projekt i logik

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Indhold

Resumé

1 Introduction

In this paper we will prove the propositions regarding S starting with proposition Prop 3.2i in mendelson [Men97]. Continuing toProp 3.4 defining numerals neded to prove Prop 3.5 . then we will define rules regarding $x < y$ needed to prove Prop 3.7 . after that we will define $x \mid y$ nedded to prove Prop 3.10 , and finally we will prove Prop 3.11.

2 Proof of Prop 3.2

A modified version of Mendelsons system S (Peano arithmetic) [Men97] may be formulated thus:

[**Theory** S]

[S **rule** MP: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A}\Rightarrow\mathcal{B}\vdash\mathcal{A}\vdash\mathcal{B}$]

[S **rule** Gen: $\Pi\mathcal{X},\mathcal{A}\colon \mathcal{A}\vdash\forall\mathcal{X}\colon\mathcal{A}$]

[S **rule** Ded: $\Pi\mathcal{A},\mathcal{B}\colon \mathrm{Ded}(\mathcal{A},\mathcal{B})\Vdash\mathcal{A}\vdash\mathcal{B}$]

[S **rule** S2: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A}=\mathcal{B}\vdash\mathcal{A}'=\mathcal{B}'$]

[S **rule** S3: $\Pi\mathcal{A}\colon \neg 0=\mathcal{A}'$]

[S **rule** S4: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A}'=\mathcal{B}'\vdash\mathcal{A}=\mathcal{B}$]

[S **rule** S5: $\Pi\mathcal{A}\colon \mathcal{A}+0=\mathcal{A}$]

[S **rule** S6: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A}+\mathcal{B}'=(\mathcal{A}+\mathcal{B})'$]

[S **rule** S7: $\Pi\mathcal{A}\colon \mathcal{A}\cdot 0=0$]

[S **rule** S8: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A}\cdot(\mathcal{B}')=(\mathcal{A}\cdot\mathcal{B})+\mathcal{A}$]

[S **rule** Neg: $\Pi\mathcal{A}\colon\Pi\mathcal{B}\colon \neg\mathcal{B}\Rightarrow\neg\mathcal{A}\vdash\neg\mathcal{B}\Rightarrow\mathcal{A}\vdash\mathcal{B}$]

[S **rule** S1: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{A}=\mathcal{B}\vdash\mathcal{A}=\mathcal{C}\vdash\mathcal{B}=\mathcal{C}$]

[S **rule** S9: $\Pi\mathcal{X},\mathcal{A},\mathcal{B},\mathcal{C}\colon \langle\mathcal{B}\equiv\mathcal{A}|\mathcal{X}{:=}0\rangle\Vdash\langle\mathcal{C}\equiv\mathcal{A}|\mathcal{X}{:=}\mathcal{X}'\rangle\Vdash\mathcal{B}\vdash\mathcal{A}\Rightarrow\mathcal{C}\vdash\mathcal{A}$]

Mendelson [Men97] defines a set of lemma's

[S **lemma** Prop 3.2a: $\Pi\mathcal{A}\colon \mathcal{A}=\mathcal{A}$]

[S **lemma** Prop 3.2b: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A}=\mathcal{B}\vdash\mathcal{B}=\mathcal{A}$]

[S **lemma** Prop 3.2c: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{A}=\mathcal{B}\vdash\mathcal{B}=\mathcal{C}\vdash\mathcal{A}=\mathcal{C}$]

[S **lemma** Prop 3.2d: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{A}=\mathcal{C}\vdash\mathcal{B}=\mathcal{C}\vdash\mathcal{A}=\mathcal{B}$]

[S **lemma** Prop 3.2e: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{A}=\mathcal{B}\vdash\mathcal{A}+\mathcal{C}=\mathcal{B}+\mathcal{C}$]

[S **lemma** Prop 3.2f: $\Pi\mathcal{A}\colon \mathcal{A}=0+\mathcal{A}$]

[S **lemma** Prop 3.2g: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A}'+\mathcal{B}=(\mathcal{A}+\mathcal{B})'$]

[S **lemma** Prop 3.2h: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A}+\mathcal{B}=\mathcal{B}+\mathcal{A}$]

These Lemma's are proved in [Gru06], and we will continue to prove the following lemma's, from Mendelson [Men97].

[S **lemma** Prop 3.2i: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{A}=\mathcal{B}\vdash\mathcal{C}+\mathcal{A}=\mathcal{C}+\mathcal{B}$]

[S **lemma** Prop 3.2j: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon (\mathcal{A}+\mathcal{B})+\mathcal{C}=\mathcal{A}+(\mathcal{B}+\mathcal{C})$]

[S **lemma** Prop 3.2k: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{A}=\mathcal{B}\vdash\mathcal{A}\cdot\mathcal{C}=\mathcal{B}\cdot\mathcal{C}$]

[S **lemma** Prop 3.2l: $\Pi\mathcal{A}\colon 0\cdot\mathcal{A}=0$]

[S **lemma** Prop 3.2m: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A}'\cdot\mathcal{B}=\mathcal{A}\cdot\mathcal{B}+\mathcal{B}$]

[S **lemma** Prop 3.2n: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A}\cdot\mathcal{B}=\mathcal{B}\cdot\mathcal{A}$]

[S **lemma** Prop 3.2o: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{A}=\mathcal{B}\vdash\mathcal{C}\cdot\mathcal{A}=\mathcal{C}\cdot\mathcal{B}$]

S **proof of** Prop 3.2i:

L01:	Arbitrary $\gg$	$\mathcal{A},\mathcal{B},\mathcal{C}$	;
L02:	Block $\gg$	Begin	;
L03:	Arbitrary $\gg$	$\mathcal{A},\mathcal{B},\mathcal{C}$	;
L04:	Premise $\gg$	$\mathcal{A}=\mathcal{B}$	;
L05:	Prop 3.2e $\rhd$ L04 $\gg$	$\mathcal{A}+\mathcal{C}=\mathcal{B}+\mathcal{C}$	;
L06:	Prop 3.2h $\gg$	$\mathcal{A}+\mathcal{C}=\mathcal{C}+\mathcal{A}$	;
L07:	Prop 3.2h $\gg$	$\mathcal{B}+\mathcal{C}=\mathcal{C}+\mathcal{B}$	;
L08:	S1 $\rhd$ L05 $\rhd$ L06 $\gg$	$\mathcal{B}+\mathcal{C}=\mathcal{C}+\mathcal{A}$	;
L09:	Prop 3.2b $\rhd$ L08 $\gg$	$\mathcal{C}+\mathcal{A}=\mathcal{B}+\mathcal{C}$	;
L10:	Prop 3.2c $\rhd$ L09 $\rhd$ L07 $\gg$	$\mathcal{C}+\mathcal{A}=\mathcal{C}+\mathcal{B}$	;
L11:	Block $\gg$	End	;
L12:	Ded $\rhd$ L11 $\gg$	$\mathcal{A}=\mathcal{B}\vdash\mathcal{C}+\mathcal{A}=\mathcal{C}+\mathcal{B}$	$\square$

[S **lemma** Prop 3.2j$_1$: $\Pi\mathcal{A},\mathcal{B}\colon (\mathcal{A}+\mathcal{B})+0=\mathcal{A}+(\mathcal{B}+0)$]

S **proof of** Prop 3.2j$_1$:

L01:	Arbitrary $\gg$	$\mathcal{A},\mathcal{B}$	;
L02:	S5 $\gg$	$(\mathcal{A}+\mathcal{B})+0=\mathcal{A}+\mathcal{B}$	;
L03:	S5 $\gg$	$\mathcal{B}+0=\mathcal{B}$	;
L04:	Prop 3.2i $\rhd$ L03 $\gg$	$\mathcal{A}+(\mathcal{B}+0)=\mathcal{A}+\mathcal{B}$	;
L05:	Prop 3.2d $\rhd$ L02 $\rhd$ L04 $\gg$	$(\mathcal{A}+\mathcal{B})+0=\mathcal{A}+(\mathcal{B}+0)$	$\square$

[S **lemma** Prop 3.2j$_2$: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon (\mathcal{A}+\mathcal{B})+\mathcal{C}=\mathcal{A}+(\mathcal{B}+\mathcal{C})\Rightarrow(\mathcal{A}+\mathcal{B})+\mathcal{C}'=\mathcal{A}+(\mathcal{B}+\mathcal{C}')$]

S **proof of** Prop 3.2j$_2$:

