¡¡¡¡¡¡¡ opgave.pyk ======= ¿¿¿¿¿¿¿ 1.6

projekt i logik

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Indhold

Resumé

1 Introduction

In this paper we will prove the propositions regarding S starting with proposition Prop 3.2i. continuing to 3.4, bevise 3.5, aksiomer, der definerer x—y, bevise 3.10, og bevise 3.11.

We chose to prove chapter 3 from mendelson, starting with Lemma 3.2, since the first ten wfs' are proved in Mendelson, we will start with Lemma 3.2.i.

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A modified version of Mendelsons system S (Peano arithmetic) [Men97] may be formulated thus:

[**Theory** S]

[S **rule** MP: $\Pi \mathcal{A}, \mathcal{B} \colon \mathcal{A} \Rightarrow \mathcal{B} \vdash \mathcal{A} \vdash \mathcal{B}$]

[S **rule** Gen: $\Pi \mathcal{X}, \mathcal{A} \colon \mathcal{A} \vdash \forall \mathcal{X} \colon \mathcal{A}$]

[S **rule** Ded: $\Pi \mathcal{A}, \mathcal{B} \colon \mathrm{Ded}(\mathcal{A}, \mathcal{B}) \Vdash \mathcal{A} \vdash \mathcal{B}$]

[S **rule** S2: $\Pi \mathcal{A}, \mathcal{B} \colon \mathcal{A} = \mathcal{B} \vdash \mathcal{A}' = \mathcal{B}'$]

[S **rule** S3: $\Pi \mathcal{A} \colon \neg 0 = \mathcal{A}'$]

[S **rule** S4: $\Pi \mathcal{A}, \mathcal{B} \colon \mathcal{A}' = \mathcal{B}' \vdash \mathcal{A} = \mathcal{B}$]

[S **rule** S5: $\Pi \mathcal{A} \colon \mathcal{A} + 0 = \mathcal{A}$]

[S **rule** S6: $\Pi \mathcal{A}, \mathcal{B} \colon \mathcal{A} + \mathcal{B}' = (\mathcal{A} + \mathcal{B})'$]

[S **rule** S7: $\Pi \mathcal{A} \colon \mathcal{A} \cdot 0 = 0$]

[S **rule** S8: $\Pi \mathcal{A}, \mathcal{B} \colon \mathcal{A} \cdot (\mathcal{B}') = (\mathcal{A} \cdot \mathcal{B}) + \mathcal{A}$]

[S **rule** Neg: $\Pi \mathcal{A} \colon \Pi \mathcal{B} \colon \neg \mathcal{B} \Rightarrow \neg \mathcal{A} \vdash \neg \mathcal{B} \Rightarrow \mathcal{A} \vdash \mathcal{B}$]

[S **rule** S1: $\Pi \mathcal{A}, \mathcal{B}, \mathcal{C}$: $\mathcal{A} = \mathcal{B} \vdash \mathcal{A} = \mathcal{C} \vdash \mathcal{B} = \mathcal{C}$]

[S **rule** S9: $\Pi \mathcal{X}, \mathcal{A}, \mathcal{B}, \mathcal{C}$: $\langle \mathcal{B} \equiv \mathcal{A} | \mathcal{X} := 0 \rangle \Vdash \langle \mathcal{C} \equiv \mathcal{A} | \mathcal{X} := \mathcal{X}' \rangle \Vdash \mathcal{B} \vdash \mathcal{A} \Rightarrow \mathcal{C} \vdash \mathcal{A}$]

[S **lemma** Prop 3.2a: $\Pi \mathcal{A}$: $\mathcal{A} = \mathcal{A}$]

[S **lemma** Prop 3.2b: $\Pi \mathcal{A}, \mathcal{B}$: $\mathcal{A} = \mathcal{B} \vdash \mathcal{B} = \mathcal{A}$]

[S **lemma** Prop 3.2c: $\Pi \mathcal{A}, \mathcal{B}, \mathcal{C}$: $\mathcal{A} = \mathcal{B} \vdash \mathcal{B} = \mathcal{C} \vdash \mathcal{A} = \mathcal{C}$]

[S **lemma** Prop 3.2d: $\Pi \mathcal{A}, \mathcal{B}, \mathcal{C}$: $\mathcal{A} = \mathcal{C} \vdash \mathcal{B} = \mathcal{C} \vdash \mathcal{A} = \mathcal{B}$]

[S **lemma** Prop 3.2e: $\Pi \mathcal{A}, \mathcal{B}, \mathcal{C}$: $\mathcal{A} = \mathcal{B} \vdash \mathcal{A} + \mathcal{C} = \mathcal{B} + \mathcal{C}$]

