



# projekt i logik

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## Indhold

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### Resumé

## 1 Introduction

In this paper we will prove the propositions regarding S starting with proposition Prop 3.2i. continuing to 3.4, bevise 3.5, aksiomer, der definerer  $x \rightarrow y$ , bevise 3.10, og bevise 3.11.

We chose to prove chapter 3 from mendelson, starting with Lemma 3.2, since the first ten wfs' are proved in Mendelson, we will start with Lemma 3.2.i.

## 2

A modified version of Mendelsons system S (Peano arithmetic) [Men97] may be formulated thus:

$$\begin{aligned}
& [S \xrightarrow{\text{stmt}} \Pi \underline{a}: \underline{a} + 0 = \underline{a} \oplus \Pi \underline{a}, \underline{b}: \underline{a} \Rightarrow \underline{b} \vdash \underline{a} \oplus \underline{b} \oplus \Pi \underline{a}, \underline{b}: \underline{a} = \underline{b} \vdash \underline{a} = \underline{c} \vdash \underline{b} = \underline{c} \oplus \Pi \underline{x}, \underline{y}: \\
& \quad \underline{b}: \underline{a} \Rightarrow \underline{b} \vdash \underline{a} \vdash \underline{b}] [MP \xrightarrow{\text{proof}} \text{Rule tactic}] \\
& [Gen \xrightarrow{\text{stmt}} S \vdash \Pi \underline{x}, \underline{a}: \underline{a} \vdash \forall \underline{x}: \underline{a}] [Gen \xrightarrow{\text{proof}} \text{Rule tactic}] \\
& \quad \underline{b}: \text{Ded}(\underline{a}, \underline{b}) \vdash \underline{a} \vdash \underline{b}] [\text{Ded} \xrightarrow{\text{proof}} \text{Rule tactic}] \\
& \quad [S2 \xrightarrow{\text{stmt}} S \vdash \Pi \underline{a}, \\
& \quad \underline{b}: \underline{a} = \underline{b} \vdash \underline{a}' = \underline{b}'] [S2 \xrightarrow{\text{proof}} \text{Rule tactic}]
\end{aligned}$$

$$[\text{S3} \xrightarrow{\text{stmt}} \text{S} \vdash \Pi \underline{a}: -0 = \underline{a}'] [\text{S3} \xrightarrow{\text{proof}} \text{Rule tactic}] [\text{S4} \xrightarrow{\text{stmt}} \text{S} \vdash \Pi \underline{a}, \underline{b}: \underline{a}' = \underline{b}' \vdash \underline{a} = \underline{b}] [\text{S4} \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[\text{S5} \xrightarrow{\text{stmt}} \text{S} \vdash \Pi \underline{a}: \underline{a} + 0 = \underline{a}] [\text{S5} \xrightarrow{\text{proof}} \text{Rule tactic}] [\text{S6} \xrightarrow{\text{stmt}} \text{S} \vdash \Pi \underline{a}, \underline{b}: \underline{a} + \underline{b}' = \underline{a} + \underline{b}'] [\text{S6} \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[\text{S7} \xrightarrow{\text{stmt}} \text{S} \vdash \Pi \underline{a}: \underline{a} \cdot 0 = 0] [\text{S7} \xrightarrow{\text{proof}} \text{Rule tactic}] [\text{S8} \xrightarrow{\text{stmt}} \text{S} \vdash \Pi \underline{a}, \underline{b}: \underline{a} \cdot \underline{b}' = \underline{a} \cdot \underline{b} + \underline{a}] [\text{S8} \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[\text{Neg} \xrightarrow{\text{stmt}} \text{S} \vdash \Pi \underline{a}: \Pi \underline{b}: \neg \underline{b} \Rightarrow \neg \underline{a} \vdash \neg \underline{b} \Rightarrow \underline{a} \vdash \underline{b}] [\text{Neg} \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[\text{S1} \xrightarrow{\text{stmt}} \text{S} \vdash \Pi \underline{a}, \underline{b}, \underline{c}: \underline{a} = \underline{b} \vdash \underline{a} = \underline{c} \vdash \underline{b} = \underline{c}] [\text{S1} \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[\text{S9} \xrightarrow{\text{stmt}} \text{S} \vdash \Pi \underline{x}, \underline{a}, \underline{b}, \underline{c}: \langle \underline{b} \equiv \underline{a} \mid \underline{x} := 0 \rangle \vdash \langle \underline{c} \equiv \underline{a} \mid \underline{x} := \underline{x}' \rangle \vdash \underline{b} \vdash \underline{a} \Rightarrow \underline{c} \vdash \underline{a}] [\text{S9} \xrightarrow{\text{proof}} \text{Rule tactic}]$$

