a}} = 0; \text{Ded} \rhd \forall \underline{\mathsf{a}} \colon \neg 0 = 0 \vdash \underline{\mathsf{a}} \cdot 0 = 0 \Rightarrow \underline{\mathsf{a}} = 0 \gg \neg 0 = 0 \Rightarrow \underline{\mathsf{a}} \cdot 0 = 0 \Rightarrow \underline{\mathsf{a}} = 0 \rceil, \text{p}_0, \mathsf{c})]$

$[\text{Prop } 3.5\text{e}_2 \stackrel{\text{stmt}}{\rightarrow} \text{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \neg \underline{\mathsf{b}} = 0 \Rightarrow \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0 \Rightarrow \underline{\mathsf{a}} = 0 \Rightarrow \neg \underline{\mathsf{b}}' = 0 \Rightarrow \underline{\mathsf{a}} \cdot \underline{\mathsf{b}}' = 0 \Rightarrow \underline{\mathsf{a}} = 0]$

$[\text{Prop } 3.5\text{e}_2 \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \neg \underline{\mathsf{b}} = 0 \Rightarrow \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0 \Rightarrow \underline{\mathsf{a}} = 0 \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \neg \underline{\mathsf{b}}' = 0 \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \cdot \underline{\mathsf{b}}' = 0 \vdash \text{S8} \gg \underline{\mathsf{a}} \cdot \underline{\mathsf{b}}' = \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} + \underline{\mathsf{a}}; \text{S1} \rhd \underline{\mathsf{a}} \cdot \underline{\mathsf{b}}' =$

$\underline{a}\cdot\underline{b}+\underline{a}\rhd\underline{a}\cdot\underline{b}'=0\gg\underline{a}\cdot\underline{b}+\underline{a}=0;\text{Prop 3.5d}\gg\underline{a}\cdot\underline{b}+\underline{a}=0\Rightarrow\neg\underline{a}\cdot\underline{b}=0\Rightarrow\neg\underline{a}=0;\text{MP}\rhd\underline{a}\cdot\underline{b}+\underline{a}=0\Rightarrow\neg\underline{a}\cdot\underline{b}=0\Rightarrow\neg\underline{a}=0\rhd\underline{a}\cdot\underline{b}+\underline{a}=0\gg\neg\underline{a}\cdot\underline{b}=0\Rightarrow\neg\underline{a}=0;\text{Con2}\rhd\neg\underline{a}\cdot\underline{b}=0\Rightarrow\neg\underline{a}=0\gg\underline{a}=0;\text{Ded}\rhd\forall\underline{a}\colon\forall\underline{b}\colon\underline{a}\cdot\underline{b}'=0\vdash\underline{a}=0\gg\underline{a}\cdot\underline{b}'=0\Rightarrow\underline{a}=0;\text{Ded}\rhd\forall\underline{a}\colon\forall\underline{b}\colon\neg\underline{b}'=0\vdash\underline{a}\cdot\underline{b}'=0\Rightarrow\underline{a}=0\gg\neg\underline{b}'=0\Rightarrow\underline{a}\cdot\underline{b}'=0\Rightarrow\underline{a}=0;\text{Ded}\rhd\forall\underline{a}\colon\forall\underline{b}\colon\neg\underline{b}=0\Rightarrow\underline{a}\cdot\underline{b}=0\Rightarrow\underline{a}=0\vdash\neg\underline{b}'=0\Rightarrow\underline{a}\cdot\underline{b}'=0\Rightarrow\underline{a}=0\gg\neg\underline{b}=0\Rightarrow\underline{a}\cdot\underline{b}=0\Rightarrow\underline{a}=0\Rightarrow\neg\underline{b}'=0\Rightarrow\underline{a}\cdot\underline{b}'=0\Rightarrow\underline{a}=0\rceil,p_0,c)]$

$[\text{Prop 3.5e}\overset{\text{stmt}}{\rightarrow}S\vdash\forall\underline{a}\colon\forall\underline{b}\colon\neg\underline{b}=0\Rightarrow\underline{a}\cdot\underline{b}=0\Rightarrow\underline{a}=0]$

$[\text{Prop 3.5e}\overset{\text{proof}}{\rightarrow}\lambda c.\lambda x.\mathcal{P}(\lceil S\vdash\forall\underline{a}\colon\forall\underline{b}\colon\text{Prop 3.5e}_1\gg\neg0=0\Rightarrow\bar{x}\cdot0=0\Rightarrow\bar{x}=0;\text{Prop 3.5e}_2\gg\neg\bar{y}=0\Rightarrow\bar{x}\cdot\bar{y}=0\Rightarrow\bar{x}=0\Rightarrow\neg\bar{y}'=0\Rightarrow\bar{x}\cdot\bar{y}'=0\Rightarrow\bar{x}=0;S9\,@\,\bar{y}\rhd\neg0=0\Rightarrow\bar{x}\cdot0=0\Rightarrow\bar{x}=0\rhd\neg\bar{y}=0\Rightarrow\bar{x}\cdot\bar{y}=0\Rightarrow\bar{x}=0\Rightarrow\neg\bar{y}'=0\Rightarrow\bar{x}\cdot\bar{y}'=0\Rightarrow\bar{x}=0\gg\neg\bar{y}=0\Rightarrow\bar{x}\cdot\bar{y}=0\Rightarrow\bar{x}=0;\text{Ded}\rhd\neg\bar{y}=0\Rightarrow\bar{x}\cdot\bar{y}=0\Rightarrow\bar{x}=0\gg\neg\underline{b}=0\Rightarrow\underline{a}\cdot\underline{b}=0\Rightarrow\underline{a}=0\rceil,p_0,c)]$

6.8 3.5f

$[\text{Prop 3.5f}_1\overset{\text{stmt}}{\rightarrow}S\vdash\forall\underline{a}\colon\underline{a}+0=0'\Rightarrow\neg\neg\underline{a}=0\Rightarrow\neg0=0'\Rightarrow\neg\underline{a}=0'\Rightarrow\neg0=0]$

$[\text{Prop 3.5f}_1\overset{\text{proof}}{\rightarrow}\lambda c.\lambda x.\mathcal{P}(\lceil S\vdash\forall\underline{a}\colon\forall\underline{a}\colon\underline{a}+0=0'\vdash S5\gg\underline{a}+0=\underline{a};S1\rhd\underline{a}+0=\underline{a}\rhd\underline{a}+0=0'\gg\underline{a}=0';\text{Prop 3.2a}\gg0=0;\text{Con}\rhd\underline{a}=0'\rhd0=0\gg\neg\underline{a}=0'\Rightarrow\neg0=0;\text{Dis2}\rhd\neg\underline{a}=0'\Rightarrow\neg0=0\gg\neg\neg\underline{a}=0\Rightarrow\neg0=0'\Rightarrow\neg\underline{a}=0'\Rightarrow\neg0=0;\text{Ded}\rhd\forall\underline{a}\colon\underline{a}+0=0'\vdash\neg\neg\underline{a}=0\Rightarrow\neg0=0'\Rightarrow\neg\underline{a}=0'\Rightarrow\neg0=0\gg\underline{a}+0=0'\Rightarrow\neg\neg\underline{a}=0\Rightarrow\neg0=0'\Rightarrow\neg\underline{a}=0'\Rightarrow\neg0=0\rceil,p_0,c)]$

$[\text{Prop 3.5f}_2\overset{\text{stmt}}{\rightarrow}S\vdash\forall\underline{a}\colon\forall\underline{b}\colon\underline{a}+\underline{b}=0'\Rightarrow\neg\neg\underline{a}=0\Rightarrow\neg\underline{b}=0'\Rightarrow\neg\underline{a}=0'\Rightarrow\neg\underline{b}=0\Rightarrow\underline{a}+\underline{b}'=0'\Rightarrow\neg\neg\underline{a}=0\Rightarrow\neg\underline{b}'=0'\Rightarrow\neg\underline{a}=0'\Rightarrow\neg\underline{b}'=0]$