[S **lemma** Prop 3.2f: $\Pi \mathcal{A}$: $\mathcal{A} = 0 + \mathcal{A}$]

[S **lemma** Prop 3.2g: $\Pi \mathcal{A}, \mathcal{B}$: $\mathcal{A}' + \mathcal{B} = (\mathcal{A} + \mathcal{B})'$]

[S **lemma** Prop 3.2h: $\Pi \mathcal{A}, \mathcal{B}$: $\mathcal{A} + \mathcal{B} = \mathcal{B} + \mathcal{A}$]

[S **lemma** Prop 3.2i: $\Pi \mathcal{A}, \mathcal{B}, \mathcal{C}$: $\mathcal{A} = \mathcal{B} \vdash \mathcal{C} + \mathcal{A} = \mathcal{C} + \mathcal{B}$]

[S **lemma** Prop 3.2j: $\Pi \mathcal{A}, \mathcal{B}, \mathcal{C}$: $(\mathcal{A} + \mathcal{B}) + \mathcal{C} = \mathcal{A} + (\mathcal{B} + \mathcal{C})$]

[S **lemma** Prop 3.2k: $\Pi \mathcal{A}, \mathcal{B}, \mathcal{C}$: $\mathcal{A} = \mathcal{B} \vdash \mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{C}$]

[S **lemma** Prop 3.2l: $\Pi \mathcal{A}$: $0 \cdot \mathcal{A} = 0$]

[S **lemma** Prop 3.2m: $\Pi \mathcal{A}, \mathcal{B}$: $\mathcal{A}' \cdot \mathcal{B} = \mathcal{A} \cdot \mathcal{B} + \mathcal{B}$]

[S **lemma** Prop 3.2n: $\Pi \mathcal{A}, \mathcal{B}$: $\mathcal{A} \cdot \mathcal{B} = \mathcal{B} \cdot \mathcal{A}$]

[S **lemma** Prop 3.2o: $\Pi \mathcal{A}, \mathcal{B}, \mathcal{C}$: $\mathcal{A} = \mathcal{B} \vdash \mathcal{C} \cdot \mathcal{A} = \mathcal{C} \cdot \mathcal{B}$]

we will start by doing Prop 3.2i

[S **lemma** Prop 3.2i: $\Pi \mathcal{A}, \mathcal{B}, \mathcal{C}$: $\mathcal{A} = \mathcal{B} \vdash \mathcal{C} + \mathcal{A} = \mathcal{C} + \mathcal{B}$]

S **proof of** Prop 3.2i:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}, \mathcal{C}$	;
L02:	Block $\gg$	Begin	;
L03:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}, \mathcal{C}$	;
L04:	Prop 3.2e $\gg$	$\mathcal{A} = \mathcal{B} \Rightarrow \mathcal{A} + \mathcal{C} = \mathcal{B} + \mathcal{C}$	;
L05:	Prop 3.2h $\gg$	$\mathcal{A} + \mathcal{C} = \mathcal{C} + \mathcal{A}$	;
L06:	Prop 3.2h $\gg$	$\mathcal{B} + \mathcal{C} = \mathcal{C} + \mathcal{B}$	;
L07:	Premise $\gg$	$\mathcal{A} = \mathcal{B}$	;
L08:	MP $\rhd$ L04 $\rhd$ L07 $\gg$	$\mathcal{A} + \mathcal{C} = \mathcal{B} + \mathcal{C}$	;
L09:	S1 $\gg$	$\mathcal{A} + \mathcal{C} = \mathcal{B} + \mathcal{C} \Rightarrow (\mathcal{A} + \mathcal{C} = \mathcal{C} + \mathcal{A} \Rightarrow \mathcal{B} + \mathcal{C} = \mathcal{C} + \mathcal{A})$	;
L10:	MP $\rhd$ L09 $\rhd$ L08 $\gg$	$\mathcal{A}+\mathcal{C} = \mathcal{C}+\mathcal{A} \Rightarrow \mathcal{B}+\mathcal{C} = \mathcal{C}+\mathcal{A}$	;
L08:	MP $\rhd$ L10 $\rhd$ L05 $\gg$	$\mathcal{B} + \mathcal{C} = \mathcal{C} + \mathcal{A}$	;
L11:	Prop 3.2b $\rhd$ L08 $\gg$	$\mathcal{C} + \mathcal{A} = \mathcal{B} + \mathcal{C}$	;

