

Udvidelse af S-reglerne

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1 Introduktion

Dette skriftlige projekt er lavet i forbindelse med Logik-kurset 2006¹.

Vores mål i denne opgave var at bevise en delmængde af udsagnene² vedrørende læresætningerne³ fra S-systemet, som er beskrevet i Mendelson [Men97] i kapitel 3.1. For lethedens skyld er alle nummereringer de samme som i Mendelson.

Mere præcist går opgaven ud på at bevise 3.2(j-o), bevise 3.4, bevise 3.5, tilføje aksiomet $x < y \Leftrightarrow \exists z : z \neq 0 \wedge z + x = y$, bevise 3.7, tilføje aksiomer, der definerer $x \mid y$, bevise 3.10 og bevise 3.11.

I afsnittene 3 til 7 er beviserne for de påviste lemmaer gennemgået. Alle trivielle hjælpelemmaer er bevist i appendix A.

2 Konklusion

Vi har ikke løst opgaven til fulde, men har dog formået at påvise 3.2j - 3.2o, 3.4, 3.5a -g. På grund af problemer med definitionen af reglen svarende til “existensial rule” side 77 i mendelson, har vi som beskrevet i afsnit 6 ikke været i stand til at kunne bevise lemmaerne fra 3.5h og frem.

Vi har dog som beskrevet i afsnit 6.10 og 6.2.2 gennemgået hvordan beviserne for 3.5h og “existensial rule” ville have set ud, hvis problemet ikke var opstået.

Vi har endvidere kort gennemgået de definitioner, som ville være nødvendige for at kunne påvise Lemma 3.7

3 S-reglerne

S-systemet er en første ordens teori, som er udviklet ud fra Peanos postulater og ved hjælp mængdelære. Det skulle være passende at bruge til at bevise basis-

¹Kursus 061004/202 Logik

²Eng.: proposition

³Eng.: theorem

resultaterne for tal-teori. Aksiomerne for S er følgende.

| | |
|--|---|
| [Theory S] | [S rule MP: $\Pi A, B: A \Rightarrow B \vdash A \vdash B$] |
| [S rule Gen: $\Pi X, A: A \vdash \forall X: A$] | [S rule Ded: $\Pi A, B: \text{Ded}(A, B) \vdash A \vdash B$] |
| | [S rule S2: $\Pi A, B: A = B \vdash A' = B'$] |
| [S rule S3: $\Pi A: -0 = A'$] | [S rule S4: $\Pi A, B: A' = B' \vdash A = B$] |
| [S rule S5: $\Pi A: A + 0 = A$] | [S rule S6: $\Pi A, B: A + B' = (A + B)'$] |
| [S rule S7: $\Pi A: A \cdot 0 = 0$] | [S rule S8: $\Pi A, B: A \cdot (B') = (A \cdot B) + A$] |
| [S rule S1: $\Pi A, B, C: A = B \vdash A = C \vdash B = C$] | |

Endvidere er reglen *S9*, som er grundlæggende for matematisk induktion, også en del af *S*-teorien. Denne regel kan på almindeligt sprog udtrykkes som:

Hvis en egenskab holder for 0 og egenskaben holder for efterfølgeren x' til et naturligt tal x , som egenskaben gælder for, så vil egenskaben gælde for alle naturlige tal.

I pyk er *S9* defineret som:

[S rule S9: $\Pi X, A, B, C: \langle B \equiv A | X := 0 \rangle \vdash \langle C \equiv A | X := X' \rangle \vdash B \vdash A \Rightarrow C \vdash A$]

Reglen for bevis ved modsætninger er den sidste grundlæggende regel, der skal bruges i beviserne. Reglen kan på almindeligt sprog udtrykkes som følgende:

Hvis man ud fra en antagelse A kan påvise at en egenskab B gælder, og at man (når antagelsen stadig gælder) kan vise at den modsatte egenskab $\neg B$ også gælder, så må det modsatte af antagelsen gælde, dvs. $\neg A$.

I pyk er reglen defineret som:

[S rule Neg: $\Pi A: \Pi B: \neg B \Rightarrow \neg A \vdash \neg B \Rightarrow A \vdash B$]

4 Udsagn 3.2

Dette udsagn indeholder de grundlæggende regneregler for tal, som opfører sig som de naturlige tal. Hjælpesætningerne eller lemmaerne er generelt set en direkte konsekvens af aksiomerne.

Alle lemmaerne i dette afsnit kan udledes ud fra de tidligere nævnte og som det er bevist i både check [Gru06] og Mendelson er der for ethvert udtryk A, B, C følgende velformulerede sætninger⁴ i systemet S:

⁴Eng.: Wellformed formulas

[S lemma Prop 3.2a: $\Pi A: A = A$]

[S lemma Prop 3.2b: $\Pi A, B: A = B \vdash B = A$]

[S lemma Prop 3.2c: $\Pi A, B, C: A = B \vdash B = C \vdash A = C$]

[S lemma Prop 3.2d: $\Pi A, B, C: A = C \vdash B = C \vdash A = B$]

[S lemma Prop 3.2e: $\Pi A, B, C: A = B \vdash A + C = B + C$]

[S lemma Prop 3.2f: $\Pi A: A = 0 + A$]

[S lemma Prop 3.2g: $\Pi A, B: A' + B = (A + B)'$]

[S lemma Prop 3.2h: $\Pi A, B: A + B = B + A$]

Da 3.2i ikke er bevist i check, (men er i Mendelson), har vi valgt for øvelsens skyld at indskrive beviset:

[S lemma Prop 3.2i: $\Pi A, B, C: A = B \vdash C + A = C + B$]

S proof of Prop 3.2i:

| | | | |
|------|---|------------------------------|---|
| L01: | Arbitrary \gg | A, B, C | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Arbitrary \gg | A, B, C | ; |
| L04: | Premise \gg | $A = B$ | ; |
| L05: | Prop 3.2e \triangleright L04 \gg | $A + C = B + C$ | ; |
| L06: | Prop 3.2h \gg | $A + C = C + A$ | ; |
| L07: | Prop 3.2h \gg | $B + C = C + B$ | ; |
| L08: | S1 \triangleright L05 \triangleright L06 \gg | $B + C = C + A$ | ; |
| L09: | Prop 3.2b \triangleright L08 \gg | $C + A = B + C$ | ; |
| L10: | Prop 3.2c \triangleright L09 \triangleright L07 \gg | $C + A = C + B$ | ; |
| L11: | Block \gg | End | ; |
| L12: | Ded \triangleright L11 \gg | $A = B \vdash C + A = C + B$ | □ |

De følgende lemmaer er opskrevet, men ikke bevist i Mendelson. Disse er blandt dem som vi ønsker at bevise.

[S lemma Prop 3.2j: $\Pi A, B, C: (A + B) + C = A + (B + C)$]

[S lemma Prop 3.2k: $\Pi A, B, C: A = B \vdash A \cdot C = B \cdot C$]

[S lemma Prop 3.2l: $\Pi A: 0 \cdot A = 0$]

[S lemma Prop 3.2m: $\Pi A, B: A' \cdot B = A \cdot B + B$]

[S lemma Prop 3.2n: $\Pi A, B: A \cdot B = B \cdot A$]

[S lemma Prop 3.2o: $\Pi A, B, C: A = B \vdash C \cdot A = C \cdot B$]

Her efter følger beviserne.

4.1 3.2j

[S lemma Prop 3.2j₁: $\Pi A, B: (A + B) + 0 = A + (B + 0)$]

S proof of Prop 3.2j₁:

| | | | |
|------|---|-----------------------------|---|
| L01: | Arbitrary \gg | A, B | ; |
| L02: | S5 \gg | $(A + B) + 0 = A + B$ | ; |
| L03: | S5 \gg | $B + 0 = B$ | ; |
| L04: | Prop 3.2i \triangleright L03 \gg | $A + (B + 0) = A + B$ | ; |
| L05: | Prop 3.2d \triangleright L02 \triangleright L04 \gg | $(A + B) + 0 = A + (B + 0)$ | □ |

[S lemma Prop 3.2j₂: $\Pi A, B, C: (A + B) + C = A + (B + C) \Rightarrow (A + B) + C' = A + (B + C')$]

S proof of Prop 3.2j₂:

| | | | |
|------|---|--|---|
| L01: | Arbitrary \gg | A, B, C | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Arbitrary \gg | A, B, C | ; |
| L04: | Premise \gg | $(A + B) + C = A + (B + C)$ | ; |
| L05: | S6 \gg | $(A + B) + C' = ((A + B) + C)'$ | ; |
| L06: | S2 \triangleright L04 \gg | $((A + B) + C)' = (A + (B + C))'$ | ; |
| L07: | Prop 3.2c \triangleright L05 \triangleright L06 \gg | $(A + B) + C' = (A + (B + C))'$ | ; |
| L08: | S6 \gg | $B + C' = (B + C)'$ | ; |
| L09: | Prop 3.2i \triangleright L08 \gg | $A + (B + C') = A + (B + C)'$ | ; |
| L10: | S6 \gg | $A + (B + C)' = (A + (B + C))'$ | ; |
| L11: | Prop 3.2c \triangleright L09 \triangleright L10 \gg | $A + (B + C') = (A + (B + C))'$ | ; |
| L12: | Prop 3.2d \triangleright L07 \triangleright L11 \gg | $(A + B) + C' = A + (B + C)'$ | ; |
| L13: | Block \gg | End | ; |
| L14: | Ded \triangleright L13 \gg | $(A + B) + C = A + (B + C) \Rightarrow$ $(A + B) + C' = A + (B + C)'$ | □ |

[S lemma Prop 3.2j: $\Pi A, B, C: (A + B) + C = A + (B + C)$]

S proof of Prop 3.2j:

| | | | |
|------|--|--|---|
| L01: | Arbitrary \gg | A, B, C | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Prop 3.2j ₁ \gg | $(x + y) + 0 = x + (y + 0)$ | ; |
| L04: | Prop 3.2j ₂ \gg | $(x + y) + z = x + (y + z) \Rightarrow$ $(x + y) + z' = x + (y + z)'$ | ; |
| L05: | S9@z \triangleright L03 \triangleright L04 \gg | $(x + y) + z = x + (y + z)$ | ; |
| L06: | Block \gg | End | ; |
| L07: | Ded \triangleright L06 \gg | $(A + B) + C = A + (B + C)$ | □ |

4.2 3.2k

[S lemma Prop 3.2k₁: $\Pi A, B: A = B \Rightarrow A \cdot 0 = B \cdot 0$]

S proof of Prop 3.2k₁:

| | | | |
|------|-----------------|--------|---|
| L01: | Arbitrary \gg | A, B | ; |
|------|-----------------|--------|---|

| | | | |
|------|---|---|-----------|
| L02: | Block \gg | Begin | ; |
| L03: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L04: | Premise \gg | $\mathcal{A} = \mathcal{B}$ | ; |
| L05: | S7 \gg | $\mathcal{A} \cdot 0 = 0$ | ; |
| L06: | S7 \gg | $\mathcal{B} \cdot 0 = 0$ | ; |
| L07: | Prop 3.2a \gg | $0 = 0$ | ; |
| L08: | Prop 3.2b \triangleright L06 \gg | $0 = \mathcal{B} \cdot 0$ | ; |
| L09: | Prop 3.2c \triangleright L05 \triangleright L08 \gg | $\mathcal{A} \cdot 0 = \mathcal{B} \cdot 0$ | ; |
| L10: | Block \gg | End | ; |
| L11: | Ded \triangleright L10 \gg | $\mathcal{A} = \mathcal{B} \Rightarrow \mathcal{A} \cdot 0 = \mathcal{B} \cdot 0$ | \square |

[S lemma Prop 3.2k₂: $\Pi \mathcal{A}, \mathcal{B}, \mathcal{C}: (\mathcal{A} = \mathcal{B} \Rightarrow \mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{C}) \Rightarrow (\mathcal{A} = \mathcal{B} \Rightarrow \mathcal{A} \cdot \mathcal{C}' = \mathcal{B} \cdot \mathcal{C}')$]

S proof of Prop 3.2k₂:

| | | | |
|------|---|---|-----------|
| L01: | Arbitrary \gg | $\mathcal{A}, \mathcal{B}, \mathcal{C}$ | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Arbitrary \gg | $\mathcal{A}, \mathcal{B}, \mathcal{C}$ | ; |
| L04: | Premise \gg | $\mathcal{A} = \mathcal{B} \Rightarrow \mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{C}$ | ; |
| L05: | Premise \gg | $\mathcal{A} = \mathcal{B}$ | ; |
| L06: | L04 \sqsubseteq L05 \gg | $\mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{C}$ | ; |
| L07: | S8 \gg | $\mathcal{A} \cdot \mathcal{C}' = \mathcal{A} \cdot \mathcal{C} + \mathcal{A}$ | ; |
| L08: | S8 \gg | $\mathcal{B} \cdot \mathcal{C}' = \mathcal{B} \cdot \mathcal{C} + \mathcal{B}$ | ; |
| L09: | Prop 3.2e \triangleright L06 \gg | $(\mathcal{A} \cdot \mathcal{C}) + \mathcal{A} = (\mathcal{B} \cdot \mathcal{C}) + \mathcal{A}$ | ; |
| L10: | Prop 3.2i \triangleright L05 \gg | $(\mathcal{B} \cdot \mathcal{C}) + \mathcal{A} = (\mathcal{B} \cdot \mathcal{C}) + \mathcal{B}$ | ; |
| L11: | Prop 3.2c \triangleright L09 \triangleright L10 \gg | $\mathcal{A} \cdot \mathcal{C} + \mathcal{A} = \mathcal{B} \cdot \mathcal{C} + \mathcal{B}$ | ; |
| L12: | Prop 3.2c \triangleright L07 \triangleright L11 \gg | $\mathcal{A} \cdot \mathcal{C}' = \mathcal{B} \cdot \mathcal{C} + \mathcal{B}$ | ; |
| L13: | Prop 3.2d \triangleright L12 \triangleright L08 \gg | $\mathcal{A} \cdot \mathcal{C}' = \mathcal{B} \cdot \mathcal{C}'$ | ; |
| L14: | Block \gg | End | ; |
| L15: | Ded \triangleright L14 \gg | $(\mathcal{A} = \mathcal{B} \Rightarrow \mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{C}) \Rightarrow (\mathcal{A} = \mathcal{B} \Rightarrow \mathcal{A} \cdot \mathcal{C}' = \mathcal{B} \cdot \mathcal{C}')$ | \square |