$[\text{Prop 3.5f}_2\overset{\text{proof}}{\rightarrow}\lambda c.\lambda x.\mathcal{P}(\lceil S\vdash\forall\underline{a}\colon\forall\underline{b}\colon\forall\underline{a}\colon\forall\underline{b}\colon\underline{a}+\underline{b}=0'\Rightarrow\neg\neg\underline{a}=0\Rightarrow\neg\underline{b}=0'\Rightarrow\neg\underline{a}=0'\Rightarrow\neg\underline{b}=0\vdash\forall\underline{a}\colon\forall\underline{b}\colon\underline{a}+\underline{b}'=0'\vdash S6\gg\underline{a}+\underline{b}'=\underline{a}+\underline{b}';S1\rhd\underline{a}+\underline{b}'=\underline{a}+\underline{b}'\rhd\underline{a}+\underline{b}'=0'\gg\underline{a}+\underline{b}'=0';S4\rhd\underline{a}+\underline{b}'=0'\gg\underline{a}+\underline{b}=0;\text{Prop 3.5d}\gg\underline{a}+\underline{b}=0\Rightarrow\neg\underline{a}=0\Rightarrow\neg\underline{b}=0;\text{MP}\rhd\underline{a}+\underline{b}=0\Rightarrow\neg\underline{a}=0\Rightarrow\neg\underline{b}=0\rhd\underline{a}+\underline{b}=0\gg\neg\underline{a}=0\Rightarrow\neg\underline{b}=0;\text{Con1}\rhd\neg\underline{a}=0\Rightarrow\neg\underline{b}=0\gg\underline{a}=0;\text{Con2}\rhd\neg\underline{a}=0\Rightarrow\neg\underline{b}=0\gg\underline{b}=0;S2\rhd\underline{b}=0\gg\underline{b}'=0';\text{Con}\rhd\underline{a}=0\rhd\underline{b}'=0'\gg\neg\underline{a}=0\Rightarrow\neg\underline{b}'=0';\text{Dis1}\rhd\neg\underline{a}=0\Rightarrow\neg\underline{b}'=0'\gg\neg\neg\underline{a}=0\Rightarrow\neg\underline{b}'=0'\Rightarrow\neg\underline{a}=0'\Rightarrow\neg\underline{b}'=0;\text{Ded}\rhd\forall\underline{a}\colon\forall\underline{b}\colon\underline{a}+\underline{b}'=0'\vdash\neg\neg\underline{a}=0\Rightarrow\neg\underline{b}'=0'\Rightarrow\neg\underline{a}=0'\Rightarrow\neg\underline{b}'=0\gg\underline{a}+\underline{b}'=0'\Rightarrow\neg\neg\underline{a}=0\Rightarrow\neg\underline{b}'=0'\Rightarrow\neg\underline{a}=0'\Rightarrow\neg\underline{b}'=0;\text{Ded}\rhd\forall\underline{a}\colon\forall\underline{b}\colon\underline{a}+\underline{b}=0'\Rightarrow\neg\neg\underline{a}=0\Rightarrow\neg\underline{b}=0'\Rightarrow\neg\underline{a}=0'\Rightarrow\neg\underline{b}=0\vdash\underline{a}+\underline{b}'=0'\Rightarrow\neg\neg\underline{a}=0\Rightarrow\neg\underline{b}'=0'\Rightarrow\neg\underline{a}=0'\Rightarrow\neg\underline{b}'=0\gg\underline{a}+\underline{b}=0'\Rightarrow\neg\neg\underline{a}=0\Rightarrow\neg\underline{b}=0'\Rightarrow\neg\underline{a}=0'\Rightarrow\neg\underline{b}=0\Rightarrow\underline{a}+\underline{b}'=0'\Rightarrow\neg\neg\underline{a}=0\Rightarrow\neg\underline{b}'=0'\Rightarrow\neg\underline{a}=0'\Rightarrow\neg\underline{b}'=0\rceil,p_0,c)]$

$[\text{Prop 3.5f}\overset{\text{stmt}}{\rightarrow}S\vdash\forall\underline{a}\colon\forall\underline{b}\colon\underline{a}+\underline{b}=0'\Rightarrow\neg\neg\underline{a}=0\Rightarrow\neg\underline{b}=0'\Rightarrow\neg\underline{a}=0'\Rightarrow\neg\underline{b}=0]$

$[\text{Prop 3.5f}\overset{\text{proof}}{\rightarrow}\lambda c.\lambda x.\mathcal{P}(\lceil S\vdash\forall\underline{a}\colon\forall\underline{b}\colon\text{Prop 3.5f}_1\gg\bar{x}+0=0'\Rightarrow\neg\neg\bar{x}=0\Rightarrow\neg0=0'\Rightarrow\neg\bar{x}=0'\Rightarrow\neg0=0;\text{Prop 3.5f}_2\gg\bar{x}+\bar{y}=0'\Rightarrow\neg\neg\bar{x}=0\Rightarrow\neg\bar{y}=$

$0' \Rightarrow \neg\bar{\mathsf{x}} = 0' \Rightarrow \neg\bar{\mathsf{y}} = 0 \Rightarrow \bar{\mathsf{x}} + \bar{\mathsf{y}}' = 0' \Rightarrow \neg\neg\bar{\mathsf{x}} = 0 \Rightarrow \neg\bar{\mathsf{y}}' = 0' \Rightarrow \neg\bar{\mathsf{x}} = 0' \Rightarrow \neg\bar{\mathsf{y}}' = 0; \mathrm{S9} @ \bar{\mathsf{y}} \rhd \bar{\mathsf{x}} + 0 = 0' \Rightarrow \neg\neg\bar{\mathsf{x}} = 0 \Rightarrow \neg 0 = 0' \Rightarrow \neg\bar{\mathsf{x}} = 0' \Rightarrow \neg 0 = 0 \rhd \bar{\mathsf{x}} + \bar{\mathsf{y}} = 0' \Rightarrow \neg\neg\bar{\mathsf{x}} = 0 \Rightarrow \neg\bar{\mathsf{y}} = 0' \Rightarrow \neg\bar{\mathsf{x}} = 0' \Rightarrow \neg\bar{\mathsf{y}} = 0 \Rightarrow \bar{\mathsf{x}} + \bar{\mathsf{y}}' = 0' \Rightarrow \neg\neg\bar{\mathsf{x}} = 0 \Rightarrow \neg\bar{\mathsf{y}}' = 0' \Rightarrow \neg\bar{\mathsf{x}} = 0' \Rightarrow \neg\bar{\mathsf{y}}' = 0 \gg \bar{\mathsf{x}} + \bar{\mathsf{y}} = 0' \Rightarrow \neg\neg\bar{\mathsf{x}} = 0 \Rightarrow \neg\bar{\mathsf{y}} = 0' \Rightarrow \neg\bar{\mathsf{x}} = 0' \Rightarrow \neg\bar{\mathsf{y}} = 0; \mathrm{Ded} \rhd \bar{\mathsf{x}} + \bar{\mathsf{y}} = 0' \Rightarrow \neg\neg\bar{\mathsf{x}} = 0 \Rightarrow \neg\bar{\mathsf{y}} = 0' \Rightarrow \neg\bar{\mathsf{x}} = 0' \Rightarrow \neg\bar{\mathsf{y}} = 0 \gg \underline{\mathsf{a}} + \underline{\mathsf{b}} = 0' \Rightarrow \neg\neg\underline{\mathsf{a}} = 0 \Rightarrow \neg\underline{\mathsf{b}} = 0' \Rightarrow \neg\underline{\mathsf{a}} = 0' \Rightarrow \neg\underline{\mathsf{b}} = 0\rceil, \mathrm{p}_0, \mathsf{c})]$