L12: Prop 3.2e $\gg$ $\mathcal{C}+\mathcal{A}=\mathcal{B}+\mathcal{C} \Rightarrow (\mathcal{B}+\mathcal{C}=\mathcal{C}+\mathcal{B} \Rightarrow \mathcal{C}+\mathcal{A}=\mathcal{C}+\mathcal{B})$;
L13: MP $\triangleright$ L12 $\triangleright$ L11 $\gg$ $\mathcal{B}+\mathcal{C}=\mathcal{C}+\mathcal{B} \Rightarrow \mathcal{C}+\mathcal{A}=\mathcal{C}+\mathcal{B}$;
L14: MP $\triangleright$ L13 $\triangleright$ L06 $\gg$ $\mathcal{C}+\mathcal{A}=\mathcal{C}+\mathcal{B}$;
L15: Block $\gg$ End ;
L16: Ded $\triangleright$ L15 $\gg$ $\mathcal{A}=\mathcal{B} \Rightarrow \mathcal{C}+\mathcal{A}=\mathcal{C}+\mathcal{B}$ □

[S **lemma** Prop 3.2j$_1$: $\Pi\mathcal{A},\mathcal{B}\colon (\mathcal{A}+\mathcal{B})+0=\mathcal{A}+(\mathcal{B}+0)$]

S **proof of** Prop 3.2j$_1$:
L01: Arbitrary $\gg$ $\mathcal{A},\mathcal{B}$;
L02: S5 $\gg$ $(\mathcal{A}+\mathcal{B})+0=\mathcal{A}+\mathcal{B}$;
L03: S5 $\gg$ $\mathcal{B}+0=\mathcal{B}$;
L04: Prop 3.2i $\triangleright$ L03 $\gg$ $\mathcal{A}+(\mathcal{B}+0)=\mathcal{A}+\mathcal{B}$;
L05: Prop 3.2d $\triangleright$ L02 $\triangleright$ L04 $\gg$ $(\mathcal{A}+\mathcal{B})+0=\mathcal{A}+(\mathcal{B}+0)$ □

[S **lemma** Prop 3.2j$_2$: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon (\mathcal{A}+\mathcal{B})+\mathcal{C}=\mathcal{A}+(\mathcal{B}+\mathcal{C}) \Rightarrow (\mathcal{A}+\mathcal{B})+\mathcal{C}'=\mathcal{A}+(\mathcal{B}+\mathcal{C}')$]

S **proof of** Prop 3.2j$_2$:
L01: Arbitrary $\gg$ $\mathcal{A},\mathcal{B},\mathcal{C}$;
L02: Block $\gg$ Begin ;
L03: Arbitrary $\gg$ $\mathcal{A},\mathcal{B},\mathcal{C}$;
L04: Premise $\gg$ $(\mathcal{A}+\mathcal{B})+\mathcal{C}=\mathcal{A}+(\mathcal{B}+\mathcal{C})$;
L05: S6 $\gg$ $(\mathcal{A}+\mathcal{B})+\mathcal{C}'=\mathcal{A}+(\mathcal{B}+\mathcal{C})'$;
L06: S2 $\triangleright$ L04 $\gg$ $((\mathcal{A}+\mathcal{B})+\mathcal{C})'=(\mathcal{A}+(\mathcal{B}+\mathcal{C}))'$;
L07: Prop 3.2c $\triangleright$ L05 $\triangleright$ L06 $\gg$ $(\mathcal{A}+\mathcal{B})+\mathcal{C}'=(\mathcal{A}+(\mathcal{B}+\mathcal{C}))'$;
L08: S6 $\gg$ $\mathcal{B}+\mathcal{C}'=(\mathcal{B}+\mathcal{C})'$;
L09: Prop 3.2i $\triangleright$ L08 $\gg$ $\mathcal{A}+(\mathcal{B}+\mathcal{C}')=\mathcal{A}+(\mathcal{B}+\mathcal{C})'$;
L10: S6 $\gg$ $\mathcal{A}+(\mathcal{B}+\mathcal{C})'=(\mathcal{A}+(\mathcal{B}+\mathcal{C}))'$;
L11: Prop 3.2c $\triangleright$ L09 $\triangleright$ L10 $\gg$ $\mathcal{A}+(\mathcal{B}+\mathcal{C}')=(\mathcal{A}+(\mathcal{B}+\mathcal{C}))'$;
L12: Prop 3.2d $\triangleright$ L07 $\triangleright$ L11 $\gg$ $(\mathcal{A}+\mathcal{B})+\mathcal{C}'=\mathcal{A}+(\mathcal{B}+\mathcal{C}')$;
L13: Block $\gg$ End ;
L14: Ded $\triangleright$ L13 $\gg$ $(\mathcal{A}+\mathcal{B})+\mathcal{C}=\mathcal{A}+(\mathcal{B}+\mathcal{C}) \Rightarrow (\mathcal{A}+\mathcal{B})+\mathcal{C}'=\mathcal{A}+(\mathcal{B}+\mathcal{C}')$ □