L01:	Arbitrary $\gg$	$\mathcal{A},\mathcal{B},\mathcal{C}$	;
L02:	Block $\gg$	Begin	;
L03:	Arbitrary $\gg$	$\mathcal{A},\mathcal{B},\mathcal{C}$	;
L04:	Premise $\gg$	$(\mathcal{A}+\mathcal{B})+\mathcal{C}=\mathcal{A}+(\mathcal{B}+\mathcal{C})$	;
L05:	S6 $\gg$	$(\mathcal{A}+\mathcal{B})+\mathcal{C}'=((\mathcal{A}+\mathcal{B})+\mathcal{C})'$	;
L06:	S2 $\rhd$ L04 $\gg$	$((\mathcal{A}+\mathcal{B})+\mathcal{C})'=(\mathcal{A}+(\mathcal{B}+\mathcal{C}))'$	;
L07:	Prop 3.2c $\rhd$ L05 $\rhd$ L06 $\gg$	$(\mathcal{A}+\mathcal{B})+\mathcal{C}'=(\mathcal{A}+(\mathcal{B}+\mathcal{C}))'$	;
L08:	S6 $\gg$	$\mathcal{B}+\mathcal{C}'=(\mathcal{B}+\mathcal{C})'$	;
L09:	Prop 3.2i $\rhd$ L08 $\gg$	$\mathcal{A}+(\mathcal{B}+\mathcal{C}')=\mathcal{A}+(\mathcal{B}+\mathcal{C})'$	;
L10:	S6 $\gg$	$\mathcal{A}+(\mathcal{B}+\mathcal{C})'=(\mathcal{A}+(\mathcal{B}+\mathcal{C}))'$	;
L11:	Prop 3.2c $\rhd$ L09 $\rhd$ L10 $\gg$	$\mathcal{A}+(\mathcal{B}+\mathcal{C}')=(\mathcal{A}+(\mathcal{B}+\mathcal{C}))'$	;
L12:	Prop 3.2d $\rhd$ L07 $\rhd$ L11 $\gg$	$(\mathcal{A}+\mathcal{B})+\mathcal{C}'=\mathcal{A}+(\mathcal{B}+\mathcal{C}')$	;
L13:	Block $\gg$	End	;
L14:	Ded $\rhd$ L13 $\gg$	$(\mathcal{A}+\mathcal{B})+\mathcal{C}=\mathcal{A}+(\mathcal{B}+\mathcal{C})\Rightarrow(\mathcal{A}+\mathcal{B})+\mathcal{C}'=\mathcal{A}+(\mathcal{B}+\mathcal{C}')$	$\square$

S **proof of** Prop 3.2j:

L01:	Arbitrary $\gg$	$\mathcal{A},\mathcal{B},\mathcal{C}$	;

L02: Block $\gg$ Begin ;
L03: Prop 3.2j$_1$ $\gg$ $(x+y)+0 = x+(y+0)$;
L04: Prop 3.2j$_2$ $\gg$ $(x+y)+z = x+(y+z) \Rightarrow (x+y)+z' = x+(y+z')$;
L05: S9 @ z $\rhd$ L03 $\rhd$ L04 $\gg$ $(x+y)+z = x+(y+z)$;
L06: Block $\gg$ End ;
L07: Ded $\rhd$ L06 $\gg$ $(\mathcal{A}+\mathcal{B})+\mathcal{C} = \mathcal{A}+(\mathcal{B}+\mathcal{C})$ $\Box$

[S **lemma** Prop 3.2k$_1$: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A}=\mathcal{B} \Rightarrow \mathcal{A}\cdot 0 = \mathcal{B}\cdot 0$]

S **proof of** Prop 3.2k$_1$:
L01: Arbitrary $\gg$ $\mathcal{A},\mathcal{B}$;
L02: Block $\gg$ Begin ;
L03: Arbitrary $\gg$ $\mathcal{A},\mathcal{B}$;
L04: Premise $\gg$ $\mathcal{A}=\mathcal{B}$;
L05: S7 $\gg$ $\mathcal{A}\cdot 0 = 0$;
L06: S7 $\gg$ $\mathcal{B}\cdot 0 = 0$;
L07: Prop 3.2a $\gg$ $0 = 0$;
L08: Prop 3.2b $\rhd$ L06 $\gg$ $0 = \mathcal{B}\cdot 0$;
L09: Prop 3.2c $\rhd$ L05 $\rhd$ L08 $\gg$ $\mathcal{A}\cdot 0 = \mathcal{B}\cdot 0$;
L10: Block $\gg$ End ;
L11: Ded $\rhd$ L10 $\gg$ $\mathcal{A}=\mathcal{B} \Rightarrow \mathcal{A}\cdot 0 = \mathcal{B}\cdot 0$ $\Box$

[S **lemma** Prop 3.2k$_2$: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon (\mathcal{A}=\mathcal{B} \Rightarrow \mathcal{A}\cdot\mathcal{C} = \mathcal{B}\cdot\mathcal{C}) \Rightarrow (\mathcal{A}=\mathcal{B} \Rightarrow \mathcal{A}\cdot\mathcal{C}' = \mathcal{B}\cdot\mathcal{C}')$]

S **proof of** Prop 3.2k$_2$:
L01: Arbitrary $\gg$ $\mathcal{A},\mathcal{B},\mathcal{C}$;
L02: Block $\gg$ Begin ;
L03: Arbitrary $\gg$ $\mathcal{A},\mathcal{B},\mathcal{C}$;
L04: Premise $\gg$ $\mathcal{A}=\mathcal{B} \Rightarrow \mathcal{A}\cdot\mathcal{C} = \mathcal{B}\cdot\mathcal{C}$;
L05: Premise $\gg$ $\mathcal{A}=\mathcal{B}$;
L06: L04 $\unrhd$ L05 $\gg$ $\mathcal{A}\cdot\mathcal{C} = \mathcal{B}\cdot\mathcal{C}$;
L07: S8 $\gg$ $\mathcal{A}\cdot\mathcal{C}' = \mathcal{A}\cdot\mathcal{C}+\mathcal{A}$;
L08: S8 $\gg$ $\mathcal{B}\cdot\mathcal{C}' = \mathcal{B}\cdot\mathcal{C}+\mathcal{B}$;
L09: Prop 3.2e $\rhd$ L06 $\gg$ $(\mathcal{A}\cdot\mathcal{C})+\mathcal{A} = (\mathcal{B}\cdot\mathcal{C})+\mathcal{A}$;
L10: Prop 3.2i $\rhd$ L05 $\gg$ $(\mathcal{B}\cdot\mathcal{C})+\mathcal{A} = (\mathcal{B}\cdot\mathcal{C})+\mathcal{B}$;
L11: Prop 3.2c $\rhd$ L09 $\rhd$ L10 $\gg$ $\mathcal{A}\cdot\mathcal{C}+\mathcal{A} = \mathcal{B}\cdot\mathcal{C}+\mathcal{B}$;
L12: Prop 3.2c $\rhd$ L07 $\rhd$ L11 $\gg$ $\mathcal{A}\cdot\mathcal{C}' = \mathcal{B}\cdot\mathcal{C}+\mathcal{B}$;
L13: Prop 3.2d $\rhd$ L12 $\rhd$ L08 $\gg$ $\mathcal{A}\cdot\mathcal{C}' = \mathcal{B}\cdot\mathcal{C}'$;
L14: Block $\gg$ End ;
L15: Ded $\rhd$ L14 $\gg$ $(\mathcal{A}=\mathcal{B} \Rightarrow \mathcal{A}\cdot\mathcal{C} = \mathcal{B}\cdot\mathcal{C}) \Rightarrow (\mathcal{A}=\mathcal{B} \Rightarrow \mathcal{A}\cdot\mathcal{C}' = \mathcal{B}\cdot\mathcal{C}')$ $\Box$

S **proof of** Prop 3.2k :
L01: Arbitrary $\gg$ $\mathcal{A},\mathcal{B},\mathcal{C}$;
L02: Premise $\gg$ $\mathcal{A}=\mathcal{B}$;
L03: Block $\gg$ Begin ;

L04:	Prop 3.2k$_1$ $\gg$	$x = y \Rightarrow x \cdot 0 = y \cdot 0$	;
L05:	Prop 3.2k$_2$ $\gg$	$(x = y \Rightarrow x \cdot z = y \cdot z) \Rightarrow (x = y \Rightarrow x \cdot z' = y \cdot z')$	;
L06:	S9 @ z $\rhd$ L04 $\rhd$ L05 $\gg$	$x = y \Rightarrow x \cdot z = y \cdot z$	;
L07:	Block $\gg$	End	;
L08:	Ded $\rhd$ L07 $\gg$	$\mathcal{A} = \mathcal{B} \Rightarrow \mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{C}$	;
L09:	L08 $\unrhd$ L02 $\gg$	$\mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{C}$	□

[S **lemma** Prop 3.2l$_2$: $\Pi\mathcal{A}\colon 0 \cdot \mathcal{A} = 0 \Rightarrow 0 \cdot \mathcal{A}' = 0$]

S **proof of** Prop 3.2l$_2$:

L01:	Arbitrary $\gg$	$\mathcal{A}$	;
L02:	Block $\gg$	Begin	;
L03:	Arbitrary $\gg$	$\mathcal{A}$	;
L04:	Premise $\gg$	$0 \cdot \mathcal{A} = 0$	;
L05:	S8 $\gg$	$0 \cdot \mathcal{A}' = 0 \cdot \mathcal{A} + 0$	;
L06:	S5 $\gg$	$0 \cdot \mathcal{A} + 0 = 0 \cdot \mathcal{A}$	;
L07:	Prop 3.2c $\rhd$ L05 $\rhd$ L06 $\gg$	$0 \cdot \mathcal{A}' = 0 \cdot \mathcal{A}$	;
L08:	Prop 3.2c $\rhd$ L07 $\rhd$ L04 $\gg$	$0 \cdot \mathcal{A}' = 0$	;
L09:	Block $\gg$	End	;
L10:	Ded $\rhd$ L09 $\gg$	$0 \cdot \mathcal{A} = 0 \Rightarrow 0 \cdot \mathcal{A}' = 0$	□

S **proof of** Prop 3.2l:

L01:	Arbitrary $\gg$	$\mathcal{A}$	;
L02:	Block $\gg$	Begin	;
L03:	S7 $\gg$	$0 \cdot 0 = 0$	;
L04:	Prop 3.2l$_2$ $\gg$	$0 \cdot x = 0 \Rightarrow 0 \cdot x' = 0$	;
L05:	S9 @ x $\rhd$ L03 $\rhd$ L04 $\gg$	$0 \cdot x = 0$	;
L06:	Block $\gg$	End	;
L07:	Ded $\rhd$ L06 $\gg$	$0 \cdot \mathcal{A} = 0$	□

