

A formal proof in Group Theory

Lasse Nielsen & Morten Ib Nielsen

Department of Computer Science, University of Copenhagen

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1 Introduction

In this report we set out to formally prove a simple result of Group Theory namely:

Theorem 1.1 *Let e be a neutral element of a Group G then e is unique. Thus we can talk about the neutral element of a Group G .*

The theorem above is very loosely formulated. In this report we set out to formalize the theorem and give a formal proof of it's correctness. In order to do this we need to do a lot of other work. First in XXTODO we define *Propositional Calculus* and then in MMTODO we define *First order Predicate Calculus*. Then using this we define ZFC set theory in YYTODO and finally having set theory available we can define a Group in ZZTODO. Then in WWTODO we restate the above theorem in a formal setting and we give a formal proof of it's correctness.

TODO mere jalla.

2 Propositional Calculus

Based on mathworld¹ and thus on Kleene (2002) we define first-order predicate calculus below. We note that the axioms 1 through 10 together with the inference rule modus ponens constitutes the propositional calculus.

Our definitions are not exactly like those found on Mathworld. The reason is that we have made \Rightarrow right associative this means that $\mathcal{F} \Rightarrow \mathcal{G} \Rightarrow \mathcal{F}$ really means $\mathcal{F} \Rightarrow (\mathcal{G} \Rightarrow \mathcal{F})$ below.

The [Theory pred calc] contains the following axioms

1. [pred calc rule pc1: $\forall \mathcal{F}: \forall \mathcal{G}: \mathcal{F} \Rightarrow \mathcal{G} \Rightarrow \mathcal{F}$]
2. [pred calc rule pc2: $\forall \mathcal{F}: \forall \mathcal{G}: \forall \mathcal{H}: (\mathcal{F} \Rightarrow \mathcal{G}) \Rightarrow (\mathcal{F} \Rightarrow \mathcal{G} \Rightarrow \mathcal{H}) \Rightarrow \mathcal{F} \Rightarrow \mathcal{H}$]
3. [pred calc rule pc3: $\forall \mathcal{F}: \forall \mathcal{G}: \mathcal{F} \Rightarrow \mathcal{G} \Rightarrow \mathcal{F} \wedge \mathcal{G}$]

¹<http://mathworld.wolfram.com/First-OrderLogic.html>.

6. [pred calc **rule** pc6: $\forall \mathcal{F}: \forall \mathcal{G}: \mathcal{F} \wedge \mathcal{G} \Rightarrow \mathcal{F}$]
7. [pred calc **rule** pc7: $\forall \mathcal{F}: \forall \mathcal{G}: \mathcal{F} \wedge \mathcal{G} \Rightarrow \mathcal{G}$]
8. [pred calc **rule** pc8: $\forall \mathcal{F}: \forall \mathcal{G}: \forall \mathcal{H}: (\mathcal{F} \Rightarrow \mathcal{G}) \Rightarrow (\mathcal{H} \Rightarrow \mathcal{G}) \Rightarrow \mathcal{F} \vee \mathcal{H} \Rightarrow \mathcal{G}$]
9. [pred calc **rule** pc9: $\forall \mathcal{F}: \forall \mathcal{G}: (\mathcal{F} \Rightarrow \mathcal{G}) \Rightarrow (\mathcal{F} \Rightarrow \neg \mathcal{G}) \Rightarrow \neg \mathcal{F}$]
10. [pred calc **rule** pc10: $\forall \mathcal{F}: \neg \neg \mathcal{F} \Rightarrow \mathcal{F}$]
11. [pred calc **rule** pc11: $\forall \mathcal{F}: \forall \mathcal{R}: \forall \mathcal{X}: \forall \mathcal{Y}: \Pi \mathcal{Y}. \langle \mathcal{F} | \mathcal{X} := \mathcal{Y} \rangle \Rightarrow \langle \mathcal{F} | \mathcal{X} := \mathcal{R} \rangle$]
12. [pred calc **rule** pc11: $\forall \mathcal{F}: \forall \mathcal{R}: \forall \mathcal{X}: \forall \mathcal{Y}: \langle \mathcal{F} | \mathcal{X} := \mathcal{R} \rangle \Rightarrow \Sigma \mathcal{Y}. \langle \mathcal{F} | \mathcal{X} := \mathcal{Y} \rangle$]

We note that in first order predicate calculus metaibles used in functions \mathcal{F} and predicates \mathcal{P} are *object metaibles*.

