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[SystemQ **lemma** RemoveOr: $\Pi\mathcal{A}\colon \mathcal{A} \mathbin{\dot\vee} \mathcal{A} \vdash \mathcal{A}$]

SystemQ **proof of** RemoveOr:

L01:	Arbitrary $\gg$	$\mathcal{A}$	;
L02:	Premise $\gg$	$\mathcal{A} \mathbin{\dot\vee} \mathcal{A}$	;
L03:	Repetition $\rhd$ L02 $\gg$	$\dot\neg\,(\mathcal{A})\mathrm{n} \Rightarrow \mathcal{A}$	;
L04:	AutoImply $\gg$	$\mathcal{A} \Rightarrow \mathcal{A}$	;
L05:	FromNegations $\rhd$ L04 $\rhd$ L03 $\gg$	$\mathcal{A}$	$\square$

[SystemQ **lemma** ToNegatedAnd: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A} \Rightarrow \dot\neg\,(\mathcal{B})\mathrm{n} \vdash \dot\neg\,((\mathcal{A} \mathbin{\dot\wedge} \mathcal{B}))\mathrm{n}$]

SystemQ **proof of** ToNegatedAnd:

L01:	Arbitrary $\gg$	$\mathcal{A},\mathcal{B}$	;
L02:	Premise $\gg$	$\mathcal{A} \Rightarrow \dot\neg\,(\mathcal{B})\mathrm{n}$	;
L03:	AddDoubleNeg $\rhd$ L02 $\gg$	$\dot\neg\,(\dot\neg\,((\mathcal{A} \Rightarrow \dot\neg\,(\mathcal{B})\mathrm{n}))\mathrm{n})\mathrm{n}$	;
L04:	Repetition $\rhd$ L03 $\gg$	$\dot\neg\,((\mathcal{A} \mathbin{\dot\wedge} \mathcal{B}))\mathrm{n}$	$\square$

[SystemQ **lemma** ToNegatedAnd(1): $\Pi\mathcal{A},\mathcal{B}\colon \dot\neg\,(\mathcal{A})\mathrm{n} \vdash \dot\neg\,((\mathcal{A} \mathbin{\dot\wedge} \mathcal{B}))\mathrm{n}$]

SystemQ **proof of** ToNegatedAnd(1):

L01:	Block $\gg$	Begin	;
L02:	Arbitrary $\gg$	$\mathcal{A},\mathcal{B}$	;
L03:	Premise $\gg$	$\dot\neg\,(\mathcal{A})\mathrm{n}$	;
L04:	Premise $\gg$	$\mathcal{A}$	;
L05:	FromContradiction $\rhd$ L04 $\rhd$ L03 $\gg$	$\dot\neg\,(\mathcal{B})\mathrm{n}$	;
L06:	Block $\gg$	End	;
L07:	Arbitrary $\gg$	$\mathcal{A},\mathcal{B}$	;
L08:	Ded $\rhd$ L06 $\gg$	$\dot\neg\,(\mathcal{A})\mathrm{n} \Rightarrow \mathcal{A} \Rightarrow \dot\neg\,(\mathcal{B})\mathrm{n}$	;
L03:	Premise $\gg$	$\dot\neg\,(\mathcal{A})\mathrm{n}$	;
L04:	MP $\rhd$ L08 $\rhd$ L03 $\gg$	$\mathcal{A} \Rightarrow \dot\neg\,(\mathcal{B})\mathrm{n}$	;
L09:	ToNegatedAnd $\rhd$ L04 $\gg$	$\dot\neg\,((\mathcal{A} \mathbin{\dot\wedge} \mathcal{B}))\mathrm{n}$	$\square$

[SystemQ **lemma** IntroExist(Helper): $\Pi\mathcal{X},\mathrm{V}_1,\mathcal{A},\mathcal{B}\colon \langle \dot\neg\,(\mathcal{A})\mathrm{n} \equiv \dot\neg\,(\mathcal{B})\mathrm{n} | \mathrm{V}_1 {:==} \mathcal{X}\rangle$
$\forall \mathrm{V}_1\colon \dot\neg\,(\mathcal{B})\mathrm{n} \Rightarrow \dot\neg\,(\mathcal{A})\mathrm{n}$]

SystemQ **proof of** IntroExist(Helper):

L01:	Block $\gg$	Begin	;
L02:	Arbitrary $\gg$	$\mathcal{X},\mathrm{V}_1,\mathcal{A},\mathcal{B}$	;
L03:	Side-condition $\gg$	$\langle \dot\neg\,(\mathcal{A})\mathrm{n} \equiv \dot\neg\,(\mathcal{B})\mathrm{n} \vert \mathrm{V}_1 {:==} \mathcal{X}\rangle_{\mathrm{Me}}$	;
L04:	Premise $\gg$	$\forall \mathrm{V}_1\colon \dot\neg\,(\mathcal{B})\mathrm{n}$	;
L05:	A4 @ $\mathcal{X}$ ⧐ L03 $\rhd$ L04 $\gg$	$\dot\neg\,(\mathcal{A})\mathrm{n}$	;
L06:	Block $\gg$	End	;
L07:	Arbitrary $\gg$	$\mathcal{X},\mathrm{V}_1,\mathcal{A},\mathcal{B}$	;
L08:	Ded $\rhd$ L06 $\gg$	$\langle \dot\neg\,(\mathcal{A})\mathrm{n} \equiv \dot\neg\,(\mathcal{B})\mathrm{n} \vert \mathrm{V}_1 {:==} \mathcal{X}\rangle_{\mathrm{Me}} \Vdash \forall \mathrm{V}_1\colon \dot\neg\,(\mathcal{B})\mathrm{n} \Rightarrow \dot\neg\,(\mathcal{A})\mathrm{n}$	$\square$

[SystemQ **lemma** IntroExist: $\Pi\mathcal{X},\mathrm{V}_1,\mathcal{A},\mathcal{B}\colon \langle \dot\neg\,(\mathcal{A})\mathrm{n} \equiv \dot\neg\,(\mathcal{B})\mathrm{n} | \mathrm{V}_1 {:==} \mathcal{X}\rangle_{\mathrm{Me}} \Vdash$
$\mathcal{A} \vdash \exists \mathrm{V}_1\colon \mathcal{B}$]

SystemQ **proof of** IntroExist:

L01:	Arbitrary $\gg$	$\mathcal{X},\mathrm{V}_1,\mathcal{A},\mathcal{B}$	;
L02:	Side-condition $\gg$	$\langle \dot\neg\,(\mathcal{A})\mathrm{n} \equiv \dot\neg\,(\mathcal{B})\mathrm{n} \vert \mathrm{V}_1 {:==} \mathcal{X}\rangle_{\mathrm{Me}}$	;

L03:	IntroExist(Helper) @ $\mathcal{X}$ $\rhd\!\!\!\rhd$ L02 $\gg$	$\forall V_1\colon \dot{\neg}\,(\mathcal{B})n \Rightarrow \dot{\neg}\,(\mathcal{A})n$	;
L04:	Premise $\gg$	$\mathcal{A}$	;
L05:	AddDoubleNeg $\rhd$ L04 $\gg$	$\dot{\neg}\,(\dot{\neg}\,(\mathcal{A})n)n$	;
L06:	MT $\rhd$ L03 $\rhd$ L05 $\gg$	$\dot{\neg}\,(\forall V_1\colon \dot{\neg}\,(\mathcal{B})n)n$	;
L07:	Repetition $\rhd$ L06 $\gg$	$\exists V_1\colon \mathcal{B}$	$\square$

[SystemQ **lemma** ExistMP: $\Pi V_1, \mathcal{A}, \mathcal{B}\colon \mathcal{A} \Rightarrow \mathcal{B} \vdash \exists V_1\colon \mathcal{A} \vdash \mathcal{B}$]

SystemQ **proof of** ExistMP:

L01:	Block $\gg$	Begin	;
L02:	Arbitrary $\gg$	$V_1, \mathcal{A}, \mathcal{B}$	;
L03:	Premise $\gg$	$\mathcal{A} \Rightarrow \mathcal{B}$	;
L04:	Premise $\gg$	$\exists V_1\colon \mathcal{A}$	;
L05:	Premise $\gg$	$\dot{\neg}\,(\mathcal{B})n$	;
L06:	MT $\rhd$ L03 $\rhd$ L05 $\gg$	$\dot{\neg}\,(\mathcal{A})n$	;
L07:	Gen $\rhd$ L06 $\gg$	$\forall V_1\colon \dot{\neg}\,(\mathcal{A})n$	;
L08:	Repetition $\rhd$ L04 $\gg$	$\dot{\neg}\,(\forall V_1\colon \dot{\neg}\,(\mathcal{A})n)n$	;
L09:	FromContradiction $\rhd$ L07 $\rhd$ L08 $\gg$	$\dot{\neg}\,(\dot{\neg}\,(\mathcal{B})n)n$	;
L10:	Block $\gg$	End	;
L11:	Arbitrary $\gg$	$V_1, \mathcal{A}, \mathcal{B}$	;
L12:	Ded $\rhd$ L10 $\gg$	$(\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow \exists V_1\colon \mathcal{A} \Rightarrow \dot{\neg}\,(\mathcal{B})n \Rightarrow \dot{\neg}\,(\dot{\neg}\,(\mathcal{B})n)n$	;
L04:	Premise $\gg$	$\mathcal{A} \Rightarrow \mathcal{B}$	;
L03:	Premise $\gg$	$\exists V_1\colon \mathcal{A}$	;
L05:	MP2 $\rhd$ L12 $\rhd$ L04 $\rhd$ L03 $\gg$	$\dot{\neg}\,(\mathcal{B})n \Rightarrow \dot{\neg}\,(\dot{\neg}\,(\mathcal{B})n)n$	;
L06:	prop lemma imply negation $\rhd$ L05 $\gg$	$\dot{\neg}\,(\dot{\neg}\,(\mathcal{B})n)n$	;
L13:	RemoveDoubleNeg $\rhd$ L06 $\gg$	$\mathcal{B}$	$\square$

[SystemQ **lemma** ExistMP2: $\Pi V_1, V_2, \mathcal{A}, \mathcal{B}, \mathcal{C}\colon \mathcal{A} \Rightarrow \mathcal{B} \Rightarrow \mathcal{C} \vdash \exists V_1\colon \mathcal{A} \vdash \exists V_2\colon \mathcal{B} \vdash \mathcal{C}$]

SystemQ **proof of** ExistMP2:

L01:	Arbitrary $\gg$	$V_1, V_2, \mathcal{A}, \mathcal{B}, \mathcal{C}$	;
L02:	Premise $\gg$	$\mathcal{A} \Rightarrow \mathcal{B} \Rightarrow \mathcal{C}$	;
L03:	Premise $\gg$	$\exists V_1\colon \mathcal{A}$	;
L04:	Premise $\gg$	$\exists V_2\colon \mathcal{B}$	;
L05:	ExistMP $\rhd$ L02 $\rhd$ L03 $\gg$	$\mathcal{B} \Rightarrow \mathcal{C}$	;
L06:	ExistMP $\rhd$ L05 $\rhd$ L04 $\gg$	$\mathcal{C}$	$\square$

[SystemQ **lemma** TwiceExistMP: $\Pi V_1, V_2, \mathcal{A}, \mathcal{B}\colon \mathcal{A} \Rightarrow \mathcal{B} \vdash \exists V_1\colon \exists V_2\colon \mathcal{A} \vdash \mathcal{B}$]

SystemQ **proof of** TwiceExistMP:

L01:	Block $\gg$	Begin	;
L02:	Arbitrary $\gg$	$V_2, \mathcal{A}, \mathcal{B}$	;
L03:	Premise $\gg$	$\mathcal{A} \Rightarrow \mathcal{B}$	;
L04:	Premise $\gg$	$\exists V_2\colon \mathcal{A}$	;
L05:	ExistMP $\rhd$ L03 $\rhd$ L04 $\gg$	$\mathcal{B}$	;
L06:	Block $\gg$	End	;

L07: Arbitrary $\gg$ $V_1, V_2, \mathcal{A}, \mathcal{B}$;

L03: Ded $\triangleright$ L06 $\gg$ $(\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow \exists V_2 \colon \mathcal{A} \Rightarrow \mathcal{B}$;

L04: Premise $\gg$ $\mathcal{A} \Rightarrow \mathcal{B}$;

L08: Premise $\gg$ $\exists V_1 \colon \exists V_2 \colon \mathcal{A}$;

L09: MP $\triangleright$ L03 $\triangleright$ L04 $\gg$ $\exists V_2 \colon \mathcal{A} \Rightarrow \mathcal{B}$;

L10: ExistMP $\triangleright$ L09 $\triangleright$ L08 $\gg$ $\mathcal{B}$ □

[SystemQ **lemma** TwiceExistMP2: $\Pi V_1, V_2, V_3, V_4, \mathcal{A}, \mathcal{B}, \mathcal{C} \colon \mathcal{A} \Rightarrow \mathcal{B} \Rightarrow \mathcal{C} \vdash \exists V_1 \colon \exists V_2 \colon \mathcal{A} \vdash \exists V_3 \colon \exists V_4 \colon \mathcal{B} \vdash \mathcal{C}$]

SystemQ **proof of** TwiceExistMP2:

L01: Arbitrary $\gg$ $V_1, V_2, V_3, V_4, \mathcal{A}, \mathcal{B}, \mathcal{C}$;

L02: Premise $\gg$ $\mathcal{A} \Rightarrow \mathcal{B} \Rightarrow \mathcal{C}$;

L03: Premise $\gg$ $\exists V_1 \colon \exists V_2 \colon \mathcal{A}$;

L04: Premise $\gg$ $\exists V_3 \colon \exists V_4 \colon \mathcal{B}$;

L05: TwiceExistMP $\triangleright$ L02 $\triangleright$ L03 $\gg$ $\mathcal{B} \Rightarrow \mathcal{C}$;

L06: TwiceExistMP $\triangleright$ L05 $\triangleright$ L04 $\gg$ $\mathcal{C}$ □

[SystemQ **lemma** AllNegated(Imply): $\Pi V_1, \mathcal{A} \colon \dot{\neg}(\forall V_1 \colon \mathcal{A})\mathrm{n} \Rightarrow \exists V_1 \colon \dot{\neg}(\mathcal{A})\mathrm{n}$]

SystemQ **proof of** AllNegated(Imply):

L01: Block $\gg$ Begin ;

L02: Arbitrary $\gg$ $V_1, \mathcal{A}$;

L03: Premise $\gg$ $\forall V_1 \colon \dot{\neg}(\dot{\neg}(\mathcal{A})\mathrm{n})\mathrm{n}$;

L04: A4 @ $\mathcal{X}$ $\triangleright$ L03 $\gg$ $\dot{\neg}(\dot{\neg}(\mathcal{A})\mathrm{n})\mathrm{n}$;

L05: RemoveDoubleNeg $\triangleright$ L04 $\gg$ $\mathcal{A}$;

L06: Gen $\triangleright$ L05 $\gg$ $\forall V_1 \colon \mathcal{A}$;

L07: Block $\gg$ End ;

L08: Arbitrary $\gg$ $V_1, \mathcal{A}$;

L09: Ded $\triangleright$ L07 $\gg$ $\forall V_1 \colon \dot{\neg}(\dot{\neg}(\mathcal{A})\mathrm{n})\mathrm{n} \Rightarrow \forall V_1 \colon \mathcal{A}$;

L10: Contrapositive $\triangleright$ L09 $\gg$ $\dot{\neg}(\forall V_1 \colon \mathcal{A})\mathrm{n} \Rightarrow \dot{\neg}(\forall V_1 \colon \dot{\neg}(\dot{\neg}(\mathcal{A})\mathrm{n})\mathrm{n})\mathrm{n}$;

L11: Repetition $\triangleright$ L10 $\gg$ $\dot{\neg}(\forall V_1 \colon \mathcal{A})\mathrm{n} \Rightarrow \exists V_1 \colon \dot{\neg}(\mathcal{A})\mathrm{n}$ □

[SystemQ **lemma** ExistNegated(Imply): $\Pi V_1, \mathcal{A} \colon \dot{\neg}(\exists V_1 \colon \mathcal{A})\mathrm{n} \Rightarrow \forall V_1 \colon \dot{\neg}(\mathcal{A})\mathrm{n}$]

SystemQ **proof of** ExistNegated(Imply):

L01: Block $\gg$ Begin ;

L02: Arbitrary $\gg$ $V_1, \mathcal{A}$;

L03: Premise $\gg$ $\dot{\neg}(\exists V_1 \colon \mathcal{A})\mathrm{n}$;

L04: Repetition $\triangleright$ L03 $\gg$ $\dot{\neg}(\dot{\neg}(\forall V_1 \colon \dot{\neg}(\mathcal{A})\mathrm{n})\mathrm{n})\mathrm{n}$;

L05: RemoveDoubleNeg $\triangleright$ L04 $\gg$ $\forall V_1 \colon \dot{\neg}(\mathcal{A})\mathrm{n}$;

L06: Block $\gg$ End ;

L07: Arbitrary $\gg$ $V_1, \mathcal{A}$;

L08: Ded $\triangleright$ L06 $\gg$ $\dot{\neg}(\exists V_1 \colon \mathcal{A})\mathrm{n} \Rightarrow \forall V_1 \colon \dot{\neg}(\mathcal{A})\mathrm{n}$ □

[SystemQ **lemma** AddAll: $\Pi V_1, \mathcal{A}, \mathcal{B} \colon \mathcal{A} \Rightarrow \mathcal{B} \vdash \forall V_1 \colon \mathcal{A} \Rightarrow \forall V_1 \colon \mathcal{B}$]

SystemQ **proof of** AddAll:

L01: Block $\gg$ Begin ;

L02: Arbitrary $\gg$ $V_1, \mathcal{A}, \mathcal{B}$;

L03: Premise $\gg$ $\mathcal{A} \Rightarrow \mathcal{B}$;

L04: Premise $\gg$ $\forall V_1 \colon \mathcal{A}$;

L05: A4 $\triangleright$ L04 $\gg$ $\mathcal{A}$;

L06:	MP $\rhd$ L03 $\rhd$ L05 $\gg$	$\mathcal{B}$	;
L07:	Gen $\rhd$ L06 $\gg$	$\forall \mathrm{V}_1 \colon \mathcal{B}$	;
L08:	Block $\gg$	End	;
L09:	Arbitrary $\gg$	$\mathrm{V}_1, \mathcal{A}, \mathcal{B}$	;
L03:	Ded $\rhd$ L08 $\gg$	$(\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow \forall \mathrm{V}_1 \colon \mathcal{A} \Rightarrow \forall \mathrm{V}_1 \colon \mathcal{B}$	;
L04:	Premise $\gg$	$\mathcal{A} \Rightarrow \mathcal{B}$	;
L10:	MP $\rhd$ L03 $\rhd$ L04 $\gg$	$\forall \mathrm{V}_1 \colon \mathcal{A} \Rightarrow \forall \mathrm{V}_1 \colon \mathcal{B}$	$\square$

[SystemQ **lemma** AddExist(Helper1): $\Pi \mathcal{Y}, \mathrm{V}_1, \mathrm{V}_2, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \colon \langle \dot{\neg}(\mathcal{B})\mathrm{n} \equiv \dot{\neg}(\mathcal{D})\mathrm{n} |$ $(\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow (\mathcal{C} \Rightarrow \mathcal{A}) \Rightarrow \exists \mathrm{V}_1 \colon \mathcal{C} \Rightarrow \forall \mathrm{V}_2 \colon \dot{\neg}(\mathcal{D})\mathrm{n} \Rightarrow \dot{\neg}(\forall \mathrm{V}_2 \colon \dot{\neg}(\mathcal{D})\mathrm{n})\mathrm{n}$]

SystemQ **proof of** AddExist(Helper1):

L01:	Block $\gg$	Begin	;
L02:	Arbitrary $\gg$	$\mathcal{Y}, \mathrm{V}_1, \mathrm{V}_2, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$	;
L03:	Side-condition $\gg$	$\langle \dot{\neg}(\mathcal{B})\mathrm{n} \equiv \dot{\neg}(\mathcal{D})\mathrm{n} \| \mathrm{V}_2 {:==} \mathcal{Y} \rangle_{\mathrm{Me}}$	;
L04:	Premise $\gg$	$\mathcal{A} \Rightarrow \mathcal{B}$	;
L03:	Premise $\gg$	$\mathcal{C} \Rightarrow \mathcal{A}$	;
L05:	Premise $\gg$	$\exists \mathrm{V}_1 \colon \mathcal{C}$	;
L06:	Premise $\gg$	$\forall \mathrm{V}_2 \colon \dot{\neg}(\mathcal{D})\mathrm{n}$	;
L07:	A4 @ $\mathcal{Y}$ $\rhd$ L06 $\gg$	$\dot{\neg}(\mathcal{B})\mathrm{n}$	;
L08:	MT $\rhd$ L04 $\rhd$ L07 $\gg$	$\dot{\neg}(\mathcal{A})\mathrm{n}$	;
L09:	MT $\rhd$ L03 $\rhd$ L08 $\gg$	$\dot{\neg}(\mathcal{C})\mathrm{n}$	;
L10:	Gen $\rhd$ L09 $\gg$	$\forall \mathrm{V}_1 \colon \dot{\neg}(\mathcal{C})\mathrm{n}$	;
L11:	Repetition $\rhd$ L05 $\gg$	$\dot{\neg}(\forall \mathrm{V}_1 \colon \dot{\neg}(\mathcal{C})\mathrm{n})\mathrm{n}$	;
L12:	FromContradiction $\rhd$ L10 $\rhd$ L11 $\gg$	$\dot{\neg}(\forall \mathrm{V}_2 \colon \dot{\neg}(\mathcal{D})\mathrm{n})\mathrm{n}$	;
L13:	Block $\gg$	End	;
L14:	Arbitrary $\gg$	$\mathcal{Y}, \mathrm{V}_1, \mathrm{V}_2, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$	;
L15:	Ded $\rhd$ L13 $\gg$	$\langle \dot{\neg}(\mathcal{B})\mathrm{n} \equiv \dot{\neg}(\mathcal{D})\mathrm{n} \| \mathrm{V}_2 {:==} \mathcal{Y} \rangle_{\mathrm{Me}} \Vdash (\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow (\mathcal{C} \Rightarrow \mathcal{A}) \Rightarrow \exists \mathrm{V}_1 \colon \mathcal{C} \Rightarrow \forall \mathrm{V}_2 \colon \dot{\neg}(\mathcal{D})\mathrm{n} \Rightarrow \dot{\neg}(\forall \mathrm{V}_2 \colon \dot{\neg}(\mathcal{D})\mathrm{n})\mathrm{n}$	$\square$

[SystemQ **lemma** AddExist(Helper2): $\Pi \mathcal{Y}, \mathrm{V}_1, \mathrm{V}_2, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \colon \langle \dot{\neg}(\mathcal{B})\mathrm{n} \equiv \dot{\neg}(\mathcal{D})\mathrm{n} |$ $(\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow (\mathcal{C} \Rightarrow \mathcal{A}) \Rightarrow \exists \mathrm{V}_1 \colon \mathcal{C} \Rightarrow \exists \mathrm{V}_2 \colon \mathcal{D}$]

SystemQ **proof of** AddExist(Helper2):

L01:	Block $\gg$	Begin	;
L02:	Arbitrary $\gg$	$\mathcal{Y}, \mathrm{V}_1, \mathrm{V}_2, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$	;
L03:	Side-condition $\gg$	$\langle \dot{\neg}(\mathcal{B})\mathrm{n} \equiv \dot{\neg}(\mathcal{D})\mathrm{n} \| \mathrm{V}_2 {:==} \mathcal{Y} \rangle_{\mathrm{Me}}$	;
L04:	Premise $\gg$	$\mathcal{A} \Rightarrow \mathcal{B}$	;
L05:	Premise $\gg$	$\mathcal{C} \Rightarrow \mathcal{A}$	;
L06:	Premise $\gg$	$\exists \mathrm{V}_1 \colon \mathcal{C}$	;
L07:	AddExist(Helper1) $\unrhd$ L03 $\gg$	$(\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow (\mathcal{C} \Rightarrow \mathcal{A}) \Rightarrow \exists \mathrm{V}_1 \colon \mathcal{C} \Rightarrow \forall \mathrm{V}_2 \colon \dot{\neg}(\mathcal{D})\mathrm{n} \Rightarrow \dot{\neg}(\forall \mathrm{V}_2 \colon \dot{\neg}(\mathcal{D})\mathrm{n})\mathrm{n}$	;
L08:	MP3 $\rhd$ L07 $\rhd$ L04 $\rhd$ L05 $\rhd$ L06 $\gg$	$\forall \mathrm{V}_2 \colon \dot{\neg}(\mathcal{D})\mathrm{n} \Rightarrow \dot{\neg}(\forall \mathrm{V}_2 \colon \dot{\neg}(\mathcal{D})\mathrm{n})\mathrm{n}$	;
L09:	prop lemma imply negation $\rhd$ L08 $\gg$	$\dot{\neg}(\forall \mathrm{V}_2 \colon \dot{\neg}(\mathcal{D})\mathrm{n})\mathrm{n}$	;

