SUPSUPSUPSUPSUPSUPSUPSUPSUPSUPSUPSUPSUPSUPSUP

[SystemQ **lemma** RemoveOr: $\Pi\mathcal{A}\colon \mathcal{A} \dot\vee \mathcal{A} \vdash \mathcal{A}$]

SystemQ **proof of** RemoveOr:

| | | | |
|---|---|---|---|
| L01: | Arbitrary $\gg$ | $\mathcal{A}$ | ; |
| L02: | Premise $\gg$ | $\mathcal{A} \dot\vee \mathcal{A}$ | ; |
| L03: | Repetition $\rhd$ L02 $\gg$ | $\dot\neg(\mathcal{A})\mathrm{n} \Rightarrow \mathcal{A}$ | ; |
| L04: | AutoImply $\gg$ | $\mathcal{A} \Rightarrow \mathcal{A}$ | ; |
| L05: | FromNegations $\rhd$ L04 $\rhd$ L03 $\gg$ | $\mathcal{A}$ | $\square$ |

[SystemQ **lemma** leqTransitivity: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z}\colon \mathcal{X} <= \mathcal{Y} \vdash \mathcal{Y} <= \mathcal{Z} \vdash \mathcal{X} <= \mathcal{Z}$]

SystemQ **proof of** leqTransitivity:

| | | | |
|---|---|---|---|
| L01: | Arbitrary $\gg$ | $\mathcal{X},\mathcal{Y},\mathcal{Z}$ | ; |
| L02: | Premise $\gg$ | $\mathcal{X} <= \mathcal{Y}$ | ; |
| L03: | Premise $\gg$ | $\mathcal{Y} <= \mathcal{Z}$ | ; |
| L04: | leqTransitivityAxiom $\gg$ | $\mathcal{X} <= \mathcal{Y} \Rightarrow \mathcal{Y} <= \mathcal{Z} \Rightarrow \mathcal{X} <= \mathcal{Z}$ | ; |
| L05: | MP2 $\rhd$ L04 $\rhd$ L02 $\rhd$ L03 $\gg$ | $\mathcal{X} <= \mathcal{Z}$ | $\square$ |

[SystemQ **lemma** leqAntisymmetry: $\Pi\mathcal{X},\mathcal{Y}\colon \mathcal{X} <= \mathcal{Y} \vdash \mathcal{Y} <= \mathcal{X} \vdash \mathcal{X} = \mathcal{Y}$]

SystemQ **proof of** leqAntisymmetry:

| | | | |
|---|---|---|---|
| L01: | Arbitrary $\gg$ | $\mathcal{X},\mathcal{Y}$ | ; |
| L02: | Premise $\gg$ | $\mathcal{X} <= \mathcal{Y}$ | ; |
| L03: | Premise $\gg$ | $\mathcal{Y} <= \mathcal{X}$ | ; |
| L04: | leqAntisymmetryAxiom $\gg$ | $\mathcal{X} <= \mathcal{Y} \Rightarrow \mathcal{Y} <= \mathcal{X} \Rightarrow \mathcal{X} = \mathcal{Y}$ | ; |
| L05: | MP2 $\rhd$ L04 $\rhd$ L02 $\rhd$ L03 $\gg$ | $\mathcal{X} = \mathcal{Y}$ | $\square$ |

[SystemQ **lemma** leqAddition: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z}\colon \mathcal{X} <= \mathcal{Y} \vdash (\mathcal{X} + \mathcal{Z}) <= (\mathcal{Y} + \mathcal{Z})$]

SystemQ **proof of** leqAddition:

| | | | |
|---|---|---|---|
| L01: | Arbitrary $\gg$ | $\mathcal{X},\mathcal{Y},\mathcal{Z}$ | ; |
| L02: | Premise $\gg$ | $\mathcal{X} <= \mathcal{Y}$ | ; |
| L03: | leqAdditionAxiom $\gg$ | $\mathcal{X} <= \mathcal{Y} \Rightarrow (\mathcal{X}+\mathcal{Z}) <= (\mathcal{Y}+\mathcal{Z})$ | ; |
| L04: | MP $\rhd$ L03 $\rhd$ L02 $\gg$ | $(\mathcal{X} + \mathcal{Z}) <= (\mathcal{Y} + \mathcal{Z})$ | $\square$ |

[SystemQ **lemma** leqMultiplication: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z}\colon 0 <= \mathcal{Z} \vdash \mathcal{X} <= \mathcal{Y} \vdash (\mathcal{X} * \mathcal{Z}) <= (\mathcal{Y} * \mathcal{Z})$]

SystemQ **proof of** leqMultiplication:

| | | | |
|---|---|---|---|
| L01: | Arbitrary $\gg$ | $\mathcal{X},\mathcal{Y},\mathcal{Z}$ | ; |
| L02: | Premise $\gg$ | $0 <= \mathcal{Z}$ | ; |
| L03: | Premise $\gg$ | $\mathcal{X} <= \mathcal{Y}$ | ; |
| L04: | leqMultiplicationAxiom $\gg$ | $0 <= \mathcal{Z} \Rightarrow \mathcal{X} <= \mathcal{Y} \Rightarrow (\mathcal{X} * \mathcal{Z}) <= (\mathcal{Y} * \mathcal{Z})$ | ; |
| L05: | MP2 $\rhd$ L04 $\rhd$ L02 $\rhd$ L03 $\gg$ | $(\mathcal{X} * \mathcal{Z}) <= (\mathcal{Y} * \mathcal{Z})$ | $\square$ |

[SystemQ **lemma** Reciprocal: $\Pi\mathcal{X}\colon \mathcal{X} \neq 0 \vdash (\mathcal{X} * \mathrm{rec}\mathcal{X}) = 1$]

SystemQ **proof of** Reciprocal:

| | | | |
|---|---|---|---|
| L01: | Arbitrary $\gg$ | $\mathcal{X}$ | ; |
| L02: | Premise $\gg$ | $\mathcal{X} \neq 0$ | ; |
| L03: | ReciprocalAxiom $\gg$ | $\mathcal{X} \neq 0 \Rightarrow (\mathcal{X} * \mathrm{rec}\mathcal{X}) = 1$ | ; |
| L04: | MP $\rhd$ L03 $\rhd$ L02 $\gg$ | $(\mathcal{X} * \mathrm{rec}\mathcal{X}) = 1$ | $\square$ |

[SystemQ **lemma** eqLeq: $\Pi\mathcal{X},\mathcal{Y}\colon \mathcal{X} = \mathcal{Y} \vdash \mathcal{X} <= \mathcal{Y}$]

SystemQ **proof of** eqLeq:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}$ ;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$ ;
L03: EqLeqAxiom $\gg$ $\mathcal{X} = \mathcal{Y} \Rightarrow \mathcal{X} <= \mathcal{Y}$ ;
L04: MP $\rhd$ L03 $\rhd$ L02 $\gg$ $\mathcal{X} <= \mathcal{Y}$ $\Box$

[SystemQ **lemma** eqAddition: $\Pi\mathcal{X}, \mathcal{Y}, \mathcal{Z}\colon \mathcal{X} = \mathcal{Y} \vdash (\mathcal{X} + \mathcal{Z}) = (\mathcal{Y} + \mathcal{Z})$]

SystemQ **proof of** eqAddition:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ ;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$ ;
L03: EqAdditionAxiom $\gg$ $\mathcal{X} = \mathcal{Y} \Rightarrow (\mathcal{X} + \mathcal{Z}) = (\mathcal{Y} + \mathcal{Z})$ ;
L04: MP $\rhd$ L03 $\rhd$ L02 $\gg$ $(\mathcal{X} + \mathcal{Z}) = (\mathcal{Y} + \mathcal{Z})$ $\Box$

[SystemQ **lemma** eqMultiplication: $\Pi\mathcal{X}, \mathcal{Y}, \mathcal{Z}\colon \mathcal{X} = \mathcal{Y} \vdash (\mathcal{X} * \mathcal{Z}) = (\mathcal{Y} * \mathcal{Z})$]

SystemQ **proof of** eqMultiplication:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ ;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$ ;
L03: EqMultiplicationAxiom $\gg$ $\mathcal{X} = \mathcal{Y} \Rightarrow (\mathcal{X} * \mathcal{Z}) = (\mathcal{Y} * \mathcal{Z})$ ;
L04: MP $\rhd$ L03 $\rhd$ L02 $\gg$ $(\mathcal{X} * \mathcal{Z}) = (\mathcal{Y} * \mathcal{Z})$ $\Box$

[SystemQ **lemma** Equality: $\Pi\mathcal{X}, \mathcal{Y}, \mathcal{Z}\colon \mathcal{X} = \mathcal{Y} \vdash \mathcal{X} = \mathcal{Z} \vdash \mathcal{Y} = \mathcal{Z}$]

SystemQ **proof of** Equality:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ ;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$ ;
L03: Premise $\gg$ $\mathcal{X} = \mathcal{Z}$ ;
L04: EqualityAxiom $\gg$ $\mathcal{X} = \mathcal{Y} \Rightarrow \mathcal{X} = \mathcal{Z} \Rightarrow \mathcal{Y} = \mathcal{Z}$ ;
L05: MP2 $\rhd$ L04 $\rhd$ L02 $\rhd$ L03 $\gg$ $\mathcal{Y} = \mathcal{Z}$ $\Box$

[SystemQ **lemma** eqReflexivity: $\Pi\mathcal{X}\colon \mathcal{X} = \mathcal{X}$]

SystemQ **proof of** eqReflexivity:

L01: Arbitrary $\gg$ $\mathcal{X}$ ;
L02: leqReflexivity $\gg$ $\mathcal{X} <= \mathcal{X}$ ;
L03: leqAntisymmetry $\rhd$ L02 $\rhd$ L02 $\gg$
$\mathcal{X} = \mathcal{X}$ $\Box$

[SystemQ **lemma** eqSymmetry: $\Pi\mathcal{X}, \mathcal{Y}\colon \mathcal{X} = \mathcal{Y} \vdash \mathcal{Y} = \mathcal{X}$]

SystemQ **proof of** eqSymmetry:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}$ ;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$ ;
L03: eqReflexivity $\gg$ $\mathcal{X} = \mathcal{X}$ ;
L04: Equality $\rhd$ L02 $\rhd$ L03 $\gg$ $\mathcal{Y} = \mathcal{X}$ $\Box$

[SystemQ **lemma** eqTransitivity: $\Pi\mathcal{X}, \mathcal{Y}, \mathcal{Z}\colon \mathcal{X} = \mathcal{Y} \vdash \mathcal{Y} = \mathcal{Z} \vdash \mathcal{X} = \mathcal{Z}$]

SystemQ **proof of** eqTransitivity:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ ;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$ ;
L03: Premise $\gg$ $\mathcal{Y} = \mathcal{Z}$ ;
L04: eqSymmetry $\rhd$ L02 $\gg$ $\mathcal{Y} = \mathcal{X}$ ;
L05: Equality $\rhd$ L04 $\rhd$ L03 $\gg$ $\mathcal{X} = \mathcal{Z}$ $\Box$

[SystemQ **lemma** eqTransitivity4: $\Pi\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}\colon \mathcal{X} = \mathcal{Y} \vdash \mathcal{Y} = \mathcal{Z} \vdash \mathcal{Z} = \mathcal{U} \vdash \mathcal{X} = \mathcal{U}$]

SystemQ **proof of** eqTransitivity4:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}$ ;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$ ;
L03: Premise $\gg$ $\mathcal{Y} = \mathcal{Z}$ ;
L04: Premise $\gg$ $\mathcal{Z} = \mathcal{U}$ ;
L05: eqTransitivity $\rhd$ L02 $\rhd$ L03 $\gg$ $\mathcal{X} = \mathcal{Z}$ ;
L06: eqTransitivity $\rhd$ L05 $\rhd$ L04 $\gg$ $\mathcal{X} = \mathcal{U}$ $\Box$

[SystemQ **lemma** eqTransitivity5: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}, \mathcal{V} \colon \mathcal{X} = \mathcal{Y} \vdash \mathcal{Y} = \mathcal{Z} \vdash \mathcal{Z} = \mathcal{U} \vdash \mathcal{U} = \mathcal{V} \vdash \mathcal{X} = \mathcal{V}$]

SystemQ **proof of** eqTransitivity5:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}, \mathcal{V}$ ;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$ ;
L03: Premise $\gg$ $\mathcal{Y} = \mathcal{Z}$ ;
L04: Premise $\gg$ $\mathcal{Z} = \mathcal{U}$ ;
L05: Premise $\gg$ $\mathcal{U} = \mathcal{V}$ ;
L06: eqTransitivity4 $\rhd$ L02 $\rhd$ L03 $\rhd$ L04 $\gg$ $\mathcal{X} = \mathcal{U}$ ;
L07: eqTransitivity $\rhd$ L06 $\rhd$ L05 $\gg$ $\mathcal{X} = \mathcal{V}$ $\Box$

[SystemQ **lemma** eqTransitivity6: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}, \mathcal{V}, \mathcal{W} \colon \mathcal{X} = \mathcal{Y} \vdash \mathcal{Y} = \mathcal{Z} \vdash \mathcal{Z} = \mathcal{U} \vdash \mathcal{U} = \mathcal{V} \vdash \mathcal{V} = \mathcal{W} \vdash \mathcal{X} = \mathcal{W}$]

SystemQ **proof of** eqTransitivity6:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}, \mathcal{V}, \mathcal{W}$ ;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$ ;
L03: Premise $\gg$ $\mathcal{Y} = \mathcal{Z}$ ;
L04: Premise $\gg$ $\mathcal{Z} = \mathcal{U}$ ;
L05: Premise $\gg$ $\mathcal{U} = \mathcal{V}$ ;
L06: Premise $\gg$ $\mathcal{V} = \mathcal{W}$ ;
L07: eqTransitivity5 $\rhd$ L02 $\rhd$ L03 $\rhd$ L04 $\rhd$ L05 $\gg$ $\mathcal{X} = \mathcal{V}$ ;
L08: eqTransitivity $\rhd$ L07 $\rhd$ L06 $\gg$ $\mathcal{X} = \mathcal{W}$ $\Box$

[SystemQ **lemma** Induction: $\Pi \mathrm{V}_1, \mathcal{A}, \mathcal{B}, \mathcal{C} \colon \langle \mathcal{B} \equiv \mathcal{A} | \mathrm{V}_1 {:==} 0 \rangle_{\mathrm{Me}} \Vdash \langle \mathcal{C} \equiv \mathcal{A} | \mathrm{V}_1 {:==} ($ $1)) \rangle_{\mathrm{Me}} \Vdash \mathcal{B} \vdash \mathcal{A} \Rightarrow \mathcal{C} \vdash \mathcal{A}$]

SystemQ **proof of** Induction:

L01: Arbitrary $\gg$ $\mathrm{V}_1, \mathcal{A}, \mathcal{B}, \mathcal{C}$ ;
L02: Side-condition $\gg$ $\langle \mathcal{B} \equiv \mathcal{A} | \mathrm{V}_1 {:==} 0 \rangle_{\mathrm{Me}}$ ;
L03: Side-condition $\gg$ $\langle \mathcal{C} \equiv \mathcal{A} | \mathrm{V}_1 {:==} (\mathrm{V}_1 + 1) \rangle_{\mathrm{Me}}$ ;
L04: Premise $\gg$ $\mathcal{B}$ ;
L05: Premise $\gg$ $\mathcal{A} \Rightarrow \mathcal{C}$ ;
L06: Gen $\rhd$ L05 $\gg$ $\forall \mathrm{V}_1 \colon (\mathcal{A} \Rightarrow \mathcal{C})$ ;
L07: InductionAxiom $\unrhd$ L02 $\unrhd$ L03 $\gg$ $\mathcal{B} \Rightarrow \forall \mathrm{V}_1 \colon (\mathcal{A} \Rightarrow \mathcal{C}) \Rightarrow \forall \mathrm{V}_1 \colon \mathcal{A}$ ;
L08: MP2 $\rhd$ L07 $\rhd$ L04 $\rhd$ L06 $\gg$ $\forall \mathrm{V}_1 \colon \mathcal{A}$ ;
L09: A4 @ $\mathrm{V}_1$ $\rhd$ L08 $\gg$ $\mathcal{A}$ $\Box$

[SystemQ **lemma** ToSeries: $\Pi \mathrm{FX}, (\mathrm{SY}) \colon \forall (\mathrm{R1ob}) \colon ((\mathrm{R1ob}) \in \mathrm{FX} \Rightarrow \mathrm{IsOrderedPa}$
$\forall (\mathrm{F1ob}), (\mathrm{F2ob}), (\mathrm{F3ob}), (\mathrm{F4ob}) \colon (\mathrm{OrderedPair}((\mathrm{F1ob}), (\mathrm{F2ob})) \in \mathrm{FX} \Rightarrow \mathrm{OrderedPa}$
$\mathrm{FX} \Rightarrow (\mathrm{F1ob}) = (\mathrm{F3ob}) \Rightarrow (\mathrm{F2ob}) = (\mathrm{F4ob})) \vdash \forall (\mathrm{S1ob}) \colon ((\mathrm{S1ob}) \in \mathrm{N} \Rightarrow$

∃(S2ob): OrderedPair((S1ob), (S2ob)) ∈ FX) ⊢ IsSeries(FX, (SY))]

SystemQ **proof of** ToSeries:

| | | | |
|---|---|---|---|
| L01: | Arbitrary ≫ | FX, (SY) | ; |
| L02: | Premise ≫ | ∀(R1ob): ((R1ob) ∈ FX ⇒ IsOrderedPair((R1ob), N, (SY))) | ; |
| L03: | Premise ≫ | ∀(F1ob), (F2ob), (F3ob), (F4ob): (OrderedPair((F1ob), (F2ob)) ∈ FX ⇒ OrderedPair((F3ob), (F4ob)) ∈ FX ⇒ (F1ob) = (F3ob) ⇒ (F2ob) = (F4ob)) | ; |
| L04: | Premise ≫ | ∀(S1ob): ((S1ob) ∈ N ⇒ ∃(S2ob): OrderedPair((S1ob), (S2ob)) ∈ FX) | ; |
| L05: | Repetition ⊳ L02 ≫ | IsRelation(FX, N, (SY)) | ; |
| L06: | JoinConjuncts ⊳ L05 ⊳ L03 ≫ | IsRelation(FX, N, (SY)) ∧̇ ∀(F1ob), (F2ob), (F3ob), (F4ob): (OrderedPair((F1ob), (F2ob)) ∈ FX ⇒ OrderedPair((F3ob), (F4ob)) ∈ FX ⇒ (F1ob) = (F3ob) ⇒ (F2ob) = (F4ob)) | ; |
| L07: | Repetition ⊳ L06 ≫ | isFunction(FX, N, (SY)) | ; |
| L08: | JoinConjuncts ⊳ L07 ⊳ L04 ≫ | isFunction(FX, N, (SY)) ∧̇ ∀(S1ob): ((S1ob) ∈ N ⇒ ∃(S2ob): OrderedPair((S1ob), (S2ob)) ∈ FX) | ; |
| L09: | Repetition ⊳ L08 ≫ | IsSeries(FX, (SY)) | □ |

[SystemQ **lemma** FromSeries: ΠFX, (SY): IsSeries(FX, (SY)) ⊢ (∀(R1ob): ((R1 FX ⇒ ∃(OP1ob): ∃(OP2ob): (OP1ob) ∈ N ∧̇ (OP2ob) ∈ (SY) ∧̇ (R1ob) = OrderedPair((OP1ob), (OP2ob)))) ∧̇ (∀(F1ob), (F2ob), (F3ob), (F4ob): (OrderedPa FX ⇒ OrderedPair((F3ob), (F4ob)) ∈ FX ⇒ (F1ob) = (F3ob) ⇒ (F2ob) = (F4ob))) ∧̇ ∀(S1ob): ((S1ob) ∈ N ⇒ ∃(S2ob): OrderedPair((S1ob), (S2ob)) ∈ FX)]

SystemQ **proof of** FromSeries:

| | | | |
|---|---|---|---|
| L01: | Arbitrary ≫ | FX, (SY) | ; |
| L02: | Premise ≫ | IsSeries(FX, (SY)) | ; |
| L03: | Repetition ⊳ L02 ≫ | isFunction(FX, N, (SY)) ∧̇ ∀(S1ob): ((S1ob) ∈ N ⇒ ∃(S2ob): OrderedPair((S1ob), (S2ob)) ∈ FX) | ; |

| | | |
|---|---|---|
| L04: | Repetition ▷ L03 ≫ | IsRelation(FX, N, (SY)) $\dot\wedge$ (∀(F1ob), (F2ob), (F3ob), (F4ob): (OrderedPair((F1ob), (F2ob)) ∈ FX ⇒ OrderedPair((F3ob), (F4ob)) ∈ FX ⇒ (F1ob) = (F3ob) ⇒ (F2ob) = (F4ob))) $\dot\wedge$ ∀(S1ob): ((S1ob) ∈ N ⇒ ∃(S2ob): OrderedPair((S1ob), (S2ob)) ∈ FX) ; |
| L05: | Repetition ▷ L04 ≫ | (∀(R1ob): ((R1ob) ∈ FX ⇒ IsOrderedPair((R1ob), N, (SY)))) $\dot\wedge$ (∀(F1ob), (F2ob), (F3ob), (F4ob): (OrderedPair((F1ob), (F2ob)) ∈ FX ⇒ OrderedPair((F3ob), (F4ob)) ∈ FX ⇒ (F1ob) = (F3ob) ⇒ (F2ob) = (F4ob))) $\dot\wedge$ ∀(S1ob): ((S1ob) ∈ N ⇒ ∃(S2ob): OrderedPair((S1ob), (S2ob)) ∈ FX) ; |
| L06: | Repetition ▷ L05 ≫ | (∀(R1ob): ((R1ob) ∈ FX ⇒ ∃(OP1ob): ∃(OP2ob): (OP1ob) ∈ N $\dot\wedge$ (OP2ob) ∈ (SY) $\dot\wedge$ (R1ob) = OrderedPair((OP1ob), (OP2ob)))) $\dot\wedge$ (∀(F1ob), (F2ob), (F3ob), (F4ob): (OrderedPair((F1ob), (F2ob)) ∈ FX ⇒ OrderedPair((F3ob), (F4ob)) ∈ FX ⇒ (F1ob) = (F3ob) ⇒ (F2ob) = (F4ob))) $\dot\wedge$ ∀(S1ob): ((S1ob) ∈ N ⇒ ∃(S2ob): OrderedPair((S1ob), (S2ob)) ∈ FX) □ |

[SystemQ **lemma** IntroExist(Helper): $\Pi\mathcal{X}, \mathrm{V}_1, \mathcal{A}, \mathcal{B}$: $\langle \dot\neg(\mathcal{A})\mathrm{n} \equiv \dot\neg(\mathcal{B})\mathrm{n} \mid \mathrm{V}_1 {:=}{=} \mathcal{X}\rangle$ $\forall \mathrm{V}_1: \dot\neg(\mathcal{B})\mathrm{n} \Rightarrow \dot\neg(\mathcal{A})\mathrm{n}$]

SystemQ **proof of** IntroExist(Helper):

| | | |
|---|---|---|
| L01: | Block ≫ | Begin ; |
| L02: | Arbitrary ≫ | $\mathcal{X}, \mathrm{V}_1, \mathcal{A}, \mathcal{B}$ ; |
| L03: | Side-condition ≫ | $\langle \dot\neg(\mathcal{A})\mathrm{n} \equiv \dot\neg(\mathcal{B})\mathrm{n} \mid \mathrm{V}_1 {:=}{=} \mathcal{X}\rangle_{\mathrm{Me}}$ ; |
| L04: | Premise ≫ | $\forall \mathrm{V}_1: \dot\neg(\mathcal{B})\mathrm{n}$ ; |
| L05: | A4 @ $\mathcal{X}$ ⧐ L03 ▷ L04 ≫ | $\dot\neg(\mathcal{A})\mathrm{n}$ ; |
| L06: | Block ≫ | End ; |
| L07: | Arbitrary ≫ | $\mathcal{X}, \mathrm{V}_1, \mathcal{A}, \mathcal{B}$ ; |