The only proof rule in [**Theory** pred calc] is Modus Ponens which says

- [pred calc **rule** pcmp: $\forall \mathcal{F}: \forall \mathcal{G}: \mathcal{F} \vdash \mathcal{F} \Rightarrow \mathcal{G} \vdash \mathcal{G}$]
- [pred calc **rule** pcia: $\forall \mathcal{F}: \forall \mathcal{G}: \forall \mathcal{X}: \forall \mathcal{Y}: \mathcal{G} \Rightarrow \langle \mathcal{F} | \mathcal{X} := \mathcal{Y} \rangle \vdash \mathcal{G} \Rightarrow \Pi \mathcal{Y}. \langle \mathcal{F} | \mathcal{X} := \mathcal{Y} \rangle$]
- [pred calc **rule** pcie: $\forall \mathcal{F}: \forall \mathcal{G}: \forall \mathcal{X}: \forall \mathcal{Y}: \langle \mathcal{F} | \mathcal{X} := \mathcal{Y} \rangle \Rightarrow \mathcal{G} \vdash \Sigma \mathcal{Y}. \langle \mathcal{F} | \mathcal{X} := \mathcal{Y} \rangle \vdash \mathcal{G}$]

2.1 A little proof

Below we make some small proofs just to demonstrate how to do it but also in order to test our definitions above.

Lemma 2.1 [*pred calc lemma trivia*: $\forall \mathcal{F}: \mathcal{F} \Rightarrow \mathcal{F}$]

pred calc **proof of** trivia:

L01:	Arbitrary \gg	\mathcal{F}	;
L02:	pc2 \gg	$(\mathcal{F} \Rightarrow \mathcal{F} \Rightarrow \mathcal{F}) \Rightarrow (\mathcal{F} \Rightarrow (\mathcal{F} \Rightarrow$	
		$\mathcal{F}) \Rightarrow \mathcal{F}) \Rightarrow \mathcal{F} \Rightarrow \mathcal{F}$;
L03:	pc1 \gg	$\mathcal{F} \Rightarrow \mathcal{F} \Rightarrow \mathcal{F}$;
L04:	pcmp \triangleright L03 \triangleright L02 \gg	$(\mathcal{F} \Rightarrow (\mathcal{F} \Rightarrow \mathcal{F}) \Rightarrow \mathcal{F}) \Rightarrow \mathcal{F} \Rightarrow$	
		\mathcal{F}	;
L05:	pc1 \gg	$\mathcal{F} \Rightarrow (\mathcal{F} \Rightarrow \mathcal{F}) \Rightarrow \mathcal{F}$;
L06:	pcmp \triangleright L05 \triangleright L04 \gg	$\mathcal{F} \Rightarrow \mathcal{F}$	□

Lemma 2.2 [*pred calc lemma iatest*: $\forall \mathcal{Y}: \forall \mathcal{G}: \mathcal{G} \Rightarrow \Pi \mathcal{Y}. \mathcal{Y} \Rightarrow \mathcal{G}$]

pred calc **proof of** iatest:

L01:	Arbitrary \gg	\mathcal{Y}	;
L02:	Arbitrary \gg	\mathcal{G}	;
L03:	pc1 \gg	$\mathcal{G} \Rightarrow \mathcal{Y} \Rightarrow \mathcal{G}$;
L04:	pcia \triangleright L03 \gg	$\mathcal{G} \Rightarrow \Pi \mathcal{Y}. \mathcal{Y} \Rightarrow \mathcal{G}$	□

Below we formally define the Zermelo-Frankel set theory. Together with the *axiom of choice* this is know as *ZFC*. Our definitions are based on those found on mathworld.wolfram.com [**Theory** zfc]