$$[\text{Prop 3.2a} \xrightarrow{\text{stmt}} \text{S} \vdash \Pi \underline{a}: \underline{a} = \underline{a}]$$

$$[\text{Prop 3.2b} \xrightarrow{\text{stmt}} \text{S} \vdash \Pi \underline{a}, \underline{b}: \underline{a} = \underline{b} \vdash \underline{b} = \underline{a}]$$

$$[\text{Prop 3.2c} \xrightarrow{\text{stmt}} \text{S} \vdash \Pi \underline{a}, \underline{b}, \underline{c}: \underline{a} = \underline{b} \vdash \underline{b} = \underline{c} \vdash \underline{a} = \underline{c}]$$

$$[\text{Prop 3.2d} \xrightarrow{\text{stmt}} \text{S} \vdash \Pi \underline{a}, \underline{b}, \underline{c}: \underline{a} = \underline{c} \vdash \underline{b} = \underline{c} \vdash \underline{a} = \underline{b}]$$

$$[\text{Prop 3.2e} \xrightarrow{\text{stmt}} \text{S} \vdash \Pi \underline{a}, \underline{b}, \underline{c}: \underline{a} = \underline{b} \vdash \underline{a} + \underline{c} = \underline{b} + \underline{c}]$$

$$[\text{Prop 3.2f} \xrightarrow{\text{stmt}} \text{S} \vdash \Pi \underline{a}: \underline{a} = 0 + \underline{a}]$$

$$[\text{Prop 3.2g} \xrightarrow{\text{stmt}} \text{S} \vdash \Pi \underline{a}, \underline{b}: \underline{a}' + \underline{b} = \underline{a} + \underline{b}']$$

$$[\text{Prop 3.2h} \xrightarrow{\text{stmt}} \text{S} \vdash \Pi \underline{a}, \underline{b}: \underline{a} + \underline{b} = \underline{b} + \underline{a}]$$

$$[\text{Prop 3.2i} \xrightarrow{\text{stmt}} \text{S} \vdash \Pi \underline{a}, \underline{b}, \underline{c}: \underline{a} = \underline{b} \vdash \underline{c} + \underline{a} = \underline{c} + \underline{b}]$$

$$[\text{Prop 3.2j} \xrightarrow{\text{stmt}} \text{S} \vdash \Pi \underline{a}, \underline{b}, \underline{c}: \underline{a} + \underline{b} + \underline{c} = \underline{a} + \underline{b} + \underline{c}]$$

$$[\text{Prop 3.2k} \xrightarrow{\text{stmt}} \text{S} \vdash \Pi \underline{a}, \underline{b}, \underline{c}: \underline{a} = \underline{b} \vdash \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c}]$$

$$[\text{Prop 3.2l} \xrightarrow{\text{stmt}} \text{S} \vdash \Pi \underline{a}: 0 \cdot \underline{a} = 0]$$

$$[\text{Prop 3.2m} \xrightarrow{\text{stmt}} \text{S} \vdash \Pi \underline{a}, \underline{b}: \underline{a}' \cdot \underline{b} = \underline{a} \cdot \underline{b} + \underline{b}]$$

$$[\text{Prop 3.2n} \xrightarrow{\text{stmt}} \text{S} \vdash \Pi \underline{a}, \underline{b}: \underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}]$$

$$[\text{Prop 3.2o} \xrightarrow{\text{stmt}} \text{S} \vdash \Pi \underline{a}, \underline{b}, \underline{c}: \underline{a} = \underline{b} \vdash \underline{c} \cdot \underline{a} = \underline{c} \cdot \underline{b}]$$

we will start by doing Prop 3.2i



$L_?: \quad S9 @ \bar{z} \triangleright \bar{x} + \bar{y} + 0 =$   
 $\quad \bar{x} + \bar{y} + 0 \triangleright \bar{x} + \bar{y} + \bar{z} = \bar{x} + \bar{y} + \bar{z} \Rightarrow$   
 $\quad \bar{x} + \bar{y} + \bar{z}' = \bar{x} + \bar{y} + \bar{z}' \gg \quad \bar{x} + \bar{y} + \bar{z} = \bar{x} + \bar{y} + \bar{z} \quad ;$   
 $L_c: \quad \text{Block} \gg \quad \text{End} \quad \text{Ded} \triangleright$   
 $L_c \gg \underline{a} + \underline{b} + \underline{c} = \underline{a} + \underline{b} + \underline{c}], p_0, c)]$

$[\text{Prop 3.2k}_1 \xrightarrow{\text{stmt}} S \vdash \Pi \underline{a}, \underline{b}: \underline{a} = \underline{b} \vdash \underline{a} \cdot 0 = \underline{b} \cdot 0]$

