



[ExistMP2  $\xrightarrow{\text{proof}}$   $\lambda c. \lambda x. \mathcal{P}(\ulcorner \text{SystemQ} \vdash \forall (v1): \forall (v2): \forall a: \forall b: \forall c: a \Rightarrow b \Rightarrow c \vdash \neg (\forall_{\text{obj}}(v1): \neg (a)n \urcorner \vdash \neg (\forall_{\text{obj}}(v2): \neg (b)n \urcorner \vdash \text{ExistMP} \triangleright a \Rightarrow b \Rightarrow c \triangleright \neg (\forall_{\text{obj}}(\underline{v1}): \neg (a)n \gg b \Rightarrow c; \text{ExistMP} \triangleright b \Rightarrow c \triangleright \neg (\forall_{\text{obj}}(\underline{v2}): \neg (b)n \gg c \urcorner, p_0, c)$ ]

[TwiceExistMP  $\xrightarrow{\text{stmt}}$   $\text{SystemQ} \vdash \forall (v1): \forall (v2): \forall a: \forall b: a \Rightarrow b \vdash \neg (\forall_{\text{obj}}(\underline{v1}): \neg (\neg (\forall_{\text{obj}}(\underline{v2}): \neg (a)n \urcorner)n \vdash \underline{b})$ ]

[TwiceExistMP  $\xrightarrow{\text{proof}}$   $\lambda c. \lambda x. \mathcal{P}(\ulcorner \text{SystemQ} \vdash \forall (v2): \forall a: \forall b: a \Rightarrow b \vdash \neg (\forall_{\text{obj}}(\underline{v2}): \neg (a)n \urcorner \vdash \text{ExistMP} \triangleright a \Rightarrow b \triangleright \neg (\forall_{\text{obj}}(\underline{v2}): \neg (a)n \gg b; \forall (v1): \forall (v2): \forall a: \forall b: \text{Ded} \triangleright \forall (v2): \forall a: \forall b: a \Rightarrow b \vdash \neg (\forall_{\text{obj}}(\underline{v2}): \neg (a)n \vdash b \gg a \Rightarrow b \Rightarrow \neg (\forall_{\text{obj}}(\underline{v2}): \neg (a)n \Rightarrow b; a \Rightarrow b \vdash \neg (\forall_{\text{obj}}(\underline{v1}): \neg (\neg (\forall_{\text{obj}}(\underline{v2}): \neg (a)n \urcorner)n \vdash \text{MP} \triangleright a \Rightarrow b \Rightarrow \neg (\forall_{\text{obj}}(\underline{v2}): \neg (a)n \Rightarrow b \triangleright a \Rightarrow b \gg \neg (\forall_{\text{obj}}(\underline{v2}): \neg (a)n \Rightarrow b; \text{ExistMP} \triangleright \neg (\forall_{\text{obj}}(\underline{v2}): \neg (a)n \Rightarrow b \triangleright \neg (\forall_{\text{obj}}(\underline{v1}): \neg (\neg (\forall_{\text{obj}}(\underline{v2}): \neg (a)n \urcorner)n \gg \underline{b} \urcorner, p_0, c)$ ]

[TwiceExistMP2  $\xrightarrow{\text{stmt}}$   $\text{SystemQ} \vdash \forall (v1): \forall (v2): \forall (v3): \forall (v4): \forall a: \forall b: \forall c: a \Rightarrow b \Rightarrow c \vdash \neg (\forall_{\text{obj}}(\underline{v1}): \neg (\neg (\forall_{\text{obj}}(\underline{v2}): \neg (a)n \urcorner)n \vdash \neg (\forall_{\text{obj}}(\underline{v3}): \neg (\neg (\forall_{\text{obj}}(\underline{v4}): \neg (b)n \urcorner)n \vdash \underline{c})$ ]

[TwiceExistMP2  $\xrightarrow{\text{proof}}$   $\lambda c. \lambda x. \mathcal{P}(\ulcorner \text{SystemQ} \vdash \forall (v1): \forall (v2): \forall (v3): \forall (v4): \forall a: \forall b: \forall c: a \Rightarrow b \Rightarrow c \vdash \neg (\forall_{\text{obj}}(\underline{v1}): \neg (\neg (\forall_{\text{obj}}(\underline{v2}): \neg (a)n \urcorner)n \vdash \neg (\forall_{\text{obj}}(\underline{v3}): \neg (\neg (\forall_{\text{obj}}(\underline{v4}): \neg (b)n \urcorner)n \vdash \text{TwiceExistMP} \triangleright a \Rightarrow b \Rightarrow c \triangleright \neg (\forall_{\text{obj}}(\underline{v1}): \neg (\neg (\forall_{\text{obj}}(\underline{v2}): \neg (a)n \urcorner)n \gg b \Rightarrow c; \text{TwiceExistMP} \triangleright b \Rightarrow c \triangleright \neg (\forall_{\text{obj}}(\underline{v3}): \neg (\neg (\forall_{\text{obj}}(\underline{v4}): \neg (b)n \urcorner)n \gg \underline{c} \urcorner, p_0, c)$ ]

[AllNegated(Impl)  $\xrightarrow{\text{stmt}}$   $\text{SystemQ} \vdash \forall (v1): \forall a: \neg (\forall_{\text{obj}}(\underline{v1}): a)n \Rightarrow \neg (\forall_{\text{obj}}(\underline{v1}): \neg (\neg (a)n \urcorner)n]$

[AllNegated(Impl)  $\xrightarrow{\text{proof}}$   $\lambda c. \lambda x. \mathcal{P}(\ulcorner \text{SystemQ} \vdash \forall (v1): \forall a: \forall_{\text{obj}}(\underline{v1}): \neg (\neg (a)n \urcorner \vdash A4 @ x \triangleright \forall_{\text{obj}}(\underline{v1}): \neg (\neg (a)n \urcorner \gg \neg (\neg (a)n \urcorner); \text{RemoveDoubleNeg} \triangleright \neg (\neg (a)n \urcorner \gg a; \text{Gen} \triangleright a \gg \forall_{\text{obj}}(\underline{v1}): a; \forall (v1): \forall a: \text{Ded} \triangleright \forall (v1): \forall a: \forall_{\text{obj}}(\underline{v1}): \neg (\neg (a)n \urcorner \vdash \forall_{\text{obj}}(\underline{v1}): a \gg \forall_{\text{obj}}(\underline{v1}): \neg (\neg (a)n \urcorner \Rightarrow \forall_{\text{obj}}(\underline{v1}): a; \text{Contrapositive} \triangleright \forall_{\text{obj}}(\underline{v1}): \neg (\neg (a)n \urcorner \Rightarrow \forall_{\text{obj}}(\underline{v1}): a \gg \neg (\forall_{\text{obj}}(\underline{v1}): a)n \Rightarrow \neg (\forall_{\text{obj}}(\underline{v1}): \neg (\neg (a)n \urcorner)n; \text{Repetition} \triangleright \neg (\forall_{\text{obj}}(\underline{v1}): a)n \Rightarrow \neg (\forall_{\text{obj}}(\underline{v1}): \neg (\neg (a)n \urcorner)n \gg \neg (\forall_{\text{obj}}(\underline{v1}): a)n \Rightarrow \neg (\forall_{\text{obj}}(\underline{v1}): \neg (\neg (a)n \urcorner)n \urcorner, p_0, c)$ ]

[ExistNegated(Impl)  $\xrightarrow{\text{stmt}}$   $\text{SystemQ} \vdash \forall (v1): \forall a: \neg (\neg (\forall_{\text{obj}}(\underline{v1}): \neg (a)n \urcorner)n \Rightarrow \forall_{\text{obj}}(\underline{v1}): \neg (a)n]$

[ExistNegated(Impl)  $\xrightarrow{\text{proof}}$   $\lambda c. \lambda x. \mathcal{P}(\ulcorner \text{SystemQ} \vdash \forall (v1): \forall a: \neg (\neg (\forall_{\text{obj}}(\underline{v1}): \neg (a)n \urcorner)n \vdash \text{Repetition} \triangleright \neg (\neg (\forall_{\text{obj}}(\underline{v1}): \neg (a)n \urcorner)n \gg \neg (\neg (\forall_{\text{obj}}(\underline{v1}): \neg (a)n \urcorner)n; \text{RemoveDoubleNeg} \triangleright \neg (\neg (\forall_{\text{obj}}(\underline{v1}): \neg (a)n \urcorner)n \gg \forall_{\text{obj}}(\underline{v1}): \neg (a)n; \forall (v1): \forall a: \text{Ded} \triangleright \forall (v1): \forall a: \neg (\neg (\forall_{\text{obj}}(\underline{v1}): \neg (a)n \urcorner)n \vdash \forall_{\text{obj}}(\underline{v1}): \neg (a)n \gg \neg (\neg (\forall_{\text{obj}}(\underline{v1}): \neg (a)n \urcorner)n \Rightarrow \forall_{\text{obj}}(\underline{v1}): \neg (a)n \urcorner, p_0, c)$ ]

[AddAll  $\xrightarrow{\text{stmt}}$   $\text{SystemQ} \vdash \forall (v1): \forall a: \forall b: a \Rightarrow b \vdash \forall_{\text{obj}}(\underline{v1}): a \Rightarrow \forall_{\text{obj}}(\underline{v1}): b]$

[AddAll  $\xrightarrow{\text{proof}}$   $\lambda c. \lambda x. \mathcal{P}(\ulcorner \text{SystemQ} \vdash \forall (v1): \forall a: \forall b: a \Rightarrow b \vdash \forall_{\text{obj}}(\underline{v1}): a \vdash$