[S **lemma** Prop 3.2m$_1$: $\Pi\mathcal{A}\colon \mathcal{A}' \cdot 0 = \mathcal{A} \cdot 0 + 0$]

S **proof of** Prop 3.2m$_1$:

L01:	Arbitrary $\gg$	$\mathcal{A}$	;
L02:	S7 $\gg$	$\mathcal{A}' \cdot 0 = 0$	;
L03:	Prop 3.2f $\gg$	$0 = 0 + 0$	;
L04:	S7 $\gg$	$\mathcal{A} \cdot 0 = 0$	;
L05:	Prop 3.2b $\rhd$ L04 $\gg$	$0 = \mathcal{A} \cdot 0$	;
L06:	Prop 3.2e $\rhd$ L05 $\gg$	$0 + 0 = \mathcal{A} \cdot 0 + 0$	;
L07:	Prop 3.2c $\rhd$ L03 $\rhd$ L06 $\gg$	$0 = \mathcal{A} \cdot 0 + 0$	;
L08:	Prop 3.2c $\rhd$ L02 $\rhd$ L07 $\gg$	$\mathcal{A}' \cdot 0 = \mathcal{A} \cdot 0 + 0$	;

[S **lemma** Prop 3.2m$_2$: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A}' \cdot \mathcal{B} = \mathcal{A} \cdot \mathcal{B} + \mathcal{B} \Rightarrow \mathcal{A}' \cdot \mathcal{B}' = \mathcal{A} \cdot \mathcal{B}' + \mathcal{B}'$]

S **proof of** Prop 3.2m$_2$:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}$	;
L02:	Block $\gg$	Begin	;

L03: Arbitrary $\gg$ $\mathcal{A}, \mathcal{B}$;
L04: Premise $\gg$ $\mathcal{A}' \cdot \mathcal{B} = \mathcal{A} \cdot \mathcal{B} + \mathcal{B}$;
L05: S8 $\gg$ $\mathcal{A}' \cdot \mathcal{B}' = \mathcal{A}' \cdot \mathcal{B} + \mathcal{A}'$;
L06: Prop 3.2e $\rhd$ L04 $\gg$ $(\mathcal{A}' \cdot \mathcal{B}) + \mathcal{A}' = (\mathcal{A} \cdot \mathcal{B} + \mathcal{B}) + \mathcal{A}'$;
L07: S6 $\gg$ $\mathcal{B} + \mathcal{A}' = (\mathcal{B} + \mathcal{A})'$;
L08: Prop 3.2g $\gg$ $\mathcal{B}' + \mathcal{A} = (\mathcal{B} + \mathcal{A})'$;
L09: Prop 3.2d $\rhd$ L07 $\rhd$ L08 $\gg$ $\mathcal{B} + \mathcal{A}' = \mathcal{B}' + \mathcal{A}$;
L10: Prop 3.2h $\gg$ $\mathcal{B}' + \mathcal{A} = \mathcal{A} + \mathcal{B}'$;
L11: Prop 3.2c $\rhd$ L09 $\rhd$ L10 $\gg$ $\mathcal{B} + \mathcal{A}' = \mathcal{A} + \mathcal{B}'$;
L12: Prop 3.2i $\rhd$ L11 $\gg$ $\mathcal{A} \cdot \mathcal{B} + (\mathcal{B} + \mathcal{A}') = \mathcal{A} \cdot \mathcal{B} + (\mathcal{A} + \mathcal{B}')$;
L13: Prop 3.2j $\gg$ $(\mathcal{A} \cdot \mathcal{B} + \mathcal{A}) + \mathcal{B}' = \mathcal{A} \cdot \mathcal{B} + (\mathcal{A} + \mathcal{B}')$;
L14: Prop 3.2d $\rhd$ L12 $\rhd$ L13 $\gg$ $\mathcal{A} \cdot \mathcal{B} + (\mathcal{B} + \mathcal{A}') = (\mathcal{A} \cdot \mathcal{B} + \mathcal{A}) + \mathcal{B}'$;
L15: S8 $\gg$ $\mathcal{A} \cdot \mathcal{B}' = \mathcal{A} \cdot \mathcal{B} + \mathcal{A}$;
L16: Prop 3.2e $\rhd$ L15 $\gg$ $\mathcal{A} \cdot \mathcal{B}' + \mathcal{B}' = (\mathcal{A} \cdot \mathcal{B} + \mathcal{A}) + \mathcal{B}'$;
L17: Prop 3.2d $\rhd$ L14 $\rhd$ L16 $\gg$ $\mathcal{A} \cdot \mathcal{B} + (\mathcal{B} + \mathcal{A}') = \mathcal{A} \cdot \mathcal{B}' + \mathcal{B}'$;
L18: Prop 3.2j $\gg$ $(\mathcal{A} \cdot \mathcal{B} + \mathcal{B}) + \mathcal{A}' = \mathcal{A} \cdot \mathcal{B} + (\mathcal{B} + \mathcal{A}')$;
L19: Prop 3.2c $\rhd$ L06 $\rhd$ L18 $\gg$ $(\mathcal{A}' \cdot \mathcal{B}) + \mathcal{A}' = \mathcal{A} \cdot \mathcal{B} + (\mathcal{B} + \mathcal{A}')$;
L20: Prop 3.2c $\rhd$ L19 $\rhd$ L17 $\gg$ $(\mathcal{A}' \cdot \mathcal{B}) + \mathcal{A}' = \mathcal{A} \cdot \mathcal{B}' + \mathcal{B}'$;
L21: Prop 3.2c $\rhd$ L05 $\rhd$ L20 $\gg$ $\mathcal{A}' \cdot \mathcal{B}' = \mathcal{A} \cdot \mathcal{B}' + \mathcal{B}'$;
L22: Block $\gg$ End ;
L23: Ded $\rhd$ L22 $\gg$ $\mathcal{A}' \cdot \mathcal{B} = \mathcal{A} \cdot \mathcal{B} + \mathcal{B} \Rightarrow \mathcal{A}' \cdot \mathcal{B}' = \mathcal{A} \cdot \mathcal{B}' + \mathcal{B}'$ □

S **proof of** Prop 3.2m:
L01: Arbitrary $\gg$ $\mathcal{A}, \mathcal{B}$;
L02: Block $\gg$ Begin ;
L03: Prop 3.2m$_1$ $\gg$ $x' \cdot 0 = x \cdot 0 + 0$;
L04: Prop 3.2m$_2$ $\gg$ $(x' \cdot y = x \cdot y + y) \Rightarrow (x' \cdot y' = x \cdot y' + y')$;
L05: S9 @ y $\rhd$ L03 $\rhd$ L04 $\gg$ $x' \cdot y = x \cdot y + y$;
L06: Block $\gg$ End ;
L07: Ded $\rhd$ L06 $\gg$ $\mathcal{A}' \cdot \mathcal{B} = \mathcal{A} \cdot \mathcal{B} + \mathcal{B}$ □

[S **lemma** Prop 3.2n$_1$: $\Pi\mathcal{A}\colon \mathcal{A} \cdot 0 = 0 \cdot \mathcal{A}$]

S **proof of** Prop 3.2n$_1$:
L01: Arbitrary $\gg$ $\mathcal{A}$;
L02: S7 $\gg$ $\mathcal{A} \cdot 0 = 0$;
L03: Prop 3.2l $\gg$ $0 \cdot \mathcal{A} = 0$;
L04: Prop 3.2d $\rhd$ L02 $\rhd$ L03 $\gg$ $\mathcal{A} \cdot 0 = 0 \cdot \mathcal{A}$ □

[S **lemma** Prop 3.2n$_2$: $\Pi\mathcal{A}, \mathcal{B}\colon \mathcal{A} \cdot \mathcal{B} = \mathcal{B} \cdot \mathcal{A} \Rightarrow \mathcal{A} \cdot \mathcal{B}' = \mathcal{B}' \cdot \mathcal{A}$]

S **proof of** Prop 3.2n$_2$:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}$	;
L02:	Block $\gg$	Begin	;
L03:	Arbitrary $\gg$	$\quad\mathcal{A}, \mathcal{B}$	;
L04:	Premise $\gg$	$\quad\mathcal{A} \cdot \mathcal{B} = \mathcal{B} \cdot \mathcal{A}$	;
L05:	S8 $\gg$	$\quad\mathcal{A} \cdot \mathcal{B}' = \mathcal{A} \cdot \mathcal{B} + \mathcal{A}$	;
L06:	Prop 3.2e $\rhd$ L04 $\gg$	$\quad\mathcal{A} \cdot \mathcal{B} + \mathcal{A} = \mathcal{B} \cdot \mathcal{A} + \mathcal{A}$	;
L07:	Prop 3.2m $\gg$	$\quad\mathcal{B}' \cdot \mathcal{A} = \mathcal{B} \cdot \mathcal{A} + \mathcal{A}$	;
L08:	Prop 3.2b $\rhd$ L07 $\gg$	$\quad\mathcal{B} \cdot \mathcal{A} + \mathcal{A} = \mathcal{B}' \cdot \mathcal{A}$	;
L09:	Prop 3.2c $\rhd$ L06 $\rhd$ L08 $\gg$	$\quad\mathcal{A} \cdot \mathcal{B} + \mathcal{A} = \mathcal{B}' \cdot \mathcal{A}$	;
L10:	Prop 3.2c $\rhd$ L05 $\rhd$ L09 $\gg$	$\quad\mathcal{A} \cdot \mathcal{B}' = \mathcal{B}' \cdot \mathcal{A}$	;
L11:	Block $\gg$	End	;
L12:	Ded $\rhd$ L11 $\gg$	$\mathcal{A} \cdot \mathcal{B} = \mathcal{B} \cdot \mathcal{A} \Rightarrow \mathcal{A} \cdot \mathcal{B}' = \mathcal{B}' \cdot \mathcal{A}$	□