[S lemma Prop 3.2k: $\Pi \mathcal{A}, \mathcal{B}, \mathcal{C}: \mathcal{A} = \mathcal{B} \vdash \mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{C}$]

S proof of Prop 3.2k:

| | | | |
|------|--|---|-----------|
| L01: | Arbitrary \gg | $\mathcal{A}, \mathcal{B}, \mathcal{C}$ | ; |
| L02: | Premise \gg | $\mathcal{A} = \mathcal{B}$ | ; |
| L03: | Block \gg | Begin | ; |
| L04: | Prop 3.2k ₁ \gg | $x = y \Rightarrow x \cdot 0 = y \cdot 0$ | ; |
| L05: | Prop 3.2k ₂ \gg | $(x = y \Rightarrow x \cdot z = y \cdot z) \Rightarrow (x = y \Rightarrow x \cdot z' = y \cdot z')$ | ; |
| L06: | S9@z \triangleright L04 \triangleright L05 \gg | $x = y \Rightarrow x \cdot z = y \cdot z$ | ; |
| L07: | Block \gg | End | ; |
| L08: | Ded \triangleright L07 \gg | $\mathcal{A} = \mathcal{B} \Rightarrow \mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{C}$ | ; |
| L09: | L08 \sqsubseteq L02 \gg | $\mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{C}$ | \square |

4.3 3.2l

[S lemma Prop 3.2l₂: $\Pi A: 0 \cdot \mathcal{A} = 0 \Rightarrow 0 \cdot \mathcal{A}' = 0$]

S proof of Prop 3.2l₂:

| | | | |
|------|---|--|---|
| L01: | Arbitrary \gg | \mathcal{A} | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Arbitrary \gg | \mathcal{A} | ; |
| L04: | Premise \gg | $0 \cdot \mathcal{A} = 0$ | ; |
| L05: | S8 \gg | $0 \cdot \mathcal{A}' = 0 \cdot \mathcal{A} + 0$ | ; |
| L06: | S5 \gg | $0 \cdot \mathcal{A} + 0 = 0 \cdot \mathcal{A}$ | ; |
| L07: | Prop 3.2c \triangleright L05 \triangleright L06 \gg | $0 \cdot \mathcal{A}' = 0 \cdot \mathcal{A}$ | ; |
| L08: | Prop 3.2c \triangleright L07 \triangleright L04 \gg | $0 \cdot \mathcal{A}' = 0$ | ; |
| L09: | Block \gg | End | ; |
| L10: | Ded \triangleright L09 \gg | $0 \cdot \mathcal{A} = 0 \Rightarrow 0 \cdot \mathcal{A}' = 0$ | □ |

[S lemma Prop 3.2l: $\Pi A: 0 \cdot \mathcal{A} = 0$]

S proof of Prop 3.2l:

| | | | |
|------|--|--|---|
| L01: | Arbitrary \gg | \mathcal{A} | ; |
| L02: | Block \gg | Begin | ; |
| L03: | S7 \gg | $0 \cdot 0 = 0$ | ; |
| L04: | Prop 3.2l ₂ \gg | $0 \cdot x = 0 \Rightarrow 0 \cdot x' = 0$ | ; |
| L05: | S9@x \triangleright L03 \triangleright L04 \gg | $0 \cdot x = 0$ | ; |
| L06: | Block \gg | End | ; |
| L07: | Ded \triangleright L06 \gg | $0 \cdot \mathcal{A} = 0$ | □ |

4.4 3.2m

[S lemma Prop 3.2m₁: $\Pi A: \mathcal{A}' \cdot 0 = \mathcal{A} \cdot 0 + 0$]

S proof of Prop 3.2m₁:

| | | | |
|------|---|--|---|
| L01: | Arbitrary \gg | \mathcal{A} | ; |
| L02: | S7 \gg | $\mathcal{A}' \cdot 0 = 0$ | ; |
| L03: | Prop 3.2f \gg | $0 = 0 + 0$ | ; |
| L04: | S7 \gg | $\mathcal{A} \cdot 0 = 0$ | ; |
| L05: | Prop 3.2b \triangleright L04 \gg | $0 = \mathcal{A} \cdot 0$ | ; |
| L06: | Prop 3.2e \triangleright L05 \gg | $0 + 0 = \mathcal{A} \cdot 0 + 0$ | ; |
| L07: | Prop 3.2c \triangleright L03 \triangleright L06 \gg | $0 = \mathcal{A} \cdot 0 + 0$ | ; |
| L08: | Prop 3.2c \triangleright L02 \triangleright L07 \gg | $\mathcal{A}' \cdot 0 = \mathcal{A} \cdot 0 + 0$ | ; |

[S lemma Prop 3.2m₂: $\Pi A, B: \mathcal{A}' \cdot B = \mathcal{A} \cdot B + B \Rightarrow \mathcal{A}' \cdot B' = \mathcal{A} \cdot B' + B'$]

S proof of Prop 3.2m₂:

| | | | |
|------|-----------------|--|---|
| L01: | Arbitrary \gg | \mathcal{A}, B | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Arbitrary \gg | \mathcal{A}, B | ; |
| L04: | Premise \gg | $\mathcal{A}' \cdot B = \mathcal{A} \cdot B + B$ | ; |

| | | | |
|------|---|--|---|
| L05: | S8 \gg | $A' \cdot B' = A' \cdot B + A'$ | ; |
| L06: | Prop 3.2e \triangleright L04 \gg | $(A' \cdot B) + A' = (A \cdot B + B) + A'$ | ; |
| L07: | S6 \gg | $B + A' = (B + A)'$ | ; |
| L08: | Prop 3.2g \gg | $B' + A = (B + A)'$ | ; |
| L09: | Prop 3.2d \triangleright L07 \triangleright L08 \gg | $B + A' = B' + A$ | ; |
| L10: | Prop 3.2h \gg | $B' + A = A + B'$ | ; |
| L11: | Prop 3.2c \triangleright L09 \triangleright L10 \gg | $B + A' = A + B'$ | ; |
| L12: | Prop 3.2i \triangleright L11 \gg | $A \cdot B + (B + A') = A \cdot B + (A + B')$ | ; |
| L13: | Prop 3.2j \gg | $(A \cdot B + A) + B' = A \cdot B + (A + B')$ | ; |
| L14: | Prop 3.2d \triangleright L12 \triangleright L13 \gg | $A \cdot B + (B + A') = (A \cdot B + A) + B'$ | ; |
| L15: | S8 \gg | $A \cdot B' = A \cdot B + A$ | ; |
| L16: | Prop 3.2e \triangleright L15 \gg | $A \cdot B' + B' = (A \cdot B + A) + B'$ | ; |
| L17: | Prop 3.2d \triangleright L14 \triangleright L16 \gg | $A \cdot B + (B + A') = A \cdot B' + B'$ | ; |
| L18: | Prop 3.2j \gg | $(A \cdot B + B) + A' = A \cdot B + (B + A')$ | ; |
| L19: | Prop 3.2c \triangleright L06 \triangleright L18 \gg | $(A' \cdot B) + A' = A \cdot B + (B + A')$ | ; |
| L20: | Prop 3.2c \triangleright L19 \triangleright L17 \gg | $(A' \cdot B) + A' = A \cdot B' + B'$ | ; |
| L21: | Prop 3.2c \triangleright L05 \triangleright L20 \gg | $A' \cdot B' = A \cdot B' + B'$ | ; |
| L22: | Block \gg | End | ; |
| L23: | Ded \triangleright L22 \gg | $A' \cdot B = A \cdot B + B \Rightarrow A' \cdot B' = A \cdot B' + B'$ | □ |

[S lemma Prop 3.2m: $\Pi A, B: A' \cdot B = A \cdot B + B$]

S proof of Prop 3.2m:

| | | | |
|------|--|--|---|
| L01: | Arbitrary \gg | A, B | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Prop 3.2m ₁ \gg | $x' \cdot 0 = x \cdot 0 + 0$ | ; |
| L04: | Prop 3.2m ₂ \gg | $(x' \cdot y = x \cdot y + y) \Rightarrow (x' \cdot y' = x \cdot y' + y')$ | ; |
| L05: | S9@y \triangleright L03 \triangleright L04 \gg | $x' \cdot y = x \cdot y + y$ | ; |
| L06: | Block \gg | End | ; |
| L07: | Ded \triangleright L06 \gg | $A' \cdot B = A \cdot B + B$ | □ |

4.5 3.2n

[S lemma Prop 3.2n₁: $\Pi A: A \cdot 0 = 0 \cdot A$]

S proof of Prop 3.2n₁:

| | | | |
|------|---|-------------------------|---|
| L01: | Arbitrary \gg | A | ; |
| L02: | S7 \gg | $A \cdot 0 = 0$ | ; |
| L03: | Prop 3.2l \gg | $0 \cdot A = 0$ | ; |
| L04: | Prop 3.2d \triangleright L02 \triangleright L03 \gg | $A \cdot 0 = 0 \cdot A$ | □ |

[S lemma Prop 3.2n₂: $\Pi A, B: A \cdot B = B \cdot A \Rightarrow A \cdot B' = B' \cdot A$]

S proof of Prop 3.2n₂:

| | | | |
|------|---|---|---|
| L01: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L04: | Premise \gg | $\mathcal{A} \cdot \mathcal{B} = \mathcal{B} \cdot \mathcal{A}$ | ; |
| L05: | S8 \gg | $\mathcal{A} \cdot \mathcal{B}' = \mathcal{A} \cdot \mathcal{B} + \mathcal{A}$ | ; |
| L06: | Prop 3.2e \triangleright L04 \gg | $\mathcal{A} \cdot \mathcal{B} + \mathcal{A} = \mathcal{B} \cdot \mathcal{A} + \mathcal{A}$ | ; |
| L07: | Prop 3.2m \gg | $\mathcal{B}' \cdot \mathcal{A} = \mathcal{B} \cdot \mathcal{A} + \mathcal{A}$ | ; |
| L08: | Prop 3.2b \triangleright L07 \gg | $\mathcal{B} \cdot \mathcal{A} + \mathcal{A} = \mathcal{B}' \cdot \mathcal{A}$ | ; |
| L09: | Prop 3.2c \triangleright L06 \triangleright L08 \gg | $\mathcal{A} \cdot \mathcal{B} + \mathcal{A} = \mathcal{B}' \cdot \mathcal{A}$ | ; |
| L10: | Prop 3.2c \triangleright L05 \triangleright L09 \gg | $\mathcal{A} \cdot \mathcal{B}' = \mathcal{B}' \cdot \mathcal{A}$ | ; |
| L11: | Block \gg | End | ; |
| L12: | Ded \triangleright L11 \gg | $\mathcal{A} \cdot \mathcal{B} = \mathcal{B} \cdot \mathcal{A} \Rightarrow \mathcal{A} \cdot \mathcal{B}' = \mathcal{B}' \cdot \mathcal{A}$ | □ |

[S lemma Prop 3.2n: $\Pi \mathcal{A}, \mathcal{B}: \mathcal{A} \cdot \mathcal{B} = \mathcal{B} \cdot \mathcal{A}$]

S proof of Prop 3.2n:

| | | | |
|------|--|---|---|
| L01: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Prop 3.2n ₁ \gg | $x \cdot 0 = 0 \cdot x$ | ; |
| L04: | Prop 3.2n ₂ \gg | $(x \cdot y = y \cdot x) \Rightarrow (x \cdot y' = y' \cdot x)$ | ; |
| L05: | S9 @ $y \triangleright$ L03 \triangleright L04 \gg | $x \cdot y = y \cdot x$ | ; |
| L06: | Block \gg | End | ; |
| L07: | Ded \triangleright L06 \gg | $\mathcal{A} \cdot \mathcal{B} = \mathcal{B} \cdot \mathcal{A}$ | □ |

4.6 3.2o

[S lemma Prop 3.2o: $\Pi \mathcal{A}, \mathcal{B}, \mathcal{C}: \mathcal{A} = \mathcal{B} \vdash \mathcal{C} \cdot \mathcal{A} = \mathcal{C} \cdot \mathcal{B}$]

S proof of Prop 3.2o:

| | | | |
|------|---|---|---|
| L01: | Arbitrary \gg | $\mathcal{A}, \mathcal{B}, \mathcal{C}$ | ; |
| L02: | Premise \gg | $\mathcal{A} = \mathcal{B}$ | ; |
| L03: | Block \gg | Begin | ; |
| L04: | Arbitrary \gg | $\mathcal{A}, \mathcal{B}, \mathcal{C}$ | ; |
| L05: | Premise \gg | $\mathcal{A} = \mathcal{B}$ | ; |
| L06: | Prop 3.2k \triangleright L05 \gg | $\mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{C}$ | ; |
| L07: | Prop 3.2n \gg | $\mathcal{A} \cdot \mathcal{C} = \mathcal{C} \cdot \mathcal{A}$ | ; |
| L08: | Prop 3.2n \gg | $\mathcal{B} \cdot \mathcal{C} = \mathcal{C} \cdot \mathcal{B}$ | ; |
| L09: | Prop 3.2c \triangleright L06 \triangleright L08 \gg | $\mathcal{A} \cdot \mathcal{C} = \mathcal{C} \cdot \mathcal{B}$ | ; |
| L10: | S1 \triangleright L07 \triangleright L09 \gg | $\mathcal{C} \cdot \mathcal{A} = \mathcal{C} \cdot \mathcal{B}$ | ; |
| L11: | Block \gg | End | ; |
| L12: | Ded \triangleright L11 \gg | $\mathcal{A} = \mathcal{B} \Rightarrow \mathcal{C} \cdot \mathcal{A} = \mathcal{C} \cdot \mathcal{B}$ | ; |
| L13: | L12 \triangleright L02 \gg | $\mathcal{C} \cdot \mathcal{A} = \mathcal{C} \cdot \mathcal{B}$ | □ |

5 Udsagn 3.4

De følgende udsagn er en udvidelse af egenskaberne ved addition og multiplikation.