$$\begin{aligned}
& \dot{\vdash} (\overline{(\text{op}2)} \in Q) n \Rightarrow \dot{\vdash} (a_{\text{Ph}} = \{\{\overline{(\text{op}1)}, \overline{(\text{op}1)}\}, \{\overline{(\text{op}1)}, \overline{(\text{op}2)}\}\} n) n) n \mid \\
& \dot{\vdash} (\forall_{\text{obj}} \underline{m}: \dot{\vdash} (d_{\text{Ph}} = \{\{\underline{m}, \underline{m}\}, \{\underline{m}, (\underline{fx})[\underline{m}] + (\underline{fy})[\underline{m}]\}\} n) n) [\underline{m}]) \\
& [\text{PlusF}(\text{Sym}) \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{m}: \forall (\underline{fx}): \forall (\underline{fy}): \text{PlusF} \gg \{\text{ph} \in \{\text{ph} \in \\
& P(P(\text{Union}(\{N, Q\})) \mid \dot{\vdash} (\forall_{\text{obj}} \overline{(\text{op}1)}: \dot{\vdash} (\dot{\vdash} (\forall_{\text{obj}} \overline{(\text{op}2)}: \dot{\vdash} (\dot{\vdash} (\overline{(\text{op}1)} \in N \Rightarrow \\
& \dot{\vdash} (\overline{(\text{op}2)} \in Q) n) n) \Rightarrow \dot{\vdash} (a_{\text{Ph}} = \{\{\overline{(\text{op}1)}, \overline{(\text{op}1)}\}, \{\overline{(\text{op}1)}, \overline{(\text{op}2)}\}\} n) n) n) n) n \mid \\
& \dot{\vdash} (\forall_{\text{obj}} \underline{m}: \dot{\vdash} (d_{\text{Ph}} = \{\{\underline{m}, \underline{m}\}, \{\underline{m}, (\underline{fx})[\underline{m}] + (\underline{fy})[\underline{m}]\}\} n) n) [\underline{m}] = \\
& (\underline{fx})[\underline{m}] + (\underline{fy})[\underline{m}]; \text{eqSymmetry} \triangleright \{\text{ph} \in \{\text{ph} \in P(P(\text{Union}(\{N, Q\})) \mid \\
& \dot{\vdash} (\forall_{\text{obj}} \overline{(\text{op}1)}: \dot{\vdash} (\dot{\vdash} (\forall_{\text{obj}} \overline{(\text{op}2)}: \dot{\vdash} (\dot{\vdash} (\overline{(\text{op}1)} \in N \Rightarrow \dot{\vdash} (\overline{(\text{op}2)} \in Q) n) n) \Rightarrow \\
& \dot{\vdash} (a_{\text{Ph}} = \{\{\overline{(\text{op}1)}, \overline{(\text{op}1)}\}, \{\overline{(\text{op}1)}, \overline{(\text{op}2)}\}\} n) n) n) n) n \mid \dot{\vdash} (\forall_{\text{obj}} \underline{m}: \dot{\vdash} (d_{\text{Ph}} = \\
& \{\{\underline{m}, \underline{m}\}, \{\underline{m}, (\underline{fx})[\underline{m}] + (\underline{fy})[\underline{m}]\}\} n) n) [\underline{m}] = (\underline{fx})[\underline{m}] + (\underline{fy})[\underline{m}] \gg \\
& (\underline{fx})[\underline{m}] + (\underline{fy})[\underline{m}] = \{\text{ph} \in \{\text{ph} \in P(P(\text{Union}(\{N, Q\})) \mid \\
& \dot{\vdash} (\forall_{\text{obj}} \overline{(\text{op}1)}: \dot{\vdash} (\dot{\vdash} (\forall_{\text{obj}} \overline{(\text{op}2)}: \dot{\vdash} (\dot{\vdash} (\overline{(\text{op}1)} \in N \Rightarrow \dot{\vdash} (\overline{(\text{op}2)} \in Q) n) n) \Rightarrow \\
& \dot{\vdash} (a_{\text{Ph}} = \{\{\overline{(\text{op}1)}, \overline{(\text{op}1)}\}, \{\overline{(\text{op}1)}, \overline{(\text{op}2)}\}\} n) n) n) n) n \mid \dot{\vdash} (\forall_{\text{obj}} \underline{m}: \dot{\vdash} (d_{\text{Ph}} = \\
& \{\{\underline{m}, \underline{m}\}, \{\underline{m}, (\underline{fx})[\underline{m}] + (\underline{fy})[\underline{m}]\}\} n) n) [\underline{m}]\rangle, p_0, c)] \\
& [\text{plus0Left} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: (0 + \underline{x}) = \underline{x}] \\
& [\text{plus0Left} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}: \text{plus0} \gg (\underline{x} + 0) = \\
& \underline{x}; \text{plusCommutativity} \gg (0 + \underline{x}) = (\underline{x} + 0); \text{eqTransitivity} \triangleright (0 + \underline{x}) = \\
& (\underline{x} + 0) \triangleright (\underline{x} + 0) = \underline{x} \gg (0 + \underline{x}) = \underline{x}\rangle, p_0, c)] \\
& [\text{times1Left} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: (1 * \underline{x}) = \underline{x}] \\
& [\text{times1Left} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}: \text{times1} \gg (\underline{x} * 1) = \\
& \underline{x}; \text{timesCommutativity} \gg (1 * \underline{x}) = (\underline{x} * 1); \text{eqTransitivity} \triangleright (1 * \underline{x}) = \\
& (\underline{x} * 1) \triangleright (\underline{x} * 1) = \underline{x} \gg (1 * \underline{x}) = \underline{x}\rangle, p_0, c)] \\
& [\text{Induction} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall (v1): \forall \underline{a}: \forall \underline{b}: \forall \underline{c}: \langle \underline{b} \equiv \underline{a} \mid (v1) : == 0 \rangle_{\text{Me}} \Vdash \\
& \langle \underline{c} \equiv \underline{a} \mid (v1) : == ((v1) + 1) \rangle_{\text{Me}} \Vdash \underline{b} \vdash \underline{a} \Rightarrow \underline{c} \vdash \underline{a}] \\
& [\text{Induction} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall (v1): \forall \underline{a}: \forall \underline{b}: \forall \underline{c}: \langle \underline{b} \equiv \underline{a} \mid (v1) : == 0 \rangle_{\text{Me}} \Vdash \\
& \langle \underline{c} \equiv \underline{a} \mid (v1) : == ((v1) + 1) \rangle_{\text{Me}} \Vdash \underline{b} \vdash \underline{a} \Rightarrow \underline{c} \vdash \text{Gen} \triangleright \underline{a} \Rightarrow \underline{c} \gg \forall_{\text{obj}} \overline{(v1)}: \underline{a} \Rightarrow \\
& \underline{c}; \text{InductionAxiom} \triangleright \langle \underline{b} \equiv \underline{a} \mid (v1) : == 0 \rangle_{\text{Me}} \triangleright \langle \underline{c} \equiv \underline{a} \mid (v1) : == ((v1) + 1) \rangle_{\text{Me}} \gg \underline{b} \Rightarrow \\
& \forall_{\text{obj}} \overline{(v1)}: \underline{a} \Rightarrow \underline{c} \Rightarrow \forall_{\text{obj}} \overline{(v1)}: \underline{a}; \text{MP2} \triangleright \underline{b} \Rightarrow \forall_{\text{obj}} \overline{(v1)}: \underline{a} \Rightarrow \underline{c} \Rightarrow \\
& \forall_{\text{obj}} \overline{(v1)}: \underline{a} \triangleright \underline{b} \triangleright \forall_{\text{obj}} \overline{(v1)}: \underline{a} \Rightarrow \underline{c} \gg \forall_{\text{obj}} \overline{(v1)}: \underline{a}; \text{A4} @ \overline{(v1)} \triangleright \forall_{\text{obj}} \overline{(v1)}: \underline{a} \gg \underline{a}\rangle, p_0, c)] \\
& [\text{ToSeries} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall (\underline{fx}): \forall (\underline{sy}): \forall_{\text{obj}} \overline{(r1)}: \overline{(r1)} \in (\underline{fx}) \Rightarrow \\
& \dot{\vdash} (\forall_{\text{obj}} \overline{(\text{op}1)}: \dot{\vdash} (\dot{\vdash} (\forall_{\text{obj}} \overline{(\text{op}2)}: \dot{\vdash} (\dot{\vdash} (\overline{(\text{op}1)} \in N \Rightarrow \dot{\vdash} (\overline{(\text{op}2)} \in (\underline{sy})) n) n) \Rightarrow \\
& \dot{\vdash} (\overline{(r1)} = \{\{\overline{(\text{op}1)}, \overline{(\text{op}1)}\}, \{\overline{(\text{op}1)}, \overline{(\text{op}2)}\}\} n) n) n) n) n \vdash \\
& \forall_{\text{obj}} \overline{(f1)}: \forall_{\text{obj}} \overline{(f2)}: \forall_{\text{obj}} \overline{(f3)}: \forall_{\text{obj}} \overline{(f4)}: \{\{\overline{(f1)}, \overline{(f1)}\}, \{\overline{(f1)}, \overline{(f2)}\}\} \in (\underline{fx}) \Rightarrow \\
& \{\{\overline{(f3)}, \overline{(f3)}\}, \{\overline{(f3)}, \overline{(f4)}\}\} \in (\underline{fx}) \Rightarrow \overline{(f1)} = \overline{(f3)} \Rightarrow \overline{(f2)} = \overline{(f4)} \vdash \forall_{\text{obj}} \overline{(s1)}: \overline{(s1)} \in \\
& N \Rightarrow \dot{\vdash} (\forall_{\text{obj}} \overline{(s2)}: \dot{\vdash} (\{\{\overline{(s1)}, \overline{(s1)}\}, \{\overline{(s1)}, \overline{(s2)}\}\} \in (\underline{fx})) n) n \vdash \dot{\vdash} (\dot{\vdash} (\forall_{\text{obj}} \overline{(r1)}: \overline{(r1)} \in \\
& (\underline{fx}) \Rightarrow \dot{\vdash} (\forall_{\text{obj}} \overline{(\text{op}1)}: \dot{\vdash} (\dot{\vdash} (\forall_{\text{obj}} \overline{(\text{op}2)}: \dot{\vdash} (\dot{\vdash} (\overline{(\text{op}1)} \in N \Rightarrow \dot{\vdash} (\overline{(\text{op}2)} \in \\
& (\underline{sy})) n) n) \Rightarrow \dot{\vdash} (\overline{(r1)} = \{\{\overline{(\text{op}1)}, \overline{(\text{op}1)}\}, \{\overline{(\text{op}1)}, \overline{(\text{op}2)}\}\} n) n) n) n) n) \Rightarrow \\
& \dot{\vdash} (\forall_{\text{obj}} \overline{(f1)}: \forall_{\text{obj}} \overline{(f2)}: \forall_{\text{obj}} \overline{(f3)}: \forall_{\text{obj}} \overline{(f4)}: \{\{\overline{(f1)}, \overline{(f1)}\}, \{\overline{(f1)}, \overline{(f2)}\}\} \in (\underline{fx}) \Rightarrow \\
& \{\{\overline{(f3)}, \overline{(f3)}\}, \{\overline{(f3)}, \overline{(f4)}\}\} \in (\underline{fx}) \Rightarrow \overline{(f1)} = \overline{(f3)} \Rightarrow \overline{(f2)} = \overline{(f4)}) n) n \Rightarrow \\
& \dot{\vdash} (\forall_{\text{obj}} \overline{(s1)}: \overline{(s1)} \in N \Rightarrow \dot{\vdash} (\forall_{\text{obj}} \overline{(s2)}: \dot{\vdash} (\{\{\overline{(s1)}, \overline{(s1)}\}, \{\overline{(s1)}, \overline{(s2)}\}\} \in (\underline{fx})) n) n) n)]
\end{aligned}$$









$$\begin{aligned}
& \dot{\dashv} (\forall_{\text{obj}} \overline{(s1)}: \overline{(s1)} \in N \Rightarrow \dot{\dashv} (\forall_{\text{obj}} \overline{(s2)}: \dot{\dashv} (\{ \{ \overline{(s1)}, \overline{(s1)} \}, \{ \overline{(s1)}, \overline{(s2)} \} \} \in \\
& \underline{(fx)} \underline{n}) \underline{n}) \underline{n} \gg \dot{\dashv} (\dot{\dashv} (\forall_{\text{obj}} \overline{(r1)}: \overline{(r1)} \in \underline{(fx)} \Rightarrow \\
& \dot{\dashv} (\forall_{\text{obj}} \overline{(op1)}: \dot{\dashv} (\dot{\dashv} (\forall_{\text{obj}} \overline{(op2)}: \dot{\dashv} (\dot{\dashv} (\overline{(op1)} \in N \Rightarrow \dot{\dashv} (\overline{(op2)} \in \underline{(sy)}) \underline{n}) \underline{n} \Rightarrow \\
& \dot{\dashv} (\overline{(r1)} = \{ \{ \overline{(op1)}, \overline{(op1)} \}, \{ \overline{(op1)}, \overline{(op2)} \} \} \underline{n}) \underline{n}) \underline{n}) \underline{n}) \underline{n} \Rightarrow \\
& \dot{\dashv} (\forall_{\text{obj}} \overline{(f1)}: \forall_{\text{obj}} \overline{(f2)}: \forall_{\text{obj}} \overline{(f3)}: \forall_{\text{obj}} \overline{(f4)}: \{ \{ \overline{(f1)}, \overline{(f1)} \}, \{ \overline{(f1)}, \overline{(f2)} \} \} \in \underline{(fx)} \Rightarrow \\
& \{ \{ \overline{(f3)}, \overline{(f3)} \}, \{ \overline{(f3)}, \overline{(f4)} \} \} \in \underline{(fx)} \Rightarrow \overline{(f1)} = \overline{(f3)} \Rightarrow \overline{(f2)} = \overline{(f4)}) \underline{n}) \underline{n} \Rightarrow \\
& \dot{\dashv} (\forall_{\text{obj}} \overline{(s1)}: \overline{(s1)} \in N \Rightarrow \dot{\dashv} (\forall_{\text{obj}} \overline{(s2)}: \dot{\dashv} (\{ \{ \overline{(s1)}, \overline{(s1)} \}, \{ \overline{(s1)}, \overline{(s2)} \} \} \in \\
& \underline{(fx)} \underline{n}) \underline{n}) \underline{n}) \underline{n}], p_0, c)] \\
[\text{NeqSymmetry} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \dot{\dashv} (\underline{x} = \underline{y}) \underline{n} \vdash \dot{\dashv} (\underline{y} = \underline{x}) \underline{n}] \\
[\text{NeqSymmetry} \xrightarrow{\text{proof}} \lambda c. \lambda x. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \underline{y} = \underline{x} \vdash \text{eqSymmetry} \triangleright \underline{y} = \\
\underline{x} \gg \underline{x} = \underline{y}; \forall \underline{x}: \forall \underline{y}: \text{Ded} \triangleright \forall \underline{x}: \forall \underline{y}: \underline{y} = \underline{x} \vdash \underline{x} = \underline{y} \gg \underline{y} = \underline{x} \Rightarrow \underline{x} = \underline{y}; \dot{\dashv} (\underline{x} = \underline{y}) \underline{n} \vdash \\
\text{MT} \triangleright \underline{y} = \underline{x} \Rightarrow \underline{x} = \underline{y} \triangleright \dot{\dashv} (\underline{x} = \underline{y}) \underline{n} \gg \dot{\dashv} (\underline{y} = \underline{x}) \underline{n}], p_0, c)] \\
[\text{PositiveNonzero} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: \dot{\dashv} (0 <= \underline{x} \Rightarrow \dot{\dashv} (\dot{\dashv} (0 = \underline{x}) \underline{n}) \underline{n}) \underline{n} \vdash \dot{\dashv} (\underline{x} = 0) \underline{n}] \\
[\text{PositiveNonzero} \xrightarrow{\text{proof}} \lambda c. \lambda x. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \forall \underline{x}: \dot{\dashv} (0 <= \underline{x} \Rightarrow \dot{\dashv} (\dot{\dashv} (0 = \\
\underline{x}) \underline{n}) \underline{n}) \underline{n} \vdash \text{Repetition} \triangleright \dot{\dashv} (0 <= \underline{x} \Rightarrow \dot{\dashv} (\dot{\dashv} (0 = \underline{x}) \underline{n}) \underline{n}) \underline{n} \gg \dot{\dashv} (0 <= \underline{x} \Rightarrow \\
\dot{\dashv} (\dot{\dashv} (0 = \underline{x}) \underline{n}) \underline{n}) \underline{n}; \text{SecondConjunct} \triangleright \dot{\dashv} (0 <= \underline{x} \Rightarrow \dot{\dashv} (\dot{\dashv} (0 = \underline{x}) \underline{n}) \underline{n}) \underline{n} \gg \dot{\dashv} (0 = \\
\underline{x}) \underline{n}; \text{NeqSymmetry} \triangleright \dot{\dashv} (0 = \underline{x}) \underline{n} \gg \dot{\dashv} (\underline{x} = 0) \underline{n}], p_0, c)] \\
[\text{SubNeqLeft} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \forall \underline{z}: \underline{x} = \underline{y} \vdash \dot{\dashv} (\underline{x} = \underline{z}) \underline{n} \vdash \dot{\dashv} (\underline{y} = \underline{z}) \underline{n}] \\
[\text{SubNeqLeft} \xrightarrow{\text{proof}} \lambda c. \lambda x. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \forall \underline{z}: \underline{x} = \underline{y} \vdash \dot{\dashv} (\underline{x} = \underline{z}) \underline{n} \vdash \\
\text{EqualityAxiom} \gg \underline{y} = \underline{x} \Rightarrow \underline{y} = \underline{z} \Rightarrow \underline{x} = \underline{z}; \text{eqSymmetry} \triangleright \underline{x} = \underline{y} \gg \underline{y} = \\
\underline{x}; \text{MP} \triangleright \underline{y} = \underline{x} \Rightarrow \underline{y} = \underline{z} \Rightarrow \underline{x} = \underline{z} \triangleright \underline{y} = \underline{x} \gg \underline{y} = \underline{z} \Rightarrow \underline{x} = \underline{z}; \text{Contrapositive} \triangleright \underline{y} = \\
\underline{z} \Rightarrow \underline{x} = \underline{z} \gg \dot{\dashv} (\underline{x} = \underline{z}) \underline{n} \Rightarrow \dot{\dashv} (\underline{y} = \underline{z}) \underline{n}; \text{MP} \triangleright \dot{\dashv} (\underline{x} = \underline{z}) \underline{n} \Rightarrow \dot{\dashv} (\underline{y} = \underline{z}) \underline{n} \triangleright \dot{\dashv} (\underline{x} = \\
\underline{z}) \underline{n} \gg \dot{\dashv} (\underline{y} = \underline{z}) \underline{n}], p_0, c)] \\
[\text{InPair}(1) \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{(sx)}: \forall \underline{(sy)}: \underline{(sx)} \in \{ \underline{(sx)}, \underline{(sy)} \}] \\
[\text{InPair}(1) \xrightarrow{\text{proof}} \lambda c. \lambda x. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \forall \underline{(sx)}: \forall \underline{(sy)}: \text{eqReflexivity} \gg \underline{(sx)} = \\
\underline{(sx)}; \text{WeakenOr2} \triangleright \underline{(sx)} = \underline{(sx)} \gg \dot{\dashv} (\underline{(sx)} = \underline{(sx)}) \underline{n} \Rightarrow \underline{(sx)} = \\
\underline{(sy)}; \text{Formula2Pair} \triangleright \dot{\dashv} (\underline{(sx)} = \underline{(sx)}) \underline{n} \Rightarrow \underline{(sx)} = \underline{(sy)} \gg \underline{(sx)} \in \\
\{ \underline{(sx)}, \underline{(sy)} \}], p_0, c)] \\
[\text{InPair}(2) \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{(sx)}: \forall \underline{(sy)}: \underline{(sy)} \in \{ \underline{(sx)}, \underline{(sy)} \}] \\
[\text{InPair}(2) \xrightarrow{\text{proof}} \lambda c. \lambda x. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \forall \underline{(sx)}: \forall \underline{(sy)}: \text{eqReflexivity} \gg \underline{(sy)} = \\
\underline{(sy)}; \text{WeakenOr1} \triangleright \underline{(sy)} = \underline{(sy)} \gg \dot{\dashv} (\underline{(sy)} = \underline{(sx)}) \underline{n} \Rightarrow \underline{(sy)} = \\
\underline{(sy)}; \text{Formula2Pair} \triangleright \dot{\dashv} (\underline{(sy)} = \underline{(sx)}) \underline{n} \Rightarrow \underline{(sy)} = \underline{(sy)} \gg \underline{(sy)} \in \\
\{ \underline{(sx)}, \underline{(sy)} \}], p_0, c)] \\
[\text{FromSingleton} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{(sx)}: \forall \underline{(sy)}: \underline{(sx)} \in \{ \underline{(sy)}, \underline{(sy)} \} \vdash \underline{(sx)} = \underline{(sy)}] \\
[\text{FromSingleton} \xrightarrow{\text{proof}} \lambda c. \lambda x. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \forall \underline{(sx)}: \forall \underline{(sy)}: \underline{(sx)} \in \{ \underline{(sy)}, \underline{(sy)} \} \vdash \\
\text{Repetition} \triangleright \underline{(sx)} \in \{ \underline{(sy)}, \underline{(sy)} \} \gg \underline{(sx)} \in \{ \underline{(sy)}, \underline{(sy)} \}; \text{Pair2Formula} \triangleright \underline{(sx)} \in \\
\{ \underline{(sy)}, \underline{(sy)} \} \gg \dot{\dashv} (\underline{(sx)} = \underline{(sy)}) \underline{n} \Rightarrow \underline{(sx)} = \underline{(sy)}; \text{RemoveOr} \triangleright \dot{\dashv} (\underline{(sx)} = \underline{(sy)}) \underline{n} \Rightarrow \\
\underline{(sx)} = \underline{(sy)} \gg \underline{(sx)} = \underline{(sy)}], p_0, c)] \\
[\text{ToSingleton} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{(sx)}: \forall \underline{(sy)}: \underline{(sx)} = \underline{(sy)} \vdash \underline{(sx)} \in \{ \underline{(sy)}, \underline{(sy)} \}]
\end{aligned}$$