L08: Ded $\rhd$ L06 $\gg$ $\langle \dot{\neg}\,(\mathcal{A})\mathrm{n} \equiv \dot{\neg}\,(\mathcal{B})\mathrm{n} | \mathrm{V}_1 {:==} \mathcal{X} \rangle_{\mathrm{Me}} \Vdash \forall \mathrm{V}_1 \colon \dot{\neg}\,(\mathcal{B})\mathrm{n} \Rightarrow \dot{\neg}\,(\mathcal{A})\mathrm{n}$ $\square$

[SystemQ **lemma** IntroExist: $\Pi \mathcal{X}, \mathrm{V}_1, \mathcal{A}, \mathcal{B} \colon \langle \dot{\neg}\,(\mathcal{A})\mathrm{n} \equiv \dot{\neg}\,(\mathcal{B})\mathrm{n} | \mathrm{V}_1 {:==} \mathcal{X} \rangle_{\mathrm{Me}} \Vdash \mathcal{A} \vdash \exists \mathrm{V}_1 \colon \mathcal{B}$]

SystemQ **proof of** IntroExist:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathrm{V}_1, \mathcal{A}, \mathcal{B}$ ;
L02: Side-condition $\gg$ $\langle \dot{\neg}\,(\mathcal{A})\mathrm{n} \equiv \dot{\neg}\,(\mathcal{B})\mathrm{n} | \mathrm{V}_1 {:==} \mathcal{X} \rangle_{\mathrm{Me}}$ ;
L03: IntroExist(Helper) @ $\mathcal{X}$ $\unrhd$ L02 $\gg$ $\forall \mathrm{V}_1 \colon \dot{\neg}\,(\mathcal{B})\mathrm{n} \Rightarrow \dot{\neg}\,(\mathcal{A})\mathrm{n}$ ;
L04: Premise $\gg$ $\mathcal{A}$ ;
L05: AddDoubleNeg $\rhd$ L04 $\gg$ $\dot{\neg}\,(\dot{\neg}\,(\mathcal{A})\mathrm{n})\mathrm{n}$ ;
L06: MT $\rhd$ L03 $\rhd$ L05 $\gg$ $\dot{\neg}\,(\forall \mathrm{V}_1 \colon \dot{\neg}\,(\mathcal{B})\mathrm{n})\mathrm{n}$ ;
L07: Repetition $\rhd$ L06 $\gg$ $\exists \mathrm{V}_1 \colon \mathcal{B}$ $\square$

[SystemQ **lemma** ExistMP: $\Pi \mathrm{V}_1, \mathcal{A}, \mathcal{B} \colon \mathcal{A} \Rightarrow \mathcal{B} \vdash \exists \mathrm{V}_1 \colon \mathcal{A} \vdash \mathcal{B}$]

SystemQ **proof of** ExistMP:

L01: Block $\gg$ Begin ;
L02: Arbitrary $\gg$ $\mathrm{V}_1, \mathcal{A}, \mathcal{B}$ ;
L03: Premise $\gg$ $\mathcal{A} \Rightarrow \mathcal{B}$ ;
L04: Premise $\gg$ $\exists \mathrm{V}_1 \colon \mathcal{A}$ ;
L05: Premise $\gg$ $\dot{\neg}\,(\mathcal{B})\mathrm{n}$ ;
L06: MT $\rhd$ L03 $\rhd$ L05 $\gg$ $\dot{\neg}\,(\mathcal{A})\mathrm{n}$ ;
L07: Gen $\rhd$ L06 $\gg$ $\forall \mathrm{V}_1 \colon \dot{\neg}\,(\mathcal{A})\mathrm{n}$ ;
L08: Repetition $\rhd$ L04 $\gg$ $\dot{\neg}\,(\forall \mathrm{V}_1 \colon \dot{\neg}\,(\mathcal{A})\mathrm{n})\mathrm{n}$ ;
L09: FromContradiction $\rhd$ L07 $\rhd$ L08 $\gg$ $\dot{\neg}\,(\dot{\neg}\,(\mathcal{B})\mathrm{n})\mathrm{n}$ ;
L10: Block $\gg$ End ;
L11: Arbitrary $\gg$ $\mathrm{V}_1, \mathcal{A}, \mathcal{B}$ ;
L12: Ded $\rhd$ L10 $\gg$ $(\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow \exists \mathrm{V}_1 \colon \mathcal{A} \Rightarrow \dot{\neg}\,(\mathcal{B})\mathrm{n} \Rightarrow \dot{\neg}\,(\dot{\neg}\,(\mathcal{B})\mathrm{n})\mathrm{n}$ ;
L04: Premise $\gg$ $\mathcal{A} \Rightarrow \mathcal{B}$ ;
L03: Premise $\gg$ $\exists \mathrm{V}_1 \colon \mathcal{A}$ ;
L05: MP2 $\rhd$ L12 $\rhd$ L04 $\rhd$ L03 $\gg$ $\dot{\neg}\,(\mathcal{B})\mathrm{n} \Rightarrow \dot{\neg}\,(\dot{\neg}\,(\mathcal{B})\mathrm{n})\mathrm{n}$ ;
L06: prop lemma imply negation $\rhd$ L05 $\gg$ $\dot{\neg}\,(\dot{\neg}\,(\mathcal{B})\mathrm{n})\mathrm{n}$ ;
L13: RemoveDoubleNeg $\rhd$ L06 $\gg$ $\mathcal{B}$ $\square$

[SystemQ **lemma** ExistMP2: $\Pi \mathrm{V}_1, \mathrm{V}_2, \mathcal{A}, \mathcal{B}, \mathcal{C} \colon \mathcal{A} \Rightarrow \mathcal{B} \Rightarrow \mathcal{C} \vdash \exists \mathrm{V}_1 \colon \mathcal{A} \vdash \exists \mathrm{V}_2 \colon \mathcal{B} \vdash \mathcal{C}$]

SystemQ **proof of** ExistMP2:

L01: Arbitrary $\gg$ $\mathrm{V}_1, \mathrm{V}_2, \mathcal{A}, \mathcal{B}, \mathcal{C}$ ;
L02: Premise $\gg$ $\mathcal{A} \Rightarrow \mathcal{B} \Rightarrow \mathcal{C}$ ;
L03: Premise $\gg$ $\exists \mathrm{V}_1 \colon \mathcal{A}$ ;
L04: Premise $\gg$ $\exists \mathrm{V}_2 \colon \mathcal{B}$ ;
L05: ExistMP $\rhd$ L02 $\rhd$ L03 $\gg$ $\mathcal{B} \Rightarrow \mathcal{C}$ ;
L06: ExistMP $\rhd$ L05 $\rhd$ L04 $\gg$ $\mathcal{C}$ $\square$

[SystemQ **lemma** TwiceExistMP: $\Pi \mathrm{V}_1, \mathrm{V}_2, \mathcal{A}, \mathcal{B} \colon \mathcal{A} \Rightarrow \mathcal{B} \vdash \exists \mathrm{V}_1 \colon \exists \mathrm{V}_2 \colon \mathcal{A} \vdash \mathcal{B}$]

SystemQ **proof of** TwiceExistMP:

| | | | |
|---|---|---|---|
| L01: | Block $\gg$ | Begin | ; |
| L02: | Arbitrary $\gg$ | $\mathrm{V}_2, \mathcal{A}, \mathcal{B}$ | ; |
| L03: | Premise $\gg$ | $\mathcal{A} \Rightarrow \mathcal{B}$ | ; |
| L04: | Premise $\gg$ | $\exists \mathrm{V}_2 \colon \mathcal{A}$ | ; |
| L05: | ExistMP $\rhd$ L03 $\rhd$ L04 $\gg$ | $\mathcal{B}$ | ; |
| L06: | Block $\gg$ | End | ; |
| L07: | Arbitrary $\gg$ | $\mathrm{V}_1, \mathrm{V}_2, \mathcal{A}, \mathcal{B}$ | ; |
| L03: | Ded $\rhd$ L06 $\gg$ | $(\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow \exists \mathrm{V}_2 \colon \mathcal{A} \Rightarrow \mathcal{B}$ | ; |
| L04: | Premise $\gg$ | $\mathcal{A} \Rightarrow \mathcal{B}$ | ; |
| L08: | Premise $\gg$ | $\exists \mathrm{V}_1 \colon \exists \mathrm{V}_2 \colon \mathcal{A}$ | ; |
| L09: | MP $\rhd$ L03 $\rhd$ L04 $\gg$ | $\exists \mathrm{V}_2 \colon \mathcal{A} \Rightarrow \mathcal{B}$ | ; |
| L10: | ExistMP $\rhd$ L09 $\rhd$ L08 $\gg$ | $\mathcal{B}$ | $\Box$ |

[SystemQ **lemma** TwiceExistMP2: $\Pi \mathrm{V}_1, \mathrm{V}_2, \mathrm{V}_3, \mathrm{V}_4, \mathcal{A}, \mathcal{B}, \mathcal{C} \colon \mathcal{A} \Rightarrow \mathcal{B} \Rightarrow \mathcal{C} \vdash \exists \mathrm{V}_1 \colon \exists \mathrm{V}_2 \colon \mathcal{A} \vdash \exists \mathrm{V}_3 \colon \exists \mathrm{V}_4 \colon \mathcal{B} \vdash \mathcal{C}$]

SystemQ **proof of** TwiceExistMP2:

| | | | |
|---|---|---|---|
| L01: | Arbitrary $\gg$ | $\mathrm{V}_1, \mathrm{V}_2, \mathrm{V}_3, \mathrm{V}_4, \mathcal{A}, \mathcal{B}, \mathcal{C}$ | ; |
| L02: | Premise $\gg$ | $\mathcal{A} \Rightarrow \mathcal{B} \Rightarrow \mathcal{C}$ | ; |
| L03: | Premise $\gg$ | $\exists \mathrm{V}_1 \colon \exists \mathrm{V}_2 \colon \mathcal{A}$ | ; |
| L04: | Premise $\gg$ | $\exists \mathrm{V}_3 \colon \exists \mathrm{V}_4 \colon \mathcal{B}$ | ; |
| L05: | TwiceExistMP $\rhd$ L02 $\rhd$ L03 $\gg$ | $\mathcal{B} \Rightarrow \mathcal{C}$ | ; |
| L06: | TwiceExistMP $\rhd$ L05 $\rhd$ L04 $\gg$ | $\mathcal{C}$ | $\Box$ |

[SystemQ **lemma** NeqSymmetry: $\Pi \mathcal{X}, \mathcal{Y} \colon \mathcal{X} \neq \mathcal{Y} \vdash \mathcal{Y} \neq \mathcal{X}$]

SystemQ **proof of** NeqSymmetry:

| | | | |
|---|---|---|---|
| L01: | Block $\gg$ | Begin | ; |
| L02: | Arbitrary $\gg$ | $\mathcal{X}, \mathcal{Y}$ | ; |
| L03: | Premise $\gg$ | $\mathcal{Y} = \mathcal{X}$ | ; |
| L04: | eqSymmetry $\rhd$ L03 $\gg$ | $\mathcal{X} = \mathcal{Y}$ | ; |
| L05: | Block $\gg$ | End | ; |
| L06: | Arbitrary $\gg$ | $\mathcal{X}, \mathcal{Y}$ | ; |
| L07: | Ded $\rhd$ L05 $\gg$ | $\mathcal{Y} = \mathcal{X} \Rightarrow \mathcal{X} = \mathcal{Y}$ | ; |
| L08: | Premise $\gg$ | $\mathcal{X} \neq \mathcal{Y}$ | ; |
| L09: | MT $\rhd$ L07 $\rhd$ L08 $\gg$ | $\mathcal{Y} \neq \mathcal{X}$ | $\Box$ |

[SystemQ **lemma** SubNeqLeft: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z} \colon \mathcal{X} = \mathcal{Y} \vdash \mathcal{X} \neq \mathcal{Z} \vdash \mathcal{Y} \neq \mathcal{Z}$]

SystemQ **proof of** SubNeqLeft:

| | | | |
|---|---|---|---|
| L01: | Arbitrary $\gg$ | $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ | ; |
| L02: | Premise $\gg$ | $\mathcal{X} = \mathcal{Y}$ | ; |
| L03: | Premise $\gg$ | $\mathcal{X} \neq \mathcal{Z}$ | ; |
| L04: | EqualityAxiom $\gg$ | $\mathcal{Y} = \mathcal{X} \Rightarrow \mathcal{Y} = \mathcal{Z} \Rightarrow \mathcal{X} = \mathcal{Z}$ | ; |
| L05: | eqSymmetry $\rhd$ L02 $\gg$ | $\mathcal{Y} = \mathcal{X}$ | ; |
| L06: | MP $\rhd$ L04 $\rhd$ L05 $\gg$ | $\mathcal{Y} = \mathcal{Z} \Rightarrow \mathcal{X} = \mathcal{Z}$ | ; |
| L07: | Contrapositive $\rhd$ L06 $\gg$ | $\mathcal{X} \neq \mathcal{Z} \Rightarrow \mathcal{Y} \neq \mathcal{Z}$ | ; |
| L08: | MP $\rhd$ L07 $\rhd$ L03 $\gg$ | $\mathcal{Y} \neq \mathcal{Z}$ | $\Box$ |

[SystemQ **lemma** InPair(1): $\Pi(\mathrm{SX}), (\mathrm{SY}) \colon (\mathrm{SX}) \in (\mathrm{p}(\mathrm{SX}), (\mathrm{SY}))$]

SystemQ **proof of** InPair(1):

| | | | |
|---|---|---|---|
| L01: | Arbitrary $\gg$ | $(\mathrm{SX}), (\mathrm{SY})$ | ; |

L02: eqReflexivity $\gg$ $(SX) = (SX)$ ;
L03: WeakenOr2 $\rhd$ L02 $\gg$ $(SX) = (SX) \dot{\vee} (SX) = (SY)$ ;
L04: Formula2Pair $\rhd$ L03 $\gg$ $(SX) \in (p(SX), (SY))$ □

[SystemQ **lemma** InPair(2): $\Pi(SX), (SY)\colon (SY) \in (p(SX), (SY))$]

SystemQ **proof of** InPair(2):

L01: Arbitrary $\gg$ $(SX), (SY)$ ;
L02: eqReflexivity $\gg$ $(SY) = (SY)$ ;
L03: WeakenOr1 $\rhd$ L02 $\gg$ $(SY) = (SX) \dot{\vee} (SY) = (SY)$ ;
L04: Formula2Pair $\rhd$ L03 $\gg$ $(SY) \in (p(SX), (SY))$ □

[SystemQ **lemma** FromSingleton: $\Pi(SX), (SY)\colon (SX) \in (s(SY)) \vdash (SX) = (SY)$]

SystemQ **proof of** FromSingleton:

L01: Arbitrary $\gg$ $(SX), (SY)$ ;
L02: Premise $\gg$ $(SX) \in (s(SY))$ ;
L03: Repetition $\rhd$ L02 $\gg$ $(SX) \in (p(SY), (SY))$ ;
L04: Pair2Formula $\rhd$ L03 $\gg$ $(SX) = (SY) \dot{\vee} (SX) = (SY)$ ;
L05: RemoveOr $\rhd$ L04 $\gg$ $(SX) = (SY)$ □

[SystemQ **lemma** ToSingleton: $\Pi(SX), (SY)\colon (SX) = (SY) \vdash (SX) \in (s(SY))$]

SystemQ **proof of** ToSingleton:

L01: Arbitrary $\gg$ $(SX), (SY)$ ;
L02: Premise $\gg$ $(SX) = (SY)$ ;
L03: WeakenOr1 $\rhd$ L02 $\gg$ $(SX) = (SY) \dot{\vee} (SX) = (SY)$ ;
L04: Formula2Pair $\rhd$ L03 $\gg$ $(SX) \in (p(SY), (SY))$ ;
L05: Repetition $\rhd$ L04 $\gg$ $(SX) \in (s(SY))$ □

[SystemQ **lemma** FromSameSingleton: $\Pi(SX), (SY)\colon (s(SX)) = (s(SY)) \vdash (SX)$ $(SY)$]

SystemQ **proof of** FromSameSingleton:

L01: Arbitrary $\gg$ $(SX), (SY)$ ;
L02: Premise $\gg$ $(s(SX)) = (s(SY))$ ;
L03: eqReflexivity $\gg$ $(SX) = (SX)$ ;
L04: ToSingleton $\rhd$ L03 $\gg$ $(SX) \in (s(SX))$ ;
L05: SENC1 $\rhd$ L02 $\rhd$ L04 $\gg$ $(SX) \in (s(SY))$ ;
L06: FromSingleton $\rhd$ L05 $\gg$ $(SX) = (SY)$ □

[SystemQ **lemma** SingletonmembersEqual: $\Pi(SX), (SY), (SZ)\colon (p(SX), (SY)) = (s(SZ)) \vdash (SX) = (SY)$]

SystemQ **proof of** SingletonmembersEqual:

L01: Arbitrary $\gg$ $(SX), (SY), (SZ)$ ;
L02: Premise $\gg$ $(p(SX), (SY)) = (s(SZ))$ ;
L03: InPair(1) $\gg$ $(SX) \in (p(SX), (SY))$ ;
L04: SENC1 $\rhd$ L02 $\rhd$ L03 $\gg$ $(SX) \in (s(SZ))$ ;
L05: FromSingleton $\rhd$ L04 $\gg$ $(SX) = (SZ)$ ;
L06: InPair(2) $\gg$ $(SY) \in (p(SX), (SY))$ ;
L07: SENC1 $\rhd$ L02 $\rhd$ L06 $\gg$ $(SY) \in (s(SZ))$ ;
L08: FromSingleton $\rhd$ L07 $\gg$ $(SY) = (SZ)$ ;
L09: eqSymmetry $\rhd$ L08 $\gg$ $(SZ) = (SY)$ ;
L10: eqTransitivity $\rhd$ L05 $\rhd$ L09 $\gg$ $(SX) = (SY)$ □

[SystemQ **lemma** UnequalsNotInSingleton: $\Pi(SX),(SY),(SZ)\colon (SX) \neq (SY) \vdash$ $(p(SX),(SY)) \neq (s(SZ))$]

SystemQ **proof of** UnequalsNotInSingleton:

| | | | |
|---|---|---|---|
| L01: | Block $\gg$ | Begin | ; |
| L02: | Arbitrary $\gg$ | $(SX),(SY),(SZ)$ | ; |
| L03: | Premise $\gg$ | $(p(SX),(SY)) = (s(SZ))$ | ; |
| L04: | SingletonmembersEqual $\rhd$ L03 $\gg$ | $(SX) = (SY)$ | ; |
| L05: | Block $\gg$ | End | ; |
| L06: | Arbitrary $\gg$ | $(SX),(SY),(SZ)$ | ; |
| L07: | Ded $\rhd$ L05 $\gg$ | $(p(SX),(SY)) = (s(SZ)) \Rightarrow (SX) = (SY)$ | ; |
| L03: | Premise $\gg$ | $(SX) \neq (SY)$ | ; |
| L08: | MT $\rhd$ L07 $\rhd$ L03 $\gg$ | $(p(SX),(SY)) \neq (s(SZ))$ | □ |

[SystemQ **lemma** NonsingletonmembersUnequal: $\Pi(SX),(SY)\colon (p(SX),(SY)) \neq$ $(s(SX)) \vdash (SX) \neq (SY)$]

SystemQ **proof of** NonsingletonmembersUnequal:

| | | | |
|---|---|---|---|
| L01: | Block $\gg$ | Begin | ; |
| L02: | Arbitrary $\gg$ | $(SX),(SY)$ | ; |
| L03: | Premise $\gg$ | $(SX) = (SY)$ | ; |
| L04: | eqReflexivity $\gg$ | $(SX) = (SX)$ | ; |
| L05: | SamePair $\rhd$ L04 $\rhd$ L03 $\gg$ | $(p(SX),(SX)) = (p(SX),(SY))$ | ; |
| L06: | Repetition $\rhd$ L05 $\gg$ | $(s(SX)) = (p(SX),(SY))$ | ; |
| L07: | eqSymmetry $\rhd$ L06 $\gg$ | $(p(SX),(SY)) = (s(SX))$ | ; |
| L08: | Block $\gg$ | End | ; |
| L09: | Arbitrary $\gg$ | $(SX),(SY)$ | ; |
| L10: | Ded $\rhd$ L08 $\gg$ | $(SX) = (SY) \Rightarrow (p(SX),(SY)) = (s(SX))$ ; | |
| L03: | Premise $\gg$ | $(p(SX),(SY)) \neq (s(SX))$ | ; |
| L11: | MT $\rhd$ L10 $\rhd$ L03 $\gg$ | $(SX) \neq (SY)$ | □ |

[SystemQ **lemma** FromOrderedPair: $\Pi(SX),(SX1),(SY),(SY1)\colon$ OrderedPair(( OrderedPair((SX1),(SY1)) $\vdash (SX) = (SX1) \dot\wedge (SY) = (SY1)$]