S **proof of** Prop 3.2n:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}$	;
L02:	Block $\gg$	Begin	;
L03:	Prop 3.2n_1 $\gg$	$\quad x \cdot 0 = 0 \cdot x$	;
L04:	Prop 3.2n_2 $\gg$	$\quad(x \cdot y = y \cdot x) \Rightarrow (x \cdot y' = y' \cdot x)$	;
L05:	S9 @ y $\rhd$ L03 $\rhd$ L04 $\gg$	$\quad x \cdot y = y \cdot x$	;
L06:	Block $\gg$	End	;
L07:	Ded $\rhd$ L06 $\gg$	$\mathcal{A} \cdot \mathcal{B} = \mathcal{B} \cdot \mathcal{A}$	□

S **proof of** Prop 3.2o:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}, \mathcal{C}$	;
L02:	Premise $\gg$	$\mathcal{A} = \mathcal{B}$	;
L03:	Block $\gg$	Begin	;
L04:	Arbitrary $\gg$	$\quad\mathcal{A}, \mathcal{B}, \mathcal{C}$	;
L05:	Premise $\gg$	$\quad\mathcal{A} = \mathcal{B}$	;
L06:	Prop 3.2k $\rhd$ L05 $\gg$	$\quad\mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{C}$	;
L07:	Prop 3.2n $\gg$	$\quad\mathcal{A} \cdot \mathcal{C} = \mathcal{C} \cdot \mathcal{A}$	;
L08:	Prop 3.2n $\gg$	$\quad\mathcal{B} \cdot \mathcal{C} = \mathcal{C} \cdot \mathcal{B}$	;
L09:	Prop 3.2c $\rhd$ L06 $\rhd$ L08 $\gg$	$\quad\mathcal{A} \cdot \mathcal{C} = \mathcal{C} \cdot \mathcal{B}$	;
L10:	S1 $\rhd$ L07 $\rhd$ L09 $\gg$	$\quad\mathcal{C} \cdot \mathcal{A} = \mathcal{C} \cdot \mathcal{B}$	;
L11:	Block $\gg$	End	;
L12:	Ded $\rhd$ L11 $\gg$	$\mathcal{A} = \mathcal{B} \Rightarrow \mathcal{C} \cdot \mathcal{A} = \mathcal{C} \cdot \mathcal{B}$	;
L13:	L12 $\unrhd$ L02 $\gg$	$\mathcal{C} \cdot \mathcal{A} = \mathcal{C} \cdot \mathcal{B}$	□

And thus we hve proved Prop 3.2.

3 **Proof of** Prop 3.4

in this section we will prove Prop 3.4 from [Men97]

[S **lemma** Prop 3.4a: $\Pi\mathcal{A}, \mathcal{B}, \mathcal{C}\colon \mathcal{A} \cdot (\mathcal{B} + \mathcal{C}) = (\mathcal{A} \cdot \mathcal{B}) + (\mathcal{A} \cdot \mathcal{C})$]

[S **lemma** Prop 3.4b: $\Pi\mathcal{A}, \mathcal{B}, \mathcal{C}\colon (\mathcal{B} + \mathcal{C}) \cdot \mathcal{A} = (\mathcal{B} \cdot \mathcal{A}) + (\mathcal{C} \cdot \mathcal{A})$]

[S **lemma** Prop 3.4c: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon (\mathcal{A}\cdot\mathcal{B})\cdot\mathcal{C} = \mathcal{A}\cdot(\mathcal{B}\cdot\mathcal{C})$]

[S **lemma** Prop 3.4d: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{A}+\mathcal{C} = \mathcal{B}+\mathcal{C} \Rightarrow \mathcal{A}=\mathcal{B}$]

[S **lemma** Prop 3.4a$_1$: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A}\cdot(\mathcal{B}+0) = \mathcal{A}\cdot\mathcal{B}+\mathcal{A}\cdot 0$]

S **proof of** Prop 3.4a$_1$:

L01:	Arbitrary $\gg$	$\mathcal{A},\mathcal{B}$	;
L02:	S5 $\gg$	$(\mathcal{B}+0) = (\mathcal{B})$	;
L03:	Prop 3.2o $\rhd$ L02 $\gg$	$\mathcal{A}\cdot(\mathcal{B}+0) = \mathcal{A}\cdot(\mathcal{B})$	;
L04:	S5 $\gg$	$\mathcal{A}\cdot\mathcal{B}+0 = \mathcal{A}\cdot\mathcal{B}$	;
L05:	Prop 3.2d $\rhd$ L03 $\rhd$ L04 $\gg$	$\mathcal{A}\cdot(\mathcal{B}+0) = \mathcal{A}\cdot\mathcal{B}+0$	;
L06:	S7 $\gg$	$\mathcal{A}\cdot 0 = 0$	;
L07:	Prop 3.2i $\rhd$ L06 $\gg$	$\mathcal{A}\cdot\mathcal{B}+\mathcal{A}\cdot 0 = \mathcal{A}\cdot\mathcal{B}+0$	;
L08:	Prop 3.2d $\rhd$ L05 $\rhd$ L07 $\gg$	$\mathcal{A}\cdot(\mathcal{B}+0) = \mathcal{A}\cdot\mathcal{B}+\mathcal{A}\cdot 0$	$\square$

[S **lemma** Prop 3.4a$_2$: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{A}\cdot(\mathcal{B}+\mathcal{C}) = \mathcal{A}\cdot\mathcal{B}+\mathcal{A}\cdot\mathcal{C} \Rightarrow \mathcal{A}\cdot(\mathcal{B}+\mathcal{C}') = \mathcal{A}\cdot\mathcal{B}+\mathcal{A}\cdot\mathcal{C}'$]

S **proof of** Prop 3.4a$_2$:

L01:	Arbitrary $\gg$	$\mathcal{A},\mathcal{B},\mathcal{C}$	;
L02:	Block $\gg$	Begin	;
L03:	Arbitrary $\gg$	$\mathcal{A},\mathcal{B},\mathcal{C}$	;
L04:	Premise $\gg$	$\mathcal{A}\cdot(\mathcal{B}+\mathcal{C}) = \mathcal{A}\cdot\mathcal{B}+\mathcal{A}\cdot\mathcal{C}$	;
L05:	S6 $\gg$	$\mathcal{B}+\mathcal{C}' = (\mathcal{B}+\mathcal{C})'$	;
L06:	Prop 3.2o $\rhd$ L05 $\gg$	$\mathcal{A}\cdot(\mathcal{B}+\mathcal{C}') = \mathcal{A}\cdot(\mathcal{B}+\mathcal{C})'$	;
L07:	S8 $\gg$	$\mathcal{A}\cdot(\mathcal{B}+\mathcal{C})' = \mathcal{A}\cdot(\mathcal{B}+\mathcal{C})+\mathcal{A}$	;
L08:	Prop 3.2e $\rhd$ L04 $\gg$	$(\mathcal{A}\cdot(\mathcal{B}+\mathcal{C}))+\mathcal{A} = (\mathcal{A}\cdot\mathcal{B}+\mathcal{A}\cdot\mathcal{C})+\mathcal{A}$	;
L09:	Prop 3.2j $\gg$	$(\mathcal{A}\cdot\mathcal{B}+\mathcal{A}\cdot\mathcal{C})+\mathcal{A} = \mathcal{A}\cdot\mathcal{B}+(\mathcal{A}\cdot\mathcal{C}+\mathcal{A})$	;
L10:	Prop 3.2c $\rhd$ L08 $\rhd$ L09 $\gg$	$(\mathcal{A}\cdot(\mathcal{B}+\mathcal{C}))+\mathcal{A} = \mathcal{A}\cdot\mathcal{B}+(\mathcal{A}\cdot\mathcal{C}+\mathcal{A})$	;
L11:	S8 $\gg$	$\mathcal{A}\cdot\mathcal{C}' = \mathcal{A}\cdot\mathcal{C}+\mathcal{A}$	;
L12:	Prop 3.2i $\rhd$ L11 $\gg$	$\mathcal{A}\cdot\mathcal{B}+\mathcal{A}\cdot\mathcal{C}' = \mathcal{A}\cdot\mathcal{B}+(\mathcal{A}\cdot\mathcal{C}+\mathcal{A})$	;
L13:	Prop 3.2d $\rhd$ L10 $\rhd$ L12 $\gg$	$\mathcal{A}\cdot(\mathcal{B}+\mathcal{C})+\mathcal{A} = \mathcal{A}\cdot\mathcal{B}+\mathcal{A}\cdot\mathcal{C}'$	;
L14:	Prop 3.2c $\rhd$ L07 $\rhd$ L13 $\gg$	$\mathcal{A}\cdot(\mathcal{B}+\mathcal{C})' = \mathcal{A}\cdot\mathcal{B}+\mathcal{A}\cdot\mathcal{C}'$	;
L15:	Prop 3.2c $\rhd$ L06 $\rhd$ L14 $\gg$	$\mathcal{A}\cdot(\mathcal{B}+\mathcal{C}') = \mathcal{A}\cdot\mathcal{B}+\mathcal{A}\cdot\mathcal{C}'$	;
L16:	Block $\gg$	End	;
L17:	Ded $\rhd$ L16 $\gg$	$(\mathcal{A}\cdot(\mathcal{B}+\mathcal{C}) = \mathcal{A}\cdot\mathcal{B}+\mathcal{A}\cdot\mathcal{C}) \Rightarrow \mathcal{A}\cdot(\mathcal{B}+\mathcal{C}') = \mathcal{A}\cdot\mathcal{B}+\mathcal{A}\cdot\mathcal{C}'$	$\square$