$[\text{Prop 3.2k}_1 \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([S \vdash$   
 $L_?: \quad \text{Arbitrary} \gg \quad \underline{a}, \underline{b} \quad S7; \gg$   
 $\underline{a} \cdot 0 = 0; S7 \gg \underline{b} \cdot 0 = 0; \text{Prop 3.2a} \gg 0 = 0; \text{Prop 3.2b} \triangleright \underline{b} \cdot 0 = 0 \gg 0 =$   
 $\underline{b} \cdot 0; \text{Prop 3.2c} \gg \underline{a} \cdot 0 = 0 \Rightarrow 0 = \underline{b} \cdot 0 \Rightarrow \underline{a} \cdot 0 = \underline{b} \cdot 0; \underline{a} \cdot 0 = 0 \Rightarrow 0 = \underline{b} \cdot 0 \Rightarrow$   
 $\underline{a} \cdot 0 = \underline{b} \cdot 0 \triangleright \underline{a} \cdot 0 = 0 \triangleright 0 = \underline{b} \cdot 0 \gg \underline{a} \cdot 0 = \underline{b} \cdot 0], p_0, c)]$

$[\text{Prop 3.2k}_2 \xrightarrow{\text{stmt}} S \vdash \Pi \underline{a}, \underline{b}, \underline{c}: \underline{a} = \underline{b} \Rightarrow \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c} \vdash \underline{a} = \underline{b} \Rightarrow \underline{a} \cdot \underline{c}' = \underline{b} \cdot \underline{c}']$

$[\text{Prop 3.2k}_2 \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([S \vdash$   
 $L_?: \quad \text{Arbitrary} \gg \quad \underline{a}, \underline{b}, \underline{c} \quad ;$   
 $L_?: \quad \text{Block} \gg \quad \text{Begin} \quad ;$   
 $L_?: \quad \text{Arbitrary} \gg \quad \underline{a}, \underline{b}, \underline{c} \quad \underline{a} \neq$   
 $\underline{b} \Rightarrow \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c} \vdash \underline{a} = \underline{b} \vdash \underline{a} = \underline{b} \Rightarrow \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c} \triangleright \underline{a} = \underline{b} \gg \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c}; S8 \gg \underline{a} \cdot \underline{c}' =$   
 $\underline{a} \cdot \underline{c} + \underline{a}; S8 \gg \underline{b} \cdot \underline{c}' = \underline{b} \cdot \underline{c} + \underline{b}; \text{Prop 3.2e} \gg \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c} \Rightarrow \underline{a} \cdot \underline{c} + \underline{a} = \underline{b} \cdot \underline{c} + \underline{a}; \underline{a} \cdot \underline{c} =$   
 $\underline{b} \cdot \underline{c} \Rightarrow \underline{a} \cdot \underline{c} + \underline{a} = \underline{b} \cdot \underline{c} + \underline{a} \triangleright \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c} \gg \underline{a} \cdot \underline{c} + \underline{a} = \underline{b} \cdot \underline{c} + \underline{a}; \text{Prop 3.2i} \gg \underline{a} =$   
 $\underline{b} \Rightarrow \underline{b} \cdot \underline{c} + \underline{a} = \underline{b} \cdot \underline{c} + \underline{b}; \underline{a} = \underline{b} \Rightarrow \underline{b} \cdot \underline{c} + \underline{a} = \underline{b} \cdot \underline{c} + \underline{b} \triangleright \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c} \gg \underline{b} \cdot \underline{c} + \underline{a} =$   
 $\underline{b} \cdot \underline{c} + \underline{b}; \text{Prop 3.2c} \gg \underline{a} \cdot \underline{c} + \underline{a} = \underline{b} \cdot \underline{c} + \underline{a} \Rightarrow \underline{b} \cdot \underline{c} + \underline{a} = \underline{b} \cdot \underline{c} + \underline{b} \Rightarrow \underline{a} \cdot \underline{c} + \underline{a} =$   
 $\underline{b} \cdot \underline{c} + \underline{b}; \underline{a} \cdot \underline{c} + \underline{a} = \underline{b} \cdot \underline{c} + \underline{a} \Rightarrow \underline{b} \cdot \underline{c} + \underline{a} = \underline{b} \cdot \underline{c} + \underline{b} \Rightarrow \underline{a} \cdot \underline{c} + \underline{a} =$   
 $\underline{b} \cdot \underline{c} + \underline{b} \triangleright \underline{a} \cdot \underline{c} + \underline{a} = \underline{b} \cdot \underline{c} + \underline{a} \triangleright \underline{b} \cdot \underline{c} + \underline{a} = \underline{b} \cdot \underline{c} + \underline{b} \gg \underline{a} \cdot \underline{c} + \underline{a} = \underline{b} \cdot \underline{c} + \underline{b};$   
 $L_?: \quad \underline{a} \cdot \underline{c} + \underline{a} = \underline{b} \cdot \underline{c} + \underline{b} \triangleright \underline{a} \cdot \underline{c}' =$   
 $\quad \underline{a} \cdot \underline{c} + \underline{a} \triangleright \underline{b} \cdot \underline{c}' = \underline{b} \cdot \underline{c} + \underline{b} \gg \quad \underline{a} \cdot \underline{c}' = \underline{b} \cdot \underline{c}' \quad ;$   
 $L_i: \quad \text{Block} \gg \quad \text{End} \quad \text{Ded} \triangleright$   
 $L_i \gg \underline{a} = \underline{b} \Rightarrow \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c} \vdash \underline{a} = \underline{b} \Rightarrow \underline{a} \cdot \underline{c}' = \underline{b} \cdot \underline{c}'], p_0, c)]$