[S lemma Prop 3.4a: $\Pi A, B, C: A \cdot (B + C) = (A \cdot B) + (A \cdot C)$]

[S lemma Prop 3.4b: $\Pi A, B, C: (B + C) \cdot A = (B \cdot A) + (C \cdot A)$]

[S lemma Prop 3.4c: $\Pi A, B, C: (A \cdot B) \cdot C = A \cdot (B \cdot C)$]

[S lemma Prop 3.4d: $\Pi A, B, C: A + C = B + C \Rightarrow A = B$]

5.1 3.4a

[S lemma Prop 3.4a₁: $\Pi A, B: A \cdot (B + 0) = A \cdot B + A \cdot 0$]

S proof of Prop 3.4a₁:

| | | | |
|------|---|---|---|
| L01: | Arbitrary \gg | A, B | ; |
| L02: | S5 \gg | $(B + 0) = (B)$ | ; |
| L03: | Prop 3.2o \triangleright L02 \gg | $A \cdot (B + 0) = A \cdot (B)$ | ; |
| L04: | S5 \gg | $A \cdot B + 0 = A \cdot B$ | ; |
| L05: | Prop 3.2d \triangleright L03 \triangleright L04 \gg | $A \cdot (B + 0) = A \cdot B + 0$ | ; |
| L06: | S7 \gg | $A \cdot 0 = 0$ | ; |
| L07: | Prop 3.2i \triangleright L06 \gg | $A \cdot B + A \cdot 0 = A \cdot B + 0$ | ; |
| L08: | Prop 3.2d \triangleright L05 \triangleright L07 \gg | $A \cdot (B + 0) = A \cdot B + A \cdot 0$ | □ |

[S lemma Prop 3.4a₂: $\Pi A, B, C: A \cdot (B + C) = A \cdot B + A \cdot C \Rightarrow A \cdot (B + C') = A \cdot B + A \cdot C'$]

S proof of Prop 3.4a₂:

| | | | |
|------|---|---|---|
| L01: | Arbitrary \gg | A, B, C | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Arbitrary \gg | A, B, C | ; |
| L04: | Premise \gg | $A \cdot (B + C) = A \cdot B + A \cdot C$ | ; |
| L05: | S6 \gg | $B + C' = (B + C)'$ | ; |
| L06: | Prop 3.2o \triangleright L05 \gg | $A \cdot (B + C') = A \cdot (B + C)'$ | ; |
| L07: | S8 \gg | $A \cdot (B + C)' = A \cdot (B + C) + A$ | ; |
| L08: | Prop 3.2e \triangleright L04 \gg | $(A \cdot (B + C)) + A = (A \cdot B + A \cdot C) + A$ | ; |
| L09: | Prop 3.2j \gg | $(A \cdot B + A \cdot C) + A = A \cdot B + (A \cdot C + A)$ | ; |
| L10: | Prop 3.2c \triangleright L08 \triangleright L09 \gg | $(A \cdot (B + C)) + A = A \cdot B + (A \cdot C + A)$ | ; |
| L11: | S8 \gg | $A \cdot C' = A \cdot C + A$ | ; |
| L12: | Prop 3.2i \triangleright L11 \gg | $A \cdot B + A \cdot C' = A \cdot B + (A \cdot C + A)$ | ; |
| L13: | Prop 3.2d \triangleright L10 \triangleright L12 \gg | $A \cdot (B + C) + A = A \cdot B + A \cdot C'$ | ; |
| L14: | Prop 3.2c \triangleright L07 \triangleright L13 \gg | $A \cdot (B + C)' = A \cdot B + A \cdot C'$ | ; |

| | | | |
|------|---|--|-----------|
| L15: | Prop 3.2c \triangleright L06 \triangleright L14 \gg | $\mathcal{A} \cdot (\mathcal{B} + \mathcal{C}') = \mathcal{A} \cdot \mathcal{B} + \mathcal{A} \cdot \mathcal{C}'$ | ; |
| L16: | Block \gg | End | ; |
| L17: | Ded \triangleright L16 \gg | $(\mathcal{A} \cdot (\mathcal{B} + \mathcal{C}) = \mathcal{A} \cdot \mathcal{B} + \mathcal{A} \cdot \mathcal{C}) \Rightarrow$ $\mathcal{A} \cdot (\mathcal{B} + \mathcal{C}') = \mathcal{A} \cdot \mathcal{B} + \mathcal{A} \cdot \mathcal{C}'$ | \square |

[S lemma Prop 3.4a: $\Pi \mathcal{A}, \mathcal{B}, \mathcal{C} : \mathcal{A} \cdot (\mathcal{B} + \mathcal{C}) = (\mathcal{A} \cdot \mathcal{B}) + (\mathcal{A} \cdot \mathcal{C})$]

S proof of Prop 3.4a:

| | | | |
|------|--|---|-----------|
| L01: | Arbitrary \gg | $\mathcal{A}, \mathcal{B}, \mathcal{C}$ | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Prop 3.4a ₁ \gg | $x \cdot (y + 0) = x \cdot y + x \cdot 0$ | ; |
| L04: | Prop 3.4a ₂ \gg | $(x \cdot (y + z) = x \cdot y + x \cdot z) \Rightarrow$ $(x \cdot (y + z') = x \cdot y + x \cdot z')$ | ; |
| L05: | S9@z \triangleright L03 \triangleright L04 \gg | $x \cdot (y + z) = x \cdot y + x \cdot z$ | ; |
| L06: | Block \gg | End | ; |
| L07: | Ded \triangleright L06 \gg | $\mathcal{A} \cdot (\mathcal{B} + \mathcal{C}) = \mathcal{A} \cdot \mathcal{B} + \mathcal{A} \cdot \mathcal{C}$ | \square |

5.2 3.4b

[S lemma Prop 3.4b: $\Pi \mathcal{A}, \mathcal{B}, \mathcal{C} : (\mathcal{B} + \mathcal{C}) \cdot \mathcal{A} = (\mathcal{B} \cdot \mathcal{A}) + (\mathcal{C} \cdot \mathcal{A})$]

S proof of Prop 3.4b:

| | | | |
|------|---|---|-----------|
| L01: | Arbitrary \gg | $\mathcal{A}, \mathcal{B}, \mathcal{C}$ | ; |
| L02: | Prop 3.4a \gg | $\mathcal{A} \cdot (\mathcal{B} + \mathcal{C}) = \mathcal{A} \cdot \mathcal{B} + \mathcal{A} \cdot \mathcal{C}$ | ; |
| L03: | Prop 3.2n \gg | $\mathcal{A} \cdot (\mathcal{B} + \mathcal{C}) = (\mathcal{B} + \mathcal{C}) \cdot \mathcal{A}$ | ; |
| L04: | Prop 3.2n \gg | $\mathcal{A} \cdot \mathcal{B} = \mathcal{B} \cdot \mathcal{A}$ | ; |
| L05: | Prop 3.2n \gg | $\mathcal{A} \cdot \mathcal{C} = \mathcal{C} \cdot \mathcal{A}$ | ; |
| L06: | S1 \triangleright L03 \triangleright L02 \gg | $(\mathcal{B} + \mathcal{C}) \cdot \mathcal{A} = \mathcal{A} \cdot \mathcal{B} + \mathcal{A} \cdot \mathcal{C}$ | ; |
| L07: | Prop 3.2e \triangleright L04 \gg | $\mathcal{A} \cdot \mathcal{B} + \mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{A} + \mathcal{A} \cdot \mathcal{C}$ | ; |
| L08: | Prop 3.2i \triangleright L05 \gg | $\mathcal{B} \cdot \mathcal{A} + \mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{A} + \mathcal{C} \cdot \mathcal{A}$ | ; |
| L09: | Prop 3.2c \triangleright L07 \triangleright L08 \gg | $\mathcal{A} \cdot \mathcal{B} + \mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{A} + \mathcal{C} \cdot \mathcal{A}$ | ; |
| L10: | Prop 3.2c \triangleright L06 \triangleright L09 \gg | $(\mathcal{B} + \mathcal{C}) \cdot \mathcal{A} = \mathcal{B} \cdot \mathcal{A} + \mathcal{C} \cdot \mathcal{A}$ | \square |

5.3 3.4c

[S lemma Prop 3.4c₁: $\Pi \mathcal{A}, \mathcal{B} : (\mathcal{A} \cdot \mathcal{B}) \cdot 0 = \mathcal{A} \cdot (\mathcal{B} \cdot 0)$]

S proof of Prop 3.4c₁:

| | | | |
|------|---|---|-----------|
| L01: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L02: | S7 \gg | $(\mathcal{A} \cdot \mathcal{B}) \cdot 0 = 0$ | ; |
| L03: | S7 \gg | $\mathcal{B} \cdot 0 = 0$ | ; |
| L04: | Prop 3.2o \triangleright L03 \gg | $\mathcal{A} \cdot (\mathcal{B} \cdot 0) = \mathcal{A} \cdot 0$ | ; |
| L05: | S7 \gg | $\mathcal{A} \cdot 0 = 0$ | ; |
| L06: | Prop 3.2c \triangleright L04 \triangleright L05 \gg | $\mathcal{A} \cdot (\mathcal{B} \cdot 0) = 0$ | ; |
| L07: | Prop 3.2d \triangleright L02 \triangleright L06 \gg | $(\mathcal{A} \cdot \mathcal{B}) \cdot 0 = \mathcal{A} \cdot (\mathcal{B} \cdot 0)$ | \square |

[S lemma Prop 3.4c₂: $\Pi \mathcal{A}, \mathcal{B}, \mathcal{C} : (\mathcal{A} \cdot \mathcal{B}) \cdot \mathcal{C} = \mathcal{A} \cdot (\mathcal{B} \cdot \mathcal{C}) \Rightarrow (\mathcal{A} \cdot \mathcal{B}) \cdot \mathcal{C}' = \mathcal{A} \cdot (\mathcal{B} \cdot \mathcal{C}')$]

S proof of Prop 3.4c₂:

| | | | |
|------|---|---|---|
| L01: | Arbitrary \gg | $\mathcal{A}, \mathcal{B}, \mathcal{C}$ | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Arbitrary \gg | $\mathcal{A}, \mathcal{B}, \mathcal{C}$ | ; |
| L04: | Premise \gg | $(\mathcal{A} \cdot \mathcal{B}) \cdot \mathcal{C} = \mathcal{A} \cdot (\mathcal{B} \cdot \mathcal{C})$ | ; |
| L05: | S8 \gg | $(\mathcal{A} \cdot \mathcal{B}) \cdot \mathcal{C}' = (\mathcal{A} \cdot \mathcal{B}) \cdot \mathcal{C} + (\mathcal{A} \cdot \mathcal{B})$ | ; |
| L06: | Prop 3.2e \triangleright L04 \gg | $((\mathcal{A} \cdot \mathcal{B}) \cdot \mathcal{C}) + (\mathcal{A} \cdot \mathcal{B}) = (\mathcal{A} \cdot (\mathcal{B} \cdot \mathcal{C})) + (\mathcal{A} \cdot \mathcal{B})$ | ; |
| L07: | Prop 3.2c \triangleright L05 \triangleright L06 \gg | $(\mathcal{A} \cdot \mathcal{B}) \cdot \mathcal{C}' = (\mathcal{A} \cdot (\mathcal{B} \cdot \mathcal{C})) + (\mathcal{A} \cdot \mathcal{B})$ | ; |
| L08: | Prop 3.4a \gg | $\mathcal{A} \cdot ((\mathcal{B} \cdot \mathcal{C}) + \mathcal{B}) = \mathcal{A} \cdot (\mathcal{B} \cdot \mathcal{C}) + \mathcal{A} \cdot \mathcal{B}$ | ; |
| L09: | Prop 3.2d \triangleright L07 \triangleright L08 \gg | $(\mathcal{A} \cdot \mathcal{B}) \cdot \mathcal{C}' = \mathcal{A} \cdot ((\mathcal{B} \cdot \mathcal{C}) + \mathcal{B})$ | ; |
| L10: | S8 \gg | $\mathcal{B} \cdot \mathcal{C}' = \mathcal{B} \cdot \mathcal{C} + \mathcal{B}$ | ; |
| L11: | Prop 3.2o \triangleright L10 \gg | $\mathcal{A} \cdot (\mathcal{B} \cdot \mathcal{C}') = \mathcal{A} \cdot (\mathcal{B} \cdot \mathcal{C} + \mathcal{B})$ | ; |
| L12: | Prop 3.2d \triangleright L09 \triangleright L11 \gg | $(\mathcal{A} \cdot \mathcal{B}) \cdot \mathcal{C}' = \mathcal{A} \cdot (\mathcal{B} \cdot \mathcal{C}')$ | ; |
| L13: | Block \gg | End | ; |
| L14: | Ded \triangleright L13 \gg | $(\mathcal{A} \cdot \mathcal{B}) \cdot \mathcal{C} = \mathcal{A} \cdot (\mathcal{B} \cdot \mathcal{C}) \Rightarrow (\mathcal{A} \cdot \mathcal{B}) \cdot \mathcal{C}' = \mathcal{A} \cdot (\mathcal{B} \cdot \mathcal{C}')$ | □ |

[S lemma Prop 3.4c: $\Pi \mathcal{A}, \mathcal{B}, \mathcal{C}: (\mathcal{A} \cdot \mathcal{B}) \cdot \mathcal{C} = \mathcal{A} \cdot (\mathcal{B} \cdot \mathcal{C})$]

S proof of Prop 3.4c:

| | | | |
|------|--|---|---|
| L01: | Arbitrary \gg | $\mathcal{A}, \mathcal{B}, \mathcal{C}$ | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Prop 3.4c ₁ \gg | $(x \cdot y) \cdot 0 = x \cdot (y \cdot 0)$ | ; |
| L04: | Prop 3.4c ₂ \gg | $(x \cdot y) \cdot z = x \cdot (y \cdot z) \Rightarrow (x \cdot y) \cdot z' = x \cdot (y \cdot z')$ | ; |
| L05: | S9@z \triangleright L03 \triangleright L04 \gg | $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ | ; |
| L06: | Block \gg | End | ; |
| L07: | Ded \triangleright L06 \gg | $(\mathcal{A} \cdot \mathcal{B}) \cdot \mathcal{C} = \mathcal{A} \cdot (\mathcal{B} \cdot \mathcal{C})$ | □ |

5.4 3.4d

[S lemma Prop 3.4d₁: $\Pi \mathcal{A}, \mathcal{B}: \mathcal{A} + 0 = \mathcal{B} + 0 \Rightarrow \mathcal{A} = \mathcal{B}$]

S proof of Prop 3.4d₁:

| | | | |
|------|---|---|---|
| L01: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L04: | Premise \gg | $\mathcal{A} + 0 = \mathcal{B} + 0$ | ; |
| L05: | S5 \gg | $\mathcal{A} + 0 = \mathcal{A}$ | ; |
| L06: | S5 \gg | $\mathcal{B} + 0 = \mathcal{B}$ | ; |
| L07: | S1 \triangleright L05 \triangleright L04 \gg | $\mathcal{A} = \mathcal{B} + 0$ | ; |
| L08: | Prop 3.2c \triangleright L07 \triangleright L06 \gg | $\mathcal{A} = \mathcal{B}$ | ; |
| L09: | Block \gg | End | ; |
| L10: | Ded \triangleright L09 \gg | $\mathcal{A} + 0 = \mathcal{B} + 0 \Rightarrow \mathcal{A} = \mathcal{B}$ | □ |