[ToSingleton  $\xrightarrow{\text{proof}}$   $\lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall(\underline{sx}): \forall(\underline{sy}): (\underline{sx}) = (\underline{sy}) \vdash$   
 WeakenOr1  $\triangleright (\underline{sx}) = (\underline{sy}) \gg \dot{\vdash} ((\underline{sx}) = (\underline{sy}))n \Rightarrow (\underline{sx}) =$   
 $(\underline{sy}); \text{Formula2Pair} \triangleright \dot{\vdash} ((\underline{sx}) = (\underline{sy}))n \Rightarrow (\underline{sx}) = (\underline{sy}) \gg (\underline{sx}) \in$   
 $\{(\underline{sy}), (\underline{sy})\}; \text{Repetition} \triangleright (\underline{sx}) \in \{(\underline{sy}), (\underline{sy})\} \gg (\underline{sx}) \in \{(\underline{sy}), (\underline{sy})\}, p_0, c \rceil]$

[FromSameSingleton  $\xrightarrow{\text{stmt}}$  SystemQ  $\vdash \forall(\underline{sx}): \forall(\underline{sy}): \{(\underline{sx}), (\underline{sx})\} = \{(\underline{sy}), (\underline{sy})\} \vdash$   
 $(\underline{sx}) = (\underline{sy})]$

[FromSameSingleton  $\xrightarrow{\text{proof}}$   $\lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall(\underline{sx}): \forall(\underline{sy}): \{(\underline{sx}), (\underline{sx})\} =$   
 $\{(\underline{sy}), (\underline{sy})\} \vdash \text{eqReflexivity} \gg (\underline{sx}) = (\underline{sx}); \text{ToSingleton} \triangleright (\underline{sx}) = (\underline{sx}) \gg (\underline{sx}) \in$   
 $\{(\underline{sx}), (\underline{sx})\}; \text{SENC1} \triangleright \{(\underline{sx}), (\underline{sx})\} = \{(\underline{sy}), (\underline{sy})\} \triangleright (\underline{sx}) \in \{(\underline{sx}), (\underline{sx})\} \gg (\underline{sx}) \in$   
 $\{(\underline{sy}), (\underline{sy})\}; \text{FromSingleton} \triangleright (\underline{sx}) \in \{(\underline{sy}), (\underline{sy})\} \gg (\underline{sx}) = (\underline{sy}), p_0, c \rceil]$

[SingletonmembersEqual  $\xrightarrow{\text{stmt}}$  SystemQ  $\vdash \forall(\underline{sx}): \forall(\underline{sy}): \forall(\underline{sz}): \{(\underline{sx}), (\underline{sy})\} =$   
 $\{(\underline{sz}), (\underline{sz})\} \vdash (\underline{sx}) = (\underline{sy})]$

[SingletonmembersEqual  $\xrightarrow{\text{proof}}$   $\lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash$   
 $\forall(\underline{sx}): \forall(\underline{sy}): \forall(\underline{sz}): \{(\underline{sx}), (\underline{sy})\} = \{(\underline{sz}), (\underline{sz})\} \vdash \text{InPair}(1) \gg (\underline{sx}) \in$   
 $\{(\underline{sx}), (\underline{sy})\}; \text{SENC1} \triangleright \{(\underline{sx}), (\underline{sy})\} = \{(\underline{sz}), (\underline{sz})\} \triangleright (\underline{sx}) \in \{(\underline{sx}), (\underline{sy})\} \gg (\underline{sx}) \in$   
 $\{(\underline{sz}), (\underline{sz})\}; \text{FromSingleton} \triangleright (\underline{sx}) \in \{(\underline{sz}), (\underline{sz})\} \gg (\underline{sx}) = (\underline{sz}); \text{InPair}(2) \gg$   
 $(\underline{sy}) \in \{(\underline{sx}), (\underline{sy})\}; \text{SENC1} \triangleright \{(\underline{sx}), (\underline{sy})\} = \{(\underline{sz}), (\underline{sz})\} \triangleright (\underline{sy}) \in \{(\underline{sx}), (\underline{sy})\} \gg$   
 $(\underline{sy}) \in \{(\underline{sz}), (\underline{sz})\}; \text{FromSingleton} \triangleright (\underline{sy}) \in \{(\underline{sz}), (\underline{sz})\} \gg (\underline{sy}) =$   
 $(\underline{sz}); \text{eqSymmetry} \triangleright (\underline{sy}) = (\underline{sz}) \gg (\underline{sz}) = (\underline{sy}); \text{eqTransitivity} \triangleright (\underline{sx}) =$   
 $(\underline{sz}) \triangleright (\underline{sz}) = (\underline{sy}) \gg (\underline{sx}) = (\underline{sy}), p_0, c \rceil]$

[UnequalsNotInSingleton  $\xrightarrow{\text{stmt}}$  SystemQ  $\vdash \forall(\underline{sx}): \forall(\underline{sy}): \forall(\underline{sz}): \dot{\vdash} ((\underline{sx}) = (\underline{sy}))n \vdash$   
 $\dot{\vdash} (\{(\underline{sx}), (\underline{sy})\} = \{(\underline{sz}), (\underline{sz})\})n]$

[UnequalsNotInSingleton  $\xrightarrow{\text{proof}}$   $\lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash$   
 $\forall(\underline{sx}): \forall(\underline{sy}): \forall(\underline{sz}): \{(\underline{sx}), (\underline{sy})\} = \{(\underline{sz}), (\underline{sz})\} \vdash$   
 SingletonmembersEqual  $\triangleright \{(\underline{sx}), (\underline{sy})\} = \{(\underline{sz}), (\underline{sz})\} \gg (\underline{sx}) =$   
 $(\underline{sy}); \forall(\underline{sx}): \forall(\underline{sy}): \forall(\underline{sz}): \text{Ded} \triangleright \forall(\underline{sx}): \forall(\underline{sy}): \forall(\underline{sz}): \{(\underline{sx}), (\underline{sy})\} = \{(\underline{sz}), (\underline{sz})\} \vdash$   
 $(\underline{sx}) = (\underline{sy}) \gg \{(\underline{sx}), (\underline{sy})\} = \{(\underline{sz}), (\underline{sz})\} \Rightarrow (\underline{sx}) = (\underline{sy}); \dot{\vdash} ((\underline{sx}) = (\underline{sy}))n \vdash$   
 MT  $\triangleright \{(\underline{sx}), (\underline{sy})\} = \{(\underline{sz}), (\underline{sz})\} \Rightarrow (\underline{sx}) = (\underline{sy}) \triangleright \dot{\vdash} ((\underline{sx}) = (\underline{sy}))n \gg$   
 $\dot{\vdash} (\{(\underline{sx}), (\underline{sy})\} = \{(\underline{sz}), (\underline{sz})\})n, p_0, c \rceil]$

[NonsingletonmembersUnequal  $\xrightarrow{\text{stmt}}$  SystemQ  $\vdash \forall(\underline{sx}): \forall(\underline{sy}): \dot{\vdash} (\{(\underline{sx}), (\underline{sy})\} =$   
 $\{(\underline{sx}), (\underline{sx})\})n \vdash \dot{\vdash} ((\underline{sx}) = (\underline{sy}))n]$

[NonsingletonmembersUnequal  $\xrightarrow{\text{proof}}$   $\lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall(\underline{sx}): \forall(\underline{sy}): (\underline{sx}) =$   
 $(\underline{sy}) \vdash \text{eqReflexivity} \gg (\underline{sx}) = (\underline{sx}); \text{SamePair} \triangleright (\underline{sx}) = (\underline{sx}) \triangleright (\underline{sx}) = (\underline{sy}) \gg$   
 $\{(\underline{sx}), (\underline{sx})\} = \{(\underline{sx}), (\underline{sy})\}; \text{Repetition} \triangleright \{(\underline{sx}), (\underline{sx})\} = \{(\underline{sx}), (\underline{sy})\} \gg$   
 $\{(\underline{sx}), (\underline{sx})\} = \{(\underline{sx}), (\underline{sy})\}; \text{eqSymmetry} \triangleright \{(\underline{sx}), (\underline{sx})\} = \{(\underline{sx}), (\underline{sy})\} \gg$   
 $\{(\underline{sx}), (\underline{sy})\} = \{(\underline{sx}), (\underline{sx})\}; \forall(\underline{sx}): \forall(\underline{sy}): \text{Ded} \triangleright \forall(\underline{sx}): \forall(\underline{sy}): (\underline{sx}) = (\underline{sy}) \vdash$   
 $\{(\underline{sx}), (\underline{sy})\} = \{(\underline{sx}), (\underline{sx})\} \gg (\underline{sx}) = (\underline{sy}) \Rightarrow \{(\underline{sx}), (\underline{sy})\} =$   
 $\{(\underline{sx}), (\underline{sx})\}; \dot{\vdash} (\{(\underline{sx}), (\underline{sy})\} = \{(\underline{sx}), (\underline{sx})\})n \vdash \text{MT} \triangleright (\underline{sx}) = (\underline{sy}) \Rightarrow \{(\underline{sx}), (\underline{sy})\} =$   
 $\{(\underline{sx}), (\underline{sx})\} \triangleright \dot{\vdash} (\{(\underline{sx}), (\underline{sy})\} = \{(\underline{sx}), (\underline{sx})\})n \gg \dot{\vdash} ((\underline{sx}) = (\underline{sy}))n, p_0, c \rceil]$

[FromOrderedPair  $\xrightarrow{\text{stmt}}$  SystemQ  $\vdash$   
 $\forall(\underline{sx}): \forall(\underline{sx1}): \forall(\underline{sy}): \forall(\underline{sy1}): \{(\underline{sx}), (\underline{sx}), \{(\underline{sx}), (\underline{sy})\}\} =$   
 $\{(\underline{sx1}), (\underline{sx1}), \{(\underline{sx1}), (\underline{sy1})\}\} \vdash \dot{\vdash} ((\underline{sx}) = (\underline{sx1}) \Rightarrow \dot{\vdash} ((\underline{sy}) = (\underline{sy1}))n)n]$



