$[\text{Prop 3.2k} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([S \vdash$   
 $L_?: \quad \text{Arbitrary} \gg \quad \underline{a}, \underline{b}, \underline{c} \quad ;$   
 $L_?: \quad \text{Block} \gg \quad \text{Begin} \quad \text{Prop 3}$   
 $\bar{y} \Rightarrow \bar{x} \cdot 0 = \bar{y} \cdot 0; \text{Prop 3.2k}_2 \gg \bar{x} = \bar{y} \Rightarrow \bar{x} \cdot \bar{z} = \bar{y} \cdot \bar{z} \Rightarrow \bar{x} = \bar{y} \Rightarrow \bar{x} \cdot \bar{z}' = \bar{y} \cdot \bar{z}';$   
 $L_?: \quad S9 @ \bar{z} \triangleright \bar{x} = \bar{y} \Rightarrow \bar{x} \cdot 0 =$   
 $\quad \bar{y} \cdot 0 \triangleright \bar{x} = \bar{y} \Rightarrow \bar{x} \cdot \bar{z} = \bar{y} \cdot \bar{z} \Rightarrow$   
 $\quad \bar{x} = \bar{y} \Rightarrow \bar{x} \cdot \bar{z}' = \bar{y} \cdot \bar{z}' \gg \quad \bar{x} = \bar{y} \Rightarrow \bar{x} \cdot \bar{z} = \bar{y} \cdot \bar{z} \quad ;$   
 $L_c: \quad \text{Block} \gg \quad \text{End} \quad \text{Ded} \triangleright$   
 $L_c \gg \underline{a} = \underline{b} \vdash \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c}], p_0, c)]$

$[\text{Prop 3.2l}_1 \xrightarrow{\text{stmt}} S \vdash \Pi \underline{a}: 0 \cdot 0 = 0]$

$[\text{Prop 3.2l}_1 \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([S \vdash S7 \gg 0 \cdot 0 = 0], p_0, c)]$

$[\text{Prop 3.2l}_2 \xrightarrow{\text{stmt}} S \vdash \Pi \underline{a}: 0 \cdot \underline{a} = 0 \Rightarrow 0 \cdot \underline{a}' = 0]$

[Prop 3.2l<sub>2</sub>  $\xrightarrow{\text{proof}}$   $\lambda c. \lambda x. \mathcal{P}([S \vdash$

L <sub>?</sub> : Arbitrary $\gg$	$\underline{a}$	;
L <sub>?</sub> : Block $\gg$	Begin	;
L <sub>?</sub> : Arbitrary $\gg$	$\underline{a}$	0 ;
$\underline{a} = 0 \vdash S8 \gg 0 \cdot \underline{a}' = 0 \cdot \underline{a} + 0; S5 \gg 0 \cdot \underline{a} + 0 = 0 \cdot \underline{a}; \text{Prop 3.2c} \triangleright 0 \cdot \underline{a} + 0 =$		
$0 \cdot \underline{a} \triangleright 0 \cdot \underline{a}' = 0 \cdot \underline{a} + 0 \gg 0 \cdot \underline{a}' = 0 \cdot \underline{a};$		
L <sub>?</sub> : Prop 3.2c $\triangleright 0 \cdot \underline{a}' =$		
	$0 \cdot \underline{a} \triangleright 0 \cdot \underline{a} = 0 \gg$	$0 \cdot \underline{a}' = 0$
L <sub>e</sub> : Block $\gg$	End	Ded $\triangleright$
L <sub>e</sub> $\gg 0 \cdot \underline{a} = 0 \Rightarrow 0 \cdot \underline{a}' = 0], p_0, c]$		

[Prop 3.2l  $\xrightarrow{\text{proof}}$   $\lambda c. \lambda x. \mathcal{P}([S \vdash$