SystemQ **proof of** FromOrderedPair:

| | | | |
|---|---|---|---|
| L01: | Block $\gg$ | Begin | ; |
| L02: | Arbitrary $\gg$ | $(SX),(SX1),(SY),(SY1)$ | ; |
| L03: | Premise $\gg$ | $(SX1) = (SY1)$ | ; |
| L04: | Premise $\gg$ | OrderedPair((SX),(SY)) = OrderedPair((SX1),(SY1)) | ; |
| L05: | Repetition $\rhd$ L04 $\gg$ | $(p(s(SX)),(p(SX),(SY))) = (p(s(SX1)),(p(SX1),(SY1)))$ | ; |
| L06: | eqReflexivity $\gg$ | $(SX1) = (SX1)$ | ; |
| L07: | SamePair $\rhd$ L06 $\rhd$ L03 $\gg$ | $(p(SX1),(SX1)) = (p(SX1),(SY1))$ | ; |
| L08: | Repetition $\rhd$ L07 $\gg$ | $(s(SX1)) = (p(SX1),(SY1))$ | ; |
| L09: | eqReflexivity $\gg$ | $(s(SX1)) = (s(SX1))$ | ; |

| | | | |
|---|---|---|---|
| L10: | SamePair $\triangleright$ L09 $\triangleright$ L08 $\gg$ | (p(s(SX1)), (s(SX1))) $=$ (p(s(SX1)), (p(SX1), (SY1))) | ; |
| L11: | Repetition $\triangleright$ L10 $\gg$ | (s(s(SX1))) $=$ (p(s(SX1)), (p(SX1), (SY1))) | ; |
| L12: | eqSymmetry $\triangleright$ L11 $\gg$ | (p(s(SX1)), (p(SX1), (SY1))) $=$ (s(s(SX1))) | ; |
| L13: | eqTransitivity $\triangleright$ L05 $\triangleright$ L12 $\gg$ | (p(s(SX)), (p(SX), (SY))) $=$ (s(s(SX1))) | ; |
| L14: | InPair(1) $\gg$ | (s(SX)) $\in$ (p(s(SX)), (p(SX), (SY))) | ; |
| L15: | SENC1 $\triangleright$ L13 $\triangleright$ L14 $\gg$ | (s(SX)) $\in$ (s(s(SX1))) | ; |
| L16: | FromSingleton $\triangleright$ L15 $\gg$ | (s(SX)) $=$ (s(SX1)) | ; |
| L17: | FromSameSingleton $\triangleright$ L16 $\gg$ | (SX) $=$ (SX1) | ; |
| L18: | eqSymmetry $\triangleright$ L16 $\gg$ | (s(SX1)) $=$ (s(SX)) | ; |
| L19: | SameSingleton $\triangleright$ L18 $\gg$ | (s(s(SX1))) $=$ (s(s(SX))) | ; |
| L20: | eqTransitivity $\triangleright$ L13 $\triangleright$ L19 $\gg$ | (p(s(SX)), (p(SX), (SY))) $=$ (s(s(SX))) | ; |
| L21: | InPair(2) $\gg$ | (p(SX), (SY)) $\in$ (p(s(SX)), (p(SX), (SY))) | ; |
| L22: | SENC1 $\triangleright$ L20 $\triangleright$ L21 $\gg$ | (p(SX), (SY)) $\in$ (s(s(SX))) | ; |
| L23: | FromSingleton $\triangleright$ L22 $\gg$ | (p(SX), (SY)) $=$ (s(SX)) | ; |
| L24: | SingletonmembersEqual $\triangleright$ L23 $\gg$ | (SX) $=$ (SY) | ; |
| L25: | eqSymmetry $\triangleright$ L24 $\gg$ | (SY) $=$ (SX) | ; |
| L26: | eqTransitivity4 $\triangleright$ L25 $\triangleright$ L17 $\triangleright$ L03 $\gg$ | (SY) $=$ (SY1) | ; |
| L27: | JoinConjuncts $\triangleright$ L17 $\triangleright$ L26 $\gg$ | (SX) $=$ (SX1) $\dot{\wedge}$ (SY) $=$ (SY1) | ; |
| L28: | Block $\gg$ | End | ; |
| L29: | Block $\gg$ | Begin | ; |
| L30: | Arbitrary $\gg$ | (SX), (SX1), (SY), (SY1) | ; |
| L03: | Premise $\gg$ | (SX1) $\neq$ (SY1) | ; |
| L04: | Premise $\gg$ | OrderedPair((SX), (SY)) $=$ OrderedPair((SX1), (SY1)) | ; |
| L05: | Repetition $\triangleright$ L04 $\gg$ | (p(s(SX)), (p(SX), (SY))) $=$ (p(s(SX1)), (p(SX1), (SY1))) | ; |
| L06: | InPair(1) $\gg$ | (s(SX)) $\in$ (p(s(SX)), (p(SX), (SY))) | ; |
| L07: | SENC1 $\triangleright$ L05 $\triangleright$ L06 $\gg$ | (s(SX)) $\in$ (p(s(SX1)), (p(SX1), (SY1))) | ; |
| L08: | Pair2Formula $\triangleright$ L07 $\gg$ | (s(SX)) $=$ (s(SX1)) $\dot{\vee}$ (s(SX)) $=$ (p(SX1), (SY1)) | ; |
| L09: | UnequalsNotInSingleton $\triangleright$ L03 $\gg$ | (p(SX1), (SY1)) $\neq$ (s(SX)) | ; |

| | | | |
|---|---|---|---|
| L10: | NeqSymmetry ▷ L09 ≫ | (s(SX)) ≠ (p(SX1), (SY1)) | ; |
| L11: | NegateDisjunct2 ▷ L08 ▷ L10 ≫ | (s(SX)) = (s(SX1)) | ; |
| L12: | FromSameSingleton ▷ L11 ≫ | (SX) = (SX1) | ; |
| L14: | InPair(2) ≫ | (p(SX1), (SY1)) ∈ (p(s(SX1)), (p(SX1), (SY1))) | ; |
| L15: | SENC2 ▷ L05 ▷ L14 ≫ | (p(SX1), (SY1)) ∈ (p(s(SX)), (p(SX), (SY))) | ; |
| L16: | Pair2Formula ▷ L15 ≫ | (p(SX1), (SY1)) = (s(SX)) $\dot\vee$ (p(SX1), (SY1)) = (p(SX), (SY)) | ; |
| L18: | NegateDisjunct1 ▷ L16 ▷ L09 ≫ | (p(SX1), (SY1)) = (p(SX), (SY)) | ; |
| L19: | InPair(2) ≫ | (SY) ∈ (p(SX), (SY)) | ; |
| L20: | SENC2 ▷ L18 ▷ L19 ≫ | (SY) ∈ (p(SX1), (SY1)) | ; |
| L21: | Pair2Formula ▷ L20 ≫ | (SY) = (SX1) $\dot\vee$ (SY) = (SY1) | ; |
| L22: | UnequalsNotInSingleton ▷ L03 ≫ | (p(SX1), (SY1)) ≠ (s(SX)) | ; |
| L23: | SubNeqLeft ▷ L18 ▷ L22 ≫ | (p(SX), (SY)) ≠ (s(SX)) | ; |
| L24: | NonsingletonmembersUnequal ▷ L23 ≫ | (SX) ≠ (SY) | ; |
| L25: | SubNeqLeft ▷ L12 ▷ L24 ≫ | (SX1) ≠ (SY) | ; |
| L26: | NeqSymmetry ▷ L25 ≫ | (SY) ≠ (SX1) | ; |
| L31: | NegateDisjunct1 ▷ L21 ▷ L26 ≫ | (SY) = (SY1) | ; |
| L32: | JoinConjuncts ▷ L12 ▷ L31 ≫ | (SX) = (SX1) $\dot\wedge$ (SY) = (SY1) | ; |
| L33: | Block ≫ | End | ; |
| L34: | Arbitrary ≫ | (SX), (SX1), (SY), (SY1) | ; |
| L35: | Ded ▷ L28 ≫ | (SX1) = (SY1) ⇒ OrderedPair((SX), (SY)) = OrderedPair((SX1), (SY1)) ⇒ (SX) = (SX1) $\dot\wedge$ (SY) = (SY1) | ; |
| L36: | Ded ▷ L33 ≫ | (SX1) ≠ (SY1) ⇒ OrderedPair((SX), (SY)) = OrderedPair((SX1), (SY1)) ⇒ (SX) = (SX1) $\dot\wedge$ (SY) = (SY1) | ; |
| L03: | Premise ≫ | OrderedPair((SX), (SY)) = OrderedPair((SX1), (SY1)) | ; |
| L04: | FromNegations ▷ L35 ▷ L36 ≫ | OrderedPair((SX), (SY)) = OrderedPair((SX1), (SY1)) ⇒ (SX) = (SX1) $\dot\wedge$ (SY) = (SY1) | ; |
| L37: | MP ▷ L04 ▷ L03 ≫ | (SX) = (SX1) $\dot\wedge$ (SY) = (SY1) | □ |

[SystemQ **lemma** FromOrderedPair(1): Π(SX), (SX1), (SY), (SY1): OrderedPai
OrderedPair((SX1), (SY1)) ⊢ (SX) = (SX1)]

SystemQ **proof of** FromOrderedPair(1):

| | | | |
|---|---|---|---|
| L01: | Arbitrary ≫ | (SX), (SX1), (SY), (SY1) | ; |
| L02: | Premise ≫ | OrderedPair((SX), (SY)) = OrderedPair((SX1), (SY1)) | ; |

L03: FromOrderedPair ▷ L02 ≫ (SX) = (SX1) ∧̇ (SY) = (SY1) ;
L04: FirstConjunct ▷ L03 ≫ (SX) = (SX1) □

[SystemQ **lemma** FromOrderedPair(2): Π(SX), (SX1), (SY), (SY1): OrderedPai
OrderedPair((SX1), (SY1)) ⊢ (SY) = (SY1)]

SystemQ **proof of** FromOrderedPair(2):

L01: Arbitrary ≫ (SX), (SX1), (SY), (SY1) ;
L02: Premise ≫ OrderedPair((SX), (SY)) = OrderedPair((SX1), (SY1)) ;
L03: FromOrderedPair ▷ L02 ≫ (SX) = (SX1) ∧̇ (SY) = (SY1) ;
L04: SecondConjunct ▷ L03 ≫ (SY) = (SY1) □

[SystemQ **lemma** SameMember(2): Π(SX), (SY), (SZ): (SX) = (SY) ⊢ (SY) ∈
(SZ) ⊢ (SX) ∈ (SZ)]

SystemQ **proof of** SameMember(2):

L01: Arbitrary ≫ (SX), (SY), (SZ) ;
L02: Premise ≫ (SX) = (SY) ;
L03: Premise ≫ (SY) ∈ (SZ) ;
L04: eqSymmetry ▷ L02 ≫ (SY) = (SX) ;
L05: SameMember ▷ L04 ▷ L03 ≫ (SX) ∈ (SZ) □

[SystemQ **lemma** ToBinaryUnion(1): Π(SX), (SY), (SZ), (SU): (SX) ∈ (SY) ⊢
(SX) ∈ binaryUnion((SY), (SZ))]

SystemQ **proof of** ToBinaryUnion(1):

L01: Arbitrary ≫ (SX), (SY), (SZ), (SU) ;
L02: Premise ≫ (SX) ∈ (SY) ;
L03: InPair(1) ≫ (SY) ∈ (p(SY), (SZ)) ;
L04: JoinConjuncts ▷ L02 ▷ L03 ≫ (SX) ∈ (SY) ∧̇ (SY) ∈ (p(SY), (SZ)) ;
L05: IntroExist @(SY) ▷ L04 ≫ ∃(SU): (SX) ∈ (SU) ∧̇ (SU) ∈ (p(SY), (SZ)) ;
L06: Formula2Union ▷ L05 ≫ (SX) ∈ Union((p(SY), (SZ))) ;
L07: Repetition ▷ L06 ≫ (SX) ∈ binaryUnion((SY), (SZ)) □

[SystemQ **lemma** ToBinaryUnion(2): Π(SX), (SY), (SZ), (SU): (SX) ∈ (SZ) ⊢
(SX) ∈ binaryUnion((SY), (SZ))]

SystemQ **proof of** ToBinaryUnion(2):

L01: Arbitrary ≫ (SX), (SY), (SZ), (SU) ;
L02: Premise ≫ (SX) ∈ (SZ) ;
L03: InPair(2) ≫ (SZ) ∈ (p(SY), (SZ)) ;
L04: JoinConjuncts ▷ L02 ▷ L03 ≫ (SX) ∈ (SZ) ∧̇ (SZ) ∈ (p(SY), (SZ)) ;
L05: IntroExist @(SZ) ▷ L04 ≫ ∃(SU): (SX) ∈ (SU) ∧̇ (SU) ∈ (p(SY), (SZ)) ;
L06: Formula2Union ▷ L05 ≫ (SX) ∈ Union((p(SY), (SZ))) ;
L07: Repetition ▷ L06 ≫ (SX) ∈ binaryUnion((SY), (SZ)) □

[SystemQ **lemma** FromOrderedPair(TwoLevels): Π(SX), (SY), (SZ), (SU): (SX
(SY) ⊢ (SY) ∈ OrderedPair((SZ), (SU)) ⊢ (SX) = (SZ) ∨̇ (SX) = (SU)]

SystemQ **proof of** FromOrderedPair(TwoLevels):

| | | | |
|---|---|---|---|
| L01: | Arbitrary $\gg$ | $(SX), (SY), (SZ), (SU)$ | ; |
| L02: | Premise $\gg$ | $(SX) \in (SY)$ | ; |
| L03: | Premise $\gg$ | $(SY) \in \mathrm{OrderedPair}((SZ), (SU))$ | ; |
| L04: | Repetition $\rhd$ L03 $\gg$ | $(SY) \in (p(s(SZ)), (p(SZ), (SU)))$ | ; |
| L05: | Pair2Formula $\rhd$ L04 $\gg$ | $(SY) = (s(SZ)) \dot{\vee} (SY) = (p(SZ), (SU))$ | ; |
| L06: | Block $\gg$ | Begin | ; |
| L07: | Arbitrary $\gg$ | $(SX), (SY), (SZ), (SU)$ | ; |
| L03: | Premise $\gg$ | $(SY) = (s(SZ))$ | ; |
| L02: | Premise $\gg$ | $(SX) \in (SY)$ | ; |
| L04: | SENC1 $\rhd$ L03 $\rhd$ L02 $\gg$ | $(SX) \in (s(SZ))$ | ; |
| L05: | FromSingleton $\rhd$ L04 $\gg$ | $(SX) = (SZ)$ | ; |
| L08: | WeakenOr2 $\rhd$ L05 $\gg$ | $(SX) = (SZ) \dot{\vee} (SX) = (SU)$ | ; |
| L09: | Block $\gg$ | End | ; |
| L10: | Block $\gg$ | Begin | ; |
| L11: | Arbitrary $\gg$ | $(SX), (SY), (SZ), (SU)$ | ; |
| L03: | Premise $\gg$ | $(SY) = (p(SZ), (SU))$ | ; |
| L02: | Premise $\gg$ | $(SX) \in (SY)$ | ; |
| L04: | SENC1 $\rhd$ L03 $\rhd$ L02 $\gg$ | $(SX) \in (p(SZ), (SU))$ | ; |
| L12: | Pair2Formula $\rhd$ L04 $\gg$ | $(SX) = (SZ) \dot{\vee} (SX) = (SU)$ | ; |
| L13: | Block $\gg$ | End | ; |
| L14: | Ded $\rhd$ L09 $\gg$ | $(SY) = (s(SZ)) \Rightarrow (SX) \in (SY) \Rightarrow (SX) = (SZ) \dot{\vee} (SX) = (SU)$ | ; |
| L15: | Ded $\rhd$ L13 $\gg$ | $(SY) = (p(SZ), (SU)) \Rightarrow (SX) \in (SY) \Rightarrow (SX) = (SZ) \dot{\vee} (SX) = (SU)$ | ; |
| L16: | FromDisjuncts $\rhd$ L05 $\rhd$ L14 $\rhd$ L15 $\gg$ | $(SX) \in (SY) \Rightarrow (SX) = (SZ) \dot{\vee} (SX) = (SU)$ | ; |
| L17: | MP $\rhd$ L16 $\rhd$ L02 $\gg$ | $(SX) = (SZ) \dot{\vee} (SX) = (SU)$ | □ |

[SystemQ **lemma** CartProdIsRelation: $\Pi(SX), (SY)$: IsRelation(cartProd((SX)

SystemQ **proof of** CartProdIsRelation:

| | | | |
|---|---|---|---|
| L01: | Block $\gg$ | Begin | ; |
| L02: | Arbitrary $\gg$ | $(SX), (SY)$ | ; |
| L03: | Premise $\gg$ | $(R1ob) \in \mathrm{cartProd}((SX))$ | ; |
| L04: | Sep2Formula $\rhd$ L03 $\gg$ | $(R1ob) \in \mathrm{Power}(\mathrm{Power}(\mathrm{binaryUnion}((SX), (SY))))$ IsOrderedPair((R1ob), (SX), (SY)) | ; |
| L05: | SecondConjunct $\rhd$ L04 $\gg$ | IsOrderedPair((R1ob), (SX), (SY)) | ; |
| L06: | Block $\gg$ | End | ; |

L07: Arbitrary ≫ (SX), (SY) ;
L03: Ded ▷ L06 ≫ (R1ob) ∈ cartProd((SX)) ⇒ IsOrderedPair((R1ob), (SX), (SY))▌;
L04: Gen ▷ L03 ≫ ∀(R1ob): ((R1ob) ∈ cartProd((SX)) ⇒ IsOrderedPair((R1ob), (SX), (SY)))▌;
L08: Repetition ▷ L04 ≫ IsRelation(cartProd((SX)), (SX), (SY))▌□

[SystemQ **lemma** FromSubset: Π(SX), (SY), (SZ): IsSubset((SX), (SY)) ⊢ (SZ) ∈ (SX) ⊢ (SZ) ∈ (SY)]

SystemQ **proof of** FromSubset:

L01: Arbitrary ≫ (SX), (SY), (SZ) ;
L02: Premise ≫ IsSubset((SX), (SY)) ;
L03: Premise ≫ (SZ) ∈ (SX) ;
L04: Repetition ▷ L02 ≫ ∀(S1ob): ((S1ob) ∈ (SX) ⇒ (S1ob) ∈ (SY)) ;
L05: A4 @(SZ) ▷ L04 ≫ (SZ) ∈ (SX) ⇒ (SZ) ∈ (SY) ;
L06: MP ▷ L05 ▷ L03 ≫ (SZ) ∈ (SY) □

[SystemQ **lemma** SubsetIsRelation: Π(SX), (SY), (SZ), (SU): IsRelation((SX), (SZ), (SU)) ⊢ IsSubset((SY), (SX)) ⊢ IsRelation((SY), (SZ), (SU))]

SystemQ **proof of** SubsetIsRelation:

L01: Block ≫ Begin ;
L02: Arbitrary ≫ (SX), (SY), (SZ), (SU) ;
L03: Premise ≫ IsRelation((SX), (SZ), (SU)) ;
L04: Premise ≫ IsSubset((SY), (SX)) ;
L05: Premise ≫ (R1ob) ∈ (SY) ;
L06: Repetition ▷ L03 ≫ ∀(R1ob): ((R1ob) ∈ (SX) ⇒ IsOrderedPair((R1ob), (SZ), (SU)))▌;
L07: A4 @(R1ob) ▷ L06 ≫ (R1ob) ∈ (SX) ⇒ IsOrderedPair((R1ob), (SZ), (SU))▌;
L08: FromSubset ▷ L04 ▷ L05 ≫ (R1ob) ∈ (SX) ;
L09: MP ▷ L07 ▷ L08 ≫ IsOrderedPair((R1ob), (SZ), (SU))▌;
L10: Block ≫ End ;
L11: Arbitrary ≫ (SX), (SY), (SZ), (SU) ;
L12: Ded ▷ L10 ≫ IsRelation((SX), (SZ), (SU)) ⇒ IsSubset((SY), (SX)) ⇒ (R1ob) ∈ (SY) ⇒ IsOrderedPair((R1ob), (SZ), (SU))▌;
L03: Premise ≫ IsRelation((SX), (SZ), (SU)) ;
L04: Premise ≫ IsSubset((SY), (SX)) ;

L05: MP2 ▷ L12 ▷ L03 ▷ L04 ≫ (R1ob) ∈ (SY) ⇒ IsOrderedPair((R1ob), (SZ), (SU)) ;
L06: Gen ▷ L05 ≫ ∀(R1ob): ((R1ob) ∈ (SY) ⇒ IsOrderedPair((R1ob), (SZ), (SU))) ;
L13: Repetition ▷ L06 ≫ IsRelation((SY), (SZ), (SU)) □

[SystemQ **lemma** CPseparationIsRelation: Π𝒜, (SX), (SY): IsRelation({ph ∈ cartProd((SX)) | 𝒜}, (SX), (SY))]

SystemQ **proof of** CPseparationIsRelation:

L01: Block ≫ Begin ;
L02: Arbitrary ≫ 𝒜, (SX), (SY) ;
L03: Premise ≫ (S1ob) ∈ {ph ∈ cartProd((SX)) | 𝒜} ;
L04: Separation2formula(1)▷L03 ≫ (S1ob) ∈ cartProd((SX)) ;
L05: Block ≫ End ;
L06: Arbitrary ≫ 𝒜, (SX), (SY) ;
L07: Ded ▷ L05 ≫ ∀(S1ob): ((S1ob) ∈ {ph ∈ cartProd((SX)) | 𝒜} ⇒ (S1ob) ∈ cartProd((SX))) ;
L08: Repetition ▷ L07 ≫ IsSubset({ph ∈ cartProd((SX)) | 𝒜}, cartProd((SX))) ;
L09: CartProdIsRelation ≫ IsRelation(cartProd((SX)), (SX), (SY)) ;
L10: SubsetIsRelation▷L09▷L08 ≫ IsRelation({ph ∈ cartProd((SX)) | 𝒜}, (SX), (SY)) □

[SystemQ **lemma** ToCartProd(Helper): Π(SX), (SX1), (SY), (SY1), (SZ): (SX) (SX1) ⊢ (SY) ∈ (SY1) ⊢ (SZ) ∈ OrderedPair((SX), (SY)) ⊢ IsSubset((SZ), binaryU

SystemQ **proof of** ToCartProd(Helper):

L01: Block ≫ Begin ;
L02: Arbitrary ≫ (SX), (SX1), (SY), (SY1), (SZ) ;
L03: Premise ≫ (SX) ∈ (SX1) ;
L04: Premise ≫ (SY) ∈ (SY1) ;
L05: Premise ≫ (SZ) ∈ OrderedPair((SX), (SY)) ;
L06: Premise ≫ (S1ob) ∈ (SZ) ;
L07: FromOrderedPair(TwoLevels)▷ L06 ▷ L05 ≫ (S1ob) = (SX) $\dot{\vee}$ (S1ob) = (SY) ;
L08: Block ≫ Begin ;
L09: Arbitrary ≫ (SX), (SX1), (SY1) ;
L04: Premise ≫ (SX) ∈ (SX1) ;
L03: Premise ≫ (S1ob) = (SX) ;
L05: SameMember(2)▷L03▷L04 ≫ (S1ob) ∈ (SX1) ;

| | | |
|---|---|---|
| L10: | ToBinaryUnion(1) ▷ L05 ≫ | $(S1ob) \in binaryUnion((SX1),(SY1))$ ; |
| L11: | Block ≫ | End ; |
| L12: | Block ≫ | Begin ; |
| L13: | Arbitrary ≫ | $(SX1),(SY),(SY1)$ ; |
| L04: | Premise ≫ | $(SY) \in (SY1)$ ; |
| L03: | Premise ≫ | $(S1ob) = (SY)$ ; |
| L05: | SameMember(2) ▷ L03 ▷ L04 ≫ | $(S1ob) \in (SY1)$ ; |
| L14: | ToBinaryUnion(2) ▷ L05 ≫ | $(S1ob) \in binaryUnion((SX1),(SY1))$ ; |
| L15: | Block ≫ | End ; |
| L16: | Ded ▷ L11 ≫ | $(SX) \in (SX1) \Rightarrow (S1ob) = (SX) \Rightarrow (S1ob) \in binaryUnion((SX1),(SY1))$ ; |
| L17: | MP ▷ L16 ▷ L03 ≫ | $(S1ob) = (SX) \Rightarrow (S1ob) \in binaryUnion((SX1),(SY1))$ ; |
| L18: | Ded ▷ L15 ≫ | $(SY) \in (SY1) \Rightarrow (S1ob) = (SY) \Rightarrow (S1ob) \in binaryUnion((SX1),(SY1))$ ; |
| L19: | MP ▷ L18 ▷ L04 ≫ | $(S1ob) = (SY) \Rightarrow (S1ob) \in binaryUnion((SX1),(SY1))$ ; |
| L20: | FromDisjuncts ▷ L07 ▷ L17 ▷ L19 ≫ | $(S1ob) \in binaryUnion((SX1),(SY1))$ ; |
| L21: | Block ≫ | End ; |
| L22: | Arbitrary ≫ | $(SX),(SX1),(SY),(SY1),(SZ)$ ; |
| L23: | Ded ▷ L21 ≫ | $(SX) \in (SX1) \Rightarrow (SY) \in (SY1) \Rightarrow (SZ) \in OrderedPair((SX),(SY)) \Rightarrow (S1ob) \in (SZ) \Rightarrow (S1ob) \in binaryUnion((SX1),(SY1))$ ; |
| L03: | Premise ≫ | $(SX) \in (SX1)$ ; |
| L04: | Premise ≫ | $(SY) \in (SY1)$ ; |
| L05: | Premise ≫ | $(SZ) \in OrderedPair((SX),(SY))$ ; |
| L06: | MP3 ▷ L23 ▷ L03 ▷ L04 ▷ L05 ≫ | $(S1ob) \in (SZ) \Rightarrow (S1ob) \in binaryUnion((SX1),(SY1))$ ; |
| L07: | Gen ▷ L06 ≫ | $\forall (S1ob): ((S1ob) \in (SZ) \Rightarrow (S1ob) \in binaryUnion((SX1),(SY1)))$ ; |

