projekt i logik

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Indhold

Resumé

1 Introduction

In this paper we will prove the propositions regarding S starting with proposition Prop 3.2i in mendelson [Men97]. Continuing toProp 3.4 defining numerals neded to prove Prop 3.5 . then we will define rules regarding $\neg\forall_{\mathrm{obj}}\mathsf{z}\colon \neg\neg\neg \mathsf{z} = 0 \Rightarrow \mathsf{z} + \bar{\mathsf{x}} = \bar{\mathsf{y}}$ needed to prove Prop 3.7 . after that we will define $\neg\forall_{\mathrm{obj}}\mathsf{z}\colon \neg\bar{\mathsf{y}} = \bar{\mathsf{x}} \cdot \mathsf{z}$ nedded to prove Prop 3.10 , and finally we will prove Prop 3.11.

2 **Proof of** Prop 3.2

A modified version of Mendelsons system S (Peano arithmetic) [Men97] may be formulated thus:

$[\mathrm{S} \overset{\text{stmt}}{\to} \mathsf{x}]$

$[\mathrm{MP} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \underline{\mathsf{a}} \vdash \underline{\mathsf{b}}][\mathrm{MP} \overset{\text{proof}}{\to} \text{Rule tactic}]$

$[\mathrm{Gen} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{a}} \colon \underline{\mathsf{a}} \vdash \forall_{\mathrm{obj}} \underline{\mathsf{x}} \colon \underline{\mathsf{a}}][\mathrm{Gen} \overset{\text{proof}}{\to} \text{Rule tactic}]$

$[\mathrm{Ded} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \lambda x.\mathrm{Ded}_0(\lceil \underline{\mathsf{a}} \rceil, \lceil \underline{\mathsf{b}} \rceil) \Vdash \underline{\mathsf{a}} \vdash \underline{\mathsf{b}}][\mathrm{Ded} \overset{\text{proof}}{\to} \text{Rule tactic}]$

$[\mathrm{S2} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} = \underline{\mathsf{b}} \vdash \underline{\mathsf{a}}' = \underline{\mathsf{b}}'][\mathrm{S2} \overset{\text{proof}}{\to} \text{Rule tactic}]$

$[\mathrm{S3} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \neg 0 = \underline{\mathsf{a}}'][\mathrm{S3} \overset{\text{proof}}{\to} \text{Rule tactic}]$

$[\mathrm{S4} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}}' = \underline{\mathsf{b}}' \vdash \underline{\mathsf{a}} = \underline{\mathsf{b}}][\mathrm{S4} \overset{\text{proof}}{\to} \text{Rule tactic}]$

$[\mathrm{S5} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \underline{\mathsf{a}} + 0 = \underline{\mathsf{a}}][\mathrm{S5} \overset{\text{proof}}{\to} \text{Rule tactic}]$

$[\mathrm{S6} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} + \underline{\mathsf{b}}' = \underline{\mathsf{a}} + \underline{\mathsf{b}}'][\mathrm{S6} \overset{\text{proof}}{\to} \text{Rule tactic}]$

$[\mathrm{S7} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \underline{\mathsf{a}} \cdot 0 = 0][\mathrm{S7} \overset{\text{proof}}{\to} \text{Rule tactic}]$

$[\mathrm{S8} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \cdot \underline{\mathsf{b}}' = \underline{\mathsf{a}} \cdot \underline{\mathsf{b}} + \underline{\mathsf{a}}][\mathrm{S8} \overset{\text{proof}}{\to} \text{Rule tactic}]$

$[\mathrm{Neg} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \neg \underline{\mathsf{b}} \Rightarrow \neg \underline{\mathsf{a}} \vdash \neg \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}} \vdash \underline{\mathsf{b}}][\mathrm{Neg} \overset{\text{proof}}{\to} \text{Rule tactic}]$

$[\mathrm{S1} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \forall \underline{\mathsf{c}} \colon \underline{\mathsf{a}} = \underline{\mathsf{b}} \vdash \underline{\mathsf{a}} = \underline{\mathsf{c}} \vdash \underline{\mathsf{b}} = \underline{\mathsf{c}}][\mathrm{S1} \overset{\text{proof}}{\to} \text{Rule tactic}]$

$[\mathrm{S9} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \forall \underline{\mathsf{c}} \colon \langle \lceil \underline{\mathsf{b}} \rceil \equiv^0 \lceil \underline{\mathsf{a}} \rceil | \lceil \underline{\mathsf{x}} \rceil := \lceil 0 \rceil \rangle \Vdash \langle \lceil \underline{\mathsf{c}} \rceil \equiv^0 \lceil \underline{\mathsf{a}} \rceil | \lceil \underline{\mathsf{x}} \rceil := \lceil \underline{\mathsf{x}}' \rceil \rangle \Vdash \underline{\mathsf{b}} \vdash \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{c}} \vdash \underline{\mathsf{a}}][\mathrm{S9} \overset{\text{proof}}{\to} \text{Rule tactic}]$

Mendelson [Men97] defines a set of lemma's

$[\text{Prop 3.2a} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \underline{\mathsf{a}} = \underline{\mathsf{a}}]$

$[\text{Prop 3.2b} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} = \underline{\mathsf{b}} \vdash \underline{\mathsf{b}} = \underline{\mathsf{a}}]$

$[\text{Prop 3.2c} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \forall \underline{\mathsf{c}} \colon \underline{\mathsf{a}} = \underline{\mathsf{b}} \vdash \underline{\mathsf{b}} = \underline{\mathsf{c}} \vdash \underline{\mathsf{a}} = \underline{\mathsf{c}}]$

$[\text{Prop 3.2d} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \forall \underline{\mathsf{c}} \colon \underline{\mathsf{a}} = \underline{\mathsf{c}} \vdash \underline{\mathsf{b}} = \underline{\mathsf{c}} \vdash \underline{\mathsf{a}} = \underline{\mathsf{b}}]$

$[\text{Prop 3.2e} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \forall \underline{\mathsf{c}} \colon \underline{\mathsf{a}} = \underline{\mathsf{b}} \vdash \underline{\mathsf{a}} + \underline{\mathsf{c}} = \underline{\mathsf{b}} + \underline{\mathsf{c}}]$

$[\text{Prop 3.2f} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \underline{\mathsf{a}} = 0 + \underline{\mathsf{a}}]$

$[\text{Prop 3.2g} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}}' + \underline{\mathsf{b}} = \underline{\mathsf{a}} + \underline{\mathsf{b}}']$

$[\text{Prop 3.2h} \overset{\text{stmt}}{\to} \mathrm{S} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} + \underline{\mathsf{b}} = \underline{\mathsf{b}} + \underline{\mathsf{a}}]$

These Lemma's are proved in [Gru06], and we will continue to prove the following lemma's, from Mendelson [Men97].

$[\text{Prop 3.2i} \overset{\text{stmt}}{\to} S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} = \underline{b} \vdash \underline{c} + \underline{a} = \underline{c} + \underline{b}]$

$[\text{Prop 3.2j} \overset{\text{stmt}}{\to} S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} + \underline{b} + \underline{c} = \underline{a} + \underline{b} + \underline{c}]$

$[\text{Prop 3.2k} \overset{\text{stmt}}{\to} S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} = \underline{b} \vdash \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c}]$

$[\text{Prop 3.2l} \overset{\text{stmt}}{\to} S \vdash \forall \underline{a}\colon 0 \cdot \underline{a} = 0]$

$[\text{Prop 3.2m} \overset{\text{stmt}}{\to} S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a}' \cdot \underline{b} = \underline{a} \cdot \underline{b} + \underline{b}]$

$[\text{Prop 3.2n} \overset{\text{stmt}}{\to} S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}]$

$[\text{Prop 3.2o} \overset{\text{stmt}}{\to} S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} = \underline{b} \vdash \underline{c} \cdot \underline{a} = \underline{c} \cdot \underline{b}]$

$[\text{Prop 3.2i} \overset{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} = \underline{b} \vdash \text{Prop 3.2e} \rhd \underline{a} = \underline{b} \gg \underline{a} + \underline{c} = \underline{b} + \underline{c}; \text{Prop 3.2h} \gg \underline{a} + \underline{c} = \underline{c} + \underline{a}; \text{Prop 3.2h} \gg \underline{b} + \underline{c} = \underline{c} + \underline{b}; \text{S1} \rhd \underline{a} + \underline{c} = \underline{b} + \underline{c} \rhd \underline{a} + \underline{c} = \underline{c} + \underline{a} \gg \underline{b} + \underline{c} = \underline{c} + \underline{a}; \text{Prop 3.2b} \rhd \underline{b} + \underline{c} = \underline{c} + \underline{a} \gg \underline{c} + \underline{a} = \underline{b} + \underline{c}; \text{Prop 3.2c} \rhd \underline{c} + \underline{a} = \underline{b} + \underline{c} \rhd \underline{b} + \underline{c} = \underline{c} + \underline{b} \gg \underline{c} + \underline{a} = \underline{c} + \underline{b}; \text{Ded} \rhd \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} = \underline{b} \vdash \underline{c} + \underline{a} = \underline{c} + \underline{b} \gg \underline{a} = \underline{b} \vdash \underline{c} + \underline{a} = \underline{c} + \underline{b}\rceil, p_0, c)]$

$[\text{Prop 3.2j}_1 \overset{\text{stmt}}{\to} S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} + \underline{b} + 0 = \underline{a} + \underline{b} + 0]$

$[\text{Prop 3.2j}_1 \overset{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \text{S5} \gg \underline{a} + \underline{b} + 0 = \underline{a} + \underline{b}; \text{S5} \gg \underline{b} + 0 = \underline{b}; \text{Prop 3.2i} \rhd \underline{b} + 0 = \underline{b} \gg \underline{a} + \underline{b} + 0 = \underline{a} + \underline{b}; \text{Prop 3.2d} \rhd \underline{a} + \underline{b} + 0 = \underline{a} + \underline{b} \rhd \underline{a} + \underline{b} + 0 = \underline{a} + \underline{b} \gg \underline{a} + \underline{b} + 0 = \underline{a} + \underline{b} + 0\rceil, p_0, c)]$

$[\text{Prop 3.2j}_2 \overset{\text{stmt}}{\to} S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} + \underline{b} + \underline{c} = \underline{a} + \underline{b} + \underline{c} \Rightarrow \underline{a} + \underline{b} + \underline{c}' = \underline{a} + \underline{b} + \underline{c}']$