S **proof of** Prop 3.4a:

L01:	Arbitrary $\gg$	$\mathcal{A},\mathcal{B},\mathcal{C}$	;
L02:	Block $\gg$	Begin	;
L03:	Prop 3.4a$_1$ $\gg$	$x\cdot(y+0) = x\cdot y+x\cdot 0$	;
L04:	Prop 3.4a$_2$ $\gg$	$(x\cdot(y+z) = x\cdot y+x\cdot z) \Rightarrow (x\cdot(y+z') = x\cdot y+x\cdot z')$	;

L05:	S9 @ z ▷ L03 ▷ L04 ≫	$\quad x \cdot (y + z) = x \cdot y + x \cdot z$	;
L06:	Block ≫	End	;
L07:	Ded ▷ L06 ≫	$\mathcal{A} \cdot (\mathcal{B} + \mathcal{C}) = \mathcal{A} \cdot \mathcal{B} + \mathcal{A} \cdot \mathcal{C}$	□

S **proof of** Prop 3.4b:

L01:	Arbitrary ≫	$\mathcal{A}, \mathcal{B}, \mathcal{C}$	;
L02:	Prop 3.4a ≫	$\mathcal{A} \cdot (\mathcal{B} + \mathcal{C}) = \mathcal{A} \cdot \mathcal{B} + \mathcal{A} \cdot \mathcal{C}$	;
L03:	Prop 3.2n ≫	$\mathcal{A} \cdot (\mathcal{B} + \mathcal{C}) = (\mathcal{B} + \mathcal{C}) \cdot \mathcal{A}$	;
L04:	Prop 3.2n ≫	$\mathcal{A} \cdot \mathcal{B} = \mathcal{B} \cdot \mathcal{A}$	;
L05:	Prop 3.2n ≫	$\mathcal{A} \cdot \mathcal{C} = \mathcal{C} \cdot \mathcal{A}$	;
L06:	S1 ▷ L03 ▷ L02 ≫	$(\mathcal{B} + \mathcal{C}) \cdot \mathcal{A} = \mathcal{A} \cdot \mathcal{B} + \mathcal{A} \cdot \mathcal{C}$	;
L07:	Prop 3.2e ▷ L04 ≫	$\mathcal{A} \cdot \mathcal{B} + \mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{A} + \mathcal{A} \cdot \mathcal{C}$	;
L08:	Prop 3.2i ▷ L05 ≫	$\mathcal{B} \cdot \mathcal{A} + \mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{A} + \mathcal{C} \cdot \mathcal{A}$	;
L09:	Prop 3.2c ▷ L07 ▷ L08 ≫	$\mathcal{A} \cdot \mathcal{B} + \mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{A} + \mathcal{C} \cdot \mathcal{A}$	;
L10:	Prop 3.2c ▷ L06 ▷ L09 ≫	$(\mathcal{B} + \mathcal{C}) \cdot \mathcal{A} = \mathcal{B} \cdot \mathcal{A} + \mathcal{C} \cdot \mathcal{A}$	□

[S **lemma** Prop 3.4c$_1$: $\Pi \mathcal{A}, \mathcal{B}\colon (\mathcal{A} \cdot \mathcal{B}) \cdot 0 = \mathcal{A} \cdot (\mathcal{B} \cdot 0)$]

S **proof of** Prop 3.4c$_1$:

L01:	Arbitrary ≫	$\mathcal{A}, \mathcal{B}$	;
L02:	S7 ≫	$(\mathcal{A} \cdot \mathcal{B}) \cdot 0 = 0$	;
L03:	S7 ≫	$\mathcal{B} \cdot 0 = 0$	;
L04:	Prop 3.2o ▷ L03 ≫	$\mathcal{A} \cdot (\mathcal{B} \cdot 0) = \mathcal{A} \cdot 0$	;
L05:	S7 ≫	$\mathcal{A} \cdot 0 = 0$	;
L06:	Prop 3.2c ▷ L04 ▷ L05 ≫	$\mathcal{A} \cdot (\mathcal{B} \cdot 0) = 0$	;
L07:	Prop 3.2d ▷ L02 ▷ L06 ≫	$(\mathcal{A} \cdot \mathcal{B}) \cdot 0 = \mathcal{A} \cdot (\mathcal{B} \cdot 0)$	□

[S **lemma** Prop 3.4c$_2$: $\Pi \mathcal{A}, \mathcal{B}, \mathcal{C}\colon (\mathcal{A} \cdot \mathcal{B}) \cdot \mathcal{C} = \mathcal{A} \cdot (\mathcal{B} \cdot \mathcal{C}) \Rightarrow (\mathcal{A} \cdot \mathcal{B}) \cdot \mathcal{C}' = \mathcal{A} \cdot (\mathcal{B} \cdot \mathcal{C}')$]

S **proof of** Prop 3.4c$_2$:

L01:	Arbitrary ≫	$\mathcal{A}, \mathcal{B}, \mathcal{C}$	;
L02:	Block ≫	Begin	;
L03:	Arbitrary ≫	$\quad \mathcal{A}, \mathcal{B}, \mathcal{C}$	;
L04:	Premise ≫	$\quad (\mathcal{A} \cdot \mathcal{B}) \cdot \mathcal{C} = \mathcal{A} \cdot (\mathcal{B} \cdot \mathcal{C})$	;
L05:	S8 ≫	$\quad (\mathcal{A} \cdot \mathcal{B}) \cdot \mathcal{C}' = (\mathcal{A} \cdot \mathcal{B}) \cdot \mathcal{C} + (\mathcal{A} \cdot \mathcal{B})$	;
L06:	Prop 3.2e ▷ L04 ≫	$\quad ((\mathcal{A} \cdot \mathcal{B}) \cdot \mathcal{C}) + (\mathcal{A} \cdot \mathcal{B}) = (\mathcal{A} \cdot (\mathcal{B} \cdot \mathcal{C})) + (\mathcal{A} \cdot \mathcal{B})$	;
L07:	Prop 3.2c ▷ L05 ▷ L06 ≫	$\quad (\mathcal{A} \cdot \mathcal{B}) \cdot \mathcal{C}' = (\mathcal{A} \cdot (\mathcal{B} \cdot \mathcal{C})) + (\mathcal{A} \cdot \mathcal{B})$	;
L08:	Prop 3.4a ≫	$\quad \mathcal{A} \cdot ((\mathcal{B} \cdot \mathcal{C}) + \mathcal{B}) = \mathcal{A} \cdot (\mathcal{B} \cdot \mathcal{C}) + \mathcal{A} \cdot \mathcal{B}$	;
L09:	Prop 3.2d ▷ L07 ▷ L08 ≫	$\quad (\mathcal{A} \cdot \mathcal{B}) \cdot \mathcal{C}' = \mathcal{A} \cdot ((\mathcal{B} \cdot \mathcal{C}) + \mathcal{B})$	;
L10:	S8 ≫	$\quad \mathcal{B} \cdot \mathcal{C}' = \mathcal{B} \cdot \mathcal{C} + \mathcal{B}$	;
L11:	Prop 3.2o ▷ L10 ≫	$\quad \mathcal{A} \cdot (\mathcal{B} \cdot \mathcal{C}') = \mathcal{A} \cdot (\mathcal{B} \cdot \mathcal{C} + \mathcal{B})$	;
L12:	Prop 3.2d ▷ L09 ▷ L11 ≫	$\quad (\mathcal{A} \cdot \mathcal{B}) \cdot \mathcal{C}' = \mathcal{A} \cdot (\mathcal{B} \cdot \mathcal{C}')$	;
L13:	Block ≫	End	;
L14:	Ded ▷ L13 ≫	$(\mathcal{A} \cdot \mathcal{B}) \cdot \mathcal{C} = \mathcal{A} \cdot (\mathcal{B} \cdot \mathcal{C}) \Rightarrow (\mathcal{A} \cdot \mathcal{B}) \cdot \mathcal{C}' = \mathcal{A} \cdot (\mathcal{B} \cdot \mathcal{C}')$	□

S **proof of** Prop 3.4c :

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}, \mathcal{C}$	;
L02:	Block $\gg$	Begin	;
L03:	Prop 3.4c$_1$ $\gg$	$(x \cdot y) \cdot 0 = x \cdot (y \cdot 0)$	;
L04:	Prop 3.4c$_2$ $\gg$	$(x \cdot y) \cdot z = x \cdot (y \cdot z) \Rightarrow (x \cdot y) \cdot z' = x \cdot (y \cdot z')$	;
L05:	S9 @ $z \triangleright$ L03 $\triangleright$ L04 $\gg$	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$	;
L06:	Block $\gg$	End	;
L07:	Ded $\triangleright$ L06 $\gg$	$(\mathcal{A} \cdot \mathcal{B}) \cdot \mathcal{C} = \mathcal{A} \cdot (\mathcal{B} \cdot \mathcal{C})$	$\square$

[S **lemma** Prop 3.4d$_1$: $\Pi \mathcal{A}, \mathcal{B} \colon \mathcal{A} + 0 = \mathcal{B} + 0 \Rightarrow \mathcal{A} = \mathcal{B}$]