6.9 3.5g

$[\mathrm{Prop\ 3.5g}_1 \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \forall\underline{\mathsf{a}}\colon \underline{\mathsf{a}} \cdot 0 = 0' \Rightarrow \neg\underline{\mathsf{a}} = 0' \Rightarrow \neg 0 = 0']$

$[\mathrm{Prop\ 3.5g}_1 \stackrel{\mathrm{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil \mathrm{S} \vdash \forall\underline{\mathsf{a}}\colon \forall\underline{\mathsf{a}}\colon \underline{\mathsf{a}} \cdot 0 = 0' \vdash \mathrm{S7} \gg \underline{\mathsf{a}} \cdot 0 = 0; \mathrm{S1} \rhd \underline{\mathsf{a}} \cdot 0 = 0 \rhd \underline{\mathsf{a}} \cdot 0 = 0' \gg 0 = 0'; \mathrm{S3} \gg \neg 0 = 0'; \mathrm{Lem1.11c} \gg \neg 0 = 0' \Rightarrow 0 = 0' \Rightarrow \neg\underline{\mathsf{a}} = 0' \Rightarrow \neg 0 = 0'; \mathrm{MP} \rhd \neg 0 = 0' \Rightarrow 0 = 0' \Rightarrow \neg\underline{\mathsf{a}} = 0' \Rightarrow \neg 0 = 0' \rhd \neg 0 = 0' \gg 0 = 0' \Rightarrow \neg\underline{\mathsf{a}} = 0' \Rightarrow \neg 0 = 0'; \mathrm{MP} \rhd 0 = 0' \Rightarrow \neg\underline{\mathsf{a}} = 0' \Rightarrow \neg 0 = 0' \rhd 0 = 0' \gg \neg\underline{\mathsf{a}} = 0' \Rightarrow \neg 0 = 0'; \mathrm{Ded} \rhd \forall\underline{\mathsf{a}}\colon \underline{\mathsf{a}} \cdot 0 = 0' \vdash \neg\underline{\mathsf{a}} = 0' \Rightarrow \neg 0 = 0' \gg \underline{\mathsf{a}} \cdot 0 = 0' \Rightarrow \neg\underline{\mathsf{a}} = 0' \Rightarrow \neg 0 = 0'\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{Prop\ 3.5g}_2 \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \forall\underline{\mathsf{a}}\colon \forall\underline{\mathsf{b}}\colon \neg\underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0 \Rightarrow \neg\underline{\mathsf{a}} = 0' \Rightarrow \neg\underline{\mathsf{a}} = 0' \Rightarrow \neg\underline{\mathsf{b}}' = 0']$

$[\mathrm{Prop\ 3.5g}_2 \stackrel{\mathrm{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil \mathrm{S} \vdash \forall\underline{\mathsf{a}}\colon \forall\underline{\mathsf{b}}\colon \forall\underline{\mathsf{a}}\colon \forall\underline{\mathsf{b}}\colon \neg\underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0 \Rightarrow \neg\underline{\mathsf{a}} = 0' \vdash \mathrm{Con2} \rhd \neg\underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0 \Rightarrow \neg\underline{\mathsf{a}} = 0' \gg \underline{\mathsf{a}} = 0'; \mathrm{Con1} \rhd \neg\underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0 \Rightarrow \neg\underline{\mathsf{a}} = 0' \gg \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0; \mathrm{S3} \gg \neg 0 = 0'; \mathrm{H10} \rhd \neg 0 = 0' \gg \neg 0' = 0; \mathrm{S1}'' \gg \underline{\mathsf{a}} = 0' \Rightarrow \underline{\mathsf{a}} = 0 \Rightarrow 0' = 0; \mathrm{H3} \rhd \underline{\mathsf{a}} = 0' \Rightarrow \underline{\mathsf{a}} = 0 \Rightarrow 0' = 0 \rhd \underline{\mathsf{a}} = 0' \rhd \neg 0' = 0 \gg \neg\underline{\mathsf{a}} = 0; \mathrm{Prop\ 3.2n} \gg \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = \underline{\mathsf{b}} \cdot \underline{\mathsf{a}}; \mathrm{S1} \rhd \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = \underline{\mathsf{b}} \cdot \underline{\mathsf{a}} \rhd \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0 \gg \underline{\mathsf{b}} \cdot \underline{\mathsf{a}} = 0; \mathrm{Prop\ 3.5e} \gg \neg\underline{\mathsf{a}} = 0 \Rightarrow \underline{\mathsf{b}} \cdot \underline{\mathsf{a}} = 0 \Rightarrow \underline{\mathsf{b}} = 0; \mathrm{MP} \rhd \neg\underline{\mathsf{a}} = 0 \Rightarrow \underline{\mathsf{b}} \cdot \underline{\mathsf{a}} = 0 \Rightarrow \underline{\mathsf{b}} = 0 \rhd \neg\underline{\mathsf{a}} = 0 \gg \underline{\mathsf{b}} \cdot \underline{\mathsf{a}} = 0 \Rightarrow \underline{\mathsf{b}} = 0; \mathrm{MP} \rhd \underline{\mathsf{b}} \cdot \underline{\mathsf{a}} = 0 \Rightarrow \underline{\mathsf{b}} = 0 \rhd \underline{\mathsf{b}} \cdot \underline{\mathsf{a}} = 0 \gg \underline{\mathsf{b}} = 0; \mathrm{S2} \rhd \underline{\mathsf{b}} = 0 \gg \underline{\mathsf{b}}' = 0'; \mathrm{Con} \rhd \underline{\mathsf{a}} = 0' \rhd \underline{\mathsf{b}}' = 0' \gg \neg\underline{\mathsf{a}} = 0' \Rightarrow \neg\underline{\mathsf{b}}' = 0'; \mathrm{Ded} \rhd \forall\underline{\mathsf{a}}\colon \forall\underline{\mathsf{b}}\colon \neg\underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0 \Rightarrow \neg\underline{\mathsf{a}} = 0' \vdash \neg\underline{\mathsf{a}} = 0' \Rightarrow \neg\underline{\mathsf{b}}' = 0' \gg \neg\underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0 \Rightarrow \neg\underline{\mathsf{a}} = 0' \Rightarrow \neg\underline{\mathsf{a}} = 0' \Rightarrow \neg\underline{\mathsf{b}}' = 0'\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{Prop\ 3.5g}_3 \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \forall\underline{\mathsf{a}}\colon \forall\underline{\mathsf{b}}\colon \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0' \Rightarrow \neg\underline{\mathsf{a}} = 0' \Rightarrow \neg\underline{\mathsf{b}} = 0' \vdash \neg\underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0' \Rightarrow \neg\underline{\mathsf{a}} = 0 \Rightarrow \neg\underline{\mathsf{a}} = 0' \Rightarrow \neg\underline{\mathsf{b}}' = 0']$