$[\text{Prop 3.2j}_2 \overset{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} + \underline{b} + \underline{c} = \underline{a} + \underline{b} + \underline{c} \vdash \text{S6} \gg \underline{a} + \underline{b} + \underline{c}' = \underline{a} + \underline{b} + \underline{c}'; \text{S2} \rhd \underline{a} + \underline{b} + \underline{c} = \underline{a} + \underline{b} + \underline{c} \gg \underline{a} + \underline{b} + \underline{c}' = \underline{a} + \underline{b} + \underline{c}'; \text{Prop 3.2c} \rhd \underline{a} + \underline{b} + \underline{c}' = \underline{a} + \underline{b} + \underline{c}' \rhd \underline{a} + \underline{b} + \underline{c}' = \underline{a} + \underline{b} + \underline{c}' \gg \underline{a} + \underline{b} + \underline{c}' = \underline{a} + \underline{b} + \underline{c}'; \text{S6} \gg \underline{b} + \underline{c}' = \underline{b} + \underline{c}'; \text{Prop 3.2i} \rhd \underline{b} + \underline{c}' = \underline{b} + \underline{c}' \gg \underline{a} + \underline{b} + \underline{c}' = \underline{a} + \underline{b} + \underline{c}'; \text{S6} \gg \underline{a} + \underline{b} + \underline{c}' = \underline{a} + \underline{b} + \underline{c}'; \text{Prop 3.2c} \rhd \underline{a} + \underline{b} + \underline{c}' = \underline{a} + \underline{b} + \underline{c}' \rhd \underline{a} + \underline{b} + \underline{c}' = \underline{a} + \underline{b} + \underline{c}' \gg \underline{a} + \underline{b} + \underline{c}' = \underline{a} + \underline{b} + \underline{c}'; \text{Prop 3.2d} \rhd \underline{a} + \underline{b} + \underline{c}' = \underline{a} + \underline{b} + \underline{c}' \rhd \underline{a} + \underline{b} + \underline{c}' = \underline{a} + \underline{b} + \underline{c}' \gg \underline{a} + \underline{b} + \underline{c}' = \underline{a} + \underline{b} + \underline{c}'; \text{Ded} \rhd \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} + \underline{b} + \underline{c} = \underline{a} + \underline{b} + \underline{c} \vdash \underline{a} + \underline{b} + \underline{c}' = \underline{a} + \underline{b} + \underline{c}' \gg \underline{a} + \underline{b} + \underline{c} = \underline{a} + \underline{b} + \underline{c} \Rightarrow \underline{a} + \underline{b} + \underline{c}' = \underline{a} + \underline{b} + \underline{c}'\rceil, p_0, c)]$

$[\text{Prop 3.2j} \overset{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \text{Prop 3.2j}_1 \gg \bar{x} + \bar{y} + 0 = \bar{x} + \bar{y} + 0; \text{Prop 3.2j}_2 \gg \bar{x} + \bar{y} + \bar{z} = \bar{x} + \bar{y} + \bar{z} \Rightarrow \bar{x} + \bar{y} + \bar{z}' = \bar{x} + \bar{y} + \bar{z}'; \text{S9} \,@\, \bar{z} \rhd \bar{x} + \bar{y} + 0 = \bar{x} + \bar{y} + 0 \rhd \bar{x} + \bar{y} + \bar{z} = \bar{x} + \bar{y} + \bar{z} \Rightarrow \bar{x} + \bar{y} + \bar{z}' = \bar{x} + \bar{y} + \bar{z}' \gg \bar{x} + \bar{y} + \bar{z} = \bar{x} + \bar{y} + \bar{z}; \text{Ded} \rhd \bar{x} + \bar{y} + \bar{z} = \bar{x} + \bar{y} + \bar{z} \gg \underline{a} + \underline{b} + \underline{c} = \underline{a} + \underline{b} + \underline{c}\rceil, p_0, c)]$

$[\text{Prop 3.2k}_1 \overset{\text{stmt}}{\to} S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} = \underline{b} \Rightarrow \underline{a} \cdot 0 = \underline{b} \cdot 0]$

$[\text{Prop 3.2k}_1 \overset{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{a} \colon \forall \underline{b} \colon \underline{a} = \underline{b} \vdash \text{S7} \gg \underline{a} \cdot 0 = 0; \text{S7} \gg \underline{b} \cdot 0 = 0; \text{Prop 3.2a} \gg 0 = 0; \text{Prop 3.2b} \rhd \underline{b} \cdot 0 = 0 \gg 0 = \underline{b} \cdot 0; \text{Prop 3.2c} \rhd \underline{a} \cdot 0 = 0 \rhd 0 = \underline{b} \cdot 0 \gg \underline{a} \cdot 0 = \underline{b} \cdot 0; \text{Ded} \rhd \forall \underline{a} \colon \forall \underline{b} \colon \underline{a} = \underline{b} \vdash \underline{a} \cdot 0 = \underline{b} \cdot 0 \gg \underline{a} = \underline{b} \Rightarrow \underline{a} \cdot 0 = \underline{b} \cdot 0 \rceil, p_0, c)]$

$[\text{Prop 3.2k}_2 \overset{\text{stmt}}{\to} S \vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \underline{a} = \underline{b} \Rightarrow \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c} \Rightarrow \underline{a} = \underline{b} \Rightarrow \underline{a} \cdot \underline{c}' = \underline{b} \cdot \underline{c}']$

$[\text{Prop 3.2k}_2 \overset{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \underline{a} = \underline{b} \Rightarrow \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c} \vdash \underline{a} = \underline{b} \vdash \text{MP} \rhd \underline{a} = \underline{b} \Rightarrow \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c} \rhd \underline{a} = \underline{b} \gg \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c}; \text{S8} \gg \underline{a} \cdot \underline{c}' = \underline{a} \cdot \underline{c} + \underline{a}; \text{S8} \gg \underline{b} \cdot \underline{c}' = \underline{b} \cdot \underline{c} + \underline{b}; \text{Prop 3.2e} \rhd \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c} \gg \underline{a} \cdot \underline{c} + \underline{a} = \underline{b} \cdot \underline{c} + \underline{a}; \text{Prop 3.2i} \rhd \underline{a} = \underline{b} \gg \underline{b} \cdot \underline{c} + \underline{a} = \underline{b} \cdot \underline{c} + \underline{b}; \text{Prop 3.2c} \rhd \underline{a} \cdot \underline{c} + \underline{a} = \underline{b} \cdot \underline{c} + \underline{a} \rhd \underline{b} \cdot \underline{c} + \underline{a} = \underline{b} \cdot \underline{c} + \underline{b} \gg \underline{a} \cdot \underline{c} + \underline{a} = \underline{b} \cdot \underline{c} + \underline{b}; \text{Prop 3.2c} \rhd \underline{a} \cdot \underline{c}' = \underline{a} \cdot \underline{c} + \underline{a} \rhd \underline{a} \cdot \underline{c} + \underline{a} = \underline{b} \cdot \underline{c} + \underline{b} \gg \underline{a} \cdot \underline{c}' = \underline{b} \cdot \underline{c} + \underline{b}; \text{Prop 3.2d} \rhd \underline{a} \cdot \underline{c}' = \underline{b} \cdot \underline{c} + \underline{b} \rhd \underline{b} \cdot \underline{c}' = \underline{b} \cdot \underline{c} + \underline{b} \gg \underline{a} \cdot \underline{c}' = \underline{b} \cdot \underline{c}'; \text{Ded} \rhd \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \underline{a} = \underline{b} \Rightarrow \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c} \vdash \underline{a} = \underline{b} \vdash \underline{a} \cdot \underline{c}' = \underline{b} \cdot \underline{c}' \gg \underline{a} = \underline{b} \Rightarrow \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c} \Rightarrow \underline{a} = \underline{b} \Rightarrow \underline{a} \cdot \underline{c}' = \underline{b} \cdot \underline{c}' \rceil, p_0, c)]$

$[\text{Prop 3.2k} \overset{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \underline{a} = \underline{b} \vdash \text{Prop 3.2k}_1 \gg \bar{x} = \bar{y} \Rightarrow \bar{x} \cdot 0 = \bar{y} \cdot 0; \text{Prop 3.2k}_2 \gg \bar{x} = \bar{y} \Rightarrow \bar{x} \cdot \bar{z} = \bar{y} \cdot \bar{z} \Rightarrow \bar{x} = \bar{y} \Rightarrow \bar{x} \cdot \bar{z}' = \bar{y} \cdot \bar{z}'; \text{S9} @ \bar{z} \rhd \bar{x} = \bar{y} \Rightarrow \bar{x} \cdot 0 = \bar{y} \cdot 0 \rhd \bar{x} = \bar{y} \Rightarrow \bar{x} \cdot \bar{z} = \bar{y} \cdot \bar{z} \Rightarrow \bar{x} = \bar{y} \Rightarrow \bar{x} \cdot \bar{z}' = \bar{y} \cdot \bar{z}' \gg \bar{x} = \bar{y} \Rightarrow \bar{x} \cdot \bar{z} = \bar{y} \cdot \bar{z}; \text{Ded} \rhd \bar{x} = \bar{y} \Rightarrow \bar{x} \cdot \bar{z} = \bar{y} \cdot \bar{z} \gg \underline{a} = \underline{b} \Rightarrow \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c}; \text{MP} \rhd \underline{a} = \underline{b} \Rightarrow \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c} \rhd \underline{a} = \underline{b} \gg \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c} \rceil, p_0, c)]$

$[\text{Prop 3.2l}_2 \overset{\text{stmt}}{\to} S \vdash \forall \underline{a} \colon 0 \cdot \underline{a} = 0 \Rightarrow 0 \cdot \underline{a}' = 0]$

$[\text{Prop 3.2l}_2 \overset{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall \underline{a} \colon \forall \underline{a} \colon 0 \cdot \underline{a} = 0 \vdash \text{S8} \gg 0 \cdot \underline{a}' = 0 \cdot \underline{a} + 0; \text{S5} \gg 0 \cdot \underline{a} + 0 = 0 \cdot \underline{a}; \text{Prop 3.2c} \rhd 0 \cdot \underline{a}' = 0 \cdot \underline{a} + 0 \rhd 0 \cdot \underline{a} + 0 = 0 \cdot \underline{a} \gg 0 \cdot \underline{a}' = 0 \cdot \underline{a}; \text{Prop 3.2c} \rhd 0 \cdot \underline{a}' = 0 \cdot \underline{a} \rhd 0 \cdot \underline{a} = 0 \gg 0 \cdot \underline{a}' = 0; \text{Ded} \rhd \forall \underline{a} \colon 0 \cdot \underline{a} = 0 \vdash 0 \cdot \underline{a}' = 0 \gg 0 \cdot \underline{a} = 0 \Rightarrow 0 \cdot \underline{a}' = 0 \rceil, p_0, c)]$

$[\text{Prop 3.2l} \overset{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall \underline{a} \colon \text{S7} \gg 0 \cdot 0 = 0; \text{Prop 3.2l}_2 \gg 0 \cdot \bar{x} = 0 \Rightarrow 0 \cdot \bar{x}' = 0; \text{S9} @ \bar{x} \rhd 0 \cdot 0 = 0 \rhd 0 \cdot \bar{x} = 0 \Rightarrow 0 \cdot \bar{x}' = 0 \gg 0 \cdot \bar{x} = 0; \text{Ded} \rhd 0 \cdot \bar{x} = 0 \gg 0 \cdot \underline{a} = 0 \rceil, p_0, c)]$