L10: Repetition ▷ L09 ≫ $\exists \mathrm{V}_2 \colon \mathcal{D}$;

L11: Block ≫ End ;

L12: Arbitrary ≫ $\mathcal{Y}, \mathrm{V}_1, \mathrm{V}_2, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$;

L13: Ded ▷ L11 ≫ $\langle \dot{\neg}(\mathcal{B})\mathrm{n} \equiv \dot{\neg}(\mathcal{D})\mathrm{n} | \mathrm{V}_2 {:==} \mathcal{Y} \rangle_{\mathrm{Me}} \Vdash (\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow (\mathcal{C} \Rightarrow \mathcal{A}) \Rightarrow \exists \mathrm{V}_1 \colon \mathcal{C} \Rightarrow \exists \mathrm{V}_2 \colon \mathcal{D}$ □

[SystemQ **lemma** AddExist: $\Pi \mathcal{Y}, \mathrm{V}_1, \mathrm{V}_2, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \colon \langle \dot{\neg}(\mathcal{B})\mathrm{n} \equiv \dot{\neg}(\mathcal{D})\mathrm{n} | \mathrm{V}_2 {:==} \mathcal{Y} \rangle$ $\mathcal{A} \Rightarrow \mathcal{B} \vdash \mathcal{C} \Rightarrow \mathcal{A} \vdash \exists \mathrm{V}_1 \colon \mathcal{C} \Rightarrow \exists \mathrm{V}_2 \colon \mathcal{D}$]

SystemQ **proof of** AddExist:

L01: Arbitrary ≫ $\mathcal{Y}, \mathrm{V}_1, \mathrm{V}_2, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$;

L02: Side-condition ≫ $\langle \dot{\neg}(\mathcal{B})\mathrm{n} \equiv \dot{\neg}(\mathcal{D})\mathrm{n} | \mathrm{V}_2 {:==} \mathcal{Y} \rangle_{\mathrm{Me}}$;

L03: Premise ≫ $\mathcal{A} \Rightarrow \mathcal{B}$;

L04: Premise ≫ $\mathcal{C} \Rightarrow \mathcal{A}$;

L05: AddExist(Helper2) ⧐ L02 ≫ $(\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow (\mathcal{C} \Rightarrow \mathcal{A}) \Rightarrow \exists \mathrm{V}_1 \colon \mathcal{C} \Rightarrow \exists \mathrm{V}_2 \colon \mathcal{D}$;

L06: MP2 ▷ L05 ▷ L03 ▷ L04 ≫ $\exists \mathrm{V}_1 \colon \mathcal{C} \Rightarrow \exists \mathrm{V}_2 \colon \mathcal{D}$ □

[SystemQ **lemma** AddExist(SimpleAnt): $\Pi \mathcal{Y}, \mathrm{V}_1, \mathrm{V}_2, \mathcal{A}, \mathcal{B}, \mathcal{D} \colon \langle \dot{\neg}(\mathcal{B})\mathrm{n} \equiv \dot{\neg}(\mathcal{D})\mathrm{n} |$ $\mathcal{A} \Rightarrow \mathcal{B} \vdash \exists \mathrm{V}_1 \colon \mathcal{A} \Rightarrow \exists \mathrm{V}_2 \colon \mathcal{D}$]

SystemQ **proof of** AddExist(SimpleAnt): $\Pi \mathcal{Y}, \mathrm{V}_1, \mathrm{V}_2, \mathcal{A}, \mathcal{B}, \mathcal{D}$:

L01: Side-condition ≫ $\langle \dot{\neg}(\mathcal{B})\mathrm{n} \equiv \dot{\neg}(\mathcal{D})\mathrm{n} | \mathrm{V}_2 {:==} \mathcal{Y} \rangle_{\mathrm{Me}}$;

L02: Premise ≫ $\mathcal{A} \Rightarrow \mathcal{B}$;

L03: AutoImply ≫ $\mathcal{A} \Rightarrow \mathcal{A}$;

L04: AddExist @ $\mathcal{Y}$ ⧐ L01 ▷ L02 ▷ L03 ≫ $\exists \mathrm{V}_1 \colon \mathcal{A} \Rightarrow \exists \mathrm{V}_2 \colon \mathcal{D}$ □

[SystemQ **lemma** AddExist(Simple): $\Pi \mathrm{V}_1, \mathrm{V}_2, \mathcal{A}, \mathcal{B} \colon \mathcal{A} \Rightarrow \mathcal{B} \vdash \exists \mathrm{V}_1 \colon \mathcal{A} \Rightarrow \exists \mathrm{V}_2 \colon \mathcal{B}$]

SystemQ **proof of** AddExist(Simple):

L01: Arbitrary ≫ $\mathrm{V}_1, \mathrm{V}_2, \mathcal{A}, \mathcal{B}$;

L02: Premise ≫ $\mathcal{A} \Rightarrow \mathcal{B}$;

L03: AutoImply ≫ $\mathcal{A} \Rightarrow \mathcal{A}$;

L04: AddExist @ V_2 ▷ L02 ▷ L03 ≫ $\exists \mathrm{V}_1 \colon \mathcal{A} \Rightarrow \exists \mathrm{V}_2 \colon \mathcal{B}$ □

[SystemQ **lemma** AEA − negated: $\Pi \mathrm{V}_1, \mathrm{V}_2, \mathrm{V}_3, \mathcal{A} \colon \dot{\neg}(\forall \mathrm{V}_1 \colon \exists \mathrm{V}_2 \colon \forall \mathrm{V}_3 \colon \mathcal{A})\mathrm{n} \vdash \exists \mathrm{V}_1 \colon \forall \mathrm{V}_2 \colon \exists \mathrm{V}_3 \colon \dot{\neg}(\mathcal{A})\mathrm{n}$]

SystemQ **proof of** AEA − negated:

L01: Arbitrary ≫ $\mathrm{V}_1, \mathrm{V}_2, \mathrm{V}_3, \mathcal{A}$;

L02: Premise ≫ $\dot{\neg}(\forall \mathrm{V}_1 \colon \exists \mathrm{V}_2 \colon \forall \mathrm{V}_3 \colon \mathcal{A})\mathrm{n}$;

L03: AllNegated(Imply) ≫ $\dot{\neg}(\forall \mathrm{V}_3 \colon \mathcal{A})\mathrm{n} \Rightarrow \exists \mathrm{V}_3 \colon \dot{\neg}(\mathcal{A})\mathrm{n}$;

L04: AddAll ▷ L03 ≫ $\forall \mathrm{V}_2 \colon \dot{\neg}(\forall \mathrm{V}_3 \colon \mathcal{A})\mathrm{n} \Rightarrow \forall \mathrm{V}_2 \colon \exists \mathrm{V}_3 \colon \dot{\neg}(\mathcal{A})\mathrm{n}$;

L05: ExistNegated(Imply) ≫ $\dot{\neg}(\exists \mathrm{V}_2 \colon \forall \mathrm{V}_3 \colon \mathcal{A})\mathrm{n} \Rightarrow \forall \mathrm{V}_2 \colon \dot{\neg}(\forall \mathrm{V}_3 \colon \mathcal{A})\mathrm{n}$;

L06: ImplyTransitivity▷L05▷L04 ≫ $\dot{\neg}(\exists \mathrm{V}_2 \colon \forall \mathrm{V}_3 \colon \mathcal{A})\mathrm{n} \Rightarrow \forall \mathrm{V}_2 \colon \exists \mathrm{V}_3 \colon \dot{\neg}(\mathcal{A})\mathrm{n}$;

L07: AddExist(Simple) ▷ L06 ≫ $\exists \mathrm{V}_1 \colon \dot{\neg}(\exists \mathrm{V}_2 \colon \forall \mathrm{V}_3 \colon \mathcal{A})\mathrm{n} \Rightarrow \exists \mathrm{V}_1 \colon \forall \mathrm{V}_2 \colon \exists \mathrm{V}_3 \colon \dot{\neg}(\mathcal{A})\mathrm{n}$;

L08: AllNegated(Imply) $\gg$ $\dot{\neg}(\forall V_1 : \exists V_2 : \forall V_3 : \mathcal{A})\mathrm{n} \Rightarrow$
$\exists V_1 : \dot{\neg}(\exists V_2 : \forall V_3 : \mathcal{A})\mathrm{n}$;

L09: ImplyTransitivity$\rhd$L08$\rhd$L07 $\gg$
$\dot{\neg}(\forall V_1 : \exists V_2 : \forall V_3 : \mathcal{A})\mathrm{n} \Rightarrow$
$\exists V_1 : \forall V_2 : \exists V_3 : \dot{\neg}(\mathcal{A})\mathrm{n}$;

L10: MP $\rhd$ L09 $\rhd$ L02 $\gg$ $\exists V_1 : \forall V_2 : \exists V_3 : \dot{\neg}(\mathcal{A})\mathrm{n}$ $\square$

[SystemQ **lemma** AddEAE: $\Pi V_1, V_2, V_3, \mathcal{A}, \mathcal{B} : \mathcal{A} \Rightarrow \mathcal{B} \vdash \exists V_1 : \forall V_2 : \exists V_3 : \mathcal{A} \Rightarrow \exists V_1 : \forall V_2 : \exists V_3 : \mathcal{B}$]

SystemQ **proof of** AddEAE:

L01: Arbitrary $\gg$ $V_1, V_2, V_3, \mathcal{A}, \mathcal{B}$;

L02: Premise $\gg$ $\mathcal{A} \Rightarrow \mathcal{B}$;

L03: AddExist(Simple) $\rhd$ L02 $\gg$ $\exists V_3 : \mathcal{A} \Rightarrow \exists V_3 : \mathcal{B}$;

L04: AddAll $\rhd$ L03 $\gg$ $\forall V_2 : \exists V_3 : \mathcal{A} \Rightarrow \forall V_2 : \exists V_3 : \mathcal{B}$;

L05: AddExist(Simple) $\rhd$ L04 $\gg$ $\exists V_1 : \forall V_2 : \exists V_3 : \mathcal{A} \Rightarrow$
$\exists V_1 : \forall V_2 : \exists V_3 : \mathcal{B}$ $\square$

[SystemQ **lemma** EAE−MP: $\Pi V_1, V_2, V_3, \mathcal{A}, \mathcal{B} : \mathcal{A} \Rightarrow \mathcal{B} \vdash \exists V_1 : \forall V_2 : \exists V_3 : \mathcal{A} \vdash \mathcal{B}$]

SystemQ **proof of** EAE − MP:

L01: Block $\gg$ Begin ;

L02: Arbitrary $\gg$ $V_2, V_3, \mathcal{A}, \mathcal{B}$;

L03: Premise $\gg$ $\mathcal{A} \Rightarrow \mathcal{B}$;

L04: Premise $\gg$ $\forall V_2 : \exists V_3 : \mathcal{A}$;

L05: A4 @ V_2 $\rhd$ L04 $\gg$ $\exists V_3 : \mathcal{A}$;

L06: ExistMP $\rhd$ L03 $\rhd$ L05 $\gg$ $\mathcal{B}$;

L07: Block $\gg$ End ;

L08: Arbitrary $\gg$ $V_1, V_2, V_3, \mathcal{A}, \mathcal{B}$;

L03: Ded $\rhd$ L07 $\gg$ $(\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow \forall V_2 : \exists V_3 : \mathcal{A} \Rightarrow \mathcal{B}$;

L04: Premise $\gg$ $\mathcal{A} \Rightarrow \mathcal{B}$;

L05: Premise $\gg$ $\exists V_1 : \forall V_2 : \exists V_3 : \mathcal{A}$;

L09: MP $\rhd$ L03 $\rhd$ L04 $\gg$ $\forall V_2 : \exists V_3 : \mathcal{A} \Rightarrow \mathcal{B}$;

L10: ExistMP $\rhd$ L09 $\rhd$ L05 $\gg$ $\mathcal{B}$ $\square$

[SystemQ **lemma** EEA − negated: $\Pi V_1, V_2, V_3, \mathcal{A} : \dot{\neg}(\exists V_1 : \exists V_2 : \forall V_3 : \mathcal{A})\mathrm{n} \vdash \forall V_1 : \forall V_2 : \exists V_3 : \dot{\neg}(\mathcal{A})\mathrm{n}$]

SystemQ **proof of** EEA − negated:

L01: Arbitrary $\gg$ $V_1, V_2, V_3, \mathcal{A}$;

L02: Premise $\gg$ $\dot{\neg}(\exists V_1 : \exists V_2 : \forall V_3 : \mathcal{A})\mathrm{n}$;

L03: AllNegated(Imply) $\gg$ $\dot{\neg}(\forall V_3 : \mathcal{A})\mathrm{n} \Rightarrow \exists V_3 : \dot{\neg}(\mathcal{A})\mathrm{n}$;

L04: AddAll $\rhd$ L03 $\gg$ $\forall V_2 : \dot{\neg}(\forall V_3 : \mathcal{A})\mathrm{n} \Rightarrow$
$\forall V_2 : \exists V_3 : \dot{\neg}(\mathcal{A})\mathrm{n}$;

L05: ExistNegated(Imply) $\gg$ $\dot{\neg}(\exists V_2 : \forall V_3 : \mathcal{A})\mathrm{n} \Rightarrow$
$\forall V_2 : \dot{\neg}(\forall V_3 : \mathcal{A})\mathrm{n}$;

L06: ImplyTransitivity$\rhd$L05$\rhd$L04 $\gg$
$\dot{\neg}(\exists V_2 : \forall V_3 : \mathcal{A})\mathrm{n} \Rightarrow$
$\forall V_2 : \exists V_3 : \dot{\neg}(\mathcal{A})\mathrm{n}$;

L07: AddAll $\rhd$ L06 $\gg$ $\forall V_1 : \dot{\neg}(\exists V_2 : \forall V_3 : \mathcal{A})\mathrm{n} \Rightarrow$
$\forall V_1 : \forall V_2 : \exists V_3 : \dot{\neg}(\mathcal{A})\mathrm{n}$;

L08: ExistNegated(Imply) ≫ $\dot\neg(\exists V_1: \exists V_2: \forall V_3: \mathcal{A})n \Rightarrow \forall V_1: \dot\neg(\exists V_2: \forall V_3: \mathcal{A})n$;

L09: ImplyTransitivity▷L08▷L07 ≫ $\dot\neg(\exists V_1: \exists V_2: \forall V_3: \mathcal{A})n \Rightarrow \forall V_1: \forall V_2: \exists V_3: \dot\neg(\mathcal{A})n$;

L10: MP ▷ L09 ▷ L02 ≫ $\forall V_1: \forall V_2: \exists V_3: \dot\neg(\mathcal{A})n$ □

[SystemQ **lemma** leqTransitivity: $\Pi\mathcal{X}, \mathcal{Y}, \mathcal{Z}: \mathcal{X} <= \mathcal{Y} \vdash \mathcal{Y} <= \mathcal{Z} \vdash \mathcal{X} <= \mathcal{Z}$]

SystemQ **proof of** leqTransitivity:

L01: Arbitrary ≫ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$;

L02: Premise ≫ $\mathcal{X} <= \mathcal{Y}$;

L03: Premise ≫ $\mathcal{Y} <= \mathcal{Z}$;

L04: leqTransitivityAxiom ≫ $\mathcal{X} <= \mathcal{Y} \Rightarrow \mathcal{Y} <= \mathcal{Z} \Rightarrow \mathcal{X} <= \mathcal{Z}$;

L05: MP2 ▷ L04 ▷ L02 ▷ L03 ≫ $\mathcal{X} <= \mathcal{Z}$ □

[SystemQ **lemma** leqAntisymmetry: $\Pi\mathcal{X}, \mathcal{Y}: \mathcal{X} <= \mathcal{Y} \vdash \mathcal{Y} <= \mathcal{X} \vdash \mathcal{X} = \mathcal{Y}$]

SystemQ **proof of** leqAntisymmetry:

L01: Arbitrary ≫ $\mathcal{X}, \mathcal{Y}$;

L02: Premise ≫ $\mathcal{X} <= \mathcal{Y}$;

L03: Premise ≫ $\mathcal{Y} <= \mathcal{X}$;

L04: leqAntisymmetryAxiom ≫ $\mathcal{X} <= \mathcal{Y} \Rightarrow \mathcal{Y} <= \mathcal{X} \Rightarrow \mathcal{X} = \mathcal{Y}$;

L05: MP2 ▷ L04 ▷ L02 ▷ L03 ≫ $\mathcal{X} = \mathcal{Y}$ □

[SystemQ **lemma** leqAddition: $\Pi\mathcal{X}, \mathcal{Y}, \mathcal{Z}: \mathcal{X} <= \mathcal{Y} \vdash (\mathcal{X} + \mathcal{Z}) <= (\mathcal{Y} + \mathcal{Z})$]

SystemQ **proof of** leqAddition:

L01: Arbitrary ≫ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$;

L02: Premise ≫ $\mathcal{X} <= \mathcal{Y}$;

L03: leqAdditionAxiom ≫ $\mathcal{X} <= \mathcal{Y} \Rightarrow (\mathcal{X}+\mathcal{Z}) <= (\mathcal{Y}+\mathcal{Z})$;

L04: MP ▷ L03 ▷ L02 ≫ $(\mathcal{X} + \mathcal{Z}) <= (\mathcal{Y} + \mathcal{Z})$ □

[SystemQ **lemma** leqMultiplication: $\Pi\mathcal{X}, \mathcal{Y}, \mathcal{Z}: 0 <= \mathcal{Z} \vdash \mathcal{X} <= \mathcal{Y} \vdash (\mathcal{X} * \mathcal{Z}) <= (\mathcal{Y} * \mathcal{Z})$]

SystemQ **proof of** leqMultiplication:

L01: Arbitrary ≫ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$;

L02: Premise ≫ $0 <= \mathcal{Z}$;

L03: Premise ≫ $\mathcal{X} <= \mathcal{Y}$;

L04: leqMultiplicationAxiom ≫ $0 <= \mathcal{Z} \Rightarrow \mathcal{X} <= \mathcal{Y} \Rightarrow (\mathcal{X} * \mathcal{Z}) <= (\mathcal{Y} * \mathcal{Z})$;

L05: MP2 ▷ L04 ▷ L02 ▷ L03 ≫ $(\mathcal{X} * \mathcal{Z}) <= (\mathcal{Y} * \mathcal{Z})$ □

[SystemQ **lemma** Reciprocal: $\Pi\mathcal{X}: \mathcal{X} \neq 0 \vdash (\mathcal{X} * \mathrm{rec}\mathcal{X}) = 1$]

SystemQ **proof of** Reciprocal:

L01: Arbitrary ≫ $\mathcal{X}$;

L02: Premise ≫ $\mathcal{X} \neq 0$;

L03: ReciprocalAxiom ≫ $\mathcal{X} \neq 0 \Rightarrow (\mathcal{X} * \mathrm{rec}\mathcal{X}) = 1$;

L04: MP ▷ L03 ▷ L02 ≫ $(\mathcal{X} * \mathrm{rec}\mathcal{X}) = 1$ □

[SystemQ **lemma** eqLeq: $\Pi\mathcal{X}, \mathcal{Y}: \mathcal{X} = \mathcal{Y} \vdash \mathcal{X} <= \mathcal{Y}$]

SystemQ **proof of** eqLeq:

L01: Arbitrary ≫ $\mathcal{X}, \mathcal{Y}$;

L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$;
L03: EqLeqAxiom $\gg$ $\mathcal{X} = \mathcal{Y} \Rightarrow \mathcal{X} <= \mathcal{Y}$;
L04: MP $\rhd$ L03 $\rhd$ L02 $\gg$ $\mathcal{X} <= \mathcal{Y}$ $\square$

[SystemQ **lemma** eqAddition: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}$: $\mathcal{X} = \mathcal{Y} \vdash (\mathcal{X} + \mathcal{Z}) = (\mathcal{Y} + \mathcal{Z})$]

SystemQ **proof of** eqAddition:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$;
L03: EqAdditionAxiom $\gg$ $\mathcal{X} = \mathcal{Y} \Rightarrow (\mathcal{X} + \mathcal{Z}) = (\mathcal{Y} + \mathcal{Z})$;
L04: MP $\rhd$ L03 $\rhd$ L02 $\gg$ $(\mathcal{X} + \mathcal{Z}) = (\mathcal{Y} + \mathcal{Z})$ $\square$

[SystemQ **lemma** eqMultiplication: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}$: $\mathcal{X} = \mathcal{Y} \vdash (\mathcal{X} * \mathcal{Z}) = (\mathcal{Y} * \mathcal{Z})$]

SystemQ **proof of** eqMultiplication:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$;
L03: EqMultiplicationAxiom $\gg$ $\mathcal{X} = \mathcal{Y} \Rightarrow (\mathcal{X} * \mathcal{Z}) = (\mathcal{Y} * \mathcal{Z})$;
L04: MP $\rhd$ L03 $\rhd$ L02 $\gg$ $(\mathcal{X} * \mathcal{Z}) = (\mathcal{Y} * \mathcal{Z})$ $\square$

[SystemQ **lemma** Equality: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}$: $\mathcal{X} = \mathcal{Y} \vdash \mathcal{X} = \mathcal{Z} \vdash \mathcal{Y} = \mathcal{Z}$]

SystemQ **proof of** Equality:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$;
L03: Premise $\gg$ $\mathcal{X} = \mathcal{Z}$;
L04: EqualityAxiom $\gg$ $\mathcal{X} = \mathcal{Y} \Rightarrow \mathcal{X} = \mathcal{Z} \Rightarrow \mathcal{Y} = \mathcal{Z}$;
L05: MP2 $\rhd$ L04 $\rhd$ L02 $\rhd$ L03 $\gg$ $\mathcal{Y} = \mathcal{Z}$ $\square$

[SystemQ **lemma** eqReflexivity: $\Pi \mathcal{X}$: $\mathcal{X} = \mathcal{X}$]

SystemQ **proof of** eqReflexivity:

L01: Arbitrary $\gg$ $\mathcal{X}$;
L02: leqReflexivity $\gg$ $\mathcal{X} <= \mathcal{X}$;
L03: leqAntisymmetry $\rhd$ L02 $\rhd$ L02 $\gg$
$\mathcal{X} = \mathcal{X}$ $\square$

[SystemQ **lemma** eqSymmetry: $\Pi \mathcal{X}, \mathcal{Y}$: $\mathcal{X} = \mathcal{Y} \vdash \mathcal{Y} = \mathcal{X}$]

SystemQ **proof of** eqSymmetry:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}$;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$;
L03: eqReflexivity $\gg$ $\mathcal{X} = \mathcal{X}$;
L04: Equality $\rhd$ L02 $\rhd$ L03 $\gg$ $\mathcal{Y} = \mathcal{X}$ $\square$

[SystemQ **lemma** eqTransitivity: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}$: $\mathcal{X} = \mathcal{Y} \vdash \mathcal{Y} = \mathcal{Z} \vdash \mathcal{X} = \mathcal{Z}$]

SystemQ **proof of** eqTransitivity:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$;
L03: Premise $\gg$ $\mathcal{Y} = \mathcal{Z}$;
L04: eqSymmetry $\rhd$ L02 $\gg$ $\mathcal{Y} = \mathcal{X}$;
L05: Equality $\rhd$ L04 $\rhd$ L03 $\gg$ $\mathcal{X} = \mathcal{Z}$ $\square$

[SystemQ **lemma** eqTransitivity4: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}$: $\mathcal{X} = \mathcal{Y} \vdash \mathcal{Y} = \mathcal{Z} \vdash \mathcal{Z} = \mathcal{U} \vdash \mathcal{X} = \mathcal{U}$]

SystemQ **proof of** eqTransitivity4:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}$;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$;

L03: Premise $\gg$ $\mathcal{Y} = \mathcal{Z}$;
L04: Premise $\gg$ $\mathcal{Z} = \mathcal{U}$;
L05: eqTransitivity $\rhd$ L02 $\rhd$ L03 $\gg$ $\mathcal{X} = \mathcal{Z}$;
L06: eqTransitivity $\rhd$ L05 $\rhd$ L04 $\gg$ $\mathcal{X} = \mathcal{U}$ $\square$