S **proof of** Prop 3.2j:
L01: Arbitrary $\gg$ $\mathcal{A},\mathcal{B},\mathcal{C}$;
L02: Block $\gg$ Begin ;
L03: Prop 3.2j$_1$ $\gg$ $(x+y)+0=x+(y+0)$;
L04: Prop 3.2j$_2$ $\gg$ $(x+y)+z=x+(y+z) \Rightarrow (x+y)+z'=x+(y+z')$;
L05: S9 @ z $\triangleright$ L03 $\triangleright$ L04 $\gg$ $(x+y)+z=x+(y+z)$;
L06: Block $\gg$ End ;
L07: Ded $\triangleright$ L06 $\gg$ $(\mathcal{A}+\mathcal{B})+\mathcal{C}=\mathcal{A}+(\mathcal{B}+\mathcal{C})$ □

[S **lemma** Prop 3.2k$_1$: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A}=\mathcal{B} \vdash \mathcal{A}\cdot 0=\mathcal{B}\cdot 0$]

S **proof of** Prop 3.2k$_1$:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}$	;
L02:	S7 $\gg$	$\mathcal{A} \cdot 0 = 0$	;
L03:	S7 $\gg$	$\mathcal{B} \cdot 0 = 0$	;
L04:	Prop 3.2a $\gg$	$0 = 0$	;
L05:	Prop 3.2b $\triangleright$ L03 $\gg$	$0 = \mathcal{B} \cdot 0$	;
L06:	Prop 3.2c $\gg$	$\mathcal{A} \cdot 0 = 0 \Rightarrow 0 = \mathcal{B} \cdot 0 \Rightarrow \mathcal{A} \cdot 0 = \mathcal{B} \cdot 0$	;
L07:	L06 $\triangleright$ L02 $\triangleright$ L05 $\gg$	$\mathcal{A} \cdot 0 = \mathcal{B} \cdot 0$	$\square$

[S **lemma** Prop 3.2k$_2$: $\Pi \mathcal{A}, \mathcal{B}, \mathcal{C}$: $\mathcal{A} = \mathcal{B} \Rightarrow \mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{C} \vdash \mathcal{A} = \mathcal{B} \Rightarrow \mathcal{A} \cdot \mathcal{C}' = \mathcal{B} \cdot \mathcal{C}'$]

S **proof of** Prop 3.2k$_2$:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}, \mathcal{C}$	;
L02:	Block $\gg$	Begin	;
L03:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}, \mathcal{C}$	;
L04:	Premise $\gg$	$\mathcal{A} = \mathcal{B} \Rightarrow \mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{C}$	;
L05:	Premise $\gg$	$\mathcal{A} = \mathcal{B}$	;
L06:	L04 $\triangleright$ L05 $\gg$	$\mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{C}$	;
L07:	S8 $\gg$	$\mathcal{A} \cdot \mathcal{C}' = \mathcal{A} \cdot \mathcal{C} + \mathcal{A}$	;
L08:	S8 $\gg$	$\mathcal{B} \cdot \mathcal{C}' = \mathcal{B} \cdot \mathcal{C} + \mathcal{B}$	;
L09:	Prop 3.2e $\gg$	$\mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{C} \Rightarrow \mathcal{A} \cdot \mathcal{C} + \mathcal{A} = \mathcal{B} \cdot \mathcal{C} + \mathcal{A}$	;
L10:	L09 $\triangleright$ L06 $\gg$	$\mathcal{A} \cdot \mathcal{C} + \mathcal{A} = \mathcal{B} \cdot \mathcal{C} + \mathcal{A}$	;
L11:	Prop 3.2i $\gg$	$\mathcal{A} = \mathcal{B} \Rightarrow \mathcal{B} \cdot \mathcal{C} + \mathcal{A} = \mathcal{B} \cdot \mathcal{C} + \mathcal{B}$	;
L12:	L11 $\triangleright$ L06 $\gg$	$\mathcal{B} \cdot \mathcal{C} + \mathcal{A} = \mathcal{B} \cdot \mathcal{C} + \mathcal{B}$	;
L13:	Prop 3.2c $\gg$	$\mathcal{A} \cdot \mathcal{C} + \mathcal{A} = \mathcal{B} \cdot \mathcal{C} + \mathcal{A} \Rightarrow \mathcal{B} \cdot \mathcal{C} + \mathcal{A} = \mathcal{B} \cdot \mathcal{C} + \mathcal{B} \Rightarrow \mathcal{A} \cdot \mathcal{C} + \mathcal{A} = \mathcal{B} \cdot \mathcal{C} + \mathcal{B}$	;
L14:	L13 $\triangleright$ L10 $\triangleright$ L12 $\gg$	$\mathcal{A} \cdot \mathcal{C} + \mathcal{A} = \mathcal{B} \cdot \mathcal{C} + \mathcal{B}$	;
L15:	L14 $\triangleright$ L07 $\triangleright$ L08 $\gg$	$\mathcal{A} \cdot \mathcal{C}' = \mathcal{B} \cdot \mathcal{C}'$	;
L12:	Block $\gg$	End	;
L16:	Ded $\triangleright$ L12 $\gg$	$\mathcal{A} = \mathcal{B} \Rightarrow \mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{C} \vdash \mathcal{A} = \mathcal{B} \Rightarrow \mathcal{A} \cdot \mathcal{C}' = \mathcal{B} \cdot \mathcal{C}'$	$\square$