$[\mathrm{Prop\ 3.5g}_3 \stackrel{\mathrm{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil \mathrm{S} \vdash \forall\underline{\mathsf{a}}\colon \forall\underline{\mathsf{b}}\colon \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0' \Rightarrow \neg\underline{\mathsf{a}} = 0' \Rightarrow \neg\underline{\mathsf{b}} = 0' \vdash \forall\underline{\mathsf{a}}\colon \forall\underline{\mathsf{b}}\colon \neg\underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0' \Rightarrow \neg\underline{\mathsf{a}} = 0 \vdash \mathrm{Con1} \rhd \neg\underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0' \Rightarrow \neg\underline{\mathsf{a}} = 0 \gg \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0'; \mathrm{MP} \rhd \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0' \Rightarrow \neg\underline{\mathsf{a}} = 0' \Rightarrow \neg\underline{\mathsf{b}} = 0' \rhd \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0' \gg \neg\underline{\mathsf{a}} = 0' \Rightarrow \neg\underline{\mathsf{b}} = 0'; \mathrm{Con1} \rhd \neg\underline{\mathsf{a}} = 0' \Rightarrow \neg\underline{\mathsf{b}} = 0' \gg \underline{\mathsf{a}} = 0'; \mathrm{Con2} \rhd \neg\underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0' \Rightarrow \neg\underline{\mathsf{a}} = 0 \gg \underline{\mathsf{a}} = 0; \mathrm{S1} \rhd \underline{\mathsf{a}} = 0 \rhd \underline{\mathsf{a}} = 0' \gg 0 = 0'; \mathrm{S3} \gg \neg 0 = 0'; \mathrm{Lem1.11c} \gg \neg 0 = 0' \Rightarrow 0 = 0' \Rightarrow \neg\underline{\mathsf{a}} = 0' \Rightarrow \neg\underline{\mathsf{b}}' = 0'; \mathrm{MP} \rhd \neg 0 = 0' \Rightarrow 0 = 0' \Rightarrow \neg\underline{\mathsf{a}} = 0' \Rightarrow \neg\underline{\mathsf{b}}' = 0' \rhd \neg 0 = 0' \gg 0 = 0' \Rightarrow \neg\underline{\mathsf{a}} = 0' \Rightarrow \neg\underline{\mathsf{b}}' = 0'; \mathrm{MP} \rhd 0 = 0' \Rightarrow \neg\underline{\mathsf{a}} = 0' \Rightarrow \neg\underline{\mathsf{b}}' = 0' \rhd 0 = 0' \gg \neg\underline{\mathsf{a}} = 0' \Rightarrow \neg\underline{\mathsf{b}}' = 0'; \mathrm{Ded} \rhd \forall\underline{\mathsf{a}}\colon \forall\underline{\mathsf{b}}\colon \neg\underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0' \Rightarrow \neg\underline{\mathsf{a}} = 0 \vdash \neg\underline{\mathsf{a}} = 0' \Rightarrow \neg\underline{\mathsf{b}}' = 0' \gg \neg\underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0' \Rightarrow \neg\underline{\mathsf{a}} = 0 \Rightarrow \neg\underline{\mathsf{a}} = 0' \Rightarrow \neg\underline{\mathsf{b}}' = 0'\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{Prop\ 3.5g}_4 \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \forall\underline{\mathsf{a}}\colon \forall\underline{\mathsf{b}}\colon \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0' \Rightarrow \neg\underline{\mathsf{a}} = 0' \Rightarrow \neg\underline{\mathsf{b}} = 0' \Rightarrow \underline{\mathsf{a}} \cdot \underline{\mathsf{b}}' = 0' \Rightarrow \neg\underline{\mathsf{a}} = 0' \Rightarrow \neg\underline{\mathsf{b}}' = 0']$

$[\mathrm{Prop\ 3.5g_4} \stackrel{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{S} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0' \Rightarrow \neg \underline{\mathsf{a}} = 0' \Rightarrow \neg \underline{\mathsf{b}} = 0' \vdash \underline{\mathsf{a}} \cdot \underline{\mathsf{b}}' = 0' \vdash \mathrm{S8} \gg \underline{\mathsf{a}} \cdot \underline{\mathsf{b}}' = \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} + \underline{\mathsf{a}}; \mathrm{S1} \rhd \underline{\mathsf{a}} \cdot \underline{\mathsf{b}}' = \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} + \underline{\mathsf{a}} \rhd \underline{\mathsf{a}} \cdot \underline{\mathsf{b}}' = 0' \gg \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} + \underline{\mathsf{a}} = 0'; \mathrm{Prop\ 3.5f} \gg \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} + \underline{\mathsf{a}} = 0' \Rightarrow \neg\neg \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0 \Rightarrow \neg \underline{\mathsf{a}} = 0' \Rightarrow \neg \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0' \Rightarrow \neg \underline{\mathsf{a}} = 0; \mathrm{MP} \rhd \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} + \underline{\mathsf{a}} = 0' \Rightarrow \neg\neg \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0 \Rightarrow \neg \underline{\mathsf{a}} = 0' \Rightarrow \neg \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0' \Rightarrow \neg \underline{\mathsf{a}} = 0 \rhd \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} + \underline{\mathsf{a}} = 0' \gg \neg\neg \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0 \Rightarrow \neg \underline{\mathsf{a}} = 0' \Rightarrow \neg \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0' \Rightarrow \neg \underline{\mathsf{a}} = 0; \mathrm{Prop\ 3.5g_2} \gg \neg \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0 \Rightarrow \neg \underline{\mathsf{a}} = 0' \Rightarrow \neg \underline{\mathsf{a}} = 0' \Rightarrow \neg \underline{\mathsf{b}}' = 0'; \mathrm{Prop\ 3.5g_3} \rhd \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0' \Rightarrow \neg \underline{\mathsf{a}} = 0' \Rightarrow \neg \underline{\mathsf{b}} = 0' \gg \neg \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0' \Rightarrow \neg \underline{\mathsf{a}} = 0 \Rightarrow \neg \underline{\mathsf{a}} = 0' \Rightarrow \neg \underline{\mathsf{b}}' = 0'; \mathrm{H11} \rhd \neg\neg \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0 \Rightarrow \neg \underline{\mathsf{a}} = 0' \Rightarrow \neg \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0' \Rightarrow \neg \underline{\mathsf{a}} = 0 \rhd \neg \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0 \Rightarrow \neg \underline{\mathsf{a}} = 0' \Rightarrow \neg \underline{\mathsf{a}} = 0' \Rightarrow \neg \underline{\mathsf{b}}' = 0' \rhd \neg \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0' \Rightarrow \neg \underline{\mathsf{a}} = 0 \Rightarrow \neg \underline{\mathsf{a}} = 0' \Rightarrow \neg \underline{\mathsf{b}}' = 0' \gg \neg \underline{\mathsf{a}} = 0' \Rightarrow \neg \underline{\mathsf{b}}' = 0'; \mathrm{Ded} \rhd \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0' \Rightarrow \neg \underline{\mathsf{a}} = 0' \Rightarrow \neg \underline{\mathsf{b}} = 0' \vdash \underline{\mathsf{a}} \cdot \underline{\mathsf{b}}' = 0' \vdash \neg \underline{\mathsf{a}} = 0' \Rightarrow \neg \underline{\mathsf{b}}' = 0' \gg \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0' \Rightarrow \neg \underline{\mathsf{a}} = 0' \Rightarrow \neg \underline{\mathsf{b}} = 0' \Rightarrow \underline{\mathsf{a}} \cdot \underline{\mathsf{b}}' = 0' \Rightarrow \neg \underline{\mathsf{a}} = 0' \Rightarrow \neg \underline{\mathsf{b}}' = 0' \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{Prop\ 3.5g} \stackrel{\mathrm{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0' \Rightarrow \neg \underline{\mathsf{a}} = 0' \Rightarrow \neg \underline{\mathsf{b}} = 0']$