S **proof of** Prop 3.4d$_1$:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}$	;
L02:	Block $\gg$	Begin	;
L03:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}$	;
L04:	Premise $\gg$	$\mathcal{A} + 0 = \mathcal{B} + 0$	;
L05:	S5 $\gg$	$\mathcal{A} + 0 = \mathcal{A}$	;
L06:	S5 $\gg$	$\mathcal{B} + 0 = \mathcal{B}$	;
L07:	S1 $\triangleright$ L05 $\triangleright$ L04 $\gg$	$\mathcal{A} = \mathcal{B} + 0$	;
L08:	Prop 3.2c $\triangleright$ L07 $\triangleright$ L06 $\gg$	$\mathcal{A} = \mathcal{B}$	;
L09:	Block $\gg$	End	;
L10:	Ded $\triangleright$ L09 $\gg$	$\mathcal{A} + 0 = \mathcal{B} + 0 \Rightarrow \mathcal{A} = \mathcal{B}$	$\square$

[S **lemma** Prop 3.4d$_2$: $\Pi \mathcal{A}, \mathcal{B}, \mathcal{C} \colon (\mathcal{A} + \mathcal{C} = \mathcal{B} + \mathcal{C} \Rightarrow \mathcal{A} = \mathcal{B}) \Rightarrow \mathcal{A} + \mathcal{C}' = \mathcal{B} + \mathcal{C}' \Rightarrow \mathcal{A} = \mathcal{B}$]

S **proof of** Prop 3.4d$_2$:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}, \mathcal{C}$	;
L02:	Block $\gg$	Begin	;
L03:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}, \mathcal{C}$	;
L04:	Premise $\gg$	$\mathcal{A} + \mathcal{C} = \mathcal{B} + \mathcal{C} \Rightarrow \mathcal{A} = \mathcal{B}$	;
L05:	Premise $\gg$	$\mathcal{A} + \mathcal{C}' = \mathcal{B} + \mathcal{C}'$	;
L06:	S6 $\gg$	$\mathcal{A} + \mathcal{C}' = (\mathcal{A} + \mathcal{C})'$	;
L07:	S6 $\gg$	$\mathcal{B} + \mathcal{C}' = (\mathcal{B} + \mathcal{C})'$	;
L08:	S1 $\triangleright$ L06 $\triangleright$ L05 $\gg$	$(\mathcal{A} + \mathcal{C})' = \mathcal{B} + \mathcal{C}'$	;
L09:	Prop 3.2c $\triangleright$ L08 $\triangleright$ L07 $\gg$	$(\mathcal{A} + \mathcal{C})' = (\mathcal{B} + \mathcal{C})'$	;
L10:	S4 $\triangleright$ L09 $\gg$	$(\mathcal{A} + \mathcal{C}) = (\mathcal{B} + \mathcal{C})$	;
L11:	L04 $\trianglerighteq$ L10 $\gg$	$\mathcal{A} = \mathcal{B}$	;
L12:	Block $\gg$	End	;
L13:	Ded $\triangleright$ L12 $\gg$	$(\mathcal{A} + \mathcal{C} = \mathcal{B} + \mathcal{C} \Rightarrow \mathcal{A} = \mathcal{B}) \Rightarrow \mathcal{A} + \mathcal{C}' = \mathcal{B} + \mathcal{C}' \Rightarrow \mathcal{A} = \mathcal{B}$	$\square$

S **proof of** Prop 3.4d :

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}, \mathcal{C}$	;
L02:	Block $\gg$	Begin	;
L03:	Prop 3.4d$_1$ $\gg$	$x + 0 = y + 0 \Rightarrow x = y$	;

L04:	Prop 3.4d$_2$ $\gg$	$(x+z=y+z \Rightarrow x=y) \Rightarrow$ $x+z'=y+z' \Rightarrow x=y$	;
L05:	S9 @ z $\rhd$ L03 $\rhd$ L04 $\gg$	$x+z=y+z \Rightarrow x=y$	;
L06:	Block $\gg$	End	;
L07:	Ded $\rhd$ L06 $\gg$	$\mathcal{A}+\mathcal{C}=\mathcal{B}+\mathcal{C} \Rightarrow \mathcal{A}=\mathcal{B}$	$\square$

4 Numerals

We will for the remainder of the project need to construct some numerals, the numerals are the names for $0, 0', 0'' \ldots$ we will define them as folows $\overline{0}, \overline{1}, \overline{2} \ldots$

With these definitions we will now continue to prove Prop 3.5 that states different things about numerals.

[S **lemma** Prop 3.5a: $\Pi\mathcal{A}\colon \mathcal{A}+\overline{1}=\mathcal{A}'$]

[S **lemma** Prop 3.5b: $\Pi\mathcal{A}\colon \mathcal{A}\cdot\overline{1}=\mathcal{A}$]

[S **lemma** Prop 3.5c: $\Pi\mathcal{A}\colon \mathcal{A}\cdot\overline{2}=\mathcal{A}+\mathcal{A}$]

[S **lemma** Prop 3.5d: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A}+\mathcal{B}=0 \Rightarrow \mathcal{A}=0 \wedge \mathcal{B}=0$]

[S **lemma** Prop 3.5e: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A}\neq 0 \Rightarrow (\mathcal{B}\cdot\mathcal{A}=0 \Rightarrow \mathcal{B}=0)$]

[S **lemma** Prop 3.5f: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A}+\mathcal{B}=\overline{1} \Rightarrow (\mathcal{A}=0 \wedge \mathcal{B}=\overline{1}) \vee (\mathcal{A}=\overline{1} \wedge \mathcal{B}=0)$]

[S **lemma** Prop 3.5g: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A}\cdot\mathcal{B}=\overline{1} \Rightarrow (\mathcal{A}=\overline{1} \wedge \mathcal{B}=\overline{1})$]

[S **lemma** Prop 3.5h: $\Pi\mathcal{A}\colon \mathcal{A}\neq 0 \Rightarrow \exists\mathcal{B}\colon \mathcal{A}=\mathcal{B}'$]

[S **lemma** Prop 3.5i: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{C}\neq 0 \Rightarrow (\mathcal{A}\cdot\mathcal{C}=\mathcal{B}\cdot\mathcal{C} \Rightarrow \mathcal{A}=\mathcal{B})$]

[S **lemma** Prop 3.5j: $\Pi\mathcal{A}\colon \mathcal{A}\neq 0 \Rightarrow \mathcal{A}\neq\overline{1} \Rightarrow \exists\mathcal{B}\colon \mathcal{A}=\mathcal{B}''$]

S **proof of** Prop 3.5a:

L01:	Arbitrary $\gg$	$\mathcal{A}$	;
L02:	S6 $\gg$	$\mathcal{A}+0'=(\mathcal{A}+0)'$	;
L03:	S5 $\gg$	$\mathcal{A}+0=\mathcal{A}$	;
L04:	S2 $\rhd$ L03 $\gg$	$(\mathcal{A}+0)'=\mathcal{A}'$	;
L05:	Prop 3.2c $\rhd$ L02 $\rhd$ L04 $\gg$	$\mathcal{A}+0'=\mathcal{A}'$	;
L06:	Prop 3.2a $\gg$	$\mathcal{A}+0'=\mathcal{A}+\overline{1}$	;
L07:	S1 $\rhd$ L06 $\rhd$ L05 $\gg$	$\mathcal{A}+\overline{1}=\mathcal{A}'$	$\square$

S **proof of** Prop 3.5b:

L01:	Arbitrary $\gg$	$\mathcal{A}$	;
L02:	S8 $\gg$	$\mathcal{A}\cdot 0'=\mathcal{A}\cdot 0+\mathcal{A}$	;
L03:	S7 $\gg$	$\mathcal{A}\cdot 0=0$	;
L04:	Prop 3.2e $\rhd$ L03 $\gg$	$\mathcal{A}\cdot 0+\mathcal{A}=0+\mathcal{A}$	;
L05:	Prop 3.2c $\rhd$ L02 $\rhd$ L04 $\gg$	$\mathcal{A}\cdot 0'=0+\mathcal{A}$	;
L06:	Prop 3.2f $\gg$	$\mathcal{A}=0+\mathcal{A}$	;

L07:	Prop 3.2b $\rhd$ L06 $\gg$	$0 + \mathcal{A} = \mathcal{A}$	;
L08:	Prop 3.2c $\rhd$ L05 $\rhd$ L07 $\gg$	$\mathcal{A} \cdot 0' = \mathcal{A}$	;
L09:	Prop 3.2a $\gg$	$\mathcal{A} \cdot 0' = \mathcal{A} \cdot \overline{1}$	;
L10:	S1 $\rhd$ L09 $\rhd$ L08 $\gg$	$\mathcal{A} \cdot \overline{1} = \mathcal{A}$	$\square$

S **proof of** Prop 3.5c:

L01:	Arbitrary $\gg$	$\mathcal{A}$	;
L02:	S8 $\gg$	$\mathcal{A} \cdot \overline{1}' = \mathcal{A} \cdot \overline{1} + \mathcal{A}$	;
L03:	Prop 3.5b $\gg$	$\mathcal{A} \cdot \overline{1} = \mathcal{A}$	;
L04:	Prop 3.2e $\rhd$ L03 $\gg$	$\mathcal{A} \cdot \overline{1} + \mathcal{A} = \mathcal{A} + \mathcal{A}$	;
L05:	Prop 3.2c $\rhd$ L02 $\rhd$ L04 $\gg$	$\mathcal{A} \cdot \overline{1}' = \mathcal{A} + \mathcal{A}$	;
L06:	Prop 3.2a $\gg$	$\mathcal{A} \cdot \overline{1}' = \mathcal{A} \cdot \overline{2}$	;
L07:	S1 $\rhd$ L06 $\rhd$ L05 $\gg$	$\mathcal{A} \cdot \overline{2} = \mathcal{A} + \mathcal{A}$	$\square$