$$\begin{aligned}
& P(P(\text{Union}(\{\underline{(sx)}, \underline{(sy)}\}))) \mid \dot{\vdash} (\forall_{\text{obj}} \overline{(\text{op1})} : \dot{\vdash} (\dot{\vdash} (\forall_{\text{obj}} \overline{(\text{op2})} : \dot{\vdash} (\dot{\vdash} (\overline{(\text{op1})} \in \\
& \underline{(sx)} \Rightarrow \dot{\vdash} (\overline{(\text{op2})} \in \underline{(sy)})) \text{n}) \Rightarrow \dot{\vdash} (\text{a}_{\text{Ph}} = \\
& \{\{\overline{(\text{op1})}, \underline{(\text{op1})}\}, \{\overline{(\text{op1})}, \underline{(\text{op2})}\}\}) \text{n}) \text{n}) \text{n}) \text{n}) \text{n}) \text{n}) \mid \underline{\mathbf{a}} \Rightarrow \\
& \dot{\vdash} (\forall_{\text{obj}} \overline{(\text{op1})} : \dot{\vdash} (\dot{\vdash} (\forall_{\text{obj}} \overline{(\text{op2})} : \dot{\vdash} (\dot{\vdash} (\overline{(\text{op1})} \in \underline{(sx)} \Rightarrow \dot{\vdash} (\overline{(\text{op2})} \in \underline{(sy)}) \text{n}) \text{n}) \Rightarrow \\
& \dot{\vdash} (\overline{(\text{r1})} = \{\{\overline{(\text{op1})}, \underline{(\text{op1})}\}, \{\overline{(\text{op1})}, \underline{(\text{op2})}\}\}) \text{n}) \text{n}) \text{n}) \text{n}) \text{n}) \text{n}], p_0, c) \\
& [\text{ToCartProd}(\text{Helper}) \xrightarrow{\text{stnt}} \text{SystemQ} \vdash \forall \underline{(sx)} : \forall \underline{(sx1)} : \forall \underline{(sy)} : \forall \underline{(sy1)} : \forall \underline{(sz)} : \underline{(sx)} \in \\
& \underline{(sx1)} \vdash \underline{(sy)} \in \underline{(sy1)} \vdash \underline{(sz)} \in \{\{\underline{(sx)}, \underline{(sx)}\}, \{\underline{(sx)}, \underline{(sy)}\}\} \vdash \forall_{\text{obj}} \underline{(s1)} : \underline{(s1)} \in \underline{(sz)} \Rightarrow \\
& \underline{(s1)} \in \text{Union}(\{\underline{(sx1)}, \underline{(sy1)}\})] \\
& [\text{ToCartProd}(\text{Helper}) \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\text{SystemQ} \vdash \\
& \forall \underline{(sx)} : \forall \underline{(sx1)} : \forall \underline{(sy)} : \forall \underline{(sy1)} : \forall \underline{(sz)} : \underline{(sx)} \in \underline{(sx1)} \vdash \underline{(sy)} \in \underline{(sy1)} \vdash \underline{(sz)} \in \\
& \{\{\underline{(sx)}, \underline{(sx)}\}, \{\underline{(sx)}, \underline{(sy)}\}\} \vdash \underline{(s1)} \in \underline{(sz)} \vdash \text{FromOrderedPair}(\text{TwoLevels}) \triangleright \underline{(s1)} \in \\
& \underline{(sz)} \triangleright \underline{(sz)} \in \{\{\underline{(sx)}, \underline{(sx)}\}, \{\underline{(sx)}, \underline{(sy)}\}\} \gg \dot{\vdash} (\overline{(s1)} = \underline{(sx)}) \text{n} \Rightarrow \overline{(s1)} = \\
& \underline{(sy)}; \forall \underline{(sx)} : \forall \underline{(sx1)} : \forall \underline{(sy1)} : \underline{(sx)} \in \underline{(sx1)} \vdash \overline{(s1)} = \underline{(sx)} \vdash \text{SameMember}(2) \triangleright \overline{(s1)} = \\
& \underline{(sx)} \triangleright \underline{(sx)} \in \underline{(sx1)} \gg \overline{(s1)} \in \underline{(sx1)}; \text{ToBinaryUnion}(1) \triangleright \overline{(s1)} \in \underline{(sx1)} \gg \overline{(s1)} \in \\
& \text{Union}(\{\underline{(sx1)}, \underline{(sy1)}\}); \forall \underline{(sx1)} : \forall \underline{(sy)} : \forall \underline{(sy1)} : \underline{(sy)} \in \underline{(sy1)} \vdash \overline{(s1)} = \underline{(sy)} \vdash \\
& \text{SameMember}(2) \triangleright \overline{(s1)} = \underline{(sy)} \triangleright \underline{(sy)} \in \underline{(sy1)} \gg \overline{(s1)} \in \\
& \underline{(sy1)}; \text{ToBinaryUnion}(2) \triangleright \overline{(s1)} \in \underline{(sy1)} \gg \overline{(s1)} \in \\
& \text{Union}(\{\underline{(sx1)}, \underline{(sy1)}\}); \text{Ded} \triangleright \forall \underline{(sx)} : \forall \underline{(sx1)} : \forall \underline{(sy1)} : \underline{(sx)} \in \underline{(sx1)} \vdash \overline{(s1)} = \underline{(sx)} \vdash \\
& \overline{(s1)} \in \text{Union}(\{\underline{(sx1)}, \underline{(sy1)}\}) \gg \underline{(sx)} \in \underline{(sx1)} \Rightarrow \overline{(s1)} = \underline{(sx)} \Rightarrow \overline{(s1)} \in \\
& \text{Union}(\{\underline{(sx1)}, \underline{(sy1)}\}); \text{MP} \triangleright \underline{(sx)} \in \underline{(sx1)} \Rightarrow \overline{(s1)} = \underline{(sx)} \Rightarrow \overline{(s1)} \in \\
& \text{Union}(\{\underline{(sx1)}, \underline{(sy1)}\}) \triangleright \underline{(sx)} \in \underline{(sx1)} \gg \overline{(s1)} = \underline{(sx)} \Rightarrow \overline{(s1)} \in \\
& \text{Union}(\{\underline{(sx1)}, \underline{(sy1)}\}); \text{Ded} \triangleright \forall \underline{(sx1)} : \forall \underline{(sy)} : \forall \underline{(sy1)} : \underline{(sy)} \in \underline{(sy1)} \vdash \overline{(s1)} = \underline{(sy)} \vdash \\
& \overline{(s1)} \in \text{Union}(\{\underline{(sx1)}, \underline{(sy1)}\}) \gg \underline{(sy)} \in \underline{(sy1)} \Rightarrow \overline{(s1)} = \underline{(sy)} \Rightarrow \overline{(s1)} \in \\
& \text{Union}(\{\underline{(sx1)}, \underline{(sy1)}\}); \text{MP} \triangleright \underline{(sy)} \in \underline{(sy1)} \Rightarrow \overline{(s1)} = \underline{(sy)} \Rightarrow \overline{(s1)} \in \\
& \text{Union}(\{\underline{(sx1)}, \underline{(sy1)}\}) \triangleright \underline{(sy)} \in \underline{(sy1)} \gg \overline{(s1)} = \underline{(sy)} \Rightarrow \overline{(s1)} \in \\
& \text{Union}(\{\underline{(sx1)}, \underline{(sy1)}\}); \text{FromDisjuncts} \triangleright \dot{\vdash} (\overline{(s1)} = \underline{(sx)}) \text{n} \Rightarrow \overline{(s1)} = \underline{(sy)} \triangleright \overline{(s1)} = \\
& \underline{(sx)} \Rightarrow \overline{(s1)} \in \text{Union}(\{\underline{(sx1)}, \underline{(sy1)}\}) \triangleright \overline{(s1)} = \underline{(sy)} \Rightarrow \overline{(s1)} \in \\
& \text{Union}(\{\underline{(sx1)}, \underline{(sy1)}\}) \gg \overline{(s1)} \in \\
& \text{Union}(\{\underline{(sx1)}, \underline{(sy1)}\}); \forall \underline{(sx)} : \forall \underline{(sx1)} : \forall \underline{(sy)} : \forall \underline{(sy1)} : \forall \underline{(sz)} : \text{Ded} \triangleright \\
& \forall \underline{(sx)} : \forall \underline{(sx1)} : \forall \underline{(sy)} : \forall \underline{(sy1)} : \forall \underline{(sz)} : \underline{(sx)} \in \underline{(sx1)} \vdash \underline{(sy)} \in \underline{(sy1)} \vdash \underline{(sz)} \in \\
& \{\{\underline{(sx)}, \underline{(sx)}\}, \{\underline{(sx)}, \underline{(sy)}\}\} \vdash \underline{(s1)} \in \underline{(sz)} \vdash \underline{(s1)} \in \text{Union}(\{\underline{(sx1)}, \underline{(sy1)}\}) \gg \underline{(sx)} \in \\
& \underline{(sx1)} \Rightarrow \underline{(sy)} \in \underline{(sy1)} \Rightarrow \underline{(sz)} \in \{\{\underline{(sx)}, \underline{(sx)}\}, \{\underline{(sx)}, \underline{(sy)}\}\} \Rightarrow \overline{(s1)} \in \underline{(sz)} \Rightarrow \\
& \overline{(s1)} \in \text{Union}(\{\underline{(sx1)}, \underline{(sy1)}\}); \underline{(sx)} \in \underline{(sx1)} \vdash \underline{(sy)} \in \underline{(sy1)} \vdash \underline{(sz)} \in \\
& \{\{\underline{(sx)}, \underline{(sx)}\}, \{\underline{(sx)}, \underline{(sy)}\}\} \vdash \text{MP3} \triangleright \underline{(sx)} \in \underline{(sx1)} \Rightarrow \underline{(sy)} \in \underline{(sy1)} \Rightarrow \underline{(sz)} \in \\
& \{\{\underline{(sx)}, \underline{(sx)}\}, \{\underline{(sx)}, \underline{(sy)}\}\} \Rightarrow \overline{(s1)} \in \underline{(sz)} \Rightarrow \overline{(s1)} \in \text{Union}(\{\underline{(sx1)}, \underline{(sy1)}\}) \triangleright \underline{(sx)} \in \\
& \underline{(sx1)} \triangleright \underline{(sy)} \in \underline{(sy1)} \triangleright \underline{(sz)} \in \{\{\underline{(sx)}, \underline{(sx)}\}, \{\underline{(sx)}, \underline{(sy)}\}\} \gg \overline{(s1)} \in \underline{(sz)} \Rightarrow \overline{(s1)} \in \\
& \text{Union}(\{\underline{(sx1)}, \underline{(sy1)}\}); \text{Gen} \triangleright \overline{(s1)} \in \underline{(sz)} \Rightarrow \overline{(s1)} \in \text{Union}(\{\underline{(sx1)}, \underline{(sy1)}\}) \gg
\end{aligned}$$





























$$(\underline{x} + (\underline{y} + \underline{z})) \triangleright (\underline{x} + (\underline{y} + \underline{z})) = (\underline{x} + \underline{u}) \gg ((\underline{x} + \underline{y}) + \underline{z}) = (\underline{x} + \underline{u}), p_0, c]$$

$$[\text{Three2threeTerms} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \forall \underline{z}: ((\underline{x} + \underline{y}) + \underline{z}) = ((\underline{x} + \underline{z}) + \underline{y})]$$

$$[\text{Three2threeTerms} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \forall \underline{z}: \text{plusCommutativity} \gg (\underline{y} + \underline{z}) = (\underline{z} + \underline{y}); \text{Three2twoTerms} \triangleright (\underline{y} + \underline{z}) = (\underline{z} + \underline{y}) \gg ((\underline{x} + \underline{y}) + \underline{z}) = (\underline{x} + (\underline{z} + \underline{y})); \text{plusAssociativity} \gg ((\underline{x} + \underline{z}) + \underline{y}) = (\underline{x} + (\underline{z} + \underline{y})); \text{eqSymmetry} \triangleright ((\underline{x} + \underline{z}) + \underline{y}) = (\underline{x} + (\underline{z} + \underline{y})) \gg (\underline{x} + (\underline{z} + \underline{y})) = ((\underline{x} + \underline{z}) + \underline{y}); \text{eqTransitivity} \triangleright ((\underline{x} + \underline{y}) + \underline{z}) = (\underline{x} + (\underline{z} + \underline{y})) \triangleright (\underline{x} + (\underline{z} + \underline{y})) = ((\underline{x} + \underline{z}) + \underline{y}) \gg ((\underline{x} + \underline{y}) + \underline{z}) = ((\underline{x} + \underline{z}) + \underline{y})], p_0, c)]$$

$$[\text{Three2twoFactors} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \forall \underline{z}: \forall \underline{u}: (\underline{y} * \underline{z}) = \underline{u} \vdash ((\underline{x} * \underline{y}) * \underline{z}) = (\underline{x} * \underline{u})]$$

$$[\text{Three2twoFactors} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \forall \underline{z}: \forall \underline{u}: (\underline{y} * \underline{z}) = \underline{u} \vdash \text{EqMultiplicationLeft} \triangleright (\underline{y} * \underline{z}) = \underline{u} \gg (\underline{x} * (\underline{y} * \underline{z})) = (\underline{x} * \underline{u}); \text{timesAssociativity} \gg ((\underline{x} * \underline{y}) * \underline{z}) = (\underline{x} * (\underline{y} * \underline{z})); \text{eqTransitivity} \triangleright ((\underline{x} * \underline{y}) * \underline{z}) = (\underline{x} * (\underline{y} * \underline{z})) \triangleright (\underline{x} * (\underline{y} * \underline{z})) = (\underline{x} * \underline{u}) \gg ((\underline{x} * \underline{y}) * \underline{z}) = (\underline{x} * \underline{u})], p_0, c)]$$

$$[\underline{x} = \underline{x} + (\underline{y} - \underline{y}) \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \underline{x} = (\underline{x} + (\underline{y} + (-\underline{u})))]$$

$$[\underline{x} = \underline{x} + (\underline{y} - \underline{y}) \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \text{plus0} \gg (\underline{x} + 0) = \underline{x}; \text{Negative} \gg (\underline{y} + (-\underline{u})) = 0; \text{eqSymmetry} \triangleright (\underline{y} + (-\underline{u})) = 0 \gg 0 = (\underline{y} + (-\underline{u})); \text{EqAdditionLeft} \triangleright 0 = (\underline{y} + (-\underline{u})) \gg (\underline{x} + 0) = (\underline{x} + (\underline{y} + (-\underline{u}))); \text{Equality} \triangleright (\underline{x} + 0) = \underline{x} \triangleright (\underline{x} + 0) = (\underline{x} + (\underline{y} + (-\underline{u}))) \gg \underline{x} = (\underline{x} + (\underline{y} + (-\underline{u}))), p_0, c)]$$

$$[\underline{x} = \underline{x} + \underline{y} - \underline{y} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \underline{x} = ((\underline{x} + \underline{y}) + (-\underline{u}))]$$

$$[\underline{x} = \underline{x} + \underline{y} - \underline{y} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \underline{x} = \underline{x} + (\underline{y} - \underline{y}) \gg \underline{x} = (\underline{x} + (\underline{y} + (-\underline{u}))); \text{plusAssociativity} \gg ((\underline{x} + \underline{y}) + (-\underline{u})) = (\underline{x} + (\underline{y} + (-\underline{u}))); \text{eqSymmetry} \triangleright ((\underline{x} + \underline{y}) + (-\underline{u})) = (\underline{x} + (\underline{y} + (-\underline{u}))) \gg (\underline{x} + (\underline{y} + (-\underline{u}))) = ((\underline{x} + \underline{y}) + (-\underline{u})); \text{eqTransitivity} \triangleright \underline{x} = (\underline{x} + (\underline{y} + (-\underline{u}))) \triangleright (\underline{x} + (\underline{y} + (-\underline{u}))) = ((\underline{x} + \underline{y}) + (-\underline{u})) \gg \underline{x} = ((\underline{x} + \underline{y}) + (-\underline{u}))), p_0, c)]$$

$$[\underline{x} = \underline{x} * \underline{y} * (1/\underline{y}) \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \dot{\vdash} (\underline{y} = 0) \text{n} \vdash \underline{x} = ((\underline{x} * \underline{y}) * \text{recy})]$$