L <sub>?</sub> : Arbitrary $\gg$	$\underline{a}$	;
L <sub>?</sub> : Block $\gg$	Begin	Prop 3
$0 = 0; \text{Prop 3.2l}_2 \gg 0 \cdot \bar{x} = 0 \Rightarrow 0 \cdot \bar{x}' = 0;$		
L <sub>?</sub> : S9 @ $\bar{x} \triangleright 0 \cdot 0 = 0 \triangleright 0 \cdot \bar{x} = 0 \Rightarrow$		
	$0 \cdot \bar{x}' = 0 \gg$	$0 \cdot \bar{x} = 0$
L <sub>c</sub> : Block $\gg$	End	Ded $\triangleright$
L <sub>c</sub> $\gg 0 \cdot \underline{a} = 0], p_0, c]$		

[Prop 3.2m<sub>1</sub>  $\xrightarrow{\text{stmt}}$   $S \vdash \Pi a. \underline{a}' \cdot 0 = \underline{t} \cdot 0 + 0]$

[Prop 3.2m<sub>1</sub>  $\xrightarrow{\text{proof}}$   $\lambda c. \lambda x. \mathcal{P}([S \vdash$

L <sub>?</sub> : Arbitrary $\gg$	$\underline{a}$	S7; $\gg$
$\underline{a}' \cdot 0 = 0; \text{Prop 3.2f} \gg 0 = 0 + 0; S7 \gg 0 = \underline{a} \cdot 0; \text{Prop 3.2e} \gg 0 = \underline{a} \cdot 0 \Rightarrow 0 + 0 =$		
$\underline{a} \cdot 0 + 0; 0 = \underline{a} \cdot 0 \Rightarrow 0 + 0 = \underline{a} \cdot 0 + 0 \triangleright 0 = \underline{a} \cdot 0 \gg 0 + 0 = \underline{a} \cdot 0 + 0; \text{Prop 3.2c} \gg$		
$0 = 0 + 0 \Rightarrow 0 + 0 = \underline{a} \cdot 0 + 0 \Rightarrow 0 = \underline{a} \cdot 0 + 0; 0 = 0 + 0 \Rightarrow 0 + 0 = \underline{a} \cdot 0 + 0 \Rightarrow 0 =$		
$\underline{a} \cdot 0 + 0 \triangleright 0 = 0 + 0 \triangleright 0 + 0 = \underline{a} \cdot 0 + 0 \gg 0 = \underline{a} \cdot 0 + 0; \text{Prop 3.2c} \gg \underline{a}' \cdot 0 = 0 \Rightarrow 0 =$		
$\underline{a} \cdot 0 + 0 \Rightarrow \underline{a}' \cdot 0 = \underline{a} \cdot 0 + 0; \underline{a}' \cdot 0 = 0 \Rightarrow 0 = \underline{a} \cdot 0 + 0 \Rightarrow \underline{a}' \cdot 0 = \underline{a} \cdot 0 + 0 \triangleright \underline{a}' \cdot 0 =$		
$0 \triangleright \underline{a}' \cdot 0 = 0 \Rightarrow 0 = \underline{a} \cdot 0 + 0 \Rightarrow \underline{a}' \cdot 0 = \underline{a} \cdot 0 + 0 \gg \underline{a}' \cdot 0 = \underline{a} \cdot 0 + 0], p_0, c]$		

[Prop 3.2m<sub>2</sub>  $\xrightarrow{\text{stmt}}$   $S \vdash \Pi a, \underline{b}. \underline{a}' \cdot \underline{b} = \underline{a} \cdot \underline{b} + \underline{b} \Rightarrow \underline{a}' \cdot \underline{b}' = \underline{a} \cdot \underline{b}' + \underline{b}']$

[Prop 3.2m<sub>2</sub>  $\xrightarrow{\text{proof}}$   $\lambda c. \lambda x. \mathcal{P}([S \vdash$