$[\mathrm{Prop\ 3.5g} \stackrel{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{S} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \mathrm{Prop\ 3.5g_1} \gg \bar{\mathsf{x}} \cdot 0 = 0' \Rightarrow \neg \bar{\mathsf{x}} = 0' \Rightarrow \neg 0 = 0'; \mathrm{Prop\ 3.5g_4} \gg \bar{\mathsf{x}} \cdot \bar{\mathsf{y}} = 0' \Rightarrow \neg \bar{\mathsf{x}} = 0' \Rightarrow \neg \bar{\mathsf{y}} = 0' \Rightarrow \bar{\mathsf{x}} \cdot \bar{\mathsf{y}}' = 0' \Rightarrow \neg \bar{\mathsf{x}} = 0' \Rightarrow \neg \bar{\mathsf{y}}' = 0'; \mathrm{S9} \mathbin{@} \bar{\mathsf{y}} \rhd \bar{\mathsf{x}} \cdot 0 = 0' \Rightarrow \neg \bar{\mathsf{x}} = 0' \Rightarrow \neg 0 = 0' \rhd \bar{\mathsf{x}} \cdot \bar{\mathsf{y}} = 0' \Rightarrow \neg \bar{\mathsf{x}} = 0' \Rightarrow \neg \bar{\mathsf{y}} = 0' \Rightarrow \bar{\mathsf{x}} \cdot \bar{\mathsf{y}}' = 0' \Rightarrow \neg \bar{\mathsf{x}} = 0' \Rightarrow \neg \bar{\mathsf{y}}' = 0' \gg \bar{\mathsf{x}} \cdot \bar{\mathsf{y}} = 0' \Rightarrow \neg \bar{\mathsf{x}} = 0' \Rightarrow \neg \bar{\mathsf{y}} = 0'; \mathrm{Ded} \rhd \bar{\mathsf{x}} \cdot \bar{\mathsf{y}} = 0' \Rightarrow \neg \bar{\mathsf{x}} = 0' \Rightarrow \neg \bar{\mathsf{y}} = 0' \gg \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} = 0' \Rightarrow \neg \underline{\mathsf{a}} = 0' \Rightarrow \neg \underline{\mathsf{b}} = 0' \rceil, \mathrm{p}_0, \mathsf{c})]$

6.10 3.5h

$[\mathrm{Prop\ 3.5h} \stackrel{\mathrm{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}}\colon \neg \underline{\mathsf{a}} = 0 \Rightarrow \neg \forall_{\mathrm{obj}} \underline{\mathsf{b}}\colon \neg \underline{\mathsf{a}} = \underline{\mathsf{b}}']$
Beviset for 3.5.h ville i pyk have set ud nogenlunde som følgende:

$$t \neq 0 \Rightarrow (\exists y)(t = y')$$

Part I

$L01$	$prem$	$0 \neq 0$
$L02$	$abbr.L01$	$\neg(0 = 0)$
$L03$	$3.2.a$	$0 = 0$
$L04$	$Lemma 1.11.c$	$\neg(0 = 0) \Rightarrow ((0 = 0) \Rightarrow (\exists w)(0 = w'))$
$L05$	$MP, L02, L03$	$(\exists w)(0 = w')$
$L06$	$Ded, L05$	$(0 \neq 0) \Rightarrow (\exists w)(0 = w')$

Part II

$L01:$	*premise*	$(x \neq 0) \Rightarrow (\exists w)(x = w')$
$L02:$	$S3'$	$x' \neq 0$
$L03:$	$Taut.: \ \mathcal{A} \vee \neg \mathcal{A}$	$(x = 0) \vee \neg(x = 0)$
$L04:$	*prem.*	$(x = 0)$
$L05:$	$S2'$	$0' = x'$
$L06:$	$E4$	$(x = 0) \Rightarrow (\exists w)(x' = w')$
$L07:$	*prem*	$\neg(x = 0)$
$L08:$	$MP, L01, L07$	$(\exists w)(x = w')$
$L09:$	*rulec*	$x = b'$
$L10:$	$S2'$	$(\exists w)(x' = w')$
$L11:$	$E4$	$\neg(x = 0) \Rightarrow (\exists w)(x' = w')$
$L12$	$Lemma\ H11, L06.L11$	$(x' \neq 0) \Rightarrow (\exists w)(x' = w')$

Tilbage er blot at bruge sætning S9 for at fuldende induktionsbeviset.

7 Udsagn 3.7

Vi har kun lavet de indledende definitioner, da hele 3.7 kræver "existential rule".

7.1 Definitioner af $\neg\forall_{\mathrm{obj}}\mathsf{z}\colon \neg\neg\neg\mathsf{z} = 0 \Rightarrow \neg\mathsf{z} + \bar{\mathsf{x}} = \bar{\mathsf{y}}$

Vi har makrodefineret følgende definitioner:

$[\mathsf{x} < \mathsf{y} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} < \mathsf{y} \doteq \exists \mathsf{z}\colon (\mathsf{z} \neq 0 \wedge \mathsf{z} + \mathsf{x} = \mathsf{y})]\rceil)]$

$[\mathsf{x} \leq \mathsf{y} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \leq \mathsf{y} \doteq \mathsf{x} < \mathsf{y} \vee \mathsf{x} = \mathsf{y}]\rceil)]$

$[\mathsf{x} > \mathsf{y} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} > \mathsf{y} \doteq \mathsf{y} < \mathsf{x}]\rceil)]$

$[\mathsf{x} \geq \mathsf{y} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \geq \mathsf{y} \doteq \mathsf{y} \leq \mathsf{x}]\rceil)]$

$[\mathsf{x} \not< \mathsf{y} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \not< \mathsf{y} \doteq \neg(\mathsf{x} < \mathsf{y})]\rceil)]$

$[\mathsf{x} \not> \mathsf{y} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \not> \mathsf{y} \doteq \mathsf{y} \not< \mathsf{x}]\rceil)]$

$[\mathsf{x} \mid \mathsf{y} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \mid \mathsf{y} \doteq \exists \mathsf{z}\colon \mathsf{y} = \mathsf{x} \cdot \mathsf{z}]\rceil)]$

Ved hjælp af disse definitioner og "existential rule" ville 3.7 kunne bevises.

A Hjælpe lemmaer

Til flere af vores beviser har vi brug for nogle hjælpe-lemmaer, disse er bevist i det følgende afsnit

$[\mathrm{Lem1.11c} \stackrel{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \neg \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}]$

$[\mathrm{Cor1.10a} \stackrel{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \forall \underline{\mathsf{c}}\colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \vdash \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{c}}]$

$[\mathrm{Cor1.10b} \stackrel{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \forall \underline{\mathsf{c}}\colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \vdash \underline{\mathsf{b}} \vdash \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{c}}]$

$[\mathrm{Lem1.11a} \stackrel{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}}\colon \neg\neg \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{a}}]$

$[\mathrm{Lem1.11b} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \forall \underline{a} \colon \underline{a} \Rightarrow \neg\neg\underline{a}]$

$[\mathrm{Prop3.2c'} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \underline{a} = \underline{b} \Rightarrow \underline{b} = \underline{c} \Rightarrow \underline{a} = \underline{c}]$

$[\mathrm{S1''} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \underline{a} = \underline{b} \Rightarrow \underline{a} = \underline{c} \Rightarrow \underline{b} = \underline{c}]$

$[\mathrm{Neg'} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \neg\underline{b} \Rightarrow \neg\underline{a} \Rightarrow \neg\underline{b} \Rightarrow \underline{a} \Rightarrow \underline{b}]$

$[\mathrm{Repetition'} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \forall \underline{a} \colon \underline{a} \Rightarrow \underline{a}]$

$[\mathrm{Lem1.11e} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \underline{a} \Rightarrow \underline{b} \Rightarrow \neg\underline{b} \Rightarrow \neg\underline{a}]$

$[\mathrm{Lem1.11d} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \neg\underline{b} \Rightarrow \neg\underline{a} \Rightarrow \underline{a} \Rightarrow \underline{b}]$

$[\mathrm{Prop3.2b'} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \underline{a} = \underline{b} \Rightarrow \underline{b} = \underline{a}]$

$[\mathrm{H10} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \neg\underline{b} = \underline{a} \vdash \neg\underline{a} = \underline{b}]$

$[\mathrm{H11} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \neg\underline{a} \Rightarrow \underline{b} \vdash \underline{a} \Rightarrow \underline{c} \vdash \underline{b} \Rightarrow \underline{c} \vdash \underline{c}]$