$$[\underline{x} = \underline{x} * \underline{y} * (1/\underline{y}) \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \dot{\vdash} (\underline{y} = 0) \text{n} \vdash \text{times1} \gg (\underline{x} * 1) = \underline{x}; \text{Reciprocal} \triangleright \dot{\vdash} (\underline{y} = 0) \text{n} \gg (\underline{y} * \text{recy}) = 1; \text{Three2twoFactors} \triangleright (\underline{y} * \text{recy}) = 1 \gg ((\underline{x} * \underline{y}) * \text{recy}) = (\underline{x} * 1); \text{eqTransitivity} \triangleright ((\underline{x} * \underline{y}) * \text{recy}) = (\underline{x} * 1) \triangleright (\underline{x} * 1) = \underline{x} \gg ((\underline{x} * \underline{y}) * \text{recy}) = \underline{x}; \text{eqSymmetry} \triangleright ((\underline{x} * \underline{y}) * \text{recy}) = \underline{x} \gg \underline{x} = ((\underline{x} * \underline{y}) * \text{recy})], p_0, c)]$$

$$[\underline{x} * 0 + \underline{x} = \underline{x} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: ((\underline{x} * 0) + \underline{x}) = \underline{x}]$$

$$[\underline{x} * 0 + \underline{x} = \underline{x} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \forall \underline{x}: \text{times1} \gg (\underline{x} * 1) = \underline{x}; \text{eqSymmetry} \triangleright (\underline{x} * 1) = \underline{x} \gg \underline{x} = (\underline{x} * 1); \text{EqAdditionLeft} \triangleright \underline{x} = (\underline{x} * 1) \gg ((\underline{x} * 0) + \underline{x}) = ((\underline{x} * 0) + (\underline{x} * 1)); \text{Distribution} \gg (\underline{x} * (0 + 1)) = ((\underline{x} * 0) + (\underline{x} * 1)); \text{eqSymmetry} \triangleright (\underline{x} * (0 + 1)) = ((\underline{x} * 0) + (\underline{x} * 1)) \gg ((\underline{x} * 0) + (\underline{x} * 1)) = (\underline{x} * (0 + 1)); \text{plus0Left} \gg (0 + 1) = 1; \text{EqMultiplicationLeft} \triangleright (0 + 1) = 1 \gg (\underline{x} * (0 + 1)) = (\underline{x} * 1); \text{eqTransitivity5} \triangleright ((\underline{x} * 0) + \underline{x}) = ((\underline{x} * 0) + (\underline{x} * 1)) \triangleright ((\underline{x} * 0) + (\underline{x} * 1)) = (\underline{x} * (0 + 1)) \triangleright (\underline{x} * (0 + 1)) = (\underline{x} * 1) \triangleright (\underline{x} * 1) = \underline{x} \gg ((\underline{x} * 0) + \underline{x}) = \underline{x}], p_0, c)]$$

$$[\underline{x} * 0 = 0 \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: (\underline{x} * 0) = 0]$$

$$[\underline{x} * 0 = 0 \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \forall \underline{x}: \underline{x} = \underline{x} + (\underline{y} - \underline{y}) \gg (\underline{x} * 0) = ((\underline{x} * 0) + (\underline{x} + (-\underline{u}))); \text{plusAssociativity} \gg (((\underline{x} * 0) + \underline{x}) + (-\underline{u})) = ((\underline{x} * 0) + (\underline{x} + (-\underline{u}))); \text{eqSymmetry} \triangleright (((\underline{x} * 0) + \underline{x}) + (-\underline{u})) =$$

$$\begin{aligned}
& ((\underline{x} * 0) + (\underline{x} + (-\underline{ux}))) \gg ((\underline{x} * 0) + (\underline{x} + (-\underline{ux}))) = (((\underline{x} * 0) + \underline{x}) + (-\underline{ux})); \underline{x} * 0 + \underline{x} = \underline{x} \gg ((\underline{x} * 0) + \underline{x}) = \underline{x}; \text{eqAddition} \triangleright ((\underline{x} * 0) + \underline{x}) = \underline{x} \gg (((\underline{x} * 0) + \underline{x}) + (-\underline{ux})) = (\underline{x} + (-\underline{ux})); \text{Negative} \gg (\underline{x} + (-\underline{ux})) = 0; \text{eqTransitivity5} \triangleright (\underline{x} * 0) = ((\underline{x} * 0) + (\underline{x} + (-\underline{ux}))) \triangleright ((\underline{x} * 0) + (\underline{x} + (-\underline{ux}))) = (((\underline{x} * 0) + \underline{x}) + (-\underline{ux})) \triangleright (((\underline{x} * 0) + \underline{x}) + (-\underline{ux})) = (\underline{x} + (-\underline{ux})) \triangleright (\underline{x} + (-\underline{ux})) = 0 \gg (\underline{x} * 0) = 0], p_0, c)] \\
& [(-1) * (-1) + (-1) * 1 = 0 \xrightarrow{\text{stmt}} \text{SystemQ} \vdash (((-u1) * (-u1)) + ((-u1) * 1)) = 0] \\
& [(-1) * (-1) + (-1) * 1 = 0 \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \text{DistributionOut} \gg (((-u1) * (-u1)) + ((-u1) * 1)) = ((-u1) * ((-u1) + 1)); \text{Negative} \gg (1 + (-u1)) = 0; \text{plusCommutativity} \gg ((-u1) + 1) = (1 + (-u1)); \text{eqTransitivity} \triangleright ((-u1) + 1) = (1 + (-u1)) \triangleright (1 + (-u1)) = 0 \gg ((-u1) + 1) = 0; \text{EqMultiplicationLeft} \triangleright ((-u1) + 1) = 0 \gg ((-u1) * ((-u1) + 1)) = ((-u1) * 0); \underline{x} * 0 = 0 \gg ((-u1) * 0) = 0; \text{eqTransitivity4} \triangleright (((-u1) * (-u1)) + ((-u1) * 1)) = ((-u1) * ((-u1) + 1)) \triangleright ((-u1) * ((-u1) + 1)) = ((-u1) * 0) \triangleright ((-u1) * 0) = 0 \gg (((-u1) * (-u1)) + ((-u1) * 1)) = 0], p_0, c)] \\
& [(-1) * (-1) = 1 \xrightarrow{\text{stmt}} \text{SystemQ} \vdash ((-u1) * (-u1)) = 1] \\
& [(-1) * (-1) = 1 \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \underline{x} = \underline{x} + (\underline{y} - \underline{y}) \gg ((-u1) * (-u1)) = (((-u1) * (-u1)) + (1 + (-u1))); \text{times1} \gg ((-u1) * 1) = (-u1); \text{eqSymmetry} \triangleright ((-u1) * 1) = (-u1) \gg (-u1) = ((-u1) * 1); \text{EqAdditionLeft} \triangleright (-u1) = ((-u1) * 1) \gg (1 + (-u1)) = (1 + ((-u1) * 1)); \text{EqAdditionLeft} \triangleright (1 + (-u1)) = (1 + ((-u1) * 1)) \gg (((-u1) * (-u1)) + (1 + (-u1))) = (((-u1) * (-u1)) + (1 + ((-u1) * 1))); \text{plusCommutativity} \gg (1 + ((-u1) * 1)) = (((-u1) * 1) + 1); \text{EqAdditionLeft} \triangleright (1 + ((-u1) * 1)) = (((-u1) * 1) + 1) \gg (((-u1) * (-u1)) + (1 + ((-u1) * 1))) = (((-u1) * (-u1)) + (((-u1) * 1) + 1)); \text{plusAssociativity} \gg ((((-u1) * (-u1)) + ((-u1) * 1)) + 1) = (((-u1) * (-u1)) + (((-u1) * 1) + 1)); \text{eqSymmetry} \triangleright ((((-u1) * (-u1)) + ((-u1) * 1)) + 1) = (((-u1) * (-u1)) + (((-u1) * 1) + 1)) \gg ((((-u1) * (-u1)) + ((-u1) * 1)) + 1); (-1) * (-1) + (-1) * 1 = 0 \gg (((-u1) * (-u1)) + ((-u1) * 1)) = 0; \text{eqAddition} \triangleright (((-u1) * (-u1)) + ((-u1) * 1)) = 0 \gg ((((-u1) * (-u1)) + ((-u1) * 1)) + 1) = (0 + 1); \text{plus0Left} \gg (0 + 1) = 1; \text{eqTransitivity5} \triangleright ((-u1) * (-u1)) = (((-u1) * (-u1)) + (1 + (-u1))) \triangleright (((-u1) * (-u1)) + (1 + (-u1))) = (((-u1) * (-u1)) + (1 + ((-u1) * 1))) \triangleright (((-u1) * (-u1)) + (1 + ((-u1) * 1))) = (((-u1) * (-u1)) + (((-u1) * 1) + 1)) \triangleright (((-u1) * (-u1)) + (((-u1) * 1) + 1)) = ((((-u1) * (-u1)) + ((-u1) * 1)) + 1) \gg ((-u1) * (-u1)) = ((((-u1) * (-u1)) + ((-u1) * 1)) + 1); \text{eqTransitivity4} \triangleright ((-u1) * (-u1)) = ((((-u1) * (-u1)) + ((-u1) * 1)) + 1) \triangleright ((((-u1) * (-u1)) + ((-u1) * 1)) + 1) = (0 + 1) \triangleright (0 + 1) = 1 \gg ((-u1) * (-u1)) = 1], p_0, c)] \\
& [\text{subLeqRight} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}. \forall \underline{y}. \forall \underline{z}. \underline{x} = \underline{y} \vdash \underline{z} \leq \underline{x} \vdash \underline{z} \leq \underline{y}] \\
& [\text{subLeqRight} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}. \forall \underline{y}. \forall \underline{z}. \underline{x} = \underline{y} \vdash \underline{z} \leq \underline{x} \vdash \text{eqLeq} \triangleright \underline{x} = \underline{y} \gg \underline{x} \leq \underline{y}; \text{leqTransitivity} \triangleright \underline{z} \leq \underline{x} \triangleright \underline{x} \leq \underline{y} \gg \underline{z} \leq \underline{y}], p_0, c)] \\
& [\text{subLeqLeft} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}. \forall \underline{y}. \forall \underline{z}. \underline{x} = \underline{y} \vdash \underline{x} \leq \underline{z} \vdash \underline{y} \leq \underline{z}] \\
& [\text{subLeqLeft} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}. \forall \underline{y}. \forall \underline{z}. \underline{x} = \underline{y} \vdash \underline{x} \leq \underline{z} \vdash \text{eqSymmetry} \triangleright \underline{x} = \underline{y} \gg \underline{y} = \underline{x}; \text{eqLeq} \triangleright \underline{y} = \underline{x} \gg \underline{y} \leq \underline{x}; \text{leqTransitivity} \triangleright \underline{y} \leq \underline{z} =
\end{aligned}$$

$$\underline{x} \triangleright \underline{x} \leq \underline{z} \gg \underline{y} \leq \underline{z}], p_0, c)]$$

$$[0 < 1 \text{Helper} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash 1 \leq 0 \Rightarrow 0 \leq 1]$$

$$[0 < 1 \text{Helper} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([\text{SystemQ} \vdash 1 \leq 0 \vdash \text{leqAddition} \triangleright 1 \leq 0 \gg (1 + (-u1)) \leq (0 + (-u1)); \text{Negative} \gg (1 + (-u1)) = 0; \text{subLeqLeft} \triangleright (1 + (-u1)) = 0 \triangleright (1 + (-u1)) \leq (0 + (-u1)) \gg 0 \leq (0 + (-u1)); \text{plus0Left} \gg (0 + (-u1)) = (-u1); \text{subLeqRight} \triangleright (0 + (-u1)) = (-u1) \triangleright 0 \leq (0 + (-u1)) \gg 0 \leq (-u1); \text{leqMultiplication} \triangleright 0 \leq (-u1) \triangleright 0 \leq (-u1) \gg (0 * (-u1)) \leq ((-u1) * (-u1)); x * 0 = 0 \gg ((-u1) * 0) = 0; \text{timesCommutativity} \gg (0 * (-u1)) = ((-u1) * 0); \text{eqTransitivity} \triangleright (0 * (-u1)) = ((-u1) * 0) \triangleright ((-u1) * 0) = 0 \gg (0 * (-u1)) = 0; \text{subLeqLeft} \triangleright (0 * (-u1)) = 0 \triangleright (0 * (-u1)) \leq ((-u1) * (-u1)) \gg 0 \leq ((-u1) * (-u1)); (-1) * (-1) = 1 \gg ((-u1) * (-u1)) = 1; \text{subLeqRight} \triangleright ((-u1) * (-u1)) = 1 \triangleright 0 \leq ((-u1) * (-u1)) \gg 0 \leq 1; \text{Ded} \triangleright 1 \leq 0 \vdash 0 \leq 1 \gg 1 \leq 0 \Rightarrow 0 \leq 1], p_0, c)]$$

$$[0 < 1 \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \dot{\vdash} (0 \leq 1 \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = 1)n)n)]$$

$$[0 < 1 \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \text{leqTotality} \gg \dot{\vdash} (0 \leq 1)n \Rightarrow 1 \leq 0; \text{AutoImPLY} \gg 0 \leq 1 \Rightarrow 0 \leq 1; 0 < 1 \text{Helper} \gg 1 \leq 0 \Rightarrow 0 \leq 1; \text{FromDisjuncts} \triangleright \dot{\vdash} (0 \leq 1)n \Rightarrow 1 \leq 0 \triangleright 0 \leq 1 \Rightarrow 0 \leq 1 \triangleright 1 \leq 0 \Rightarrow 0 \leq 1 \gg 0 \leq 1; 0 \text{not} 1 \gg \dot{\vdash} (0 = 1)n; \text{JoinConjuncts} \triangleright 0 \leq 1 \triangleright \dot{\vdash} (0 = 1)n \gg \dot{\vdash} (0 \leq 1 \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = 1)n)n)], p_0, c)]$$

$$[\text{AddEquations} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}. \forall \underline{y}. \forall \underline{z}. \forall \underline{u}. \underline{x} = \underline{y} \vdash \underline{z} = \underline{u} \vdash (\underline{x} + \underline{z}) = (\underline{y} + \underline{u})]$$

$$[\text{AddEquations} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \forall \underline{x}. \forall \underline{y}. \forall \underline{z}. \forall \underline{u}. \underline{x} = \underline{y} \vdash \underline{z} = \underline{u} \vdash \text{eqAddition} \triangleright \underline{x} = \underline{y} \gg (\underline{x} + \underline{z}) = (\underline{y} + \underline{z}); \text{EqAdditionLeft} \triangleright \underline{z} = \underline{u} \gg (\underline{y} + \underline{z}) = (\underline{y} + \underline{u}); \text{eqTransitivity} \triangleright (\underline{x} + \underline{z}) = (\underline{y} + \underline{z}) \triangleright (\underline{y} + \underline{z}) = (\underline{y} + \underline{u}) \gg (\underline{x} + \underline{z}) = (\underline{y} + \underline{u})], p_0, c)]$$