[S **lemma** Prop 3.5d$_1$: $\Pi\mathcal{A}\colon \mathcal{A} + 0 \Rightarrow \mathcal{A} = 0 \wedge 0 = 0$]

[S **lemma** Prop 3.5d$_2$: $\Pi\mathcal{A},\mathcal{B}\colon (\mathcal{A} + \mathcal{B} = 0 \Rightarrow \mathcal{A} = 0 \wedge \mathcal{B} = 0) \Rightarrow \mathcal{A} + \mathcal{B}' = 0 \Rightarrow \mathcal{A} = 0 \wedge \mathcal{B}' = 0$]

5 Lemma Prop 3.7

5.1 Definitions of $x < y$

in the folowing we need some definitions, we will macro define them as:

[$\mathsf{x} < \mathsf{y} \ddot{=} \exists \mathsf{z}\colon (\mathsf{z} \neq 0 \wedge \mathsf{z} + \mathsf{x} = \mathsf{y})$]
[$\mathsf{x} \leq \mathsf{y} \ddot{=} \mathsf{x} < \mathsf{y} \vee \mathsf{x} = \mathsf{y}$]
[$\mathsf{x} > \mathsf{y} \ddot{=} \mathsf{y} < \mathsf{x}$]
[$\mathsf{x} \geq \mathsf{y} \ddot{=} \mathsf{y} \leq \mathsf{x}$]
[$\mathsf{x} \not< \mathsf{y} \ddot{=} \neg(\mathsf{x} < \mathsf{y})$]
[$\mathsf{x} \not> \mathsf{y} \ddot{=} \mathsf{y} \not< \mathsf{x}$]
[$\mathsf{x} \mid \mathsf{y} \ddot{=} \exists \mathsf{z}\colon \mathsf{y} = \mathsf{x} \cdot \mathsf{z}$]
with these definitions, we will continue to prove the following statements:

[S **lemma** Prop 3.7a: $\Pi\mathcal{A}\colon \mathcal{A} \not< \mathcal{A}$]

[S **lemma** Prop 3.7b: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{A} < \mathcal{B} \Rightarrow \mathcal{B} < \mathcal{C} \Rightarrow \mathcal{A} < \mathcal{C}$]

[S **lemma** Prop 3.7c: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A} < \mathcal{B} \Rightarrow \mathcal{B} \not< \mathcal{A}$]

[S **lemma** Prop 3.7d: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{A} < \mathcal{B} \Rightarrow \mathcal{A} + \mathcal{C} < \mathcal{B} + \mathcal{C}$]

[S **lemma** Prop 3.7e: $\Pi\mathcal{A}\colon \mathcal{A} \leq \mathcal{A}$]

[S **lemma** Prop 3.7f: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{A} \leq \mathcal{B} \Rightarrow \mathcal{B} \leq \mathcal{C} \Rightarrow \mathcal{A} \leq \mathcal{C}$]

[S **lemma** Prop 3.7g: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{A} \leq \mathcal{B} \Rightarrow \mathcal{A} + \mathcal{C} \leq \mathcal{B} + \mathcal{C}$]

[S **lemma** Prop 3.7g$'$: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{A} + \mathcal{C} \leq \mathcal{B} + \mathcal{C} \Rightarrow \mathcal{A} \leq \mathcal{B}$]

[S **lemma** Prop 3.7h: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{A} \leq \mathcal{B} \Rightarrow \mathcal{B} < \mathcal{C} \Rightarrow \mathcal{A} < \mathcal{C}$]

[S **lemma** Prop 3.7i: $\Pi\mathcal{A}\colon 0 \leq \mathcal{A}$]

[S **lemma** Prop 3.7j: $\Pi\mathcal{A}\colon 0 < \mathcal{A}'$]

[S **lemma** Prop 3.7k: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A} < \mathcal{B} \Rightarrow \mathcal{A}' \leq \mathcal{B}$]

[S **lemma** Prop 3.7k′: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A}' \leq \mathcal{B} \Rightarrow \mathcal{A} < \mathcal{B}$]

[S **lemma** Prop 3.7l: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A} \leq \mathcal{B} \Rightarrow \mathcal{A} < \mathcal{B}'$]

[S **lemma** Prop 3.7l′: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A} < \mathcal{B}' \Rightarrow \mathcal{A} \leq \mathcal{B}$]

[S **lemma** Prop 3.7m: $\Pi\mathcal{A}\colon \mathcal{A} < \mathcal{A}'$]

[S **lemma** Prop 3.7o: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A} \neq \mathcal{B} \Rightarrow (\mathcal{A} < \mathcal{B} \vee \mathcal{B} < \mathcal{A})$]

[S **lemma** Prop 3.7p: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A} = \mathcal{B} \vee \mathcal{A} < \mathcal{B} \vee \mathcal{B} < \mathcal{A}$]

[S **lemma** Prop 3.7q: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A} \leq \mathcal{B} \vee \mathcal{B} \leq \mathcal{A}$]

[S **lemma** Prop 3.7r: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A} + \mathcal{B} \geq \mathcal{A}$]

[S **lemma** Prop 3.7s: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{B} \neq 0 \Rightarrow \mathcal{A} + \mathcal{B} > \mathcal{A}$]

[S **lemma** Prop 3.7t: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{B} \neq 0 \Rightarrow \mathcal{A} \cdot \mathcal{B} \geq \mathcal{A}$]

[S **lemma** Prop 3.7u: $\Pi\mathcal{A}\colon \mathcal{A} \neq 0 \Rightarrow \mathcal{A} > 0$]

[S **lemma** Prop 3.7u′: $\Pi\mathcal{A}\colon \mathcal{A} > 0 \Rightarrow \mathcal{A} \neq 0$]

[S **lemma** Prop 3.7v: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A} > 0 \Rightarrow \mathcal{B} > 0 \Rightarrow \mathcal{A} \cdot \mathcal{B} > 0$]

[S **lemma** Prop 3.7w: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A} \neq 0 \Rightarrow \mathcal{B} > \overline{1} \Rightarrow \mathcal{B} \cdot \mathcal{A} > \mathcal{A}$]

[S **lemma** Prop 3.7x: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{A} \neq 0 \Rightarrow \mathcal{B} < \mathcal{C} \Rightarrow \mathcal{B} \cdot \mathcal{A} < \mathcal{C} \cdot \mathcal{A}$]

[S **lemma** Prop 3.7x′: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{A} \neq 0 \Rightarrow \mathcal{B} \cdot \mathcal{A} < \mathcal{C} \cdot \mathcal{A} \Rightarrow \mathcal{B} < \mathcal{C}$]

[S **lemma** Prop 3.7y: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{A} \neq 0 \Rightarrow \mathcal{B} \leq \mathcal{C} \Rightarrow \mathcal{B} \cdot \mathcal{A} \leq \mathcal{C} \cdot \mathcal{A}$]

[S **lemma** Prop 3.7y′: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{A} \neq 0 \Rightarrow \mathcal{B} \cdot \mathcal{A} \leq \mathcal{C} \cdot \mathcal{A} \Rightarrow \mathcal{B} \leq \mathcal{C}$]

[S **lemma** Prop 3.7z: $\Pi\mathcal{A}\colon \mathcal{A} \not< 0$]

[S **lemma** Prop 3.7z′: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A} \leq \mathcal{B} \wedge \mathcal{B} \leq \mathcal{A} \Rightarrow \mathcal{A} = \mathcal{B}$]

5.2 **proof of** Prop 3.7

6 **Lemma** Prop 3.10

[S **lemma** Prop 3.10a: $\Pi\mathcal{A}\colon \mathcal{A} \mid \mathcal{A}$]

[S **lemma** Prop 3.10b: $\Pi\mathcal{A}\colon \bar{1} \mid \mathcal{A}$]

[S **lemma** Prop 3.10c: $\Pi\mathcal{A}\colon \mathcal{A} \mid 0$]

[S **lemma** Prop 3.10d: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{A} \mid \mathcal{B} \wedge \mathcal{B} \mid \mathcal{C} \Rightarrow \mathcal{A} \mid \mathcal{C}$]

[S **lemma** Prop 3.10e: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A} \neq 0 \wedge \mathcal{B} \mid \mathcal{B} \Rightarrow \mathcal{A} \leq \mathcal{A}$]

[S **lemma** Prop 3.10f: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A} \mid \mathcal{B} \wedge \mathcal{B} \mid \mathcal{A} \Rightarrow \mathcal{A} = \mathcal{B}$]

[S **lemma** Prop 3.10g: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{A} \mid \mathcal{B} \Rightarrow \mathcal{A} \mid (\mathcal{B} \cdot \mathcal{C})$]

[S **lemma** Prop 3.10h: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{A} \mid \mathcal{B} \wedge \mathcal{A} \mid \mathcal{C} \Rightarrow \mathcal{A} \mid (\mathcal{B} + \mathcal{C})$]

7 **Lemma** Prop 3.11

[S **lemma** Prop 3.11: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A} \neq 0 \Rightarrow \exists\mathcal{C},\mathcal{D}\colon (\mathcal{B} = \mathcal{A} \cdot \mathcal{C} + \mathcal{D} \wedge \mathcal{D} < \mathcal{A} \wedge \forall\mathcal{E}, \mathcal{F}\colon ((\mathcal{B} = \mathcal{A} \cdot \mathcal{E} + \mathcal{F} \wedge \mathcal{F} < \mathcal{A}) \Rightarrow \mathcal{C} = \mathcal{E} \wedge \mathcal{D} = \mathcal{F}))$]

A Helping lemmas

for some of our proofs we need to prove some lemmas concerning $\wedge$ and $\vee$ these will be proved here.