$[\text{Prop 3.2m}_1 \overset{\text{stmt}}{\to} S \vdash \forall \underline{a} \colon \underline{a}' \cdot 0 = \underline{a} \cdot 0 + 0]$

$[\text{Prop 3.2m}_1 \overset{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall \underline{a} \colon \text{S7} \gg \underline{a}' \cdot 0 = 0; \text{Prop 3.2f} \gg 0 = 0 + 0; \text{S7} \gg \underline{a} \cdot 0 = 0; \text{Prop 3.2b} \rhd \underline{a} \cdot 0 = 0 \gg 0 = \underline{a} \cdot 0; \text{Prop 3.2e} \rhd 0 = \underline{a} \cdot 0 \gg 0 + 0 = \underline{a} \cdot 0 + 0; \text{Prop 3.2c} \rhd 0 = 0 + 0 \rhd 0 + 0 = \underline{a} \cdot 0 + 0 \gg 0 = \underline{a} \cdot 0 + 0; \text{Prop 3.2c} \rhd \underline{a}' \cdot 0 = 0 \rhd 0 = \underline{a} \cdot 0 + 0 \gg \underline{a}' \cdot 0 = \underline{a} \cdot 0 + 0 \rceil, p_0, c)]$

$[\text{Prop 3.2m}_2 \overset{\text{stmt}}{\to} S \vdash \forall \underline{a} \colon \forall \underline{b} \colon \underline{a}' \cdot \underline{b} = \underline{a} \cdot \underline{b} + \underline{b} \Rightarrow \underline{a}' \cdot \underline{b}' = \underline{a} \cdot \underline{b}' + \underline{b}']$

$[\text{Prop 3.2m}_2 \overset{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{a} \colon \forall \underline{b} \colon \underline{a}' \cdot \underline{b} = \underline{a} \cdot \underline{b} + \underline{b} \vdash \text{S8} \gg \underline{a}' \cdot \underline{b}' = \underline{a}' \cdot \underline{b} + \underline{a}'; \text{Prop 3.2e} \rhd \underline{a}' \cdot \underline{b} = \underline{a} \cdot \underline{b} + \underline{b} \gg \underline{a}' \cdot \underline{b} + \underline{a}' = \underline{a} \cdot \underline{b} + \underline{b} + \underline{a}'; \text{S6} \gg \underline{b} + \underline{a}' = \underline{b} + \underline{a}'; \text{Prop 3.2g} \gg \underline{b}' + \underline{a} = \underline{b} + \underline{a}'; \text{Prop 3.2d} \rhd \underline{b} + \underline{a}' = \underline{b} + \underline{a}' \rhd \underline{b}' + \underline{a} = \underline{b} + \underline{a}' \gg \underline{b} + \underline{a}' = \underline{b}' + \underline{a}; \text{Prop 3.2h} \gg \underline{b}' + \underline{a} = \underline{a} + \underline{b}'; \text{Prop 3.2c} \rhd \underline{b} + \underline{a}' = \underline{b}' + \underline{a} \rhd \underline{b}' + \underline{a} = \underline{a} + \underline{b}' \gg \underline{b} + \underline{a}' = \underline{a} + \underline{b}'; \text{Prop 3.2i} \rhd \underline{b} + \underline{a}' = \underline{a} + \underline{b}' \gg \underline{a} \cdot \underline{b} + \underline{b} + \underline{a}' =$

$\underline{a} \cdot \underline{b} + \underline{a} + \underline{b}'$; Prop 3.2j $\gg \underline{a} \cdot \underline{b} + \underline{a} + \underline{b}' = \underline{a} \cdot \underline{b} + \underline{a} + \underline{b}'$; Prop 3.2d $\rhd \underline{a} \cdot \underline{b} + \underline{b} + \underline{a}' = \underline{a} \cdot \underline{b} + \underline{a} + \underline{b}' \rhd \underline{a} \cdot \underline{b} + \underline{a} + \underline{b}' = \underline{a} \cdot \underline{b} + \underline{a} + \underline{b}' \gg \underline{a} \cdot \underline{b} + \underline{b} + \underline{a}' = \underline{a} \cdot \underline{b} + \underline{a} + \underline{b}'$; S8 $\gg \underline{a} \cdot \underline{b}' = \underline{a} \cdot \underline{b} + \underline{a}$; Prop 3.2e $\rhd \underline{a} \cdot \underline{b}' = \underline{a} \cdot \underline{b} + \underline{a} \gg \underline{a} \cdot \underline{b}' + \underline{b}' = \underline{a} \cdot \underline{b} + \underline{a} + \underline{b}'$; Prop 3.2d $\rhd \underline{a} \cdot \underline{b} + \underline{b} + \underline{a}' = \underline{a} \cdot \underline{b} + \underline{a} + \underline{b}' \rhd \underline{a} \cdot \underline{b}' + \underline{b}' = \underline{a} \cdot \underline{b} + \underline{a} + \underline{b}' \gg \underline{a} \cdot \underline{b} + \underline{b} + \underline{a}' = \underline{a} \cdot \underline{b}' + \underline{b}'$; Prop 3.2j $\gg \underline{a} \cdot \underline{b} + \underline{b} + \underline{a}' = \underline{a} \cdot \underline{b} + \underline{b} + \underline{a}'$; Prop 3.2c $\rhd \underline{a}' \cdot \underline{b} + \underline{a}' = \underline{a} \cdot \underline{b} + \underline{b} + \underline{a}' \rhd \underline{a} \cdot \underline{b} + \underline{b} + \underline{a}' = \underline{a} \cdot \underline{b} + \underline{b} + \underline{a}' \gg \underline{a}' \cdot \underline{b} + \underline{a}' = \underline{a} \cdot \underline{b} + \underline{b} + \underline{a}'$; Prop 3.2c $\rhd \underline{a}' \cdot \underline{b} + \underline{a}' = \underline{a} \cdot \underline{b} + \underline{b} + \underline{a}' \rhd \underline{a} \cdot \underline{b} + \underline{b} + \underline{a}' = \underline{a} \cdot \underline{b}' + \underline{b}' \gg \underline{a}' \cdot \underline{b} + \underline{a}' = \underline{a} \cdot \underline{b}' + \underline{b}'$; Prop 3.2c $\rhd \underline{a}' \cdot \underline{b}' = \underline{a}' \cdot \underline{b} + \underline{a}' \rhd \underline{a}' \cdot \underline{b} + \underline{a}' = \underline{a} \cdot \underline{b}' + \underline{b}' \gg \underline{a}' \cdot \underline{b}' = \underline{a} \cdot \underline{b}' + \underline{b}'$; Ded $\rhd \forall \underline{a} \colon \forall \underline{b} \colon \underline{a}' \cdot \underline{b} = \underline{a} \cdot \underline{b} + \underline{b} \vdash \underline{a}' \cdot \underline{b}' = \underline{a} \cdot \underline{b}' + \underline{b}' \gg \underline{a}' \cdot \underline{b} = \underline{a} \cdot \underline{b} + \underline{b} \Rightarrow \underline{a}' \cdot \underline{b}' = \underline{a} \cdot \underline{b}' + \underline{b}' \rceil, p_0, \mathsf{c})]$

[Prop 3.2m $\overset{\text{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil$S $\vdash \forall \underline{a} \colon \forall \underline{b} \colon$ Prop 3.2m$_1 \gg \bar{x}' \cdot 0 = \bar{x} \cdot 0 + 0$; Prop 3.2m$_2 \gg \bar{x}' \cdot \bar{y} = \bar{x} \cdot \bar{y} + \bar{y} \Rightarrow \bar{x}' \cdot \bar{y}' = \bar{x} \cdot \bar{y}' + \bar{y}'$; S9 @ $\bar{y} \rhd \bar{x}' \cdot 0 = \bar{x} \cdot 0 + 0 \rhd \bar{x}' \cdot \bar{y} = \bar{x} \cdot \bar{y} + \bar{y} \Rightarrow \bar{x}' \cdot \bar{y}' = \bar{x} \cdot \bar{y}' + \bar{y}' \gg \bar{x}' \cdot \bar{y} = \bar{x} \cdot \bar{y} + \bar{y}$; Ded $\rhd \bar{x}' \cdot \bar{y} = \bar{x} \cdot \bar{y} + \bar{y} \gg \underline{a}' \cdot \underline{b} = \underline{a} \cdot \underline{b} + \underline{b} \rceil, p_0, \mathsf{c})]$

[Prop 3.2n$_1 \overset{\text{stmt}}{\to}$ S $\vdash \forall \underline{a} \colon \underline{a} \cdot 0 = 0 \cdot \underline{a}]$

[Prop 3.2n$_1 \overset{\text{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil$S $\vdash \forall \underline{a} \colon$ S7 $\gg \underline{a} \cdot 0 = 0$; Prop 3.2l $\gg 0 \cdot \underline{a} = 0$; Prop 3.2d $\rhd \underline{a} \cdot 0 = 0 \rhd 0 \cdot \underline{a} = 0 \gg \underline{a} \cdot 0 = 0 \cdot \underline{a} \rceil, p_0, \mathsf{c})]$

[Prop 3.2n$_2 \overset{\text{stmt}}{\to}$ S $\vdash \forall \underline{a} \colon \forall \underline{b} \colon \underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a} \Rightarrow \underline{a} \cdot \underline{b}' = \underline{b}' \cdot \underline{a}]$

[Prop 3.2n$_2 \overset{\text{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil$S $\vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{a} \colon \forall \underline{b} \colon \underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a} \vdash$ S8 $\gg \underline{a} \cdot \underline{b}' = \underline{a} \cdot \underline{b} + \underline{a}$; Prop 3.2e $\rhd \underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a} \gg \underline{a} \cdot \underline{b} + \underline{a} = \underline{b} \cdot \underline{a} + \underline{a}$; Prop 3.2m $\gg \underline{b}' \cdot \underline{a} = \underline{b} \cdot \underline{a} + \underline{a}$; Prop 3.2b $\rhd \underline{b}' \cdot \underline{a} = \underline{b} \cdot \underline{a} + \underline{a} \gg \underline{b} \cdot \underline{a} + \underline{a} = \underline{b}' \cdot \underline{a}$; Prop 3.2c $\rhd \underline{a} \cdot \underline{b} + \underline{a} = \underline{b} \cdot \underline{a} + \underline{a} \rhd \underline{b} \cdot \underline{a} + \underline{a} = \underline{b}' \cdot \underline{a} \gg \underline{a} \cdot \underline{b} + \underline{a} = \underline{b}' \cdot \underline{a}$; Prop 3.2c $\rhd \underline{a} \cdot \underline{b}' = \underline{a} \cdot \underline{b} + \underline{a} \rhd \underline{a} \cdot \underline{b} + \underline{a} = \underline{b}' \cdot \underline{a} \gg \underline{a} \cdot \underline{b}' = \underline{b}' \cdot \underline{a}$; Ded $\rhd \forall \underline{a} \colon \forall \underline{b} \colon \underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a} \vdash \underline{a} \cdot \underline{b}' = \underline{b}' \cdot \underline{a} \gg \underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a} \Rightarrow \underline{a} \cdot \underline{b}' = \underline{b}' \cdot \underline{a} \rceil, p_0, \mathsf{c})]$