$$[\text{PositiveToRight}(\text{Eq}) \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}. \forall \underline{y}. \forall \underline{z}. (\underline{x} + \underline{y}) = \underline{z} \vdash \underline{x} = (\underline{z} + (-\underline{u}))]$$

$$[\text{PositiveToRight}(\text{Eq}) \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \forall \underline{x}. \forall \underline{y}. \forall \underline{z}. (\underline{x} + \underline{y}) = \underline{z} \vdash \text{eqAddition} \triangleright (\underline{x} + \underline{y}) = \underline{z} \gg ((\underline{x} + \underline{y}) + (-\underline{u})) = (\underline{z} + (-\underline{u})); x = x + y - y \gg x = ((\underline{x} + \underline{y}) + (-\underline{u})); \text{eqTransitivity} \triangleright x = ((\underline{x} + \underline{y}) + (-\underline{u})) \triangleright ((\underline{x} + \underline{y}) + (-\underline{u})) = (\underline{z} + (-\underline{u})) \gg x = (\underline{z} + (-\underline{u}))], p_0, c)]$$

$$[\text{PositiveToLeft}(\text{Eq})(1\text{term}) \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}. \forall \underline{y}. \underline{x} = \underline{y} \vdash (\underline{x} + (-\underline{u})) = 0]$$

$$[\text{PositiveToLeft}(\text{Eq})(1\text{term}) \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \forall \underline{x}. \forall \underline{y}. \underline{x} = \underline{y} \vdash \text{eqAddition} \triangleright \underline{x} = \underline{y} \gg (\underline{x} + (-\underline{u})) = (\underline{y} + (-\underline{u})); \text{Negative} \gg (\underline{y} + (-\underline{u})) = 0; \text{eqTransitivity} \triangleright (\underline{x} + (-\underline{u})) = (\underline{y} + (-\underline{u})) \triangleright (\underline{y} + (-\underline{u})) = 0 \gg (\underline{x} + (-\underline{u})) = 0], p_0, c)]$$

$$[\text{PositiveToRight}(\text{Leq}) \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}. \forall \underline{y}. \forall \underline{z}. (\underline{x} + \underline{y}) \leq \underline{z} \vdash \underline{x} \leq (\underline{z} + (-\underline{u}))]$$

$$[\text{PositiveToRight}(\text{Leq}) \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \forall \underline{x}. \forall \underline{y}. \forall \underline{z}. (\underline{x} + \underline{y}) \leq \underline{z} \vdash \text{leqAddition} \triangleright (\underline{x} + \underline{y}) \leq \underline{z} \gg ((\underline{x} + \underline{y}) + (-\underline{u})) \leq (\underline{z} + (-\underline{u})); x = x + y - y \gg x = ((\underline{x} + \underline{y}) + (-\underline{u})); \text{eqSymmetry} \triangleright x = ((\underline{x} + \underline{y}) + (-\underline{u})) \gg ((\underline{x} + \underline{y}) + (-\underline{u})) = x; \text{subLeqLeft} \triangleright ((\underline{x} + \underline{y}) + (-\underline{u})) = x \triangleright ((\underline{x} + \underline{y}) + (-\underline{u})) \leq (\underline{z} + (-\underline{u})) \gg x \leq (\underline{z} + (-\underline{u}))], p_0, c)]$$

$$[\text{PositiveToRight}(\text{Leq})(1\text{term}) \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{y}. \forall \underline{z}. \underline{y} \leq \underline{z} \vdash 0 \leq$$

$(\underline{z} + (-\underline{uy}))]$

$[\text{PositiveToRight}(\text{Leq})(1\text{term}) \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{y}. \forall \underline{z}. \underline{y} \leq \underline{z} \vdash$   
 $\text{plus0Left} \gg (0 + \underline{y}) = \underline{y}; \text{eqSymmetry} \triangleright (0 + \underline{y}) = \underline{y} \gg \underline{y} =$   
 $(0 + \underline{y}); \text{subLeqLeft} \triangleright \underline{y} = (0 + \underline{y}) \triangleright \underline{y} \leq \underline{z} \gg (0 + \underline{y}) \leq$   
 $\underline{z}; \text{PositiveToRight}(\text{Leq}) \triangleright (0 + \underline{y}) \leq \underline{z} \gg 0 \leq (\underline{z} + (-\underline{uy})) \rceil, p_0, c)]$

$[\text{NegativeToLeft}(\text{Eq}) \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}. \forall \underline{y}. \forall \underline{z}. \underline{x} = (\underline{y} + (-\underline{uz})) \vdash (\underline{x} + \underline{z}) = \underline{y}]$

$[\text{NegativeToLeft}(\text{Eq}) \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}. \forall \underline{y}. \forall \underline{z}. \underline{x} = (\underline{y} + (-\underline{uz})) \vdash$   
 $\text{eqAddition} \triangleright \underline{x} = (\underline{y} + (-\underline{uz})) \gg (\underline{x} + \underline{z}) = ((\underline{y} + (-\underline{uz})) + \underline{z}); \text{Three2threeTerms} \gg$   
 $((\underline{y} + (-\underline{uz})) + \underline{z}) = ((\underline{y} + \underline{z}) + (-\underline{uz})); \underline{x} = \underline{x} + \underline{y} - \underline{y} \gg \underline{y} =$   
 $((\underline{y} + \underline{z}) + (-\underline{uz})); \text{eqSymmetry} \triangleright \underline{y} = ((\underline{y} + \underline{z}) + (-\underline{uz})) \gg ((\underline{y} + \underline{z}) + (-\underline{uz})) =$   
 $\underline{y}; \text{eqTransitivity4} \triangleright (\underline{x} + \underline{z}) = ((\underline{y} + (-\underline{uz})) + \underline{z}) \triangleright ((\underline{y} + (-\underline{uz})) + \underline{z}) =$   
 $((\underline{y} + \underline{z}) + (-\underline{uz})) \triangleright ((\underline{y} + \underline{z}) + (-\underline{uz})) = \underline{y} \gg (\underline{x} + \underline{z}) = \underline{y} \rceil, p_0, c)]$

$[\text{SubtractEquations} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}. \forall \underline{y}. \forall \underline{z}. \forall \underline{u}. (\underline{x} + \underline{z}) = (\underline{y} + \underline{u}) \vdash \underline{z} = \underline{u} \vdash$   
 $\underline{x} = \underline{y}]$

$[\text{SubtractEquations} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}. \forall \underline{y}. \forall \underline{z}. \forall \underline{u}. (\underline{x} + \underline{z}) = (\underline{y} + \underline{u}) \vdash$   
 $\underline{z} = \underline{u} \vdash \text{eqAddition} \triangleright (\underline{x} + \underline{z}) = (\underline{y} + \underline{u}) \gg ((\underline{x} + \underline{z}) + (-\underline{uz})) =$   
 $((\underline{y} + \underline{u}) + (-\underline{uz})); \text{plus0Left} \gg (0 + \underline{z}) = \underline{z}; \text{eqTransitivity} \triangleright (0 + \underline{z}) = \underline{z} \triangleright \underline{z} =$   
 $\underline{u} \gg (0 + \underline{z}) = \underline{u}; \text{PositiveToRight}(\text{Eq}) \triangleright (0 + \underline{z}) = \underline{u} \gg 0 =$   
 $(\underline{u} + (-\underline{uz})); \text{eqSymmetry} \triangleright 0 = (\underline{u} + (-\underline{uz})) \gg (\underline{u} + (-\underline{uz})) =$   
 $0; \text{EqAdditionLeft} \triangleright (\underline{u} + (-\underline{uz})) = 0 \gg (\underline{y} + (\underline{u} + (-\underline{uz}))) =$   
 $(\underline{y} + 0); \text{plusAssociativity} \gg ((\underline{y} + \underline{u}) + (-\underline{uz})) = (\underline{y} + (\underline{u} + (-\underline{uz}))); \text{plus0} \gg (\underline{y} + 0) =$   
 $\underline{y}; \text{eqTransitivity4} \triangleright ((\underline{y} + \underline{u}) + (-\underline{uz})) = (\underline{y} + (\underline{u} + (-\underline{uz}))) \triangleright (\underline{y} + (\underline{u} + (-\underline{uz}))) =$   
 $(\underline{y} + 0) \triangleright (\underline{y} + 0) = \underline{y} \gg ((\underline{y} + \underline{u}) + (-\underline{uz})) = \underline{y}; \underline{x} = \underline{x} + \underline{y} - \underline{y} \gg \underline{x} =$   
 $((\underline{x} + \underline{z}) + (-\underline{uz})); \text{eqTransitivity4} \triangleright \underline{x} = ((\underline{x} + \underline{z}) + (-\underline{uz})) \triangleright ((\underline{x} + \underline{z}) + (-\underline{uz})) =$   
 $((\underline{y} + \underline{u}) + (-\underline{uz})) \triangleright ((\underline{y} + \underline{u}) + (-\underline{uz})) = \underline{y} \gg \underline{x} = \underline{y} \rceil, p_0, c)]$

$[\text{SubtractEquationsLeft} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}. \forall \underline{y}. \forall \underline{z}. \forall \underline{u}. (\underline{x} + \underline{z}) = (\underline{y} + \underline{u}) \vdash \underline{x} =$   
 $\underline{y} \vdash \underline{z} = \underline{u}]$

$[\text{SubtractEquationsLeft} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}. \forall \underline{y}. \forall \underline{z}. \forall \underline{u}. (\underline{x} + \underline{z}) =$   
 $(\underline{y} + \underline{u}) \vdash \underline{x} = \underline{y} \vdash \text{plusCommutativity} \gg (\underline{z} + \underline{x}) =$   
 $(\underline{x} + \underline{z}); \text{plusCommutativity} \gg (\underline{y} + \underline{u}) = (\underline{u} + \underline{y}); \text{eqTransitivity4} \triangleright (\underline{z} + \underline{x}) =$   
 $(\underline{x} + \underline{z}) \triangleright (\underline{x} + \underline{z}) = (\underline{y} + \underline{u}) \triangleright (\underline{y} + \underline{u}) = (\underline{u} + \underline{y}) \gg (\underline{z} + \underline{x}) =$   
 $(\underline{u} + \underline{y}); \text{SubtractEquations} \triangleright (\underline{z} + \underline{x}) = (\underline{u} + \underline{y}) \triangleright \underline{x} = \underline{y} \gg \underline{z} = \underline{u} \rceil, p_0, c)]$

$[\text{EqNegated} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}. \forall \underline{y}. \underline{x} = \underline{y} \vdash (-\underline{ux}) = (-\underline{uy})]$

$[\text{EqNegated} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}. \forall \underline{y}. \underline{x} = \underline{y} \vdash \text{Negative} \gg$   
 $(\underline{x} + (-\underline{ux})) = 0; \text{Negative} \gg (\underline{y} + (-\underline{uy})) = 0; \text{eqSymmetry} \triangleright (\underline{y} + (-\underline{uy})) =$   
 $0 \gg 0 = (\underline{y} + (-\underline{uy})); \text{eqTransitivity} \triangleright (\underline{x} + (-\underline{ux})) = 0 \triangleright 0 = (\underline{y} + (-\underline{uy})) \gg$   
 $(\underline{x} + (-\underline{ux})) = (\underline{y} + (-\underline{uy})); \text{SubtractEquationsLeft} \triangleright (\underline{x} + (-\underline{ux})) =$   
 $(\underline{y} + (-\underline{uy})) \triangleright \underline{x} = \underline{y} \gg (-\underline{ux}) = (-\underline{uy}) \rceil, p_0, c)]$

**(\*\*\* NO EQUALITY \*\*\*)**

$[\text{LessNeq} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}. \forall \underline{y}. \dot{\vdash} (\underline{x} \leq \underline{y}) \Rightarrow \dot{\vdash} (\dot{\vdash} (\underline{x} = \underline{y})n)n \vdash \dot{\vdash} (\underline{x} = \underline{y})n]$

$[\text{LessNeq} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}. \forall \underline{y}. \dot{\vdash} (\underline{x} \leq \underline{y}) \Rightarrow \dot{\vdash} (\dot{\vdash} (\underline{x} = \underline{y})n)n \vdash$   
 $\text{Repetition} \triangleright \dot{\vdash} (\underline{x} \leq \underline{y}) \Rightarrow \dot{\vdash} (\dot{\vdash} (\underline{x} = \underline{y})n)n \gg \dot{\vdash} (\underline{x} \leq \underline{y}) \Rightarrow \dot{\vdash} (\dot{\vdash} (\underline{x} =$