[S lemma Prop 3.4d₂: $\Pi A, B, C: (A + C = B + C \Rightarrow A = B) \Rightarrow A + C' = B + C' \Rightarrow A = B$]

S proof of Prop 3.4d₂:

| | | | |
|------|---|--|-----------|
| L01: | Arbitrary \gg | A, B, C | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Arbitrary \gg | A, B, C | ; |
| L04: | Premise \gg | $A + C = B + C \Rightarrow A = B$ | ; |
| L05: | Premise \gg | $A + C' = B + C'$ | ; |
| L06: | S6 \gg | $A + C' = (A + C)'$ | ; |
| L07: | S6 \gg | $B + C' = (B + C)'$ | ; |
| L08: | S1 \triangleright L06 \triangleright L05 \gg | $(A + C)' = B + C'$ | ; |
| L09: | Prop 3.2c \triangleright L08 \triangleright L07 \gg | $(A + C)' = (B + C)'$ | ; |
| L10: | S4 \triangleright L09 \gg | $(A + C) = (B + C)$ | ; |
| L11: | L04 \triangleright L10 \gg | $A = B$ | ; |
| L12: | Block \gg | End | ; |
| L13: | Ded \triangleright L12 \gg | $(A + C = B + C \Rightarrow A = B) \Rightarrow$ $A + C' = B + C' \Rightarrow A = B$ | \square |

[S lemma Prop 3.4d: $\Pi A, B, C: A + C = B + C \Rightarrow A = B$]

S proof of Prop 3.4d:

| | | | |
|------|--|--|-----------|
| L01: | Arbitrary \gg | A, B, C | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Prop 3.4d ₁ \gg | $x + 0 = y + 0 \Rightarrow x = y$ | ; |
| L04: | Prop 3.4d ₂ \gg | $(x + z = y + z \Rightarrow x = y) \Rightarrow$ $x + z' = y + z' \Rightarrow x = y$ | ; |
| L05: | S9@z \triangleright L03 \triangleright L04 \gg | $x + z = y + z \Rightarrow x = y$ | ; |
| L06: | Block \gg | End | ; |
| L07: | Ded \triangleright L06 \gg | $A + C = B + C \Rightarrow A = B$ | \square |

6 Udsagn 3.5

I denne del har vi kun kunne løse indtil h, da denne og de efterfølgende kræver reglen “existensial rule”. Dog er der i afsnit 6.2.2 gennemgået hvilket problem der forhindrer os i at bevise ”existencial rule”, og hvordan vi ville have bevist reglen, hvis problemet ikke var opstået. Endvidere er der i 6.2.2 gennemgået hvordan vi ville have løst 3.5h.

I de følgende udsagn indgår numeraler opfører sig som de naturlige tal og er defineret i forhold til 0 på følgende måde:

$$\begin{aligned}
 0 &= \bar{0} \\
 0' &= \bar{1} \\
 0'' &= \bar{2} \\
 \vdots &\quad \quad \quad \vdots \\
 0^{n*'} &= \bar{n}
 \end{aligned}$$

Dvs. hvis $\bar{0}$ er et numeral, og hvis \bar{n} er et numeral, så er \bar{n}' også.

[S lemma Prop 3.5a: $\Pi\mathcal{A}: \mathcal{A} + \bar{1} = \mathcal{A}'$]

[S lemma Prop 3.5b: $\Pi\mathcal{A}: \mathcal{A} \cdot \bar{1} = \mathcal{A}$]

[S lemma Prop 3.5c: $\Pi\mathcal{A}: \mathcal{A} \cdot \bar{2} = \mathcal{A} + \mathcal{A}$]

[S lemma Prop 3.5d: $\Pi\mathcal{A}, \mathcal{B}: \mathcal{A} + \mathcal{B} = 0 \Rightarrow \mathcal{A} = 0 \wedge \mathcal{B} = 0$]

[S lemma Prop 3.5e: $\Pi\mathcal{A}, \mathcal{B}: \mathcal{B} \neq 0 \Rightarrow (\mathcal{A} \cdot \mathcal{B} = 0 \Rightarrow \mathcal{A} = 0)$]

[S lemma Prop 3.5f: $\Pi\mathcal{A}, \mathcal{B}: \mathcal{A} + \mathcal{B} = \bar{1} \Rightarrow (\mathcal{A} = 0 \wedge \mathcal{B} = \bar{1}) \vee (\mathcal{A} = \bar{1} \wedge \mathcal{B} = 0)$]

[S lemma Prop 3.5g: $\Pi\mathcal{A}, \mathcal{B}: \mathcal{A} \cdot \mathcal{B} = \bar{1} \Rightarrow (\mathcal{A} = \bar{1} \wedge \mathcal{B} = \bar{1})$]

[S lemma Prop 3.5h: $\Pi\mathcal{A}: \mathcal{A} \neq 0 \Rightarrow \exists\mathcal{B}: \mathcal{A} = \mathcal{B}'$]

[S lemma Prop 3.5i: $\Pi\mathcal{A}, \mathcal{B}, \mathcal{C}: \mathcal{C} \neq 0 \Rightarrow (\mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{C} \Rightarrow \mathcal{A} = \mathcal{B})$]

[S lemma Prop 3.5j: $\Pi\mathcal{A}: \mathcal{A} \neq 0 \Rightarrow \mathcal{A} \neq \bar{1} \Rightarrow \exists\mathcal{B}: \mathcal{A} = \mathcal{B}''$]

Efterfølgende vil vi gennemgå de nødvendige hjælpesætninger og definitioner for at kunne bevise 3.5a-3.5g.

6.1 Definitioner i forbindelse med \wedge og \vee

Fra 3.5d bruges \wedge og \vee , som begge er makrodefinerede udtryk.

$[x \wedge y \doteq \neg(x \Rightarrow \neg y)]$

$[x \vee y \doteq (\neg x) \Rightarrow y]$

Endvidere er det klart at reglerne for introduktion og eliminering af \wedge og \vee dermed også skal bruges. Alle disse er bevist i de følgende afsnit.

6.1.1 Introduktion af \wedge

[S lemma Con: $\Pi\mathcal{A}, \mathcal{B}: \mathcal{A} \vdash \mathcal{B} \vdash \mathcal{A} \wedge \mathcal{B}$]

S proof of Con:

| | | | |
|------|---------------------------------------|---|---|
| L01: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L02: | Premise \gg | \mathcal{A} | ; |
| L03: | Premise \gg | \mathcal{B} | ; |
| L04: | Block \gg | Begin | ; |
| L05: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L06: | Premise \gg | $\mathcal{A} \Rightarrow \neg\mathcal{B}$ | ; |
| L07: | Repetition \triangleright L02 \gg | \mathcal{A} | ; |
| L08: | L06 \triangleright L07 \gg | $\neg\mathcal{B}$ | ; |

| | | | |
|------|--|---|---|
| L09: | Block \gg | End | ; |
| L10: | Ded \triangleright L09 \gg | $(\mathcal{A} \Rightarrow \neg \mathcal{B}) \Rightarrow \neg \mathcal{B}$ | ; |
| L11: | Lem1.11b \gg | $\mathcal{B} \Rightarrow \neg \neg \mathcal{B}$ | ; |
| L12: | L11 \triangleright L03 \gg | $\neg \neg \mathcal{B}$ | ; |
| L13: | MT \triangleright L10 \triangleright L12 \gg | $\neg(\mathcal{A} \Rightarrow \neg \mathcal{B})$ | □ |

6.1.2 Elimination af \wedge 1

[S lemma Con1: $\Pi \mathcal{A}, \mathcal{B}: \mathcal{A} \wedge \mathcal{B} \vdash \mathcal{A}$]

S proof of Con1:

| | | | |
|------|--|---|---|
| L01: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L02: | Premise \gg | $\neg(\mathcal{A} \Rightarrow \neg \mathcal{B})$ | ; |
| L03: | Block \gg | Begin | ; |
| L04: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L05: | Premise \gg | $\neg \mathcal{A}$ | ; |
| L06: | A1' \gg | $\neg \mathcal{A} \Rightarrow \neg \neg \mathcal{B} \Rightarrow \neg \mathcal{A}$ | ; |
| L07: | L06 \triangleright L05 \gg | $\neg \neg \mathcal{B} \Rightarrow \neg \mathcal{A}$ | ; |
| L08: | Lem1.11d \gg | $(\neg \neg \mathcal{B} \Rightarrow \neg \mathcal{A}) \Rightarrow \mathcal{A} \Rightarrow \neg \mathcal{B}$ | ; |
| L09: | L08 \triangleright L07 \gg | $\mathcal{A} \Rightarrow \neg \mathcal{B}$ | ; |
| L10: | Block \gg | End | ; |
| L11: | Ded \triangleright L10 \gg | $(\neg \mathcal{A}) \Rightarrow (\mathcal{A} \Rightarrow \neg \mathcal{B})$ | ; |
| L12: | MT \triangleright L11 \triangleright L02 \gg | $\neg \neg \mathcal{A}$ | ; |
| L13: | Lem1.11a \gg | $\neg \neg \mathcal{A} \Rightarrow \mathcal{A}$ | ; |
| L14: | L13 \triangleright L12 \gg | \mathcal{A} | □ |

6.1.3 Elimination af \wedge 2

[S lemma Con2: $\Pi \mathcal{A}, \mathcal{B}: \mathcal{A} \wedge \mathcal{B} \vdash \mathcal{B}$]

S proof of Con2:

| | | | |
|------|--|---|---|
| L01: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L02: | Premise \gg | $\neg(\mathcal{A} \Rightarrow \neg \mathcal{B})$ | ; |
| L03: | Block \gg | Begin | ; |
| L04: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L05: | Premise \gg | $\neg \mathcal{B}$ | ; |
| L06: | A1' \gg | $\neg \mathcal{B} \Rightarrow \mathcal{A} \Rightarrow \neg \mathcal{B}$ | ; |
| L07: | L06 \triangleright L05 \gg | $\mathcal{A} \Rightarrow \neg \mathcal{B}$ | ; |
| L08: | Block \gg | End | ; |
| L09: | Ded \triangleright L08 \gg | $\neg \mathcal{B} \Rightarrow \mathcal{A} \Rightarrow \neg \mathcal{B}$ | ; |
| L10: | MT \triangleright L09 \triangleright L02 \gg | $\neg \neg \mathcal{B}$ | ; |
| L11: | Lem1.11a \gg | $\neg \neg \mathcal{B} \Rightarrow \mathcal{B}$ | ; |
| L12: | L11 \triangleright L10 \gg | \mathcal{B} | □ |

6.1.4 Introduktion af \vee 1

[S lemma Dis1: $\Pi \mathcal{A}, \mathcal{B}: \mathcal{A} \vdash \mathcal{A} \vee \mathcal{B}$]

S proof of Dis1:

| | | | |
|------|---------------------------|---|---|
| L01: | Arbitrary \gg | A, B | ; |
| L02: | Premise \gg | A | ; |
| L03: | Lem1.11c \gg | $\neg\neg A \Rightarrow (\neg A \Rightarrow B)$ | ; |
| L04: | Lem1.11b \gg | $A \Rightarrow \neg\neg A$ | ; |
| L05: | L04 \supseteq L02 \gg | $\neg\neg A$ | ; |
| L06: | L03 \supseteq L05 \gg | $\neg A \Rightarrow B$ | □ |

6.1.5 Introduktion af \vee 2

[S lemma Dis2: $\Pi A, B: B \vdash A \vee B$]

S proof of Dis2:

| | | | |
|------|---------------------------|--------------------------------------|---|
| L01: | Arbitrary \gg | A, B | ; |
| L02: | Premise \gg | B | ; |
| L03: | $A1'$ \gg | $B \Rightarrow \neg A \Rightarrow B$ | ; |
| L04: | L03 \supseteq L02 \gg | $\neg A \Rightarrow B$ | □ |

6.1.6 Elimination af \vee

Da vi ikke har haft brug for at fjerne \vee , har vi ikke bevist denne regel.

6.2 Andre hjælpesætninger

Vi har til beviset af 3.5a-3.5g haft brug for nogle af reglerne fra [Gru06], dog med **imply** istedet for **infer**. Sådanne regler er navngivet med det <oprindelige navn>' og beviserne for disse kan ses i Appendix A.

6.2.1 $A \neq B$

En ny hjælperegul er **regel H3**, som skal bruges i forbindelse med $A \neq B$ for at kunne konkludere sammenhænge i mellem $(x = y) \wedge (x \neq 3) \Rightarrow (y \neq 3)$.

[S lemma H3: $\Pi A, B, C: A \Rightarrow (B \Rightarrow C) \vdash A \vdash \neg C \vdash \neg B$]

S proof of H3:

| | | | |
|------|--|-----------------------------------|---|
| L01: | Arbitrary \gg | A, B, C | ; |
| L02: | Premise \gg | $A \Rightarrow (B \Rightarrow C)$ | ; |
| L03: | Premise \gg | A | ; |
| L04: | Premise \gg | $\neg C$ | ; |
| L05: | L02 \supseteq L03 \gg | $B \Rightarrow C$ | ; |
| L06: | MT \supseteq L05 \supseteq L04 \gg | $\neg B$ | □ |

6.2.2 $\exists x$

Selv om vi ikke har nået at implementere fra opgave 3.5h, hvor der gøres brug af \exists , har vi alligevel makrodefineret kvantoren som følger.

$$[\exists x: y \doteq \neg(\forall x: \neg y)]$$

Vores problem har været, at vi ikke har kunne udtrykke " $\mathcal{B}(x, t)$, hvor t kan indsættes i stedet for x 'erne uden at blive bundet til en alkvantor", i pyk.

Beviset for hjælpesætningen, der indfører eksistenskvantoren, ville dog have været opbygget nogenlunde som følger.

Følgende tautologi,

$$(\mathcal{A} \Rightarrow \neg \mathcal{B}) \Rightarrow (\mathcal{B} \Rightarrow \neg \mathcal{A})$$

antages vist nogenlunde som Modus Tollens, dog med variation.