S **proof of** Prop 3.2k:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}, \mathcal{C}$	;
L02:	Block $\gg$	Begin	;
L03:	Prop 3.2k$_1$ $\gg$	$x = y \Rightarrow x \cdot 0 = y \cdot 0$	;
L04:	Prop 3.2k$_2$ $\gg$	$x = y \Rightarrow x \cdot z = y \cdot z \Rightarrow x = y \Rightarrow x \cdot z' = y \cdot z'$	;
L05:	S9 @ z $\triangleright$ L03 $\triangleright$ L04 $\gg$	$x = y \Rightarrow x \cdot z = y \cdot z$	;
L06:	Block $\gg$	End	;
L07:	Ded $\triangleright$ L06 $\gg$	$\mathcal{A} = \mathcal{B} \vdash \mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{C}$	$\square$

[S **lemma** Prop 3.2l$_1$: $\Pi \mathcal{A}$: $0 \cdot 0 = 0$]

S **proof of** Prop 3.2l$_1$:

L01:	S7 $\gg$	$0 \cdot 0 = 0$	$\square$

[S **lemma** Prop 3.2l$_2$: $\Pi\mathcal{A}\colon 0\cdot\mathcal{A}=0 \Rightarrow 0\cdot\mathcal{A}'=0$]

S **proof of** Prop 3.2l$_2$:

L01:	Arbitrary $\gg$	$\mathcal{A}$	;
L02:	Block $\gg$	Begin	;
L03:	Arbitrary $\gg$	$\mathcal{A}$	;
L04:	Premise $\gg$	$0\cdot\mathcal{A}=0$	;
L05:	S8 $\gg$	$0\cdot\mathcal{A}'=0\cdot\mathcal{A}+0$	;
L06:	S5 $\gg$	$0\cdot\mathcal{A}+0=0\cdot\mathcal{A}$	;
L07:	Prop 3.2c $\rhd$ L06 $\rhd$ L05 $\gg$	$0\cdot\mathcal{A}'=0\cdot\mathcal{A}$	;
L08:	Prop 3.2c $\rhd$ L07 $\rhd$ L04 $\gg$	$0\cdot\mathcal{A}'=0$	;
L09:	Block $\gg$	End	;
L10:	Ded $\rhd$ L09 $\gg$	$0\cdot\mathcal{A}=0 \Rightarrow 0\cdot\mathcal{A}'=0$	$\square$

S **proof of** Prop 3.2l:

L01:	Arbitrary $\gg$	$\mathcal{A}$	;
L02:	Block $\gg$	Begin	;
L03:	Prop 3.2l$_1$ $\gg$	$0\cdot 0=0$	;
L04:	Prop 3.2l$_2$ $\gg$	$0\cdot x=0 \Rightarrow 0\cdot x'=0$	;
L05:	S9 @ x $\rhd$ L03 $\rhd$ L04 $\gg$	$0\cdot x=0$	;
L06:	Block $\gg$	End	;
L07:	Ded $\rhd$ L06 $\gg$	$0\cdot\mathcal{A}=0$	$\square$

[S **lemma** Prop 3.2m$_1$: $\Pi\mathcal{A}\colon \mathcal{A}'\cdot 0=\mathcal{T}\cdot 0+0$]

S **proof of** Prop 3.2m$_1$:

L01:	Arbitrary $\gg$	$\mathcal{A}$	;
L02:	S7 $\gg$	$\mathcal{A}'\cdot 0=0$	;
L03:	Prop 3.2f $\gg$	$0=0+0$	;
L04:	S7 $\gg$	$0=\mathcal{A}\cdot 0$	;
L05:	Prop 3.2e $\gg$	$0=\mathcal{A}\cdot 0 \Rightarrow 0+0=\mathcal{A}\cdot 0+0$	;
L06:	L05 $\rhd$ L04 $\gg$	$0+0=\mathcal{A}\cdot 0+0$	;
L07:	Prop 3.2c $\gg$	$0=0+0 \Rightarrow 0+0=\mathcal{A}\cdot 0+0 \Rightarrow 0=\mathcal{A}\cdot 0+0$	;
L08:	L07 $\rhd$ L03 $\rhd$ L06 $\gg$	$0=\mathcal{A}\cdot 0+0$	;
L09:	Prop 3.2c $\gg$	$\mathcal{A}'\cdot 0=0 \Rightarrow 0=\mathcal{A}\cdot 0+0 \Rightarrow \mathcal{A}'\cdot 0=\mathcal{A}\cdot 0+0$	;
L10:	L09 $\rhd$ L02 $\rhd$ L09 $\gg$	$\mathcal{A}'\cdot 0=\mathcal{A}\cdot 0+0$	$\square$

[S **lemma** Prop 3.2m$_2$: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A}'\cdot\mathcal{B}=\mathcal{A}\cdot\mathcal{B}+\mathcal{B} \Rightarrow \mathcal{A}'\cdot\mathcal{B}'=\mathcal{A}\cdot\mathcal{B}'+\mathcal{B}'$]

S **proof of** Prop 3.2m$_2$:

L01:	Arbitrary $\gg$	$\mathcal{A},\mathcal{B}$	;
L02:	Block $\gg$	Begin	;
L03:	Arbitrary $\gg$	$\mathcal{A},\mathcal{B}$	;
L04:	Premise $\gg$	$\mathcal{A}'\cdot\mathcal{B}=\mathcal{A}\cdot\mathcal{B}+\mathcal{B}$	;
L05:	S8 $\gg$	$\mathcal{A}'\cdot\mathcal{B}'=\mathcal{A}'\cdot\mathcal{B}'+\mathcal{A}'$	;
L06:	Prop 3.2e $\gg$	$\mathcal{A}'\cdot\mathcal{B}=\mathcal{A}\cdot\mathcal{B}+\mathcal{B} \Rightarrow \mathcal{A}'\cdot\mathcal{B}+\mathcal{A}'=\mathcal{A}\cdot\mathcal{B}+\mathcal{B}+\mathcal{A}'$	;

L07:	L06 $\rhd$ L04 $\gg$	$\quad\mathcal{A}' \cdot \mathcal{B} + \mathcal{A}' = \mathcal{A} \cdot \mathcal{B} + \mathcal{B} + \mathcal{A}'$	;
L08:	S6 $\gg$	$\quad\mathcal{B} + \mathcal{A}' = (\mathcal{B} + \mathcal{A})'$	;
L09:	Prop 3.2g $\gg$	$\mathcal{B}' + \mathcal{A} = (\mathcal{B} + \mathcal{A})'$	;
L10:	Prop 3.2d $\rhd$ L08 $\rhd$ L09 $\gg$	$\quad\mathcal{B} + \mathcal{A}' = \mathcal{B}' + \mathcal{A}$	;
L11:	Prop 3.2h $\gg$	$\quad\mathcal{B}' + \mathcal{A} = \mathcal{A} + \mathcal{B}'$	;
L12:	Prop 3.2c $\rhd$ L10 $\rhd$ L11 $\gg$	$\quad\mathcal{B} + \mathcal{A}' = \mathcal{A} + \mathcal{B}'$	;
L13:	Prop 3.2i $\rhd$ L12 $\gg$	$\quad\mathcal{A} \cdot \mathcal{B} + \mathcal{B} + \mathcal{A}' = \mathcal{A} \cdot \mathcal{B} + \mathcal{A} + \mathcal{B}'$	;
L14:	S8 $\gg$	$\quad\mathcal{A} \cdot \mathcal{B}' = \mathcal{A} \cdot \mathcal{B} + \mathcal{A}$	;
L15:	Prop 3.2e $\rhd$ L14 $\gg$	$\quad\mathcal{A} \cdot \mathcal{B}' + \mathcal{B}' = \mathcal{A} \cdot \mathcal{B} + \mathcal{A} + \mathcal{B}'$	;
L16:	Prop 3.2d $\rhd$ L15 $\rhd$ L13 $\gg$	$\quad\mathcal{A} \cdot \mathcal{B} + \mathcal{B} + \mathcal{A}' = \mathcal{A} \cdot \mathcal{B}' + \mathcal{B}'$	;
L17:	Prop 3.2c $\rhd$ L07 $\rhd$ L16 $\gg$	$\quad\mathcal{A}' \cdot \mathcal{B} + \mathcal{A}' = \mathcal{A} \cdot \mathcal{B}' + \mathcal{B}'$	;
L18:	Prop 3.2c $\rhd$ L05 $\rhd$ L17 $\gg$	$\quad\mathcal{A}' \cdot \mathcal{B}' = \mathcal{A} \cdot \mathcal{B}' + \mathcal{B}'$	;
L19:	Block $\gg$	End	;
L20:	Ded $\rhd$ L19 $\gg$	$\mathcal{A}' \cdot \mathcal{B} = \mathcal{A} \cdot \mathcal{B} + \mathcal{B} \Rightarrow \mathcal{A}' \cdot \mathcal{B}' = \mathcal{A} \cdot \mathcal{B}' + \mathcal{B}'$	□