L <sub>?</sub> : Arbitrary $\gg$	$\underline{a}, \underline{b}$	;
L <sub>?</sub> : Block $\gg$	Begin	;
L <sub>?</sub> : Arbitrary $\gg$	$\underline{a}, \underline{b}$	$\underline{a}'$ ;
$\underline{b} = \underline{a} \cdot \underline{b} + \underline{b} \vdash S8 \gg \underline{a}' \cdot \underline{b}' = \underline{a}' \cdot \underline{b}' + \underline{a}'; \text{Prop 3.2e} \gg \underline{a}' \cdot \underline{b} = \underline{a} \cdot \underline{b} + \underline{b} \Rightarrow$		
$\underline{a}' \cdot \underline{b} + \underline{a}' = \underline{a} \cdot \underline{b} + \underline{b} + \underline{a}'; \underline{a}' \cdot \underline{b} = \underline{a} \cdot \underline{b} + \underline{b} \Rightarrow \underline{a}' \cdot \underline{b} + \underline{a}' = \underline{a} \cdot \underline{b} + \underline{b} + \underline{a}' \triangleright \underline{a}' \cdot \underline{b} =$		
$\underline{a} \cdot \underline{b} + \underline{b} \gg \underline{a}' \cdot \underline{b} + \underline{a}' = \underline{a} \cdot \underline{b} + \underline{b} + \underline{a}'; S6 \gg \underline{b} + \underline{a}' = \underline{b} + \underline{a}'; \text{Prop 3.2g} \gg$		
$\underline{b}' + \underline{a} = \underline{b} + \underline{a}'; \text{Prop 3.2d} \triangleright \underline{b} + \underline{a}' = \underline{b} + \underline{a}' \triangleright \underline{b}' + \underline{a} = \underline{b} + \underline{a}' \gg \underline{b} + \underline{a}' =$		
$\underline{b}' + \underline{a}; \text{Prop 3.2h} \gg \underline{b}' + \underline{a} = \underline{a} + \underline{b}'; \text{Prop 3.2c} \triangleright \underline{b} + \underline{a}' = \underline{b}' + \underline{a} \triangleright \underline{b}' + \underline{a} =$		
$\underline{a} + \underline{b}' \gg \underline{b} + \underline{a}' = \underline{a} + \underline{b}'; \text{Prop 3.2i} \triangleright \underline{b} + \underline{a}' = \underline{a} + \underline{b}' \gg \underline{a} \cdot \underline{b} + \underline{b} + \underline{a}' =$		
$\underline{a} \cdot \underline{b} + \underline{a} + \underline{b}'; S8 \gg \underline{a} \cdot \underline{b}' = \underline{a} \cdot \underline{b} + \underline{a}; \text{Prop 3.2e} \triangleright \underline{a} \cdot \underline{b}' = \underline{a} \cdot \underline{b} + \underline{a} \gg \underline{a} \cdot \underline{b}' + \underline{b}' =$		
$\underline{a} \cdot \underline{b} + \underline{a} + \underline{b}'; \text{Prop 3.2d} \triangleright \underline{a} \cdot \underline{b}' + \underline{b}' = \underline{a} \cdot \underline{b} + \underline{a} + \underline{b}' \triangleright \underline{a} \cdot \underline{b} + \underline{b} + \underline{a}' =$		
$\underline{a} \cdot \underline{b} + \underline{a} + \underline{b}' \gg \underline{a} \cdot \underline{b} + \underline{b} + \underline{a}' = \underline{a} \cdot \underline{b}' + \underline{b}'; \text{Prop 3.2c} \triangleright \underline{a}' \cdot \underline{b} + \underline{a}' =$		

$$\underline{a} \cdot \underline{b} + \underline{b} + \underline{a}' \triangleright \underline{a} \cdot \underline{b} + \underline{b} + \underline{a}' = \underline{a} \cdot \underline{b}' + \underline{b}' \gg \underline{a}' \cdot \underline{b} + \underline{a}' = \underline{a} \cdot \underline{b}' + \underline{b}';$$

$$\begin{array}{l} L_? : \quad \text{Prop 3.2c} \triangleright \underline{a}' \cdot \underline{b}' = \underline{a}' \cdot \underline{b}' + \\ \quad \underline{a}' \triangleright \underline{a}' \cdot \underline{b} + \underline{a}' = \underline{a} \cdot \underline{b}' + \underline{b}' \gg \quad \underline{a}' \cdot \underline{b}' = \underline{a} \cdot \underline{b}' + \underline{b}' \quad ; \\ L_o : \quad \text{Block} \gg \quad \text{End} \quad \text{Ded} \triangleright \\ L_o \gg \underline{a}' \cdot \underline{b} = \underline{a} \cdot \underline{b} + \underline{b} \Rightarrow \underline{a}' \cdot \underline{b}' = \underline{a} \cdot \underline{b}' + \underline{b}', p_0, c) \end{array}$$

$$[\text{Prop 3.2m} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil S \vdash$$