Ved axiom $A4'$ kan vi få instansen:

$$(\forall x) \neg \mathcal{A}(x, t) \Rightarrow \neg \mathcal{A}(t, t)$$

Idet man kan få følgende instans af tautologien:

$$((\forall x) \neg \mathcal{A}(x, t) \Rightarrow \neg \mathcal{A}(t, t)) \Rightarrow (\mathcal{A}(t, t) \Rightarrow \neg(\forall x) \neg \mathcal{A}(x, t))$$

kan man vha. MP på instansen og axiomet få:

$$(\mathcal{A}(t, t) \Rightarrow \neg(\forall x) \neg \mathcal{A}(x, t))$$

og pga. af makrodefinitionen er denne nu vist.

Herefter vises Lemmaerne 3.5a-3.5g

6.3 3.5a

[S lemma Prop 3.5a: $\Pi \mathcal{A}: \mathcal{A} + \bar{1} = \mathcal{A}'$]

S proof of Prop 3.5a:

| | | | |
|------|---|--|---|
| L01: | Arbitrary \gg | \mathcal{A} | ; |
| L02: | S6 \gg | $\mathcal{A} + 0' = (\mathcal{A} + 0)'$ | ; |
| L03: | S5 \gg | $\mathcal{A} + 0 = \mathcal{A}$ | ; |
| L04: | S2 \triangleright L03 \gg | $(\mathcal{A} + 0)' = \mathcal{A}'$ | ; |
| L05: | Prop 3.2c \triangleright L02 \triangleright L04 \gg | $\mathcal{A} + 0' = \mathcal{A}'$ | ; |
| L06: | Prop 3.2a \gg | $\mathcal{A} + 0' = \mathcal{A} + \bar{1}$ | ; |
| L07: | S1 \triangleright L06 \triangleright L05 \gg | $\mathcal{A} + \bar{1} = \mathcal{A}'$ | □ |

6.4 3.5b

[S lemma Prop 3.5b: $\Pi \mathcal{A}: \mathcal{A} \cdot \bar{1} = \mathcal{A}$]

S proof of Prop 3.5b:

| | | | |
|------|---|--|---|
| L01: | Arbitrary \gg | \mathcal{A} | ; |
| L02: | S8 \gg | $\mathcal{A} \cdot 0' = \mathcal{A} \cdot 0 + \mathcal{A}$ | ; |
| L03: | S7 \gg | $\mathcal{A} \cdot 0 = 0$ | ; |
| L04: | Prop 3.2e \triangleright L03 \gg | $\mathcal{A} \cdot 0 + \mathcal{A} = 0 + \mathcal{A}$ | ; |
| L05: | Prop 3.2c \triangleright L02 \triangleright L04 \gg | $\mathcal{A} \cdot 0' = 0 + \mathcal{A}$ | ; |
| L06: | Prop 3.2f \gg | $\mathcal{A} = 0 + \mathcal{A}$ | ; |
| L07: | Prop 3.2b \triangleright L06 \gg | $0 + \mathcal{A} = \mathcal{A}$ | ; |
| L08: | Prop 3.2c \triangleright L05 \triangleright L07 \gg | $\mathcal{A} \cdot 0' = \mathcal{A}$ | ; |
| L09: | Prop 3.2a \gg | $\mathcal{A} \cdot 0' = \mathcal{A} \cdot \bar{1}$ | ; |
| L10: | S1 \triangleright L09 \triangleright L08 \gg | $\mathcal{A} \cdot \bar{1} = \mathcal{A}$ | □ |

6.5 3.5c

[S lemma Prop 3.5c: $\Pi \mathcal{A}: \mathcal{A} \cdot \bar{2} = \mathcal{A} + \mathcal{A}$]

S proof of Prop 3.5c:

| | | | |
|------|---|--|---|
| L01: | Arbitrary \gg | \mathcal{A} | ; |
| L02: | S8 \gg | $\mathcal{A} \cdot \bar{1}' = \mathcal{A} \cdot \bar{1} + \mathcal{A}$ | ; |
| L03: | Prop 3.5b \gg | $\mathcal{A} \cdot \bar{1} = \mathcal{A}$ | ; |
| L04: | Prop 3.2e \triangleright L03 \gg | $\mathcal{A} \cdot \bar{1} + \mathcal{A} = \mathcal{A} + \mathcal{A}$ | ; |
| L05: | Prop 3.2c \triangleright L02 \triangleright L04 \gg | $\mathcal{A} \cdot \bar{1}' = \mathcal{A} + \mathcal{A}$ | ; |
| L06: | Prop 3.2a \gg | $\mathcal{A} \cdot \bar{1}' = \mathcal{A} \cdot \bar{2}$ | ; |
| L07: | S1 \triangleright L06 \triangleright L05 \gg | $\mathcal{A} \cdot \bar{2} = \mathcal{A} + \mathcal{A}$ | □ |

6.6 3.5d

[S lemma Prop 3.5d₁: $\Pi \mathcal{A}: \mathcal{A} + 0 = 0 \Rightarrow (\mathcal{A} = 0) \wedge (0 = 0)$]

S proof of Prop 3.5d₁:

| | | | |
|------|---|--|---|
| L01: | Arbitrary \gg | \mathcal{A} | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Arbitrary \gg | \mathcal{A} | ; |
| L04: | Premise \gg | $\mathcal{A} + 0 = 0$ | ; |
| L05: | S5 \gg | $\mathcal{A} + 0 = \mathcal{A}$ | ; |
| L06: | S1 \triangleright L05 \triangleright L04 \gg | $\mathcal{A} = 0$ | ; |
| L07: | Prop 3.2a \gg | $0 = 0$ | ; |
| L08: | Con \triangleright L06 \triangleright L07 \gg | $\mathcal{A} = 0 \wedge 0 = 0$ | ; |
| L09: | Block \gg | End | ; |
| L10: | Ded \triangleright L09 \gg | $\mathcal{A} + 0 = 0 \Rightarrow (\mathcal{A} = 0) \wedge (0 = 0)$ | □ |

[S lemma Prop 3.5d₂: $\Pi \mathcal{A}, \mathcal{B}: (\mathcal{A} + \mathcal{B} = 0 \Rightarrow \mathcal{A} = 0 \wedge \mathcal{B} = 0) \Rightarrow \mathcal{A} + \mathcal{B}' = 0 \Rightarrow \mathcal{A} = 0 \wedge \mathcal{B}' = 0$]

S proof of Prop 3.5d₂:

| | | | |
|------|-----------------|----------------------------|---|
| L01: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L02: | Block \gg | Begin | ; |

| | | | |
|------|--|---|---|
| L03: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L04: | Premise \gg | $\mathcal{A} + \mathcal{B} = 0 \Rightarrow \mathcal{A} = 0 \wedge \mathcal{B} = 0$ | ; |
| L05: | Block \gg | Begin | ; |
| L06: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L07: | Premise \gg | $\mathcal{A} + \mathcal{B}' = 0$ | ; |
| L08: | S3 \gg | $0 \neq (\mathcal{A} + \mathcal{B})'$ | ; |
| L09: | Prop3.2b' \gg | $(\mathcal{A} + \mathcal{B})' = 0 \Rightarrow 0 = (\mathcal{A} + \mathcal{B})'$ | ; |
| L10: | Lem1.11e \gg | $((\mathcal{A} + \mathcal{B})' = 0 \Rightarrow 0 = (\mathcal{A} + \mathcal{B})') \Rightarrow 0 \neq (\mathcal{A} + \mathcal{B})' \Rightarrow (\mathcal{A} + \mathcal{B})' \neq 0$ | ; |
| L11: | L10 \supseteq L09 \gg | $0 \neq (\mathcal{A} + \mathcal{B})' \Rightarrow (\mathcal{A} + \mathcal{B})' \neq 0$ | ; |
| L12: | L11 \supseteq L08 \gg | $(\mathcal{A} + \mathcal{B})' \neq 0$ | ; |
| L13: | S6 \gg | $(\mathcal{A} + \mathcal{B}') = (\mathcal{A} + \mathcal{B})'$ | ; |
| L14: | S1'' \gg | $\mathcal{A} + \mathcal{B}' = (\mathcal{A} + \mathcal{B})' \Rightarrow \mathcal{A} + \mathcal{B}' = 0 \Rightarrow (\mathcal{A} + \mathcal{B})' = 0$ | ; |
| L15: | H3 \supset L14 \supset L13 \supset L12 \gg | $\mathcal{A} + \mathcal{B}' \neq 0$ | ; |
| L16: | Lem1.11c \gg | $\mathcal{A} + \mathcal{B}' \neq 0 \Rightarrow \mathcal{A} + \mathcal{B}' = 0 \Rightarrow \mathcal{A} = 0 \wedge \mathcal{B}' = 0$ | ; |
| L17: | L16 \supseteq L15 \gg | $\mathcal{A} + \mathcal{B}' = 0 \Rightarrow \mathcal{A} = 0 \wedge \mathcal{B}' = 0$ | ; |
| L18: | L17 \supseteq L07 \gg | $\mathcal{A} = 0 \wedge \mathcal{B}' = 0$ | ; |
| L19: | Block \gg | End | ; |
| L20: | Ded \supset L19 \gg | $\mathcal{A} + \mathcal{B}' = 0 \Rightarrow \mathcal{A} = 0 \wedge \mathcal{B}' = 0$ | ; |
| L21: | Block \gg | End | ; |
| L22: | Ded \supset L21 \gg | $(\mathcal{A} + \mathcal{B} = 0 \Rightarrow \mathcal{A} = 0 \wedge \mathcal{B} = 0) \Rightarrow \mathcal{A} + \mathcal{B}' = 0 \Rightarrow \mathcal{A} = 0 \wedge \mathcal{B}' = 0$ | □ |

[S lemma Prop 3.5d: $\Pi \mathcal{A}, \mathcal{B}: \mathcal{A} + \mathcal{B} = 0 \Rightarrow \mathcal{A} = 0 \wedge \mathcal{B} = 0$]

S proof of Prop 3.5d:

| | | | |
|------|--|---|---|
| L01: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Prop 3.5d ₁ \gg | $x + 0 = 0 \Rightarrow x = 0 \wedge 0 = 0$ | ; |
| L04: | Prop 3.5d ₂ \gg | $(x + y = 0 \Rightarrow x = 0 \wedge y = 0) \Rightarrow x + y' = 0 \Rightarrow x = 0 \wedge y' = 0$ | ; |
| L05: | S9@y \supset L03 \supset L04 \gg | $x + y = 0 \Rightarrow x = 0 \wedge y = 0$ | ; |
| L06: | Block \gg | End | ; |
| L07: | Ded \supset L06 \gg | $\mathcal{A} + \mathcal{B} = 0 \Rightarrow \mathcal{A} = 0 \wedge \mathcal{B} = 0$ | □ |

6.7 3.5e

[S lemma Prop 3.5e₁: $\Pi \mathcal{A}: 0 \neq 0 \Rightarrow (\mathcal{A} \cdot 0 = 0 \Rightarrow \mathcal{A} = 0)$]

S proof of Prop 3.5e₁:

| | | | |
|------|-----------------|---------------|---|
| L01: | Arbitrary \gg | \mathcal{A} | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Arbitrary \gg | \mathcal{A} | ; |

| | | | |
|------|--------------------------------|---|-----------|
| L04: | Premise \gg | $0 \neq 0$ | ; |
| L05: | Prop 3.2a \gg | $0 = 0$ | ; |
| L06: | Lem1.11c \gg | $\neg(0 = 0) \Rightarrow ((0 = 0) \Rightarrow (\mathcal{A} \cdot 0 = 0 \Rightarrow \mathcal{A} = 0))$ | ; |
| L07: | L06 \supseteq L04 \gg | $(0 = 0) \Rightarrow (\mathcal{A} \cdot 0 = 0 \Rightarrow \mathcal{A} = 0)$ | ; |
| L08: | L07 \supseteq L05 \gg | $\mathcal{A} \cdot 0 = 0 \Rightarrow \mathcal{A} = 0$ | ; |
| L09: | Block \gg | End | ; |
| L10: | Ded \triangleright L09 \gg | $0 \neq 0 \Rightarrow \mathcal{A} \cdot 0 = 0 \Rightarrow \mathcal{A} = 0$ | \square |

[S lemma Prop 3.5e₂: $\Pi \mathcal{A}, \mathcal{B}: (\mathcal{B} \neq 0 \Rightarrow (\mathcal{A} \cdot \mathcal{B} = 0 \Rightarrow \mathcal{A} = 0)) \Rightarrow \mathcal{B}' \neq 0 \Rightarrow (\mathcal{A} \cdot \mathcal{B}' = 0 \Rightarrow \mathcal{A} = 0)$]

S proof of Prop 3.5e₂:

| | | | |
|------|--|---|-----------|
| L01: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L04: | Premise \gg | $\mathcal{B} \neq 0 \Rightarrow (\mathcal{A} \cdot \mathcal{B} = 0 \Rightarrow \mathcal{A} = 0)$ | ; |
| L05: | Block \gg | Begin | ; |
| L06: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L07: | Premise \gg | $\mathcal{B}' \neq 0$ | ; |
| L08: | Block \gg | Begin | ; |
| L09: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L10: | Premise \gg | $\mathcal{A} \cdot \mathcal{B}' = 0$ | ; |
| L11: | S8 \gg | $\mathcal{A} \cdot \mathcal{B}' = \mathcal{A} \cdot \mathcal{B} + \mathcal{A}$ | ; |
| L12: | S1 \triangleright L11 \triangleright L10 \gg | $(\mathcal{A} \cdot \mathcal{B}) + \mathcal{A} = 0$ | ; |
| L13: | Prop 3.5d \gg | $(\mathcal{A} \cdot \mathcal{B}) + \mathcal{A} = 0 \Rightarrow (\mathcal{A} \cdot \mathcal{B}) = 0 \wedge \mathcal{A} = 0$ | ; |
| L14: | L13 \supseteq L12 \gg | $(\mathcal{A} \cdot \mathcal{B}) = 0 \wedge \mathcal{A} = 0$ | ; |
| L15: | Con2 \triangleright L14 \gg | $\mathcal{A} = 0$ | ; |
| L16: | Block \gg | End | ; |
| L17: | Ded \triangleright L16 \gg | $\mathcal{A} \cdot \mathcal{B}' = 0 \Rightarrow \mathcal{A} = 0$ | ; |
| L18: | Block \gg | End | ; |
| L19: | Ded \triangleright L18 \gg | $\mathcal{B}' \neq 0 \Rightarrow (\mathcal{A} \cdot \mathcal{B}' = 0 \Rightarrow \mathcal{A} = 0)$ | ; |
| L20: | Block \gg | End | ; |
| L21: | Ded \triangleright L20 \gg | $(\mathcal{B} \neq 0 \Rightarrow (\mathcal{A} \cdot \mathcal{B} = 0 \Rightarrow \mathcal{A} = 0)) \Rightarrow \mathcal{B}' \neq 0 \Rightarrow (\mathcal{A} \cdot \mathcal{B}' = 0 \Rightarrow \mathcal{A} = 0)$ | \square |