$\underline{y})n)n$ ; SecondConjunct  $\triangleright \dot{\vdash} (\underline{x} \leq \underline{y} \Rightarrow \dot{\vdash} (\dot{\vdash} (\underline{x} = \underline{y})n)n) \gg \dot{\vdash} (\underline{x} = \underline{y})n$ ,  $p_0, c$ )  
 $[\underline{x} + \underline{y} = \underline{z}$ Backwards  $\xrightarrow{\text{stmt}}$  SystemQ  $\vdash \forall \underline{x}: \forall \underline{y}: \forall \underline{z}: (\underline{x} + \underline{y}) = \underline{z} \vdash \underline{z} = (\underline{y} + \underline{x})$ ]  
 $[\underline{x} + \underline{y} = \underline{z}$ Backwards  $\xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \forall \underline{z}: (\underline{x} + \underline{y}) = \underline{z} \vdash$   
plusCommutativity  $\gg (\underline{x} + \underline{y}) = (\underline{y} + \underline{x});$  Equality  $\triangleright (\underline{x} + \underline{y}) = \underline{z} \gg \underline{z} =$   
 $(\underline{y} + \underline{x}) \rceil, p_0, c)$ ]  
 $[\underline{x} * \underline{y} = \underline{z}$ Backwards  $\xrightarrow{\text{stmt}}$  SystemQ  $\vdash \forall \underline{x}: \forall \underline{y}: \forall \underline{z}: (\underline{x} * \underline{y}) = \underline{z} \vdash \underline{z} = (\underline{y} * \underline{x})$ ]  
 $[\underline{x} * \underline{y} = \underline{z}$ Backwards  $\xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \forall \underline{z}: (\underline{x} * \underline{y}) = \underline{z} \vdash$   
timesCommutativity  $\gg (\underline{x} * \underline{y}) = (\underline{y} * \underline{x});$  Equality  $\triangleright (\underline{x} * \underline{y}) = \underline{z} \gg \underline{z} = (\underline{y} * \underline{x}) \rceil, p_0, c)$ ]  
 $[\text{DoubleMinus} \xrightarrow{\text{stmt}}$  SystemQ  $\vdash \forall \underline{x}: (-u(-u\underline{x})) = \underline{x}]$   
 $[\text{DoubleMinus} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}: \text{Negative} \gg$   
 $((-u\underline{x}) + (-u(-u\underline{x}))) = 0; \underline{x} + \underline{y} = \underline{z}$ Backwards  $\triangleright ((-u\underline{x}) + (-u(-u\underline{x}))) = 0 \gg$   
 $0 = ((-u(-u\underline{x})) + (-u\underline{x}));$  NegativeToLeft(Eq)  $\triangleright 0 = ((-u(-u\underline{x})) + (-u\underline{x})) \gg$   
 $(0 + \underline{x}) = (-u(-u\underline{x}));$  plus0Left  $\gg (0 + \underline{x}) = \underline{x};$  Equality  $\triangleright (0 + \underline{x}) =$   
 $(-u(-u\underline{x})) \triangleright (0 + \underline{x}) = \underline{x} \gg (-u(-u\underline{x})) = \underline{x} \rceil, p_0, c)$ ]  
 $[\text{NeqNegated} \xrightarrow{\text{stmt}}$  SystemQ  $\vdash \forall \underline{x}: \forall \underline{y}: \dot{\vdash} (\underline{x} = \underline{y})n \vdash \dot{\vdash} ((-u\underline{x}) = (-u\underline{y}))n$ ]  
 $[\text{NeqNegated} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \dot{\vdash} (\underline{x} = \underline{y})n \vdash (-u\underline{x}) = (-u\underline{y}) \vdash$   
EqNegated  $\triangleright (-u\underline{x}) = (-u\underline{y}) \gg (-u(-u\underline{x})) = (-u(-u\underline{y}));$  DoubleMinus  $\gg$   
 $(-u(-u\underline{x})) = \underline{x};$  eqSymmetry  $\triangleright (-u(-u\underline{x})) = \underline{x} \gg \underline{x} =$   
 $(-u(-u\underline{x}));$  DoubleMinus  $\gg (-u(-u\underline{y})) = \underline{y};$  eqTransitivity4  $\triangleright \underline{x} =$   
 $(-u(-u\underline{x})) \triangleright (-u(-u\underline{x})) = (-u(-u\underline{y})) \triangleright (-u(-u\underline{y})) = \underline{y} \gg \underline{x} =$   
 $\underline{y};$  FromContradiction  $\triangleright \underline{x} = \underline{y} \triangleright \dot{\vdash} (\underline{x} = \underline{y})n \gg \dot{\vdash} ((-u\underline{x}) =$   
 $(-u\underline{y}))n; \forall \underline{x}: \forall \underline{y}: \text{Ded} \triangleright \forall \underline{x}: \forall \underline{y}: \dot{\vdash} (\underline{x} = \underline{y})n \vdash (-u\underline{x}) = (-u\underline{y}) \vdash \dot{\vdash} ((-u\underline{x}) =$   
 $(-u\underline{y}))n \gg \dot{\vdash} (\underline{x} = \underline{y})n \Rightarrow (-u\underline{x}) = (-u\underline{y}) \Rightarrow \dot{\vdash} ((-u\underline{x}) = (-u\underline{y}))n; \dot{\vdash} (\underline{x} = \underline{y})n \vdash$   
MP  $\triangleright \dot{\vdash} (\underline{x} = \underline{y})n \Rightarrow (-u\underline{x}) = (-u\underline{y}) \Rightarrow \dot{\vdash} ((-u\underline{x}) = (-u\underline{y}))n \triangleright \dot{\vdash} (\underline{x} = \underline{y})n \gg$   
 $(-u\underline{x}) = (-u\underline{y}) \Rightarrow \dot{\vdash} ((-u\underline{x}) = (-u\underline{y}))n; \text{prop lemma imply negation} \triangleright (-u\underline{x}) =$   
 $(-u\underline{y}) \Rightarrow \dot{\vdash} ((-u\underline{x}) = (-u\underline{y}))n \gg \dot{\vdash} ((-u\underline{x}) = (-u\underline{y}))n \rceil, p_0, c)$ ]  
 $[\text{SubNeqRight} \xrightarrow{\text{stmt}}$  SystemQ  $\vdash \forall \underline{x}: \forall \underline{y}: \forall \underline{z}: \underline{x} = \underline{y} \vdash \dot{\vdash} (\underline{z} = \underline{x})n \vdash \dot{\vdash} (\underline{z} = \underline{y})n$ ]  
 $[\text{SubNeqRight} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \forall \underline{z}: \underline{x} = \underline{y} \vdash \dot{\vdash} (\underline{z} = \underline{x})n \vdash$   
NeqSymmetry  $\triangleright \dot{\vdash} (\underline{z} = \underline{x})n \gg \dot{\vdash} (\underline{x} = \underline{z})n; \text{SubNeqLeft} \triangleright \underline{x} = \underline{y} \triangleright \dot{\vdash} (\underline{x} = \underline{z})n \gg$   
 $\dot{\vdash} (\underline{y} = \underline{z})n; \text{NeqSymmetry} \triangleright \dot{\vdash} (\underline{y} = \underline{z})n \gg \dot{\vdash} (\underline{z} = \underline{y})n \rceil, p_0, c)$ ]  
 $[\text{NeqAddition} \xrightarrow{\text{stmt}}$  SystemQ  $\vdash \forall \underline{x}: \forall \underline{y}: \forall \underline{z}: \dot{\vdash} (\underline{x} = \underline{y})n \vdash \dot{\vdash} ((\underline{x} + \underline{z}) = (\underline{y} + \underline{z}))n$ ]  
 $[\text{NeqAddition} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \forall \underline{z}: \dot{\vdash} (\underline{x} = \underline{y})n \vdash (\underline{x} + \underline{z}) =$   
 $(\underline{y} + \underline{z}) \vdash \text{eqReflexivity} \gg \underline{z} = \underline{z}; \text{SubtractEquations} \triangleright (\underline{x} + \underline{z}) = (\underline{y} + \underline{z}) \triangleright \underline{z} =$   
 $\underline{z} \gg \underline{x} = \underline{y}; \text{FromContradiction} \triangleright \underline{x} = \underline{y} \triangleright \dot{\vdash} (\underline{x} = \underline{y})n \gg \dot{\vdash} ((\underline{x} + \underline{z}) =$   
 $(\underline{y} + \underline{z}))n; \forall \underline{x}: \forall \underline{y}: \forall \underline{z}: \text{Ded} \triangleright \forall \underline{x}: \forall \underline{y}: \forall \underline{z}: \dot{\vdash} (\underline{x} = \underline{y})n \vdash (\underline{x} + \underline{z}) = (\underline{y} + \underline{z}) \vdash \dot{\vdash} ((\underline{x} + \underline{z}) =$   
 $(\underline{y} + \underline{z}))n \gg \dot{\vdash} (\underline{x} = \underline{y})n \Rightarrow (\underline{x} + \underline{z}) = (\underline{y} + \underline{z}) \Rightarrow \dot{\vdash} ((\underline{x} + \underline{z}) = (\underline{y} + \underline{z}))n; \dot{\vdash} (\underline{x} = \underline{y})n \vdash$   
MP  $\triangleright \dot{\vdash} (\underline{x} = \underline{y})n \Rightarrow (\underline{x} + \underline{z}) = (\underline{y} + \underline{z}) \Rightarrow \dot{\vdash} ((\underline{x} + \underline{z}) = (\underline{y} + \underline{z}))n \triangleright \dot{\vdash} (\underline{x} = \underline{y})n \gg$   
 $(\underline{x} + \underline{z}) = (\underline{y} + \underline{z}) \Rightarrow \dot{\vdash} ((\underline{x} + \underline{z}) = (\underline{y} + \underline{z}))n; \text{prop lemma imply negation} \triangleright (\underline{x} + \underline{z}) =$   
 $(\underline{y} + \underline{z}) \Rightarrow \dot{\vdash} ((\underline{x} + \underline{z}) = (\underline{y} + \underline{z}))n \gg \dot{\vdash} ((\underline{x} + \underline{z}) = (\underline{y} + \underline{z}))n \rceil, p_0, c)$ ]  
 $[\text{NeqMultiplication} \xrightarrow{\text{stmt}}$  SystemQ  $\vdash \forall \underline{x}: \forall \underline{y}: \forall \underline{z}: \dot{\vdash} (\underline{z} = 0)n \vdash \dot{\vdash} (\underline{x} = \underline{y})n \vdash$   
 $\dot{\vdash} ((\underline{x} * \underline{z}) = (\underline{y} * \underline{z}))n$ ]











[negativeToLeft(Leq)(1term)  $\xrightarrow{\text{stmt}}$  SystemQ  $\vdash \forall \underline{y}. \forall \underline{z}. 0 \leq (\underline{y} + (-\underline{uz})) \vdash \underline{z} \leq \underline{y}$ ]

[negativeToLeft(Leq)(1term)  $\xrightarrow{\text{proof}}$   $\lambda c. \lambda x. \lambda \mathcal{P}(\ulcorner \text{SystemQ} \vdash \forall \underline{y}. \forall \underline{z}. 0 \leq (\underline{y} + (-\underline{uz})) \vdash \text{negativeToLeft(Leq)} \triangleright 0 \leq (\underline{y} + (-\underline{uz})) \gg (0 + \underline{z}) \leq \underline{y}; \text{plus0Left} \gg (0 + \underline{z}) = \underline{z}; \text{subLeqLeft} \triangleright (0 + \underline{z}) = \underline{z} \gg \underline{z} \leq \underline{y} \urcorner, p_0, c)$ ]

[PositiveToLeft(Leq)  $\xrightarrow{\text{stmt}}$  SystemQ  $\vdash \forall \underline{x}. \forall \underline{y}. \forall \underline{z}. \underline{x} \leq (\underline{y} + \underline{z}) \vdash (\underline{x} + (-\underline{uz})) \leq \underline{y}$ ]

[PositiveToLeft(Leq)  $\xrightarrow{\text{proof}}$   $\lambda c. \lambda x. \lambda \mathcal{P}(\ulcorner \text{SystemQ} \vdash \forall \underline{x}. \forall \underline{y}. \forall \underline{z}. \underline{x} \leq (\underline{y} + \underline{z}) \vdash \text{leqAddition} \triangleright \underline{x} \leq (\underline{y} + \underline{z}) \gg (\underline{x} + (-\underline{uz})) \leq ((\underline{y} + \underline{z}) + (-\underline{uz})); \underline{x} = \underline{x} + \underline{y} - \underline{y} \gg \underline{y} = ((\underline{y} + \underline{z}) + (-\underline{uz})); \text{eqSymmetry} \triangleright \underline{y} = ((\underline{y} + \underline{z}) + (-\underline{uz})) \gg ((\underline{y} + \underline{z}) + (-\underline{uz})) = \underline{y}; \text{subLeqRight} \triangleright ((\underline{y} + \underline{z}) + (-\underline{uz})) = \underline{y} \triangleright (\underline{x} + (-\underline{uz})) \leq ((\underline{y} + \underline{z}) + (-\underline{uz})) \gg (\underline{x} + (-\underline{uz})) \leq \underline{y} \urcorner, p_0, c)$ ]

[thirdGeq  $\xrightarrow{\text{stmt}}$  SystemQ  $\vdash \forall \underline{x}. \forall \underline{y}. \dot{\vdash} (\underline{x} \leq c_{\text{Ex}} \Rightarrow \dot{\vdash} (\underline{y} \leq c_{\text{Ex}})n)n]$

[thirdGeq  $\xrightarrow{\text{proof}}$   $\lambda c. \lambda x. \lambda \mathcal{P}(\ulcorner \text{SystemQ} \vdash \forall \underline{x}. \forall \underline{y}. \underline{x} \leq \underline{y} \vdash \text{leqReflexivity} \gg \underline{y} \leq \underline{y}; \text{JoinConjuncts} \triangleright \underline{x} \leq \underline{y} \triangleright \underline{y} \leq \underline{y} \gg \dot{\vdash} (\underline{x} \leq \underline{y} \Rightarrow \dot{\vdash} (\underline{y} \leq \underline{y})n)n; \text{ExistIntro} @ c_{\text{Ex}} @ \underline{y} \triangleright \dot{\vdash} (\underline{x} \leq \underline{y} \Rightarrow \dot{\vdash} (\underline{y} \leq \underline{y})n)n \gg \dot{\vdash} (\underline{x} \leq c_{\text{Ex}} \Rightarrow \dot{\vdash} (\underline{y} \leq c_{\text{Ex}})n)n; \forall \underline{x}. \forall \underline{y}. \underline{y} \leq \underline{x} \vdash \text{leqReflexivity} \gg \underline{x} \leq \underline{x}; \text{JoinConjuncts} \triangleright \underline{x} \leq \underline{x} \triangleright \underline{y} \leq \underline{x} \gg \dot{\vdash} (\underline{x} \leq \underline{x} \Rightarrow \dot{\vdash} (\underline{y} \leq \underline{x})n)n; \text{ExistIntro} @ c_{\text{Ex}} @ \underline{x} \triangleright \dot{\vdash} (\underline{x} \leq \underline{x} \Rightarrow \dot{\vdash} (\underline{y} \leq \underline{x})n)n \gg \dot{\vdash} (\underline{x} \leq c_{\text{Ex}} \Rightarrow \dot{\vdash} (\underline{y} \leq c_{\text{Ex}})n)n; \forall \underline{x}. \forall \underline{y}. \text{Ded} \triangleright \forall \underline{x}. \forall \underline{y}. \underline{x} \leq \underline{y} \vdash \dot{\vdash} (\underline{x} \leq c_{\text{Ex}} \Rightarrow \dot{\vdash} (\underline{y} \leq c_{\text{Ex}})n)n \gg \underline{x} \leq \underline{y} \Rightarrow \dot{\vdash} (\underline{x} \leq c_{\text{Ex}} \Rightarrow \dot{\vdash} (\underline{y} \leq c_{\text{Ex}})n)n; \text{Ded} \triangleright \forall \underline{x}. \forall \underline{y}. \underline{y} \leq \underline{x} \vdash \dot{\vdash} (\underline{x} \leq c_{\text{Ex}} \Rightarrow \dot{\vdash} (\underline{y} \leq c_{\text{Ex}})n)n \gg \underline{y} \leq \underline{x} \Rightarrow \dot{\vdash} (\underline{x} \leq c_{\text{Ex}} \Rightarrow \dot{\vdash} (\underline{y} \leq c_{\text{Ex}})n)n; \text{leqTotality} \gg \dot{\vdash} (\underline{x} \leq \underline{y})n \Rightarrow \underline{y} \leq \underline{x}; \text{FromDisjuncts} \triangleright \dot{\vdash} (\underline{x} \leq \underline{y})n \Rightarrow \underline{y} \leq \underline{x} \triangleright \underline{x} \leq \underline{y} \Rightarrow \dot{\vdash} (\underline{x} \leq c_{\text{Ex}} \Rightarrow \dot{\vdash} (\underline{y} \leq c_{\text{Ex}})n)n \triangleright \underline{y} \leq \underline{x} \Rightarrow \dot{\vdash} (\underline{x} \leq c_{\text{Ex}} \Rightarrow \dot{\vdash} (\underline{y} \leq c_{\text{Ex}})n)n \gg \dot{\vdash} (\underline{x} \leq c_{\text{Ex}} \Rightarrow \dot{\vdash} (\underline{y} \leq c_{\text{Ex}})n)n \urcorner, p_0, c)$ ]