A Pyk definitions

$[* \Rightarrow * \xrightarrow{\text{pyk}} \text{" " imply " "}]$
 $[\text{pred calc} \xrightarrow{\text{pyk}} \text{"pred calc"}]$
 $[\text{pc1} \xrightarrow{\text{pyk}} \text{"pc1"}]$
 $[\text{pc2} \xrightarrow{\text{pyk}} \text{"pc2"}]$
 $[\text{pc3} \xrightarrow{\text{pyk}} \text{"pc3"}]$
 $[\text{pc4} \xrightarrow{\text{pyk}} \text{"pc4"}]$
 $[\text{pc5} \xrightarrow{\text{pyk}} \text{"pc5"}]$
 $[\text{pc6} \xrightarrow{\text{pyk}} \text{"pc6"}]$
 $[\text{pc7} \xrightarrow{\text{pyk}} \text{"pc7"}]$
 $[\text{pc8} \xrightarrow{\text{pyk}} \text{"pc8"}]$
 $[\text{pc9} \xrightarrow{\text{pyk}} \text{"pc9"}]$
 $[\text{pc10} \xrightarrow{\text{pyk}} \text{"pc10"}]$
 $[\text{pc11} \xrightarrow{\text{pyk}} \text{"pc11"}]$
 $[\text{pc12} \xrightarrow{\text{pyk}} \text{"pc12"}]$
 $[\text{pcmp} \xrightarrow{\text{pyk}} \text{"pcmp"}]$
 $[\text{pcia} \xrightarrow{\text{pyk}} \text{"pcia"}]$
 $[\text{pcie} \xrightarrow{\text{pyk}} \text{"pcie"}]$
 $[\Pi * .* \xrightarrow{\text{pyk}} \text{"forall " dot " "}]$
 $[\Sigma * .* \xrightarrow{\text{pyk}} \text{"exists " " "}]$
 $[\neg * \xrightarrow{\text{pyk}} \text{"not " "}]$
 $[* \wedge * \xrightarrow{\text{pyk}} \text{" " land " "}]$
 $[* \vee * \xrightarrow{\text{pyk}} \text{" " lor " "}]$
 $[* \Leftrightarrow * \xrightarrow{\text{pyk}} \text{" " bimplify " "}]$
 $[\text{trivia} \xrightarrow{\text{pyk}} \text{"trivia"}]$
 $[\text{iatest} \xrightarrow{\text{pyk}} \text{"iatest"}]$
 $[\text{zfc} \xrightarrow{\text{pyk}} \text{"zfc"}]$
 $[\text{test} \xrightarrow{\text{pyk}} \text{"test"}]$

B TeX definitions

- $[-x \stackrel{\text{tex}}{=} "\backslash\text{neg \#1.}"]$
- $[x \wedge y \stackrel{\text{tex}}{=} "\#1. \backslash\text{wedge \#2.}"]$
- $[x \vee y \stackrel{\text{tex}}{=} "\#1. \backslash\text{vee \#2.}"]$
- $[x \Rightarrow y \stackrel{\text{tex}}{=} "\#1. \backslash\text{Rrightarrow \#2.}"]$
- $[x \Leftrightarrow y \stackrel{\text{tex}}{=} "\#1. \backslash\text{Leftrightarrow \#2.}"]$
- $[\Pi y.b \stackrel{\text{tex}}{=} "\backslash\text{Pi~\#1. . \#2.}"]$
- $[\Sigma y.b \stackrel{\text{tex}}{=} "\backslash\text{Sigma~\#1. . \#2.}"]$