[S lemma Prop 3.5e: $\Pi \mathcal{A}, \mathcal{B}: \mathcal{B} \neq 0 \Rightarrow (\mathcal{A} \cdot \mathcal{B} = 0 \Rightarrow \mathcal{A} = 0)$]

S proof of Prop 3.5e:

| | | | |
|------|------------------------------|--|---|
| L01: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Prop 3.5e ₁ \gg | $0 \neq 0 \Rightarrow (x \cdot 0 = 0 \Rightarrow x = 0)$ | ; |

| | | | |
|------|--|---|---|
| L04: | Prop 3.5e ₂ \gg | $(y \neq 0 \Rightarrow (x \cdot y = 0 \Rightarrow x = 0)) \Rightarrow y' \neq 0 \Rightarrow (x \cdot y' = 0 \Rightarrow x = 0)$ | ; |
| L05: | S9 @ $y \triangleright$ L03 \triangleright L04 \gg | $y \neq 0 \Rightarrow (x \cdot y = 0 \Rightarrow x = 0)$ | ; |
| L06: | Block \gg | End | ; |
| L07: | Ded \triangleright L06 \gg | $\mathcal{B} \neq 0 \Rightarrow (\mathcal{A} \cdot \mathcal{B} = 0 \Rightarrow \mathcal{A} = 0)$ | □ |

6.8 3.5f

[S lemma Prop 3.5f₁ : $\Pi \mathcal{A} : \mathcal{A} + 0 = \bar{1} \Rightarrow (\mathcal{A} = 0 \wedge 0 = \bar{1}) \vee (\mathcal{A} = \bar{1} \wedge 0 = 0)$]

S proof of Prop 3.5f₁ :

| | | | |
|------|---|--|---|
| L01: | Arbitrary \gg | \mathcal{A} | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Arbitrary \gg | \mathcal{A} | ; |
| L04: | Premise \gg | $\mathcal{A} + 0 = \bar{1}$ | ; |
| L05: | S5 \gg | $\mathcal{A} + 0 = \mathcal{A}$ | ; |
| L06: | S1 \triangleright L05 \triangleright L04 \gg | $\mathcal{A} = \bar{1}$ | ; |
| L07: | Prop 3.2a \gg | $0 = 0$ | ; |
| L08: | Con \triangleright L06 \triangleright L07 \gg | $\mathcal{A} = \bar{1} \wedge 0 = 0$ | ; |
| L09: | Dis2 \triangleright L08 \gg | $(\mathcal{A} = 0 \wedge 0 = \bar{1}) \vee (\mathcal{A} = \bar{1} \wedge 0 = 0)$ | ; |
| L10: | Block \gg | End | ; |
| L11: | Ded \triangleright L10 \gg | $\mathcal{A} + 0 = \bar{1} \Rightarrow (\mathcal{A} = 0 \wedge 0 = \bar{1}) \vee (\mathcal{A} = \bar{1} \wedge 0 = 0)$ | □ |

[S lemma Prop 3.5f₂ : $\Pi \mathcal{A}, \mathcal{B} : (\mathcal{A} + \mathcal{B} = \bar{1} \Rightarrow ((\mathcal{A} = 0 \wedge \mathcal{B} = \bar{1}) \vee (\mathcal{A} = \bar{1} \wedge \mathcal{B} = 0))) \Rightarrow (\mathcal{A} + \mathcal{B}' = \bar{1} \Rightarrow ((\mathcal{A} = 0 \wedge \mathcal{B}' = \bar{1}) \vee (\mathcal{A} = \bar{1} \wedge \mathcal{B}' = 0)))$]

S proof of Prop 3.5f₂ :

| | | | |
|------|---|--|---|
| L01: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L04: | Premise \gg | $\mathcal{A} + \mathcal{B} = \bar{1} \Rightarrow ((\mathcal{A} = 0 \wedge \mathcal{B} = \bar{1}) \vee (\mathcal{A} = \bar{1} \wedge \mathcal{B} = 0))$ | ; |
| L05: | Block \gg | Begin | ; |
| L06: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L07: | Premise \gg | $\mathcal{A} + \mathcal{B}' = \bar{1}$ | ; |
| L08: | S6 \gg | $\mathcal{A} + \mathcal{B}' = (\mathcal{A} + \mathcal{B})'$ | ; |
| L09: | S1 \triangleright L08 \triangleright L07 \gg | $(\mathcal{A} + \mathcal{B})' = \bar{1}$ | ; |
| L10: | S4 \triangleright L09 \gg | $\mathcal{A} + \mathcal{B} = 0$ | ; |
| L11: | Prop 3.5d \gg | $\mathcal{A} + \mathcal{B} = 0 \Rightarrow \mathcal{A} = 0 \wedge \mathcal{B} = 0$ | ; |
| L12: | L11 \triangleright L10 \gg | $\mathcal{A} = 0 \wedge \mathcal{B} = 0$ | ; |
| L13: | Con1 \triangleright L12 \gg | $\mathcal{A} = 0$ | ; |
| L14: | Con2 \triangleright L12 \gg | $\mathcal{B} = 0$ | ; |
| L15: | S2 \triangleright L14 \gg | $\mathcal{B}' = \bar{1}$ | ; |
| L16: | Con \triangleright L13 \triangleright L15 \gg | $\mathcal{A} = 0 \wedge \mathcal{B}' = \bar{1}$ | ; |

| | | | |
|------|---------------------------------|--|---|
| L17: | Dis1 \triangleright L16 \gg | $(\mathcal{A} = 0 \wedge \mathcal{B}' = \bar{1}) \vee (\mathcal{A} = \bar{1} \wedge \mathcal{B}' = 0)$ | ; |
| L18: | Block \gg | End | ; |
| L19: | Ded \triangleright L18 \gg | $\mathcal{A} + \mathcal{B}' = \bar{1} \Rightarrow (\mathcal{A} = 0 \wedge \mathcal{B}' = \bar{1}) \vee (\mathcal{A} = \bar{1} \wedge \mathcal{B}' = 0)$ | ; |
| L20: | Block \gg | End | ; |
| L21: | Ded \triangleright L20 \gg | $(\mathcal{A} + \mathcal{B} = \bar{1} \Rightarrow ((\mathcal{A} = 0 \wedge \mathcal{B} = \bar{1}) \vee (\mathcal{A} = \bar{1} \wedge \mathcal{B} = 0))) \Rightarrow \mathcal{A} + \mathcal{B}' = \bar{1} \Rightarrow (\mathcal{A} = 0 \wedge \mathcal{B}' = \bar{1}) \vee (\mathcal{A} = \bar{1} \wedge \mathcal{B}' = 0)$ | □ |

[S lemma Prop 3.5f: $\Pi \mathcal{A}, \mathcal{B}: \mathcal{A} + \mathcal{B} = \bar{1} \Rightarrow (\mathcal{A} = 0 \wedge \mathcal{B} = \bar{1}) \vee (\mathcal{A} = \bar{1} \wedge \mathcal{B} = 0)$]

S proof of Prop 3.5f:

| | | | |
|------|--|--|---|
| L01: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Prop 3.5f ₁ \gg | $x + 0 = \bar{1} \Rightarrow (x = 0 \wedge 0 = \bar{1}) \vee (x = \bar{1} \wedge 0 = 0)$ | ; |
| L04: | Prop 3.5f ₂ \gg | $(x + y = \bar{1} \Rightarrow ((x = 0 \wedge y = \bar{1}) \vee (x = \bar{1} \wedge y = 0))) \Rightarrow x + y' = \bar{1} \Rightarrow (x = 0 \wedge y' = \bar{1}) \vee (x = \bar{1} \wedge y' = 0)$ | ; |
| L05: | S9@y \triangleright L03 \triangleright L04 \gg | $x + y = \bar{1} \Rightarrow ((x = 0 \wedge y = \bar{1}) \vee (x = \bar{1} \wedge y = 0))$ | ; |
| L06: | Block \gg | End | ; |
| L07: | Ded \triangleright L06 \gg | $\mathcal{A} + \mathcal{B} = \bar{1} \Rightarrow (\mathcal{A} = 0 \wedge \mathcal{B} = \bar{1}) \vee (\mathcal{A} = \bar{1} \wedge \mathcal{B} = 0)$ | □ |

6.9 3.5g

[S lemma Prop 3.5g₁: $\Pi \mathcal{A}: \mathcal{A} \cdot 0 = \bar{1} \Rightarrow (\mathcal{A} = \bar{1}) \wedge (0 = \bar{1})$]

S proof of Prop 3.5g₁:

| | | | |
|------|--|--|---|
| L01: | Arbitrary \gg | \mathcal{A} | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Arbitrary \gg | \mathcal{A} | ; |
| L04: | Premise \gg | $\mathcal{A} \cdot 0 = \bar{1}$ | ; |
| L05: | S7 \gg | $\mathcal{A} \cdot 0 = 0$ | ; |
| L06: | S1 \triangleright L05 \triangleright L04 \gg | $0 = \bar{1}$ | ; |
| L07: | S3 \gg | $0 \neq \bar{1}$ | ; |
| L08: | Lem1.11c \gg | $\neg(0 = \bar{1}) \Rightarrow ((0 = \bar{1}) \Rightarrow (\mathcal{A} = \bar{1} \wedge 0 = \bar{1}))$ | ; |
| L09: | L08 $\underline{\triangleright}$ L07 \gg | $(0 = \bar{1}) \Rightarrow (\mathcal{A} = \bar{1} \wedge 0 = \bar{1})$ | ; |
| L10: | L09 $\underline{\triangleright}$ L06 \gg | $(\mathcal{A} = \bar{1} \wedge 0 = \bar{1})$ | ; |
| L11: | Block \gg | End | ; |
| L12: | Ded \triangleright L11 \gg | $\mathcal{A} \cdot 0 = \bar{1} \Rightarrow (\mathcal{A} = \bar{1} \wedge 0 = \bar{1})$ | □ |

[S lemma Prop 3.5g₂: $\Pi \mathcal{A}, \mathcal{B}: (\mathcal{A} \cdot \mathcal{B} = 0 \wedge \mathcal{A} = \bar{1}) \Rightarrow (\mathcal{A} = \bar{1} \wedge \mathcal{B}' = \bar{1})$]

S proof of Prop 3.5g₂:

| | | | |
|------|---|--|---|
| L01: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L04: | Premise \gg | $\mathcal{A} \cdot \mathcal{B} = 0 \wedge \mathcal{A} = \bar{1}$ | ; |
| L05: | Con2 \triangleright L04 \gg | $\mathcal{A} = \bar{1}$ | ; |
| L06: | Con1 \triangleright L04 \gg | $\mathcal{A} \cdot \mathcal{B} = 0$ | ; |
| L07: | S3 \gg | $0 \neq \bar{1}$ | ; |
| L08: | H10 \triangleright L07 \gg | $\bar{1} \neq 0$ | ; |
| L09: | S1'' \gg | $\mathcal{A} = \bar{1} \Rightarrow \mathcal{A} = 0 \Rightarrow \bar{1} = 0$ | ; |
| L10: | H3 \triangleright L09 \triangleright L05 \triangleright L08 \gg | $\mathcal{A} \neq 0$ | ; |
| L11: | Prop 3.2n \gg | $\mathcal{A} \cdot \mathcal{B} = \mathcal{B} \cdot \mathcal{A}$ | ; |
| L12: | S1 \triangleright L11 \triangleright L06 \gg | $\mathcal{B} \cdot \mathcal{A} = 0$ | ; |
| L13: | Prop 3.5e \gg | $\mathcal{A} \neq 0 \Rightarrow \mathcal{B} \cdot \mathcal{A} = 0 \Rightarrow \mathcal{B} = 0$ | ; |
| L14: | L13 \supseteq L10 \gg | $\mathcal{B} \cdot \mathcal{A} = 0 \Rightarrow \mathcal{B} = 0$ | ; |
| L15: | L14 \supseteq L12 \gg | $\mathcal{B} = 0$ | ; |
| L16: | S2 \triangleright L15 \gg | $\mathcal{B}' = \bar{1}$ | ; |
| L17: | Con \triangleright L05 \triangleright L16 \gg | $\mathcal{A} = \bar{1} \wedge \mathcal{B}' = \bar{1}$ | ; |
| L18: | Block \gg | End | ; |
| L19: | Ded \triangleright L18 \gg | $(\mathcal{A} \cdot \mathcal{B} = 0 \wedge \mathcal{A} = \bar{1}) \Rightarrow (\mathcal{A} = \bar{1} \wedge \mathcal{B}' = \bar{1})$ | □ |

[S lemma Prop 3.5g₃: $\Pi \mathcal{A}, \mathcal{B}: (\mathcal{A} \cdot \mathcal{B} = \bar{1} \Rightarrow (\mathcal{A} = \bar{1} \wedge \mathcal{B} = \bar{1})) \vdash (\mathcal{A} \cdot \mathcal{B} = \bar{1} \wedge \mathcal{A} = 0) \Rightarrow (\mathcal{A} = \bar{1} \wedge \mathcal{B}' = \bar{1})$]