[Prop 3.2n $\overset{\text{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil$S $\vdash \forall \underline{a} \colon \forall \underline{b} \colon$ Prop 3.2n$_1 \gg \bar{x} \cdot 0 = 0 \cdot \bar{x}$; Prop 3.2n$_2 \gg \bar{x} \cdot \bar{y} = \bar{y} \cdot \bar{x} \Rightarrow \bar{x} \cdot \bar{y}' = \bar{y}' \cdot \bar{x}$; S9 @ $\bar{y} \rhd \bar{x} \cdot 0 = 0 \cdot \bar{x} \rhd \bar{x} \cdot \bar{y} = \bar{y} \cdot \bar{x} \Rightarrow \bar{x} \cdot \bar{y}' = \bar{y}' \cdot \bar{x} \gg \bar{x} \cdot \bar{y} = \bar{y} \cdot \bar{x}$; Ded $\rhd \bar{x} \cdot \bar{y} = \bar{y} \cdot \bar{x} \gg \underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a} \rceil, p_0, \mathsf{c})]$

[Prop 3.2o $\overset{\text{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil$S $\vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \underline{a} = \underline{b} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \underline{a} = \underline{b} \vdash$ Prop 3.2k $\rhd \underline{a} = \underline{b} \gg \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c}$; Prop 3.2n $\gg \underline{a} \cdot \underline{c} = \underline{c} \cdot \underline{a}$; Prop 3.2n $\gg \underline{b} \cdot \underline{c} = \underline{c} \cdot \underline{b}$; Prop 3.2c $\rhd \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c} \rhd \underline{b} \cdot \underline{c} = \underline{c} \cdot \underline{b} \gg \underline{a} \cdot \underline{c} = \underline{c} \cdot \underline{b}$; S1 $\rhd \underline{a} \cdot \underline{c} = \underline{c} \cdot \underline{a} \rhd \underline{a} \cdot \underline{c} = \underline{c} \cdot \underline{b} \gg \underline{c} \cdot \underline{a} = \underline{c} \cdot \underline{b}$; Ded $\rhd \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \underline{a} = \underline{b} \vdash \underline{c} \cdot \underline{a} = \underline{c} \cdot \underline{b} \gg \underline{a} = \underline{b} \Rightarrow \underline{c} \cdot \underline{a} = \underline{c} \cdot \underline{b}$; MP $\rhd \underline{a} = \underline{b} \Rightarrow \underline{c} \cdot \underline{a} = \underline{c} \cdot \underline{b} \rhd \underline{a} = \underline{b} \gg \underline{c} \cdot \underline{a} = \underline{c} \cdot \underline{b} \rceil, p_0, \mathsf{c})]$

And thus we hve proved Prop 3.2.

3 Proof of Prop 3.4

in this section we will prove Prop 3.4 from [Men97]

[Prop 3.4a $\overset{\text{stmt}}{\to}$ S $\vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \underline{a} \cdot \underline{b} + \underline{c} = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c}]$

[Prop 3.4b $\overset{\text{stmt}}{\to}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{b}+\underline{c}\cdot\underline{a} = \underline{b}\cdot\underline{a}+\underline{c}\cdot\underline{a}$]

[Prop 3.4c $\overset{\text{stmt}}{\to}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a}\cdot\underline{b}\cdot\underline{c} = \underline{a}\cdot\underline{b}\cdot\underline{c}$]

[Prop 3.4d $\overset{\text{stmt}}{\to}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a}+\underline{c} = \underline{b}+\underline{c} \Rightarrow \underline{a} = \underline{b}$]

[Prop 3.4a$_1$ $\overset{\text{stmt}}{\to}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a}\cdot\underline{b}+0 = \underline{a}\cdot\underline{b}+\underline{a}\cdot 0$]

[Prop 3.4a$_1$ $\overset{\text{proof}}{\to}$ $\lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \text{S5} \gg \underline{b}+0=\underline{b}; \text{Prop 3.2o} \rhd \underline{b}+0=\underline{b} \gg \underline{a}\cdot\underline{b}+0=\underline{a}\cdot\underline{b}; \text{S5} \gg \underline{a}\cdot\underline{b}+0=\underline{a}\cdot\underline{b}; \text{Prop 3.2d} \rhd \underline{a}\cdot\underline{b}+0=\underline{a}\cdot\underline{b} \rhd \underline{a}\cdot\underline{b}+0=\underline{a}\cdot\underline{b} \gg \underline{a}\cdot\underline{b}+0=\underline{a}\cdot\underline{b}+0; \text{S7} \gg \underline{a}\cdot 0=0; \text{Prop 3.2i} \rhd \underline{a}\cdot 0=0 \gg \underline{a}\cdot\underline{b}+\underline{a}\cdot 0=\underline{a}\cdot\underline{b}+0; \text{Prop 3.2d} \rhd \underline{a}\cdot\underline{b}+0=\underline{a}\cdot\underline{b}+0 \rhd \underline{a}\cdot\underline{b}+\underline{a}\cdot 0=\underline{a}\cdot\underline{b}+0 \gg \underline{a}\cdot\underline{b}+0=\underline{a}\cdot\underline{b}+\underline{a}\cdot 0\rceil, p_0, c)$]

[Prop 3.4a$_2$ $\overset{\text{stmt}}{\to}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a}\cdot\underline{b}+\underline{c} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c} \Rightarrow \underline{a}\cdot\underline{b}+\underline{c}' = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}'$]

[Prop 3.4a$_2$ $\overset{\text{proof}}{\to}$ $\lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a}\cdot\underline{b}+\underline{c} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c} \vdash \text{S6} \gg \underline{b}+\underline{c}' = \underline{b}+\underline{c}'; \text{Prop 3.2o} \rhd \underline{b}+\underline{c}' = \underline{b}+\underline{c}' \gg \underline{a}\cdot\underline{b}+\underline{c}' = \underline{a}\cdot\underline{b}+\underline{c}'; \text{S8} \gg \underline{a}\cdot\underline{b}+\underline{c}' = \underline{a}\cdot\underline{b}+\underline{c}+\underline{a}; \text{Prop 3.2e} \rhd \underline{a}\cdot\underline{b}+\underline{c} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c} \gg \underline{a}\cdot\underline{b}+\underline{c}+\underline{a} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}+\underline{a}; \text{Prop 3.2j} \gg \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}+\underline{a} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}+\underline{a}; \text{Prop 3.2c} \rhd \underline{a}\cdot\underline{b}+\underline{c}+\underline{a} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}+\underline{a} \rhd \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}+\underline{a} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}+\underline{a} \gg \underline{a}\cdot\underline{b}+\underline{c}+\underline{a} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}+\underline{a}; \text{S8} \gg \underline{a}\cdot\underline{c}' = \underline{a}\cdot\underline{c}+\underline{a}; \text{Prop 3.2i} \rhd \underline{a}\cdot\underline{c}' = \underline{a}\cdot\underline{c}+\underline{a} \gg \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}' = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}+\underline{a}; \text{Prop 3.2d} \rhd \underline{a}\cdot\underline{b}+\underline{c}+\underline{a} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}+\underline{a} \rhd \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}' = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}+\underline{a} \gg \underline{a}\cdot\underline{b}+\underline{c}+\underline{a} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}'; \text{Prop 3.2c} \rhd \underline{a}\cdot\underline{b}+\underline{c}' = \underline{a}\cdot\underline{b}+\underline{c}+\underline{a} \rhd \underline{a}\cdot\underline{b}+\underline{c}+\underline{a} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}' \gg \underline{a}\cdot\underline{b}+\underline{c}' = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}'; \text{Prop 3.2c} \rhd \underline{a}\cdot\underline{b}+\underline{c}' = \underline{a}\cdot\underline{b}+\underline{c}' \rhd \underline{a}\cdot\underline{b}+\underline{c}' = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}' \gg \underline{a}\cdot\underline{b}+\underline{c}' = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}'; \text{Ded} \rhd \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a}\cdot\underline{b}+\underline{c} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c} \vdash \underline{a}\cdot\underline{b}+\underline{c}' = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}' \gg \underline{a}\cdot\underline{b}+\underline{c} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c} \Rightarrow \underline{a}\cdot\underline{b}+\underline{c}' = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}'\rceil, p_0, c)$]

[Prop 3.4a $\overset{\text{proof}}{\to}$ $\lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \text{Prop 3.4a}_1 \gg \bar{x}\cdot\bar{y}+0 = \bar{x}\cdot\bar{y}+\bar{x}\cdot 0; \text{Prop 3.4a}_2 \gg \bar{x}\cdot\bar{y}+\bar{z} = \bar{x}\cdot\bar{y}+\bar{x}\cdot\bar{z} \Rightarrow \bar{x}\cdot\bar{y}+\bar{z}' = \bar{x}\cdot\bar{y}+\bar{x}\cdot\bar{z}'; \text{S9} @ \bar{z} \rhd \bar{x}\cdot\bar{y}+0 = \bar{x}\cdot\bar{y}+\bar{x}\cdot 0 \rhd \bar{x}\cdot\bar{y}+\bar{z} = \bar{x}\cdot\bar{y}+\bar{x}\cdot\bar{z} \Rightarrow \bar{x}\cdot\bar{y}+\bar{z}' = \bar{x}\cdot\bar{y}+\bar{x}\cdot\bar{z}' \gg \bar{x}\cdot\bar{y}+\bar{z} = \bar{x}\cdot\bar{y}+\bar{x}\cdot\bar{z}; \text{Ded} \rhd \bar{x}\cdot\bar{y}+\bar{z} = \bar{x}\cdot\bar{y}+\bar{x}\cdot\bar{z} \gg \underline{a}\cdot\underline{b}+\underline{c} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}\rceil, p_0, c)$]