$$\begin{array}{l} L_? : \quad \text{Arbitrary} \gg \quad \underline{a}, \underline{b} \quad ; \\ L_? : \quad \text{Block} \gg \quad \text{Begin} \quad \text{Prpp 3} \\ 0 = \bar{x} \cdot 0 + 0; \text{Prop 3.2m}_2 \gg \bar{x}' \cdot \bar{y} = \bar{x} \cdot \bar{y} + \bar{y} \Rightarrow \bar{x}' \cdot \bar{y}' = \bar{x} \cdot \bar{y}' + \bar{y}; \\ L_? : \quad S9 @ \bar{y} \triangleright \bar{x}' \cdot 0 = \bar{x} \cdot 0 + 0 \triangleright \bar{x}' \cdot \bar{y} = \\ \quad \bar{x} \cdot \bar{y} + \bar{y} \Rightarrow \bar{x}' \cdot \bar{y}' = \bar{x} \cdot \bar{y}' + \bar{y} \gg \quad \bar{x}' \cdot \bar{y} = \bar{x} \cdot \bar{y} + \bar{y} \quad ; \\ L_c : \quad \text{Block} \gg \quad \text{End} \quad \text{Ded} \triangleright \\ L_c \gg \underline{a}' \cdot \underline{b} = \underline{a} \cdot \underline{b} + \underline{b}], p_0, c) \end{array}$$

$$[\text{Prop 3.2n}_1 \xrightarrow{\text{stmt}} S \vdash \Pi a. \underline{a} \cdot 0 = 0 \cdot \underline{a}]$$

$$[\text{Prop 3.2n}_1 \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil S \vdash$$

$$\begin{array}{l} L_? : \quad \text{Arbitrary} \gg \quad \underline{a} \quad S7; \gg \\ \underline{a} \cdot 0 = 0; \text{Prop 3.2l} \gg 0 \cdot \underline{a} = 0; \text{Prop 3.2c} \triangleright \underline{a} \cdot 0 = 0 \triangleright 0 \cdot \underline{a} = 0 \gg \underline{a} \cdot 0 = 0 \cdot \underline{a}], p_0, c) \end{array}$$

$$[\text{Prop 3.2n}_2 \xrightarrow{\text{stmt}} S \vdash \Pi a, b. \underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a} \Rightarrow \underline{a} \cdot \underline{b}' = \underline{b}' \cdot \underline{a}]$$

$$[\text{Prop 3.2n}_2 \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil S \vdash$$

$$\begin{array}{l} L_? : \quad \text{Arbitrary} \gg \quad \underline{a}, \underline{b} \quad ; \\ L_? : \quad \text{Block} \gg \quad \text{Begin} \\ L_? : \quad \text{Arbitrary} \gg \quad \underline{a}, \underline{b} \quad \underline{a} ; \\ \underline{b} = \underline{b} \cdot \underline{a} \vdash S8 \gg \underline{a} \cdot \underline{b}' = \underline{a} \cdot \underline{b} + \underline{a}; \text{Prop 3.2e} \triangleright \underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a} \gg \underline{a} \cdot \underline{b} + \underline{a} = \\ \underline{b} \cdot \underline{a} + \underline{a}; \text{Prop 3.2b} \triangleright \text{Prop 3.2m} \gg \underline{b} \cdot \underline{a} + \underline{a} = \underline{b}' \cdot \underline{a}; \text{Prop 3.2c} \triangleright L_e \triangleright \underline{b} \cdot \underline{a} + \underline{a} = \\ \underline{b}' \cdot \underline{a} \gg \underline{a} \cdot \underline{b} + \underline{a} = \underline{b}' \cdot \underline{a}; \\ L_? : \quad \text{Prop 3.2c} \triangleright \underline{a} \cdot \underline{b}' = \\ \quad \underline{a} \cdot \underline{b} + \underline{a} \triangleright \underline{a} \cdot \underline{b} + \underline{a} = \underline{b}' \cdot \underline{a} \gg \quad \underline{a} \cdot \underline{b}' = \underline{b}' \cdot \underline{a} \quad ; \\ L_f : \quad \text{Block} \gg \quad \text{End} \quad \text{Ded} \triangleright \\ L_f \gg \underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a} \Rightarrow \underline{a} \cdot \underline{b}' = \underline{b}' \cdot \underline{a}], p_0, c) \end{array}$$

$$[\text{Prop 3.2n} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil S \vdash$$

$$\begin{array}{l} L_? : \quad \text{Arbitrary} \gg \quad \underline{a}, \underline{b} \quad ; \\ L_? : \quad \text{Block} \gg \quad \text{Begin} \quad \text{Prpp 3} \\ 0 = 0 \cdot \bar{x}; \text{Prop 3.2n}_2 \gg \bar{x} \cdot \bar{y} = \bar{y} \cdot \bar{x} \Rightarrow \bar{x} \cdot \bar{y}' = \bar{y}' \cdot \bar{x}; \\ L_? : \quad S9 @ \bar{y} \triangleright \bar{x} \cdot 0 = 0 \cdot \bar{x} \triangleright \bar{x} \cdot \bar{y} = \\ \quad \bar{y} \cdot \bar{x} \Rightarrow \bar{x} \cdot \bar{y}' = \bar{y}' \cdot \bar{x} \gg \quad \bar{x} \cdot \bar{y} = \bar{y} \cdot \bar{x} \quad ; \\ L_c : \quad \text{Block} \gg \quad \text{End} \quad \text{Ded} \triangleright \\ L_c \gg \underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}], p_0, c) \end{array}$$

$$[\text{Prop 3.2o} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil S \vdash$$

$$\begin{array}{l} L_? : \quad \text{Arbitrary} \gg \quad \underline{a}, \underline{b}, \underline{c} \quad ; \\ L_? : \quad \text{Block} \gg \quad \text{Begin} \quad ; \end{array}$$