[SystemQ **lemma** eqTransitivity5: $\Pi\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}, \mathcal{V}$: $\mathcal{X} = \mathcal{Y} \vdash \mathcal{Y} = \mathcal{Z} \vdash \mathcal{Z} = \mathcal{U} \vdash \mathcal{U} = \mathcal{V} \vdash \mathcal{X} = \mathcal{V}$]

SystemQ **proof of** eqTransitivity5:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}, \mathcal{V}$;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$;
L03: Premise $\gg$ $\mathcal{Y} = \mathcal{Z}$;
L04: Premise $\gg$ $\mathcal{Z} = \mathcal{U}$;
L05: Premise $\gg$ $\mathcal{U} = \mathcal{V}$;
L06: eqTransitivity4 $\rhd$ L02 $\rhd$ L03 $\rhd$ L04 $\gg$ $\mathcal{X} = \mathcal{U}$;
L07: eqTransitivity $\rhd$ L06 $\rhd$ L05 $\gg$ $\mathcal{X} = \mathcal{V}$ $\square$

[SystemQ **lemma** eqTransitivity6: $\Pi\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}, \mathcal{V}, \mathcal{W}$: $\mathcal{X} = \mathcal{Y} \vdash \mathcal{Y} = \mathcal{Z} \vdash \mathcal{Z} = \mathcal{U} \vdash \mathcal{U} = \mathcal{V} \vdash \mathcal{V} = \mathcal{W} \vdash \mathcal{X} = \mathcal{W}$]

SystemQ **proof of** eqTransitivity6:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}, \mathcal{V}, \mathcal{W}$;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$;
L03: Premise $\gg$ $\mathcal{Y} = \mathcal{Z}$;
L04: Premise $\gg$ $\mathcal{Z} = \mathcal{U}$;
L05: Premise $\gg$ $\mathcal{U} = \mathcal{V}$;
L06: Premise $\gg$ $\mathcal{V} = \mathcal{W}$;
L07: eqTransitivity5 $\rhd$ L02 $\rhd$ L03 $\rhd$ L04 $\rhd$ L05 $\gg$ $\mathcal{X} = \mathcal{V}$;
L08: eqTransitivity $\rhd$ L07 $\rhd$ L06 $\gg$ $\mathcal{X} = \mathcal{W}$ $\square$

[SystemQ **lemma** EqAdditionLeft: $\Pi\mathcal{X}, \mathcal{Y}, \mathcal{Z}$: $\mathcal{X} = \mathcal{Y} \vdash (\mathcal{Z} + \mathcal{X}) = (\mathcal{Z} + \mathcal{Y})$]

SystemQ **proof of** EqAdditionLeft:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$;
L03: eqAddition $\rhd$ L02 $\gg$ $(\mathcal{X} + \mathcal{Z}) = (\mathcal{Y} + \mathcal{Z})$;
L04: plusCommutativity $\gg$ $(\mathcal{Z} + \mathcal{X}) = (\mathcal{X} + \mathcal{Z})$;
L05: plusCommutativity $\gg$ $(\mathcal{Y} + \mathcal{Z}) = (\mathcal{Z} + \mathcal{Y})$;
L06: eqTransitivity4 $\rhd$ L04 $\rhd$ L03 $\rhd$ L05 $\gg$ $(\mathcal{Z} + \mathcal{X}) = (\mathcal{Z} + \mathcal{Y})$ $\square$

[SystemQ **lemma** EqMultiplicationLeft: $\Pi\mathcal{X}, \mathcal{Y}, \mathcal{Z}$: $\mathcal{X} = \mathcal{Y} \vdash (\mathcal{Z} * \mathcal{X}) = (\mathcal{Z} * \mathcal{Y})$]

SystemQ **proof of** EqMultiplicationLeft:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$;
L03: eqMultiplication $\rhd$ L02 $\gg$ $(\mathcal{X} * \mathcal{Z}) = (\mathcal{Y} * \mathcal{Z})$;
L04: timesCommutativity $\gg$ $(\mathcal{Z} * \mathcal{X}) = (\mathcal{X} * \mathcal{Z})$;
L05: timesCommutativity $\gg$ $(\mathcal{Y} * \mathcal{Z}) = (\mathcal{Z} * \mathcal{Y})$;
L06: eqTransitivity4 $\rhd$ L04 $\rhd$ L03 $\rhd$ L05 $\gg$ $(\mathcal{Z} * \mathcal{X}) = (\mathcal{Z} * \mathcal{Y})$ $\square$

[SystemQ **lemma** PlusF(Sym): $\Pi\mathcal{M}, \mathrm{FX}, \mathrm{FY}\colon (\mathrm{FX}[\mathcal{M}] + \mathrm{FY}[\mathcal{M}]) = \mathrm{FX} +_{\mathrm{f}} \mathrm{FY}[\mathcal{M}]$]

SystemQ **proof of** PlusF(Sym):

L01:	Arbitrary $\gg$	$\mathcal{M}, \mathrm{FX}, \mathrm{FY}$	;
L02:	PlusF $\gg$	$\mathrm{FX} +_{\mathrm{f}} \mathrm{FY}[\mathcal{M}] = (\mathrm{FX}[\mathcal{M}] + \mathrm{FY}[\mathcal{M}])$	;
L03:	eqSymmetry $\rhd$ L02 $\gg$	$(\mathrm{FX}[\mathcal{M}] + \mathrm{FY}[\mathcal{M}]) = \mathrm{FX} +_{\mathrm{f}} \mathrm{FY}[\mathcal{M}]$	$\square$

[SystemQ **lemma** plus0Left: $\Pi\mathcal{X}\colon (0 + \mathcal{X}) = \mathcal{X}$]

SystemQ **proof of** plus0Left:

L01:	Arbitrary $\gg$	$\mathcal{X}$	;
L02:	plus0 $\gg$	$(\mathcal{X} + 0) = \mathcal{X}$	;
L03:	plusCommutativity $\gg$	$(0 + \mathcal{X}) = (\mathcal{X} + 0)$	;
L04:	eqTransitivity $\rhd$ L03 $\rhd$ L02 $\gg$	$(0 + \mathcal{X}) = \mathcal{X}$	$\square$

[SystemQ **lemma** times1Left: $\Pi\mathcal{X}\colon (1 * \mathcal{X}) = \mathcal{X}$]

SystemQ **proof of** times1Left:

L01:	Arbitrary $\gg$	$\mathcal{X}$	;
L02:	times1 $\gg$	$(\mathcal{X} * 1) = \mathcal{X}$	;
L03:	timesCommutativity $\gg$	$(1 * \mathcal{X}) = (\mathcal{X} * 1)$	;
L04:	eqTransitivity $\rhd$ L03 $\rhd$ L02 $\gg$	$(1 * \mathcal{X}) = \mathcal{X}$	$\square$

[SystemQ **lemma** Induction: $\Pi\mathrm{V}_1, \mathcal{A}, \mathcal{B}, \mathcal{C}\colon \langle\mathcal{B}{\equiv}\mathcal{A}|\mathrm{V}_1{:==}0\rangle_{\mathrm{Me}} \Vdash \langle\mathcal{C}{\equiv}\mathcal{A}|\mathrm{V}_1{:==}($
$1)\rangle_{\mathrm{Me}} \Vdash \mathcal{B} \vdash \mathcal{A} \Rightarrow \mathcal{C} \vdash \mathcal{A}$]

SystemQ **proof of** Induction:

L01:	Arbitrary $\gg$	$\mathrm{V}_1, \mathcal{A}, \mathcal{B}, \mathcal{C}$	;
L02:	Side-condition $\gg$	$\langle\mathcal{B}{\equiv}\mathcal{A}|\mathrm{V}_1{:==}0\rangle_{\mathrm{Me}}$	;
L03:	Side-condition $\gg$	$\langle\mathcal{C}{\equiv}\mathcal{A}|\mathrm{V}_1{:==}(\mathrm{V}_1 + 1)\rangle_{\mathrm{Me}}$	;
L04:	Premise $\gg$	$\mathcal{B}$	;
L05:	Premise $\gg$	$\mathcal{A} \Rightarrow \mathcal{C}$	;
L06:	Gen $\rhd$ L05 $\gg$	$\forall\mathrm{V}_1\colon (\mathcal{A} \Rightarrow \mathcal{C})$	;
L07:	InductionAxiom $\unrhd$ L02 $\unrhd$ L03 $\gg$	$\mathcal{B} \Rightarrow \forall\mathrm{V}_1\colon (\mathcal{A} \Rightarrow \mathcal{C}) \Rightarrow \forall\mathrm{V}_1\colon \mathcal{A}$	;
L08:	MP2 $\rhd$ L07 $\rhd$ L04 $\rhd$ L06 $\gg$	$\forall\mathrm{V}_1\colon \mathcal{A}$	;
L09:	A4 @ V_1 $\rhd$ L08 $\gg$	$\mathcal{A}$	$\square$

[SystemQ **lemma** ToSeries: $\Pi\mathrm{FX}, (\mathrm{SY})\colon \forall(\mathrm{R1ob})\colon ((\mathrm{R1ob}) \in \mathrm{FX} \Rightarrow \mathrm{IsOrderedPa}$
$\forall(\mathrm{F1ob}), (\mathrm{F2ob}), (\mathrm{F3ob}), (\mathrm{F4ob})\colon (\mathrm{OrderedPair}((\mathrm{F1ob}), (\mathrm{F2ob})) \in \mathrm{FX} \Rightarrow \mathrm{OrderedPa}$
$\mathrm{FX} \Rightarrow (\mathrm{F1ob}) = (\mathrm{F3ob}) \Rightarrow (\mathrm{F2ob}) = (\mathrm{F4ob})) \vdash \forall(\mathrm{S1ob})\colon ((\mathrm{S1ob}) \in \mathrm{N} \Rightarrow$
$\exists(\mathrm{S2ob})\colon \mathrm{OrderedPair}((\mathrm{S1ob}), (\mathrm{S2ob})) \in \mathrm{FX}) \vdash \mathrm{IsSeries}(\mathrm{FX}, (\mathrm{SY}))$]

SystemQ **proof of** ToSeries:

L01:	Arbitrary $\gg$	$\mathrm{FX}, (\mathrm{SY})$	;
L02:	Premise $\gg$	$\forall(\mathrm{R1ob})\colon ((\mathrm{R1ob}) \in \mathrm{FX} \Rightarrow \mathrm{IsOrderedPair}((\mathrm{R1ob}), \mathrm{N}, (\mathrm{SY})))$;	

L03: Premise ≫ ∀(F1ob), (F2ob), (F3ob), (F4ob): (OrderedPair((F1ob), (F2ob)) ∈ FX ⇒ OrderedPair((F3ob), (F4ob)) ∈ FX ⇒ (F1ob) = (F3ob) ⇒ (F2ob) = (F4ob)) ;

L04: Premise ≫ ∀(S1ob): ((S1ob) ∈ N ⇒ ∃(S2ob): OrderedPair((S1ob), (S2ob)) ∈ FX) ;

L05: Repetition ▷ L02 ≫ IsRelation(FX, N, (SY)) ;

L06: JoinConjuncts ▷ L05 ▷ L03 ≫ IsRelation(FX, N, (SY)) ∧̇ ∀(F1ob), (F2ob), (F3ob), (F4ob): (OrderedPair((F1ob), (F2ob)) ∈ FX ⇒ OrderedPair((F3ob), (F4ob)) ∈ FX ⇒ (F1ob) = (F3ob) ⇒ (F2ob) = (F4ob)) ;

L07: Repetition ▷ L06 ≫ isFunction(FX, N, (SY)) ;

L08: JoinConjuncts ▷ L07 ▷ L04 ≫ isFunction(FX, N, (SY)) ∧̇ ∀(S1ob): ((S1ob) ∈ N ⇒ ∃(S2ob): OrderedPair((S1ob), (S2ob)) ∈ FX) ;

L09: Repetition ▷ L08 ≫ IsSeries(FX, (SY)) □

[SystemQ **lemma** FromSeries: ΠFX, (SY): IsSeries(FX, (SY)) ⊢ (∀(R1ob): ((R1
FX ⇒ ∃(OP1ob): ∃(OP2ob): (OP1ob) ∈ N ∧̇ (OP2ob) ∈ (SY) ∧̇ (R1ob) = OrderedPair((OP1ob), (OP2ob)))) ∧̇ (∀(F1ob), (F2ob), (F3ob), (F4ob): (OrderedPa
FX ⇒ OrderedPair((F3ob), (F4ob)) ∈ FX ⇒ (F1ob) = (F3ob) ⇒ (F2ob) = (F4ob))) ∧̇ ∀(S1ob): ((S1ob) ∈ N ⇒ ∃(S2ob): OrderedPair((S1ob), (S2ob)) ∈ FX)]

SystemQ **proof of** FromSeries:

L01: Arbitrary ≫ FX, (SY) ;

L02: Premise ≫ IsSeries(FX, (SY)) ;

L03: Repetition ▷ L02 ≫ isFunction(FX, N, (SY)) ∧̇ ∀(S1ob): ((S1ob) ∈ N ⇒ ∃(S2ob): OrderedPair((S1ob), (S2ob)) ∈ FX) ;

L04: Repetition ▷ L03 ≫ IsRelation(FX, N, (SY)) ∧̇ (∀(F1ob), (F2ob), (F3ob), (F4ob): (OrderedPair((F1ob), (F2ob)) ∈ FX ⇒ OrderedPair((F3ob), (F4ob)) ∈ FX ⇒ (F1ob) = (F3ob) ⇒ (F2ob) = (F4ob))) ∧̇ ∀(S1ob): ((S1ob) ∈ N ⇒ ∃(S2ob): OrderedPair((S1ob), (S2ob)) ∈ FX) ;

L05: Repetition $\rhd$ L04 $\gg$ $(\forall(\text{R1ob})\colon((\text{R1ob}) \in \text{FX} \Rightarrow \text{IsOrderedPair}((\text{R1ob}), \text{N}, (\text{SY})))) \dot\wedge (\forall(\text{F1ob}), (\text{F2ob}), (\text{F3ob}), (\text{F4ob})\colon(\text{OrderedPair}((\text{F1ob}), (\text{F2ob})) \in \text{FX} \Rightarrow \text{OrderedPair}((\text{F3ob}), (\text{F4ob})) \in \text{FX} \Rightarrow (\text{F1ob}) = (\text{F3ob}) \Rightarrow (\text{F2ob}) = (\text{F4ob}))) \dot\wedge \forall(\text{S1ob})\colon((\text{S1ob}) \in \text{N} \Rightarrow \exists(\text{S2ob})\colon\text{OrderedPair}((\text{S1ob}), (\text{S2ob})) \in \text{FX})$;

L06: Repetition $\rhd$ L05 $\gg$ $(\forall(\text{R1ob})\colon((\text{R1ob}) \in \text{FX} \Rightarrow \exists(\text{OP1ob})\colon\exists(\text{OP2ob})\colon(\text{OP1ob}) \in \text{N} \dot\wedge (\text{OP2ob}) \in (\text{SY}) \dot\wedge (\text{R1ob}) = \text{OrderedPair}((\text{OP1ob}), (\text{OP2ob})))) \dot\wedge (\forall(\text{F1ob}), (\text{F2ob}), (\text{F3ob}), (\text{F4ob})\colon(\text{OrderedPair}((\text{F1ob}), (\text{F2ob})) \in \text{FX} \Rightarrow \text{OrderedPair}((\text{F3ob}), (\text{F4ob})) \in \text{FX} \Rightarrow (\text{F1ob}) = (\text{F3ob}) \Rightarrow (\text{F2ob}) = (\text{F4ob}))) \dot\wedge \forall(\text{S1ob})\colon((\text{S1ob}) \in \text{N} \Rightarrow \exists(\text{S2ob})\colon\text{OrderedPair}((\text{S1ob}), (\text{S2ob})) \in \text{FX})$ □

[SystemQ **lemma** NeqSymmetry: $\Pi\mathcal{X}, \mathcal{Y}\colon \mathcal{X} \neq \mathcal{Y} \vdash \mathcal{Y} \neq \mathcal{X}$]

SystemQ **proof of** NeqSymmetry:

L01: Block $\gg$ Begin ;
L02: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}$;
L03: Premise $\gg$ $\mathcal{Y} = \mathcal{X}$;
L04: eqSymmetry $\rhd$ L03 $\gg$ $\mathcal{X} = \mathcal{Y}$;
L05: Block $\gg$ End ;
L06: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}$;
L07: Ded $\rhd$ L05 $\gg$ $\mathcal{Y} = \mathcal{X} \Rightarrow \mathcal{X} = \mathcal{Y}$;
L08: Premise $\gg$ $\mathcal{X} \neq \mathcal{Y}$;
L09: MT $\rhd$ L07 $\rhd$ L08 $\gg$ $\mathcal{Y} \neq \mathcal{X}$ □

[SystemQ **lemma** PositiveNonzero: $\Pi\mathcal{X}\colon 0 < \mathcal{X} \vdash \mathcal{X} \neq 0$]

SystemQ **proof of** PositiveNonzero:

L01: Arbitrary $\gg$ $\mathcal{X}$;
L02: Premise $\gg$ $0 < \mathcal{X}$;
L03: Repetition $\rhd$ L02 $\gg$ $0 <= \mathcal{X} \dot\wedge 0 \neq \mathcal{X}$;
L04: SecondConjunct $\rhd$ L03 $\gg$ $0 \neq \mathcal{X}$;
L05: NeqSymmetry $\rhd$ L04 $\gg$ $\mathcal{X} \neq 0$ □

[SystemQ **lemma** SubNeqLeft: $\Pi\mathcal{X}, \mathcal{Y}, \mathcal{Z}\colon \mathcal{X} = \mathcal{Y} \vdash \mathcal{X} \neq \mathcal{Z} \vdash \mathcal{Y} \neq \mathcal{Z}$]

SystemQ **proof of** SubNeqLeft:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$;

L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$;
L03: Premise $\gg$ $\mathcal{X} \neq \mathcal{Z}$;
L04: EqualityAxiom $\gg$ $\mathcal{Y} = \mathcal{X} \Rightarrow \mathcal{Y} = \mathcal{Z} \Rightarrow \mathcal{X} = \mathcal{Z}$;
L05: eqSymmetry $\rhd$ L02 $\gg$ $\mathcal{Y} = \mathcal{X}$;
L06: MP $\rhd$ L04 $\rhd$ L05 $\gg$ $\mathcal{Y} = \mathcal{Z} \Rightarrow \mathcal{X} = \mathcal{Z}$;
L07: Contrapositive $\rhd$ L06 $\gg$ $\mathcal{X} \neq \mathcal{Z} \Rightarrow \mathcal{Y} \neq \mathcal{Z}$;
L08: MP $\rhd$ L07 $\rhd$ L03 $\gg$ $\mathcal{Y} \neq \mathcal{Z}$ $\Box$

[SystemQ **lemma** InPair(1): Π(SX), (SY): (SX) $\in$ (p(SX), (SY))]

SystemQ **proof of** InPair(1):

L01: Arbitrary $\gg$ (SX), (SY) ;
L02: eqReflexivity $\gg$ (SX) = (SX) ;
L03: WeakenOr2 $\rhd$ L02 $\gg$ (SX) = (SX) $\dot{\vee}$ (SX) = (SY) ;
L04: Formula2Pair $\rhd$ L03 $\gg$ (SX) $\in$ (p(SX), (SY)) $\Box$

[SystemQ **lemma** InPair(2): Π(SX), (SY): (SY) $\in$ (p(SX), (SY))]

SystemQ **proof of** InPair(2):

L01: Arbitrary $\gg$ (SX), (SY) ;
L02: eqReflexivity $\gg$ (SY) = (SY) ;
L03: WeakenOr1 $\rhd$ L02 $\gg$ (SY) = (SX) $\dot{\vee}$ (SY) = (SY) ;
L04: Formula2Pair $\rhd$ L03 $\gg$ (SY) $\in$ (p(SX), (SY)) $\Box$

[SystemQ **lemma** FromSingleton: Π(SX), (SY): (SX) $\in$ (s(SY)) $\vdash$ (SX) = (SY)]

SystemQ **proof of** FromSingleton:

L01: Arbitrary $\gg$ (SX), (SY) ;
L02: Premise $\gg$ (SX) $\in$ (s(SY)) ;
L03: Repetition $\rhd$ L02 $\gg$ (SX) $\in$ (p(SY), (SY)) ;
L04: Pair2Formula $\rhd$ L03 $\gg$ (SX) = (SY) $\dot{\vee}$ (SX) = (SY) ;
L05: RemoveOr $\rhd$ L04 $\gg$ (SX) = (SY) $\Box$

[SystemQ **lemma** ToSingleton: Π(SX), (SY): (SX) = (SY) $\vdash$ (SX) $\in$ (s(SY))]

SystemQ **proof of** ToSingleton:

L01: Arbitrary $\gg$ (SX), (SY) ;
L02: Premise $\gg$ (SX) = (SY) ;
L03: WeakenOr1 $\rhd$ L02 $\gg$ (SX) = (SY) $\dot{\vee}$ (SX) = (SY) ;
L04: Formula2Pair $\rhd$ L03 $\gg$ (SX) $\in$ (p(SY), (SY)) ;
L05: Repetition $\rhd$ L04 $\gg$ (SX) $\in$ (s(SY)) $\Box$

[SystemQ **lemma** FromSameSingleton: Π(SX), (SY): (s(SX)) = (s(SY)) $\vdash$ (SX)
(SY)]

SystemQ **proof of** FromSameSingleton:

L01: Arbitrary $\gg$ (SX), (SY) ;
L02: Premise $\gg$ (s(SX)) = (s(SY)) ;
L03: eqReflexivity $\gg$ (SX) = (SX) ;
L04: ToSingleton $\rhd$ L03 $\gg$ (SX) $\in$ (s(SX)) ;
L05: SENC1 $\rhd$ L02 $\rhd$ L04 $\gg$ (SX) $\in$ (s(SY)) ;
L06: FromSingleton $\rhd$ L05 $\gg$ (SX) = (SY) $\Box$

[SystemQ **lemma** SingletonmembersEqual: Π(SX), (SY), (SZ): (p(SX), (SY)) = (s(SZ)) $\vdash$ (SX) = (SY)]

SystemQ **proof of** SingletonmembersEqual:

L01:	Arbitrary $\gg$	(SX), (SY), (SZ)	;
L02:	Premise $\gg$	$(\mathrm{p(SX)}, \mathrm{(SY)}) = (\mathrm{s(SZ)})$	;
L03:	InPair(1) $\gg$	$(\mathrm{SX}) \in (\mathrm{p(SX)}, \mathrm{(SY)})$	;
L04:	SENC1 $\rhd$ L02 $\rhd$ L03 $\gg$	$(\mathrm{SX}) \in (\mathrm{s(SZ)})$	;
L05:	FromSingleton $\rhd$ L04 $\gg$	$(\mathrm{SX}) = (\mathrm{SZ})$	;
L06:	InPair(2) $\gg$	$(\mathrm{SY}) \in (\mathrm{p(SX)}, \mathrm{(SY)})$	;
L07:	SENC1 $\rhd$ L02 $\rhd$ L06 $\gg$	$(\mathrm{SY}) \in (\mathrm{s(SZ)})$	;
L08:	FromSingleton $\rhd$ L07 $\gg$	$(\mathrm{SY}) = (\mathrm{SZ})$	;
L09:	eqSymmetry $\rhd$ L08 $\gg$	$(\mathrm{SZ}) = (\mathrm{SY})$	;
L10:	eqTransitivity $\rhd$ L05 $\rhd$ L09 $\gg$	$(\mathrm{SX}) = (\mathrm{SY})$	$\square$

[SystemQ **lemma** UnequalsNotInSingleton: $\Pi(\mathrm{SX}), (\mathrm{SY}), (\mathrm{SZ})\colon (\mathrm{SX}) \neq (\mathrm{SY}) \vdash$
$(\mathrm{p(SX)}, \mathrm{(SY)}) \neq (\mathrm{s(SZ)})$]

SystemQ **proof of** UnequalsNotInSingleton:

L01:	Block $\gg$	Begin	;
L02:	Arbitrary $\gg$	(SX), (SY), (SZ)	;
L03:	Premise $\gg$	$(\mathrm{p(SX)}, \mathrm{(SY)}) = (\mathrm{s(SZ)})$	;
L04:	SingletonmembersEqual $\rhd$ L03 $\gg$	$(\mathrm{SX}) = (\mathrm{SY})$	;
L05:	Block $\gg$	End	;
L06:	Arbitrary $\gg$	(SX), (SY), (SZ)	;
L07:	Ded $\rhd$ L05 $\gg$	$(\mathrm{p(SX)}, \mathrm{(SY)}) = (\mathrm{s(SZ)}) \Rightarrow (\mathrm{SX}) = (\mathrm{SY})$	;
L03:	Premise $\gg$	$(\mathrm{SX}) \neq (\mathrm{SY})$	;
L08:	MT $\rhd$ L07 $\rhd$ L03 $\gg$	$(\mathrm{p(SX)}, \mathrm{(SY)}) \neq (\mathrm{s(SZ)})$	$\square$