[Prop 3.4b $\overset{\text{proof}}{\to}$ $\lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \text{Prop 3.4a} \gg \underline{a}\cdot\underline{b}+\underline{c} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}; \text{Prop 3.2n} \gg \underline{a}\cdot\underline{b}+\underline{c} = \underline{b}+\underline{c}\cdot\underline{a}; \text{Prop 3.2n} \gg \underline{a}\cdot\underline{b} = \underline{b}\cdot\underline{a}; \text{Prop 3.2n} \gg \underline{a}\cdot\underline{c} = \underline{c}\cdot\underline{a}; \text{S1} \rhd \underline{a}\cdot\underline{b}+\underline{c} = \underline{b}+\underline{c}\cdot\underline{a} \rhd \underline{a}\cdot\underline{b}+\underline{c} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c} \gg \underline{b}+\underline{c}\cdot\underline{a} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}; \text{Prop 3.2e} \rhd \underline{a}\cdot\underline{b} = \underline{b}\cdot\underline{a} \gg \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c} = \underline{b}\cdot\underline{a}+\underline{a}\cdot\underline{c}; \text{Prop 3.2i} \rhd \underline{a}\cdot\underline{c} = \underline{c}\cdot\underline{a} \gg \underline{b}\cdot\underline{a}+\underline{a}\cdot\underline{c} = \underline{b}\cdot\underline{a}+\underline{c}\cdot\underline{a}; \text{Prop 3.2c} \rhd \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c} = \underline{b}\cdot\underline{a}+\underline{a}\cdot\underline{c} \rhd \underline{b}\cdot\underline{a}+\underline{a}\cdot\underline{c} = \underline{b}\cdot\underline{a}+\underline{c}\cdot\underline{a} \gg \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c} = \underline{b}\cdot\underline{a}+\underline{c}\cdot\underline{a}; \text{Prop 3.2c} \rhd \underline{b}+\underline{c}\cdot\underline{a} = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c} \rhd \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c} = \underline{b}\cdot\underline{a}+\underline{c}\cdot\underline{a} \gg \underline{b}+\underline{c}\cdot\underline{a} = \underline{b}\cdot\underline{a}+\underline{c}\cdot\underline{a}\rceil, p_0, c)$]

[Prop 3.4c$_1$ $\overset{\text{stmt}}{\to}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a}\cdot\underline{b}\cdot 0 = \underline{a}\cdot\underline{b}\cdot 0$]

[Prop 3.4c$_1$ $\overset{\text{proof}}{\to}$ $\lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \text{S7} \gg \underline{a}\cdot\underline{b}\cdot 0 = 0; \text{S7} \gg \underline{b}\cdot 0 = 0; \text{Prop 3.2o} \rhd \underline{b}\cdot 0 = 0 \gg \underline{a}\cdot\underline{b}\cdot 0 = \underline{a}\cdot 0; \text{S7} \gg \underline{a}\cdot 0 = 0; \text{Prop 3.2c} \rhd \underline{a}\cdot\underline{b}\cdot 0 = \underline{a}\cdot 0 \rhd \underline{a}\cdot 0 = 0 \gg \underline{a}\cdot\underline{b}\cdot 0 = 0; \text{Prop 3.2d} \rhd \underline{a}\cdot\underline{b}\cdot 0 = 0 \rhd \underline{a}\cdot\underline{b}\cdot 0 = 0 \gg \underline{a}\cdot\underline{b}\cdot 0 = \underline{a}\cdot\underline{b}\cdot 0\rceil, p_0, c)$]

[Prop 3.4c$_2$ $\overset{\text{stmt}}{\to}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a}\cdot\underline{b}\cdot\underline{c} = \underline{a}\cdot\underline{b}\cdot\underline{c} \Rightarrow \underline{a}\cdot\underline{b}\cdot\underline{c}' = \underline{a}\cdot\underline{b}\cdot\underline{c}'$]

[Prop $3.4c_2 \overset{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a}\cdot\underline{b}\cdot\underline{c} = \underline{a}\cdot\underline{b}\cdot\underline{c} \vdash S8 \gg \underline{a}\cdot\underline{b}\cdot\underline{c}' = \underline{a}\cdot\underline{b}\cdot\underline{c} + \underline{a}\cdot\underline{b};$ Prop $3.2e \rhd \underline{a}\cdot\underline{b}\cdot\underline{c} = \underline{a}\cdot\underline{b}\cdot\underline{c} \gg \underline{a}\cdot\underline{b}\cdot\underline{c} + \underline{a}\cdot\underline{b} = \underline{a}\cdot\underline{b}\cdot\underline{c} + \underline{a}\cdot\underline{b};$ Prop $3.2c \rhd \underline{a}\cdot\underline{b}\cdot\underline{c}' = \underline{a}\cdot\underline{b}\cdot\underline{c} + \underline{a}\cdot\underline{b} \rhd \underline{a}\cdot\underline{b}\cdot\underline{c} + \underline{a}\cdot\underline{b} = \underline{a}\cdot\underline{b}\cdot\underline{c} + \underline{a}\cdot\underline{b} \gg \underline{a}\cdot\underline{b}\cdot\underline{c}' = \underline{a}\cdot\underline{b}\cdot\underline{c} + \underline{a}\cdot\underline{b};$ Prop $3.4a \gg \underline{a}\cdot\underline{b}\cdot\underline{c} + \underline{b} = \underline{a}\cdot\underline{b}\cdot\underline{c} + \underline{a}\cdot\underline{b};$ Prop $3.2d \rhd \underline{a}\cdot\underline{b}\cdot\underline{c}' = \underline{a}\cdot\underline{b}\cdot\underline{c} + \underline{a}\cdot\underline{b} \rhd \underline{a}\cdot\underline{b}\cdot\underline{c} + \underline{b} = \underline{a}\cdot\underline{b}\cdot\underline{c} + \underline{a}\cdot\underline{b} \gg \underline{a}\cdot\underline{b}\cdot\underline{c}' = \underline{a}\cdot\underline{b}\cdot\underline{c} + \underline{b}; S8 \gg \underline{b}\cdot\underline{c}' = \underline{b}\cdot\underline{c} + \underline{b};$ Prop $3.2o \rhd \underline{b}\cdot\underline{c}' = \underline{b}\cdot\underline{c} + \underline{b} \gg \underline{a}\cdot\underline{b}\cdot\underline{c}' = \underline{a}\cdot\underline{b}\cdot\underline{c} + \underline{b};$ Prop $3.2d \rhd \underline{a}\cdot\underline{b}\cdot\underline{c}' = \underline{a}\cdot\underline{b}\cdot\underline{c} + \underline{b} \rhd \underline{a}\cdot\underline{b}\cdot\underline{c}' = \underline{a}\cdot\underline{b}\cdot\underline{c} + \underline{b} \gg \underline{a}\cdot\underline{b}\cdot\underline{c}' = \underline{a}\cdot\underline{b}\cdot\underline{c}';$ Ded $\rhd \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a}\cdot\underline{b}\cdot\underline{c} = \underline{a}\cdot\underline{b}\cdot\underline{c} \vdash \underline{a}\cdot\underline{b}\cdot\underline{c}' = \underline{a}\cdot\underline{b}\cdot\underline{c}' \gg \underline{a}\cdot\underline{b}\cdot\underline{c} = \underline{a}\cdot\underline{b}\cdot\underline{c} \Rightarrow \underline{a}\cdot\underline{b}\cdot\underline{c}' = \underline{a}\cdot\underline{b}\cdot\underline{c}'\rceil, p_0, c)$]

[Prop $3.4c \overset{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon$ Prop $3.4c_1 \gg \bar{x}\cdot\bar{y}\cdot 0 = \bar{x}\cdot\bar{y}\cdot 0;$ Prop $3.4c_2 \gg \bar{x}\cdot\bar{y}\cdot\bar{z} = \bar{x}\cdot\bar{y}\cdot\bar{z} \Rightarrow \bar{x}\cdot\bar{y}\cdot\bar{z}' = \bar{x}\cdot\bar{y}\cdot\bar{z}'; S9 \mathbin{@} \bar{z} \rhd \bar{x}\cdot\bar{y}\cdot 0 = \bar{x}\cdot\bar{y}\cdot 0 \rhd \bar{x}\cdot\bar{y}\cdot\bar{z} = \bar{x}\cdot\bar{y}\cdot\bar{z} \Rightarrow \bar{x}\cdot\bar{y}\cdot\bar{z}' = \bar{x}\cdot\bar{y}\cdot\bar{z}' \gg \bar{x}\cdot\bar{y}\cdot\bar{z} = \bar{x}\cdot\bar{y}\cdot\bar{z};$ Ded $\rhd \bar{x}\cdot\bar{y}\cdot\bar{z} = \bar{x}\cdot\bar{y}\cdot\bar{z} \gg \underline{a}\cdot\underline{b}\cdot\underline{c} = \underline{a}\cdot\underline{b}\cdot\underline{c}\rceil, p_0, c)$]

[Prop $3.4d_1 \overset{\text{stmt}}{\to} S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} + 0 = \underline{b} + 0 \Rightarrow \underline{a} = \underline{b}$]

[Prop $3.4d_1 \overset{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} + 0 = \underline{b} + 0 \vdash S5 \gg \underline{a} + 0 = \underline{a}; S5 \gg \underline{b} + 0 = \underline{b}; S1 \rhd \underline{a} + 0 = \underline{a} \rhd \underline{a} + 0 = \underline{b} + 0 \gg \underline{a} = \underline{b} + 0;$ Prop $3.2c \rhd \underline{a} = \underline{b} + 0 \rhd \underline{b} + 0 = \underline{b} \gg \underline{a} = \underline{b};$ Ded $\rhd \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} + 0 = \underline{b} + 0 \vdash \underline{a} = \underline{b} \gg \underline{a} + 0 = \underline{b} + 0 \Rightarrow \underline{a} = \underline{b}\rceil, p_0, c)$]

[Prop $3.4d_2 \overset{\text{stmt}}{\to} S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a}+\underline{c} = \underline{b}+\underline{c} \Rightarrow \underline{a} = \underline{b} \Rightarrow \underline{a}+\underline{c}' = \underline{b}+\underline{c}' \Rightarrow \underline{a} = \underline{b}$]

[Prop $3.4d_2 \overset{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} + \underline{c} = \underline{b} + \underline{c} \Rightarrow \underline{a} = \underline{b} \vdash \underline{a} + \underline{c}' = \underline{b} + \underline{c}' \vdash S6 \gg \underline{a} + \underline{c}' = \underline{a} + \underline{c}'; S6 \gg \underline{b} + \underline{c}' = \underline{b} + \underline{c}'; S1 \rhd \underline{a} + \underline{c}' = \underline{a} + \underline{c}' \rhd \underline{a} + \underline{c}' = \underline{b} + \underline{c}' \gg \underline{a} + \underline{c}' = \underline{b} + \underline{c}';$ Prop $3.2c \rhd \underline{a} + \underline{c}' = \underline{b} + \underline{c}' \rhd \underline{b} + \underline{c}' = \underline{b} + \underline{c}' \gg \underline{a} + \underline{c}' = \underline{b} + \underline{c}'; S4 \rhd \underline{a} + \underline{c}' = \underline{b} + \underline{c}' \gg \underline{a} + \underline{c} = \underline{b} + \underline{c};$ MP $\rhd \underline{a} + \underline{c} = \underline{b} + \underline{c} \Rightarrow \underline{a} = \underline{b} \rhd \underline{a} + \underline{c} = \underline{b} + \underline{c} \gg \underline{a} = \underline{b};$ Ded $\rhd \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} + \underline{c} = \underline{b} + \underline{c} \Rightarrow \underline{a} = \underline{b} \vdash \underline{a} + \underline{c}' = \underline{b} + \underline{c}' \vdash \underline{a} = \underline{b} \gg \underline{a} + \underline{c} = \underline{b} + \underline{c} \Rightarrow \underline{a} = \underline{b} \Rightarrow \underline{a} + \underline{c}' = \underline{b} + \underline{c}' \Rightarrow \underline{a} = \underline{b}\rceil, p_0, c)$]