[S **lemma** T0: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{B} \Rightarrow (\mathcal{A} \Rightarrow \mathcal{B})$]

[S **lemma** T1: $\Pi\mathcal{A},\mathcal{B}\colon \neg\mathcal{A} \Rightarrow (\mathcal{A} \Rightarrow \mathcal{B})$]

[S **lemma** H0a: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon (\mathcal{A} \Rightarrow \mathcal{B}) \vdash (\mathcal{B} \Rightarrow \mathcal{C}) \vdash \mathcal{A} \Rightarrow \mathcal{C}$]

[S **lemma** H0b: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{A} \Rightarrow (\mathcal{B} \Rightarrow \mathcal{C}) \vdash \mathcal{B} \vdash \mathcal{A} \Rightarrow \mathcal{C}$]

[S **lemma** H1: $\Pi\mathcal{A}\colon \neg\neg\mathcal{A} \Rightarrow \mathcal{A}$]

[S **lemma** H2: $\Pi\mathcal{A}\colon \mathcal{A} \Rightarrow \neg\neg\mathcal{A}$]

[S **lemma** Con1: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A} \wedge \mathcal{B} \vdash \mathcal{A}$]

[S **lemma** Con2: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A} \wedge \mathcal{B} \vdash \mathcal{B}$]

[S **lemma** Con: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A} \vdash \mathcal{B} \vdash \mathcal{A} \wedge \mathcal{B}$]

[S **lemma** Dis1: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A} \vdash \mathcal{A} \vee \mathcal{B}$]

[S **lemma** Dis2: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{B} \vdash \mathcal{A} \vee \mathcal{B}$]

S **proof of** T0:

L01:	Arbitrary $\gg$	$\mathcal{A},\mathcal{B}$	;
L02:	Block $\gg$	Begin	;
L03:	Arbitrary $\gg$	$\mathcal{A},\mathcal{B}$	;

L04:	Premise $\gg$	$\mathcal{B}$	;
L05:	Block $\gg$	Begin	;
L06:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}$	;
L07:	Premise $\gg$	$\mathcal{A}$	;
L08:	Repetition $\triangleright$ L04 $\gg$	$\mathcal{B}$	;
L09:	Block $\gg$	End	;
L10:	Ded $\triangleright$ L09 $\gg$	$\mathcal{A} \Rightarrow \mathcal{B}$	;
L11:	Block $\gg$	End	;
L12:	Ded $\triangleright$ L11 $\gg$	$\mathcal{B} \Rightarrow (\mathcal{A} \Rightarrow \mathcal{B})$	$\square$

S **proof of** T1:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}$	;
L02:	Block $\gg$	Begin	;
L03:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}$	;
L04:	Premise $\gg$	$\neg\mathcal{A}$	;
L05:	Block $\gg$	Begin	;
L06:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}$	;
L07:	Premise $\gg$	$\mathcal{A}$	;
L08:	Repetition $\triangleright$ L04 $\gg$	$\neg\mathcal{A}$	;
L09:	T0 $\gg$	$\mathcal{A} \Rightarrow (\neg\mathcal{B} \Rightarrow \mathcal{A})$	;
L10:	L09 $\trianglerighteq$ L07 $\gg$	$\neg\mathcal{B} \Rightarrow \mathcal{A}$	;
L11:	T0 $\gg$	$\neg\mathcal{A} \Rightarrow (\neg\mathcal{B} \Rightarrow \neg\mathcal{A})$	;
L12:	L11 $\trianglerighteq$ L08 $\gg$	$\neg\mathcal{B} \Rightarrow \neg\mathcal{A}$	;
L13:	Neg $\triangleright$ L12 $\triangleright$ L10 $\gg$	$\mathcal{B}$	;
L14:	Block $\gg$	End	;
L15:	Ded $\triangleright$ L14 $\gg$	$\mathcal{A} \Rightarrow \mathcal{B}$	;
L16:	Block $\gg$	End	;
L17:	Ded $\triangleright$ L16 $\gg$	$\neg\mathcal{A} \Rightarrow (\mathcal{A} \Rightarrow \mathcal{B})$	$\square$

S **proof of** H0a:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}, \mathcal{C}$	;
L02:	Premise $\gg$	$\mathcal{A} \Rightarrow \mathcal{B}$	;
L03:	Premise $\gg$	$\mathcal{B} \Rightarrow \mathcal{C}$	;
L04:	Block $\gg$	Begin	;
L05:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}, \mathcal{C}$	;
L06:	Premise $\gg$	$\mathcal{A} \Rightarrow \mathcal{B}$	;
L07:	Premise $\gg$	$\mathcal{B} \Rightarrow \mathcal{C}$	;
L08:	Premise $\gg$	$\mathcal{A}$	;
L09:	L06 $\trianglerighteq$ L08 $\gg$	$\mathcal{B}$	;
L10:	L07 $\trianglerighteq$ L09 $\gg$	$\mathcal{C}$	;
L11:	Block $\gg$	End	;
L12:	Ded $\triangleright$ L11 $\gg$	$(\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow (\mathcal{B} \Rightarrow \mathcal{C}) \Rightarrow \mathcal{A} \Rightarrow \mathcal{C}$	;
L13:	L12 $\trianglerighteq$ L02 $\gg$	$(\mathcal{B} \Rightarrow \mathcal{C}) \Rightarrow \mathcal{A} \Rightarrow \mathcal{C}$	;
L14:	L13 $\trianglerighteq$ L03 $\gg$	$\mathcal{A} \Rightarrow \mathcal{C}$	$\square$

S **proof of** H0b:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}, \mathcal{C}$	;

L02:	Premise $\gg$	$\mathcal{A} \Rightarrow \mathcal{B} \Rightarrow \mathcal{C}$	;
L03:	Premise $\gg$	$\mathcal{B}$	;
L04:	A2′ $\gg$	$(\mathcal{A} \Rightarrow \mathcal{B} \Rightarrow \mathcal{C}) \Rightarrow (\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow \mathcal{A} \Rightarrow \mathcal{C}$	;
L05:	L04 $\trianglerighteq$ L02 $\gg$	$(\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow \mathcal{A} \Rightarrow \mathcal{C}$	;
L06:	T0 $\gg$	$\mathcal{B} \Rightarrow \mathcal{A} \Rightarrow \mathcal{B}$	;
L07:	L06 $\trianglerighteq$ L03 $\gg$	$\mathcal{A} \Rightarrow \mathcal{B}$	;
L08:	L05 $\trianglerighteq$ L07 $\gg$	$\mathcal{A} \Rightarrow \mathcal{C}$	□

S **proof of** H1:

L01:	Arbitrary $\gg$	$\mathcal{A}$	;
L02:	Neg $\gg$	$(\neg\mathcal{A} \Rightarrow \neg(\neg\mathcal{A})) \Rightarrow ((\neg\mathcal{A} \Rightarrow (\neg\mathcal{A})) \Rightarrow \mathcal{A})$	;
L03:	Repetition $\gg$	$\neg\mathcal{A} \Rightarrow \neg\mathcal{A}$	;
L04:	H0b $\triangleright$ L02 $\triangleright$ L03 $\gg$	$(\neg\mathcal{A} \Rightarrow \neg(\neg\mathcal{A})) \Rightarrow \mathcal{A}$	;
L05:	A1′ $\gg$	$\neg(\neg\mathcal{A}) \Rightarrow (\neg\mathcal{A} \Rightarrow \neg(\neg\mathcal{A}))$	;
L06:	H0a $\triangleright$ L05 $\triangleright$ L04 $\gg$	$\neg(\neg\mathcal{A}) \Rightarrow \mathcal{A}$	□

S **proof of** H2:

L01:	Arbitrary $\gg$	$\mathcal{A}$	;
L02:	Neg $\gg$	$(\neg\neg\neg\mathcal{A} \Rightarrow \neg\mathcal{A}) \Rightarrow ((\neg\neg\neg\mathcal{A} \Rightarrow \mathcal{A}) \Rightarrow \neg\neg\mathcal{A})$	;
L03:	H1 $\gg$	$\neg\neg\neg\mathcal{A} \Rightarrow \neg\mathcal{A}$	;
L04:	L02 $\trianglerighteq$ L03 $\gg$	$(\neg\neg\neg\mathcal{A} \Rightarrow \mathcal{A}) \Rightarrow \neg\neg\mathcal{A}$	;
L05:	A1′ $\gg$	$\mathcal{A} \Rightarrow (\neg\neg\neg\mathcal{A} \Rightarrow \mathcal{A})$	;
L06:	H1 $\triangleright$ L04 $\triangleright$ L05 $\gg$	$\mathcal{A} \Rightarrow \neg\neg\mathcal{A}$	□

Litteratur

[Gru06] K. Grue. A logiweb base page. Technical report, Logiweb, 2006. http://www.diku.dk/cgi-bin/cginetd/grue/relay/go/01693B66F0C393B695F3F15770898897A683141109C083AC838ADCA20806/2/.

[Men97] E. Mendelson. *Introduction to Mathematical Logic*. Chapmann & Hall, 4. edition, 1997.