L24: Repetition ▷ L07 ≫ IsSubset((SZ), binaryUnion((SX1), (SY1

□

[SystemQ **lemma** ToCartProd: Π(SX), (SX1), (SY), (SY1): (SX) ∈ (SX1) ⊢ (SY) ∈ (SY1) ⊢ OrderedPair((SX), (SY)) ∈ cartProd((SX1))]

SystemQ **proof of** ToCartProd:

L01: Block ≫ Begin ;
L02: Arbitrary ≫ (SX), (SX1), (SY), (SY1) ;
L03: Premise ≫ (SX) ∈ (SX1) ;
L04: Premise ≫ (SY) ∈ (SY1) ;
L05: Premise ≫ (S1ob) ∈ OrderedPair((SX), (SY)) ;
L06: ToCartProd(Helper) ▷ L03 ▷ L04 ▷ L05 ≫ IsSubset((S1ob), binaryUnion((SX1), (SY ;
L07: Formula2Power ▷ L06 ≫ (S1ob) ∈ Power(binaryUnion((SX1), (SY1))) ;
L08: Block ≫ End ;
L09: Arbitrary ≫ (SX), (SX1), (SY), (SY1) ;
L10: Ded ▷ L08 ≫ (SX) ∈ (SX1) ⇒ (SY) ∈ (SY1) ⇒ (S1ob) ∈ OrderedPair((SX), (SY)) ⇒ (S1ob) ∈ Power(binaryUnion((SX1), (SY1))) ;
L03: Premise ≫ (SX) ∈ (SX1) ;
L04: Premise ≫ (SY) ∈ (SY1) ;
L06: MP2 ▷ L10 ▷ L03 ▷ L04 ≫ (S1ob) ∈ OrderedPair((SX), (SY)) ⇒ (S1ob) ∈ Power(binaryUnion((SX1), (SY1))) ;
L11: Gen ▷ L06 ≫ ∀(S1ob): ((S1ob) ∈ OrderedPair((SX), (SY)) ⇒ (S1ob) ∈ Power(binaryUnion((SX1), (SY1)))) ;
L12: Repetition ▷ L11 ≫ IsSubset(OrderedPair((SX), (SY)), Powe ;
L13: Formula2Power ▷ L12 ≫ OrderedPair((SX), (SY)) ∈ Power(Power(binaryUnion((SX1), (SY1) ;
L14: eqReflexivity ≫ OrderedPair((SX), (SY)) = OrderedPair((SX), (SY)) ;
L15: JoinConjuncts ▷ L03 ▷ L04 ≫ (SX) ∈ (SX1) ∧̇ (SY) ∈ (SY1) ;

L16: JoinConjuncts ▷ L15 ▷ L14 ≫ (SX) ∈ (SX1) $\dot\wedge$ (SY) ∈ (SY1) $\dot\wedge$ OrderedPair((SX), (SY)) = OrderedPair((SX), (SY)) ;

L17: IntroExist @(SY) ▷ L16 ≫ ∃(OP2ob): (SX) ∈ (SX1) $\dot\wedge$ (OP2ob) ∈ (SY1) $\dot\wedge$ OrderedPair((SX), (SY)) = OrderedPair((SX), (OP2ob)) ;

L18: IntroExist @(SX) ▷ L17 ≫ ∃(OP1ob): ∃(OP2ob): (OP1ob) ∈ (SX1) $\dot\wedge$ (OP2ob) ∈ (SY1) $\dot\wedge$ OrderedPair((SX), (SY)) = OrderedPair((OP1ob), (OP2ob)) ;

L19: Repetition ▷ L18 ≫ IsOrderedPair(OrderedPair((SX), (SY)),
;

L20: Formula2Sep ▷ L13 ▷ L19 ≫ OrderedPair((SX), (SY)) ∈ {ph ∈ Power(Power(binaryUnion((SX1), (SY1) IsOrderedPair($ph_1$, (SX1), (SY1))} ;

L21: Repetition ▷ L20 ≫ OrderedPair((SX), (SY)) ∈ cartProd((SX1)) □

[SystemQ **lemma** CrsIsRelation: Π𝒳: IsRelation(constantRationalSeries(𝒳), N

SystemQ **proof of** CrsIsRelation:

L01: Block ≫ Begin ;

L02: Arbitrary ≫ 𝒳 ;

L03: Premise ≫ (S1ob) ∈ constantRationalSeries(𝒳) ;

L04: Repetition ▷ L03 ≫ (S1ob) ∈ {ph ∈ cartProd(N) | ∃(CRS1ob): $ph_3$ = OrderedPair((CRS1ob), 𝒳)} ;

L05: Sep2Formula ▷ L04 ≫ (S1ob) ∈ cartProd(N) $\dot\wedge$ ∃(CRS1ob): (S1ob) = OrderedPair((CRS1ob), 𝒳) ;

L06: FirstConjunct ▷ L05 ≫ (S1ob) ∈ cartProd(N) ;

L07: Block ≫ End ;

L08: Arbitrary ≫ 𝒳 ;

L03: Ded ▷ L07 ≫ (S1ob) ∈ constantRationalSeries(𝒳) ⇒ (S1ob) ∈ cartProd(N) ;

L04: Gen ▷ L03 ≫ ∀(S1ob): ((S1ob) ∈ constantRationalSeries(𝒳) ⇒ (S1ob) ∈ cartProd(N)) ;

L05: Repetition ▷ L04 ≫ IsSubset(constantRationalSeries(𝒳), car
;

L09: CartProdIsRelation ≫ IsRelation(cartProd(N), N, Q) ;
L10: SubsetIsRelation ▷ L09 ▷ L05 ≫ IsRelation(constantRationalSeries($\mathcal{X}$), N

□

[SystemQ **lemma** CrsIsFunction: Π$\mathcal{X}$: isFunction(constantRationalSeries($\mathcal{X}$), N
SystemQ **proof of** CrsIsFunction:

L01: Block ≫ Begin ;
L02: Arbitrary ≫ $\mathcal{X}$ ;
L03: Premise ≫ OrderedPair((F1ob), (F2ob)) = OrderedPair((CRS1ob), $\mathcal{X}$) ;
L04: Premise ≫ OrderedPair((F3ob), (F4ob)) = OrderedPair((CRS1ob), $\mathcal{X}$) ;
L05: FromOrderedPair ▷ L03 ≫ (F1ob) = (CRS1ob) $\dot{\wedge}$ (F2ob) = $\mathcal{X}$ ;
L06: SecondConjunct ▷ L05 ≫ (F2ob) = $\mathcal{X}$ ;
L07: FromOrderedPair ▷ L04 ≫ (F3ob) = (CRS1ob) $\dot{\wedge}$ (F4ob) = $\mathcal{X}$ ;
L08: SecondConjunct ▷ L07 ≫ (F4ob) = $\mathcal{X}$ ;
L09: eqSymmetry ▷ L08 ≫ $\mathcal{X}$ = (F4ob) ;
L10: eqTransitivity ▷ L06 ▷ L09 ≫ (F2ob) = (F4ob) ;
L11: Block ≫ End ;
L12: Block ≫ Begin ;
L13: Arbitrary ≫ $\mathcal{X}$ ;
L14: Ded ▷ L11 ≫ OrderedPair((F1ob), (F2ob)) = OrderedPair((CRS1ob), $\mathcal{X}$) ⇒ OrderedPair((F3ob), (F4ob)) = OrderedPair((CRS1ob), $\mathcal{X}$) ⇒ (F2ob) = (F4ob) ;
L03: Premise ≫ OrderedPair((F1ob), (F2ob)) ∈ constantRationalSeries($\mathcal{X}$) ;
L04: Premise ≫ OrderedPair((F3ob), (F4ob)) ∈ constantRationalSeries($\mathcal{X}$) ;
L05: Premise ≫ (F1ob) = (F3ob) ;
L06: Sep2Formula ▷ L03 ≫ OrderedPair((F1ob), (F2ob)) ∈ cartProd(N) $\dot{\wedge}$ ∃(CRS1ob): OrderedPair((F1ob), (F2ob)
OrderedPair((CRS1ob), $\mathcal{X}$) ;
L07: SecondConjunct ▷ L06 ≫ ∃(CRS1ob): OrderedPair((F1ob), (F2ob)
OrderedPair((CRS1ob), $\mathcal{X}$) ;
L08: Sep2Formula ▷ L04 ≫ OrderedPair((F3ob), (F4ob)) ∈ cartProd(N) $\dot{\wedge}$ ∃(CRS1ob): OrderedPair((F3ob), (F4ob)
OrderedPair((CRS1ob), $\mathcal{X}$) ;
L09: SecondConjunct ▷ L08 ≫ ∃(CRS1ob): OrderedPair((F3ob), (F4ob)
OrderedPair((CRS1ob), $\mathcal{X}$) ;
L15: ExistMP2 ▷ L14 ▷ L07 ▷ L09 ≫ (F2ob) = (F4ob) ;
L16: Block ≫ End ;
L17: Arbitrary ≫ $\mathcal{X}$ ;

L03: Ded $\rhd$ L16 $\gg$ $\forall(\mathrm{F1ob}), (\mathrm{F2ob}), (\mathrm{F3ob}), (\mathrm{F4ob})\colon (\mathrm{OrderedPair}((\mathrm{F1ob}), (\mathrm{F2ob})) \in \mathrm{constantRationalSeries}(\mathcal{X}) \Rightarrow \mathrm{OrderedPair}((\mathrm{F3ob}), (\mathrm{F4ob})) \in \mathrm{constantRationalSeries}(\mathcal{X}) \Rightarrow (\mathrm{F1ob}) = (\mathrm{F3ob}) \Rightarrow (\mathrm{F2ob}) = (\mathrm{F4ob}))$ ;

L04: CrsIsRelation $\gg$ $\mathrm{IsRelation}(\mathrm{constantRationalSeries}(\mathcal{X}), \mathrm{N}$ ;

L18: JoinConjuncts $\rhd$ L04 $\rhd$ L03 $\gg$ $\mathrm{isFunction}(\mathrm{constantRationalSeries}(\mathcal{X}), \mathrm{N}$

□

[SystemQ **lemma** CrsIsTotal: $\Pi \mathcal{M}, \mathcal{X}\colon \mathrm{TypeRational}(\mathcal{X}) \Vdash \mathcal{M} \in \mathrm{N} \vdash \mathrm{OrderedP}$ $\mathrm{constantRationalSeries}(\mathcal{X})$]

SystemQ **proof of** CrsIsTotal:

L01: Arbitrary $\gg$ $\mathcal{M}, \mathcal{X}$ ;

L02: Side-condition $\gg$ $\mathrm{TypeRational}(\mathcal{X})$ ;

L03: Premise $\gg$ $\mathcal{M} \in \mathrm{N}$ ;

L04: RationalType $\unrhd$ L02 $\gg$ $\mathcal{X} \in \mathrm{Q}$ ;

L05: ToCartProd $\rhd$ L03 $\rhd$ L04 $\gg$ $\mathrm{OrderedPair}(\mathcal{M}, \mathcal{X}) \in \mathrm{cartProd}(\mathrm{N})$ ;

L06: eqReflexivity $\gg$ $\mathrm{OrderedPair}(\mathcal{M}, \mathcal{X}) = \mathrm{OrderedPair}(\mathcal{M}, \mathcal{X})$ ;

L07: IntroExist @ $\mathcal{M}$ $\rhd$ L06 $\gg$ $\exists(\mathrm{CRS1ob})\colon \mathrm{OrderedPair}(\mathcal{M}, \mathcal{X}) = \mathrm{OrderedPair}((\mathrm{CRS1ob}), \mathcal{X})$ ;

L08: Formula2Sep $\rhd$ L05 $\rhd$ L07 $\gg$ $\mathrm{OrderedPair}(\mathcal{M}, \mathcal{X}) \in \mathrm{constantRationalSeries}(\mathcal{X})$

□

[SystemQ **lemma** CrsIsSeries: $\Pi \mathcal{X}\colon \mathrm{IsSeries}(\mathrm{constantRationalSeries}(\mathcal{X}), \mathrm{Q})$]

SystemQ **proof of** CrsIsSeries:

L01: Block $\gg$ Begin ;

L02: Arbitrary $\gg$ $\mathcal{X}$ ;

L03: Premise $\gg$ $(\mathrm{S1ob}) \in \mathrm{N}$ ;

L04: CrsIsTotal $\rhd$ L03 $\gg$ $\mathrm{OrderedPair}((\mathrm{S1ob}), \mathcal{X}) \in \mathrm{constantRationalSeries}(\mathcal{X})$ ;

L05: IntroExist @ $\mathcal{X}$ $\rhd$ L03 $\gg$ $\exists(\mathrm{S2ob})\colon \mathrm{OrderedPair}((\mathrm{S1ob}), (\mathrm{S2ob})) \in \mathrm{constantRationalSeries}(\mathcal{X})$ ;

L06: Block $\gg$ End ;

L07: Arbitrary $\gg$ $\mathcal{X}$ ;

L03: Ded $\rhd$ L06 $\gg$ $(\mathrm{S1ob}) \in \mathrm{N} \Rightarrow \exists(\mathrm{S2ob})\colon \mathrm{OrderedPair}((\mathrm{S1ob}), (\mathrm{S2ob})) \in \mathrm{constantRationalSeries}(\mathcal{X})$ ;

L08: Gen $\rhd$ L03 $\gg$ $\forall(\mathrm{S1ob})\colon ((\mathrm{S1ob}) \in \mathrm{N} \Rightarrow \exists(\mathrm{S2ob})\colon \mathrm{OrderedPair}((\mathrm{S1ob}), (\mathrm{S2ob})) \in \mathrm{constantRationalSeries}(\mathcal{X}))$ ;

L09: CrsIsFunction $\gg$ $\mathrm{isFunction}(\mathrm{constantRationalSeries}(\mathcal{X}), \mathrm{N}$ ;

L10: JoinConjuncts $\rhd$ L09 $\rhd$ L08 $\gg$ IsSeries(constantRationalSeries($\mathcal{X}$), Q) $\square$

[SystemQ **lemma** CrsLookup: $\Pi \mathcal{M}, \mathcal{X}$: $\mathcal{M} \in$ N $\vdash$ constantRationalSeries($\mathcal{X}$)[$\mathcal{M}$
$\mathcal{X}$]

SystemQ **proof of** CrsLookup:

L01: Arbitrary $\gg$ $\mathcal{M}, \mathcal{X}$ ;
L02: Premise $\gg$ $\mathcal{M} \in$ N ;
L03: CrsIsSeries $\gg$ IsSeries(constantRationalSeries($\mathcal{X}$), Q) ;
L04: MemberOfSeries $\rhd$ L02 $\rhd$ L03 $\gg$ OrderedPair($\mathcal{M}$, constantRationalSeries
constantRationalSeries($\mathcal{X}$) ;
L05: CrsIsTotal $\rhd$ L02 $\gg$ OrderedPair($\mathcal{M}, \mathcal{X}$) $\in$ constantRationalSeries($\mathcal{X}$) ;
L06: eqReflexivity $\gg$ $\mathcal{M} = \mathcal{M}$ ;
L07: UniqueMember $\rhd$ L03 $\rhd$ L04 $\rhd$ L05 $\rhd$ L06 $\gg$ constantRationalSeries($\mathcal{X}$)[$\mathcal{M}$] = $\mathcal{X}$ $\square$

[SystemQ **lemma** 0f: $\Pi \mathcal{M}$: $\mathcal{M} \in$ N $\vdash$ 0f[$\mathcal{M}$] = 0]

SystemQ **proof of** 0f:

L01: Arbitrary $\gg$ $\mathcal{M}$ ;
L02: Premise $\gg$ $\mathcal{M} \in$ N ;
L03: CrsLookup $\rhd$ L02 $\gg$ constantRationalSeries(0)[$\mathcal{M}$] = 0 ;
L04: Repetition $\rhd$ L03 $\gg$ 0f[$\mathcal{M}$] = 0 $\square$

[SystemQ **lemma** 1f: $\Pi \mathcal{M}$: $\mathcal{M} \in$ N $\vdash$ 1f[$\mathcal{M}$] = 1]

SystemQ **proof of** 1f:

L01: Arbitrary $\gg$ $\mathcal{M}$ ;
L02: Premise $\gg$ $\mathcal{M} \in$ N ;
L03: CrsLookup $\rhd$ L02 $\gg$ constantRationalSeries(1)[$\mathcal{M}$] = 1 ;
L04: Repetition $\rhd$ L03 $\gg$ 1f[$\mathcal{M}$] = 1 $\square$

———-(6.11.06, lemmaer fra kvanti, mod kronologien)

[SystemQ **lemma** DistributionOut: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}$: $((\mathcal{X} * \mathcal{Y}) + (\mathcal{X} * \mathcal{Z})) = (\mathcal{X} * ((\mathcal{Y} + \mathcal{Z})))$]

SystemQ **proof of** DistributionOut:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ ;
L02: Distribution $\gg$ $(\mathcal{X} * ((\mathcal{Y} + \mathcal{Z}))) = ((\mathcal{X} * \mathcal{Y}) + (\mathcal{X} * \mathcal{Z}))$ ;
L03: eqSymmetry $\rhd$ L02 $\gg$ $((\mathcal{X} * \mathcal{Y}) + (\mathcal{X} * \mathcal{Z})) = (\mathcal{X} * ((\mathcal{Y} + \mathcal{Z})))$ $\square$

[SystemQ **lemma** Three2twoTerms: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}$: $(\mathcal{Y} + \mathcal{Z}) = \mathcal{U} \vdash ((\mathcal{X} + \mathcal{Y}) + \mathcal{Z}) = (\mathcal{X} + \mathcal{U})$]

SystemQ **proof of** Three2twoTerms:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}$ ;
L02: Premise $\gg$ $(\mathcal{Y} + \mathcal{Z}) = \mathcal{U}$ ;
L03: lemma eqAdditionLeft $\rhd$ L02 $\gg$ $(\mathcal{X} + ((\mathcal{Y} + \mathcal{Z}))) = (\mathcal{X} + \mathcal{U})$ ;

L04: plusAssociativity ≫ $((\mathcal{X}+\mathcal{Y})+\mathcal{Z}) = (\mathcal{X}+((\mathcal{Y}+\mathcal{Z})))$ ;
L05: eqTransitivity ▷ L04 ▷ L03 ≫ $((\mathcal{X}+\mathcal{Y})+\mathcal{Z}) = (\mathcal{X}+\mathcal{U})$ □

[SystemQ **lemma** Three2threeTerms: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z}\colon ((\mathcal{X}+\mathcal{Y})+\mathcal{Z}) = ((\mathcal{X}+\mathcal{Z})+\mathcal{Y})$]

SystemQ **proof of** Three2threeTerms:

L01: Arbitrary ≫ $\mathcal{X},\mathcal{Y},\mathcal{Z}$ ;
L02: plusCommutativity ≫ $(\mathcal{Y}+\mathcal{Z}) = (\mathcal{Z}+\mathcal{Y})$ ;
L03: Three2twoTerms ▷ L02 ≫ $((\mathcal{X}+\mathcal{Y})+\mathcal{Z}) = (\mathcal{X}+((\mathcal{Z}+\mathcal{Y})))$ ;
L04: plusAssociativity ≫ $((\mathcal{X}+\mathcal{Z})+\mathcal{Y}) = (\mathcal{X}+((\mathcal{Z}+\mathcal{Y})))$ ;
L05: eqSymmetry ▷ L04 ≫ $(\mathcal{X}+((\mathcal{Z}+\mathcal{Y}))) = ((\mathcal{X}+\mathcal{Z})+\mathcal{Y})$ ;
L06: eqTransitivity ▷ L03 ▷ L05 ≫ $((\mathcal{X}+\mathcal{Y})+\mathcal{Z}) = ((\mathcal{X}+\mathcal{Z})+\mathcal{Y})$ □

[SystemQ **lemma** Three2twoFactors: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{U}\colon (\mathcal{Y}*\mathcal{Z}) = \mathcal{U} \vdash ((\mathcal{X}*\mathcal{Y})*\mathcal{Z}) = (\mathcal{X}*\mathcal{U})$]

SystemQ **proof of** Three2twoFactors:

L01: Arbitrary ≫ $\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{U}$ ;
L02: Premise ≫ $(\mathcal{Y}*\mathcal{Z}) = \mathcal{U}$ ;
L03: lemma eqMultiplicationLeft ▷
L02 ≫ $(\mathcal{X}*((\mathcal{Y}*\mathcal{Z}))) = (\mathcal{X}*\mathcal{U})$ ;
L04: timesAssociativity ≫ $((\mathcal{X}*\mathcal{Y})*\mathcal{Z}) = (\mathcal{X}*((\mathcal{Y}*\mathcal{Z})))$ ;
L05: eqTransitivity ▷ L04 ▷ L03 ≫ $((\mathcal{X}*\mathcal{Y})*\mathcal{Z}) = (\mathcal{X}*\mathcal{U})$ □

[SystemQ **lemma** x = x + (y − y): $\Pi\mathcal{X},\mathcal{Y}\colon \mathcal{X} = (\mathcal{X}+((\mathcal{Y}-\mathcal{Y})))$]

SystemQ **proof of** x = x + (y − y):

L01: Arbitrary ≫ $\mathcal{X},\mathcal{Y}$ ;
L02: plus0 ≫ $(\mathcal{X}+0) = \mathcal{X}$ ;
L03: Negative ≫ $(\mathcal{Y}-\mathcal{Y}) = 0$ ;
L04: eqSymmetry ▷ L03 ≫ $0 = (\mathcal{Y}-\mathcal{Y})$ ;
L05: lemma eqAdditionLeft▷L04 ≫ $(\mathcal{X}+0) = (\mathcal{X}+((\mathcal{Y}-\mathcal{Y})))$ ;
L06: Equality ▷ L02 ▷ L05 ≫ $\mathcal{X} = (\mathcal{X}+((\mathcal{Y}-\mathcal{Y})))$ □

[SystemQ **lemma** x = x + y − y: $\Pi\mathcal{X},\mathcal{Y}\colon \mathcal{X} = ((\mathcal{X}+\mathcal{Y})-\mathcal{Y})$]

SystemQ **proof of** x = x + y − y:

L01: Arbitrary ≫ $\mathcal{X},\mathcal{Y}$ ;
L02: x = x + (y − y) ≫ $\mathcal{X} = (\mathcal{X}+((\mathcal{Y}-\mathcal{Y})))$ ;
L03: plusAssociativity ≫ $((\mathcal{X}+\mathcal{Y})-\mathcal{Y}) = (\mathcal{X}+((\mathcal{Y}-\mathcal{Y})))$ ;
L04: eqSymmetry ▷ L03 ≫ $(\mathcal{X}+((\mathcal{Y}-\mathcal{Y}))) = ((\mathcal{X}+\mathcal{Y})-\mathcal{Y})$ ;
L05: eqTransitivity ▷ L02 ▷ L04 ≫ $\mathcal{X} = ((\mathcal{X}+\mathcal{Y})-\mathcal{Y})$ □

[SystemQ **lemma** x = x ∗ y ∗ (1/y): $\Pi\mathcal{X},\mathcal{Y}\colon \mathcal{Y} \neq 0 \vdash \mathcal{X} = ((\mathcal{X}*\mathcal{Y})*\mathrm{rec}\mathcal{Y})$]

SystemQ **proof of** x = x ∗ y ∗ (1/y):