$[\mathrm{Lem1.11g} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \underline{a} \Rightarrow \underline{b} \vdash \neg\underline{a} \Rightarrow \underline{b} \vdash \underline{b}]$

$[\mathrm{MT} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \underline{a} \Rightarrow \underline{b} \vdash \neg\underline{b} \vdash \neg\underline{a}]$

$[\mathrm{Lem1.11c} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \neg\underline{a} \Rightarrow \underline{a} \Rightarrow \underline{b}]$

$[\mathrm{Lem1.11c} \stackrel{\mathrm{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{S} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{a} \colon \forall \underline{b} \colon \neg\underline{a} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \underline{a} \vdash \mathrm{Repetition} \rhd \neg\underline{a} \gg \neg\underline{a}; \mathrm{A1'} \gg \underline{a} \Rightarrow \neg\underline{b} \Rightarrow \underline{a}; \mathrm{MP} \rhd \underline{a} \Rightarrow \neg\underline{b} \Rightarrow \underline{a} \rhd \underline{a} \gg \neg\underline{b} \Rightarrow \underline{a}; \mathrm{A1'} \gg \neg\underline{a} \Rightarrow \neg\underline{b} \Rightarrow \neg\underline{a}; \mathrm{MP} \rhd \neg\underline{a} \Rightarrow \neg\underline{b} \Rightarrow \neg\underline{a} \rhd \neg\underline{a} \gg \neg\underline{b} \Rightarrow \neg\underline{a}; \mathrm{Neg} \rhd \neg\underline{b} \Rightarrow \neg\underline{a} \rhd \neg\underline{b} \Rightarrow \underline{a} \gg \underline{b}; \mathrm{Ded} \rhd \forall \underline{a} \colon \forall \underline{b} \colon \underline{a} \vdash \underline{b} \gg \underline{a} \Rightarrow \underline{b}; \mathrm{Ded} \rhd \forall \underline{a} \colon \forall \underline{b} \colon \neg\underline{a} \vdash \underline{a} \Rightarrow \underline{b} \gg \neg\underline{a} \Rightarrow \underline{a} \Rightarrow \underline{b}\rceil, \mathrm{p}_0, c)]$

$[\mathrm{Cor1.10a} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \underline{a} \Rightarrow \underline{b} \vdash \underline{b} \Rightarrow \underline{c} \vdash \underline{a} \Rightarrow \underline{c}]$

$[\mathrm{Cor1.10a} \stackrel{\mathrm{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{S} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \underline{a} \Rightarrow \underline{b} \vdash \underline{b} \Rightarrow \underline{c} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \underline{a} \Rightarrow \underline{b} \vdash \underline{b} \Rightarrow \underline{c} \vdash \underline{a} \vdash \mathrm{MP} \rhd \underline{a} \Rightarrow \underline{b} \rhd \underline{a} \gg \underline{b}; \mathrm{MP} \rhd \underline{b} \Rightarrow \underline{c} \rhd \underline{b} \gg \underline{c}; \mathrm{Ded} \rhd \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \underline{a} \Rightarrow \underline{b} \vdash \underline{b} \Rightarrow \underline{c} \vdash \underline{a} \vdash \underline{c} \gg \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{a} \Rightarrow \underline{c}; \mathrm{MP} \rhd \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{a} \Rightarrow \underline{c} \rhd \underline{a} \Rightarrow \underline{b} \gg \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{a} \Rightarrow \underline{c}; \mathrm{MP} \rhd \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{a} \Rightarrow \underline{c} \rhd \underline{b} \Rightarrow \underline{c} \gg \underline{a} \Rightarrow \underline{c}\rceil, \mathrm{p}_0, c)]$

$[\mathrm{Cor1.10b} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \vdash \underline{b} \vdash \underline{a} \Rightarrow \underline{c}]$

$[\mathrm{Cor1.10b} \stackrel{\mathrm{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{S} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \vdash \underline{b} \vdash \mathrm{A2'} \gg \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a} \Rightarrow \underline{c}; \mathrm{MP} \rhd \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a} \Rightarrow \underline{c} \rhd \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \gg \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a} \Rightarrow \underline{c}; \mathrm{A1'} \gg \underline{b} \Rightarrow \underline{a} \Rightarrow \underline{b}; \mathrm{MP} \rhd \underline{b} \Rightarrow \underline{a} \Rightarrow \underline{b} \rhd \underline{b} \gg \underline{a} \Rightarrow \underline{b}; \mathrm{MP} \rhd \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a} \Rightarrow \underline{c} \rhd \underline{a} \Rightarrow \underline{b} \gg \underline{a} \Rightarrow \underline{c}\rceil, \mathrm{p}_0, c)]$

$[\mathrm{Lem1.11a} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \forall \underline{a} \colon \neg\neg\underline{a} \Rightarrow \underline{a}]$

$[\mathrm{Lem1.11a} \stackrel{\mathrm{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{S} \vdash \forall \underline{a}\colon \mathrm{Neg}' \gg \neg\underline{a} \Rightarrow \neg\neg\underline{a} \Rightarrow \neg\underline{a} \Rightarrow \neg\underline{a} \Rightarrow \underline{a}; \mathrm{Repetition}' \gg \neg\underline{a} \Rightarrow \neg\underline{a}; \mathrm{Cor1.10b} \rhd \neg\underline{a} \Rightarrow \neg\neg\underline{a} \Rightarrow \neg\underline{a} \Rightarrow \neg\underline{a} \Rightarrow \underline{a} \rhd \neg\underline{a} \Rightarrow \neg\underline{a} \gg \neg\underline{a} \Rightarrow \neg\neg\underline{a} \Rightarrow \underline{a}; \mathrm{A1}' \gg \neg\neg\underline{a} \Rightarrow \neg\underline{a} \Rightarrow \neg\neg\underline{a}; \mathrm{Cor1.10a} \rhd \neg\neg\underline{a} \Rightarrow \neg\underline{a} \Rightarrow \neg\neg\underline{a} \rhd \neg\underline{a} \Rightarrow \neg\neg\underline{a} \Rightarrow \underline{a} \gg \neg\neg\underline{a} \Rightarrow \underline{a}\rceil, \mathrm{p}_0, c)]$

$[\mathrm{Lem1.11b} \stackrel{\mathrm{stmt}}{\to} \mathrm{S} \vdash \forall \underline{a}\colon \underline{a} \Rightarrow \neg\neg\underline{a}]$

$[\mathrm{Lem1.11b} \stackrel{\mathrm{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{S} \vdash \forall \underline{a}\colon \mathrm{Neg}' \gg \neg\neg\neg\underline{a} \Rightarrow \neg\underline{a} \Rightarrow \neg\neg\neg\underline{a} \Rightarrow \underline{a} \Rightarrow \neg\neg\underline{a}; \mathrm{Lem1.11a} \gg \neg\neg\neg\underline{a} \Rightarrow \neg\underline{a}; \mathrm{MP} \rhd \neg\neg\neg\underline{a} \Rightarrow \neg\underline{a} \Rightarrow \neg\neg\neg\underline{a} \Rightarrow \underline{a} \Rightarrow \neg\neg\underline{a} \rhd \neg\neg\neg\underline{a} \Rightarrow \neg\underline{a} \gg \neg\neg\neg\underline{a} \Rightarrow \underline{a} \Rightarrow \neg\neg\underline{a}; \mathrm{A1}' \gg \underline{a} \Rightarrow \neg\neg\neg\underline{a} \Rightarrow \underline{a}; \mathrm{Cor1.10a} \rhd \underline{a} \Rightarrow \neg\neg\neg\underline{a} \Rightarrow \underline{a} \rhd \neg\neg\neg\underline{a} \Rightarrow \underline{a} \Rightarrow \neg\neg\underline{a} \gg \underline{a} \Rightarrow \neg\neg\underline{a}\rceil, \mathrm{p}_0, c)]$