S proof of Prop 3.5g₃:

| | | | |
|------|--|--|---|
| L01: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L02: | Premise \gg | $\mathcal{A} \cdot \mathcal{B} = \bar{1} \Rightarrow (\mathcal{A} = \bar{1} \wedge \mathcal{B} = \bar{1})$ | ; |
| L03: | Block \gg | Begin | ; |
| L04: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L05: | Premise \gg | $\mathcal{A} \cdot \mathcal{B} = \bar{1} \wedge \mathcal{A} = 0$ | ; |
| L06: | Con1 \triangleright L05 \gg | $\mathcal{A} \cdot \mathcal{B} = \bar{1}$ | ; |
| L07: | L02 \supseteq L06 \gg | $(\mathcal{A} = \bar{1} \wedge \mathcal{B} = \bar{1})$ | ; |
| L08: | Con1 \triangleright L07 \gg | $\mathcal{A} = \bar{1}$ | ; |
| L09: | Con2 \triangleright L05 \gg | $\mathcal{A} = 0$ | ; |
| L10: | S1 \triangleright L09 \triangleright L08 \gg | $0 = \bar{1}$ | ; |
| L11: | S3 \gg | $0 \neq \bar{1}$ | ; |
| L12: | Lem1.11c \gg | $0 \neq \bar{1} \Rightarrow ((0 = \bar{1}) \Rightarrow (\mathcal{A} = \bar{1} \wedge \mathcal{B}' = \bar{1}))$ | ; |
| L13: | L12 \supseteq L11 \gg | $(0 = \bar{1}) \Rightarrow (\mathcal{A} = \bar{1} \wedge \mathcal{B}' = \bar{1})$ | ; |
| L14: | L13 \supseteq L10 \gg | $(\mathcal{A} = \bar{1} \wedge \mathcal{B}' = \bar{1})$ | ; |
| L15: | Block \gg | End | ; |
| L16: | Ded \triangleright L15 \gg | $(\mathcal{A} \cdot \mathcal{B} = \bar{1} \wedge \mathcal{A} = 0) \Rightarrow (\mathcal{A} = \bar{1} \wedge \mathcal{B}' = \bar{1})$ | □ |

[S lemma Prop 3.5g₄: $\Pi \mathcal{A}, \mathcal{B}: (\mathcal{A} \cdot \mathcal{B} = \bar{1} \Rightarrow (\mathcal{A} = \bar{1} \wedge \mathcal{B} = \bar{1})) \Rightarrow (\mathcal{A} \cdot \mathcal{B}' = \bar{1} \Rightarrow (\mathcal{A} = \bar{1} \wedge \mathcal{B}' = \bar{1}))$]

S proof of Prop 3.5g₄:

| | | | |
|------|--|---|---|
| L01: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L04: | Premise \gg | $\mathcal{A} \cdot \mathcal{B} = \bar{1} \Rightarrow (\mathcal{A} = \bar{1} \wedge \mathcal{B} = \bar{1})$ | ; |
| L05: | Premise \gg | $\mathcal{A} \cdot \mathcal{B}' = \bar{1}$ | ; |
| L06: | S8 \gg | $\mathcal{A} \cdot \mathcal{B}' = \mathcal{A} \cdot \mathcal{B} + \mathcal{A}$ | ; |
| L07: | S1 \triangleright L06 \triangleright L05 \gg | $\mathcal{A} \cdot \mathcal{B} + \mathcal{A} = \bar{1}$ | ; |
| L08: | Prop 3.5f \gg | $(\mathcal{A} \cdot \mathcal{B} + \mathcal{A} = \bar{1}) \Rightarrow (\mathcal{A} \cdot \mathcal{B} = 0 \wedge \mathcal{A} = \bar{1}) \vee (\mathcal{A} \cdot \mathcal{B} = \bar{1} \wedge \mathcal{A} = 0)$ | ; |
| L09: | L08 \sqsupseteq L07 \gg | $(\mathcal{A} \cdot \mathcal{B} = 0 \wedge \mathcal{A} = \bar{1}) \vee (\mathcal{A} \cdot \mathcal{B} = \bar{1} \wedge \mathcal{A} = 0)$ | ; |
| L10: | Prop 3.5g ₂ \gg | $(\mathcal{A} \cdot \mathcal{B} = 0 \wedge \mathcal{A} = \bar{1}) \Rightarrow (\mathcal{A} = \bar{1} \wedge \mathcal{B}' = \bar{1})$ | ; |
| L11: | Prop 3.5g ₃ \triangleright L04 \gg | $(\mathcal{A} \cdot \mathcal{B} = \bar{1} \wedge \mathcal{A} = 0) \Rightarrow (\mathcal{A} = \bar{1} \wedge \mathcal{B}' = \bar{1})$ | ; |
| L12: | H11 \triangleright L09 \triangleright L10 \triangleright L11 \gg | $\mathcal{A} = \bar{1} \wedge \mathcal{B}' = \bar{1}$ | ; |
| L13: | Block \gg | End | ; |
| L14: | Ded \triangleright L13 \gg | $(\mathcal{A} \cdot \mathcal{B} = \bar{1} \Rightarrow (\mathcal{A} = \bar{1} \wedge \mathcal{B} = \bar{1})) \Rightarrow \mathcal{A} \cdot \mathcal{B}' = \bar{1} \Rightarrow (\mathcal{A} = \bar{1} \wedge \mathcal{B}' = \bar{1})$ | □ |

[S lemma Prop 3.5g: $\Pi \mathcal{A}, \mathcal{B}: \mathcal{A} \cdot \mathcal{B} = \bar{1} \Rightarrow (\mathcal{A} = \bar{1} \wedge \mathcal{B} = \bar{1})$]

S proof of Prop 3.5g:

| | | | |
|------|--|---|---|
| L01: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Prop 3.5g ₁ \gg | $x \cdot 0 = \bar{1} \Rightarrow (x = \bar{1} \wedge 0 = \bar{1})$ | ; |
| L04: | Prop 3.5g ₄ \gg | $(x \cdot y = \bar{1} \Rightarrow (x = \bar{1} \wedge y = \bar{1})) \Rightarrow (x \cdot y' = \bar{1} \Rightarrow (x = \bar{1} \wedge y' = \bar{1}))$ | ; |
| L05: | S9@y \triangleright L03 \triangleright L04 \gg | $x \cdot y = \bar{1} \Rightarrow (x = \bar{1} \wedge y = \bar{1})$ | ; |
| L06: | Block \gg | End | ; |
| L07: | Ded \triangleright L06 \gg | $\mathcal{A} \cdot \mathcal{B} = \bar{1} \Rightarrow (\mathcal{A} = \bar{1} \wedge \mathcal{B} = \bar{1})$ | □ |

6.10 3.5h

[S lemma Prop 3.5h: $\Pi \mathcal{A}: \mathcal{A} \neq 0 \Rightarrow \exists \mathcal{B}: \mathcal{A} = \mathcal{B}'$]

Beviset for 3.5.h ville i pyk have set ud nogenlunde som følgende:

$$t \neq 0 \Rightarrow (\exists y)(t = y')$$

Part I

| | | |
|-----|---------------------|---|
| L01 | <i>prem</i> | $0 \neq 0$ |
| L02 | <i>abbr.L01</i> | $\neg(0 = 0)$ |
| L03 | <i>3.2.a</i> | $0 = 0$ |
| L04 | <i>Lemma1.11.c</i> | $\neg(0 = 0) \Rightarrow ((0 = 0) \Rightarrow (\exists w)(0 = w'))$ |
| L05 | <i>MP, L02, L03</i> | $(\exists w)(0 = w')$ |
| L06 | <i>Ded, L05</i> | $(0 \neq 0) \Rightarrow (\exists w)(0 = w')$ |

Part II

| | | |
|-------|--|--|
| L01 : | <i>premise</i> | $(x \neq 0) \Rightarrow (\exists w)(x = w')$ |
| L02 : | <i>S3'</i> | $x' \neq 0$ |
| L03 : | <i>Taut. : $\mathcal{A} \vee \neg\mathcal{A}$</i> | $(x = 0) \vee \neg(x = 0)$ |
| L04 : | <i>prem.</i> | $(x = 0)$ |
| L05 : | <i>S2'</i> | $0' = x'$ |
| L06 : | <i>E4</i> | $(x = 0) \Rightarrow (\exists w)(x' = w')$ |
| L07 : | <i>prem</i> | $\neg(x = 0)$ |
| L08 : | <i>MP, L01, L07</i> | $(\exists w)(x = w')$ |
| L09 : | <i>rulec</i> | $x = b'$ |
| L10 : | <i>S2'</i> | $(\exists w)(x' = w')$ |
| L11 : | <i>E4</i> | $\neg(x = 0) \Rightarrow (\exists w)(x' = w')$ |
| L12 | <i>Lemma H11, L06.L11</i> | $(x' \neq 0) \Rightarrow (\exists w)(x' = w')$ |

Tilbage er blot at bruge sætning S9 for at fuldende induktionsbeviset.

7 Udsagn 3.7

Vi har kun lavet de indledende definitioner, da hele 3.7 kræver ”existential rule”.

7.1 Definitioner af $x < y$

Vi har makrodefineret følgende definitioner:

$$[x < y \doteq \exists z: (z \neq 0 \wedge z + x = y)]$$

$$[x \leq y \doteq x < y \vee x = y]$$

$$[x > y \doteq y < x]$$

$$[x \geq y \doteq y \leq x]$$

$$[x \not< y \doteq \neg(x < y)]$$

$$[x \not\leq y \doteq y \not< x]$$

$$[x \mid y \doteq \exists z: y = x \cdot z]$$

Ved hjælp af disse definitioner og ”existential rule” ville 3.7 kunne bevises.

A Hjælpe lemmaer

Til flere af vores beviser har vi brug for nogle hjælpe-lemmaer, disse er bevist i det følgende afsnit

[S lemma Lem1.11c: $\Pi A, B: \neg A \Rightarrow (A \Rightarrow B)$]

[S lemma Cor1.10a: $\Pi A, B, C: (A \Rightarrow B) \vdash (B \Rightarrow C) \vdash A \Rightarrow C$]

[S lemma Cor1.10b: $\Pi A, B, C: A \Rightarrow (B \Rightarrow C) \vdash B \vdash A \Rightarrow C$]

[S lemma Lem1.11a: $\Pi A: \neg\neg A \Rightarrow A$]

[S lemma Lem1.11b: $\Pi A: A \Rightarrow \neg\neg A$]

[S lemma Prop3.2c': $\Pi A, B, C: A = B \Rightarrow B = C \Rightarrow A = C$]

[S lemma S1'': $\Pi A, B, C: A = B \Rightarrow A = C \Rightarrow B = C$]

[S lemma Neg': $\Pi A, B: (\neg B \Rightarrow \neg A) \Rightarrow (\neg B \Rightarrow A) \Rightarrow B$]

[S lemma Repetition': $\Pi A: A \Rightarrow A$]

[S lemma Lem1.11e: $\Pi A, B: (A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$]

[S lemma Lem1.11d: $\Pi A, B: (\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)$]

[S lemma Prop3.2b': $\Pi A, B: A = B \Rightarrow B = A$]

[S lemma H10: $\Pi A, B: B \neq A \vdash A \neq B$]

[S lemma H11: $\Pi A, B, C: A \vee B \vdash A \Rightarrow C \vdash B \Rightarrow C \vdash C$]

[S lemma Lem1.11g: $\Pi A, B: (A \Rightarrow B) \vdash (\neg A \Rightarrow B) \vdash B$]

[S lemma MT: $\Pi A, B: (A \Rightarrow B) \vdash \neg B \vdash \neg A$]

[S lemma Lem1.11c: $\Pi A, B: \neg A \Rightarrow (A \Rightarrow B)$]

S **proof of** Lem1.11c:

| | | | |
|------|---|--|---|
| L01: | Arbitrary \gg | A, B | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Arbitrary \gg | A, B | ; |
| L04: | Premise \gg | $\neg A$ | ; |
| L05: | Block \gg | Begin | ; |
| L06: | Arbitrary \gg | A, B | ; |
| L07: | Premise \gg | A | ; |
| L08: | Repetition \triangleright L04 \gg | $\neg A$ | ; |
| L09: | A1' \gg | $A \Rightarrow (\neg B \Rightarrow A)$ | ; |
| L10: | L09 \triangleright L07 \gg | $\neg B \Rightarrow A$ | ; |
| L11: | A1' \gg | $\neg A \Rightarrow (\neg B \Rightarrow \neg A)$ | ; |
| L12: | L11 \triangleright L08 \gg | $\neg B \Rightarrow \neg A$ | ; |
| L13: | Neg \triangleright L12 \triangleright L10 \gg | B | ; |
| L14: | Block \gg | End | ; |
| L15: | Ded \triangleright L14 \gg | $A \Rightarrow B$ | ; |
| L16: | Block \gg | End | ; |
| L17: | Ded \triangleright L16 \gg | $\neg A \Rightarrow (A \Rightarrow B)$ | □ |

[S lemma Cor1.10a: $\Pi A, B, C: (A \Rightarrow B) \vdash (B \Rightarrow C) \vdash A \Rightarrow C$]

S proof of Cor1.10a:

| | | | |
|------|--------------------------------|---|---|
| L01: | Arbitrary \gg | A, B, C | ; |
| L02: | Premise \gg | $A \Rightarrow B$ | ; |
| L03: | Premise \gg | $B \Rightarrow C$ | ; |
| L04: | Block \gg | Begin | ; |
| L05: | Arbitrary \gg | A, B, C | ; |
| L06: | Premise \gg | $A \Rightarrow B$ | ; |
| L07: | Premise \gg | $B \Rightarrow C$ | ; |
| L08: | Premise \gg | A | ; |
| L09: | L06 \supseteq L08 \gg | B | ; |
| L10: | L07 \supseteq L09 \gg | C | ; |
| L11: | Block \gg | End | ; |
| L12: | Ded \triangleright L11 \gg | $(A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$ | ; |
| L13: | L12 \supseteq L02 \gg | $(B \Rightarrow C) \Rightarrow A \Rightarrow C$ | ; |
| L14: | L13 \supseteq L03 \gg | $A \Rightarrow C$ | □ |

[S lemma Cor1.10b: $\Pi A, B, C: A \Rightarrow (B \Rightarrow C) \vdash B \vdash A \Rightarrow C$]

S proof of Cor1.10b:

| | | | |
|------|---------------------------|--|---|
| L01: | Arbitrary \gg | A, B, C | ; |
| L02: | Premise \gg | $A \Rightarrow B \Rightarrow C$ | ; |
| L03: | Premise \gg | B | ; |
| L04: | A2' \gg | $(A \Rightarrow B \Rightarrow C) \Rightarrow (A \Rightarrow B) \Rightarrow$ $A \Rightarrow C$ | ; |
| L05: | L04 \supseteq L02 \gg | $(A \Rightarrow B) \Rightarrow A \Rightarrow C$ | ; |
| L06: | A1' \gg | $B \Rightarrow A \Rightarrow B$ | ; |
| L07: | L06 \supseteq L03 \gg | $A \Rightarrow B$ | ; |
| L08: | L05 \supseteq L07 \gg | $A \Rightarrow C$ | □ |

[S lemma Lem1.11a: $\Pi A: \neg\neg A \Rightarrow A$]