L01: Arbitrary ≫ $\mathcal{X},\mathcal{Y}$ ;
L02: Premise ≫ $\mathcal{Y} \neq 0$ ;
L03: times1 ≫ $(\mathcal{X}*1) = \mathcal{X}$ ;
L04: Reciprocal ▷ L02 ≫ $(\mathcal{Y}*\mathrm{rec}\mathcal{Y}) = 1$ ;
L05: Three2twoFactors ▷ L04 ≫ $((\mathcal{X}*\mathcal{Y})*\mathrm{rec}\mathcal{Y}) = (\mathcal{X}*1)$ ;
L06: eqTransitivity ▷ L05 ▷ L03 ≫ $((\mathcal{X}*\mathcal{Y})*\mathrm{rec}\mathcal{Y}) = \mathcal{X}$ ;
L07: eqSymmetry ▷ L06 ≫ $\mathcal{X} = ((\mathcal{X}*\mathcal{Y})*\mathrm{rec}\mathcal{Y})$ □

[SystemQ **lemma** x ∗ 0 + x = x: $\Pi\mathcal{X}\colon ((\mathcal{X}*0)+\mathcal{X}) = \mathcal{X}$]

SystemQ **proof of** x ∗ 0 + x = x:

L01: Arbitrary $\gg$ $\mathcal{X}$ ;
L02: times1 $\gg$ $(\mathcal{X} * 1) = \mathcal{X}$ ;
L03: eqSymmetry $\rhd$ L02 $\gg$ $\mathcal{X} = (\mathcal{X} * 1)$ ;
L04: lemma eqAdditionLeft$\rhd$L03 $\gg$ $((\mathcal{X}*0)+\mathcal{X}) = ((\mathcal{X}*0)+(\mathcal{X}*1))$ ;
L05: Distribution $\gg$ $(\mathcal{X}*((0+1))) = ((\mathcal{X}*0)+(\mathcal{X}*1))$ ;
L06: eqSymmetry $\rhd$ L05 $\gg$ $((\mathcal{X}*0)+(\mathcal{X}*1)) = (\mathcal{X}*((0+1)))$ ;
L07: lemma plus0Left $\gg$ $(0 + 1) = 1$ ;
L08: lemma eqMultiplicationLeft $\rhd$ L07 $\gg$ $(\mathcal{X} * ((0 + 1))) = (\mathcal{X} * 1)$ ;
L09: eqTransitivity5 $\rhd$ L04 $\rhd$ L06 $\rhd$ L08 $\rhd$ L02 $\gg$ $((\mathcal{X} * 0) + \mathcal{X}) = \mathcal{X}$ □

[SystemQ **lemma** $x * 0 = 0$: $\Pi\mathcal{X}$: $(\mathcal{X} * 0) = 0$]

SystemQ **proof of** $x * 0 = 0$:

L01: Arbitrary $\gg$ $\mathcal{X}$ ;
L02: $x = x + (y - y)$ $\gg$ $(\mathcal{X} * 0) = ((\mathcal{X} * 0) + ((\mathcal{X} - \mathcal{X})))$ ;
L03: plusAssociativity $\gg$ $(((\mathcal{X}*0)+\mathcal{X})-\mathcal{X}) = ((\mathcal{X}*0)+((\mathcal{X} - \mathcal{X})))$ ;
L04: eqSymmetry $\rhd$ L03 $\gg$ $((\mathcal{X} * 0) + ((\mathcal{X} - \mathcal{X}))) = (((\mathcal{X} * 0) + \mathcal{X}) - \mathcal{X})$ ;
L05: $x * 0 + x = x$ $\gg$ $((\mathcal{X} * 0) + \mathcal{X}) = \mathcal{X}$ ;
L06: eqAddition $\rhd$ L05 $\gg$ $(((\mathcal{X} * 0) + \mathcal{X}) - \mathcal{X}) = (\mathcal{X} - \mathcal{X})$ ;
L07: Negative $\gg$ $(\mathcal{X} - \mathcal{X}) = 0$ ;
L08: eqTransitivity5 $\rhd$ L02 $\rhd$ L04 $\rhd$ L06 $\rhd$ L07 $\gg$ $(\mathcal{X} * 0) = 0$ □

[SystemQ **lemma** $(-1)*(-1)+(-1)*1 = 0$: $(((-1)*(-1))+((-1)*1)) = 0$]

SystemQ **proof of** $(-1) * (-1) + (-1) * 1 = 0$:

L01: DistributionOut $\gg$ $(((-1) * (-1)) + ((-1) * 1)) = ((-1) * (((-1) + 1)))$ ;
L02: Negative $\gg$ $(1 + (-1)) = 0$ ;
L03: plusCommutativity $\gg$ $((-1) + 1) = (1 + (-1))$ ;
L04: eqTransitivity $\rhd$ L03 $\rhd$ L02 $\gg$ $((-1) + 1) = 0$ ;
L05: lemma eqMultiplicationLeft $\rhd$ L04 $\gg$ $((-1)*(((-1)+1))) = ((-1)*0)$ ;
L06: $x * 0 = 0$ $\gg$ $((-1) * 0) = 0$ ;
L07: eqTransitivity4 $\rhd$ L01 $\rhd$ L05 $\rhd$ L06 $\gg$ $(((-1) * (-1)) + ((-1) * 1)) = 0$ □

[SystemQ **lemma** $(-1) * (-1) = 1$: $((-1) * (-1)) = 1$]

SystemQ **proof of** $(-1) * (-1) = 1$:

L01: $x = x + (y - y)$ $\gg$ $((-1)*(-1)) = (((-1)*(-1))+((1 - 1)))$ ;
L02: times1 $\gg$ $((-1) * 1) = (-1)$ ;
L03: eqSymmetry $\rhd$ L02 $\gg$ $(-1) = ((-1) * 1)$ ;
L04: lemma eqAdditionLeft$\rhd$L03 $\gg$ $(1 - 1) = (1 + ((-1) * 1))$ ;
L05: lemma eqAdditionLeft$\rhd$L04 $\gg$ $(((-1) * (-1)) + ((1 - 1))) = (((-1)*(-1))+((1+((-1)*1))))$ ;
L06: plusCommutativity $\gg$ $(1+((-1)*1)) = (((-1)*1)+1)$ ;

L07: lemma eqAdditionLeft ▷ L06 ≫ $(((-1) * (-1)) + ((1 + ((-1) * 1)))) = (((-1)*(-1))+((((-1)* 1) + 1)))$ ;
L08: plusAssociativity ≫ $((((-1) * (-1)) + ((-1) * 1)) + 1) = (((-1) * (-1)) + ((((-1) * 1) + 1)))$ ;
L09: eqSymmetry ▷ L08 ≫ $(((-1) * (-1)) + ((((-1) * 1) + 1))) = ((((-1) * (-1)) + ((-1) * 1)) + 1)$ ;
L10: $(-1) * (-1) + (-1) * 1 = 0$ ≫ $(((-1) * (-1)) + ((-1) * 1)) = 0$ ;
L11: eqAddition ▷ L10 ≫ $((((-1) * (-1)) + ((-1) * 1)) + 1) = (0 + 1)$ ;
L12: lemma plus0Left ≫ $(0 + 1) = 1$ ;
L13: eqTransitivity5 ▷ L01 ▷ L05 ▷ L07 ▷ L09 ≫ $((-1)*(-1)) = ((((-1)*(-1))+ ((-1) * 1)) + 1)$ ;
L14: eqTransitivity4 ▷ L13 ▷ L11 ▷ L12 ≫ $((-1) * (-1)) = 1$ □

[SystemQ **lemma** subLeqRight: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}\colon \mathcal{X} = \mathcal{Y} \vdash \mathcal{Z} <= \mathcal{X} \vdash \mathcal{Z} <= \mathcal{Y}$]

SystemQ **proof of** subLeqRight:

L01: Arbitrary ≫ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ ;
L02: Premise ≫ $\mathcal{X} = \mathcal{Y}$ ;
L03: Premise ≫ $\mathcal{Z} <= \mathcal{X}$ ;
L04: eqLeq ▷ L02 ≫ $\mathcal{X} <= \mathcal{Y}$ ;
L05: leqTransitivity ▷ L03 ▷ L04 ≫ $\mathcal{Z} <= \mathcal{Y}$ □

[SystemQ **lemma** subLeqLeft: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}\colon \mathcal{X} = \mathcal{Y} \vdash \mathcal{X} <= \mathcal{Z} \vdash \mathcal{Y} <= \mathcal{Z}$]

SystemQ **proof of** subLeqLeft:

L01: Arbitrary ≫ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ ;
L02: Premise ≫ $\mathcal{X} = \mathcal{Y}$ ;
L03: Premise ≫ $\mathcal{X} <= \mathcal{Z}$ ;
L04: eqSymmetry ▷ L02 ≫ $\mathcal{Y} = \mathcal{X}$ ;
L05: eqLeq ▷ L04 ≫ $\mathcal{Y} <= \mathcal{X}$ ;
L06: leqTransitivity ▷ L05 ▷ L03 ≫ $\mathcal{Y} <= \mathcal{Z}$ □

[SystemQ **lemma** $0 < 1$Helper: $1 <= 0 \Rightarrow 0 <= 1$]

SystemQ **proof of** $0 < 1$Helper:

L01: Block ≫ Begin ;
L02: Premise ≫ $1 <= 0$ ;
L03: leqAddition ▷ L02 ≫ $(1 + (-1)) <= (0 + (-1))$ ;
L04: Negative ≫ $(1 + (-1)) = 0$ ;
L05: subLeqLeft ▷ L04 ▷ L03 ≫ $0 <= (0 + (-1))$ ;
L06: lemma plus0Left ≫ $(0 + (-1)) = (-1)$ ;
L07: subLeqRight ▷ L06 ▷ L05 ≫ $0 <= (-1)$ ;
L08: leqMultiplication ▷ L07 ▷ L07 ≫ $(0 * (-1)) <= ((-1) * (-1))$ ;
L09: $x * 0 = 0$ ≫ $((-1) * 0) = 0$ ;
L10: timesCommutativity ≫ $(0 * (-1)) = ((-1) * 0)$ ;

L11: eqTransitivity $\triangleright$ L10 $\triangleright$ L09 $\gg$ $(0 * (-1)) = 0$ ;
L12: subLeqLeft $\triangleright$ L11 $\triangleright$ L08 $\gg$ $0 <= ((-1) * (-1))$ ;
L13: $(-1) * (-1) = 1 \gg$ $((-1) * (-1)) = 1$ ;
L14: subLeqRight $\triangleright$ L13 $\triangleright$ L12 $\gg$ $0 <= 1$ ;
L15: Block $\gg$ End ;
L16: Ded $\triangleright$ L15 $\gg$ $1 <= 0 \Rightarrow 0 <= 1$ □

[SystemQ **lemma** $0 < 1$: $0 < 1$]

SystemQ **proof of** $0 < 1$:

L01: leqTotality $\gg$ $0 <= 1 \dot{\vee} 1 <= 0$ ;
L02: AutoImply $\gg$ $0 <= 1 \Rightarrow 0 <= 1$ ;
L03: $0 <$ 1Helper $\gg$ $1 <= 0 \Rightarrow 0 <= 1$ ;
L04: FromDisjuncts $\triangleright$ L01 $\triangleright$ L02 $\triangleright$ L03 $\gg$ $0 <= 1$ ;
L05: 0not1 $\gg$ $0 \neq 1$ ;
L06: JoinConjuncts $\triangleright$ L04 $\triangleright$ L05 $\gg$ $0 < 1$ □

[SystemQ **lemma** AddEquations: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}$: $\mathcal{X} = \mathcal{Y} \vdash \mathcal{Z} = \mathcal{U} \vdash (\mathcal{X} + \mathcal{Z}) = (\mathcal{Y} + \mathcal{U})$]

SystemQ **proof of** AddEquations:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}$ ;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$ ;
L03: Premise $\gg$ $\mathcal{Z} = \mathcal{U}$ ;
L04: eqAddition $\triangleright$ L02 $\gg$ $(\mathcal{X} + \mathcal{Z}) = (\mathcal{Y} + \mathcal{Z})$ ;
L05: lemma eqAdditionLeft$\triangleright$L03 $\gg$ $(\mathcal{Y} + \mathcal{Z}) = (\mathcal{Y} + \mathcal{U})$ ;
L06: eqTransitivity $\triangleright$ L04 $\triangleright$ L05 $\gg$ $(\mathcal{X} + \mathcal{Z}) = (\mathcal{Y} + \mathcal{U})$ □

[SystemQ **lemma** SubtractEquations: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}$: $(\mathcal{X} + \mathcal{Z}) = (\mathcal{Y} + \mathcal{U}) \vdash \mathcal{Z} = \mathcal{U} \vdash \mathcal{X} = \mathcal{Y}$]

SystemQ **proof of** SubtractEquations:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}$ ;
L02: Premise $\gg$ $(\mathcal{X} + \mathcal{Z}) = (\mathcal{Y} + \mathcal{U})$ ;
L03: Premise $\gg$ $\mathcal{Z} = \mathcal{U}$ ;
L04: eqAddition $\triangleright$ L02 $\gg$ $((\mathcal{X} + \mathcal{Z}) - \mathcal{Z}) = ((\mathcal{Y} + \mathcal{U}) - \mathcal{Z})$ ;
L05: lemma plus0Left $\gg$ $(0 + \mathcal{Z}) = \mathcal{Z}$ ;
L06: eqTransitivity $\triangleright$ L05 $\triangleright$ L03 $\gg$ $(0 + \mathcal{Z}) = \mathcal{U}$ ;
L07: PositiveToRight(Eq) $\triangleright$ L06 $\gg$ $0 = (\mathcal{U} - \mathcal{Z})$ ;
L08: eqSymmetry $\triangleright$ L07 $\gg$ $(\mathcal{U} - \mathcal{Z}) = 0$ ;
L09: lemma eqAdditionLeft$\triangleright$L08 $\gg$ $(\mathcal{Y} + ((\mathcal{U} - \mathcal{Z}))) = (\mathcal{Y} + 0)$ ;
L10: plusAssociativity $\gg$ $((\mathcal{Y} + \mathcal{U}) - \mathcal{Z}) = (\mathcal{Y} + ((\mathcal{U} - \mathcal{Z})))$ ;
L11: plus0 $\gg$ $(\mathcal{Y} + 0) = \mathcal{Y}$ ;
L12: eqTransitivity4 $\triangleright$ L10 $\triangleright$ L09 $\triangleright$ L11 $\gg$ $((\mathcal{Y} + \mathcal{U}) - \mathcal{Z}) = \mathcal{Y}$ ;
L13: $\mathrm{x = x + y - y} \gg$ $\mathcal{X} = ((\mathcal{X} + \mathcal{Z}) - \mathcal{Z})$ ;
L14: eqTransitivity4 $\triangleright$ L13 $\triangleright$ L04 $\triangleright$ L12 $\gg$ $\mathcal{X} = \mathcal{Y}$ □

[SystemQ **lemma** SubtractEquationsLeft: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}$: $(\mathcal{X} + \mathcal{Z}) = (\mathcal{Y} + \mathcal{U}) \vdash \mathcal{X} = \mathcal{Y} \vdash \mathcal{Z} = \mathcal{U}$]

SystemQ **proof of** SubtractEquationsLeft:

| | | | |
|---|---|---|---|
| L01: | Arbitrary $\gg$ | $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}$ | ; |
| L02: | Premise $\gg$ | $(\mathcal{X} + \mathcal{Z}) = (\mathcal{Y} + \mathcal{U})$ | ; |
| L03: | Premise $\gg$ | $\mathcal{X} = \mathcal{Y}$ | ; |
| L04: | plusCommutativity $\gg$ | $(\mathcal{Z} + \mathcal{X}) = (\mathcal{X} + \mathcal{Z})$ | ; |
| L05: | plusCommutativity $\gg$ | $(\mathcal{Y} + \mathcal{U}) = (\mathcal{U} + \mathcal{Y})$ | ; |
| L06: | eqTransitivity4 $\triangleright$ L04 $\triangleright$ L02 $\triangleright$ L05 $\gg$ | $(\mathcal{Z} + \mathcal{X}) = (\mathcal{U} + \mathcal{Y})$ | ; |
| L07: | SubtractEquations $\triangleright$ L06 $\triangleright$ L03 $\gg$ | $\mathcal{Z} = \mathcal{U}$ | $\square$ |

[SystemQ **lemma** EqNegated: $\Pi \mathcal{X}, \mathcal{Y}$: $\mathcal{X} = \mathcal{Y} \vdash (-\mathrm{u}\mathcal{X}) = (-\mathrm{u}\mathcal{Y})$]

SystemQ **proof of** EqNegated:

| | | | |
|---|---|---|---|
| L01: | Arbitrary $\gg$ | $\mathcal{X}, \mathcal{Y}$ | ; |
| L02: | Premise $\gg$ | $\mathcal{X} = \mathcal{Y}$ | ; |
| L03: | Negative $\gg$ | $(\mathcal{X} - \mathcal{X}) = 0$ | ; |
| L04: | Negative $\gg$ | $(\mathcal{Y} - \mathcal{Y}) = 0$ | ; |
| L05: | eqSymmetry $\triangleright$ L04 $\gg$ | $0 = (\mathcal{Y} - \mathcal{Y})$ | ; |
| L06: | eqTransitivity $\triangleright$ L03 $\triangleright$ L05 $\gg$ | $(\mathcal{X} - \mathcal{X}) = (\mathcal{Y} - \mathcal{Y})$ | ; |
| L07: | SubtractEquationsLeft $\triangleright$ L06 $\triangleright$ L02 $\gg$ | $(-\mathrm{u}\mathcal{X}) = (-\mathrm{u}\mathcal{Y})$ | $\square$ |

[SystemQ **lemma** PositiveToRight(Eq): $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}$: $(\mathcal{X} + \mathcal{Y}) = \mathcal{Z} \vdash \mathcal{X} = (\mathcal{Z} - \mathcal{Y})$]

SystemQ **proof of** PositiveToRight(Eq):

| | | | |
|---|---|---|---|
| L01: | Arbitrary $\gg$ | $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ | ; |
| L02: | Premise $\gg$ | $(\mathcal{X} + \mathcal{Y}) = \mathcal{Z}$ | ; |
| L03: | eqAddition $\triangleright$ L02 $\gg$ | $((\mathcal{X} + \mathcal{Y}) - \mathcal{Y}) = (\mathcal{Z} - \mathcal{Y})$ | ; |
| L04: | x = x + y − y $\gg$ | $\mathcal{X} = ((\mathcal{X} + \mathcal{Y}) - \mathcal{Y})$ | ; |
| L05: | eqTransitivity $\triangleright$ L04 $\triangleright$ L03 $\gg$ | $\mathcal{X} = (\mathcal{Z} - \mathcal{Y})$ | $\square$ |

[SystemQ **lemma** PositiveToLeft(Eq)(1term): $\Pi \mathcal{X}, \mathcal{Y}$: $\mathcal{X} = \mathcal{Y} \vdash (\mathcal{X} - \mathcal{Y}) = 0$]

SystemQ **proof of** PositiveToLeft(Eq)(1term):

| | | | |
|---|---|---|---|
| L01: | Arbitrary $\gg$ | $\mathcal{X}, \mathcal{Y}$ | ; |
| L02: | Premise $\gg$ | $\mathcal{X} = \mathcal{Y}$ | ; |
| L03: | eqAddition $\triangleright$ L02 $\gg$ | $(\mathcal{X} - \mathcal{Y}) = (\mathcal{Y} - \mathcal{Y})$ | ; |
| L04: | Negative $\gg$ | $(\mathcal{Y} - \mathcal{Y}) = 0$ | ; |
| L05: | eqTransitivity $\triangleright$ L03 $\triangleright$ L04 $\gg$ | $(\mathcal{X} - \mathcal{Y}) = 0$ | $\square$ |

[SystemQ **lemma** PositiveToRight(Leq): $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}$: $(\mathcal{X} + \mathcal{Y}) <= \mathcal{Z} \vdash \mathcal{X} <= (\mathcal{Z} - \mathcal{Y})$]

SystemQ **proof of** PositiveToRight(Leq):

| | | | |
|---|---|---|---|
| L01: | Arbitrary $\gg$ | $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ | ; |
| L02: | Premise $\gg$ | $(\mathcal{X} + \mathcal{Y}) <= \mathcal{Z}$ | ; |
| L03: | leqAddition $\triangleright$ L02 $\gg$ | $((\mathcal{X} + \mathcal{Y}) - \mathcal{Y}) <= (\mathcal{Z} - \mathcal{Y})$ | ; |
| L04: | x = x + y − y $\gg$ | $\mathcal{X} = ((\mathcal{X} + \mathcal{Y}) - \mathcal{Y})$ | ; |
| L05: | eqSymmetry $\triangleright$ L04 $\gg$ | $((\mathcal{X} + \mathcal{Y}) - \mathcal{Y}) = \mathcal{X}$ | ; |
| L06: | subLeqLeft $\triangleright$ L05 $\triangleright$ L03 $\gg$ | $\mathcal{X} <= (\mathcal{Z} - \mathcal{Y})$ | $\square$ |

[SystemQ **lemma** PositiveToRight(Leq)(1term): $\Pi \mathcal{Y}, \mathcal{Z}$: $\mathcal{Y} <= \mathcal{Z} \vdash 0 <= (\mathcal{Z} - \mathcal{Y})$]

SystemQ **proof of** PositiveToRight(Leq)(1term):

| | | | |
|---|---|---|---|
| L01: | Arbitrary ≫ | $\mathcal{Y}, \mathcal{Z}$ | ; |
| L02: | Premise ≫ | $\mathcal{Y} <= \mathcal{Z}$ | ; |
| L03: | lemma plus0Left ≫ | $(0 + \mathcal{Y}) = \mathcal{Y}$ | ; |
| L04: | eqSymmetry ⊳ L03 ≫ | $\mathcal{Y} = (0 + \mathcal{Y})$ | ; |
| L05: | subLeqLeft ⊳ L04 ⊳ L02 ≫ | $(0 + \mathcal{Y}) <= \mathcal{Z}$ | ; |
| L06: | PositiveToRight(Leq) ⊳ L05 ≫ | $0 <= (\mathcal{Z} - \mathcal{Y})$ | □ |

[SystemQ **lemma** NegativeToLeft(Eq): $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}$: $\mathcal{X} = (\mathcal{Y} - \mathcal{Z}) \vdash (\mathcal{X} + \mathcal{Z}) = \mathcal{Y}$]

SystemQ **proof of** NegativeToLeft(Eq):

| | | | |
|---|---|---|---|
| L01: | Arbitrary ≫ | $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ | ; |
| L02: | Premise ≫ | $\mathcal{X} = (\mathcal{Y} - \mathcal{Z})$ | ; |
| L03: | eqAddition ⊳ L02 ≫ | $(\mathcal{X} + \mathcal{Z}) = ((\mathcal{Y} - \mathcal{Z}) + \mathcal{Z})$ | ; |
| L04: | Three2threeTerms ≫ | $((\mathcal{Y} - \mathcal{Z}) + \mathcal{Z}) = ((\mathcal{Y} + \mathcal{Z}) - \mathcal{Z})$ | ; |
| L05: | x = x + y − y ≫ | $\mathcal{Y} = ((\mathcal{Y} + \mathcal{Z}) - \mathcal{Z})$ | ; |
| L06: | eqSymmetry ⊳ L05 ≫ | $((\mathcal{Y} + \mathcal{Z}) - \mathcal{Z}) = \mathcal{Y}$ | ; |
| L07: | eqTransitivity4 ⊳ L03 ⊳ L04 ⊳ L06 ≫ | $(\mathcal{X} + \mathcal{Z}) = \mathcal{Y}$ | □ |

(*** NO EQUALITY ***)

[SystemQ **lemma** LessNeq: $\Pi \mathcal{X}, \mathcal{Y}$: $\mathcal{X} < \mathcal{Y} \vdash \mathcal{X} \neq \mathcal{Y}$]