S **proof of** Prop 3.2m:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}$	;
L02:	Block $\gg$	Begin	;
L03:	Prop 3.2m$_1$ $\gg$	$\quad x' \cdot 0 = x \cdot 0 + 0$	;
L04:	Prop 3.2m$_2$ $\gg$	$\quad x' \cdot y = x \cdot y + y \Rightarrow x' \cdot y' = x \cdot y' + y$	;
L05:	S9 @ y $\rhd$ L03 $\rhd$ L04 $\gg$	$\quad x' \cdot y = x \cdot y + y$	;
L06:	Block $\gg$	End	;
L07:	Ded $\rhd$ L06 $\gg$	$\mathcal{A}' \cdot \mathcal{B} = \mathcal{A} \cdot \mathcal{B} + \mathcal{B}$	□

[S **lemma** Prop 3.2n$_1$: $\Pi\mathcal{A}\colon \mathcal{A} \cdot 0 = 0 \cdot \mathcal{A}$]

S **proof of** Prop 3.2n$_1$:

L01:	Arbitrary $\gg$	$\mathcal{A}$	;
L02:	S7 $\gg$	$\mathcal{A} \cdot 0 = 0$	;
L03:	Prop 3.2l $\gg$	$0 \cdot \mathcal{A} = 0$	;
L04:	Prop 3.2c $\rhd$ L02 $\rhd$ L03 $\gg$	$\mathcal{A} \cdot 0 = 0 \cdot \mathcal{A}$	□

[S **lemma** Prop 3.2n$_2$: $\Pi\mathcal{A}, \mathcal{B}\colon \mathcal{A} \cdot \mathcal{B} = \mathcal{B} \cdot \mathcal{A} \Rightarrow \mathcal{A} \cdot \mathcal{B}' = \mathcal{B}' \cdot \mathcal{A}$]

S **proof of** Prop 3.2n$_2$:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}$	;
L02:	Block $\gg$	Begin	;
L03:	Arbitrary $\gg$	$\quad\mathcal{A}, \mathcal{B}$	;
L04:	Premise $\gg$	$\quad\mathcal{A} \cdot \mathcal{B} = \mathcal{B} \cdot \mathcal{A}$	;
L05:	S8 $\gg$	$\quad\mathcal{A} \cdot \mathcal{B}' = \mathcal{A} \cdot \mathcal{B} + \mathcal{A}$	;
L06:	Prop 3.2e $\rhd$ L04 $\gg$	$\quad\mathcal{A} \cdot \mathcal{B} + \mathcal{A} = \mathcal{B} \cdot \mathcal{A} + \mathcal{A}$	;
L07:	Prop 3.2b $\rhd$ Prop 3.2m $\gg$	$\quad\mathcal{B} \cdot \mathcal{A} + \mathcal{A} = \mathcal{B}' \cdot \mathcal{A}$	;
L08:	Prop 3.2c $\rhd$ L08 $\rhd$ L07 $\gg$	$\quad\mathcal{A} \cdot \mathcal{B} + \mathcal{A} = \mathcal{B}' \cdot \mathcal{A}$	;
L09:	Prop 3.2c $\rhd$ L05 $\rhd$ L08 $\gg$	$\quad\mathcal{A} \cdot \mathcal{B}' = \mathcal{B}' \cdot \mathcal{A}$	;
L10:	Block $\gg$	End	;
L11:	Ded $\rhd$ L10 $\gg$	$\mathcal{A} \cdot \mathcal{B} = \mathcal{B} \cdot \mathcal{A} \Rightarrow \mathcal{A} \cdot \mathcal{B}' = \mathcal{B}' \cdot \mathcal{S}$	□