$[\mathrm{Prop3.2c}' \stackrel{\mathrm{stmt}}{\to} \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} = \underline{b} \Rightarrow \underline{b} = \underline{c} \Rightarrow \underline{a} = \underline{c}]$

$[\mathrm{Prop3.2c}' \stackrel{\mathrm{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} = \underline{b} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{b} = \underline{c} \vdash \mathrm{Prop\ 3.2c} \rhd \underline{a} = \underline{b} \rhd \underline{b} = \underline{c} \gg \underline{a} = \underline{c}; \mathrm{Ded} \rhd \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{b} = \underline{c} \vdash \underline{a} = \underline{c} \gg \underline{b} = \underline{c} \Rightarrow \underline{a} = \underline{c}; \mathrm{Ded} \rhd \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} = \underline{b} \vdash \underline{b} = \underline{c} \Rightarrow \underline{a} = \underline{c} \gg \underline{a} = \underline{b} \Rightarrow \underline{b} = \underline{c} \Rightarrow \underline{a} = \underline{c}\rceil, \mathrm{p}_0, c)]$

$[\mathrm{S1}'' \stackrel{\mathrm{stmt}}{\to} \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} = \underline{b} \Rightarrow \underline{a} = \underline{c} \Rightarrow \underline{b} = \underline{c}]$

$[\mathrm{S1}'' \stackrel{\mathrm{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} = \underline{b} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} = \underline{c} \vdash \mathrm{S1} \rhd \underline{a} = \underline{b} \rhd \underline{a} = \underline{c} \gg \underline{b} = \underline{c}; \mathrm{Ded} \rhd \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} = \underline{c} \vdash \underline{b} = \underline{c} \gg \underline{a} = \underline{c} \Rightarrow \underline{b} = \underline{c}; \mathrm{Ded} \rhd \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} = \underline{b} \vdash \underline{a} = \underline{c} \Rightarrow \underline{b} = \underline{c} \gg \underline{a} = \underline{b} \Rightarrow \underline{a} = \underline{c} \Rightarrow \underline{b} = \underline{c}\rceil, \mathrm{p}_0, c)]$

$[\mathrm{Neg}' \stackrel{\mathrm{stmt}}{\to} \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \neg\underline{b} \Rightarrow \neg\underline{a} \Rightarrow \neg\underline{b} \Rightarrow \underline{a} \Rightarrow \underline{b}]$

$[\mathrm{Neg}' \stackrel{\mathrm{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{a}\colon \forall \underline{b}\colon \neg\underline{b} \Rightarrow \neg\underline{a} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \neg\underline{b} \Rightarrow \underline{a} \vdash \mathrm{Neg} \rhd \neg\underline{b} \Rightarrow \neg\underline{a} \rhd \neg\underline{b} \Rightarrow \underline{a} \gg \underline{b}; \mathrm{Ded} \rhd \forall \underline{a}\colon \forall \underline{b}\colon \neg\underline{b} \Rightarrow \underline{a} \vdash \underline{b} \gg \neg\underline{b} \Rightarrow \underline{a} \Rightarrow \underline{b}; \mathrm{Ded} \rhd \forall \underline{a}\colon \forall \underline{b}\colon \neg\underline{b} \Rightarrow \neg\underline{a} \vdash \neg\underline{b} \Rightarrow \underline{a} \Rightarrow \underline{b} \gg \neg\underline{b} \Rightarrow \neg\underline{a} \Rightarrow \neg\underline{b} \Rightarrow \underline{a} \Rightarrow \underline{b}\rceil, \mathrm{p}_0, c)]$

$[\mathrm{Repetition}' \stackrel{\mathrm{stmt}}{\to} \mathrm{S} \vdash \forall \underline{a}\colon \underline{a} \Rightarrow \underline{a}]$

$[\mathrm{Repetition}' \stackrel{\mathrm{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{a}\colon \underline{a} \vdash \mathrm{Repetition} \rhd \underline{a} \gg \underline{a}; \mathrm{Ded} \rhd \forall \underline{a}\colon \underline{a} \vdash \underline{a} \gg \underline{a} \Rightarrow \underline{a}\rceil, \mathrm{p}_0, c)]$

$[\mathrm{H10} \stackrel{\mathrm{stmt}}{\to} \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \neg\underline{b} = \underline{a} \vdash \neg\underline{a} = \underline{b}]$

$[\mathrm{H10} \stackrel{\mathrm{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \neg\underline{b} = \underline{a} \vdash \mathrm{Prop3.2b}' \gg \underline{a} = \underline{b} \Rightarrow \underline{b} = \underline{a}; \mathrm{Lem1.11e} \gg \underline{a} = \underline{b} \Rightarrow \underline{b} = \underline{a} \Rightarrow \neg\underline{b} = \underline{a} \Rightarrow \neg\underline{a} = \underline{b}; \mathrm{MP} \rhd \underline{a} = \underline{b} \Rightarrow \underline{b} = \underline{a} \Rightarrow \neg\underline{b} = \underline{a} \Rightarrow \neg\underline{a} = \underline{b} \rhd \underline{a} = \underline{b} \Rightarrow \underline{b} = \underline{a} \gg \neg\underline{b} = \underline{a} \Rightarrow \neg\underline{a} = \underline{b}; \mathrm{MP} \rhd \neg\underline{b} = \underline{a} \Rightarrow \neg\underline{a} = \underline{b} \rhd \neg\underline{b} = \underline{a} \gg \neg\underline{a} = \underline{b}\rceil, \mathrm{p}_0, c)]$

$[\mathrm{MT} \stackrel{\mathrm{stmt}}{\to} \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \Rightarrow \underline{b} \vdash \neg\underline{b} \vdash \neg\underline{a}]$

$[\mathrm{MT} \stackrel{\mathrm{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \Rightarrow \underline{b} \vdash \neg\underline{b} \vdash \mathrm{Lem1.11a} \gg \neg\neg\underline{a} \Rightarrow \underline{a}; \mathrm{Cor1.10a} \rhd \neg\neg\underline{a} \Rightarrow \underline{a} \rhd \underline{a} \Rightarrow \underline{b} \gg \neg\neg\underline{a} \Rightarrow \underline{b}; \forall \underline{a}\colon \forall \underline{b}\colon \neg\neg\underline{a} \Rightarrow \underline{b} \vdash \neg\underline{b} \vdash \mathrm{A1}' \gg$

$\neg\underline{b} \Rightarrow \neg\neg\underline{a} \Rightarrow \neg\underline{b}; \mathrm{MP} \rhd \neg\underline{b} \Rightarrow \neg\neg\underline{a} \Rightarrow \neg\underline{b} \rhd \neg\underline{b} \gg \neg\neg\underline{a} \Rightarrow \neg\underline{b}; \mathrm{Neg} \rhd \neg\neg\underline{a} \Rightarrow \neg\underline{b} \rhd \neg\neg\underline{a} \Rightarrow \underline{b} \gg \neg\underline{a}; \mathrm{Ded} \rhd \forall\underline{a}\colon \forall\underline{b}\colon \neg\neg\underline{a} \Rightarrow \underline{b} \vdash \neg\underline{b} \vdash \neg\underline{a} \gg \neg\neg\underline{a} \Rightarrow \underline{b} \Rightarrow \neg\underline{b} \Rightarrow \neg\underline{a}; \mathrm{MP} \rhd \neg\neg\underline{a} \Rightarrow \underline{b} \Rightarrow \neg\underline{b} \Rightarrow \neg\underline{a} \rhd \neg\neg\underline{a} \Rightarrow \underline{b} \gg \neg\underline{b} \Rightarrow \neg\underline{a}; \mathrm{MP} \rhd \neg\underline{b} \Rightarrow \neg\underline{a} \rhd \neg\underline{b} \gg \neg\underline{a}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{Lem1.11e} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \forall\underline{a}\colon \forall\underline{b}\colon \underline{a} \Rightarrow \underline{b} \Rightarrow \neg\underline{b} \Rightarrow \neg\underline{a}]$