[SystemQ **lemma** NonsingletonmembersUnequal: $\Pi(\mathrm{SX}), (\mathrm{SY})\colon (\mathrm{p(SX)}, \mathrm{(SY)}) \neq$
$(\mathrm{s(SX)}) \vdash (\mathrm{SX}) \neq (\mathrm{SY})$]

SystemQ **proof of** NonsingletonmembersUnequal:

L01:	Block $\gg$	Begin	;
L02:	Arbitrary $\gg$	(SX), (SY)	;
L03:	Premise $\gg$	$(\mathrm{SX}) = (\mathrm{SY})$	;
L04:	eqReflexivity $\gg$	$(\mathrm{SX}) = (\mathrm{SX})$	;
L05:	SamePair $\rhd$ L04 $\rhd$ L03 $\gg$	$(\mathrm{p(SX)}, \mathrm{(SX)}) = (\mathrm{p(SX)}, \mathrm{(SY)})$	;
L06:	Repetition $\rhd$ L05 $\gg$	$(\mathrm{s(SX)}) = (\mathrm{p(SX)}, \mathrm{(SY)})$	;
L07:	eqSymmetry $\rhd$ L06 $\gg$	$(\mathrm{p(SX)}, \mathrm{(SY)}) = (\mathrm{s(SX)})$	;
L08:	Block $\gg$	End	;
L09:	Arbitrary $\gg$	(SX), (SY)	;
L10:	Ded $\rhd$ L08 $\gg$	$(\mathrm{SX}) = (\mathrm{SY}) \Rightarrow (\mathrm{p(SX)}, \mathrm{(SY)}) = (\mathrm{s(SX)})$;	
L03:	Premise $\gg$	$(\mathrm{p(SX)}, \mathrm{(SY)}) \neq (\mathrm{s(SX)})$	;
L11:	MT $\rhd$ L10 $\rhd$ L03 $\gg$	$(\mathrm{SX}) \neq (\mathrm{SY})$	$\square$

[SystemQ **lemma** FromOrderedPair: $\Pi(\mathrm{SX}), (\mathrm{SX1}), (\mathrm{SY}), (\mathrm{SY1})\colon$ OrderedPair((
OrderedPair((SX1), (SY1)) $\vdash (\mathrm{SX}) = (\mathrm{SX1}) \dot{\wedge} (\mathrm{SY}) = (\mathrm{SY1})$]

SystemQ **proof of** FromOrderedPair:

L01:	Block $\gg$	Begin	;

L02:	Arbitrary $\gg$	(SX), (SX1), (SY), (SY1) ;
L03:	Premise $\gg$	(SX1) = (SY1) ;
L04:	Premise $\gg$	OrderedPair((SX), (SY)) = OrderedPair((SX1), (SY1)) ;
L05:	Repetition $\triangleright$ L04 $\gg$	(p(s(SX)), (p(SX), (SY))) = (p(s(SX1)), (p(SX1), (SY1))) ;
L06:	eqReflexivity $\gg$	(SX1) = (SX1) ;
L07:	SamePair $\triangleright$ L06 $\triangleright$ L03 $\gg$	(p(SX1), (SX1)) = (p(SX1), (SY1)) ;
L08:	Repetition $\triangleright$ L07 $\gg$	(s(SX1)) = (p(SX1), (SY1)) ;
L09:	eqReflexivity $\gg$	(s(SX1)) = (s(SX1)) ;
L10:	SamePair $\triangleright$ L09 $\triangleright$ L08 $\gg$	(p(s(SX1)), (s(SX1))) = (p(s(SX1)), (p(SX1), (SY1))) ;
L11:	Repetition $\triangleright$ L10 $\gg$	(s(s(SX1))) = (p(s(SX1)), (p(SX1), (SY1))) ;
L12:	eqSymmetry $\triangleright$ L11 $\gg$	(p(s(SX1)), (p(SX1), (SY1))) = (s(s(SX1))) ;
L13:	eqTransitivity $\triangleright$ L05 $\triangleright$ L12 $\gg$	(p(s(SX)), (p(SX), (SY))) = (s(s(SX1))) ;
L14:	InPair(1) $\gg$	(s(SX)) $\in$ (p(s(SX)), (p(SX), (SY))) ;
L15:	SENC1 $\triangleright$ L13 $\triangleright$ L14 $\gg$	(s(SX)) $\in$ (s(s(SX1))) ;
L16:	FromSingleton $\triangleright$ L15 $\gg$	(s(SX)) = (s(SX1)) ;
L17:	FromSameSingleton $\triangleright$ L16 $\gg$	(SX) = (SX1) ;
L18:	eqSymmetry $\triangleright$ L16 $\gg$	(s(SX1)) = (s(SX)) ;
L19:	SameSingleton $\triangleright$ L18 $\gg$	(s(s(SX1))) = (s(s(SX))) ;
L20:	eqTransitivity $\triangleright$ L13 $\triangleright$ L19 $\gg$	(p(s(SX)), (p(SX), (SY))) = (s(s(SX))) ;
L21:	InPair(2) $\gg$	(p(SX), (SY)) $\in$ (p(s(SX)), (p(SX), (SY))) ;
L22:	SENC1 $\triangleright$ L20 $\triangleright$ L21 $\gg$	(p(SX), (SY)) $\in$ (s(s(SX))) ;
L23:	FromSingleton $\triangleright$ L22 $\gg$	(p(SX), (SY)) = (s(SX)) ;
L24:	SingletonmembersEqual $\triangleright$ L23 $\gg$	(SX) = (SY) ;
L25:	eqSymmetry $\triangleright$ L24 $\gg$	(SY) = (SX) ;
L26:	eqTransitivity4 $\triangleright$ L25 $\triangleright$ L17 $\triangleright$ L03 $\gg$	(SY) = (SY1) ;
L27:	JoinConjuncts $\triangleright$ L17 $\triangleright$ L26 $\gg$	(SX) = (SX1) $\dot{\wedge}$ (SY) = (SY1) ;
L28:	Block $\gg$	End ;
L29:	Block $\gg$	Begin ;
L30:	Arbitrary $\gg$	(SX), (SX1), (SY), (SY1) ;
L03:	Premise $\gg$	(SX1) $\neq$ (SY1) ;
L04:	Premise $\gg$	OrderedPair((SX), (SY)) = OrderedPair((SX1), (SY1)) ;

L05:	Repetition ▷ L04 ≫	$(p(s(SX)),(p(SX),(SY))) = (p(s(SX1)),(p(SX1),(SY1)))$;
L06:	InPair(1) ≫	$(s(SX)) \in (p(s(SX)),(p(SX),(SY)))$;
L07:	SENC1 ▷ L05 ▷ L06 ≫	$(s(SX)) \in (p(s(SX1)),(p(SX1),(SY1)))$;
L08:	Pair2Formula ▷ L07 ≫	$(s(SX)) = (s(SX1)) \dot{\vee} (s(SX)) = (p(SX1),(SY1))$;
L09:	UnequalsNotInSingleton ▷ L03 ≫	$(p(SX1),(SY1)) \neq (s(SX))$;
L10:	NeqSymmetry ▷ L09 ≫	$(s(SX)) \neq (p(SX1),(SY1))$;
L11:	NegateDisjunct2 ▷ L08 ▷ L10 ≫	$(s(SX)) = (s(SX1))$;
L12:	FromSameSingleton ▷ L11 ≫	$(SX) = (SX1)$;
L14:	InPair(2) ≫	$(p(SX1),(SY1)) \in (p(s(SX1)),(p(SX1),(SY1)))$;
L15:	SENC2 ▷ L05 ▷ L14 ≫	$(p(SX1),(SY1)) \in (p(s(SX)),(p(SX),(SY)))$;
L16:	Pair2Formula ▷ L15 ≫	$(p(SX1),(SY1)) = (s(SX)) \dot{\vee} (p(SX1),(SY1)) = (p(SX),(SY))$;
L18:	NegateDisjunct1 ▷ L16 ▷ L09 ≫	$(p(SX1),(SY1)) = (p(SX),(SY))$;
L19:	InPair(2) ≫	$(SY) \in (p(SX),(SY))$;
L20:	SENC2 ▷ L18 ▷ L19 ≫	$(SY) \in (p(SX1),(SY1))$;
L21:	Pair2Formula ▷ L20 ≫	$(SY) = (SX1) \dot{\vee} (SY) = (SY1)$;
L22:	UnequalsNotInSingleton ▷ L03 ≫	$(p(SX1),(SY1)) \neq (s(SX))$;
L23:	SubNeqLeft ▷ L18 ▷ L22 ≫	$(p(SX),(SY)) \neq (s(SX))$;
L24:	NonsingletonmembersUnequal ▷ L23 ≫	$(SX) \neq (SY)$;
L25:	SubNeqLeft ▷ L12 ▷ L24 ≫	$(SX1) \neq (SY)$;
L26:	NeqSymmetry ▷ L25 ≫	$(SY) \neq (SX1)$;
L31:	NegateDisjunct1 ▷ L21 ▷ L26 ≫	$(SY) = (SY1)$;
L32:	JoinConjuncts ▷ L12 ▷ L31 ≫	$(SX) = (SX1) \dot{\wedge} (SY) = (SY1)$;
L33:	Block ≫	End ;
L34:	Arbitrary ≫	$(SX),(SX1),(SY),(SY1)$;
L35:	Ded ▷ L28 ≫	$(SX1) = (SY1) \Rightarrow \mathrm{OrderedPair}((SX),(SY)) = \mathrm{OrderedPair}((SX1),(SY1)) \Rightarrow (SX) = (SX1) \dot{\wedge} (SY) = (SY1)$;
L36:	Ded ▷ L33 ≫	$(SX1) \neq (SY1) \Rightarrow \mathrm{OrderedPair}((SX),(SY)) = \mathrm{OrderedPair}((SX1),(SY1)) \Rightarrow (SX) = (SX1) \dot{\wedge} (SY) = (SY1)$;

L03: Premise $\gg$ $\mathrm{OrderedPair}((SX),(SY)) = \mathrm{OrderedPair}((SX1),(SY1))$;

L04: FromNegations $\rhd$ L35 $\rhd$ L36 $\gg$ $\mathrm{OrderedPair}((SX),(SY)) = \mathrm{OrderedPair}((SX1),(SY1)) \Rightarrow (SX)=(SX1) \dot\wedge (SY)=(SY1)$;

L37: MP $\rhd$ L04 $\rhd$ L03 $\gg$ $(SX)=(SX1) \dot\wedge (SY)=(SY1)$ $\square$

[SystemQ **lemma** FromOrderedPair(1): $\Pi(SX),(SX1),(SY),(SY1)$: OrderedPai
$\mathrm{OrderedPair}((SX1),(SY1)) \vdash (SX)=(SX1)$]

SystemQ **proof of** FromOrderedPair(1):

L01: Arbitrary $\gg$ $(SX),(SX1),(SY),(SY1)$;

L02: Premise $\gg$ $\mathrm{OrderedPair}((SX),(SY)) = \mathrm{OrderedPair}((SX1),(SY1))$;

L03: FromOrderedPair $\rhd$ L02 $\gg$ $(SX)=(SX1) \dot\wedge (SY)=(SY1)$;

L04: FirstConjunct $\rhd$ L03 $\gg$ $(SX)=(SX1)$ $\square$

[SystemQ **lemma** FromOrderedPair(2): $\Pi(SX),(SX1),(SY),(SY1)$: OrderedPai
$\mathrm{OrderedPair}((SX1),(SY1)) \vdash (SY)=(SY1)$]

SystemQ **proof of** FromOrderedPair(2):

L01: Arbitrary $\gg$ $(SX),(SX1),(SY),(SY1)$;

L02: Premise $\gg$ $\mathrm{OrderedPair}((SX),(SY)) = \mathrm{OrderedPair}((SX1),(SY1))$;

L03: FromOrderedPair $\rhd$ L02 $\gg$ $(SX)=(SX1) \dot\wedge (SY)=(SY1)$;

L04: SecondConjunct $\rhd$ L03 $\gg$ $(SY)=(SY1)$ $\square$

[SystemQ **lemma** SameMember(2): $\Pi(SX),(SY),(SZ)$: $(SX)=(SY) \vdash (SY) \in$
$(SZ) \vdash (SX) \in (SZ)$]

SystemQ **proof of** SameMember(2):

L01: Arbitrary $\gg$ $(SX),(SY),(SZ)$;

L02: Premise $\gg$ $(SX)=(SY)$;

L03: Premise $\gg$ $(SY) \in (SZ)$;

L04: eqSymmetry $\rhd$ L02 $\gg$ $(SY)=(SX)$;

L05: SameMember $\rhd$ L04 $\rhd$ L03 $\gg$ $(SX) \in (SZ)$ $\square$

[SystemQ **lemma** ToBinaryUnion(1): $\Pi(SX),(SY),(SZ),(SU)$: $(SX) \in (SY) \vdash$
$(SX) \in \mathrm{binaryUnion}((SY),(SZ))$]

SystemQ **proof of** ToBinaryUnion(1):

L01: Arbitrary $\gg$ $(SX),(SY),(SZ),(SU)$;

L02: Premise $\gg$ $(SX) \in (SY)$;

L03: InPair(1) $\gg$ $(SY) \in (\mathrm{p}(SY),(SZ))$;

L04: JoinConjuncts $\rhd$ L02 $\rhd$ L03 $\gg$ $(SX) \in (SY) \dot\wedge (SY) \in (\mathrm{p}(SY),(SZ))$;

L05: IntroExist @(SY) $\rhd$ L04 $\gg$ $\exists(SU)\colon (SX) \in (SU) \dot\wedge (SU) \in (\mathrm{p}(SY),(SZ))$;

L06: Formula2Union $\rhd$ L05 $\gg$ $(SX) \in \mathrm{Union}((\mathrm{p}(SY),(SZ)))$;

L07: Repetition $\rhd$ L06 $\gg$ $(SX) \in \mathrm{binaryUnion}((SY),(SZ))$ $\square$

[SystemQ **lemma** ToBinaryUnion(2): $\Pi(SX),(SY),(SZ),(SU)$: $(SX) \in (SZ) \vdash$
$(SX) \in \mathrm{binaryUnion}((SY),(SZ))$]

SystemQ **proof of** ToBinaryUnion(2):

L01:	Arbitrary $\gg$	(SX), (SY), (SZ), (SU)	;
L02:	Premise $\gg$	(SX) $\in$ (SZ)	;
L03:	InPair(2) $\gg$	(SZ) $\in$ (p(SY), (SZ))	;
L04:	JoinConjuncts $\rhd$ L02 $\rhd$ L03 $\gg$	(SX) $\in$ (SZ) $\dot{\wedge}$ (SZ) $\in$ (p(SY), (SZ))	;
L05:	IntroExist @(SZ) $\rhd$ L04 $\gg$	$\exists$(SU): (SX) $\in$ (SU) $\dot{\wedge}$ (SU) $\in$ (p(SY), (SZ))	;
L06:	Formula2Union $\rhd$ L05 $\gg$	(SX) $\in$ Union((p(SY), (SZ)))	;
L07:	Repetition $\rhd$ L06 $\gg$	(SX) $\in$ binaryUnion((SY), (SZ))	$\square$

[SystemQ **lemma** FromOrderedPair(TwoLevels): Π(SX), (SY), (SZ), (SU): (SX)
(SY) $\vdash$ (SY) $\in$ OrderedPair((SZ), (SU)) $\vdash$ (SX) = (SZ) $\dot{\vee}$ (SX) = (SU)]

SystemQ **proof of** FromOrderedPair(TwoLevels):

L01:	Arbitrary $\gg$	(SX), (SY), (SZ), (SU)	;
L02:	Premise $\gg$	(SX) $\in$ (SY)	;
L03:	Premise $\gg$	(SY) $\in$ OrderedPair((SZ), (SU))	;
L04:	Repetition $\rhd$ L03 $\gg$	(SY) $\in$ (p(s(SZ)), (p(SZ), (SU)))	;
L05:	Pair2Formula $\rhd$ L04 $\gg$	(SY) = (s(SZ)) $\dot{\vee}$ (SY) = (p(SZ), (SU))	;
L06:	Block $\gg$	Begin	;
L07:	Arbitrary $\gg$	(SX), (SY), (SZ), (SU)	;
L03:	Premise $\gg$	(SY) = (s(SZ))	;
L02:	Premise $\gg$	(SX) $\in$ (SY)	;
L04:	SENC1 $\rhd$ L03 $\rhd$ L02 $\gg$	(SX) $\in$ (s(SZ))	;
L05:	FromSingleton $\rhd$ L04 $\gg$	(SX) = (SZ)	;
L08:	WeakenOr2 $\rhd$ L05 $\gg$	(SX) = (SZ) $\dot{\vee}$ (SX) = (SU)	;
L09:	Block $\gg$	End	;
L10:	Block $\gg$	Begin	;
L11:	Arbitrary $\gg$	(SX), (SY), (SZ), (SU)	;
L03:	Premise $\gg$	(SY) = (p(SZ), (SU))	;
L02:	Premise $\gg$	(SX) $\in$ (SY)	;
L04:	SENC1 $\rhd$ L03 $\rhd$ L02 $\gg$	(SX) $\in$ (p(SZ), (SU))	;
L12:	Pair2Formula $\rhd$ L04 $\gg$	(SX) = (SZ) $\dot{\vee}$ (SX) = (SU)	;
L13:	Block $\gg$	End	;
L14:	Ded $\rhd$ L09 $\gg$	(SY) = (s(SZ)) $\Rightarrow$ (SX) $\in$ (SY) $\Rightarrow$ (SX) = (SZ) $\dot{\vee}$ (SX) = (SU)	;
L15:	Ded $\rhd$ L13 $\gg$	(SY) = (p(SZ), (SU)) $\Rightarrow$ (SX) $\in$ (SY) $\Rightarrow$ (SX) = (SZ) $\dot{\vee}$ (SX) = (SU)	;
L16:	FromDisjuncts $\rhd$ L05 $\rhd$ L14 $\rhd$ L15 $\gg$	(SX) $\in$ (SY) $\Rightarrow$ (SX) = (SZ) $\dot{\vee}$ (SX) = (SU)	;
L17:	MP $\rhd$ L16 $\rhd$ L02 $\gg$	(SX) = (SZ) $\dot{\vee}$ (SX) = (SU)	$\square$

[SystemQ **lemma** CartProdIsRelation: Π(SX), (SY): IsRelation(cartProd((SX)

SystemQ **proof of** CartProdIsRelation:

L01: Block ≫ Begin ;
L02: Arbitrary ≫ (SX), (SY) ;
L03: Premise ≫ (R1ob) ∈ cartProd((SX)) ;
L04: Sep2Formula ▷ L03 ≫ (R1ob) ∈ Power(Power(binaryUnion((SX), (SY))) IsOrderedPair((R1ob), (SX), (SY))▌ ;
L05: SecondConjunct ▷ L04 ≫ IsOrderedPair((R1ob), (SX), (SY))▌ ;
L06: Block ≫ End ;
L07: Arbitrary ≫ (SX), (SY) ;
L03: Ded ▷ L06 ≫ (R1ob) ∈ cartProd((SX)) ⇒ IsOrderedPair((R1ob), (SX), (SY))▌ ;
L04: Gen ▷ L03 ≫ ∀(R1ob): ((R1ob) ∈ cartProd((SX)) ⇒ IsOrderedPair((R1ob), (SX), (SY)))▌ ;
L08: Repetition ▷ L04 ≫ IsRelation(cartProd((SX)), (SX), (SY))▌ □

[SystemQ **lemma** FromSubset: Π(SX), (SY), (SZ): IsSubset((SX), (SY)) ⊢ (SZ)
(SX) ⊢ (SZ) ∈ (SY)]

SystemQ **proof of** FromSubset:

L01: Arbitrary ≫ (SX), (SY), (SZ) ;
L02: Premise ≫ IsSubset((SX), (SY)) ;
L03: Premise ≫ (SZ) ∈ (SX) ;
L04: Repetition ▷ L02 ≫ ∀(S1ob): ((S1ob) ∈ (SX) ⇒ (S1ob) ∈ (SY)) ;
L05: A4 @(SZ) ▷ L04 ≫ (SZ) ∈ (SX) ⇒ (SZ) ∈ (SY) ;
L06: MP ▷ L05 ▷ L03 ≫ (SZ) ∈ (SY) □

[SystemQ **lemma** SubsetIsRelation: Π(SX), (SY), (SZ), (SU): IsRelation((SX), (
IsSubset((SY), (SX)) ⊢ IsRelation((SY), (SZ), (SU))]

SystemQ **proof of** SubsetIsRelation:

L01: Block ≫ Begin ;
L02: Arbitrary ≫ (SX), (SY), (SZ), (SU) ;
L03: Premise ≫ IsRelation((SX), (SZ), (SU)) ;
L04: Premise ≫ IsSubset((SY), (SX)) ;
L05: Premise ≫ (R1ob) ∈ (SY) ;
L06: Repetition ▷ L03 ≫ ∀(R1ob): ((R1ob) ∈ (SX) ⇒ IsOrderedPair((R1ob), (SZ), (SU)))▌ ;
L07: A4 @(R1ob) ▷ L06 ≫ (R1ob) ∈ (SX) ⇒ IsOrderedPair((R1ob), (SZ), (SU))▌ ;
L08: FromSubset ▷ L04 ▷ L05 ≫ (R1ob) ∈ (SX) ;

L09:	MP ⊳ L07 ⊳ L08 ≫	$\mathrm{IsOrderedPair}((\mathrm{R1ob}), (\mathrm{SZ}), (\mathrm{SU}))$;
L10:	Block ≫	End ;
L11:	Arbitrary ≫	$(\mathrm{SX}), (\mathrm{SY}), (\mathrm{SZ}), (\mathrm{SU})$;
L12:	Ded ⊳ L10 ≫	$\mathrm{IsRelation}((\mathrm{SX}), (\mathrm{SZ}), (\mathrm{SU})) \Rightarrow \mathrm{IsSubset}((\mathrm{SY}), (\mathrm{SX})) \Rightarrow (\mathrm{R1ob}) \in (\mathrm{SY}) \Rightarrow \mathrm{IsOrderedPair}((\mathrm{R1ob}), (\mathrm{SZ}), (\mathrm{SU}))$;
L03:	Premise ≫	$\mathrm{IsRelation}((\mathrm{SX}), (\mathrm{SZ}), (\mathrm{SU}))$;
L04:	Premise ≫	$\mathrm{IsSubset}((\mathrm{SY}), (\mathrm{SX}))$;
L05:	MP2 ⊳ L12 ⊳ L03 ⊳ L04 ≫	$(\mathrm{R1ob}) \in (\mathrm{SY}) \Rightarrow \mathrm{IsOrderedPair}((\mathrm{R1ob}), (\mathrm{SZ}), (\mathrm{SU}))$;
L06:	Gen ⊳ L05 ≫	$\forall (\mathrm{R1ob})\colon ((\mathrm{R1ob}) \in (\mathrm{SY}) \Rightarrow \mathrm{IsOrderedPair}((\mathrm{R1ob}), (\mathrm{SZ}), (\mathrm{SU})))$;
L13:	Repetition ⊳ L06 ≫	$\mathrm{IsRelation}((\mathrm{SY}), (\mathrm{SZ}), (\mathrm{SU}))$ □

[SystemQ **lemma** CPseparationIsRelation: $\Pi \mathcal{A}, (\mathrm{SX}), (\mathrm{SY})\colon \mathrm{IsRelation}(\{\mathrm{ph} \in \mathrm{cartProd}((\mathrm{SX})) \mid \mathcal{A}\}, (\mathrm{SX}), (\mathrm{SY}))$]

SystemQ **proof of** CPseparationIsRelation:

L01:	Block ≫	Begin ;
L02:	Arbitrary ≫	$\mathcal{A}, (\mathrm{SX}), (\mathrm{SY})$;
L03:	Premise ≫	$(\mathrm{S1ob}) \in \{\mathrm{ph} \in \mathrm{cartProd}((\mathrm{SX})) \mid \mathcal{A}\}$;
L04:	Separation2formula(1) ⊳ L03 ≫	$(\mathrm{S1ob}) \in \mathrm{cartProd}((\mathrm{SX}))$;
L05:	Block ≫	End ;
L06:	Arbitrary ≫	$\mathcal{A}, (\mathrm{SX}), (\mathrm{SY})$;
L07:	Ded ⊳ L05 ≫	$\forall (\mathrm{S1ob})\colon ((\mathrm{S1ob}) \in \{\mathrm{ph} \in \mathrm{cartProd}((\mathrm{SX})) \mid \mathcal{A}\} \Rightarrow (\mathrm{S1ob}) \in \mathrm{cartProd}((\mathrm{SX})))$;
L08:	Repetition ⊳ L07 ≫	$\mathrm{IsSubset}(\{\mathrm{ph} \in \mathrm{cartProd}((\mathrm{SX})) \mid \mathcal{A}\}, \mathrm{cartProd}((\mathrm{SX})))$;
L09:	CartProdIsRelation ≫	$\mathrm{IsRelation}(\mathrm{cartProd}((\mathrm{SX})), (\mathrm{SX}), (\mathrm{SY}))$;
L10:	SubsetIsRelation ⊳ L09 ⊳ L08 ≫	$\mathrm{IsRelation}(\{\mathrm{ph} \in \mathrm{cartProd}((\mathrm{SX})) \mid \mathcal{A}\}, (\mathrm{SX}), (\mathrm{SY}))$ □

[SystemQ **lemma** ToCartProd(Helper): $\Pi (\mathrm{SX}), (\mathrm{SX1}), (\mathrm{SY}), (\mathrm{SY1}), (\mathrm{SZ})\colon (\mathrm{SX})$
$(\mathrm{SX1}) \vdash (\mathrm{SY}) \in (\mathrm{SY1}) \vdash (\mathrm{SZ}) \in \mathrm{OrderedPair}((\mathrm{SX}), (\mathrm{SY})) \vdash \mathrm{IsSubset}((\mathrm{SZ}), \mathrm{binaryU}$

SystemQ **proof of** ToCartProd(Helper):

L01:	Block ≫	Begin ;
L02:	Arbitrary ≫	$(\mathrm{SX}), (\mathrm{SX1}), (\mathrm{SY}), (\mathrm{SY1}), (\mathrm{SZ})$;
L03:	Premise ≫	$(\mathrm{SX}) \in (\mathrm{SX1})$;
L04:	Premise ≫	$(\mathrm{SY}) \in (\mathrm{SY1})$;

L05:	Premise ≫	(SZ) ∈ OrderedPair((SX), (SY)) ;
L06:	Premise ≫	(S1ob) ∈ (SZ) ;
L07:	FromOrderedPair(TwoLevels) ▷ L06 ▷ L05 ≫	(S1ob) = (SX) ∨̇ (S1ob) = (SY) ;
L08:	Block ≫	Begin ;
L09:	Arbitrary ≫	(SX), (SX1), (SY1) ;
L04:	Premise ≫	(SX) ∈ (SX1) ;
L03:	Premise ≫	(S1ob) = (SX) ;
L05:	SameMember(2) ▷ L03 ▷ L04 ≫	(S1ob) ∈ (SX1) ;
L10:	ToBinaryUnion(1) ▷ L05 ≫	(S1ob) ∈ binaryUnion((SX1), (SY1)) ;
L11:	Block ≫	End ;
L12:	Block ≫	Begin ;
L13:	Arbitrary ≫	(SX1), (SY), (SY1) ;
L04:	Premise ≫	(SY) ∈ (SY1) ;
L03:	Premise ≫	(S1ob) = (SY) ;
L05:	SameMember(2) ▷ L03 ▷ L04 ≫	(S1ob) ∈ (SY1) ;
L14:	ToBinaryUnion(2) ▷ L05 ≫	(S1ob) ∈ binaryUnion((SX1), (SY1)) ;
L15:	Block ≫	End ;
L16:	Ded ▷ L11 ≫	(SX) ∈ (SX1) ⇒ (S1ob) = (SX) ⇒ (S1ob) ∈ binaryUnion((SX1), (SY1)) ;
L17:	MP ▷ L16 ▷ L03 ≫	(S1ob) = (SX) ⇒ (S1ob) ∈ binaryUnion((SX1), (SY1)) ;
L18:	Ded ▷ L15 ≫	(SY) ∈ (SY1) ⇒ (S1ob) = (SY) ⇒ (S1ob) ∈ binaryUnion((SX1), (SY1)) ;
L19:	MP ▷ L18 ▷ L04 ≫	(S1ob) = (SY) ⇒ (S1ob) ∈ binaryUnion((SX1), (SY1)) ;
L20:	FromDisjuncts ▷ L07 ▷ L17 ▷ L19 ≫	(S1ob) ∈ binaryUnion((SX1), (SY1)) ;
L21:	Block ≫	End ;
L22:	Arbitrary ≫	(SX), (SX1), (SY), (SY1), (SZ) ;
L23:	Ded ▷ L21 ≫	(SX) ∈ (SX1) ⇒ (SY) ∈ (SY1) ⇒ (SZ) ∈ OrderedPair((SX), (SY)) ⇒ (S1ob) ∈ (SZ) ⇒ (S1ob) ∈ binaryUnion((SX1), (SY1)) ;

L03:	Premise ≫	(SX) ∈ (SX1) ;
L04:	Premise ≫	(SY) ∈ (SY1) ;
L05:	Premise ≫	(SZ) ∈ OrderedPair((SX), (SY)) ;
L06:	MP3 ▷ L23 ▷ L03 ▷ L04 ▷ L05 ≫	(S1ob) ∈ (SZ) ⇒ (S1ob) ∈ binaryUnion((SX1), (SY1)) ;
L07:	Gen ▷ L06 ≫	∀(S1ob): ((S1ob) ∈ (SZ) ⇒ (S1ob) ∈ binaryUnion((SX1), (SY1))) ;
L24:	Repetition ▷ L07 ≫	IsSubset((SZ), binaryUnion((SX1), (SY1 □

[SystemQ **lemma** ToCartProd: Π(SX), (SX1), (SY), (SY1): (SX) ∈ (SX1) ⊢ (SY) ∈ (SY1) ⊢ OrderedPair((SX), (SY)) ∈ cartProd((SX1))]

SystemQ **proof of** ToCartProd:

L01:	Block ≫	Begin ;
L02:	Arbitrary ≫	(SX), (SX1), (SY), (SY1) ;
L03:	Premise ≫	(SX) ∈ (SX1) ;
L04:	Premise ≫	(SY) ∈ (SY1) ;
L05:	Premise ≫	(S1ob) ∈ OrderedPair((SX), (SY)) ;
L06:	ToCartProd(Helper) ▷ L03 ▷ L04 ▷ L05 ≫	IsSubset((S1ob), binaryUnion((SX1), (SY ;
L07:	Formula2Power ▷ L06 ≫	(S1ob) ∈ Power(binaryUnion((SX1), (SY1))) ;
L08:	Block ≫	End ;
L09:	Arbitrary ≫	(SX), (SX1), (SY), (SY1) ;
L10:	Ded ▷ L08 ≫	(SX) ∈ (SX1) ⇒ (SY) ∈ (SY1) ⇒ (S1ob) ∈ OrderedPair((SX), (SY)) ⇒ (S1ob) ∈ Power(binaryUnion((SX1), (SY1))) ;
L03:	Premise ≫	(SX) ∈ (SX1) ;
L04:	Premise ≫	(SY) ∈ (SY1) ;
L06:	MP2 ▷ L10 ▷ L03 ▷ L04 ≫	(S1ob) ∈ OrderedPair((SX), (SY)) ⇒ (S1ob) ∈ Power(binaryUnion((SX1), (SY1))) ;
L11:	Gen ▷ L06 ≫	∀(S1ob): ((S1ob) ∈ OrderedPair((SX), (SY)) ⇒ (S1ob) ∈ Power(binaryUnion((SX1), (SY1)))) ;

L12:	Repetition ▷ L11 ≫	IsSubset(OrderedPair((SX), (SY)), Powe ;
L13:	Formula2Power ▷ L12 ≫	OrderedPair((SX), (SY)) ∈ Power(Power(binaryUnion((SX1), (SY1) ;
L14:	eqReflexivity ≫	OrderedPair((SX), (SY)) = OrderedPair((SX), (SY)) ;
L15:	JoinConjuncts ▷ L03 ▷ L04 ≫	(SX) ∈ (SX1) $\dot\wedge$ (SY) ∈ (SY1) ;
L16:	JoinConjuncts ▷ L15 ▷ L14 ≫	(SX) ∈ (SX1) $\dot\wedge$ (SY) ∈ (SY1) $\dot\wedge$ OrderedPair((SX), (SY)) = OrderedPair((SX), (SY)) ;
L17:	IntroExist @(SY) ▷ L16 ≫	∃(OP2ob): (SX) ∈ (SX1) $\dot\wedge$ (OP2ob) ∈ (SY1) $\dot\wedge$ OrderedPair((SX), (SY)) = OrderedPair((SX), (OP2ob)) ;
L18:	IntroExist @(SX) ▷ L17 ≫	∃(OP1ob): ∃(OP2ob): (OP1ob) ∈ (SX1) $\dot\wedge$ (OP2ob) ∈ (SY1) $\dot\wedge$ OrderedPair((SX), (SY)) = OrderedPair((OP1ob), (OP2ob)) ;
L19:	Repetition ▷ L18 ≫	IsOrderedPair(OrderedPair((SX), (SY)), ;
L20:	Formula2Sep ▷ L13 ▷ L19 ≫	OrderedPair((SX), (SY)) ∈ {ph ∈ Power(Power(binaryUnion((SX1), (SY1) IsOrderedPair(ph$_1$, (SX1), (SY1))} ;
L21:	Repetition ▷ L20 ≫	OrderedPair((SX), (SY)) ∈ cartProd((SX1)) □

[SystemQ **lemma** CrsIsRelation: Π𝒳: IsRelation(constantRationalSeries(𝒳), N

SystemQ **proof of** CrsIsRelation:

L01:	Block ≫	Begin ;
L02:	Arbitrary ≫	𝒳 ;
L03:	Premise ≫	(S1ob) ∈ constantRationalSeries(𝒳) ;
L04:	Repetition ▷ L03 ≫	(S1ob) ∈ {ph ∈ cartProd(N) | ∃(CRS1ob): ph$_3$ = OrderedPair((CRS1ob), 𝒳)} ;
L05:	Sep2Formula ▷ L04 ≫	(S1ob) ∈ cartProd(N) $\dot\wedge$ ∃(CRS1ob): (S1ob) = OrderedPair((CRS1ob), 𝒳) ;
L06:	FirstConjunct ▷ L05 ≫	(S1ob) ∈ cartProd(N) ;
L07:	Block ≫	End ;
L08:	Arbitrary ≫	𝒳 ;

L03: Ded $\rhd$ L07 $\gg$ $(\mathrm{S1ob}) \in \mathrm{constantRationalSeries}(\mathcal{X}) \Rightarrow (\mathrm{S1ob}) \in \mathrm{cartProd}(\mathrm{N})$;

L04: Gen $\rhd$ L03 $\gg$ $\forall(\mathrm{S1ob})\colon ((\mathrm{S1ob}) \in \mathrm{constantRationalSeries}(\mathcal{X}) \Rightarrow (\mathrm{S1ob}) \in \mathrm{cartProd}(\mathrm{N}))$;

L05: Repetition $\rhd$ L04 $\gg$ $\mathrm{IsSubset}(\mathrm{constantRationalSeries}(\mathcal{X}), \mathrm{car}$;

L09: CartProdIsRelation $\gg$ $\mathrm{IsRelation}(\mathrm{cartProd}(\mathrm{N}), \mathrm{N}, \mathrm{Q})$;

L10: SubsetIsRelation$\rhd$L09$\rhd$L05 $\gg$ $\mathrm{IsRelation}(\mathrm{constantRationalSeries}(\mathcal{X}), \mathrm{N}$

$\square$

[SystemQ **lemma** CrsIsFunction: $\Pi\mathcal{X}\colon \mathrm{isFunction}(\mathrm{constantRationalSeries}(\mathcal{X}), \mathrm{N}$

SystemQ **proof of** CrsIsFunction:

L01: Block $\gg$ Begin ;

L02: Arbitrary $\gg$ $\mathcal{X}$;

L03: Premise $\gg$ $\mathrm{OrderedPair}((\mathrm{F1ob}), (\mathrm{F2ob})) = \mathrm{OrderedPair}((\mathrm{CRS1ob}), \mathcal{X})$;

L04: Premise $\gg$ $\mathrm{OrderedPair}((\mathrm{F3ob}), (\mathrm{F4ob})) = \mathrm{OrderedPair}((\mathrm{CRS1ob}), \mathcal{X})$;

L05: FromOrderedPair $\rhd$ L03 $\gg$ $(\mathrm{F1ob}) = (\mathrm{CRS1ob}) \dot{\wedge} (\mathrm{F2ob}) = \mathcal{X}$;

L06: SecondConjunct $\rhd$ L05 $\gg$ $(\mathrm{F2ob}) = \mathcal{X}$;

L07: FromOrderedPair $\rhd$ L04 $\gg$ $(\mathrm{F3ob}) = (\mathrm{CRS1ob}) \dot{\wedge} (\mathrm{F4ob}) = \mathcal{X}$;

L08: SecondConjunct $\rhd$ L07 $\gg$ $(\mathrm{F4ob}) = \mathcal{X}$;

L09: eqSymmetry $\rhd$ L08 $\gg$ $\mathcal{X} = (\mathrm{F4ob})$;

L10: eqTransitivity $\rhd$ L06 $\rhd$ L09 $\gg$ $(\mathrm{F2ob}) = (\mathrm{F4ob})$;

L11: Block $\gg$ End ;

L12: Block $\gg$ Begin ;

L13: Arbitrary $\gg$ $\mathcal{X}$;

L14: Ded $\rhd$ L11 $\gg$ $\mathrm{OrderedPair}((\mathrm{F1ob}), (\mathrm{F2ob})) = \mathrm{OrderedPair}((\mathrm{CRS1ob}), \mathcal{X}) \Rightarrow \mathrm{OrderedPair}((\mathrm{F3ob}), (\mathrm{F4ob})) = \mathrm{OrderedPair}((\mathrm{CRS1ob}), \mathcal{X}) \Rightarrow (\mathrm{F2ob}) = (\mathrm{F4ob})$;

L03: Premise $\gg$ $\mathrm{OrderedPair}((\mathrm{F1ob}), (\mathrm{F2ob})) \in \mathrm{constantRationalSeries}(\mathcal{X})$;

L04: Premise $\gg$ $\mathrm{OrderedPair}((\mathrm{F3ob}), (\mathrm{F4ob})) \in \mathrm{constantRationalSeries}(\mathcal{X})$;

L05: Premise $\gg$ $(\mathrm{F1ob}) = (\mathrm{F3ob})$;

L06: Sep2Formula $\rhd$ L03 $\gg$ $\mathrm{OrderedPair}((\mathrm{F1ob}), (\mathrm{F2ob})) \in \mathrm{cartProd}(\mathrm{N}) \dot{\wedge} \exists(\mathrm{CRS1ob})\colon \mathrm{OrderedPair}((\mathrm{F1ob}), (\mathrm{F2ob})$ $\mathrm{OrderedPair}((\mathrm{CRS1ob}), \mathcal{X})$;

L07: SecondConjunct $\rhd$ L06 $\gg$ $\exists(\mathrm{CRS1ob})\colon \mathrm{OrderedPair}((\mathrm{F1ob}), (\mathrm{F2ob})$ $\mathrm{OrderedPair}((\mathrm{CRS1ob}), \mathcal{X})$;

L08:	Sep2Formula ▷ L04 ≫	OrderedPair((F3ob), (F4ob)) ∈ cartProd(N) $\dot{\wedge}$ ∃(CRS1ob): OrderedPair((F3ob), (F4ob) OrderedPair((CRS1ob), $\mathcal{X}$) ;
L09:	SecondConjunct ▷ L08 ≫	∃(CRS1ob): OrderedPair((F3ob), (F4ob) OrderedPair((CRS1ob), $\mathcal{X}$) ;
L15:	ExistMP2 ▷ L14 ▷ L07 ▷ L09 ≫	(F2ob) = (F4ob) ;
L16:	Block ≫	End ;
L17:	Arbitrary ≫	$\mathcal{X}$;
L03:	Ded ▷ L16 ≫	∀(F1ob), (F2ob), (F3ob), (F4ob): (OrderedPair((F1ob), (F2ob)) ∈ constantRationalSeries($\mathcal{X}$) ⇒ OrderedPair((F3ob), (F4ob)) ∈ constantRationalSeries($\mathcal{X}$) ⇒ (F1ob) = (F3ob) ⇒ (F2ob) = (F4ob)) ;
L04:	CrsIsRelation ≫	IsRelation(constantRationalSeries($\mathcal{X}$), N ;
L18:	JoinConjuncts ▷ L04 ▷ L03 ≫	isFunction(constantRationalSeries($\mathcal{X}$), N □

[SystemQ **lemma** CrsIsTotal: Π$\mathcal{M}$, $\mathcal{X}$: TypeRational($\mathcal{X}$) ⊩ $\mathcal{M}$ ∈ N ⊢ OrderedP constantRationalSeries($\mathcal{X}$)]

SystemQ **proof of** CrsIsTotal:

L01:	Arbitrary ≫	$\mathcal{M}$, $\mathcal{X}$;
L02:	Side-condition ≫	TypeRational($\mathcal{X}$) ;
L03:	Premise ≫	$\mathcal{M}$ ∈ N ;
L04:	RationalType ⧐ L02 ≫	$\mathcal{X}$ ∈ Q ;
L05:	ToCartProd ▷ L03 ▷ L04 ≫	OrderedPair($\mathcal{M}$, $\mathcal{X}$) ∈ cartProd(N) ;
L06:	eqReflexivity ≫	OrderedPair($\mathcal{M}$, $\mathcal{X}$) = OrderedPair($\mathcal{M}$, $\mathcal{X}$) ;
L07:	IntroExist @ $\mathcal{M}$ ▷ L06 ≫	∃(CRS1ob): OrderedPair($\mathcal{M}$, $\mathcal{X}$) = OrderedPair((CRS1ob), $\mathcal{X}$) ;
L08:	Formula2Sep ▷ L05 ▷ L07 ≫	OrderedPair($\mathcal{M}$, $\mathcal{X}$) ∈ constantRationalSeries($\mathcal{X}$) □

[SystemQ **lemma** CrsIsSeries: Π$\mathcal{X}$: IsSeries(constantRationalSeries($\mathcal{X}$), Q)]

SystemQ **proof of** CrsIsSeries:

L01:	Block ≫	Begin ;
L02:	Arbitrary ≫	$\mathcal{X}$;
L03:	Premise ≫	(S1ob) ∈ N ;
L04:	CrsIsTotal ▷ L03 ≫	OrderedPair((S1ob), $\mathcal{X}$) ∈ constantRationalSeries($\mathcal{X}$) ;
L05:	IntroExist @ $\mathcal{X}$ ▷ L03 ≫	∃(S2ob): OrderedPair((S1ob), (S2ob)) ∈ constantRationalSeries($\mathcal{X}$) ;
L06:	Block ≫	End ;
L07:	Arbitrary ≫	$\mathcal{X}$;

L03: Ded ⊳ L06 ≫ (S1ob) ∈ N ⇒ ∃(S2ob): OrderedPair((S1ob), (S2ob)) ∈ constantRationalSeries($\mathcal{X}$) ;

L08: Gen ⊳ L03 ≫ ∀(S1ob): ((S1ob) ∈ N ⇒ ∃(S2ob): OrderedPair((S1ob), (S2ob)) ∈ constantRationalSeries($\mathcal{X}$)) ;

L09: CrsIsFunction ≫ isFunction(constantRationalSeries($\mathcal{X}$), N ;

L10: JoinConjuncts ⊳ L09 ⊳ L08 ≫ IsSeries(constantRationalSeries($\mathcal{X}$), Q) □

[SystemQ **lemma** CrsLookup: Π$\mathcal{M}$, $\mathcal{X}$: $\mathcal{M}$ ∈ N ⊢ constantRationalSeries($\mathcal{X}$)[$\mathcal{M}$
$\mathcal{X}$]

SystemQ **proof of** CrsLookup:

L01: Arbitrary ≫ $\mathcal{M}$, $\mathcal{X}$;

L02: Premise ≫ $\mathcal{M}$ ∈ N ;

L03: CrsIsSeries ≫ IsSeries(constantRationalSeries($\mathcal{X}$), Q) ;

L04: MemberOfSeries ⊳ L02 ⊳ L03 ≫ OrderedPair($\mathcal{M}$, constantRationalSeries
constantRationalSeries($\mathcal{X}$) ;

L05: CrsIsTotal ⊳ L02 ≫ OrderedPair($\mathcal{M}$, $\mathcal{X}$) ∈ constantRationalSeries($\mathcal{X}$) ;

L06: eqReflexivity ≫ $\mathcal{M} = \mathcal{M}$;

L07: UniqueMember ⊳ L03 ⊳ L04 ⊳ L05 ⊳ L06 ≫ constantRationalSeries($\mathcal{X}$)[$\mathcal{M}$] = $\mathcal{X}$ □

[SystemQ **lemma** 0f: Π$\mathcal{M}$: $\mathcal{M}$ ∈ N ⊢ 0f[$\mathcal{M}$] = 0]

SystemQ **proof of** 0f:

L01: Arbitrary ≫ $\mathcal{M}$;

L02: Premise ≫ $\mathcal{M}$ ∈ N ;

L03: CrsLookup ⊳ L02 ≫ constantRationalSeries(0)[$\mathcal{M}$] = 0 ;

L04: Repetition ⊳ L03 ≫ 0f[$\mathcal{M}$] = 0 □

[SystemQ **lemma** 1f: Π$\mathcal{M}$: $\mathcal{M}$ ∈ N ⊢ 1f[$\mathcal{M}$] = 1]

SystemQ **proof of** 1f:

L01: Arbitrary ≫ $\mathcal{M}$;

L02: Premise ≫ $\mathcal{M}$ ∈ N ;

L03: CrsLookup ⊳ L02 ≫ constantRationalSeries(1)[$\mathcal{M}$] = 1 ;

L04: Repetition ⊳ L03 ≫ 1f[$\mathcal{M}$] = 1 □

———-(6.11.06, lemmaer fra kvanti, mod kronologien)

[SystemQ **lemma** DistributionOut: Π$\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$: $((\mathcal{X} * \mathcal{Y}) + (\mathcal{X} * \mathcal{Z})) = (\mathcal{X} * ((\mathcal{Y} + \mathcal{Z})))$]

SystemQ **proof of** DistributionOut:

L01: Arbitrary ≫ $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$;

L02: Distribution ≫ $(\mathcal{X} * ((\mathcal{Y} + \mathcal{Z}))) = ((\mathcal{X} * \mathcal{Y}) + (\mathcal{X} * \mathcal{Z}))$;

L03: eqSymmetry ▷ L02 ≫ $((\mathcal{X}*\mathcal{Y})+(\mathcal{X}*\mathcal{Z})) = (\mathcal{X}*((\mathcal{Y}+\mathcal{Z})))$ □

[SystemQ **lemma** Three2twoTerms: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{U}\colon (\mathcal{Y}+\mathcal{Z}) = \mathcal{U} \vdash ((\mathcal{X}+\mathcal{Y})+\mathcal{Z}) = (\mathcal{X}+\mathcal{U})$]

SystemQ **proof of** Three2twoTerms:

L01: Arbitrary ≫ $\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{U}$;
L02: Premise ≫ $(\mathcal{Y}+\mathcal{Z}) = \mathcal{U}$;
L03: EqAdditionLeft ▷ L02 ≫ $(\mathcal{X}+((\mathcal{Y}+\mathcal{Z}))) = (\mathcal{X}+\mathcal{U})$;
L04: plusAssociativity ≫ $((\mathcal{X}+\mathcal{Y})+\mathcal{Z}) = (\mathcal{X}+((\mathcal{Y}+\mathcal{Z})))$;
L05: eqTransitivity ▷ L04 ▷ L03 ≫ $((\mathcal{X}+\mathcal{Y})+\mathcal{Z}) = (\mathcal{X}+\mathcal{U})$ □

[SystemQ **lemma** Three2threeTerms: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z}\colon ((\mathcal{X}+\mathcal{Y})+\mathcal{Z}) = ((\mathcal{X}+\mathcal{Z})+\mathcal{Y})$]

SystemQ **proof of** Three2threeTerms:

L01: Arbitrary ≫ $\mathcal{X},\mathcal{Y},\mathcal{Z}$;
L02: plusCommutativity ≫ $(\mathcal{Y}+\mathcal{Z}) = (\mathcal{Z}+\mathcal{Y})$;
L03: Three2twoTerms ▷ L02 ≫ $((\mathcal{X}+\mathcal{Y})+\mathcal{Z}) = (\mathcal{X}+((\mathcal{Z}+\mathcal{Y})))$;
L04: plusAssociativity ≫ $((\mathcal{X}+\mathcal{Z})+\mathcal{Y}) = (\mathcal{X}+((\mathcal{Z}+\mathcal{Y})))$;
L05: eqSymmetry ▷ L04 ≫ $(\mathcal{X}+((\mathcal{Z}+\mathcal{Y}))) = ((\mathcal{X}+\mathcal{Z})+\mathcal{Y})$;
L06: eqTransitivity ▷ L03 ▷ L05 ≫ $((\mathcal{X}+\mathcal{Y})+\mathcal{Z}) = ((\mathcal{X}+\mathcal{Z})+\mathcal{Y})$ □

[SystemQ **lemma** Three2twoFactors: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{U}\colon (\mathcal{Y}*\mathcal{Z}) = \mathcal{U} \vdash ((\mathcal{X}*\mathcal{Y})*\mathcal{Z}) = (\mathcal{X}*\mathcal{U})$]

SystemQ **proof of** Three2twoFactors:

L01: Arbitrary ≫ $\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{U}$;
L02: Premise ≫ $(\mathcal{Y}*\mathcal{Z}) = \mathcal{U}$;
L03: EqMultiplicationLeft ▷ L02 ≫ $(\mathcal{X}*((\mathcal{Y}*\mathcal{Z}))) = (\mathcal{X}*\mathcal{U})$;
L04: timesAssociativity ≫ $((\mathcal{X}*\mathcal{Y})*\mathcal{Z}) = (\mathcal{X}*((\mathcal{Y}*\mathcal{Z})))$;
L05: eqTransitivity ▷ L04 ▷ L03 ≫ $((\mathcal{X}*\mathcal{Y})*\mathcal{Z}) = (\mathcal{X}*\mathcal{U})$ □

[SystemQ **lemma** x = x + (y − y): $\Pi\mathcal{X},\mathcal{Y}\colon \mathcal{X} = (\mathcal{X}+((\mathcal{Y}-\mathcal{Y})))$]

SystemQ **proof of** x = x + (y − y):

L01: Arbitrary ≫ $\mathcal{X},\mathcal{Y}$;
L02: plus0 ≫ $(\mathcal{X}+0) = \mathcal{X}$;
L03: Negative ≫ $(\mathcal{Y}-\mathcal{Y}) = 0$;
L04: eqSymmetry ▷ L03 ≫ $0 = (\mathcal{Y}-\mathcal{Y})$;
L05: EqAdditionLeft ▷ L04 ≫ $(\mathcal{X}+0) = (\mathcal{X}+((\mathcal{Y}-\mathcal{Y})))$;
L06: Equality ▷ L02 ▷ L05 ≫ $\mathcal{X} = (\mathcal{X}+((\mathcal{Y}-\mathcal{Y})))$ □

[SystemQ **lemma** x = x + y − y: $\Pi\mathcal{X},\mathcal{Y}\colon \mathcal{X} = ((\mathcal{X}+\mathcal{Y})-\mathcal{Y})$]

SystemQ **proof of** x = x + y − y:

L01: Arbitrary ≫ $\mathcal{X},\mathcal{Y}$;
L02: x = x + (y − y) ≫ $\mathcal{X} = (\mathcal{X}+((\mathcal{Y}-\mathcal{Y})))$;
L03: plusAssociativity ≫ $((\mathcal{X}+\mathcal{Y})-\mathcal{Y}) = (\mathcal{X}+((\mathcal{Y}-\mathcal{Y})))$;
L04: eqSymmetry ▷ L03 ≫ $(\mathcal{X}+((\mathcal{Y}-\mathcal{Y}))) = ((\mathcal{X}+\mathcal{Y})-\mathcal{Y})$;
L05: eqTransitivity ▷ L02 ▷ L04 ≫ $\mathcal{X} = ((\mathcal{X}+\mathcal{Y})-\mathcal{Y})$ □

[SystemQ **lemma** x = x ∗ y ∗ (1/y): $\Pi\mathcal{X},\mathcal{Y}\colon \mathcal{Y} \neq 0 \vdash \mathcal{X} = ((\mathcal{X}*\mathcal{Y})*\mathrm{rec}\mathcal{Y})$]

SystemQ **proof of** x = x ∗ y ∗ (1/y):

L01: Arbitrary ≫ $\mathcal{X},\mathcal{Y}$;
L02: Premise ≫ $\mathcal{Y} \neq 0$;

L03: times1 ≫ $(\mathcal{X} * 1) = \mathcal{X}$;
L04: Reciprocal ▷ L02 ≫ $(\mathcal{Y} * \mathrm{rec}\mathcal{Y}) = 1$;
L05: Three2twoFactors ▷ L04 ≫ $((\mathcal{X} * \mathcal{Y}) * \mathrm{rec}\mathcal{Y}) = (\mathcal{X} * 1)$;
L06: eqTransitivity ▷ L05 ▷ L03 ≫ $((\mathcal{X} * \mathcal{Y}) * \mathrm{rec}\mathcal{Y}) = \mathcal{X}$;
L07: eqSymmetry ▷ L06 ≫ $\mathcal{X} = ((\mathcal{X} * \mathcal{Y}) * \mathrm{rec}\mathcal{Y})$ □

[SystemQ **lemma** x ∗ 0 + x = x: $\Pi\mathcal{X}$: $((\mathcal{X} * 0) + \mathcal{X}) = \mathcal{X}$]

SystemQ **proof of** x ∗ 0 + x = x:

L01: Arbitrary ≫ $\mathcal{X}$;
L02: times1 ≫ $(\mathcal{X} * 1) = \mathcal{X}$;
L03: eqSymmetry ▷ L02 ≫ $\mathcal{X} = (\mathcal{X} * 1)$;
L04: EqAdditionLeft ▷ L03 ≫ $((\mathcal{X}*0)+\mathcal{X}) = ((\mathcal{X}*0)+(\mathcal{X}*1))$;
L05: Distribution ≫ $(\mathcal{X}*((0+1))) = ((\mathcal{X}*0)+(\mathcal{X}*1))$;
L06: eqSymmetry ▷ L05 ≫ $((\mathcal{X}*0)+(\mathcal{X}*1)) = (\mathcal{X}*((0+1)))$;
L07: plus0Left ≫ $(0 + 1) = 1$;
L08: EqMultiplicationLeft ▷ L07 ≫ $(\mathcal{X} * ((0 + 1))) = (\mathcal{X} * 1)$;
L09: eqTransitivity5 ▷ L04 ▷ L06 ▷ L08 ▷ L02 ≫ $((\mathcal{X} * 0) + \mathcal{X}) = \mathcal{X}$ □

[SystemQ **lemma** x ∗ 0 = 0: $\Pi\mathcal{X}$: $(\mathcal{X} * 0) = 0$]

SystemQ **proof of** x ∗ 0 = 0:

L01: Arbitrary ≫ $\mathcal{X}$;
L02: x = x + (y − y) ≫ $(\mathcal{X} * 0) = ((\mathcal{X} * 0) + ((\mathcal{X} - \mathcal{X})))$;
L03: plusAssociativity ≫ $(((\mathcal{X} * 0) + \mathcal{X}) - \mathcal{X}) = ((\mathcal{X} * 0) + ((\mathcal{X} - \mathcal{X})))$;
L04: eqSymmetry ▷ L03 ≫ $((\mathcal{X} * 0) + ((\mathcal{X} - \mathcal{X}))) = (((\mathcal{X} * 0) + \mathcal{X}) - \mathcal{X})$;
L05: x ∗ 0 + x = x ≫ $((\mathcal{X} * 0) + \mathcal{X}) = \mathcal{X}$;
L06: eqAddition ▷ L05 ≫ $(((\mathcal{X} * 0) + \mathcal{X}) - \mathcal{X}) = (\mathcal{X} - \mathcal{X})$;
L07: Negative ≫ $(\mathcal{X} - \mathcal{X}) = 0$;
L08: eqTransitivity5 ▷ L02 ▷ L04 ▷ L06 ▷ L07 ≫ $(\mathcal{X} * 0) = 0$ □

[SystemQ **lemma** (−1) ∗ (−1) + (−1) ∗ 1 = 0: $(((-1) * (-1)) + ((-1) * 1)) = 0$]

SystemQ **proof of** (−1) ∗ (−1) + (−1) ∗ 1 = 0:

L01: DistributionOut ≫ $(((-1) * (-1)) + ((-1) * 1)) = ((-1) * (((-1) + 1)))$;
L02: Negative ≫ $(1 + (-1)) = 0$;
L03: plusCommutativity ≫ $((-1) + 1) = (1 + (-1))$;
L04: eqTransitivity ▷ L03 ▷ L02 ≫ $((-1) + 1) = 0$;
L05: EqMultiplicationLeft ▷ L04 ≫ $((-1)*(((-1)+1))) = ((-1)*0)$;
L06: x ∗ 0 = 0 ≫ $((-1) * 0) = 0$;
L07: eqTransitivity4 ▷ L01 ▷ L05 ▷ L06 ≫ $(((-1) * (-1)) + ((-1) * 1)) = 0$ □

[SystemQ **lemma** (−1) ∗ (−1) = 1: $((-1) * (-1)) = 1$]

SystemQ **proof of** (−1) ∗ (−1) = 1:

L01: x = x + (y − y) ≫ $((-1)*(-1)) = (((-1)*(-1)) + ((1 - 1)))$;
L02: times1 ≫ $((-1) * 1) = (-1)$;

L03:	eqSymmetry $\rhd$ L02 $\gg$	$(-1) = ((-1) * 1)$	;
L04:	EqAdditionLeft $\rhd$ L03 $\gg$	$(1 - 1) = (1 + ((-1) * 1))$	;
L05:	EqAdditionLeft $\rhd$ L04 $\gg$	$(((-1) * (-1)) + ((1 - 1))) = (((-1)*(-1))+((1+((-1)*1))))$	;
L06:	plusCommutativity $\gg$	$(1+((-1)*1)) = (((-1)*1)+1)$	;
L07:	EqAdditionLeft $\rhd$ L06 $\gg$	$(((-1) * (-1)) + ((1 + ((-1) * 1)))) = (((-1)*(-1))+((((-1)* 1) + 1)))$	;
L08:	plusAssociativity $\gg$	$((((-1) * (-1)) + ((-1) * 1)) + 1) = (((-1) * (-1)) + ((((-1) * 1) + 1)))$	;
L09:	eqSymmetry $\rhd$ L08 $\gg$	$(((-1) * (-1)) + ((((-1) * 1) + 1))) = ((((-1) * (-1)) + ((-1) * 1)) + 1)$	;
L10:	$(-1) * (-1) + (-1) * 1 = 0$ $\gg$	$(((-1) * (-1)) + ((-1) * 1)) = 0$	;
L11:	eqAddition $\rhd$ L10 $\gg$	$((((-1) * (-1)) + ((-1) * 1)) + 1) = (0 + 1)$	;
L12:	plus0Left $\gg$	$(0 + 1) = 1$	;
L13:	eqTransitivity5 $\rhd$ L01 $\rhd$ L05 $\rhd$ L07 $\rhd$ L09 $\gg$	$((-1)*(-1)) = ((((-1)*(-1))+ ((-1) * 1)) + 1)$	;
L14:	eqTransitivity4 $\rhd$ L13 $\rhd$ L11 $\rhd$ L12 $\gg$	$((-1) * (-1)) = 1$	$\square$

[SystemQ **lemma** subLeqRight: $\Pi\mathcal{X}, \mathcal{Y}, \mathcal{Z}\colon \mathcal{X} = \mathcal{Y} \vdash \mathcal{Z} <= \mathcal{X} \vdash \mathcal{Z} <= \mathcal{Y}$]

SystemQ **proof of** subLeqRight:

L01:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}, \mathcal{Z}$	;
L02:	Premise $\gg$	$\mathcal{X} = \mathcal{Y}$	;
L03:	Premise $\gg$	$\mathcal{Z} <= \mathcal{X}$	;
L04:	eqLeq $\rhd$ L02 $\gg$	$\mathcal{X} <= \mathcal{Y}$	;
L05:	leqTransitivity $\rhd$ L03 $\rhd$ L04 $\gg$	$\mathcal{Z} <= \mathcal{Y}$	$\square$

[SystemQ **lemma** subLeqLeft: $\Pi\mathcal{X}, \mathcal{Y}, \mathcal{Z}\colon \mathcal{X} = \mathcal{Y} \vdash \mathcal{X} <= \mathcal{Z} \vdash \mathcal{Y} <= \mathcal{Z}$]

SystemQ **proof of** subLeqLeft:

L01:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}, \mathcal{Z}$	;
L02:	Premise $\gg$	$\mathcal{X} = \mathcal{Y}$	;
L03:	Premise $\gg$	$\mathcal{X} <= \mathcal{Z}$	;
L04:	eqSymmetry $\rhd$ L02 $\gg$	$\mathcal{Y} = \mathcal{X}$	;
L05:	eqLeq $\rhd$ L04 $\gg$	$\mathcal{Y} <= \mathcal{X}$	;
L06:	leqTransitivity $\rhd$ L05 $\rhd$ L03 $\gg$	$\mathcal{Y} <= \mathcal{Z}$	$\square$

[SystemQ **lemma** $0 < 1$Helper: $1 <= 0 \Rightarrow 0 <= 1$]

SystemQ **proof of** $0 < 1$Helper:

L01:	Block $\gg$	Begin	;
L02:	Premise $\gg$	$1 <= 0$	;
L03:	leqAddition $\rhd$ L02 $\gg$	$(1 + (-1)) <= (0 + (-1))$	;
L04:	Negative $\gg$	$(1 + (-1)) = 0$	;
L05:	subLeqLeft $\rhd$ L04 $\rhd$ L03 $\gg$	$0 <= (0 + (-1))$	;
L06:	plus0Left $\gg$	$(0 + (-1)) = (-1)$	;

L07:	subLeqRight ▷ L06 ▷ L05 ≫	$0 <= (-1)$	;
L08:	leqMultiplication ▷ L07 ▷ L07 ≫		
		$(0 * (-1)) <= ((-1) * (-1))$	;
L09:	$x * 0 = 0$ ≫	$((-1) * 0) = 0$	;
L10:	timesCommutativity ≫	$(0 * (-1)) = ((-1) * 0)$	;
L11:	eqTransitivity ▷ L10 ▷ L09 ≫	$(0 * (-1)) = 0$	;
L12:	subLeqLeft ▷ L11 ▷ L08 ≫	$0 <= ((-1) * (-1))$	;
L13:	$(-1) * (-1) = 1$ ≫	$((-1) * (-1)) = 1$	;
L14:	subLeqRight ▷ L13 ▷ L12 ≫	$0 <= 1$	;
L15:	Block ≫	End	;
L16:	Ded ▷ L15 ≫	$1 <= 0 \Rightarrow 0 <= 1$	□

[SystemQ **lemma** $0 < 1$: $0 < 1$]

SystemQ **proof of** $0 < 1$:

L01:	leqTotality ≫	$0 <= 1 \dot{\vee} 1 <= 0$	;
L02:	AutoImply ≫	$0 <= 1 \Rightarrow 0 <= 1$	;
L03:	$0 < 1$Helper ≫	$1 <= 0 \Rightarrow 0 <= 1$	;
L04:	FromDisjuncts ▷ L01 ▷ L02 ▷ L03 ≫	$0 <= 1$	;
L05:	0not1 ≫	$0 \neq 1$	;
L06:	JoinConjuncts ▷ L04 ▷ L05 ≫	$0 < 1$	□

[SystemQ **lemma** AddEquations: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}$: $\mathcal{X} = \mathcal{Y} \vdash \mathcal{Z} = \mathcal{U} \vdash (\mathcal{X} + \mathcal{Z}) = (\mathcal{Y} + \mathcal{U})$]

SystemQ **proof of** AddEquations:

L01:	Arbitrary ≫	$\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}$	;
L02:	Premise ≫	$\mathcal{X} = \mathcal{Y}$	;
L03:	Premise ≫	$\mathcal{Z} = \mathcal{U}$	;
L04:	eqAddition ▷ L02 ≫	$(\mathcal{X} + \mathcal{Z}) = (\mathcal{Y} + \mathcal{Z})$	;
L05:	EqAdditionLeft ▷ L03 ≫	$(\mathcal{Y} + \mathcal{Z}) = (\mathcal{Y} + \mathcal{U})$	;
L06:	eqTransitivity ▷ L04 ▷ L05 ≫	$(\mathcal{X} + \mathcal{Z}) = (\mathcal{Y} + \mathcal{U})$	□

[SystemQ **lemma** PositiveToRight(Eq): $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}$: $(\mathcal{X} + \mathcal{Y}) = \mathcal{Z} \vdash \mathcal{X} = (\mathcal{Z} - \mathcal{Y})$]

SystemQ **proof of** PositiveToRight(Eq):

L01:	Arbitrary ≫	$\mathcal{X}, \mathcal{Y}, \mathcal{Z}$	;
L02:	Premise ≫	$(\mathcal{X} + \mathcal{Y}) = \mathcal{Z}$	;
L03:	eqAddition ▷ L02 ≫	$((\mathcal{X} + \mathcal{Y}) - \mathcal{Y}) = (\mathcal{Z} - \mathcal{Y})$	;
L04:	$x = x + y - y$ ≫	$\mathcal{X} = ((\mathcal{X} + \mathcal{Y}) - \mathcal{Y})$	;
L05:	eqTransitivity ▷ L04 ▷ L03 ≫	$\mathcal{X} = (\mathcal{Z} - \mathcal{Y})$	□

[SystemQ **lemma** PositiveToLeft(Eq)(1term): $\Pi \mathcal{X}, \mathcal{Y}$: $\mathcal{X} = \mathcal{Y} \vdash (\mathcal{X} - \mathcal{Y}) = 0$]

SystemQ **proof of** PositiveToLeft(Eq)(1term):

L01:	Arbitrary ≫	$\mathcal{X}, \mathcal{Y}$	;
L02:	Premise ≫	$\mathcal{X} = \mathcal{Y}$	;
L03:	eqAddition ▷ L02 ≫	$(\mathcal{X} - \mathcal{Y}) = (\mathcal{Y} - \mathcal{Y})$	;
L04:	Negative ≫	$(\mathcal{Y} - \mathcal{Y}) = 0$	;
L05:	eqTransitivity ▷ L03 ▷ L04 ≫	$(\mathcal{X} - \mathcal{Y}) = 0$	□

[SystemQ **lemma** PositiveToRight(Leq): $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}$: $(\mathcal{X} + \mathcal{Y}) <= \mathcal{Z} \vdash \mathcal{X} <= (\mathcal{Z} - \mathcal{Y})$]

SystemQ **proof of** PositiveToRight(Leq):

L01:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}, \mathcal{Z}$	;
L02:	Premise $\gg$	$(\mathcal{X} + \mathcal{Y}) <= \mathcal{Z}$	;
L03:	leqAddition $\rhd$ L02 $\gg$	$((\mathcal{X} + \mathcal{Y}) - \mathcal{Y}) <= (\mathcal{Z} - \mathcal{Y})$	;
L04:	x = x + y − y $\gg$	$\mathcal{X} = ((\mathcal{X} + \mathcal{Y}) - \mathcal{Y})$	;
L05:	eqSymmetry $\rhd$ L04 $\gg$	$((\mathcal{X} + \mathcal{Y}) - \mathcal{Y}) = \mathcal{X}$	;
L06:	subLeqLeft $\rhd$ L05 $\rhd$ L03 $\gg$	$\mathcal{X} <= (\mathcal{Z} - \mathcal{Y})$	$\square$

[SystemQ **lemma** PositiveToRight(Leq)(1term): $\Pi \mathcal{Y}, \mathcal{Z}\colon \mathcal{Y} <= \mathcal{Z} \vdash 0 <= (\mathcal{Z} - \mathcal{Y})$]

SystemQ **proof of** PositiveToRight(Leq)(1term):

L01:	Arbitrary $\gg$	$\mathcal{Y}, \mathcal{Z}$	;
L02:	Premise $\gg$	$\mathcal{Y} <= \mathcal{Z}$	;
L03:	plus0Left $\gg$	$(0 + \mathcal{Y}) = \mathcal{Y}$	;
L04:	eqSymmetry $\rhd$ L03 $\gg$	$\mathcal{Y} = (0 + \mathcal{Y})$	;
L05:	subLeqLeft $\rhd$ L04 $\rhd$ L02 $\gg$	$(0 + \mathcal{Y}) <= \mathcal{Z}$	;
L06:	PositiveToRight(Leq) $\rhd$ L05 $\gg$	$0 <= (\mathcal{Z} - \mathcal{Y})$	$\square$

[SystemQ **lemma** NegativeToLeft(Eq): $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}\colon \mathcal{X} = (\mathcal{Y} - \mathcal{Z}) \vdash (\mathcal{X} + \mathcal{Z}) = \mathcal{Y}$]

SystemQ **proof of** NegativeToLeft(Eq):

L01:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}, \mathcal{Z}$	;
L02:	Premise $\gg$	$\mathcal{X} = (\mathcal{Y} - \mathcal{Z})$	;
L03:	eqAddition $\rhd$ L02 $\gg$	$(\mathcal{X} + \mathcal{Z}) = ((\mathcal{Y} - \mathcal{Z}) + \mathcal{Z})$	;
L04:	Three2threeTerms $\gg$	$((\mathcal{Y} - \mathcal{Z}) + \mathcal{Z}) = ((\mathcal{Y} + \mathcal{Z}) - \mathcal{Z})$	;
L05:	x = x + y − y $\gg$	$\mathcal{Y} = ((\mathcal{Y} + \mathcal{Z}) - \mathcal{Z})$	;
L06:	eqSymmetry $\rhd$ L05 $\gg$	$((\mathcal{Y} + \mathcal{Z}) - \mathcal{Z}) = \mathcal{Y}$	;
L07:	eqTransitivity4 $\rhd$ L03 $\rhd$ L04 $\rhd$ L06 $\gg$	$(\mathcal{X} + \mathcal{Z}) = \mathcal{Y}$	$\square$

[SystemQ **lemma** SubtractEquations: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}\colon (\mathcal{X} + \mathcal{Z}) = (\mathcal{Y} + \mathcal{U}) \vdash \mathcal{Z} = \mathcal{U} \vdash \mathcal{X} = \mathcal{Y}$]

SystemQ **proof of** SubtractEquations:

L01:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}$	;
L02:	Premise $\gg$	$(\mathcal{X} + \mathcal{Z}) = (\mathcal{Y} + \mathcal{U})$	;
L03:	Premise $\gg$	$\mathcal{Z} = \mathcal{U}$	;
L04:	eqAddition $\rhd$ L02 $\gg$	$((\mathcal{X} + \mathcal{Z}) - \mathcal{Z}) = ((\mathcal{Y} + \mathcal{U}) - \mathcal{Z})$	;
L05:	plus0Left $\gg$	$(0 + \mathcal{Z}) = \mathcal{Z}$	;
L06:	eqTransitivity $\rhd$ L05 $\rhd$ L03 $\gg$	$(0 + \mathcal{Z}) = \mathcal{U}$	;
L07:	PositiveToRight(Eq) $\rhd$ L06 $\gg$	$0 = (\mathcal{U} - \mathcal{Z})$	;
L08:	eqSymmetry $\rhd$ L07 $\gg$	$(\mathcal{U} - \mathcal{Z}) = 0$	;
L09:	EqAdditionLeft $\rhd$ L08 $\gg$	$(\mathcal{Y} + ((\mathcal{U} - \mathcal{Z}))) = (\mathcal{Y} + 0)$	;
L10:	plusAssociativity $\gg$	$((\mathcal{Y} + \mathcal{U}) - \mathcal{Z}) = (\mathcal{Y} + ((\mathcal{U} - \mathcal{Z})))$	;
L11:	plus0 $\gg$	$(\mathcal{Y} + 0) = \mathcal{Y}$	;
L12:	eqTransitivity4 $\rhd$ L10 $\rhd$ L09 $\rhd$ L11 $\gg$	$((\mathcal{Y} + \mathcal{U}) - \mathcal{Z}) = \mathcal{Y}$	;
L13:	x = x + y − y $\gg$	$\mathcal{X} = ((\mathcal{X} + \mathcal{Z}) - \mathcal{Z})$	;
L14:	eqTransitivity4 $\rhd$ L13 $\rhd$ L04 $\rhd$ L12 $\gg$	$\mathcal{X} = \mathcal{Y}$	$\square$