C Priority table

Priority table

Preassociative

[test], [base], [bracket * end bracket], [big bracket * end bracket], [\$ * \$],
[flush left [*]], [x], [y], [z], [[* \bowtie *]], [[* $\xrightarrow{*}$ *]], [pyk], [tex], [name], [prio], [*], [T],
[if(*, *, *)], [[* $\xrightarrow{*}$ *]], [val], [claim], [\perp], [f(*)], [(*)^I], [F], [0], [1], [2], [3], [4], [5], [6],
[7], [8], [9], [0], [1], [2], [3], [4], [5], [6], [7], [8], [9], [a], [b], [c], [d], [e], [f], [g], [h], [i], [j],
[k], [l], [m], [n], [o], [p], [q], [r], [s], [t], [u], [v], [w], [(*)^M], [If(*, *, *)],
[array{*} * end array], [l], [c], [r], [empty], [$\langle * | * := * \rangle$], [$\mathcal{M}(*)$], [$\tilde{\mathcal{U}}(*)$], [$\mathcal{U}(*)$],
[$\mathcal{U}^M(*)$], [apply(*, *)], [apply₁(*, *)], [identifier(*)], [identifier₁(*, *)], [array-
plus(*, *)], [array-remove(*, *, *)], [array-put(*, *, *, *)], [array-add(*, *, *, *, *)],
[bit(*, *)], [bit₁(*, *)], [rack], ["vector"], ["bibliography"], ["dictionary"],
["body"], ["codex"], ["expansion"], ["code"], ["cache"], ["diagnose"], ["pyk"],
["tex"], ["texname"], ["value"], ["message"], ["macro"], ["definition"],
["unpack"], ["claim"], ["priority"], ["lambda"], ["apply"], ["true"], ["if"],
["quote"], ["proclaim"], ["define"], ["introduce"], ["hide"], ["pre"], ["post"],
[$\mathcal{E}(*, *, *)$], [$\mathcal{E}_2(*, *, *, *, *)$], [$\mathcal{E}_3(*, *, *, *, *)$], [$\mathcal{E}_4(*, *, *, *, *)$], [look_{up}(*, *, *)],
[abstract(*, *, *, *)], [[*]], [$\mathcal{M}(*, *, *)$], [$\mathcal{M}_2(*, *, *, *, *)$], [$\mathcal{M}^*(*, *, *, *)$], [macro],
[s₀], [zip(*, *)], [assoc₁(*, *, *)], [(*)^P], [self], [[* \doteq *]], [[* $\dot{=}$ *]], [[* $\dot{=}$ *]],
[[* $\stackrel{\text{pyk}}{=}$ *]], [[* $\stackrel{\text{tex}}{=}$ *]], [[* $\stackrel{\text{name}}{=}$ *]], [Priority table[*]], [$\tilde{\mathcal{M}}_1$], [$\tilde{\mathcal{M}}_2(*)$], [$\tilde{\mathcal{M}}_3(*)$],
[$\tilde{\mathcal{M}}_4(*, *, *, *, *)$], [$\mathcal{M}(*, *, *, *)$], [$\mathcal{Q}(*, *, *, *)$], [$\tilde{\mathcal{Q}}_2(*, *, *, *)$], [$\tilde{\mathcal{Q}}_3(*, *, *, *, *)$], [$\tilde{\mathcal{Q}}^*(*, *, *, *)$],
[(*)], [(*)], [display(*)], [statement(*)], [(*)[.]], [(*)⁻], [aspect(*, *)],
[aspect(*, *, *)], [$\langle * \rangle$], [tuple₁(*)], [tuple₂(*)], [let₂(*, *)], [let₁(*, *)],
[[* $\stackrel{\text{claim}}{=}$ *]], [checker], [check(*, *)], [check₂(*, *, *)], [check₃(*, *, *, *)],
[check^{*}(*, *)], [check₂^{*}(*, *, *)], [(*)[.]], [(*)⁻], [(*)^o], [msg], [[* $\stackrel{\text{msg}}{=}$ *]], [$\langle \text{stmt} \rangle$],