$[\mathrm{Lem1.11e} \stackrel{\mathrm{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil \mathrm{S} \vdash \forall\underline{a}\colon \forall\underline{b}\colon \forall\underline{a}\colon \forall\underline{b}\colon \underline{a} \Rightarrow \underline{b} \vdash \mathrm{Lem1.11a} \gg \neg\neg\underline{a} \Rightarrow \underline{a}; \mathrm{Cor1.10a} \rhd \neg\neg\underline{a} \Rightarrow \underline{a} \rhd \underline{a} \Rightarrow \underline{b} \gg \neg\neg\underline{a} \Rightarrow \underline{b}; \mathrm{Lem1.11b} \gg \underline{b} \Rightarrow \neg\neg\underline{b}; \mathrm{Cor1.10a} \rhd \neg\neg\underline{a} \Rightarrow \underline{b} \rhd \underline{b} \Rightarrow \neg\neg\underline{b} \gg \neg\neg\underline{a} \Rightarrow \neg\neg\underline{b}; \mathrm{Lem1.11d} \gg \neg\neg\underline{a} \Rightarrow \neg\neg\underline{b} \Rightarrow \neg\underline{b} \Rightarrow \neg\underline{a}; \mathrm{MP} \rhd \neg\neg\underline{a} \Rightarrow \neg\neg\underline{b} \Rightarrow \neg\underline{b} \Rightarrow \neg\underline{a} \rhd \neg\neg\underline{a} \Rightarrow \neg\neg\underline{b} \gg \neg\underline{b} \Rightarrow \neg\underline{a}; \mathrm{Ded} \rhd \forall\underline{a}\colon \forall\underline{b}\colon \underline{a} \Rightarrow \underline{b} \vdash \neg\underline{b} \Rightarrow \neg\underline{a} \gg \underline{a} \Rightarrow \underline{b} \Rightarrow \neg\underline{b} \Rightarrow \neg\underline{a}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{Lem1.11d} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \forall\underline{a}\colon \forall\underline{b}\colon \neg\underline{b} \Rightarrow \neg\underline{a} \Rightarrow \underline{a} \Rightarrow \underline{b}]$

$[\mathrm{Lem1.11d} \stackrel{\mathrm{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil \mathrm{S} \vdash \forall\underline{a}\colon \forall\underline{b}\colon \forall\underline{a}\colon \forall\underline{b}\colon \neg\underline{b} \Rightarrow \neg\underline{a} \vdash \mathrm{Neg}' \gg \neg\underline{b} \Rightarrow \neg\underline{a} \Rightarrow \neg\underline{b} \Rightarrow \underline{a} \Rightarrow \underline{b}; \mathrm{A1}' \gg \underline{a} \Rightarrow \neg\underline{b} \Rightarrow \underline{a}; \mathrm{MP} \rhd \neg\underline{b} \Rightarrow \neg\underline{a} \Rightarrow \neg\underline{b} \Rightarrow \underline{a} \Rightarrow \underline{b} \rhd \neg\underline{b} \Rightarrow \neg\underline{a} \gg \neg\underline{b} \Rightarrow \underline{a} \Rightarrow \underline{b}; \mathrm{Cor1.10a} \rhd \underline{a} \Rightarrow \neg\underline{b} \Rightarrow \underline{a} \rhd \neg\underline{b} \Rightarrow \underline{a} \Rightarrow \underline{b} \gg \underline{a} \Rightarrow \underline{b}; \mathrm{Ded} \rhd \forall\underline{a}\colon \forall\underline{b}\colon \neg\underline{b} \Rightarrow \neg\underline{a} \vdash \underline{a} \Rightarrow \underline{b} \gg \neg\underline{b} \Rightarrow \neg\underline{a} \Rightarrow \underline{a} \Rightarrow \underline{b}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{Prop3.2b}' \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \forall\underline{a}\colon \forall\underline{b}\colon \underline{a} = \underline{b} \Rightarrow \underline{b} = \underline{a}]$

$[\mathrm{Prop3.2b}' \stackrel{\mathrm{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil \mathrm{S} \vdash \forall\underline{a}\colon \forall\underline{b}\colon \forall\underline{a}\colon \forall\underline{b}\colon \underline{a} = \underline{b} \vdash \mathrm{Prop\ 3.2b} \rhd \underline{a} = \underline{b} \gg \underline{b} = \underline{a}; \mathrm{Ded} \rhd \forall\underline{a}\colon \forall\underline{b}\colon \underline{a} = \underline{b} \vdash \underline{b} = \underline{a} \gg \underline{a} = \underline{b} \Rightarrow \underline{b} = \underline{a}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{Lem1.11g} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \forall\underline{a}\colon \forall\underline{b}\colon \underline{a} \Rightarrow \underline{b} \vdash \neg\underline{a} \Rightarrow \underline{b} \vdash \underline{b}]$

$[\mathrm{Lem1.11g} \stackrel{\mathrm{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil \mathrm{S} \vdash \forall\underline{a}\colon \forall\underline{b}\colon \underline{a} \Rightarrow \underline{b} \vdash \neg\underline{a} \Rightarrow \underline{b} \vdash \mathrm{Lem1.11e} \gg \underline{a} \Rightarrow \underline{b} \Rightarrow \neg\underline{b} \Rightarrow \neg\underline{a}; \mathrm{MP} \rhd \underline{a} \Rightarrow \underline{b} \Rightarrow \neg\underline{b} \Rightarrow \neg\underline{a} \rhd \underline{a} \Rightarrow \underline{b} \gg \neg\underline{b} \Rightarrow \neg\underline{a}; \mathrm{Lem1.11e} \gg \neg\underline{a} \Rightarrow \underline{b} \Rightarrow \neg\underline{b} \Rightarrow \neg\neg\underline{a}; \mathrm{MP} \rhd \neg\underline{a} \Rightarrow \underline{b} \Rightarrow \neg\underline{b} \Rightarrow \neg\neg\underline{a} \rhd \neg\underline{a} \Rightarrow \underline{b} \gg \neg\underline{b} \Rightarrow \neg\neg\underline{a}; \mathrm{Neg} \rhd \neg\underline{b} \Rightarrow \neg\neg\underline{a} \rhd \neg\underline{b} \Rightarrow \neg\underline{a} \gg \underline{b}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{H11} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \forall\underline{a}\colon \forall\underline{b}\colon \forall\underline{c}\colon \neg\underline{a} \Rightarrow \underline{b} \vdash \underline{a} \Rightarrow \underline{c} \vdash \underline{b} \Rightarrow \underline{c} \vdash \underline{c}]$

$[\mathrm{H11} \stackrel{\mathrm{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil \mathrm{S} \vdash \forall\underline{a}\colon \forall\underline{b}\colon \forall\underline{c}\colon \neg\underline{a} \Rightarrow \underline{b} \vdash \underline{a} \Rightarrow \underline{c} \vdash \underline{b} \Rightarrow \underline{c} \vdash \mathrm{Cor1.10a} \rhd \neg\underline{a} \Rightarrow \underline{b} \rhd \underline{b} \Rightarrow \underline{c} \gg \neg\underline{a} \Rightarrow \underline{c}; \mathrm{Lem1.11g} \rhd \underline{a} \Rightarrow \underline{c} \rhd \neg\underline{a} \Rightarrow \underline{c} \gg \underline{c}\rceil, \mathrm{p}_0, \mathsf{c})]$

Litteratur

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