SystemQ **proof of** LessNeq:

| | | | |
|---|---|---|---|
| L01: | Arbitrary ≫ | $\mathcal{X}, \mathcal{Y}$ | ; |
| L02: | Premise ≫ | $\mathcal{X} < \mathcal{Y}$ | ; |
| L03: | Repetition ⊳ L02 ≫ | $\mathcal{X} <= \mathcal{Y} \dot{\wedge} \dot{\neg} ((\mathcal{X} = \mathcal{Y}))$n | ; |
| L04: | SecondConjunct ⊳ L03 ≫ | $\mathcal{X} \neq \mathcal{Y}$ | □ |

[SystemQ **lemma** x + y = zBackwards: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}$: $(\mathcal{X} + \mathcal{Y}) = \mathcal{Z} \vdash \mathcal{Z} = (\mathcal{Y} + \mathcal{X})$]

SystemQ **proof of** x + y = zBackwards:

| | | | |
|---|---|---|---|
| L01: | Arbitrary ≫ | $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ | ; |
| L02: | Premise ≫ | $(\mathcal{X} + \mathcal{Y}) = \mathcal{Z}$ | ; |
| L03: | plusCommutativity ≫ | $(\mathcal{X} + \mathcal{Y}) = (\mathcal{Y} + \mathcal{X})$ | ; |
| L04: | Equality ⊳ L02 ≫ | $\mathcal{Z} = (\mathcal{Y} + \mathcal{X})$ | □ |

[SystemQ **lemma** x∗y = zBackwards: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}$: $(\mathcal{X} * \mathcal{Y}) = \mathcal{Z} \vdash \mathcal{Z} = (\mathcal{Y} * \mathcal{X})$]

SystemQ **proof of** x ∗ y = zBackwards:

| | | | |
|---|---|---|---|
| L01: | Arbitrary ≫ | $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ | ; |
| L02: | Premise ≫ | $(\mathcal{X} * \mathcal{Y}) = \mathcal{Z}$ | ; |
| L03: | timesCommutativity ≫ | $(\mathcal{X} * \mathcal{Y}) = (\mathcal{Y} * \mathcal{X})$ | ; |
| L04: | Equality ⊳ L02 ≫ | $\mathcal{Z} = (\mathcal{Y} * \mathcal{X})$ | □ |

[SystemQ **lemma** DoubleMinus: $\Pi \mathcal{X}$: $(-\mathrm{u}(-\mathrm{u}\mathcal{X})) = \mathcal{X}$]

SystemQ **proof of** DoubleMinus:

| | | | |
|---|---|---|---|
| L01: | Arbitrary ≫ | $\mathcal{X}$ | ; |
| L02: | Negative ≫ | $((-\mathrm{u}\mathcal{X}) - (-\mathrm{u}\mathcal{X})) = 0$ | ; |
| L03: | x + y = zBackwards ⊳ L02 ≫ | $0 = ((-\mathrm{u}(-\mathrm{u}\mathcal{X})) - \mathcal{X})$ | ; |
| L04: | NegativeToLeft(Eq) ⊳ L03 ≫ | $(0 + \mathcal{X}) = (-\mathrm{u}(-\mathrm{u}\mathcal{X}))$ | ; |
| L05: | lemma plus0Left ≫ | $(0 + \mathcal{X}) = \mathcal{X}$ | ; |
| L06: | Equality ⊳ L04 ⊳ L05 ≫ | $(-\mathrm{u}(-\mathrm{u}\mathcal{X})) = \mathcal{X}$ | □ |

[SystemQ **lemma** NeqNegated: $\Pi \mathcal{X}, \mathcal{Y}$: $\mathcal{X} \neq \mathcal{Y} \vdash (-\mathrm{u}\mathcal{X}) \neq (-\mathrm{u}\mathcal{Y})$]

SystemQ **proof of** NeqNegated:

L01: Block $\gg$ Begin ;
L02: Arbitrary $\gg$ $\mathcal{X},\mathcal{Y}$ ;
L03: Premise $\gg$ $\mathcal{X} \neq \mathcal{Y}$ ;
L04: Premise $\gg$ $(-\mathrm{u}\mathcal{X}) = (-\mathrm{u}\mathcal{Y})$ ;
L05: EqNegated $\rhd$ L04 $\gg$ $(-\mathrm{u}(-\mathrm{u}\mathcal{X})) = (-\mathrm{u}(-\mathrm{u}\mathcal{Y}))$ ;
L06: DoubleMinus $\gg$ $(-\mathrm{u}(-\mathrm{u}\mathcal{X})) = \mathcal{X}$ ;
L07: eqSymmetry $\rhd$ L06 $\gg$ $\mathcal{X} = (-\mathrm{u}(-\mathrm{u}\mathcal{X}))$ ;
L08: DoubleMinus $\gg$ $(-\mathrm{u}(-\mathrm{u}\mathcal{Y})) = \mathcal{Y}$ ;
L09: eqTransitivity4 $\rhd$ L07 $\rhd$ L05 $\rhd$ L08 $\gg$ $\mathcal{X} = \mathcal{Y}$ ;
L10: FromContradiction $\rhd$ L09 $\rhd$ L03 $\gg$ $(-\mathrm{u}\mathcal{X}) \neq (-\mathrm{u}\mathcal{Y})$ ;
L11: Block $\gg$ End ;
L12: Arbitrary $\gg$ $\mathcal{X},\mathcal{Y}$ ;
L13: Ded $\rhd$ L11 $\gg$ $\mathcal{X} \neq \mathcal{Y} \Rightarrow (-\mathrm{u}\mathcal{X}) = (-\mathrm{u}\mathcal{Y}) \Rightarrow \dot{\neg}((-\mathrm{u}\mathcal{X}) = (-\mathrm{u}\mathcal{Y}))\mathrm{n}$ ;
L14: Premise $\gg$ $\mathcal{X} \neq \mathcal{Y}$ ;
L15: MP $\rhd$ L13 $\rhd$ L14 $\gg$ $(-\mathrm{u}\mathcal{X}) = (-\mathrm{u}\mathcal{Y}) \Rightarrow \dot{\neg}((-\mathrm{u}\mathcal{X}) = (-\mathrm{u}\mathcal{Y}))\mathrm{n}$ ;
L16: prop lemma imply negation $\rhd$ L15 $\gg$ $\dot{\neg}((-\mathrm{u}\mathcal{X}) = (-\mathrm{u}\mathcal{Y}))\mathrm{n}$ □

[SystemQ **lemma** SubNeqRight: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z}\colon \mathcal{X} = \mathcal{Y} \vdash \mathcal{Z} \neq \mathcal{X} \vdash \mathcal{Z} \neq \mathcal{Y}$]

SystemQ **proof of** SubNeqRight:

L01: Arbitrary $\gg$ $\mathcal{X},\mathcal{Y},\mathcal{Z}$ ;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$ ;
L03: Premise $\gg$ $\mathcal{Z} \neq \mathcal{X}$ ;
L04: NeqSymmetry $\rhd$ L03 $\gg$ $\mathcal{X} \neq \mathcal{Z}$ ;
L05: SubNeqLeft $\rhd$ L02 $\rhd$ L04 $\gg$ $\mathcal{Y} \neq \mathcal{Z}$ ;
L06: NeqSymmetry $\rhd$ L05 $\gg$ $\mathcal{Z} \neq \mathcal{Y}$ □

[SystemQ **lemma** NeqAddition: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z}\colon \mathcal{X} \neq \mathcal{Y} \vdash (\mathcal{X}+\mathcal{Z}) \neq (\mathcal{Y}+\mathcal{Z})$]

SystemQ **proof of** NeqAddition:

L01: Block $\gg$ Begin ;
L02: Arbitrary $\gg$ $\mathcal{X},\mathcal{Y},\mathcal{Z}$ ;
L03: Premise $\gg$ $\mathcal{X} \neq \mathcal{Y}$ ;
L04: Premise $\gg$ $(\mathcal{X}+\mathcal{Z}) = (\mathcal{Y}+\mathcal{Z})$ ;
L05: eqReflexivity $\gg$ $\mathcal{Z} = \mathcal{Z}$ ;
L06: SubtractEquations $\rhd$ L04 $\rhd$ L05 $\gg$ $\mathcal{X} = \mathcal{Y}$ ;
L07: FromContradiction $\rhd$ L06 $\rhd$ L03 $\gg$ $(\mathcal{X}+\mathcal{Z}) \neq (\mathcal{Y}+\mathcal{Z})$ ;
L08: Block $\gg$ End ;
L09: Arbitrary $\gg$ $\mathcal{X},\mathcal{Y},\mathcal{Z}$ ;
L10: Ded $\rhd$ L08 $\gg$ $\mathcal{X} \neq \mathcal{Y} \Rightarrow (\mathcal{X}+\mathcal{Z}) = (\mathcal{Y}+\mathcal{Z}) \Rightarrow (\mathcal{X}+\mathcal{Z}) \neq (\mathcal{Y}+\mathcal{Z})$ ;
L11: Premise $\gg$ $\mathcal{X} \neq \mathcal{Y}$ ;

L12: MP ▷ L10 ▷ L11 ≫ $(\mathcal{X}+\mathcal{Z}) = (\mathcal{Y}+\mathcal{Z}) \Rightarrow (\mathcal{X}+\mathcal{Z}) \neq (\mathcal{Y}+\mathcal{Z})$ ;
L13: prop lemma imply negation ▷ L12 ≫ $(\mathcal{X}+\mathcal{Z}) \neq (\mathcal{Y}+\mathcal{Z})$ □

[SystemQ **lemma** NeqMultiplication: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z}\colon \mathcal{Z} \neq 0 \vdash \mathcal{X} \neq \mathcal{Y} \vdash (\mathcal{X}*\mathcal{Z}) \neq (\mathcal{Y}*\mathcal{Z})$]

SystemQ **proof of** NeqMultiplication:

L01: Block ≫ Begin ;
L02: Arbitrary ≫ $\mathcal{X},\mathcal{Y},\mathcal{Z}$ ;
L03: Premise ≫ $\mathcal{Z} \neq 0$ ;
L04: Premise ≫ $\mathcal{X} \neq \mathcal{Y}$ ;
L05: Premise ≫ $(\mathcal{X}*\mathcal{Z}) = (\mathcal{Y}*\mathcal{Z})$ ;
L06: x = x ∗ y ∗ (1/y) ▷ L03 ≫ $\mathcal{X} = ((\mathcal{X}*\mathcal{Z}) * \mathrm{rec}\mathcal{Z})$ ;
L07: eqMultiplication ▷ L05 ≫ $((\mathcal{X}*\mathcal{Z}) * \mathrm{rec}\mathcal{Z}) = ((\mathcal{Y}*\mathcal{Z}) * \mathrm{rec}\mathcal{Z})$ ;
L08: x = x ∗ y ∗ (1/y) ▷ L03 ≫ $\mathcal{Y} = ((\mathcal{Y}*\mathcal{Z}) * \mathrm{rec}\mathcal{Z})$ ;
L09: eqSymmetry ▷ L08 ≫ $((\mathcal{Y}*\mathcal{Z}) * \mathrm{rec}\mathcal{Z}) = \mathcal{Y}$ ;
L10: eqTransitivity4 ▷ L06 ▷ L07 ▷ L09 ≫ $\mathcal{X} = \mathcal{Y}$ ;
L11: FromContradiction ▷ L10 ▷ L04 ≫ $(\mathcal{X}*\mathcal{Z}) \neq (\mathcal{Y}*\mathcal{Z})$ ;
L12: Block ≫ End ;
L13: Arbitrary ≫ $\mathcal{X},\mathcal{Y},\mathcal{Z}$ ;
L14: Ded ▷ L12 ≫ $\mathcal{Z} \neq 0 \Rightarrow \mathcal{X} \neq \mathcal{Y} \Rightarrow (\mathcal{X}*\mathcal{Z}) = (\mathcal{Y}*\mathcal{Z}) \Rightarrow (\mathcal{X}*\mathcal{Z}) \neq (\mathcal{Y}*\mathcal{Z})$ ;
L15: Premise ≫ $\mathcal{Z} \neq 0$ ;
L16: Premise ≫ $\mathcal{X} \neq \mathcal{Y}$ ;
L17: MP2 ▷ L14 ▷ L15 ▷ L16 ≫ $(\mathcal{X}*\mathcal{Z}) = (\mathcal{Y}*\mathcal{Z}) \Rightarrow (\mathcal{X}*\mathcal{Z}) \neq (\mathcal{Y}*\mathcal{Z})$ ;
L18: prop lemma imply negation ▷ L17 ≫ $(\mathcal{X}*\mathcal{Z}) \neq (\mathcal{Y}*\mathcal{Z})$ □

(*** NEGATIVE ***)

[SystemQ **lemma** UniqueNegative: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z}\colon (\mathcal{X}+\mathcal{Y}) = 0 \vdash (\mathcal{X}+\mathcal{Z}) = 0 \vdash \mathcal{Y} = \mathcal{Z}$]

SystemQ **proof of** UniqueNegative:

L01: Arbitrary ≫ $\mathcal{X},\mathcal{Y},\mathcal{Z}$ ;
L02: Premise ≫ $(\mathcal{X}+\mathcal{Y}) = 0$ ;
L03: Premise ≫ $(\mathcal{X}+\mathcal{Z}) = 0$ ;
L04: plusCommutativity ≫ $(\mathcal{Y}+\mathcal{X}) = (\mathcal{X}+\mathcal{Y})$ ;
L05: eqTransitivity ▷ L04 ▷ L02 ≫ $(\mathcal{Y}+\mathcal{X}) = 0$ ;
L06: PositiveToRight(Eq) ▷ L05 ≫ $\mathcal{Y} = (0 - \mathcal{X})$ ;
L07: plusCommutativity ≫ $(\mathcal{Z}+\mathcal{X}) = (\mathcal{X}+\mathcal{Z})$ ;
L08: eqTransitivity ▷ L07 ▷ L03 ≫ $(\mathcal{Z}+\mathcal{X}) = 0$ ;
L09: PositiveToRight(Eq) ▷ L08 ≫ $\mathcal{Z} = (0 - \mathcal{X})$ ;
L10: eqSymmetry ▷ L09 ≫ $(0 - \mathcal{X}) = \mathcal{Z}$ ;
L11: eqTransitivity ▷ L06 ▷ L10 ≫ $\mathcal{Y} = \mathcal{Z}$ □

[SystemQ **lemma** FromLess: $\Pi\mathcal{X},\mathcal{Y}\colon \mathcal{X} < \mathcal{Y} \vdash \dot\neg\,(\mathcal{Y} <= \mathcal{X})\mathrm{n}$]

SystemQ **proof of** FromLess:

| | | | |
|---|---|---|---|
| L01: | Block $\gg$ | Begin | ; |
| L02: | Arbitrary $\gg$ | $\mathcal{X},\mathcal{Y}$ | ; |
| L03: | Premise $\gg$ | $\mathcal{Y} <= \mathcal{X}$ | ; |
| L04: | toNotLess $\rhd$ L03 $\gg$ | $\dot\neg\,(\mathcal{X} < \mathcal{Y})\mathrm{n}$ | ; |
| L05: | Block $\gg$ | End | ; |
| L06: | Arbitrary $\gg$ | $\mathcal{X},\mathcal{Y}$ | ; |
| L07: | Ded $\rhd$ L05 $\gg$ | $\mathcal{Y} <= \mathcal{X} \Rightarrow \dot\neg\,(\mathcal{X} < \mathcal{Y})\mathrm{n}$ | ; |
| L08: | Premise $\gg$ | $\mathcal{X} < \mathcal{Y}$ | ; |
| L09: | AddDoubleNeg $\rhd$ L08 $\gg$ | $\dot\neg\,(\dot\neg\,(\mathcal{X} < \mathcal{Y})\mathrm{n})\mathrm{n}$ | ; |
| L10: | MT $\rhd$ L07 $\rhd$ L09 $\gg$ | $\dot\neg\,(\mathcal{Y} <= \mathcal{X})\mathrm{n}$ | $\Box$ |

[SystemQ **lemma** ToLess: $\Pi\mathcal{X},\mathcal{Y}\colon \dot\neg\,(\mathcal{X} <= \mathcal{Y})\mathrm{n} \vdash \mathcal{Y} < \mathcal{X}$]

SystemQ **proof of** ToLess:

| | | | |
|---|---|---|---|
| L01: | Block $\gg$ | Begin | ; |
| L02: | Arbitrary $\gg$ | $\mathcal{X},\mathcal{Y}$ | ; |
| L03: | Premise $\gg$ | $\dot\neg\,(\mathcal{Y} < \mathcal{X})\mathrm{n}$ | ; |
| L04: | fromNotLess $\rhd$ L03 $\gg$ | $\mathcal{X} <= \mathcal{Y}$ | ; |
| L05: | Block $\gg$ | End | ; |
| L06: | Arbitrary $\gg$ | $\mathcal{X},\mathcal{Y}$ | ; |
| L07: | Ded $\rhd$ L05 $\gg$ | $\dot\neg\,(\mathcal{Y} < \mathcal{X})\mathrm{n} \Rightarrow \mathcal{X} <= \mathcal{Y}$ | ; |
| L08: | Premise $\gg$ | $\dot\neg\,(\mathcal{X} <= \mathcal{Y})\mathrm{n}$ | ; |
| L09: | NegativeMT $\rhd$ L07 $\rhd$ L08 $\gg$ | $\mathcal{Y} < \mathcal{X}$ | $\Box$ |

[SystemQ **lemma** fromNotLess: $\Pi\mathcal{X},\mathcal{Y}\colon \dot\neg\,((\mathcal{X} < \mathcal{Y}))\mathrm{n} \vdash \mathcal{Y} <= \mathcal{X}$]

SystemQ **proof of** fromNotLess:

| | | | |
|---|---|---|---|
| L01: | Block $\gg$ | Begin | ; |
| L02: | Arbitrary $\gg$ | $\mathcal{X},\mathcal{Y}$ | ; |
| L03: | Premise $\gg$ | $\dot\neg\,((\mathcal{X} < \mathcal{Y}))\mathrm{n}$ | ; |
| L04: | Premise $\gg$ | $\mathcal{X} <= \mathcal{Y}$ | ; |
| L05: | Repetition $\rhd$ L03 $\gg$ | $\dot\neg\,(\dot\neg\,((\mathcal{X} <= \mathcal{Y} \Rightarrow \dot\neg\,(\mathcal{X} \neq \mathcal{Y})\mathrm{n}))\mathrm{n})\mathrm{n}$ | ; |
| L06: | RemoveDoubleNeg $\rhd$ L05 $\gg$ | $\mathcal{X} <= \mathcal{Y} \Rightarrow \dot\neg\,(\mathcal{X} \neq \mathcal{Y})\mathrm{n}$ | ; |
| L07: | MP $\rhd$ L06 $\rhd$ L04 $\gg$ | $\dot\neg\,(\mathcal{X} \neq \mathcal{Y})\mathrm{n}$ | ; |
| L08: | RemoveDoubleNeg $\rhd$ L07 $\gg$ | $\mathcal{X} = \mathcal{Y}$ | ; |
| L09: | eqSymmetry $\rhd$ L08 $\gg$ | $\mathcal{Y} = \mathcal{X}$ | ; |
| L10: | eqLeq $\rhd$ L09 $\gg$ | $\mathcal{Y} <= \mathcal{X}$ | ; |
| L11: | Block $\gg$ | End | ; |
| L12: | Arbitrary $\gg$ | $\mathcal{X},\mathcal{Y}$ | ; |
| L13: | Ded $\rhd$ L11 $\gg$ | $\dot\neg\,(\mathcal{X} < \mathcal{Y})\mathrm{n} \Rightarrow \mathcal{X} <= \mathcal{Y} \Rightarrow \mathcal{Y} <= \mathcal{X}$ | ; |
| L14: | Premise $\gg$ | $\dot\neg\,(\mathcal{X} < \mathcal{Y})\mathrm{n}$ | ; |
| L15: | MP $\rhd$ L13 $\rhd$ L14 $\gg$ | $\mathcal{X} <= \mathcal{Y} \Rightarrow \mathcal{Y} <= \mathcal{X}$ | ; |
| L16: | AutoImply $\gg$ | $\mathcal{Y} <= \mathcal{X} \Rightarrow \mathcal{Y} <= \mathcal{X}$ | ; |
| L17: | leqTotality $\gg$ | $\mathcal{X} <= \mathcal{Y} \dot\vee \mathcal{Y} <= \mathcal{X}$ | ; |
| L18: | FromDisjuncts $\rhd$ L17 $\rhd$ L15 $\rhd$ L16 $\gg$ | $\mathcal{Y} <= \mathcal{X}$ | $\Box$ |

[SystemQ **lemma** toNotLess: $\Pi\mathcal{X},\mathcal{Y}: \mathcal{X} <= \mathcal{Y} \vdash \dot{\neg}(\mathcal{Y} < \mathcal{X})\text{n}$]

SystemQ **proof of** toNotLess:

L01: Block $\gg$ Begin ;
L02: Arbitrary $\gg$ $\mathcal{X},\mathcal{Y}$ ;
L03: Premise $\gg$ $\mathcal{X} <= \mathcal{Y}$ ;
L04: Premise $\gg$ $\mathcal{Y} <= \mathcal{X}$ ;
L05: leqAntisymmetry $\triangleright$ L04 $\triangleright$ L03 $\gg$ $\mathcal{Y} = \mathcal{X}$ ;
L06: AddDoubleNeg $\triangleright$ L05 $\gg$ $\dot{\neg}(\dot{\neg}(\mathcal{Y} = \mathcal{X})\text{n})\text{n}$ ;
L07: Block $\gg$ End ;
L08: Arbitrary $\gg$ $\mathcal{X},\mathcal{Y}$ ;
L09: Ded $\triangleright$ L07 $\gg$ $\mathcal{X} <= \mathcal{Y} \Rightarrow \mathcal{Y} <= \mathcal{X} \Rightarrow \dot{\neg}(\dot{\neg}(\mathcal{Y} = \mathcal{X})\text{n})\text{n}$ ;
L10: Premise $\gg$ $\mathcal{X} <= \mathcal{Y}$ ;
L11: MP $\triangleright$ L09 $\triangleright$ L10 $\gg$ $\mathcal{Y} <= \mathcal{X} \Rightarrow \dot{\neg}(\dot{\neg}(\mathcal{Y} = \mathcal{X})\text{n})\text{n}$ ;
L12: AddDoubleNeg $\triangleright$ L11 $\gg$ $\dot{\neg}(\dot{\neg}((\mathcal{Y} <= \mathcal{X} \Rightarrow \dot{\neg}(\dot{\neg}(\mathcal{Y} = \mathcal{X})\text{n})\text{n}))\text{n})\text{n}$ ;
L13: Repetition $\triangleright$ L12 $\gg$ $\dot{\neg}((\mathcal{Y} <= \mathcal{X} \dot{\wedge} \dot{\neg}(\mathcal{Y} = \mathcal{X})\text{n}))\text{n}$ ;
L14: Repetition $\triangleright$ L13 $\gg$ $\dot{\neg}(\mathcal{Y} < \mathcal{X})\text{n}$ $\Box$

(*** LEQ ***)

[SystemQ **lemma** LeqLessEq: $\Pi\mathcal{X},\mathcal{Y}: \mathcal{X} <= \mathcal{Y} \vdash \mathcal{X} < \mathcal{Y} \dot{\vee} \mathcal{X} = \mathcal{Y}$]

SystemQ **proof of** LeqLessEq:

L01: Block $\gg$ Begin ;
L02: Arbitrary $\gg$ $\mathcal{X},\mathcal{Y}$ ;
L03: Premise $\gg$ $\mathcal{X} <= \mathcal{Y}$ ;
L04: Premise $\gg$ $\dot{\neg}(\mathcal{X} < \mathcal{Y})\text{n}$ ;
L05: fromNotLess $\triangleright$ L04 $\gg$ $\mathcal{Y} <= \mathcal{X}$ ;
L06: leqAntisymmetry $\triangleright$ L03 $\triangleright$ L05 $\gg$ $\mathcal{X} = \mathcal{Y}$ ;
L07: Block $\gg$ End ;
L08: Arbitrary $\gg$ $\mathcal{X},\mathcal{Y}$ ;
L09: Ded $\triangleright$ L07 $\gg$ $\mathcal{X} <= \mathcal{Y} \Rightarrow \dot{\neg}(\mathcal{X} < \mathcal{Y})\text{n} \Rightarrow \mathcal{X} = \mathcal{Y}$ ;
L10: Premise $\gg$ $\mathcal{X} <= \mathcal{Y}$ ;
L11: MP $\triangleright$ L09 $\triangleright$ L10 $\gg$ $\dot{\neg}(\mathcal{X} < \mathcal{Y})\text{n} \Rightarrow \mathcal{X} = \mathcal{Y}$ ;
L12: Repetition $\triangleright$ L11 $\gg$ $\mathcal{X} < \mathcal{Y} \dot{\vee} \mathcal{X} = \mathcal{Y}$ $\Box$

[SystemQ **lemma** LessLeq: $\Pi\mathcal{X},\mathcal{Y}: \mathcal{X} < \mathcal{Y} \vdash \mathcal{X} <= \mathcal{Y}$]

SystemQ **proof of** LessLeq:

L01: Arbitrary $\gg$ $\mathcal{X},\mathcal{Y}$ ;
L02: Premise $\gg$ $\mathcal{X} < \mathcal{Y}$ ;
L03: Repetition $\triangleright$ L02 $\gg$ $\mathcal{X} <= \mathcal{Y} \dot{\wedge} \dot{\neg}((\mathcal{X} = \mathcal{Y}))\text{n}$ ;
L04: FirstConjunct $\triangleright$ L03 $\gg$ $\mathcal{X} <= \mathcal{Y}$ $\Box$

[SystemQ **lemma** FromLeqGeq: $\Pi\mathcal{A},\mathcal{X},\mathcal{Y}: \mathcal{X} <= \mathcal{Y} \Rightarrow \mathcal{A} \vdash \mathcal{Y} <= \mathcal{X} \Rightarrow \mathcal{A} \vdash \mathcal{A}$]

SystemQ **proof of** FromLeqGeq:

L01: Arbitrary $\gg$ $\mathcal{A},\mathcal{X},\mathcal{Y}$ ;

L02: Premise $\gg$ $\mathcal{X} <= \mathcal{Y} \Rightarrow \mathcal{A}$ ;
L03: Premise $\gg$ $\mathcal{Y} <= \mathcal{X} \Rightarrow \mathcal{A}$ ;
L04: leqTotality $\gg$ $\mathcal{X} <= \mathcal{Y} \dot{\vee} \mathcal{Y} <= \mathcal{X}$ ;
L05: FromDisjuncts $\rhd$ L04 $\rhd$ L02 $\rhd$ L03 $\gg$ $\mathcal{A}$ □

[SystemQ **lemma** SubLessRight: $\Pi\mathcal{X}, \mathcal{Y}, \mathcal{Z}\colon \mathcal{X} = \mathcal{Y} \vdash \mathcal{Z} < \mathcal{X} \vdash \mathcal{Z} < \mathcal{Y}$]

SystemQ **proof of** SubLessRight:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ ;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$ ;
L03: Premise $\gg$ $\mathcal{Z} < \mathcal{X}$ ;
L04: Repetition $\rhd$ L03 $\gg$ $\mathcal{Z} <= \mathcal{X} \dot{\wedge} \mathcal{Z} \neq \mathcal{X}$ ;
L05: FirstConjunct $\rhd$ L04 $\gg$ $\mathcal{Z} <= \mathcal{X}$ ;
L06: subLeqRight $\rhd$ L02 $\rhd$ L05 $\gg$ $\mathcal{Z} <= \mathcal{Y}$ ;
L07: SecondConjunct $\rhd$ L04 $\gg$ $\mathcal{Z} \neq \mathcal{X}$ ;
L08: SubNeqRight $\rhd$ L02 $\rhd$ L07 $\gg$ $\mathcal{Z} \neq \mathcal{Y}$ ;
L09: JoinConjuncts $\rhd$ L06 $\rhd$ L08 $\gg$ $\mathcal{Z} < \mathcal{Y}$ □

[SystemQ **lemma** SubLessLeft: $\Pi\mathcal{X}, \mathcal{Y}, \mathcal{Z}\colon \mathcal{X} = \mathcal{Y} \vdash \mathcal{X} < \mathcal{Z} \vdash \mathcal{Y} < \mathcal{Z}$]

SystemQ **proof of** SubLessLeft:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ ;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$ ;
L03: Premise $\gg$ $\mathcal{X} < \mathcal{Z}$ ;
L04: Repetition $\rhd$ L03 $\gg$ $\mathcal{X} <= \mathcal{Z} \dot{\wedge} \mathcal{X} \neq \mathcal{Z}$ ;
L05: FirstConjunct $\rhd$ L04 $\gg$ $\mathcal{X} <= \mathcal{Z}$ ;
L06: subLeqLeft $\rhd$ L02 $\rhd$ L05 $\gg$ $\mathcal{Y} <= \mathcal{Z}$ ;
L07: SecondConjunct $\rhd$ L04 $\gg$ $\mathcal{X} \neq \mathcal{Z}$ ;
L08: SubNeqLeft $\rhd$ L02 $\rhd$ L07 $\gg$ $\mathcal{Y} \neq \mathcal{Z}$ ;
L09: JoinConjuncts $\rhd$ L06 $\rhd$ L08 $\gg$ $\mathcal{Y} < \mathcal{Z}$ □

[SystemQ **lemma** leqLessTransitivity: $\Pi\mathcal{X}, \mathcal{Y}, \mathcal{Z}\colon \mathcal{X} <= \mathcal{Y} \vdash \mathcal{Y} < \mathcal{Z} \vdash \mathcal{X} < \mathcal{Z}$]

SystemQ **proof of** leqLessTransitivity:

L01: Block $\gg$ Begin ;
L02: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ ;
L03: Premise $\gg$ $\mathcal{X} <= \mathcal{Y}$ ;
L04: Premise $\gg$ $\mathcal{Y} < \mathcal{Z}$ ;
L05: Premise $\gg$ $\mathcal{X} = \mathcal{Z}$ ;
L06: FirstConjunct $\rhd$ L04 $\gg$ $\mathcal{Y} <= \mathcal{Z}$ ;
L07: SecondConjunct $\rhd$ L04 $\gg$ $\mathcal{Y} \neq \mathcal{Z}$ ;
L08: subLeqLeft $\rhd$ L05 $\rhd$ L03 $\gg$ $\mathcal{Z} <= \mathcal{Y}$ ;
L09: leqAntisymmetry $\rhd$ L06 $\rhd$ L08 $\gg$ $\mathcal{Y} = \mathcal{Z}$ ;
L10: FromContradiction $\rhd$ L09 $\rhd$ L07 $\gg$ $\mathcal{X} \neq \mathcal{Z}$ ;
L11: Block $\gg$ End ;
L12: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ ;
L13: Ded $\rhd$ L11 $\gg$ $\mathcal{X} <= \mathcal{Y} \Rightarrow \mathcal{Y} < \mathcal{Z} \Rightarrow \mathcal{X} = \mathcal{Z} \Rightarrow \mathcal{X} \neq \mathcal{Z}$ ;

L14: Premise $\gg$ $\mathcal{X} <= \mathcal{Y}$ ;
L15: Premise $\gg$ $\mathcal{Y} < \mathcal{Z}$ ;
L16: MP2 $\rhd$ L13 $\rhd$ L14 $\rhd$ L15 $\gg$ $\mathcal{X} = \mathcal{Z} \Rightarrow \mathcal{X} \neq \mathcal{Z}$ ;
L17: prop lemma imply negation $\rhd$
L16 $\gg$ $\mathcal{X} \neq \mathcal{Z}$ ;
L18: FirstConjunct $\rhd$ L15 $\gg$ $\mathcal{Y} <= \mathcal{Z}$ ;
L19: leqTransitivity $\rhd$ L14 $\rhd$ L18 $\gg$ $\mathcal{X} <= \mathcal{Z}$ ;
L20: JoinConjuncts $\rhd$ L19 $\rhd$ L17 $\gg$ $\mathcal{X} < \mathcal{Z}$ □

[SystemQ **lemma** LessAddition: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z}\colon \mathcal{X} < \mathcal{Y} \vdash (\mathcal{X}+\mathcal{Z}) < (\mathcal{Y}+\mathcal{Z})$]

SystemQ **proof of** LessAddition:

L01: Arbitrary $\gg$ $\mathcal{X},\mathcal{Y},\mathcal{Z}$ ;
L02: Premise $\gg$ $\mathcal{X} < \mathcal{Y}$ ;
L03: LessLeq $\rhd$ L02 $\gg$ $\mathcal{X} <= \mathcal{Y}$ ;
L04: leqAddition $\rhd$ L03 $\gg$ $(\mathcal{X}+\mathcal{Z}) <= (\mathcal{Y}+\mathcal{Z})$ ;
L05: LessNeq $\rhd$ L02 $\gg$ $\mathcal{X} \neq \mathcal{Y}$ ;
L06: NeqAddition $\rhd$ L05 $\gg$ $(\mathcal{X}+\mathcal{Z}) \neq (\mathcal{Y}+\mathcal{Z})$ ;
L07: JoinConjuncts $\rhd$ L04 $\rhd$ L06 $\gg$ $(\mathcal{X}+\mathcal{Z}) < (\mathcal{Y}+\mathcal{Z})$ □

[SystemQ **lemma** LessAdditionLeft: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z}\colon \mathcal{X} < \mathcal{Y} \vdash (\mathcal{Z}+\mathcal{X}) < (\mathcal{Z}+\mathcal{Y})$]

SystemQ **proof of** LessAdditionLeft:

L01: Arbitrary $\gg$ $\mathcal{X},\mathcal{Y},\mathcal{Z}$ ;
L02: Premise $\gg$ $\mathcal{X} < \mathcal{Y}$ ;
L03: LessAddition $\rhd$ L02 $\gg$ $(\mathcal{X}+\mathcal{Z}) < (\mathcal{Y}+\mathcal{Z})$ ;
L04: plusCommutativity $\gg$ $(\mathcal{X}+\mathcal{Z}) = (\mathcal{Z}+\mathcal{X})$ ;
L05: SubLessLeft $\rhd$ L04 $\rhd$ L03 $\gg$ $(\mathcal{Z}+\mathcal{X}) < (\mathcal{Y}+\mathcal{Z})$ ;
L06: plusCommutativity $\gg$ $(\mathcal{Y}+\mathcal{Z}) = (\mathcal{Z}+\mathcal{Y})$ ;
L07: SubLessRight $\rhd$ L06 $\rhd$ L05 $\gg$ $(\mathcal{Z}+\mathcal{X}) < (\mathcal{Z}+\mathcal{Y})$ □

[SystemQ **lemma** Leq + 1: $\Pi\mathcal{X},\mathcal{Y}\colon \mathcal{X} <= \mathcal{Y} \vdash \mathcal{X} < (\mathcal{Y}+1)$]

SystemQ **proof of** Leq + 1:

L01: Arbitrary $\gg$ $\mathcal{X},\mathcal{Y}$ ;
L02: Premise $\gg$ $\mathcal{X} <= \mathcal{Y}$ ;
L03: $0 < 1 \gg$ $0 < 1$ ;
L04: LessAdditionLeft $\rhd$ L03 $\gg$ $(\mathcal{Y}+0) < (\mathcal{Y}+1)$ ;
L05: plus0 $\gg$ $(\mathcal{Y}+0) = \mathcal{Y}$ ;
L06: SubLessLeft $\rhd$ L05 $\rhd$ L04 $\gg$ $\mathcal{Y} < (\mathcal{Y}+1)$ ;
L07: leqLessTransitivity $\rhd$ L02 $\rhd$
L06 $\gg$ $\mathcal{X} < (\mathcal{Y}+1)$ □

[SystemQ **lemma** LeqAdditionLeft: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z}\colon \mathcal{X} <= \mathcal{Y} \vdash (\mathcal{Z}+\mathcal{X}) <= (\mathcal{Z}+\mathcal{Y})$]

SystemQ **proof of** LeqAdditionLeft:

L01: Arbitrary $\gg$ $\mathcal{X},\mathcal{Y},\mathcal{Z}$ ;
L02: Premise $\gg$ $\mathcal{X} <= \mathcal{Y}$ ;
L03: leqAddition $\rhd$ L02 $\gg$ $(\mathcal{X}+\mathcal{Z}) <= (\mathcal{Y}+\mathcal{Z})$ ;
L04: plusCommutativity $\gg$ $(\mathcal{X}+\mathcal{Z}) = (\mathcal{Z}+\mathcal{X})$ ;
L05: plusCommutativity $\gg$ $(\mathcal{Y}+\mathcal{Z}) = (\mathcal{Z}+\mathcal{Y})$ ;
L06: subLeqLeft $\rhd$ L04 $\rhd$ L03 $\gg$ $(\mathcal{Z}+\mathcal{X}) <= (\mathcal{Y}+\mathcal{Z})$ ;
L07: subLeqRight $\rhd$ L05 $\rhd$ L06 $\gg$ $(\mathcal{Z}+\mathcal{X}) <= (\mathcal{Z}+\mathcal{Y})$ □

[SystemQ **lemma** leqSubtraction: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z}\colon (\mathcal{X}+\mathcal{Z}) <= (\mathcal{Y}+\mathcal{Z}) \vdash \mathcal{X} <= \mathcal{Y}$]

SystemQ **proof of** leqSubtraction:

| | | | |
|---|---|---|---|
| L01: | Arbitrary $\gg$ | $\mathcal{X},\mathcal{Y},\mathcal{Z}$ | ; |
| L02: | Premise $\gg$ | $(\mathcal{X}+\mathcal{Z}) <= (\mathcal{Y}+\mathcal{Z})$ | ; |
| L03: | leqAddition $\rhd$ L02 $\gg$ | $((\mathcal{X}+\mathcal{Z})-\mathcal{Z}) <= ((\mathcal{Y}+\mathcal{Z})-\mathcal{Z})$ | ; |
| L04: | $\mathrm{x} = \mathrm{x}+\mathrm{y}-\mathrm{y}$ $\gg$ | $\mathcal{X} = ((\mathcal{X}+\mathcal{Z})-\mathcal{Z})$ | ; |
| L05: | eqSymmetry $\rhd$ L04 $\gg$ | $((\mathcal{X}+\mathcal{Z})-\mathcal{Z}) = \mathcal{X}$ | ; |
| L06: | $\mathrm{x} = \mathrm{x}+\mathrm{y}-\mathrm{y}$ $\gg$ | $\mathcal{Y} = ((\mathcal{Y}+\mathcal{Z})-\mathcal{Z})$ | ; |
| L07: | eqSymmetry $\rhd$ L06 $\gg$ | $((\mathcal{Y}+\mathcal{Z})-\mathcal{Z}) = \mathcal{Y}$ | ; |
| L08: | subLeqLeft $\rhd$ L05 $\rhd$ L03 $\gg$ | $\mathcal{X} <= ((\mathcal{Y}+\mathcal{Z})-\mathcal{Z})$ | ; |
| L09: | subLeqRight $\rhd$ L07 $\rhd$ L08 $\gg$ | $\mathcal{X} <= \mathcal{Y}$ | $\Box$ |

[SystemQ **lemma** leqSubtractionLeft: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z}\colon (\mathcal{Z}+\mathcal{X}) <= (\mathcal{Z}+\mathcal{Y}) \vdash \mathcal{X} <= \mathcal{Y}$]

SystemQ **proof of** leqSubtractionLeft:

| | | | |
|---|---|---|---|
| L01: | Arbitrary $\gg$ | $\mathcal{X},\mathcal{Y},\mathcal{Z}$ | ; |
| L02: | Premise $\gg$ | $(\mathcal{Z}+\mathcal{X}) <= (\mathcal{Z}+\mathcal{Y})$ | ; |
| L03: | plusCommutativity $\gg$ | $(\mathcal{Z}+\mathcal{X}) = (\mathcal{X}+\mathcal{Z})$ | ; |
| L04: | plusCommutativity $\gg$ | $(\mathcal{Z}+\mathcal{Y}) = (\mathcal{Y}+\mathcal{Z})$ | ; |
| L05: | subLeqLeft $\rhd$ L03 $\rhd$ L02 $\gg$ | $(\mathcal{X}+\mathcal{Z}) <= (\mathcal{Z}+\mathcal{Y})$ | ; |
| L06: | subLeqRight $\rhd$ L04 $\rhd$ L05 $\gg$ | $(\mathcal{X}+\mathcal{Z}) <= (\mathcal{Y}+\mathcal{Z})$ | ; |
| L07: | leqSubtraction $\rhd$ L06 $\gg$ | $\mathcal{X} <= \mathcal{Y}$ | $\Box$ |

[SystemQ **lemma** negativeToLeft(Leq): $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z}\colon \mathcal{X} <= (\mathcal{Y}-\mathcal{Z}) \vdash (\mathcal{X}+\mathcal{Z}) <= \mathcal{Y}$]

SystemQ **proof of** negativeToLeft(Leq):

| | | | |
|---|---|---|---|
| L01: | Arbitrary $\gg$ | $\mathcal{X},\mathcal{Y},\mathcal{Z}$ | ; |
| L02: | Premise $\gg$ | $\mathcal{X} <= (\mathcal{Y}-\mathcal{Z})$ | ; |
| L03: | leqAddition $\rhd$ L02 $\gg$ | $(\mathcal{X}+\mathcal{Z}) <= ((\mathcal{Y}-\mathcal{Z})+\mathcal{Z})$ | ; |
| L04: | $\mathrm{x} = \mathrm{x}+\mathrm{y}-\mathrm{y}$ $\gg$ | $\mathcal{Y} = ((\mathcal{Y}+\mathcal{Z})-\mathcal{Z})$ | ; |
| L05: | Three2threeTerms $\gg$ | $((\mathcal{Y}+\mathcal{Z})-\mathcal{Z}) = ((\mathcal{Y}-\mathcal{Z})+\mathcal{Z})$ | ; |
| L06: | eqTransitivity $\rhd$ L04 $\rhd$ L05 $\gg$ | $\mathcal{Y} = ((\mathcal{Y}-\mathcal{Z})+\mathcal{Z})$ | ; |
| L07: | eqSymmetry $\rhd$ L06 $\gg$ | $((\mathcal{Y}-\mathcal{Z})+\mathcal{Z}) = \mathcal{Y}$ | ; |
| L08: | subLeqRight $\rhd$ L07 $\rhd$ L03 $\gg$ | $(\mathcal{X}+\mathcal{Z}) <= \mathcal{Y}$ | $\Box$ |

[SystemQ **lemma** thirdGeq: $\Pi\mathcal{X},\mathcal{Y}\colon \mathcal{X} <= \mathrm{Ex3} \mathbin{\dot\wedge} \mathcal{Y} <= \mathrm{Ex3}$]

SystemQ **proof of** thirdGeq:

| | | | |
|---|---|---|---|
| L01: | Block $\gg$ | Begin | ; |
| L02: | Arbitrary $\gg$ | $\mathcal{X},\mathcal{Y}$ | ; |
| L03: | Premise $\gg$ | $\mathcal{X} <= \mathcal{Y}$ | ; |
| L04: | leqReflexivity $\gg$ | $\mathcal{Y} <= \mathcal{Y}$ | ; |
| L05: | JoinConjuncts $\rhd$ L03 $\rhd$ L04 $\gg$ | $\mathcal{X} <= \mathcal{Y} \mathbin{\dot\wedge} \mathcal{Y} <= \mathcal{Y}$ | ; |
| L06: | ExistIntro @ Ex3 @ $\mathcal{Y}$ $\rhd$ L05 $\gg$ | $\mathcal{X} <= \mathrm{Ex3} \mathbin{\dot\wedge} \mathcal{Y} <= \mathrm{Ex3}$ | ; |
| L07: | Block $\gg$ | End | ; |
| L08: | Block $\gg$ | Begin | ; |
| L09: | Arbitrary $\gg$ | $\mathcal{X},\mathcal{Y}$ | ; |
| L10: | Premise $\gg$ | $\mathcal{Y} <= \mathcal{X}$ | ; |
| L11: | leqReflexivity $\gg$ | $\mathcal{X} <= \mathcal{X}$ | ; |

L12: JoinConjuncts $\rhd$ L11 $\rhd$ L10 $\gg$ $\mathcal{X} <= \mathcal{X} \mathrel{\dot\wedge} \mathcal{Y} <= \mathcal{X}$ ;

L13: ExistIntro @ Ex3 @ $\mathcal{X}$ $\rhd$ L12 $\gg$ $\mathcal{X} <= \mathrm{Ex3} \mathrel{\dot\wedge} \mathcal{Y} <= \mathrm{Ex3}$ ;

L14: Block $\gg$ End ;

L15: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}$ ;

L16: Ded $\rhd$ L07 $\gg$ $\mathcal{X} <= \mathcal{Y} \Rightarrow \mathcal{X} <= \mathrm{Ex3} \mathrel{\dot\wedge} \mathcal{Y} <= \mathrm{Ex3}$ ;

L17: Ded $\rhd$ L14 $\gg$ $\mathcal{Y} <= \mathcal{X} \Rightarrow \mathcal{X} <= \mathrm{Ex3} \mathrel{\dot\wedge} \mathcal{Y} <= \mathrm{Ex3}$ ;

L18: leqTotality $\gg$ $\mathcal{X} <= \mathcal{Y} \mathrel{\dot\vee} \mathcal{Y} <= \mathcal{X}$ ;

L19: FromDisjuncts $\rhd$ L18 $\rhd$ L16 $\rhd$ L17 $\gg$ $\mathcal{X} <= \mathrm{Ex3} \mathrel{\dot\wedge} \mathcal{Y} <= \mathrm{Ex3}$ $\Box$

[SystemQ **lemma** LeqNegated: $\Pi\mathcal{X}, \mathcal{Y}$: $\mathcal{X} <= \mathcal{Y} \vdash (-\mathrm{u}\mathcal{Y}) <= (-\mathrm{u}\mathcal{X})$]

SystemQ **proof of** LeqNegated:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}$ ;

L02: Premise $\gg$ $\mathcal{X} <= \mathcal{Y}$ ;

L03: leqAddition $\rhd$ L02 $\gg$ $(\mathcal{X} - \mathcal{X}) <= (\mathcal{Y} - \mathcal{X})$ ;

L04: Negative $\gg$ $(\mathcal{X} - \mathcal{X}) = 0$ ;

L05: subLeqLeft $\rhd$ L04 $\rhd$ L03 $\gg$ $0 <= (\mathcal{Y} - \mathcal{X})$ ;

L06: plusCommutativity $\gg$ $(\mathcal{Y} - \mathcal{X}) = ((-\mathrm{u}\mathcal{X}) + \mathcal{Y})$ ;

L07: subLeqRight $\rhd$ L06 $\rhd$ L05 $\gg$ $0 <= ((-\mathrm{u}\mathcal{X}) + \mathcal{Y})$ ;

L08: leqAddition $\rhd$ L07 $\gg$ $(0 - \mathcal{Y}) <= (((-\mathrm{u}\mathcal{X}) + \mathcal{Y}) - \mathcal{Y})$ ;

L09: lemma plus0Left $\gg$ $(0 - \mathcal{Y}) = (-\mathrm{u}\mathcal{Y})$ ;