S proof of Lem1.11a:

| | | | |
|------|--|--|---|
| L01: | Arbitrary \gg | A | ; |
| L02: | Neg' \gg | $(\neg(A) \Rightarrow \neg(\neg A)) \Rightarrow ((\neg(A) \Rightarrow$ $(\neg A)) \Rightarrow A)$ | ; |
| L03: | Repetition' \gg | $\neg A \Rightarrow \neg A$ | ; |
| L04: | Cor1.10b \triangleright L02 \triangleright L03 \gg | $(\neg A \Rightarrow \neg(\neg A)) \Rightarrow A$ | ; |
| L05: | A1' \gg | $\neg(\neg A) \Rightarrow (\neg A \Rightarrow \neg(\neg A))$ | ; |
| L06: | Cor1.10a \triangleright L05 \triangleright L04 \gg | $\neg\neg A \Rightarrow A$ | □ |

[S lemma Lem1.11b: $\Pi A: A \Rightarrow \neg\neg A$]

S proof of Lem1.11b:

| | | | |
|------|-----------------|--|---|
| L01: | Arbitrary \gg | A | ; |
| L02: | Neg' \gg | $(\neg\neg\neg A \Rightarrow \neg A) \Rightarrow ((\neg\neg\neg A \Rightarrow$ $A) \Rightarrow \neg\neg A)$ | ; |

| | | | |
|------|--|---|---|
| L03: | Lem1.11a \gg | $\neg\neg(\neg A) \Rightarrow (\neg A)$ | ; |
| L04: | L02 \supseteq L03 \gg | $(\neg\neg\neg A \Rightarrow A) \Rightarrow \neg\neg A$ | ; |
| L05: | A1' \gg | $A \Rightarrow (\neg\neg\neg A \Rightarrow A)$ | ; |
| L06: | Cor1.10a \supseteq L05 \supseteq L04 \gg | $A \Rightarrow \neg\neg A$ | □ |

[S lemma Prop3.2c': $\Pi A, B, C: A = B \Rightarrow B = C \Rightarrow A = C$]

S proof of Prop3.2c':

| | | | |
|------|---|---|---|
| L01: | Arbitrary \gg | A, B, C | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Arbitrary \gg | A, B, C | ; |
| L04: | Premise \gg | $A = B$ | ; |
| L05: | Block \gg | Begin | ; |
| L06: | Arbitrary \gg | A, B, C | ; |
| L07: | Premise \gg | $B = C$ | ; |
| L08: | Prop 3.2c \supseteq L04 \supseteq L07 \gg | $A = C$ | ; |
| L09: | Block \gg | End | ; |
| L10: | Ded \supseteq L09 \gg | $B = C \Rightarrow A = C$ | ; |
| L11: | Block \gg | End | ; |
| L12: | Ded \supseteq L11 \gg | $A = B \Rightarrow B = C \Rightarrow A = C$ | □ |

[S lemma S1'': $\Pi A, B, C: A = B \Rightarrow A = C \Rightarrow B = C$]

S proof of S1'':

| | | | |
|------|--|---|---|
| L01: | Arbitrary \gg | A, B, C | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Arbitrary \gg | A, B, C | ; |
| L04: | Premise \gg | $A = B$ | ; |
| L05: | Block \gg | Begin | ; |
| L06: | Arbitrary \gg | A, B, C | ; |
| L07: | Premise \gg | $A = C$ | ; |
| L08: | S1 \supseteq L04 \supseteq L07 \gg | $B = C$ | ; |
| L09: | Block \gg | End | ; |
| L10: | Ded \supseteq L09 \gg | $A = C \Rightarrow B = C$ | ; |
| L11: | Block \gg | End | ; |
| L12: | Ded \supseteq L11 \gg | $A = B \Rightarrow A = C \Rightarrow B = C$ | □ |

[S lemma Neg': $\Pi A, B: (\neg B \Rightarrow \neg A) \Rightarrow (\neg B \Rightarrow A) \Rightarrow B$]

S proof of Neg':

| | | | |
|------|---|-----------------------------|---|
| L01: | Arbitrary \gg | A, B | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Arbitrary \gg | A, B | ; |
| L04: | Premise \gg | $\neg B \Rightarrow \neg A$ | ; |
| L05: | Block \gg | Begin | ; |
| L06: | Arbitrary \gg | A, B | ; |
| L07: | Premise \gg | $\neg B \Rightarrow A$ | ; |
| L08: | Neg \supseteq L04 \supseteq L07 \gg | B | ; |

| | | | |
|------|--------------------------------|---|-----------|
| L09: | Block \gg | End | ; |
| L10: | Ded \triangleright L09 \gg | $(\neg\mathcal{B} \Rightarrow \mathcal{A}) \Rightarrow \mathcal{B}$ | ; |
| L11: | Block \gg | End | ; |
| L12: | Ded \triangleright L11 \gg | $(\neg\mathcal{B} \Rightarrow \neg\mathcal{A}) \Rightarrow (\neg\mathcal{B} \Rightarrow \mathcal{A}) \Rightarrow \mathcal{B}$ | \square |

[S lemma Repetition': $\Pi\mathcal{A}: \mathcal{A} \Rightarrow \mathcal{A}$]

S proof of Repetition':

| | | | |
|------|---------------------------------------|---------------------------------------|-----------|
| L01: | Arbitrary \gg | \mathcal{A} | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Arbitrary \gg | \mathcal{A} | ; |
| L04: | Premise \gg | \mathcal{A} | ; |
| L05: | Repetition \triangleright L04 \gg | \mathcal{A} | ; |
| L06: | Block \gg | End | ; |
| L07: | Ded \triangleright L06 \gg | $\mathcal{A} \Rightarrow \mathcal{A}$ | \square |

[S lemma H10: $\Pi\mathcal{A}, \mathcal{B}: \mathcal{B} \neq \mathcal{A} \vdash \mathcal{A} \neq \mathcal{B}$]

S proof of H10:

| | | | |
|------|--------------------------------|--|-----------|
| L01: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L02: | Premise \gg | $\mathcal{B} \neq \mathcal{A}$ | ; |
| L03: | Prop3.2b' \gg | $\mathcal{A} = \mathcal{B} \Rightarrow \mathcal{B} = \mathcal{A}$ | ; |
| L04: | Lem1.11e \gg | $(\mathcal{A} = \mathcal{B} \Rightarrow \mathcal{B} = \mathcal{A}) \Rightarrow \mathcal{B} \neq \mathcal{A} \Rightarrow$ $\mathcal{A} \neq \mathcal{B}$ | ; |
| L05: | L04 \triangleright L03 \gg | $\mathcal{B} \neq \mathcal{A} \Rightarrow \mathcal{A} \neq \mathcal{B}$ | ; |
| L06: | L05 \triangleright L02 \gg | $\mathcal{A} \neq \mathcal{B}$ | \square |

[S lemma MT: $\Pi\mathcal{A}, \mathcal{B}: (\mathcal{A} \Rightarrow \mathcal{B}) \vdash \neg\mathcal{B} \vdash \neg\mathcal{A}$]

S proof of MT:

| | | | |
|------|--|---|-----------|
| L01: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L02: | Premise \gg | $\mathcal{A} \Rightarrow \mathcal{B}$ | ; |
| L03: | Premise \gg | $\neg\mathcal{B}$ | ; |
| L04: | Lem1.11a \gg | $\neg\neg\mathcal{A} \Rightarrow \mathcal{A}$ | ; |
| L05: | Cor1.10a \triangleright L04 \triangleright L02 \gg | $\neg\neg\mathcal{A} \Rightarrow \mathcal{B}$ | ; |
| L06: | Block \gg | Begin | ; |
| L07: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L08: | Premise \gg | $\neg\neg\mathcal{A} \Rightarrow \mathcal{B}$ | ; |
| L09: | Premise \gg | $\neg\mathcal{B}$ | ; |
| L10: | A1' \gg | $\neg\mathcal{B} \Rightarrow \neg\neg\mathcal{A} \Rightarrow \neg\mathcal{B}$ | ; |
| L11: | L10 \triangleright L09 \gg | $\neg\neg\mathcal{A} \Rightarrow \neg\mathcal{B}$ | ; |
| L12: | Neg \triangleright L11 \triangleright L08 \gg | $\neg\mathcal{A}$ | ; |
| L13: | Block \gg | End | ; |
| L14: | Ded \triangleright L13 \gg | $(\neg\neg\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow \neg\mathcal{B} \Rightarrow \neg\mathcal{A}$ | ; |
| L15: | L14 \triangleright L05 \gg | $\neg\mathcal{B} \Rightarrow \neg\mathcal{A}$ | ; |
| L16: | L15 \triangleright L03 \gg | $\neg\mathcal{A}$ | \square |

[S lemma Lem1.11e: $\Pi\mathcal{A}, \mathcal{B}: (\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow (\neg\mathcal{B} \Rightarrow \neg\mathcal{A})$]

S proof of Lem1.11e:

| | | | |
|------|--|---|---|
| L01: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L04: | Premise \gg | $\mathcal{A} \Rightarrow \mathcal{B}$ | ; |
| L05: | Lem1.11a \gg | $\neg\neg\mathcal{A} \Rightarrow \mathcal{A}$ | ; |
| L06: | Cor1.10a \triangleright L05 \triangleright L04 \gg | $\neg\neg\mathcal{A} \Rightarrow \mathcal{B}$ | ; |
| L07: | Lem1.11b \gg | $\mathcal{B} \Rightarrow \neg\neg\mathcal{B}$ | ; |
| L08: | Cor1.10a \triangleright L06 \triangleright L07 \gg | $\neg\neg\mathcal{A} \Rightarrow \neg\neg\mathcal{B}$ | ; |
| L09: | Lem1.11d \gg | $(\neg\neg\mathcal{A} \Rightarrow \neg\neg\mathcal{B}) \Rightarrow \neg\mathcal{B} \Rightarrow \neg\mathcal{A}$ | ; |
| L10: | L09 \sqsupseteq L08 \gg | $\neg\mathcal{B} \Rightarrow \neg\mathcal{A}$ | ; |
| L11: | Block \gg | End | ; |
| L12: | Ded \triangleright L11 \gg | $(\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow \neg\mathcal{B} \Rightarrow \neg\mathcal{A}$ | □ |

[S lemma Lem1.11d: $\Pi\mathcal{A}, \mathcal{B}: (\neg\mathcal{B} \Rightarrow \neg\mathcal{A}) \Rightarrow (\mathcal{A} \Rightarrow \mathcal{B})$]

S proof of Lem1.11d:

| | | | |
|------|--|---|---|
| L01: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L04: | Premise \gg | $\neg\mathcal{B} \Rightarrow \neg\mathcal{A}$ | ; |
| L05: | Neg' \gg | $(\neg\mathcal{B} \Rightarrow \neg\mathcal{A}) \Rightarrow ((\neg\mathcal{B} \Rightarrow \mathcal{A}) \Rightarrow \mathcal{B})$ | ; |
| L06: | A1' \gg | $\mathcal{A} \Rightarrow (\neg\mathcal{B} \Rightarrow \mathcal{A})$ | ; |
| L07: | L05 \sqsupseteq L04 \gg | $(\neg\mathcal{B} \Rightarrow \mathcal{A}) \Rightarrow \mathcal{B}$ | ; |
| L08: | Cor1.10a \triangleright L06 \triangleright L07 \gg | $\mathcal{A} \Rightarrow \mathcal{B}$ | ; |
| L09: | Block \gg | End | ; |
| L10: | Ded \triangleright L09 \gg | $(\neg\mathcal{B} \Rightarrow \neg\mathcal{A}) \Rightarrow (\mathcal{A} \Rightarrow \mathcal{B})$ | □ |

[S lemma Prop3.2b': $\Pi\mathcal{A}, \mathcal{B}: \mathcal{A} = \mathcal{B} \Rightarrow \mathcal{B} = \mathcal{A}$]

S proof of Prop3.2b':

| | | | |
|------|--------------------------------------|---|---|
| L01: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L02: | Block \gg | Begin | ; |
| L03: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L04: | Premise \gg | $\mathcal{A} = \mathcal{B}$ | ; |
| L05: | Prop 3.2b \triangleright L04 \gg | $\mathcal{B} = \mathcal{A}$ | ; |
| L06: | Block \gg | End | ; |
| L07: | Ded \triangleright L06 \gg | $\mathcal{A} = \mathcal{B} \Rightarrow \mathcal{B} = \mathcal{A}$ | □ |

[S lemma Lem1.11g: $\Pi\mathcal{A}, \mathcal{B}: (\mathcal{A} \Rightarrow \mathcal{B}) \vdash (\neg\mathcal{A} \Rightarrow \mathcal{B}) \vdash \mathcal{B}$]

S proof of Lem1.11g:

| | | | |
|------|-----------------------------|---|---|
| L01: | Arbitrary \gg | \mathcal{A}, \mathcal{B} | ; |
| L02: | Premise \gg | $\mathcal{A} \Rightarrow \mathcal{B}$ | ; |
| L03: | Premise \gg | $\neg\mathcal{A} \Rightarrow \mathcal{B}$ | ; |
| L04: | Lem1.11e \gg | $(\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow \neg\mathcal{B} \Rightarrow \neg\mathcal{A}$ | ; |
| L05: | L04 \sqsupseteq L02 \gg | $\neg\mathcal{B} \Rightarrow \neg\mathcal{A}$ | ; |

| | | | |
|------|---|--|-----------|
| L06: | Lem1.11e \gg | $(\neg A \Rightarrow B) \Rightarrow \neg B \Rightarrow \neg\neg A$ | ; |
| L07: | L06 \supseteq L03 \gg | $\neg B \Rightarrow \neg\neg A$ | ; |
| L08: | Neg \triangleright L07 \triangleright L05 \gg | B | \square |

[S lemma H11: $\Pi A, B, C: \mathcal{A} \vee \mathcal{B} \vdash \mathcal{A} \Rightarrow \mathcal{C} \vdash \mathcal{B} \Rightarrow \mathcal{C} \vdash \mathcal{C}$]

S proof of H11:

| | | | |
|------|--|------------------------|-----------|
| L01: | Arbitrary \gg | A, B, C | ; |
| L02: | Premise \gg | $A \vee B$ | ; |
| L03: | Premise \gg | $A \Rightarrow C$ | ; |
| L04: | Premise \gg | $B \Rightarrow C$ | ; |
| L05: | Cor1.10a \triangleright L02 \triangleright L04 \gg | $\neg A \Rightarrow C$ | ; |
| L06: | Lem1.11g \triangleright L03 \triangleright L05 \gg | C | \square |

Litteratur

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