$[(*)^\vee]$, $[\text{intro}(*, *, *, *)]$, $[\text{intro}(*, *, *)]$, $[\text{error}(*, *)]$, $[\text{error}_2(*, *)]$, $[\text{proof}(*, *, *)]$,
 $[\text{proof}_2(*, *)]$, $[\mathcal{S}(*, *)]$, $[\mathcal{S}^I(*, *)]$, $[\mathcal{S}^\triangleright(*, *)]$, $[\mathcal{S}_1^\triangleright(*, *, *)]$, $[\mathcal{S}^E(*, *)]$, $[\mathcal{S}_1^E(*, *, *)]$,
 $[\mathcal{S}^+(*, *)]$, $[\mathcal{S}_1^+(*, *, *)]$, $[\mathcal{S}^-(*, *)]$, $[\mathcal{S}_1^-(*, *, *)]$, $[\mathcal{S}^*(*)]$, $[\mathcal{S}_1^*(*)]$,
 $[\mathcal{S}_2^*(*)]$, $[\mathcal{S}^\otimes(*, *)]$, $[\mathcal{S}_1^\otimes(*, *, *)]$, $[\mathcal{S}^+(*, *)]$, $[\mathcal{S}_1^+(*, *, *, *)]$, $[\mathcal{S}^{\#}(*, *)]$,
 $[\mathcal{S}_1^{\#}(*, *, *, *)]$, $[\mathcal{S}^{\text{i.e.}}(*, *)]$, $[\mathcal{S}_1^{\text{i.e.}}(*, *, *, *)]$, $[\mathcal{S}_2^{\text{i.e.}}(*, *, *, *, *)]$, $[\mathcal{S}^\vee(*, *)]$,
 $[\mathcal{S}_1^\vee(*, *, *, *)]$, $[\mathcal{S}^i(*, *)]$, $[\mathcal{S}_1^i(*, *, *, *)]$, $[\mathcal{S}_2^i(*, *, *, *, *)]$, $[\mathcal{T}(*)]$, $[\text{claims}(*, *, *)]$,
 $[\text{claims}_2(*, *, *)]$, $[\text{<proof>}]$, $[\text{proof}]$, $[[\text{Lemma } *: *]]$, $[[\text{Proof of } *: *]]$,
 $[[* \text{ lemma } *: *]]$, $[[* \text{ antilemma } *: *]]$, $[[* \text{ rule } *: *]]$, $[[* \text{ antirule } *: *]]$,
 $[\text{verifier}]$, $[\mathcal{V}_1(*)]$, $[\mathcal{V}_2(*, *)]$, $[\mathcal{V}_3(*, *, *, *)]$, $[\mathcal{V}_4(*, *)]$, $[\mathcal{V}_5(*, *, *, *, *)]$, $[\mathcal{V}_6(*, *, *, *, *)]$,
 $[\mathcal{V}_7(*, *, *, *, *)]$, $[\text{Cut}(*, *)]$, $[\text{Head}_\oplus(*)]$, $[\text{Tail}_\oplus(*)]$, $[\text{rule}_1(*, *)]$, $[\text{rule}(*, *)]$,
 $[\text{Rule tactic}]$, $[\text{Plus}(*, *)]$, $[[\text{Theory } *]]$, $[\text{theory}_2(*, *)]$, $[\text{theory}_3(*, *, *)]$,
 $[\text{theory}_4(*, *, *, *)]$, $[\text{HeadNil}''']$, $[\text{HeadPair}''']$, $[\text{Transitivity}''']$, $[\text{Contra}''']$, $[\text{HeadNil}]$,
 $[\text{HeadPair}]$, $[\text{Transitivity}]$, $[\text{Contra}]$, $[\text{T}_E]$, $[\text{ragged right}]$,
 $[\text{ragged right expansion}]$, $[\text{parm}(*, *, *)]$, $[\text{parm}^*(*)]$, $[\text{inst}(*, *)]$,
 $[\text{inst}^*(*)]$, $[\text{occur}(*, *, *)]$, $[\text{occur}^*(*)]$, $[\text{unify}(* = *, *)]$, $[\text{unify}^*(*)]$,
 $[\text{unify}_2(* = *, *)]$, $[\text{L}_a]$, $[\text{L}_b]$, $[\text{L}_c]$, $[\text{L}_d]$, $[\text{L}_e]$, $[\text{L}_f]$, $[\text{L}_g]$, $[\text{L}_h]$, $[\text{L}_i]$, $[\text{L}_j]$, $[\text{L}_k]$, $[\text{L}_l]$, $[\text{L}_m]$,
 $[\text{L}_n]$, $[\text{L}_o]$, $[\text{L}_p]$, $[\text{L}_q]$, $[\text{L}_r]$, $[\text{L}_s]$, $[\text{L}_t]$, $[\text{L}_u]$, $[\text{L}_v]$, $[\text{L}_w]$, $[\text{L}_x]$, $[\text{L}_y]$, $[\text{L}_z]$, $[\text{L}_A]$, $[\text{L}_B]$, $[\text{L}_C]$,
 $[\text{L}_D]$, $[\text{L}_E]$, $[\text{L}_F]$, $[\text{L}_G]$, $[\text{L}_H]$, $[\text{L}_I]$, $[\text{L}_J]$, $[\text{L}_K]$, $[\text{L}_L]$, $[\text{L}_M]$, $[\text{L}_N]$, $[\text{L}_O]$, $[\text{L}_P]$, $[\text{L}_Q]$, $[\text{L}_R]$,
 $[\text{L}_S]$, $[\text{L}_T]$, $[\text{L}_U]$, $[\text{L}_V]$, $[\text{L}_W]$, $[\text{L}_X]$, $[\text{L}_Y]$, $[\text{L}_Z]$, $[\text{L}_?]$, $[\text{Reflexivity}]$, $[\text{Reflexivity}_1]$,
 $[\text{Commutativity}]$, $[\text{Commutativity}_1]$, $[\text{<tactic>}]$, $[\text{tactic}]$, $[[* \text{ tactic}^*]]$, $[\mathcal{P}(*, *, *)]$,
 $[\mathcal{P}^*(*)]$, $[\text{p}_0]$, $[\text{conclude}_1(*, *)]$, $[\text{conclude}_2(*, *, *)]$, $[\text{conclude}_3(*, *, *, *)]$,
 $[\text{conclude}_4(*, *)]$;