S **proof of** Prop 3.2n:

L01:	Arbitrary ≫	$\mathcal{A}, \mathcal{B}$	;
L02:	Block ≫	Begin	;
L03:	Prop 3.2n$_1$ ≫	$x \cdot 0 = 0 \cdot x$	;
L04:	Prop 3.2n$_2$ ≫	$x \cdot y = y \cdot x \Rightarrow x \cdot y' = y' \cdot x$	;
L05:	S9 @ y ▷ L03 ▷ L04 ≫	$x \cdot y = y = x$	;
L06:	Block ≫	End	;
L07:	Ded ▷ L06 ≫	$\mathcal{A} \cdot \mathcal{B} = \mathcal{B} \cdot \mathcal{A}$	□

S **proof of** Prop 3.2o:

L01:	Arbitrary ≫	$\mathcal{A}, \mathcal{B}, \mathcal{C}$	;
L02:	Block ≫	Begin	;
L03:	Arbitrary ≫	$\mathcal{A}, \mathcal{B}, \mathcal{C}$	;
L04:	Premise ≫	$\mathcal{A} = \mathcal{B}$	;
L05:	Prop 3.2k ≫	$\mathcal{A} = \mathcal{B} \Rightarrow \mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{C}$	;
L06:	L05 ▷ L04 ≫	$\mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{C}$	;
L07:	Prop 3.2n ≫	$\mathcal{A} \cdot \mathcal{C} = \mathcal{C} \cdot \mathcal{A}$	;
L08:	Prop 3.2n ≫	$\mathcal{B} \cdot \mathcal{C} = \mathcal{C} \cdot \mathcal{B}$	;
L09:	Prop 3.2c ≫	$\mathcal{A} \cdot \mathcal{C} = \mathcal{B} \cdot \mathcal{C} \Rightarrow \mathcal{B} \cdot \mathcal{C} = \mathcal{C} \cdot \mathcal{B} \Rightarrow \mathcal{A} \cdot \mathcal{C} = \mathcal{C} \cdot \mathcal{B}$	;
L10:	L09 ▷ L06 ▷ L08 ≫	$\mathcal{A} \cdot \mathcal{C} = \mathcal{C} \cdot \mathcal{B}$	;
L11:	S1 ≫	$\mathcal{A} \cdot \mathcal{C} = \mathcal{C} \cdot \mathcal{B} \Rightarrow \mathcal{A} \cdot \mathcal{C} = \mathcal{C} \cdot \mathcal{A} \Rightarrow \mathcal{C} \cdot \mathcal{A} = \mathcal{C} \cdot \mathcal{B}$	;
L12:	L11 ▷ L10 ▷ L07 ≫	$\mathcal{C} \cdot \mathcal{A} = \mathcal{C} \cdot \mathcal{B}$	;
L13:	Block ≫	End	;
L14:	Ded ▷ L13 ≫	$\mathcal{A} = \mathcal{B} \Rightarrow \mathcal{C} \cdot \mathcal{A} = \mathcal{C} \cdot \mathcal{B}$	□

[S **lemma** Prop 3.4a: $\Pi \mathcal{A}, \mathcal{B}, \mathcal{C} \colon \mathcal{A} \cdot (\mathcal{B} + \mathcal{C}) = (\mathcal{A} \cdot \mathcal{B}) + (\mathcal{A} \cdot \mathcal{C})$]

[S **lemma** Prop 3.4b: $\Pi \mathcal{A}, \mathcal{B}, \mathcal{C} \colon (\mathcal{B} + \mathcal{C}) \cdot \mathcal{A} = (\mathcal{B} \cdot \mathcal{A}) + (\mathcal{C} \cdot \mathcal{B})$]

[S **lemma** Prop 3.4c: $\Pi \mathcal{A}, \mathcal{B}, \mathcal{C} \colon (\mathcal{A} \cdot \mathcal{B}) \cdot \mathcal{C} = \mathcal{A} \cdot (\mathcal{B} \cdot \mathcal{C})$]

[S **lemma** Prop 3.4d: $\Pi \mathcal{A}, \mathcal{B}, \mathcal{C} \colon \mathcal{A} + \mathcal{C} = \mathcal{B} + \mathcal{C} \Rightarrow \mathcal{A} = \mathcal{B}$]

Litteratur

[Men97] E. Mendelson. *Introduction to Mathematical Logic*. Chapmann & Hall, 4. edition, 1997.