[SystemQ **lemma** SubtractEquationsLeft: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{U}\colon (\mathcal{X}+\mathcal{Z}) = (\mathcal{Y}+\mathcal{U}) \vdash \mathcal{X} = \mathcal{Y} \vdash \mathcal{Z} = \mathcal{U}$]

SystemQ **proof of** SubtractEquationsLeft:

L01:	Arbitrary $\gg$	$\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{U}$	;
L02:	Premise $\gg$	$(\mathcal{X}+\mathcal{Z}) = (\mathcal{Y}+\mathcal{U})$	;
L03:	Premise $\gg$	$\mathcal{X} = \mathcal{Y}$	;
L04:	plusCommutativity $\gg$	$(\mathcal{Z}+\mathcal{X}) = (\mathcal{X}+\mathcal{Z})$	;
L05:	plusCommutativity $\gg$	$(\mathcal{Y}+\mathcal{U}) = (\mathcal{U}+\mathcal{Y})$	;
L06:	eqTransitivity4 $\rhd$ L04 $\rhd$ L02 $\rhd$ L05 $\gg$	$(\mathcal{Z}+\mathcal{X}) = (\mathcal{U}+\mathcal{Y})$	;
L07:	SubtractEquations $\rhd$ L06 $\rhd$ L03 $\gg$	$\mathcal{Z} = \mathcal{U}$	$\square$

[SystemQ **lemma** EqNegated: $\Pi\mathcal{X},\mathcal{Y}\colon \mathcal{X} = \mathcal{Y} \vdash (-\mathrm{u}\mathcal{X}) = (-\mathrm{u}\mathcal{Y})$]

SystemQ **proof of** EqNegated:

L01:	Arbitrary $\gg$	$\mathcal{X},\mathcal{Y}$	;
L02:	Premise $\gg$	$\mathcal{X} = \mathcal{Y}$	;
L03:	Negative $\gg$	$(\mathcal{X}-\mathcal{X}) = 0$	;
L04:	Negative $\gg$	$(\mathcal{Y}-\mathcal{Y}) = 0$	;
L05:	eqSymmetry $\rhd$ L04 $\gg$	$0 = (\mathcal{Y}-\mathcal{Y})$	;
L06:	eqTransitivity $\rhd$ L03 $\rhd$ L05 $\gg$	$(\mathcal{X}-\mathcal{X}) = (\mathcal{Y}-\mathcal{Y})$	;
L07:	SubtractEquationsLeft $\rhd$ L06 $\rhd$ L02 $\gg$	$(-\mathrm{u}\mathcal{X}) = (-\mathrm{u}\mathcal{Y})$	$\square$

(*** NO EQUALITY ***)

[SystemQ **lemma** LessNeq: $\Pi\mathcal{X},\mathcal{Y}\colon \mathcal{X} < \mathcal{Y} \vdash \mathcal{X} \neq \mathcal{Y}$]

SystemQ **proof of** LessNeq:

L01:	Arbitrary $\gg$	$\mathcal{X},\mathcal{Y}$	;
L02:	Premise $\gg$	$\mathcal{X} < \mathcal{Y}$	;
L03:	Repetition $\rhd$ L02 $\gg$	$\mathcal{X} <= \mathcal{Y} \mathbin{\dot\wedge} \dot\neg((\mathcal{X} = \mathcal{Y}))\mathrm{n}$	;
L04:	SecondConjunct $\rhd$ L03 $\gg$	$\mathcal{X} \neq \mathcal{Y}$	$\square$

[SystemQ **lemma** x + y = zBackwards: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z}\colon (\mathcal{X}+\mathcal{Y}) = \mathcal{Z} \vdash \mathcal{Z} = (\mathcal{Y}+\mathcal{X})$]

SystemQ **proof of** x + y = zBackwards:

L01:	Arbitrary $\gg$	$\mathcal{X},\mathcal{Y},\mathcal{Z}$	;
L02:	Premise $\gg$	$(\mathcal{X}+\mathcal{Y}) = \mathcal{Z}$	;
L03:	plusCommutativity $\gg$	$(\mathcal{X}+\mathcal{Y}) = (\mathcal{Y}+\mathcal{X})$	;
L04:	Equality $\rhd$ L02 $\gg$	$\mathcal{Z} = (\mathcal{Y}+\mathcal{X})$	$\square$

[SystemQ **lemma** x∗y = zBackwards: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z}\colon (\mathcal{X}*\mathcal{Y}) = \mathcal{Z} \vdash \mathcal{Z} = (\mathcal{Y}*\mathcal{X})$]

SystemQ **proof of** x ∗ y = zBackwards:

L01:	Arbitrary $\gg$	$\mathcal{X},\mathcal{Y},\mathcal{Z}$	;
L02:	Premise $\gg$	$(\mathcal{X}*\mathcal{Y}) = \mathcal{Z}$	;
L03:	timesCommutativity $\gg$	$(\mathcal{X}*\mathcal{Y}) = (\mathcal{Y}*\mathcal{X})$	;
L04:	Equality $\rhd$ L02 $\gg$	$\mathcal{Z} = (\mathcal{Y}*\mathcal{X})$	$\square$

[SystemQ **lemma** DoubleMinus: $\Pi\mathcal{X}\colon (-\mathrm{u}(-\mathrm{u}\mathcal{X})) = \mathcal{X}$]

SystemQ **proof of** DoubleMinus:

L01:	Arbitrary $\gg$	$\mathcal{X}$	;
L02:	Negative $\gg$	$((-\mathrm{u}\mathcal{X}) - (-\mathrm{u}\mathcal{X})) = 0$	;

L03: x + y = zBackwards ▷ L02 ≫ $0 = ((-\mathrm{u}(-\mathrm{u}\mathcal{X})) - \mathcal{X})$;
L04: NegativeToLeft(Eq) ▷ L03 ≫ $(0 + \mathcal{X}) = (-\mathrm{u}(-\mathrm{u}\mathcal{X}))$;
L05: plus0Left ≫ $(0 + \mathcal{X}) = \mathcal{X}$;
L06: Equality ▷ L04 ▷ L05 ≫ $(-\mathrm{u}(-\mathrm{u}\mathcal{X})) = \mathcal{X}$ □

[SystemQ **lemma** NeqNegated: $\Pi\mathcal{X}, \mathcal{Y}: \mathcal{X} \neq \mathcal{Y} \vdash (-\mathrm{u}\mathcal{X}) \neq (-\mathrm{u}\mathcal{Y})$]

SystemQ **proof of** NeqNegated:

L01: Block ≫ Begin ;
L02: Arbitrary ≫ $\mathcal{X}, \mathcal{Y}$;
L03: Premise ≫ $\mathcal{X} \neq \mathcal{Y}$;
L04: Premise ≫ $(-\mathrm{u}\mathcal{X}) = (-\mathrm{u}\mathcal{Y})$;
L05: EqNegated ▷ L04 ≫ $(-\mathrm{u}(-\mathrm{u}\mathcal{X})) = (-\mathrm{u}(-\mathrm{u}\mathcal{Y}))$;
L06: DoubleMinus ≫ $(-\mathrm{u}(-\mathrm{u}\mathcal{X})) = \mathcal{X}$;
L07: eqSymmetry ▷ L06 ≫ $\mathcal{X} = (-\mathrm{u}(-\mathrm{u}\mathcal{X}))$;
L08: DoubleMinus ≫ $(-\mathrm{u}(-\mathrm{u}\mathcal{Y})) = \mathcal{Y}$;
L09: eqTransitivity4 ▷ L07 ▷ L05 ▷ L08 ≫ $\mathcal{X} = \mathcal{Y}$;
L10: FromContradiction ▷ L09 ▷ L03 ≫ $(-\mathrm{u}\mathcal{X}) \neq (-\mathrm{u}\mathcal{Y})$;
L11: Block ≫ End ;
L12: Arbitrary ≫ $\mathcal{X}, \mathcal{Y}$;
L13: Ded ▷ L11 ≫ $\mathcal{X} \neq \mathcal{Y} \Rightarrow (-\mathrm{u}\mathcal{X}) = (-\mathrm{u}\mathcal{Y}) \Rightarrow \dot{\neg}((-\mathrm{u}\mathcal{X}) = (-\mathrm{u}\mathcal{Y}))\mathrm{n}$;
L14: Premise ≫ $\mathcal{X} \neq \mathcal{Y}$;
L15: MP ▷ L13 ▷ L14 ≫ $(-\mathrm{u}\mathcal{X}) = (-\mathrm{u}\mathcal{Y}) \Rightarrow \dot{\neg}((-\mathrm{u}\mathcal{X}) = (-\mathrm{u}\mathcal{Y}))\mathrm{n}$;
L16: prop lemma imply negation ▷ L15 ≫ $\dot{\neg}((-\mathrm{u}\mathcal{X}) = (-\mathrm{u}\mathcal{Y}))\mathrm{n}$ □

[SystemQ **lemma** SubNeqRight: $\Pi\mathcal{X}, \mathcal{Y}, \mathcal{Z}: \mathcal{X} = \mathcal{Y} \vdash \mathcal{Z} \neq \mathcal{X} \vdash \mathcal{Z} \neq \mathcal{Y}$]

SystemQ **proof of** SubNeqRight:

L01: Arbitrary ≫ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$;
L02: Premise ≫ $\mathcal{X} = \mathcal{Y}$;
L03: Premise ≫ $\mathcal{Z} \neq \mathcal{X}$;
L04: NeqSymmetry ▷ L03 ≫ $\mathcal{X} \neq \mathcal{Z}$;
L05: SubNeqLeft ▷ L02 ▷ L04 ≫ $\mathcal{Y} \neq \mathcal{Z}$;
L06: NeqSymmetry ▷ L05 ≫ $\mathcal{Z} \neq \mathcal{Y}$ □

[SystemQ **lemma** NeqAddition: $\Pi\mathcal{X}, \mathcal{Y}, \mathcal{Z}: \mathcal{X} \neq \mathcal{Y} \vdash (\mathcal{X} + \mathcal{Z}) \neq (\mathcal{Y} + \mathcal{Z})$]

SystemQ **proof of** NeqAddition:

L01: Block ≫ Begin ;
L02: Arbitrary ≫ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$;
L03: Premise ≫ $\mathcal{X} \neq \mathcal{Y}$;
L04: Premise ≫ $(\mathcal{X} + \mathcal{Z}) = (\mathcal{Y} + \mathcal{Z})$;
L05: eqReflexivity ≫ $\mathcal{Z} = \mathcal{Z}$;
L06: SubtractEquations ▷ L04 ▷ L05 ≫ $\mathcal{X} = \mathcal{Y}$;
L07: FromContradiction ▷ L06 ▷ L03 ≫ $(\mathcal{X} + \mathcal{Z}) \neq (\mathcal{Y} + \mathcal{Z})$;

L08:	Block $\gg$	End	;
L09:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}, \mathcal{Z}$	;
L10:	Ded $\rhd$ L08 $\gg$	$\mathcal{X} \neq \mathcal{Y} \Rightarrow (\mathcal{X}+\mathcal{Z}) = (\mathcal{Y}+\mathcal{Z}) \Rightarrow (\mathcal{X}+\mathcal{Z}) \neq (\mathcal{Y}+\mathcal{Z})$	;
L11:	Premise $\gg$	$\mathcal{X} \neq \mathcal{Y}$	;
L12:	MP $\rhd$ L10 $\rhd$ L11 $\gg$	$(\mathcal{X}+\mathcal{Z}) = (\mathcal{Y}+\mathcal{Z}) \Rightarrow (\mathcal{X}+\mathcal{Z}) \neq (\mathcal{Y}+\mathcal{Z})$	;
L13:	prop lemma imply negation $\rhd$ L12 $\gg$	$(\mathcal{X}+\mathcal{Z}) \neq (\mathcal{Y}+\mathcal{Z})$	$\square$

[SystemQ **lemma** NeqMultiplication: $\Pi\mathcal{X}, \mathcal{Y}, \mathcal{Z}\colon \mathcal{Z} \neq 0 \vdash \mathcal{X} \neq \mathcal{Y} \vdash (\mathcal{X}*\mathcal{Z}) \neq (\mathcal{Y}*\mathcal{Z})$]

SystemQ **proof of** NeqMultiplication:

L01:	Block $\gg$	Begin	;
L02:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}, \mathcal{Z}$	;
L03:	Premise $\gg$	$\mathcal{Z} \neq 0$	;
L04:	Premise $\gg$	$\mathcal{X} \neq \mathcal{Y}$	;
L05:	Premise $\gg$	$(\mathcal{X}*\mathcal{Z}) = (\mathcal{Y}*\mathcal{Z})$	;
L06:	x = x * y * (1/y) $\rhd$ L03 $\gg$	$\mathcal{X} = ((\mathcal{X}*\mathcal{Z}) * \mathrm{rec}\mathcal{Z})$	;
L07:	eqMultiplication $\rhd$ L05 $\gg$	$((\mathcal{X}*\mathcal{Z}) * \mathrm{rec}\mathcal{Z}) = ((\mathcal{Y}*\mathcal{Z}) * \mathrm{rec}\mathcal{Z})$	;
L08:	x = x * y * (1/y) $\rhd$ L03 $\gg$	$\mathcal{Y} = ((\mathcal{Y}*\mathcal{Z}) * \mathrm{rec}\mathcal{Z})$	;
L09:	eqSymmetry $\rhd$ L08 $\gg$	$((\mathcal{Y}*\mathcal{Z}) * \mathrm{rec}\mathcal{Z}) = \mathcal{Y}$	;
L10:	eqTransitivity4 $\rhd$ L06 $\rhd$ L07 $\rhd$ L09 $\gg$	$\mathcal{X} = \mathcal{Y}$	;
L11:	FromContradiction $\rhd$ L10 $\rhd$ L04 $\gg$	$(\mathcal{X}*\mathcal{Z}) \neq (\mathcal{Y}*\mathcal{Z})$	;
L12:	Block $\gg$	End	;
L13:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}, \mathcal{Z}$	;
L14:	Ded $\rhd$ L12 $\gg$	$\mathcal{Z} \neq 0 \Rightarrow \mathcal{X} \neq \mathcal{Y} \Rightarrow (\mathcal{X}*\mathcal{Z}) = (\mathcal{Y}*\mathcal{Z}) \Rightarrow (\mathcal{X}*\mathcal{Z}) \neq (\mathcal{Y}*\mathcal{Z})$	;
L15:	Premise $\gg$	$\mathcal{Z} \neq 0$	;
L16:	Premise $\gg$	$\mathcal{X} \neq \mathcal{Y}$	;
L17:	MP2 $\rhd$ L14 $\rhd$ L15 $\rhd$ L16 $\gg$	$(\mathcal{X}*\mathcal{Z}) = (\mathcal{Y}*\mathcal{Z}) \Rightarrow (\mathcal{X}*\mathcal{Z}) \neq (\mathcal{Y}*\mathcal{Z})$	;
L18:	prop lemma imply negation $\rhd$ L17 $\gg$	$(\mathcal{X}*\mathcal{Z}) \neq (\mathcal{Y}*\mathcal{Z})$	$\square$

(*** NEGATIVE ***)

[SystemQ **lemma** UniqueNegative: $\Pi\mathcal{X}, \mathcal{Y}, \mathcal{Z}\colon (\mathcal{X}+\mathcal{Y}) = 0 \vdash (\mathcal{X}+\mathcal{Z}) = 0 \vdash \mathcal{Y} = \mathcal{Z}$]

SystemQ **proof of** UniqueNegative:

L01:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}, \mathcal{Z}$	;
L02:	Premise $\gg$	$(\mathcal{X}+\mathcal{Y}) = 0$	;
L03:	Premise $\gg$	$(\mathcal{X}+\mathcal{Z}) = 0$	;
L04:	plusCommutativity $\gg$	$(\mathcal{Y}+\mathcal{X}) = (\mathcal{X}+\mathcal{Y})$	;
L05:	eqTransitivity $\rhd$ L04 $\rhd$ L02 $\gg$	$(\mathcal{Y}+\mathcal{X}) = 0$	;
L06:	PositiveToRight(Eq) $\rhd$ L05 $\gg$	$\mathcal{Y} = (0 - \mathcal{X})$	;

L07: plusCommutativity $\gg$ $(\mathcal{Z} + \mathcal{X}) = (\mathcal{X} + \mathcal{Z})$;

L08: eqTransitivity $\rhd$ L07 $\rhd$ L03 $\gg$ $(\mathcal{Z} + \mathcal{X}) = 0$;

L09: PositiveToRight(Eq) $\rhd$ L08 $\gg$ $\mathcal{Z} = (0 - \mathcal{X})$;

L10: eqSymmetry $\rhd$ L09 $\gg$ $(0 - \mathcal{X}) = \mathcal{Z}$;

L11: eqTransitivity $\rhd$ L06 $\rhd$ L10 $\gg$ $\mathcal{Y} = \mathcal{Z}$ □

[SystemQ **lemma** toNotLess: $\Pi\mathcal{X}, \mathcal{Y}\colon \mathcal{X} <= \mathcal{Y} \vdash \dot{\neg}(\mathcal{Y} < \mathcal{X})$n]

SystemQ **proof of** toNotLess:

L01: Block $\gg$ Begin ;

L02: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}$;

L03: Premise $\gg$ $\mathcal{X} <= \mathcal{Y}$;

L04: Premise $\gg$ $\mathcal{Y} <= \mathcal{X}$;

L05: leqAntisymmetry $\rhd$ L04 $\rhd$ L03 $\gg$ $\mathcal{Y} = \mathcal{X}$;

L06: AddDoubleNeg $\rhd$ L05 $\gg$ $\dot{\neg}(\dot{\neg}(\mathcal{Y} = \mathcal{X})\text{n})$n ;

L07: Block $\gg$ End ;

L08: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}$;

L09: Ded $\rhd$ L07 $\gg$ $\mathcal{X} <= \mathcal{Y} \Rightarrow \mathcal{Y} <= \mathcal{X} \Rightarrow \dot{\neg}(\dot{\neg}(\mathcal{Y} = \mathcal{X})\text{n})$n ;

L10: Premise $\gg$ $\mathcal{X} <= \mathcal{Y}$;

L11: MP $\rhd$ L09 $\rhd$ L10 $\gg$ $\mathcal{Y} <= \mathcal{X} \Rightarrow \dot{\neg}(\dot{\neg}(\mathcal{Y} = \mathcal{X})\text{n})$n ;

L12: AddDoubleNeg $\rhd$ L11 $\gg$ $\dot{\neg}(\dot{\neg}((\mathcal{Y} <= \mathcal{X} \Rightarrow \dot{\neg}(\dot{\neg}(\mathcal{Y} = \mathcal{X})\text{n})\text{n}))\text{n})$n ;

L13: Repetition $\rhd$ L12 $\gg$ $\dot{\neg}((\mathcal{Y} <= \mathcal{X} \dot{\wedge} \dot{\neg}(\mathcal{Y} = \mathcal{X})\text{n}))$n ;

L14: Repetition $\rhd$ L13 $\gg$ $\dot{\neg}(\mathcal{Y} < \mathcal{X})$n □

[SystemQ **lemma** FromLess: $\Pi\mathcal{X}, \mathcal{Y}\colon \mathcal{X} < \mathcal{Y} \vdash \dot{\neg}(\mathcal{Y} <= \mathcal{X})$n]

SystemQ **proof of** FromLess:

L01: Block $\gg$ Begin ;

L02: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}$;

L03: Premise $\gg$ $\mathcal{Y} <= \mathcal{X}$;

L04: toNotLess $\rhd$ L03 $\gg$ $\dot{\neg}(\mathcal{X} < \mathcal{Y})$n ;

L05: Block $\gg$ End ;

L06: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}$;

L07: Ded $\rhd$ L05 $\gg$ $\mathcal{Y} <= \mathcal{X} \Rightarrow \dot{\neg}(\mathcal{X} < \mathcal{Y})$n ;

L08: Premise $\gg$ $\mathcal{X} < \mathcal{Y}$;

L09: AddDoubleNeg $\rhd$ L08 $\gg$ $\dot{\neg}(\dot{\neg}(\mathcal{X} < \mathcal{Y})\text{n})$n ;

L10: MT $\rhd$ L07 $\rhd$ L09 $\gg$ $\dot{\neg}(\mathcal{Y} <= \mathcal{X})$n □

[SystemQ **lemma** fromNotLess: $\Pi\mathcal{X}, \mathcal{Y}\colon \dot{\neg}((\mathcal{X} < \mathcal{Y}))\text{n} \vdash \mathcal{Y} <= \mathcal{X}$]

SystemQ **proof of** fromNotLess:

L01: Block $\gg$ Begin ;

L02: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}$;

L03: Premise $\gg$ $\dot{\neg}((\mathcal{X} < \mathcal{Y}))$n ;

L04: Premise $\gg$ $\mathcal{X} <= \mathcal{Y}$;

L05: Repetition $\rhd$ L03 $\gg$ $\dot{\neg}(\dot{\neg}((\mathcal{X} <= \mathcal{Y} \Rightarrow \dot{\neg}(\mathcal{X} \neq \mathcal{Y})\text{n}))\text{n})$n ;

L06: RemoveDoubleNeg $\rhd$ L05 $\gg$ $\mathcal{X} <= \mathcal{Y} \Rightarrow \dot{\neg}(\mathcal{X} \neq \mathcal{Y})$n ;

L07: MP $\rhd$ L06 $\rhd$ L04 $\gg$ $\dot{\neg}(\mathcal{X} \neq \mathcal{Y})$n ;

L08:	RemoveDoubleNeg $\rhd$ L07 $\gg$	$\mathcal{X} = \mathcal{Y}$	;
L09:	eqSymmetry $\rhd$ L08 $\gg$	$\mathcal{Y} = \mathcal{X}$	;
L10:	eqLeq $\rhd$ L09 $\gg$	$\mathcal{Y} <= \mathcal{X}$	;
L11:	Block $\gg$	End	;
L12:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}$	;
L13:	Ded $\rhd$ L11 $\gg$	$\dot{\neg}(\mathcal{X} < \mathcal{Y})\mathrm{n} \Rightarrow \mathcal{X} <= \mathcal{Y} \Rightarrow \mathcal{Y} <= \mathcal{X}$	;
L14:	Premise $\gg$	$\dot{\neg}(\mathcal{X} < \mathcal{Y})\mathrm{n}$	;
L15:	MP $\rhd$ L13 $\rhd$ L14 $\gg$	$\mathcal{X} <= \mathcal{Y} \Rightarrow \mathcal{Y} <= \mathcal{X}$	;
L16:	AutoImply $\gg$	$\mathcal{Y} <= \mathcal{X} \Rightarrow \mathcal{Y} <= \mathcal{X}$	;
L17:	leqTotality $\gg$	$\mathcal{X} <= \mathcal{Y} \dot{\vee} \mathcal{Y} <= \mathcal{X}$	;
L18:	FromDisjuncts $\rhd$ L17 $\rhd$ L15 $\rhd$ L16 $\gg$	$\mathcal{Y} <= \mathcal{X}$	□

[SystemQ **lemma** ToLess: $\Pi \mathcal{X}, \mathcal{Y}\colon \dot{\neg}(\mathcal{X} <= \mathcal{Y})\mathrm{n} \vdash \mathcal{Y} < \mathcal{X}$]

SystemQ **proof of** ToLess:

L01:	Block $\gg$	Begin	;
L02:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}$	;
L03:	Premise $\gg$	$\dot{\neg}(\mathcal{Y} < \mathcal{X})\mathrm{n}$	;
L04:	fromNotLess $\rhd$ L03 $\gg$	$\mathcal{X} <= \mathcal{Y}$	;
L05:	Block $\gg$	End	;
L06:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}$	;
L07:	Ded $\rhd$ L05 $\gg$	$\dot{\neg}(\mathcal{Y} < \mathcal{X})\mathrm{n} \Rightarrow \mathcal{X} <= \mathcal{Y}$	;
L08:	Premise $\gg$	$\dot{\neg}(\mathcal{X} <= \mathcal{Y})\mathrm{n}$	;
L09:	NegativeMT $\rhd$ L07 $\rhd$ L08 $\gg$	$\mathcal{Y} < \mathcal{X}$	□

(*** LEQ ***)

[SystemQ **lemma** LeqLessEq: $\Pi \mathcal{X}, \mathcal{Y}\colon \mathcal{X} <= \mathcal{Y} \vdash \mathcal{X} < \mathcal{Y} \dot{\vee} \mathcal{X} = \mathcal{Y}$]

SystemQ **proof of** LeqLessEq:

L01:	Block $\gg$	Begin	;
L02:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}$	;
L03:	Premise $\gg$	$\mathcal{X} <= \mathcal{Y}$	;
L04:	Premise $\gg$	$\dot{\neg}(\mathcal{X} < \mathcal{Y})\mathrm{n}$	;
L05:	fromNotLess $\rhd$ L04 $\gg$	$\mathcal{Y} <= \mathcal{X}$	;
L06:	leqAntisymmetry $\rhd$ L03 $\rhd$ L05 $\gg$	$\mathcal{X} = \mathcal{Y}$	;
L07:	Block $\gg$	End	;
L08:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}$	;
L09:	Ded $\rhd$ L07 $\gg$	$\mathcal{X} <= \mathcal{Y} \Rightarrow \dot{\neg}(\mathcal{X} < \mathcal{Y})\mathrm{n} \Rightarrow \mathcal{X} = \mathcal{Y}$	;
L10:	Premise $\gg$	$\mathcal{X} <= \mathcal{Y}$	;
L11:	MP $\rhd$ L09 $\rhd$ L10 $\gg$	$\dot{\neg}(\mathcal{X} < \mathcal{Y})\mathrm{n} \Rightarrow \mathcal{X} = \mathcal{Y}$	;
L12:	Repetition $\rhd$ L11 $\gg$	$\mathcal{X} < \mathcal{Y} \dot{\vee} \mathcal{X} = \mathcal{Y}$	□

[SystemQ **lemma** LessLeq: $\Pi \mathcal{X}, \mathcal{Y}\colon \mathcal{X} < \mathcal{Y} \vdash \mathcal{X} <= \mathcal{Y}$]

SystemQ **proof of** LessLeq:

L01:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}$	;
L02:	Premise $\gg$	$\mathcal{X} < \mathcal{Y}$	;
L03:	Repetition $\rhd$ L02 $\gg$	$\mathcal{X} <= \mathcal{Y} \dot{\wedge} \dot{\neg}((\mathcal{X} = \mathcal{Y}))\mathrm{n}$	;

L04: FirstConjunct $\triangleright$ L03 $\gg$ $\mathcal{X} <= \mathcal{Y}$ $\Box$

[SystemQ **lemma** FromLeqGeq: $\Pi\mathcal{A},\mathcal{X},\mathcal{Y}\colon \mathcal{X} <= \mathcal{Y} \Rightarrow \mathcal{A} \vdash \mathcal{Y} <= \mathcal{X} \Rightarrow \mathcal{A} \vdash \mathcal{A}$]

SystemQ **proof of** FromLeqGeq:

L01: Arbitrary $\gg$ $\mathcal{A},\mathcal{X},\mathcal{Y}$;
L02: Premise $\gg$ $\mathcal{X} <= \mathcal{Y} \Rightarrow \mathcal{A}$;
L03: Premise $\gg$ $\mathcal{Y} <= \mathcal{X} \Rightarrow \mathcal{A}$;
L04: leqTotality $\gg$ $\mathcal{X} <= \mathcal{Y} \mathbin{\dot{\vee}} \mathcal{Y} <= \mathcal{X}$;
L05: FromDisjuncts $\triangleright$ L04 $\triangleright$ L02 $\triangleright$ L03 $\gg$ $\mathcal{A}$ $\Box$

[SystemQ **lemma** SubLessRight: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z}\colon \mathcal{X} = \mathcal{Y} \vdash \mathcal{Z} < \mathcal{X} \vdash \mathcal{Z} < \mathcal{Y}$]

SystemQ **proof of** SubLessRight:

L01: Arbitrary $\gg$ $\mathcal{X},\mathcal{Y},\mathcal{Z}$;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$;
L03: Premise $\gg$ $\mathcal{Z} < \mathcal{X}$;
L04: Repetition $\triangleright$ L03 $\gg$ $\mathcal{Z} <= \mathcal{X} \mathbin{\dot{\wedge}} \mathcal{Z} \neq \mathcal{X}$;
L05: FirstConjunct $\triangleright$ L04 $\gg$ $\mathcal{Z} <= \mathcal{X}$;
L06: subLeqRight $\triangleright$ L02 $\triangleright$ L05 $\gg$ $\mathcal{Z} <= \mathcal{Y}$;
L07: SecondConjunct $\triangleright$ L04 $\gg$ $\mathcal{Z} \neq \mathcal{X}$;
L08: SubNeqRight $\triangleright$ L02 $\triangleright$ L07 $\gg$ $\mathcal{Z} \neq \mathcal{Y}$;
L09: JoinConjuncts $\triangleright$ L06 $\triangleright$ L08 $\gg$ $\mathcal{Z} < \mathcal{Y}$ $\Box$

[SystemQ **lemma** SubLessLeft: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z}\colon \mathcal{X} = \mathcal{Y} \vdash \mathcal{X} < \mathcal{Z} \vdash \mathcal{Y} < \mathcal{Z}$]

SystemQ **proof of** SubLessLeft:

L01: Arbitrary $\gg$ $\mathcal{X},\mathcal{Y},\mathcal{Z}$;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$;
L03: Premise $\gg$ $\mathcal{X} < \mathcal{Z}$;
L04: Repetition $\triangleright$ L03 $\gg$ $\mathcal{X} <= \mathcal{Z} \mathbin{\dot{\wedge}} \mathcal{X} \neq \mathcal{Z}$;
L05: FirstConjunct $\triangleright$ L04 $\gg$ $\mathcal{X} <= \mathcal{Z}$;
L06: subLeqLeft $\triangleright$ L02 $\triangleright$ L05 $\gg$ $\mathcal{Y} <= \mathcal{Z}$;
L07: SecondConjunct $\triangleright$ L04 $\gg$ $\mathcal{X} \neq \mathcal{Z}$;
L08: SubNeqLeft $\triangleright$ L02 $\triangleright$ L07 $\gg$ $\mathcal{Y} \neq \mathcal{Z}$;
L09: JoinConjuncts $\triangleright$ L06 $\triangleright$ L08 $\gg$ $\mathcal{Y} < \mathcal{Z}$ $\Box$

[SystemQ **lemma** leqLessTransitivity: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z}\colon \mathcal{X} <= \mathcal{Y} \vdash \mathcal{Y} < \mathcal{Z} \vdash \mathcal{X} < \mathcal{Z}$]

SystemQ **proof of** leqLessTransitivity:

L01: Block $\gg$ Begin ;
L02: Arbitrary $\gg$ $\mathcal{X},\mathcal{Y},\mathcal{Z}$;
L03: Premise $\gg$ $\mathcal{X} <= \mathcal{Y}$;
L04: Premise $\gg$ $\mathcal{Y} < \mathcal{Z}$;
L05: Premise $\gg$ $\mathcal{X} = \mathcal{Z}$;
L06: FirstConjunct $\triangleright$ L04 $\gg$ $\mathcal{Y} <= \mathcal{Z}$;
L07: SecondConjunct $\triangleright$ L04 $\gg$ $\mathcal{Y} \neq \mathcal{Z}$;
L08: subLeqLeft $\triangleright$ L05 $\triangleright$ L03 $\gg$ $\mathcal{Z} <= \mathcal{Y}$;
L09: leqAntisymmetry $\triangleright$ L06 $\triangleright$ L08 $\gg$ $\mathcal{Y} = \mathcal{Z}$;

L10:	FromContradiction ▷ L09 ▷ L07 ≫	$\mathcal{X} \neq \mathcal{Z}$	;
L11:	Block ≫	End	;
L12:	Arbitrary ≫	$\mathcal{X}, \mathcal{Y}, \mathcal{Z}$	;
L13:	Ded ▷ L11 ≫	$\mathcal{X} <= \mathcal{Y} \Rightarrow \mathcal{Y} < \mathcal{Z} \Rightarrow \mathcal{X} = \mathcal{Z} \Rightarrow \mathcal{X} \neq \mathcal{Z}$	;
L14:	Premise ≫	$\mathcal{X} <= \mathcal{Y}$	;
L15:	Premise ≫	$\mathcal{Y} < \mathcal{Z}$	;
L16:	MP2 ▷ L13 ▷ L14 ▷ L15 ≫	$\mathcal{X} = \mathcal{Z} \Rightarrow \mathcal{X} \neq \mathcal{Z}$	;
L17:	prop lemma imply negation ▷ L16 ≫	$\mathcal{X} \neq \mathcal{Z}$	;
L18:	FirstConjunct ▷ L15 ≫	$\mathcal{Y} <= \mathcal{Z}$	;
L19:	leqTransitivity ▷ L14 ▷ L18 ≫	$\mathcal{X} <= \mathcal{Z}$	;
L20:	JoinConjuncts ▷ L19 ▷ L17 ≫	$\mathcal{X} < \mathcal{Z}$	□

[SystemQ **lemma** LessAddition: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z} \colon \mathcal{X} < \mathcal{Y} \vdash (\mathcal{X} + \mathcal{Z}) < (\mathcal{Y} + \mathcal{Z})$]

SystemQ **proof of** LessAddition:

L01:	Arbitrary ≫	$\mathcal{X}, \mathcal{Y}, \mathcal{Z}$	;
L02:	Premise ≫	$\mathcal{X} < \mathcal{Y}$	;
L03:	LessLeq ▷ L02 ≫	$\mathcal{X} <= \mathcal{Y}$	;
L04:	leqAddition ▷ L03 ≫	$(\mathcal{X} + \mathcal{Z}) <= (\mathcal{Y} + \mathcal{Z})$	;
L05:	LessNeq ▷ L02 ≫	$\mathcal{X} \neq \mathcal{Y}$	;
L06:	NeqAddition ▷ L05 ≫	$(\mathcal{X} + \mathcal{Z}) \neq (\mathcal{Y} + \mathcal{Z})$	;
L07:	JoinConjuncts ▷ L04 ▷ L06 ≫	$(\mathcal{X} + \mathcal{Z}) < (\mathcal{Y} + \mathcal{Z})$	□

[SystemQ **lemma** LessAdditionLeft: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z} \colon \mathcal{X} < \mathcal{Y} \vdash (\mathcal{Z} + \mathcal{X}) < (\mathcal{Z} + \mathcal{Y})$]

SystemQ **proof of** LessAdditionLeft:

L01:	Arbitrary ≫	$\mathcal{X}, \mathcal{Y}, \mathcal{Z}$	;
L02:	Premise ≫	$\mathcal{X} < \mathcal{Y}$	;
L03:	LessAddition ▷ L02 ≫	$(\mathcal{X} + \mathcal{Z}) < (\mathcal{Y} + \mathcal{Z})$	;
L04:	plusCommutativity ≫	$(\mathcal{X} + \mathcal{Z}) = (\mathcal{Z} + \mathcal{X})$	;
L05:	SubLessLeft ▷ L04 ▷ L03 ≫	$(\mathcal{Z} + \mathcal{X}) < (\mathcal{Y} + \mathcal{Z})$	;
L06:	plusCommutativity ≫	$(\mathcal{Y} + \mathcal{Z}) = (\mathcal{Z} + \mathcal{Y})$	;
L07:	SubLessRight ▷ L06 ▷ L05 ≫	$(\mathcal{Z} + \mathcal{X}) < (\mathcal{Z} + \mathcal{Y})$	□

[SystemQ **lemma** Leq + 1: $\Pi \mathcal{X}, \mathcal{Y} \colon \mathcal{X} <= \mathcal{Y} \vdash \mathcal{X} < (\mathcal{Y} + 1)$]

SystemQ **proof of** Leq + 1:

L01:	Arbitrary ≫	$\mathcal{X}, \mathcal{Y}$	;
L02:	Premise ≫	$\mathcal{X} <= \mathcal{Y}$	;
L03:	$0 < 1$ ≫	$0 < 1$	;
L04:	LessAdditionLeft ▷ L03 ≫	$(\mathcal{Y} + 0) < (\mathcal{Y} + 1)$	;
L05:	plus0 ≫	$(\mathcal{Y} + 0) = \mathcal{Y}$	;
L06:	SubLessLeft ▷ L05 ▷ L04 ≫	$\mathcal{Y} < (\mathcal{Y} + 1)$	;
L07:	leqLessTransitivity ▷ L02 ▷ L06 ≫	$\mathcal{X} < (\mathcal{Y} + 1)$	□

[SystemQ **lemma** LeqAdditionLeft: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z} \colon \mathcal{X} <= \mathcal{Y} \vdash (\mathcal{Z} + \mathcal{X}) <= (\mathcal{Z} + \mathcal{Y})$]

SystemQ **proof of** LeqAdditionLeft:

L01:	Arbitrary ≫	$\mathcal{X}, \mathcal{Y}, \mathcal{Z}$	;

L02: Premise $\gg$ $\mathcal{X} <= \mathcal{Y}$;
L03: leqAddition $\rhd$ L02 $\gg$ $(\mathcal{X} + \mathcal{Z}) <= (\mathcal{Y} + \mathcal{Z})$;
L04: plusCommutativity $\gg$ $(\mathcal{X} + \mathcal{Z}) = (\mathcal{Z} + \mathcal{X})$;
L05: plusCommutativity $\gg$ $(\mathcal{Y} + \mathcal{Z}) = (\mathcal{Z} + \mathcal{Y})$;
L06: subLeqLeft $\rhd$ L04 $\rhd$ L03 $\gg$ $(\mathcal{Z} + \mathcal{X}) <= (\mathcal{Y} + \mathcal{Z})$;
L07: subLeqRight $\rhd$ L05 $\rhd$ L06 $\gg$ $(\mathcal{Z} + \mathcal{X}) <= (\mathcal{Z} + \mathcal{Y})$ □

[SystemQ **lemma** leqSubtraction: $\Pi\mathcal{X}, \mathcal{Y}, \mathcal{Z}: (\mathcal{X} + \mathcal{Z}) <= (\mathcal{Y} + \mathcal{Z}) \vdash \mathcal{X} <= \mathcal{Y}$]

SystemQ **proof of** leqSubtraction:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$;
L02: Premise $\gg$ $(\mathcal{X} + \mathcal{Z}) <= (\mathcal{Y} + \mathcal{Z})$;
L03: leqAddition $\rhd$ L02 $\gg$ $((\mathcal{X}+\mathcal{Z})-\mathcal{Z}) <= ((\mathcal{Y}+\mathcal{Z})-\mathcal{Z})$;
L04: x = x + y − y $\gg$ $\mathcal{X} = ((\mathcal{X} + \mathcal{Z}) - \mathcal{Z})$;
L05: eqSymmetry $\rhd$ L04 $\gg$ $((\mathcal{X} + \mathcal{Z}) - \mathcal{Z}) = \mathcal{X}$;
L06: x = x + y − y $\gg$ $\mathcal{Y} = ((\mathcal{Y} + \mathcal{Z}) - \mathcal{Z})$;
L07: eqSymmetry $\rhd$ L06 $\gg$ $((\mathcal{Y} + \mathcal{Z}) - \mathcal{Z}) = \mathcal{Y}$;
L08: subLeqLeft $\rhd$ L05 $\rhd$ L03 $\gg$ $\mathcal{X} <= ((\mathcal{Y} + \mathcal{Z}) - \mathcal{Z})$;
L09: subLeqRight $\rhd$ L07 $\rhd$ L08 $\gg$ $\mathcal{X} <= \mathcal{Y}$ □

[SystemQ **lemma** leqSubtractionLeft: $\Pi\mathcal{X}, \mathcal{Y}, \mathcal{Z}: (\mathcal{Z} + \mathcal{X}) <= (\mathcal{Z} + \mathcal{Y}) \vdash \mathcal{X} <= \mathcal{Y}$]

SystemQ **proof of** leqSubtractionLeft:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$;
L02: Premise $\gg$ $(\mathcal{Z} + \mathcal{X}) <= (\mathcal{Z} + \mathcal{Y})$;
L03: plusCommutativity $\gg$ $(\mathcal{Z} + \mathcal{X}) = (\mathcal{X} + \mathcal{Z})$;
L04: plusCommutativity $\gg$ $(\mathcal{Z} + \mathcal{Y}) = (\mathcal{Y} + \mathcal{Z})$;
L05: subLeqLeft $\rhd$ L03 $\rhd$ L02 $\gg$ $(\mathcal{X} + \mathcal{Z}) <= (\mathcal{Z} + \mathcal{Y})$;
L06: subLeqRight $\rhd$ L04 $\rhd$ L05 $\gg$ $(\mathcal{X} + \mathcal{Z}) <= (\mathcal{Y} + \mathcal{Z})$;
L07: leqSubtraction $\rhd$ L06 $\gg$ $\mathcal{X} <= \mathcal{Y}$ □

[SystemQ **lemma** negativeToLeft(Leq): $\Pi\mathcal{X}, \mathcal{Y}, \mathcal{Z}: \mathcal{X} <= (\mathcal{Y} - \mathcal{Z}) \vdash (\mathcal{X} + \mathcal{Z}) <= \mathcal{Y}$]

SystemQ **proof of** negativeToLeft(Leq):

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$;
L02: Premise $\gg$ $\mathcal{X} <= (\mathcal{Y} - \mathcal{Z})$;
L03: leqAddition $\rhd$ L02 $\gg$ $(\mathcal{X} + \mathcal{Z}) <= ((\mathcal{Y} - \mathcal{Z}) + \mathcal{Z})$;
L04: x = x + y − y $\gg$ $\mathcal{Y} = ((\mathcal{Y} + \mathcal{Z}) - \mathcal{Z})$;
L05: Three2threeTerms $\gg$ $((\mathcal{Y} + \mathcal{Z}) - \mathcal{Z}) = ((\mathcal{Y} - \mathcal{Z}) + \mathcal{Z})$;
L06: eqTransitivity $\rhd$ L04 $\rhd$ L05 $\gg$ $\mathcal{Y} = ((\mathcal{Y} - \mathcal{Z}) + \mathcal{Z})$;
L07: eqSymmetry $\rhd$ L06 $\gg$ $((\mathcal{Y} - \mathcal{Z}) + \mathcal{Z}) = \mathcal{Y}$;
L08: subLeqRight $\rhd$ L07 $\rhd$ L03 $\gg$ $(\mathcal{X} + \mathcal{Z}) <= \mathcal{Y}$ □

[SystemQ **lemma** negativeToLeft(Leq)(1term): $\Pi\mathcal{Y}, \mathcal{Z}: 0 <= (\mathcal{Y}-\mathcal{Z}) \vdash \mathcal{Z} <= \mathcal{Y}$]

SystemQ **proof of** negativeToLeft(Leq)(1term):

L01: Arbitrary $\gg$ $\mathcal{Y}, \mathcal{Z}$;
L02: Premise $\gg$ $0 <= (\mathcal{Y} - \mathcal{Z})$;
L03: negativeToLeft(Leq) $\rhd$ L02 $\gg$ $(0 + \mathcal{Z}) <= \mathcal{Y}$;
L04: plus0Left $\gg$ $(0 + \mathcal{Z}) = \mathcal{Z}$;

L05:	subLeqLeft ▷ L04 ≫	$\mathcal{Z} <= \mathcal{Y}$	□

[SystemQ **lemma** PositiveToLeft(Leq): $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z}{:}\,\mathcal{X} <= (\mathcal{Y}+\mathcal{Z}) \vdash (\mathcal{X} - \mathcal{Z}) <= \mathcal{Y}$]

SystemQ **proof of** PositiveToLeft(Leq):

L01:	Arbitrary ≫	$\mathcal{X},\mathcal{Y},\mathcal{Z}$	;
L02:	Premise ≫	$\mathcal{X} <= (\mathcal{Y}+\mathcal{Z})$	;
L03:	leqAddition ▷ L02 ≫	$(\mathcal{X}-\mathcal{Z}) <= ((\mathcal{Y}+\mathcal{Z})-\mathcal{Z})$	;
L04:	x = x + y − y ≫	$\mathcal{Y} = ((\mathcal{Y}+\mathcal{Z})-\mathcal{Z})$	;
L05:	eqSymmetry ▷ L04 ≫	$((\mathcal{Y}+\mathcal{Z})-\mathcal{Z}) = \mathcal{Y}$	;
L06:	subLeqRight ▷ L05 ▷ L03 ≫	$(\mathcal{X}-\mathcal{Z}) <= \mathcal{Y}$	□

[SystemQ **lemma** thirdGeq: $\Pi\mathcal{X},\mathcal{Y}{:}\,\mathcal{X} <= \mathrm{Ex3} \dot{\wedge} \mathcal{Y} <= \mathrm{Ex3}$]

SystemQ **proof of** thirdGeq:

L01:	Block ≫	Begin	;
L02:	Arbitrary ≫	$\mathcal{X},\mathcal{Y}$	;
L03:	Premise ≫	$\mathcal{X} <= \mathcal{Y}$	;
L04:	leqReflexivity ≫	$\mathcal{Y} <= \mathcal{Y}$	;
L05:	JoinConjuncts ▷ L03 ▷ L04 ≫	$\mathcal{X} <= \mathcal{Y} \dot{\wedge} \mathcal{Y} <= \mathcal{Y}$	;
L06:	ExistIntro @ Ex3 @ $\mathcal{Y}$ ▷ L05 ≫	$\mathcal{X} <= \mathrm{Ex3} \dot{\wedge} \mathcal{Y} <= \mathrm{Ex3}$	;
L07:	Block ≫	End	;
L08:	Block ≫	Begin	;
L09:	Arbitrary ≫	$\mathcal{X},\mathcal{Y}$	;
L10:	Premise ≫	$\mathcal{Y} <= \mathcal{X}$	;
L11:	leqReflexivity ≫	$\mathcal{X} <= \mathcal{X}$	;
L12:	JoinConjuncts ▷ L11 ▷ L10 ≫	$\mathcal{X} <= \mathcal{X} \dot{\wedge} \mathcal{Y} <= \mathcal{X}$	;
L13:	ExistIntro @ Ex3 @ $\mathcal{X}$ ▷ L12 ≫	$\mathcal{X} <= \mathrm{Ex3} \dot{\wedge} \mathcal{Y} <= \mathrm{Ex3}$	;
L14:	Block ≫	End	;
L15:	Arbitrary ≫	$\mathcal{X},\mathcal{Y}$	;
L16:	Ded ▷ L07 ≫	$\mathcal{X} <= \mathcal{Y} \Rightarrow \mathcal{X} <= \mathrm{Ex3} \dot{\wedge} \mathcal{Y} <= \mathrm{Ex3}$	;
L17:	Ded ▷ L14 ≫	$\mathcal{Y} <= \mathcal{X} \Rightarrow \mathcal{X} <= \mathrm{Ex3} \dot{\wedge} \mathcal{Y} <= \mathrm{Ex3}$	;
L18:	leqTotality ≫	$\mathcal{X} <= \mathcal{Y} \dot{\vee} \mathcal{Y} <= \mathcal{X}$	;
L19:	FromDisjuncts ▷ L18 ▷ L16 ▷ L17 ≫	$\mathcal{X} <= \mathrm{Ex3} \dot{\wedge} \mathcal{Y} <= \mathrm{Ex3}$	□

[SystemQ **lemma** LeqNegated: $\Pi\mathcal{X},\mathcal{Y}{:}\,\mathcal{X} <= \mathcal{Y} \vdash (-\mathrm{u}\mathcal{Y}) <= (-\mathrm{u}\mathcal{X})$]

SystemQ **proof of** LeqNegated:

L01:	Arbitrary ≫	$\mathcal{X},\mathcal{Y}$	;
L02:	Premise ≫	$\mathcal{X} <= \mathcal{Y}$	;
L03:	leqAddition ▷ L02 ≫	$(\mathcal{X}-\mathcal{X}) <= (\mathcal{Y}-\mathcal{X})$	;
L04:	Negative ≫	$(\mathcal{X}-\mathcal{X}) = 0$	;
L05:	subLeqLeft ▷ L04 ▷ L03 ≫	$0 <= (\mathcal{Y}-\mathcal{X})$	;
L06:	plusCommutativity ≫	$(\mathcal{Y}-\mathcal{X}) = ((-\mathrm{u}\mathcal{X})+\mathcal{Y})$	;
L07:	subLeqRight ▷ L06 ▷ L05 ≫	$0 <= ((-\mathrm{u}\mathcal{X})+\mathcal{Y})$	;
L08:	leqAddition ▷ L07 ≫	$(0-\mathcal{Y}) <= (((-\mathrm{u}\mathcal{X})+\mathcal{Y})-\mathcal{Y})$	;
L09:	plus0Left ≫	$(0-\mathcal{Y}) = (-\mathrm{u}\mathcal{Y})$	;
L10:	x = x + y − y ≫	$(-\mathrm{u}\mathcal{X}) = (((-\mathrm{u}\mathcal{X})+\mathcal{Y})-\mathcal{Y})$	;