[Prop $3.4d \overset{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon$ Prop $3.4d_1 \gg \bar{x} + 0 = \bar{y} + 0 \Rightarrow \bar{x} = \bar{y};$ Prop $3.4d_2 \gg \bar{x} + \bar{z} = \bar{y} + \bar{z} \Rightarrow \bar{x} = \bar{y} \Rightarrow \bar{x} + \bar{z}' = \bar{y} + \bar{z}' \Rightarrow \bar{x} = \bar{y}; S9 \mathbin{@} \bar{z} \rhd \bar{x} + 0 = \bar{y} + 0 \Rightarrow \bar{x} = \bar{y} \rhd \bar{x} + \bar{z} = \bar{y} + \bar{z} \Rightarrow \bar{x} = \bar{y} \Rightarrow \bar{x} + \bar{z}' = \bar{y} + \bar{z}' \Rightarrow \bar{x} = \bar{y} \gg \bar{x} + \bar{z} = \bar{y} + \bar{z} \Rightarrow \bar{x} = \bar{y};$ Ded $\rhd \bar{x} + \bar{z} = \bar{y} + \bar{z} \Rightarrow \bar{x} = \bar{y} \gg \underline{a} + \underline{c} = \underline{b} + \underline{c} \Rightarrow \underline{a} = \underline{b}\rceil, p_0, c)$]

4 Numerals

We will for the remainder of the project need to construct some numerals, the numerals are the names for $0, 0', 0'' \ldots$ we will define them as folows $0, 0', 0'' \ldots$ With these definitions we will now continue to prove Prop 3.5 that states different things about numerals.

[Prop $3.5a \overset{\text{stmt}}{\to} S \vdash \forall \underline{a}\colon \underline{a} + 0' = \underline{a}'$]

[Prop $3.5b \overset{\text{stmt}}{\to} S \vdash \forall \underline{a}\colon \underline{a} \cdot 0' = \underline{a}$]

[Prop $3.5c \overset{\text{stmt}}{\to} S \vdash \forall \underline{a}\colon \underline{a} \cdot 0'' = \underline{a} + \underline{a}$]

[Prop 3.5d $\stackrel{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a} \colon \forall \underline{b} \colon \underline{a} + \underline{b} = 0 \Rightarrow \neg \underline{a} = 0 \Rightarrow \underline{b} = 0$]

[Prop 3.5e $\stackrel{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a} \colon \forall \underline{b} \colon \neg \underline{a} = 0 \Rightarrow \underline{b} \cdot \underline{a} = 0 \Rightarrow \underline{b} = 0$]

[Prop 3.5f $\stackrel{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a} \colon \forall \underline{b} \colon \underline{a} + \underline{b} = 0' \Rightarrow \neg\neg \underline{a} = 0 \Rightarrow \underline{b} = 0' \Rightarrow \neg \underline{a} = 0' \Rightarrow \underline{b} = 0$]

[Prop 3.5g $\stackrel{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a} \colon \forall \underline{b} \colon \underline{a} \cdot \underline{b} = 0' \Rightarrow \neg \underline{a} = 0' \Rightarrow \underline{b} = 0'$]

[Prop 3.5h $\stackrel{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a} \colon \neg \underline{a} = 0 \Rightarrow \neg \forall_{\text{obj}} \underline{b} \colon \neg \underline{a} = \underline{b}'$]

[Prop 3.5i $\stackrel{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \neg \underline{c} = 0 \Rightarrow \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c} \Rightarrow \underline{a} = \underline{b}$]

[Prop 3.5j $\stackrel{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a} \colon \neg \underline{a} = 0 \Rightarrow \neg \underline{a} = 0' \Rightarrow \neg \forall_{\text{obj}} \underline{b} \colon \neg \underline{a} = \underline{b}''$]

[Prop 3.5a $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall \underline{a} \colon$ S6 $\gg \underline{a} + 0' = \underline{a} + 0'$; S5 $\gg \underline{a} + 0 = \underline{a}$; S2 $\rhd \underline{a} + 0 = \underline{a} \gg \underline{a} + 0' = \underline{a}'$; Prop 3.2c $\rhd \underline{a} + 0' = \underline{a} + 0' \rhd \underline{a} + 0' = \underline{a}' \gg \underline{a} + 0' = \underline{a}'$; Prop 3.2a $\gg \underline{a} + 0' = \underline{a} + 0'$; S1 $\rhd \underline{a} + 0' = \underline{a} + 0' \rhd \underline{a} + 0' = \underline{a}' \gg \underline{a} + 0' = \underline{a}' \rceil, p_0, c)$]

[Prop 3.5b $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall \underline{a} \colon$ S8 $\gg \underline{a} \cdot 0' = \underline{a} \cdot 0 + \underline{a}$; S7 $\gg \underline{a} \cdot 0 = 0$; Prop 3.2e $\rhd \underline{a} \cdot 0 = 0 \gg \underline{a} \cdot 0 + \underline{a} = 0 + \underline{a}$; Prop 3.2c $\rhd \underline{a} \cdot 0' = \underline{a} \cdot 0 + \underline{a} \rhd \underline{a} \cdot 0 + \underline{a} = 0 + \underline{a} \gg \underline{a} \cdot 0' = 0 + \underline{a}$; Prop 3.2f $\gg \underline{a} = 0 + \underline{a}$; Prop 3.2b $\rhd \underline{a} = 0 + \underline{a} \gg 0 + \underline{a} = \underline{a}$; Prop 3.2c $\rhd \underline{a} \cdot 0' = 0 + \underline{a} \rhd 0 + \underline{a} = \underline{a} \gg \underline{a} \cdot 0' = \underline{a}$; Prop 3.2a $\gg \underline{a} \cdot 0' = \underline{a} \cdot 0'$; S1 $\rhd \underline{a} \cdot 0' = \underline{a} \cdot 0' \rhd \underline{a} \cdot 0' = \underline{a} \gg \underline{a} \cdot 0' = \underline{a} \rceil, p_0, c)$]

[Prop 3.5c $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil S \vdash \forall \underline{a} \colon$ S8 $\gg \underline{a} \cdot 0'' = \underline{a} \cdot 0' + \underline{a}$; Prop 3.5b $\gg \underline{a} \cdot 0' = \underline{a}$; Prop 3.2e $\rhd \underline{a} \cdot 0' = \underline{a} \gg \underline{a} \cdot 0' + \underline{a} = \underline{a} + \underline{a}$; Prop 3.2c $\rhd \underline{a} \cdot 0'' = \underline{a} \cdot 0' + \underline{a} \rhd \underline{a} \cdot 0' + \underline{a} = \underline{a} + \underline{a} \gg \underline{a} \cdot 0'' = \underline{a} + \underline{a}$; Prop 3.2a $\gg \underline{a} \cdot 0'' = \underline{a} \cdot 0''$; S1 $\rhd \underline{a} \cdot 0'' = \underline{a} \cdot 0'' \rhd \underline{a} \cdot 0'' = \underline{a} + \underline{a} \gg \underline{a} \cdot 0'' = \underline{a} + \underline{a} \rceil, p_0, c)$]

[Prop 3.5d$_1$ $\stackrel{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a} \colon \underline{a} + 0 \Rightarrow \underline{a} = 0 \wedge 0 = 0$]

[Prop 3.5d$_2$ $\stackrel{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a} \colon \forall \underline{b} \colon \underline{a} + \underline{b} = 0 \Rightarrow \neg \underline{a} = 0 \Rightarrow \underline{b} = 0 \Rightarrow \underline{a} + \underline{b}' = 0 \Rightarrow \neg \underline{a} = 0 \Rightarrow \underline{b}' = 0$]

5 **Lemma** Prop 3.7

5.1 Definitions of $\neg \forall_{\text{obj}} z \colon \neg\neg\neg z = 0 \Rightarrow z + \overline{x} = \overline{y}$

in the folowing we need some definitions, we will macro define them as:
[$x < y \stackrel{\text{macro}}{\rightarrow} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, \lceil [x < y \ddot{=} \exists z \colon (z \neq 0 \wedge z + x = y)] \rceil)$]
[$x \leq y \stackrel{\text{macro}}{\rightarrow} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, \lceil [x \leq y \ddot{=} x < y \vee x = y] \rceil)$]
[$x > y \stackrel{\text{macro}}{\rightarrow} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, \lceil [x > y \ddot{=} y < x] \rceil)$]
[$x \geq y \stackrel{\text{macro}}{\rightarrow} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, \lceil [x \geq y \ddot{=} y \leq x] \rceil)$]
[$x \not< y \stackrel{\text{macro}}{\rightarrow} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, \lceil [x \not< y \ddot{=} \neg(x < y)] \rceil)$]
[$x \not> y \stackrel{\text{macro}}{\rightarrow} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, \lceil [x \not> y \ddot{=} y \not< x] \rceil)$]
[$x \mid y \stackrel{\text{macro}}{\rightarrow} \lambda t.\lambda s.\lambda c.\tilde{\mathcal{M}}_4(t, s, c, \lceil [x \mid y \ddot{=} \exists z \colon y = x \cdot z] \rceil)$]

with these definitions, we will continue to prove the following statements:

[Prop 3.7a $\overset{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a}\colon \neg\neg\forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{a} = \underline{a}$]

[Prop 3.7b $\overset{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \neg\forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{a} = \underline{b} \Rightarrow \neg\forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{b} = \underline{c} \Rightarrow \neg\forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{a} = \underline{c}$]

[Prop 3.7c $\overset{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \neg\forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{a} = \underline{b} \Rightarrow \neg\neg\forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{b} = \underline{a}$]

[Prop 3.7d $\overset{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \neg\forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{a} = \underline{b} \Rightarrow \neg\forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{a} + \underline{c} = \underline{b} + \underline{c}$]

[Prop 3.7e $\overset{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a}\colon \neg\neg\forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{a} = \underline{a} \Rightarrow \underline{a} = \underline{a}$]

[Prop 3.7f $\overset{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \neg\neg\forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{a} = \underline{b} \Rightarrow \underline{a} = \underline{b} \Rightarrow \neg\neg\forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{b} = \underline{c} \Rightarrow \underline{b} = \underline{c} \Rightarrow \neg\neg\forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{a} = \underline{c} \Rightarrow \underline{a} = \underline{c}$]

[Prop 3.7g $\overset{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \neg\neg\forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{a} = \underline{b} \Rightarrow \underline{a} = \underline{b} \Rightarrow \neg\neg\forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{a} + \underline{c} = \underline{b} + \underline{c} \Rightarrow \underline{a} + \underline{c} = \underline{b} + \underline{c}$]

[Prop 3.7g′ $\overset{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \neg\neg\forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{a} + \underline{c} = \underline{b} + \underline{c} \Rightarrow \underline{a} + \underline{c} = \underline{b} + \underline{c} \Rightarrow \neg\neg\forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{a} = \underline{b} \Rightarrow \underline{a} = \underline{b}$]