$L_7$ : Arbitrary  $\gg$   $\underline{a}, \underline{b}, \underline{c}$   $\underline{a} \neq$   
 $\underline{b} \vdash \text{Prop 3.2k} \gg \underline{a} = \underline{b} \Rightarrow \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c}; \underline{a} = \underline{b} \Rightarrow \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c} \triangleright \underline{a} = \underline{b} \gg \underline{a} \cdot \underline{c} =$   
 $\underline{b} \cdot \underline{c}; \text{Prop 3.2n} \gg \underline{a} \cdot \underline{c} = \underline{c} \cdot \underline{a}; \text{Prop 3.2n} \gg \underline{b} \cdot \underline{c} = \underline{c} \cdot \underline{b}; \text{Prop 3.2c} \gg \underline{a} \cdot \underline{c} =$   
 $\underline{b} \cdot \underline{c} \Rightarrow \underline{b} \cdot \underline{c} = \underline{c} \cdot \underline{b} \Rightarrow \underline{a} \cdot \underline{c} = \underline{c} \cdot \underline{b}; \underline{a} \cdot \underline{c} = \underline{b} \cdot \underline{c} \Rightarrow \underline{b} \cdot \underline{c} = \underline{c} \cdot \underline{b} \Rightarrow \underline{a} \cdot \underline{c} = \underline{c} \cdot \underline{b} \triangleright \underline{a} \cdot \underline{c} =$   
 $\underline{b} \cdot \underline{c} \triangleright \underline{b} \cdot \underline{c} = \underline{c} \cdot \underline{b} \gg \underline{a} \cdot \underline{c} = \underline{c} \cdot \underline{b}; S1 \gg \underline{a} \cdot \underline{c} = \underline{c} \cdot \underline{b} \Rightarrow \underline{a} \cdot \underline{c} = \underline{c} \cdot \underline{a} \Rightarrow \underline{c} \cdot \underline{a} = \underline{c} \cdot \underline{b};$   
 $L_7$ :  $\underline{a} \cdot \underline{c} = \underline{c} \cdot \underline{b} \Rightarrow \underline{a} \cdot \underline{c} = \underline{c} \cdot \underline{a} \Rightarrow$   
 $\underline{c} \cdot \underline{a} = \underline{c} \cdot \underline{b} \triangleright \underline{a} \cdot \underline{c} =$   
 $\underline{c} \cdot \underline{b} \triangleright \underline{a} \cdot \underline{c} = \underline{c} \cdot \underline{a} \gg$   $\underline{c} \cdot \underline{a} = \underline{c} \cdot \underline{b}$  ;  
 $L_i$ : Block  $\gg$  End  $\text{Ded} \triangleright$   
 $L_i \gg \underline{a} = \underline{b} \Rightarrow \underline{c} \cdot \underline{a} = \underline{c} \cdot \underline{b}], (p_0, c)]$

[Prop 3.4a  $\xrightarrow{\text{stmt}}$   $S \vdash \Pi \underline{a}, \underline{b}, \underline{c}: \underline{a} \cdot \underline{b} + \underline{c} = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c}$ ]

[Prop 3.4b  $\xrightarrow{\text{stmt}}$   $S \vdash \Pi \underline{a}, \underline{b}, \underline{c}: \underline{b} + \underline{c} \cdot \underline{a} = \underline{b} \cdot \underline{a} + \underline{c} \cdot \underline{b}$ ]

[Prop 3.4c  $\xrightarrow{\text{stmt}}$   $S \vdash \Pi \underline{a}, \underline{b}, \underline{c}: \underline{a} \cdot \underline{b} \cdot \underline{c} = \underline{a} \cdot \underline{b} \cdot \underline{c}$ ]

[Prop 3.4d  $\xrightarrow{\text{stmt}}$   $S \vdash \Pi \underline{a}, \underline{b}, \underline{c}: \underline{a} + \underline{c} = \underline{b} + \underline{c} \Rightarrow \underline{a} = \underline{b}$ ]

## Litteratur

[Men97] E. Mendelson. *Introduction to Mathematical Logic*. Chapman & Hall, 4. edition, 1997.