L10: $\mathrm{x} = \mathrm{x} + \mathrm{y} - \mathrm{y}$ $\gg$ $(-\mathrm{u}\mathcal{X}) = (((-\mathrm{u}\mathcal{X}) + \mathcal{Y}) - \mathcal{Y})$ ;

L11: eqSymmetry $\rhd$ L10 $\gg$ $(((-\mathrm{u}\mathcal{X}) + \mathcal{Y}) - \mathcal{Y}) = (-\mathrm{u}\mathcal{X})$ ;

L12: subLeqLeft $\rhd$ L09 $\rhd$ L08 $\gg$ $(-\mathrm{u}\mathcal{Y}) <= (((-\mathrm{u}\mathcal{X}) + \mathcal{Y}) - \mathcal{Y})$ ;

L13: subLeqRight $\rhd$ L11 $\rhd$ L12 $\gg$ $(-\mathrm{u}\mathcal{Y}) <= (-\mathrm{u}\mathcal{X})$ $\Box$

[SystemQ **lemma** AddEquations(Leq): $\Pi\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}$: $\mathcal{X} <= \mathcal{Y} \vdash \mathcal{Z} <= \mathcal{U} \vdash (\mathcal{X} + \mathcal{Z}) <= (\mathcal{Y} + \mathcal{U})$]

SystemQ **proof of** AddEquations(Leq):

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}$ ;

L02: Premise $\gg$ $\mathcal{X} <= \mathcal{Y}$ ;

L03: Premise $\gg$ $\mathcal{Z} <= \mathcal{U}$ ;

L04: leqAddition $\rhd$ L02 $\gg$ $(\mathcal{X} + \mathcal{Z}) <= (\mathcal{Y} + \mathcal{Z})$ ;

L05: LeqAdditionLeft $\rhd$ L03 $\gg$ $(\mathcal{Y} + \mathcal{Z}) <= (\mathcal{Y} + \mathcal{U})$ ;

L06: leqTransitivity $\rhd$ L04 $\rhd$ L05 $\gg$ $(\mathcal{X} + \mathcal{Z}) <= (\mathcal{Y} + \mathcal{U})$ $\Box$

(*** LESS ***)

[SystemQ **lemma** LeqNeqLess: $\Pi\mathcal{X}, \mathcal{Y}$: $\mathcal{X} <= \mathcal{Y} \vdash \mathcal{X} \neq \mathcal{Y} \vdash \mathcal{X} < \mathcal{Y}$]

SystemQ **proof of** LeqNeqLess:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}$ ;

L02: Premise $\gg$ $\mathcal{X} <= \mathcal{Y}$ ;

L03: Premise $\gg$ $\mathcal{X} \neq \mathcal{Y}$ ;

L04: JoinConjuncts $\rhd$ L02 $\rhd$ L03 $\gg$ $\mathcal{X} <= \mathcal{Y} \mathrel{\dot\wedge} \mathcal{X} \neq \mathcal{Y}$ ;

L05: Repetition $\rhd$ L04 $\gg$ $\mathcal{X} < \mathcal{Y}$ $\Box$

[SystemQ **lemma** LessMultiplication: $\Pi\mathcal{X}, \mathcal{Y}, \mathcal{Z}$: $0 < \mathcal{Z} \vdash \mathcal{X} < \mathcal{Y} \vdash (\mathcal{X} * \mathcal{Z}) < (\mathcal{Y} * \mathcal{Z})$]

SystemQ **proof of** LessMultiplication:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ ;
L02: Premise $\gg$ $0 < \mathcal{Z}$ ;
L03: Premise $\gg$ $\mathcal{X} < \mathcal{Y}$ ;
L04: LessLeq $\triangleright$ L03 $\gg$ $\mathcal{X} <= \mathcal{Y}$ ;
L05: LessLeq $\triangleright$ L02 $\gg$ $0 <= \mathcal{Z}$ ;
L06: leqMultiplication $\triangleright$ L05 $\triangleright$ L04 $\gg$ $(\mathcal{X} * \mathcal{Z}) <= (\mathcal{Y} * \mathcal{Z})$ ;
L07: LessNeq $\triangleright$ L03 $\gg$ $\mathcal{X} \neq \mathcal{Y}$ ;
L08: LessNeq $\triangleright$ L02 $\gg$ $0 \neq \mathcal{Z}$ ;
L09: NeqSymmetry $\triangleright$ L08 $\gg$ $\mathcal{Z} \neq 0$ ;
L10: NeqMultiplication $\triangleright$ L09 $\triangleright$ L07 $\gg$ $(\mathcal{X} * \mathcal{Z}) \neq (\mathcal{Y} * \mathcal{Z})$ ;
L11: LeqNeqLess $\triangleright$ L06 $\triangleright$ L10 $\gg$ $(\mathcal{X} * \mathcal{Z}) < (\mathcal{Y} * \mathcal{Z})$ $\square$

[SystemQ **lemma** LessMultiplicationLeft: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}\colon 0 < \mathcal{Z} \vdash \mathcal{X} < \mathcal{Y} \vdash (\mathcal{Z} * \mathcal{X}) < (\mathcal{Z} * \mathcal{Y})$]

SystemQ **proof of** LessMultiplicationLeft:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ ;
L02: Premise $\gg$ $0 < \mathcal{Z}$ ;
L03: Premise $\gg$ $\mathcal{X} < \mathcal{Y}$ ;
L04: LessMultiplication $\triangleright$ L02 $\triangleright$ L03 $\gg$ $(\mathcal{X} * \mathcal{Z}) < (\mathcal{Y} * \mathcal{Z})$ ;
L05: timesCommutativity $\gg$ $(\mathcal{X} * \mathcal{Z}) = (\mathcal{Z} * \mathcal{X})$ ;
L06: timesCommutativity $\gg$ $(\mathcal{Y} * \mathcal{Z}) = (\mathcal{Z} * \mathcal{Y})$ ;
L07: SubLessLeft $\triangleright$ L05 $\triangleright$ L04 $\gg$ $(\mathcal{Z} * \mathcal{X}) < (\mathcal{Y} * \mathcal{Z})$ ;
L08: SubLessRight $\triangleright$ L06 $\triangleright$ L07 $\gg$ $(\mathcal{Z} * \mathcal{X}) < (\mathcal{Z} * \mathcal{Y})$ $\square$

[SystemQ **lemma** LessDivision: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}\colon 0 <= \mathcal{Z} \vdash (\mathcal{X} * \mathcal{Z}) < (\mathcal{Y} * \mathcal{Z}) \vdash \mathcal{X} < \mathcal{Y}$]

SystemQ **proof of** LessDivision:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ ;
L02: Premise $\gg$ $0 <= \mathcal{Z}$ ;
L03: Premise $\gg$ $(\mathcal{X} * \mathcal{Z}) < (\mathcal{Y} * \mathcal{Z})$ ;
L04: FromLess $\triangleright$ L03 $\gg$ $\dot\neg\, ((\mathcal{Y} * \mathcal{Z}) <= (\mathcal{X} * \mathcal{Z}))\mathrm{n}$ ;
L05: leqMultiplicationAxiom $\gg$ $0 <= \mathcal{Z} \Rightarrow \mathcal{Y} <= \mathcal{X} \Rightarrow (\mathcal{Y} * \mathcal{Z}) <= (\mathcal{X} * \mathcal{Z})$ ;
L06: MP $\triangleright$ L05 $\triangleright$ L02 $\gg$ $\mathcal{Y} <= \mathcal{X} \Rightarrow (\mathcal{Y} * \mathcal{Z}) <= (\mathcal{X} * \mathcal{Z})$ ;
L07: Contrapositive $\triangleright$ L06 $\gg$ $\dot\neg\, ((\mathcal{Y} * \mathcal{Z}) <= (\mathcal{X} * \mathcal{Z}))\mathrm{n} \Rightarrow \dot\neg\, (\mathcal{Y} <= \mathcal{X})\mathrm{n}$ ;
L08: MP $\triangleright$ L07 $\triangleright$ L04 $\gg$ $\dot\neg\, (\mathcal{Y} <= \mathcal{X})\mathrm{n}$ ;
L09: ToLess $\triangleright$ L08 $\gg$ $\mathcal{X} < \mathcal{Y}$ $\square$

[SystemQ **lemma** LessLeqTransitivity: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}\colon \mathcal{X} < \mathcal{Y} \vdash \mathcal{Y} <= \mathcal{Z} \vdash \mathcal{X} < \mathcal{Z}$]

SystemQ **proof of** LessLeqTransitivity:

L01: Block $\gg$ Begin ;
L02: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ ;
L03: Premise $\gg$ $\mathcal{X} < \mathcal{Y}$ ;
L04: Premise $\gg$ $\mathcal{Y} <= \mathcal{Z}$ ;

L05: Premise $\gg$ $\mathcal{Z} = \mathcal{X}$ ;
L06: FirstConjunct $\rhd$ L03 $\gg$ $\mathcal{X} <= \mathcal{Y}$ ;
L07: SecondConjunct $\rhd$ L03 $\gg$ $\mathcal{X} \neq \mathcal{Y}$ ;
L08: subLeqRight $\rhd$ L05 $\rhd$ L04 $\gg$ $\mathcal{Y} <= \mathcal{X}$ ;
L09: leqAntisymmetry $\rhd$ L06 $\rhd$ L08 $\gg$ $\mathcal{X} = \mathcal{Y}$ ;
L10: FromContradiction $\rhd$ L09 $\rhd$ L07 $\gg$ $\mathcal{Z} \neq \mathcal{X}$ ;
L11: Block $\gg$ End ;
L12: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ ;
L13: Ded $\rhd$ L11 $\gg$ $\mathcal{X} < \mathcal{Y} \Rightarrow \mathcal{Y} <= \mathcal{Z} \Rightarrow \mathcal{Z} = \mathcal{X} \Rightarrow \mathcal{Z} \neq \mathcal{X}$ ;
L14: Premise $\gg$ $\mathcal{X} < \mathcal{Y}$ ;
L15: Premise $\gg$ $\mathcal{Y} <= \mathcal{Z}$ ;
L16: MP2 $\rhd$ L13 $\rhd$ L14 $\rhd$ L15 $\gg$ $\mathcal{Z} = \mathcal{X} \Rightarrow \mathcal{Z} \neq \mathcal{X}$ ;
L17: prop lemma imply negation $\rhd$ L16 $\gg$ $\mathcal{Z} \neq \mathcal{X}$ ;
L18: NeqSymmetry $\rhd$ L17 $\gg$ $\mathcal{X} \neq \mathcal{Z}$ ;
L19: FirstConjunct $\rhd$ L14 $\gg$ $\mathcal{X} <= \mathcal{Y}$ ;
L20: leqTransitivity $\rhd$ L19 $\rhd$ L15 $\gg$ $\mathcal{X} <= \mathcal{Z}$ ;
L21: JoinConjuncts $\rhd$ L20 $\rhd$ L18 $\gg$ $\mathcal{X} < \mathcal{Z}$ □

[SystemQ **lemma** LessTransitivity: $\Pi\mathcal{X}, \mathcal{Y}, \mathcal{Z}\colon \mathcal{X} < \mathcal{Y} \vdash \mathcal{Y} < \mathcal{Z} \vdash \mathcal{X} < \mathcal{Z}$]

SystemQ **proof of** LessTransitivity:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ ;
L02: Premise $\gg$ $\mathcal{X} < \mathcal{Y}$ ;
L03: Premise $\gg$ $\mathcal{Y} < \mathcal{Z}$ ;
L04: FirstConjunct $\rhd$ L03 $\gg$ $\mathcal{Y} <= \mathcal{Z}$ ;
L05: LessLeqTransitivity $\rhd$ L02 $\rhd$ L04 $\gg$ $\mathcal{X} < \mathcal{Z}$ □

[SystemQ **lemma** AddEquations(Less): $\Pi\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}\colon \mathcal{X} < \mathcal{Y} \vdash \mathcal{Z} < \mathcal{U} \vdash (\mathcal{X} + \mathcal{Z}) < (\mathcal{Y} + \mathcal{U})$]

SystemQ **proof of** AddEquations(Less):

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}$ ;
L02: Premise $\gg$ $\mathcal{X} < \mathcal{Y}$ ;
L03: Premise $\gg$ $\mathcal{Z} < \mathcal{U}$ ;
L04: LessAddition $\rhd$ L02 $\gg$ $(\mathcal{X} + \mathcal{Z}) < (\mathcal{Y} + \mathcal{Z})$ ;
L05: LessAdditionLeft $\rhd$ L03 $\gg$ $(\mathcal{Y} + \mathcal{Z}) < (\mathcal{Y} + \mathcal{U})$ ;
L06: LessTransitivity $\rhd$ L04 $\rhd$ L05 $\gg$ $(\mathcal{X} + \mathcal{Z}) < (\mathcal{Y} + \mathcal{U})$ □

[SystemQ **lemma** LessTotality: $\Pi\mathcal{X}, \mathcal{Y}\colon \mathcal{X} < \mathcal{Y} \dot{\vee} \mathcal{X} = \mathcal{Y} \dot{\vee} \mathcal{Y} < \mathcal{X}$]

SystemQ **proof of** LessTotality:

L01: Block $\gg$ Begin ;
L02: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}$ ;
L03: Premise $\gg$ $\dot{\neg}(\mathcal{X} < \mathcal{Y})$n ;
L04: Premise $\gg$ $\mathcal{X} \neq \mathcal{Y}$ ;
L05: fromNotLess $\rhd$ L03 $\gg$ $\mathcal{Y} <= \mathcal{X}$ ;
L06: NeqSymmetry $\rhd$ L04 $\gg$ $\mathcal{Y} \neq \mathcal{X}$ ;

L07: LeqNeqLess $\rhd$ L05 $\rhd$ L06 $\gg$ $\mathcal{Y} < \mathcal{X}$ ;
L08: Block $\gg$ End ;
L09: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}$ ;
L10: Ded $\rhd$ L08 $\gg$ $\dot{\neg}(\mathcal{X} < \mathcal{Y})\mathrm{n} \Rightarrow \mathcal{X} \neq \mathcal{Y} \Rightarrow \mathcal{Y} < \mathcal{X}$ ;
L11: Repetition $\rhd$ L10 $\gg$ $\mathcal{X} < \mathcal{Y} \dot{\vee} \mathcal{X} = \mathcal{Y} \dot{\vee} \mathcal{Y} < \mathcal{X}$ □

[SystemQ **lemma** NegativeLessPositive: $\Pi\mathcal{X}\colon 0 < \mathcal{X} \vdash (-\mathrm{u}\mathcal{X}) < \mathcal{X}$]

SystemQ **proof of** NegativeLessPositive:

L01: Arbitrary $\gg$ $\mathcal{X}$ ;
L02: Premise $\gg$ $0 < \mathcal{X}$ ;
L03: FirstConjunct $\rhd$ L02 $\gg$ $0 <= \mathcal{X}$ ;
L04: leqAddition $\rhd$ L03 $\gg$ $(0 - \mathcal{X}) <= (\mathcal{X} - \mathcal{X})$ ;
L05: lemma plus0Left $\gg$ $(0 - \mathcal{X}) = (-\mathrm{u}\mathcal{X})$ ;
L06: Negative $\gg$ $(\mathcal{X} - \mathcal{X}) = 0$ ;
L07: subLeqLeft $\rhd$ L05 $\rhd$ L04 $\gg$ $(-\mathrm{u}\mathcal{X}) <= (\mathcal{X} - \mathcal{X})$ ;
L08: subLeqRight $\rhd$ L06 $\rhd$ L07 $\gg$ $(-\mathrm{u}\mathcal{X}) <= 0$ ;
L09: leqLessTransitivity $\rhd$ L08 $\rhd$ L02 $\gg$ $(-\mathrm{u}\mathcal{X}) < \mathcal{X}$ □

[SystemQ **lemma** LessNegated: $\Pi\mathcal{X}, \mathcal{Y}\colon \mathcal{X} < \mathcal{Y} \vdash (-\mathrm{u}\mathcal{Y}) < (-\mathrm{u}\mathcal{X})$]

SystemQ **proof of** LessNegated:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}$ ;
L02: Premise $\gg$ $\mathcal{X} < \mathcal{Y}$ ;
L03: LessLeq $\rhd$ L02 $\gg$ $\mathcal{X} <= \mathcal{Y}$ ;
L04: LeqNegated $\rhd$ L03 $\gg$ $(-\mathrm{u}\mathcal{Y}) <= (-\mathrm{u}\mathcal{X})$ ;
L05: LessNeq $\rhd$ L02 $\gg$ $\mathcal{X} \neq \mathcal{Y}$ ;
L06: NeqNegated $\rhd$ L05 $\gg$ $\dot{\neg}((-\mathrm{u}\mathcal{X}) = (-\mathrm{u}\mathcal{Y}))\mathrm{n}$ ;
L07: NeqSymmetry $\rhd$ L06 $\gg$ $\dot{\neg}((-\mathrm{u}\mathcal{Y}) = (-\mathrm{u}\mathcal{X}))\mathrm{n}$ ;
L08: LeqNeqLess $\rhd$ L04 $\rhd$ L07 $\gg$ $(-\mathrm{u}\mathcal{Y}) < (-\mathrm{u}\mathcal{X})$ □

[SystemQ **lemma** $-0 = 0$: $(-\mathrm{u}0) = 0$]

SystemQ **proof of** $-0 = 0$:

L01: Negative $\gg$ $(0 - 0) = 0$ ;
L02: plus0 $\gg$ $(0 + 0) = 0$ ;
L03: UniqueNegative $\rhd$ L01 $\rhd$ L02 $\gg$ $(-\mathrm{u}0) = 0$ □

[SystemQ **lemma** PositiveNegated: $\Pi\mathcal{X}\colon 0 < \mathcal{X} \vdash (-\mathrm{u}\mathcal{X}) < 0$]

SystemQ **proof of** PositiveNegated:

L01: Arbitrary $\gg$ $\mathcal{X}$ ;
L02: Premise $\gg$ $0 < \mathcal{X}$ ;
L03: LessNegated $\rhd$ L02 $\gg$ $(-\mathrm{u}\mathcal{X}) < (-\mathrm{u}0)$ ;
L04: $-0 = 0$ $\gg$ $(-\mathrm{u}0) = 0$ ;
L05: SubLessRight $\rhd$ L04 $\rhd$ L03 $\gg$ $(-\mathrm{u}\mathcal{X}) < 0$ □

[SystemQ **lemma** NonpositiveNegated: $\Pi\mathcal{X}\colon \mathcal{X} <= 0 \vdash 0 <= (-\mathrm{u}\mathcal{X})$]

SystemQ **proof of** NonpositiveNegated:

L01: Arbitrary $\gg$ $\mathcal{X}$ ;
L02: Premise $\gg$ $\mathcal{X} <= 0$ ;
L03: LeqNegated $\rhd$ L02 $\gg$ $(-\mathrm{u}0) <= (-\mathrm{u}\mathcal{X})$ ;
L04: $-0 = 0$ $\gg$ $(-\mathrm{u}0) = 0$ ;

L05: subLeqLeft $\triangleright$ L04 $\triangleright$ L03 $\gg$ $0 <= (-\mathrm{u}\mathcal{X})$ $\Box$

[SystemQ **lemma** NegativeNegated: $\Pi\mathcal{X}\colon \mathcal{X} < 0 \vdash 0 < (-\mathrm{u}\mathcal{X})$]

SystemQ **proof of** NegativeNegated:

L01: Arbitrary $\gg$ $\mathcal{X}$ ;

L02: Premise $\gg$ $\mathcal{X} < 0$ ;

L03: LessNegated $\triangleright$ L02 $\gg$ $(-\mathrm{u}0) < (-\mathrm{u}\mathcal{X})$ ;

L04: $-0 = 0 \gg$ $(-\mathrm{u}0) = 0$ ;

L05: SubLessLeft $\triangleright$ L04 $\triangleright$ L03 $\gg$ $0 < (-\mathrm{u}\mathcal{X})$ $\Box$

[SystemQ **lemma** NonnegativeNegated: $\Pi\mathcal{X}\colon 0 <= \mathcal{X} \vdash (-\mathrm{u}\mathcal{X}) <= 0$]

SystemQ **proof of** NonnegativeNegated:

L01: Arbitrary $\gg$ $\mathcal{X}$ ;

L02: Premise $\gg$ $0 <= \mathcal{X}$ ;

L03: LeqNegated $\triangleright$ L02 $\gg$ $(-\mathrm{u}\mathcal{X}) <= (-\mathrm{u}0)$ ;

L04: $-0 = 0 \gg$ $(-\mathrm{u}0) = 0$ ;

L05: subLeqRight $\triangleright$ L04 $\triangleright$ L03 $\gg$ $(-\mathrm{u}\mathcal{X}) <= 0$ $\Box$

[SystemQ **lemma** $0 < 2$: $0 < 2$]

SystemQ **proof of** $0 < 2$:

L01: $0 < 1 \gg$ $0 < 1$ ;

L02: LessAddition $\triangleright$ L01 $\gg$ $(0 + 1) < (1 + 1)$ ;

L03: lemma plus0Left $\gg$ $(0 + 1) = 1$ ;

L04: SubLessLeft $\triangleright$ L03 $\triangleright$ L02 $\gg$ $1 < (1 + 1)$ ;

L05: LessTransitivity $\triangleright$ L01 $\triangleright$ L04 $\gg$ $0 < 2$ $\Box$

[SystemQ **lemma** $0 < 1/2$: $0 < 1/2$]

SystemQ **proof of** $0 < 1/2$:

L01: $0 < 2 \gg$ $0 < 2$ ;

L02: FirstConjunct $\triangleright$ L01 $\gg$ $0 <= 2$ ;

L03: SecondConjunct $\triangleright$ L01 $\gg$ $0 \neq 2$ ;

L04: NeqSymmetry $\triangleright$ L03 $\gg$ $2 \neq 0$ ;

L05: $0 < 1 \gg$ $0 < 1$ ;

L06: $\mathrm{x} * 0 = 0 \gg$ $(2 * 0) = 0$ ;

L07: $\mathrm{x} * \mathrm{y} =$ zBackwards $\triangleright$ L06 $\gg$ $0 = (0 * 2)$ ;

L08: SubLessLeft $\triangleright$ L07 $\triangleright$ L05 $\gg$ $(0 * 2) < 1$ ;

L09: Reciprocal $\triangleright$ L04 $\gg$ $(2 * 1/2) = 1$ ;

L10: $\mathrm{x} * \mathrm{y} =$ zBackwards $\triangleright$ L09 $\gg$ $1 = (1/2 * 2)$ ;

L11: SubLessRight $\triangleright$ L10 $\triangleright$ L08 $\gg$ $(0 * 2) < (1/2 * 2)$ ;

L12: LessDivision $\triangleright$ L02 $\triangleright$ L11 $\gg$ $0 < 1/2$ $\Box$

[SystemQ **lemma** PositiveHalved: $\Pi\mathcal{X}\colon 0 < \mathcal{X} \vdash 0 < (1/2 * \mathcal{X})$]

SystemQ **proof of** PositiveHalved:

L01: Arbitrary $\gg$ $\mathcal{X}$ ;

L02: Premise $\gg$ $0 < \mathcal{X}$ ;

L03: $0 < 1/2 \gg$ $0 < 1/2$ ;

L04: LessMultiplicationLeft $\triangleright$ L03 $\triangleright$ L02 $\gg$ $(1/2 * 0) < (1/2 * \mathcal{X})$ ;

L05: $\mathrm{x} * 0 = 0 \gg$ $(1/2 * 0) = 0$ ;

L06: SubLessLeft $\triangleright$ L05 $\triangleright$ L04 $\gg$ $0 < (1/2 * \mathcal{X})$ $\Box$