[Prop 3.7h $\overset{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \neg\neg\forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{a} = \underline{b} \Rightarrow \underline{a} = \underline{b} \Rightarrow \neg\forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{b} = \underline{c} \Rightarrow \neg\forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{a} = \underline{c}$]

[Prop 3.7i $\overset{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a}\colon \neg\neg\forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + 0 = \underline{a} \Rightarrow 0 = \underline{a}$]

[Prop 3.7j $\overset{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a}\colon \neg\forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + 0 = \underline{a}'$]

[Prop 3.7k $\overset{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \neg\forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{a} = \underline{b} \Rightarrow \neg\neg\forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{a}' = \underline{b} \Rightarrow \underline{a}' = \underline{b}$]

[Prop 3.7k′ $\overset{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \neg\neg\forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{a}' = \underline{b} \Rightarrow \underline{a}' = \underline{b} \Rightarrow \neg\forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{a} = \underline{b}$]

[Prop 3.7l $\overset{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \neg\neg\forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{a} = \underline{b} \Rightarrow \underline{a} = \underline{b} \Rightarrow \neg\forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{a} = \underline{b}'$]

[Prop 3.7l′ $\overset{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \neg\forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{a} = \underline{b}' \Rightarrow \neg\neg\forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{a} = \underline{b} \Rightarrow \underline{a} = \underline{b}$]

[Prop 3.7m $\overset{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a}\colon \neg\forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{a} = \underline{a}'$]

[Prop 3.7o $\overset{\text{stmt}}{\rightarrow}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \neg \underline{a} = \underline{b} \Rightarrow \neg\neg\forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{a} = \underline{b} \Rightarrow \neg\forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{b} = \underline{a}$]

[Prop 3.7p $\stackrel{\text{stmt}}{\to}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \neg\neg \underline{a} = \underline{b} \Rightarrow \neg \forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{a} = \underline{b} \Rightarrow \neg \forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{b} = \underline{a}$]

[Prop 3.7q $\stackrel{\text{stmt}}{\to}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \neg\neg\neg \forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{a} = \underline{b} \Rightarrow \underline{a} = \underline{b} \Rightarrow \neg\neg \forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{b} = \underline{a} \Rightarrow \underline{b} = \underline{a}$]

[Prop 3.7r $\stackrel{\text{stmt}}{\to}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \neg\neg \forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{a} = \underline{a} + \underline{b} \Rightarrow \underline{a} = \underline{a} + \underline{b}$]

[Prop 3.7s $\stackrel{\text{stmt}}{\to}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \neg \underline{b} = 0 \Rightarrow \neg \forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{a} = \underline{a} + \underline{b}$]

[Prop 3.7t $\stackrel{\text{stmt}}{\to}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \neg \underline{b} = 0 \Rightarrow \neg\neg \forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{a} = \underline{a} \cdot \underline{b} \Rightarrow \underline{a} = \underline{a} \cdot \underline{b}$]

[Prop 3.7u $\stackrel{\text{stmt}}{\to}$ $S \vdash \forall \underline{a}\colon \neg \underline{a} = 0 \Rightarrow \neg \forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + 0 = \underline{a}$]

[Prop 3.7u′ $\stackrel{\text{stmt}}{\to}$ $S \vdash \forall \underline{a}\colon \neg \forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + 0 = \underline{a} \Rightarrow \neg \underline{a} = 0$]

[Prop 3.7v $\stackrel{\text{stmt}}{\to}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \neg \forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + 0 = \underline{a} \Rightarrow \neg \forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + 0 = \underline{b} \Rightarrow \neg \forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + 0 = \underline{a} \cdot \underline{b}$]

[Prop 3.7w $\stackrel{\text{stmt}}{\to}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \neg \underline{a} = 0 \Rightarrow \neg \forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + 0' = \underline{b} \Rightarrow \neg \forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{a} = \underline{b} \cdot \underline{a}$]

[Prop 3.7x $\stackrel{\text{stmt}}{\to}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \neg \underline{a} = 0 \Rightarrow \neg \forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{b} = \underline{c} \Rightarrow \neg \forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{b} \cdot \underline{a} = \underline{c} \cdot \underline{a}$]

[Prop 3.7x′ $\stackrel{\text{stmt}}{\to}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \neg \underline{a} = 0 \Rightarrow \neg \forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{b} \cdot \underline{a} = \underline{c} \cdot \underline{a} \Rightarrow \neg \forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{b} = \underline{c}$]

[Prop 3.7y $\stackrel{\text{stmt}}{\to}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \neg \underline{a} = 0 \Rightarrow \neg\neg \forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{b} = \underline{c} \Rightarrow \underline{b} = \underline{c} \Rightarrow \neg\neg \forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{b} \cdot \underline{a} = \underline{c} \cdot \underline{a} \Rightarrow \underline{b} \cdot \underline{a} = \underline{c} \cdot \underline{a}$]

[Prop 3.7y′ $\stackrel{\text{stmt}}{\to}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \neg \underline{a} = 0 \Rightarrow \neg\neg \forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{b} \cdot \underline{a} = \underline{c} \cdot \underline{a} \Rightarrow \underline{b} \cdot \underline{a} = \underline{c} \cdot \underline{a} \Rightarrow \neg\neg \forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{b} = \underline{c} \Rightarrow \underline{b} = \underline{c}$]

[Prop 3.7z $\stackrel{\text{stmt}}{\to}$ $S \vdash \forall \underline{a}\colon \neg\neg \forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{a} = 0$]

[Prop 3.7z′ $\stackrel{\text{stmt}}{\to}$ $S \vdash \forall \underline{a}\colon \forall \underline{b}\colon \neg\neg\neg \forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{a} = \underline{b} \Rightarrow \underline{a} = \underline{b} \Rightarrow \neg\neg \forall_{\text{obj}} z\colon \neg\neg\neg z = 0 \Rightarrow z + \underline{b} = \underline{a} \Rightarrow \underline{b} = \underline{a} \Rightarrow \underline{a} = \underline{b}$]

5.2 **proof of** Prop 3.7

6 **Lemma** Prop 3.10

[Prop 3.10a $\stackrel{\text{stmt}}{\to}$ $S \vdash \neg \forall_{\text{obj}} z\colon \neg \underline{a} = \forall \underline{a}\colon \underline{a} \cdot z$]

[Prop 3.10b $\stackrel{\text{stmt}}{\to}$ $S \vdash \neg \forall_{\text{obj}} z\colon \neg \underline{a} = \forall \underline{a}\colon 0' \cdot z$]

[Prop 3.10c $\stackrel{\text{stmt}}{\rightarrow}$ S $\vdash \neg\forall_{\text{obj}}\mathsf{z}\colon \neg 0 = \forall\underline{\mathsf{a}}\colon \underline{\mathsf{a}} \cdot \mathsf{z}$]

[Prop 3.10d $\stackrel{\text{stmt}}{\rightarrow}$ S $\vdash \neg\forall_{\text{obj}}\mathsf{z}\colon \neg\underline{\mathsf{c}} = \neg\forall_{\text{obj}}\mathsf{z}\colon \neg\underline{\mathsf{c}} \Rightarrow \underline{\mathsf{a}} = \neg\forall_{\text{obj}}\mathsf{z}\colon \neg\underline{\mathsf{b}} \wedge \underline{\mathsf{b}} = \forall\underline{\mathsf{a}}\colon \forall\underline{\mathsf{b}}\colon \forall\underline{\mathsf{c}}\colon \underline{\mathsf{a}} \cdot \mathsf{z} \cdot \mathsf{z} \cdot \mathsf{z}$]

[Prop 3.10e $\stackrel{\text{stmt}}{\rightarrow}$ S $\vdash \neg\forall_{\text{obj}}\mathsf{z}\colon \neg\underline{\mathsf{b}} \Rightarrow \neg\neg\forall_{\text{obj}}\mathsf{z}\colon \neg\neg\neg\mathsf{z} = 0 \Rightarrow \mathsf{z} + \underline{\mathsf{a}} = \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{a}} = \underline{\mathsf{a}} = \forall\underline{\mathsf{a}}\colon \forall\underline{\mathsf{b}}\colon \neg\neg\underline{\mathsf{a}} = 0 \Rightarrow \underline{\mathsf{b}} \cdot \mathsf{z}$]

[Prop 3.10f $\stackrel{\text{stmt}}{\rightarrow}$ S $\vdash \neg\forall_{\text{obj}}\mathsf{z}\colon \neg\underline{\mathsf{a}} \Rightarrow \underline{\mathsf{a}} = \underline{\mathsf{b}} = \neg\forall_{\text{obj}}\mathsf{z}\colon \neg\neg\underline{\mathsf{b}} \Rightarrow \underline{\mathsf{b}} = \forall\underline{\mathsf{a}}\colon \forall\underline{\mathsf{b}}\colon \underline{\mathsf{a}} \cdot \mathsf{z} \cdot \mathsf{z}$]

[Prop 3.10g $\stackrel{\text{stmt}}{\rightarrow}$ S $\vdash \neg\forall_{\text{obj}}\mathsf{z}\colon \neg\underline{\mathsf{b}} \cdot \underline{\mathsf{c}} = \neg\forall_{\text{obj}}\mathsf{z}\colon \neg\underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}} = \forall\underline{\mathsf{a}}\colon \forall\underline{\mathsf{b}}\colon \forall\underline{\mathsf{c}}\colon \underline{\mathsf{a}} \cdot \mathsf{z} \cdot \mathsf{z}$]

[Prop 3.10h $\stackrel{\text{stmt}}{\rightarrow}$ S $\vdash \neg\forall_{\text{obj}}\mathsf{z}\colon \neg\underline{\mathsf{b}} + \underline{\mathsf{c}} = \neg\forall_{\text{obj}}\mathsf{z}\colon \neg\underline{\mathsf{c}} \Rightarrow \underline{\mathsf{a}} = \neg\forall_{\text{obj}}\mathsf{z}\colon \neg\underline{\mathsf{b}} \wedge \underline{\mathsf{a}} = \forall\underline{\mathsf{a}}\colon \forall\underline{\mathsf{b}}\colon \forall\underline{\mathsf{c}}\colon \underline{\mathsf{a}} \cdot \mathsf{z} \cdot \mathsf{z} \cdot \mathsf{z}$]

7 **Lemma** Prop 3.11

[Prop 3.11 $\stackrel{\text{stmt}}{\rightarrow}$ S $\vdash \forall\underline{\mathsf{a}}\colon \forall\underline{\mathsf{b}}\colon \neg\underline{\mathsf{a}} = 0 \Rightarrow \neg\forall_{\text{obj}}\underline{\mathsf{c}}\colon \forall_{\text{obj}}\underline{\mathsf{d}}\colon \neg\neg\neg\underline{\mathsf{b}} = \underline{\mathsf{a}} \cdot \underline{\mathsf{c}} + \underline{\mathsf{d}} \Rightarrow \neg\forall_{\text{obj}}\mathsf{z}\colon \neg\neg\neg\mathsf{z} = 0 \Rightarrow \mathsf{z} + \underline{\mathsf{d}} = \underline{\mathsf{a}} \Rightarrow \forall_{\text{obj}}\underline{\mathsf{e}}\colon \forall_{\text{obj}}\underline{\mathsf{f}}\colon \neg\underline{\mathsf{b}} = \underline{\mathsf{a}} \cdot \underline{\mathsf{e}} + \underline{\mathsf{f}} \Rightarrow \neg\forall_{\text{obj}}\mathsf{z}\colon \neg\neg\neg\mathsf{z} = 0 \Rightarrow \mathsf{z} + \underline{\mathsf{f}} = \underline{\mathsf{a}} \Rightarrow \neg\underline{\mathsf{c}} = \underline{\mathsf{e}} \Rightarrow \underline{\mathsf{d}} = \underline{\mathsf{f}}$]

A Helping lemmas

for some of our proofs we need to prove some lemmas concerning $\wedge$ and $\vee$ these will be proved here.