Preassociative

$[* \{ * \}]$, $[/indexintro(*, *, *, *)]$, $[/intro(*, *, *)]$, $[/bothintro(*, *, *, *, *)]$,
 $[/nameintro(*, *, *, *)]$, $[*']$, $[* [*]]$, $[* [* \rightarrow *]]$, $[* [* \Rightarrow *]]$, $[* 0]$, $[* 1]$, $[0b]$, $[* \text{-color}(*)]$,
 $[* \text{-color}^*(*)]$, $[*^H]$, $[*^T]$, $[*^U]$, $[*^h]$, $[*^t]$, $[*^s]$, $[*^c]$, $[*^d]$, $[*^a]$, $[*^C]$, $[*^M]$, $[*^B]$, $[*^F]$, $[*^i]$,
 $[*^d]$, $[*^R]$, $[*^0]$, $[*^1]$, $[*^2]$, $[*^3]$, $[*^4]$, $[*^5]$, $[*^6]$, $[*^7]$, $[*^8]$, $[*^9]$, $[*^E]$, $[*^\nu]$, $[*^C]$, $[*^C^*]$;

Preassociative

$[" * "]$, $[\]$, $[(*)^t]$, $[\text{string}(*) + *]$, $[\text{string}(*) ++ *]$, $[\]$,
 $[*]$, $[*]$, $[! *]$, $[# *]$, $[\$ *]$, $[\% *]$, $[\& *]$, $[* *]$, $[(*)]$, $[\]$, $[* *]$, $[+ *]$, $[, *]$, $[- *]$, $[. *]$, $[/ *]$,
 $[0 *]$, $[1 *]$, $[2 *]$, $[3 *]$, $[4 *]$, $[5 *]$, $[6 *]$, $[7 *]$, $[8 *]$, $[9 *]$, $[: *]$, $[; *]$, $[< *]$, $[= *]$, $[> *]$, $[? *]$,
 $[@ *]$, $[A *]$, $[B *]$, $[C *]$, $[D *]$, $[E *]$, $[F *]$, $[G *]$, $[H *]$, $[I *]$, $[J *]$, $[K *]$, $[L *]$, $[M *]$, $[N *]$,
 $[O *]$, $[P *]$, $[Q *]$, $[R *]$, $[S *]$, $[T *]$, $[U *]$, $[V *]$, $[W *]$, $[X *]$, $[Y *]$, $[Z *]$, $[[*]]$, $[\ \backslash *]$, $[\] *]$, $[\ ^ *]$,
 $[_ *]$, $[' *]$, $[a *]$, $[b *]$, $[c *]$, $[d *]$, $[e *]$, $[f *]$, $[g *]$, $[h *]$, $[i *]$, $[j *]$, $[k *]$, $[l *]$, $[m *]$, $[n *]$, $[o *]$,
 $[p *]$, $[q *]$, $[r *]$, $[s *]$, $[t *]$, $[u *]$, $[v *]$, $[w *]$, $[x *]$, $[y *]$, $[z *]$, $[\{ *]$, $[| *]$, $[} *]$, $[\sim *]$,
 $[\text{Preassociative } *; *]$, $[\text{Postassociative } *; *]$, $[[*], *]$, $[\text{priority } * \text{ end}]$,
 $[\text{newline } *]$, $[\text{macro newline } *]$;

Preassociative

$[* ' *]$, $[* ' *]$;

Preassociative

$[* \cdot *]$, $[* \cdot 0 *]$;

Preassociative

Preassociative

[**], [* linebreak[4] *];

Preassociative

[pred calc], [pc1], [pc2], [pc3], [pc4], [pc5], [pc6], [pc7], [pc8], [pc9], [pc10], [pc11],
[pc12], [pcmp], [pcia], [pcie];

Postassociative

[Π * .*], [Σ * .*];

Preassociative

[\neg *];

Preassociative

[* \wedge *];

Preassociative

[* \vee *];

Preassociative

[* \Leftrightarrow *];

Preassociative

[trivia], [iatest];

Preassociative

[zfc]; **End table**