[T0 $\stackrel{\text{stmt}}{\rightarrow}$ S $\vdash \forall\underline{\mathsf{a}}\colon \forall\underline{\mathsf{b}}\colon \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}$]

[T1 $\stackrel{\text{stmt}}{\rightarrow}$ S $\vdash \forall\underline{\mathsf{a}}\colon \forall\underline{\mathsf{b}}\colon \neg\underline{\mathsf{a}} \Rightarrow \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}$]

[H0a $\stackrel{\text{stmt}}{\rightarrow}$ S $\vdash \forall\underline{\mathsf{a}}\colon \forall\underline{\mathsf{b}}\colon \forall\underline{\mathsf{c}}\colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \vdash \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{c}}$]

[H0b $\stackrel{\text{stmt}}{\rightarrow}$ S $\vdash \forall\underline{\mathsf{a}}\colon \forall\underline{\mathsf{b}}\colon \forall\underline{\mathsf{c}}\colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \vdash \underline{\mathsf{b}} \vdash \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{c}}$]

[H1 $\stackrel{\text{stmt}}{\rightarrow}$ S $\vdash \forall\underline{\mathsf{a}}\colon \neg\neg\underline{\mathsf{a}} \Rightarrow \underline{\mathsf{a}}$]

[H2 $\stackrel{\text{stmt}}{\rightarrow}$ S $\vdash \forall\underline{\mathsf{a}}\colon \underline{\mathsf{a}} \Rightarrow \neg\neg\underline{\mathsf{a}}$]

[Con1 $\stackrel{\text{stmt}}{\rightarrow}$ S $\vdash \forall\underline{\mathsf{a}}\colon \forall\underline{\mathsf{b}}\colon \neg\underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \underline{\mathsf{a}}$]

[Con2 $\stackrel{\text{stmt}}{\rightarrow}$ S $\vdash \forall\underline{\mathsf{a}}\colon \forall\underline{\mathsf{b}}\colon \neg\underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \underline{\mathsf{b}}$]

[Con $\stackrel{\text{stmt}}{\rightarrow}$ S $\vdash \forall\underline{\mathsf{a}}\colon \forall\underline{\mathsf{b}}\colon \underline{\mathsf{a}} \vdash \underline{\mathsf{b}} \vdash \neg\underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}$]

[Dis1 $\stackrel{\text{stmt}}{\rightarrow}$ S $\vdash \forall\underline{\mathsf{a}}\colon \forall\underline{\mathsf{b}}\colon \underline{\mathsf{a}} \vdash \neg\underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}$]

$[\mathrm{Dis2} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{b} \vdash \neg \underline{a} \Rightarrow \underline{b}]$

$[\mathrm{T0} \stackrel{\mathrm{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{a}\colon \forall \underline{b}\colon \underline{b} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \vdash \mathrm{Repetition} \triangleright \underline{b} \gg \underline{b}; \mathrm{Ded} \triangleright \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \vdash \underline{b} \gg \underline{a} \Rightarrow \underline{b}; \mathrm{Ded} \triangleright \forall \underline{a}\colon \forall \underline{b}\colon \underline{b} \vdash \underline{a} \Rightarrow \underline{b} \gg \underline{b} \Rightarrow \underline{a} \Rightarrow \underline{b} \rceil, p_0, c)]$

$[\mathrm{T1} \stackrel{\mathrm{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{a}\colon \forall \underline{b}\colon \neg \underline{a} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \vdash \mathrm{Repetition} \triangleright \neg \underline{a} \gg \neg \underline{a}; \mathrm{T0} \gg \underline{a} \Rightarrow \neg \underline{b} \Rightarrow \underline{a}; \mathrm{MP} \triangleright \underline{a} \Rightarrow \neg \underline{b} \Rightarrow \underline{a} \triangleright \underline{a} \gg \neg \underline{b} \Rightarrow \underline{a}; \mathrm{T0} \gg \neg \underline{a} \Rightarrow \neg \underline{b} \Rightarrow \neg \underline{a}; \mathrm{MP} \triangleright \neg \underline{a} \Rightarrow \neg \underline{b} \Rightarrow \neg \underline{a} \triangleright \neg \underline{a} \gg \neg \underline{b} \Rightarrow \neg \underline{a}; \mathrm{Neg} \triangleright \neg \underline{b} \Rightarrow \neg \underline{a} \triangleright \neg \underline{b} \Rightarrow \underline{a} \gg \underline{b}; \mathrm{Ded} \triangleright \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \vdash \underline{b} \gg \underline{a} \Rightarrow \underline{b}; \mathrm{Ded} \triangleright \forall \underline{a}\colon \forall \underline{b}\colon \neg \underline{a} \vdash \underline{a} \Rightarrow \underline{b} \gg \neg \underline{a} \Rightarrow \underline{a} \Rightarrow \underline{b} \rceil, p_0, c)]$

$[\mathrm{H0a} \stackrel{\mathrm{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} \Rightarrow \underline{b} \vdash \underline{b} \Rightarrow \underline{c} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} \Rightarrow \underline{b} \vdash \underline{b} \Rightarrow \underline{c} \vdash \underline{a} \vdash \mathrm{MP} \triangleright \underline{a} \Rightarrow \underline{b} \triangleright \underline{a} \gg \underline{b}; \mathrm{MP} \triangleright \underline{b} \Rightarrow \underline{c} \triangleright \underline{b} \gg \underline{c}; \mathrm{Ded} \triangleright \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} \Rightarrow \underline{b} \vdash \underline{b} \Rightarrow \underline{c} \vdash \underline{a} \vdash \underline{c} \gg \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{a} \Rightarrow \underline{c}; \mathrm{MP} \triangleright \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{a} \Rightarrow \underline{c} \triangleright \underline{a} \Rightarrow \underline{b} \gg \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{a} \Rightarrow \underline{c}; \mathrm{MP} \triangleright \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{a} \Rightarrow \underline{c} \triangleright \underline{b} \Rightarrow \underline{c} \gg \underline{a} \Rightarrow \underline{c} \rceil, p_0, c)]$

$[\mathrm{H0b} \stackrel{\mathrm{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{S} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \vdash \underline{b} \vdash \mathrm{A2}' \gg \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a} \Rightarrow \underline{c}; \mathrm{MP} \triangleright \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a} \Rightarrow \underline{c} \triangleright \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \gg \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a} \Rightarrow \underline{c}; \mathrm{T0} \gg \underline{b} \Rightarrow \underline{a} \Rightarrow \underline{b}; \mathrm{MP} \triangleright \underline{b} \Rightarrow \underline{a} \Rightarrow \underline{b} \triangleright \underline{b} \gg \underline{a} \Rightarrow \underline{b}; \mathrm{MP} \triangleright \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{a} \Rightarrow \underline{c} \triangleright \underline{a} \Rightarrow \underline{b} \gg \underline{a} \Rightarrow \underline{c} \rceil, p_0, c)]$

$[\mathrm{H1} \stackrel{\mathrm{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{S} \vdash \forall \underline{a}\colon \mathrm{Neg} \gg \neg \underline{a} \Rightarrow \neg\neg \underline{a} \Rightarrow \neg \underline{a} \Rightarrow \neg \underline{a} \Rightarrow \underline{a}; \mathrm{Repetition} \gg \neg \underline{a} \Rightarrow \neg \underline{a}; \mathrm{H0b} \triangleright \neg \underline{a} \Rightarrow \neg\neg \underline{a} \Rightarrow \neg \underline{a} \Rightarrow \neg \underline{a} \Rightarrow \underline{a} \triangleright \neg \underline{a} \Rightarrow \neg \underline{a} \gg \neg \underline{a} \Rightarrow \neg\neg \underline{a} \Rightarrow \underline{a}; \mathrm{A1}' \gg \neg\neg \underline{a} \Rightarrow \neg \underline{a} \Rightarrow \neg\neg \underline{a}; \mathrm{H0a} \triangleright \neg\neg \underline{a} \Rightarrow \neg \underline{a} \Rightarrow \neg\neg \underline{a} \triangleright \neg \underline{a} \Rightarrow \neg\neg \underline{a} \Rightarrow \underline{a} \gg \neg\neg \underline{a} \Rightarrow \underline{a} \rceil, p_0, c)]$

$[\mathrm{H2} \stackrel{\mathrm{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{S} \vdash \forall \underline{a}\colon \mathrm{Neg} \gg \neg\neg\neg \underline{a} \Rightarrow \neg \underline{a} \Rightarrow \neg\neg\neg \underline{a} \Rightarrow \underline{a} \Rightarrow \neg\neg \underline{a}; \mathrm{H1} \gg \neg\neg\neg \underline{a} \Rightarrow \neg \underline{a}; \mathrm{MP} \triangleright \neg\neg\neg \underline{a} \Rightarrow \neg \underline{a} \Rightarrow \neg\neg\neg \underline{a} \Rightarrow \underline{a} \Rightarrow \neg\neg \underline{a} \triangleright \neg\neg\neg \underline{a} \Rightarrow \neg \underline{a} \gg \neg\neg\neg \underline{a} \Rightarrow \underline{a} \Rightarrow \neg\neg \underline{a}; \mathrm{A1}' \gg \underline{a} \Rightarrow \neg\neg\neg \underline{a} \Rightarrow \underline{a}; \mathrm{H1} \triangleright \neg\neg\neg \underline{a} \Rightarrow \underline{a} \Rightarrow \neg\neg \underline{a} \triangleright \underline{a} \Rightarrow \neg\neg\neg \underline{a} \Rightarrow \underline{a} \gg \underline{a} \Rightarrow \neg\neg \underline{a} \rceil, p_0, c)]$

Litteratur

[Gru06] K. Grue. A logiweb base page. Technical report, Logiweb, 2006. http://www.diku.dk/cgi-bin/cginetd/grue/relay/go/01693B66F0C393B695F3F15770898897A683141109C083AC838ADCA20806/2/.

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