[LeqNegated  $\xrightarrow{\text{stmt}}$  SystemQ  $\vdash \forall \underline{x}. \forall \underline{y}. \underline{x} \leq \underline{y} \vdash (-\underline{uy}) \leq (-\underline{ux})]$

[LeqNegated  $\xrightarrow{\text{proof}}$   $\lambda c. \lambda x. \lambda \mathcal{P}(\ulcorner \text{SystemQ} \vdash \forall \underline{x}. \forall \underline{y}. \underline{x} \leq \underline{y} \vdash \text{leqAddition} \triangleright \underline{x} \leq \underline{y} \gg (\underline{x} + (-\underline{ux})) \leq (\underline{y} + (-\underline{ux})); \text{Negative} \gg (\underline{x} + (-\underline{ux})) = 0; \text{subLeqLeft} \triangleright (\underline{x} + (-\underline{ux})) = 0 \triangleright (\underline{x} + (-\underline{ux})) \leq (\underline{y} + (-\underline{ux})) \gg 0 \leq (\underline{y} + (-\underline{ux})); \text{plusCommutativity} \gg (\underline{y} + (-\underline{ux})) = ((-\underline{ux}) + \underline{y}); \text{subLeqRight} \triangleright (\underline{y} + (-\underline{ux})) = ((-\underline{ux}) + \underline{y}) \triangleright 0 \leq (\underline{y} + (-\underline{ux})) \gg 0 \leq ((-\underline{ux}) + \underline{y}); \text{leqAddition} \triangleright 0 \leq ((-\underline{ux}) + \underline{y}) \gg (0 + (-\underline{uy})) \leq (((-\underline{ux}) + \underline{y}) + (-\underline{uy})); \text{plus0Left} \gg (0 + (-\underline{uy})) = (-\underline{uy}); \underline{x} = \underline{x} + \underline{y} - \underline{y} \gg (-\underline{ux}) = (((-\underline{ux}) + \underline{y}) + (-\underline{uy})); \text{eqSymmetry} \triangleright (-\underline{ux}) = (((-\underline{ux}) + \underline{y}) + (-\underline{uy})) \gg (((-\underline{ux}) + \underline{y}) + (-\underline{uy})) = (-\underline{ux}); \text{subLeqLeft} \triangleright (0 + (-\underline{uy})) = (-\underline{uy}) \triangleright (0 + (-\underline{uy})) \leq (((-\underline{ux}) + \underline{y}) + (-\underline{uy})) \gg (-\underline{uy}) \leq (((-\underline{ux}) + \underline{y}) + (-\underline{uy})); \text{subLeqRight} \triangleright (((-\underline{ux}) + \underline{y}) + (-\underline{uy})) = (-\underline{ux}) \triangleright (-\underline{uy}) \leq (((-\underline{ux}) + \underline{y}) + (-\underline{uy})) \gg (-\underline{uy}) \leq (-\underline{ux}) \urcorner, p_0, c)$ ]

[AddEquations(Leq)  $\xrightarrow{\text{stmt}}$  SystemQ  $\vdash \forall \underline{x}. \forall \underline{y}. \forall \underline{z}. \forall \underline{u}. \underline{x} \leq \underline{y} \vdash \underline{z} \leq \underline{u} \vdash (\underline{x} + \underline{z}) \leq (\underline{y} + \underline{u})]$

[AddEquations(Leq)  $\xrightarrow{\text{proof}}$   $\lambda c. \lambda x. \lambda \mathcal{P}(\ulcorner \text{SystemQ} \vdash \forall \underline{x}. \forall \underline{y}. \forall \underline{z}. \forall \underline{u}. \underline{x} \leq \underline{y} \vdash \underline{z} \leq \underline{u} \vdash \text{leqAddition} \triangleright \underline{x} \leq \underline{y} \gg (\underline{x} + \underline{z}) \leq (\underline{y} + \underline{z}); \text{LeqAdditionLeft} \triangleright \underline{z} \leq \underline{u} \gg (\underline{y} + \underline{z}) \leq (\underline{y} + \underline{u}); \text{leqTransitivity} \triangleright (\underline{x} + \underline{z}) \leq (\underline{y} + \underline{z}) \triangleright (\underline{y} + \underline{z}) \leq (\underline{y} + \underline{u}) \gg (\underline{x} + \underline{z}) \leq (\underline{y} + \underline{u}) \urcorner, p_0, c)$ ]





$$\begin{aligned}
& [\text{NegativeLessPositive} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}: \dot{\vdash} (0 \leq \underline{x} \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \underline{x})n)n) \vdash \text{FirstConjunct} \triangleright \dot{\vdash} (0 \leq \underline{x} \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \underline{x})n)n) \gg 0 \leq \underline{x}; \text{leqAddition} \triangleright 0 \leq \underline{x} \gg (0 + (-\underline{ux})) \leq (\underline{x} + (-\underline{ux})); \text{plus0Left} \gg (0 + (-\underline{ux})) = (-\underline{ux}); \text{Negative} \gg (\underline{x} + (-\underline{ux})) = 0; \text{subLeqLeft} \triangleright (0 + (-\underline{ux})) = (-\underline{ux}) \triangleright (0 + (-\underline{ux})) \leq (\underline{x} + (-\underline{ux})) \gg (-\underline{ux}) \leq (\underline{x} + (-\underline{ux})); \text{subLeqRight} \triangleright (\underline{x} + (-\underline{ux})) = 0 \triangleright (-\underline{ux}) \leq (\underline{x} + (-\underline{ux})) \gg (-\underline{ux}) \leq 0; \text{leqLessTransitivity} \triangleright (-\underline{ux}) \leq 0 \triangleright \dot{\vdash} (0 \leq \underline{x} \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \underline{x})n)n) \gg \dot{\vdash} ((-\underline{ux}) \leq \underline{x} \Rightarrow \dot{\vdash} (\dot{\vdash} ((-\underline{ux}) = \underline{x})n)n) \rceil, p_0, c)] \\
& [\text{LessNegated} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \dot{\vdash} (\underline{x} \leq \underline{y} \Rightarrow \dot{\vdash} (\dot{\vdash} (\underline{x} = \underline{y})n)n) \vdash \dot{\vdash} ((-\underline{uy}) \leq (-\underline{ux}) \Rightarrow \dot{\vdash} (\dot{\vdash} ((-\underline{uy}) = (-\underline{ux}))n)n)] \\
& [\text{LessNegated} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \dot{\vdash} (\underline{x} \leq \underline{y} \Rightarrow \dot{\vdash} (\dot{\vdash} (\underline{x} = \underline{y})n)n) \vdash \text{LessLeq} \triangleright \dot{\vdash} (\underline{x} \leq \underline{y} \Rightarrow \dot{\vdash} (\dot{\vdash} (\underline{x} = \underline{y})n)n) \gg \underline{x} \leq \underline{y}; \text{LeqNegated} \triangleright \underline{x} \leq \underline{y} \gg (-\underline{uy}) \leq (-\underline{ux}); \text{LessNeq} \triangleright \dot{\vdash} (\underline{x} \leq \underline{y} \Rightarrow \dot{\vdash} (\dot{\vdash} (\underline{x} = \underline{y})n)n) \gg \dot{\vdash} (\underline{x} = \underline{y})n; \text{NeqNegated} \triangleright \dot{\vdash} (\underline{x} = \underline{y})n \gg \dot{\vdash} ((-\underline{ux}) = (-\underline{uy}))n; \text{NeqSymmetry} \triangleright \dot{\vdash} ((-\underline{ux}) = (-\underline{uy}))n \gg \dot{\vdash} ((-\underline{uy}) = (-\underline{ux}))n; \text{LeqNeqLess} \triangleright (-\underline{uy}) \leq (-\underline{ux}) \triangleright \dot{\vdash} ((-\underline{uy}) = (-\underline{ux}))n \gg \dot{\vdash} ((-\underline{uy}) \leq (-\underline{ux}) \Rightarrow \dot{\vdash} (\dot{\vdash} ((-\underline{uy}) = (-\underline{ux}))n)n) \rceil, p_0, c)] \\
& [-0 = 0 \xrightarrow{\text{stmt}} \text{SystemQ} \vdash (-u0) = 0] \\
& [-0 = 0 \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \text{Negative} \gg (0 + (-u0)) = 0; \text{plus0} \gg (0 + 0) = 0; \text{UniqueNegative} \triangleright (0 + (-u0)) = 0 \triangleright (0 + 0) = 0 \gg (-u0) = 0 \rceil, p_0, c)] \\
& [\text{PositiveNegated} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: \dot{\vdash} (0 \leq \underline{x} \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \underline{x})n)n) \vdash \dot{\vdash} ((-\underline{ux}) \leq 0 \Rightarrow \dot{\vdash} (\dot{\vdash} ((-\underline{ux}) = 0)n)n)] \\
& [\text{PositiveNegated} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}: \dot{\vdash} (0 \leq \underline{x} \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \underline{x})n)n) \vdash \text{LessNegated} \triangleright \dot{\vdash} (0 \leq \underline{x} \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \underline{x})n)n) \gg \dot{\vdash} ((-\underline{ux}) \leq (-u0) \Rightarrow \dot{\vdash} (\dot{\vdash} ((-\underline{ux}) = (-u0))n)n); -0 = 0 \gg (-u0) = 0; \text{SubLessRight} \triangleright (-u0) = 0 \triangleright \dot{\vdash} ((-\underline{ux}) \leq (-u0) \Rightarrow \dot{\vdash} (\dot{\vdash} ((-\underline{ux}) = (-u0))n)n) \gg \dot{\vdash} ((-\underline{ux}) \leq 0 \Rightarrow \dot{\vdash} (\dot{\vdash} ((-\underline{ux}) = 0)n)n) \rceil, p_0, c)] \\
& [\text{NonpositiveNegated} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: \underline{x} \leq 0 \vdash 0 \leq (-\underline{ux})] \\
& [\text{NonpositiveNegated} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}: \underline{x} \leq 0 \vdash \text{LeqNegated} \triangleright \underline{x} \leq 0 \gg (-u0) \leq (-\underline{ux}); -0 = 0 \gg (-u0) = 0; \text{subLeqLeft} \triangleright (-u0) = 0 \triangleright (-u0) \leq (-\underline{ux}) \gg 0 \leq (-\underline{ux}) \rceil, p_0, c)] \\
& [\text{NegativeNegated} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: \dot{\vdash} (\underline{x} \leq 0 \Rightarrow \dot{\vdash} (\dot{\vdash} (\underline{x} = 0)n)n) \vdash \dot{\vdash} (0 \leq (-\underline{ux}) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = (-\underline{ux}))n)n)] \\
& [\text{NegativeNegated} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}: \dot{\vdash} (\underline{x} \leq 0 \Rightarrow \dot{\vdash} (\dot{\vdash} (\underline{x} = 0)n)n) \vdash \text{LessNegated} \triangleright \dot{\vdash} (\underline{x} \leq 0 \Rightarrow \dot{\vdash} (\dot{\vdash} (\underline{x} = 0)n)n) \gg \dot{\vdash} ((-\underline{u0}) \leq (-\underline{ux}) \Rightarrow \dot{\vdash} (\dot{\vdash} ((-\underline{u0}) = (-\underline{ux}))n)n); -0 = 0 \gg (-u0) = 0; \text{SubLessLeft} \triangleright (-u0) = 0 \triangleright \dot{\vdash} ((-\underline{u0}) \leq (-\underline{ux}) \Rightarrow \dot{\vdash} (\dot{\vdash} ((-\underline{u0}) = (-\underline{ux}))n)n) \gg \dot{\vdash} (0 \leq (-\underline{ux}) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = (-\underline{ux}))n)n) \rceil, p_0, c)] \\
& [\text{NonnegativeNegated} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: 0 \leq \underline{x} \vdash (-\underline{ux}) \leq 0] \\
& [\text{NonnegativeNegated} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}: 0 \leq \underline{x} \vdash \text{LeqNegated} \triangleright 0 \leq \underline{x} \gg (-\underline{ux}) \leq (-u0); -0 = 0 \gg (-u0) = 0; \text{subLeqRight} \triangleright (-u0) = 0 \triangleright (-\underline{ux}) \leq (-u0) \gg (-\underline{ux}) \leq 0 \rceil, p_0, c)]
\end{aligned}$$

$[0 < 2 \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \dot{\vdash} (0 \leq (1 + 1) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = (1 + 1))n)n)n]$   
 $[0 < 2 \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash 0 < 1 \gg \dot{\vdash} (0 \leq 1 \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = 1)n)n)n; \text{LessAddition} \triangleright \dot{\vdash} (0 \leq 1 \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = 1)n)n)n \gg \dot{\vdash} ((0 + 1) \leq (1 + 1) \Rightarrow \dot{\vdash} (\dot{\vdash} ((0 + 1) = (1 + 1))n)n)n; \text{plus0Left} \gg (0 + 1) = 1; \text{SubLessLeft} \triangleright (0 + 1) = 1 \triangleright \dot{\vdash} ((0 + 1) \leq (1 + 1) \Rightarrow \dot{\vdash} (\dot{\vdash} ((0 + 1) = (1 + 1))n)n)n \gg \dot{\vdash} (1 < = (1 + 1) \Rightarrow \dot{\vdash} (\dot{\vdash} (1 = (1 + 1))n)n)n; \text{LessTransitivity} \triangleright \dot{\vdash} (0 \leq 1 \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = 1)n)n)n \triangleright \dot{\vdash} (1 \leq (1 + 1) \Rightarrow \dot{\vdash} (\dot{\vdash} (1 = (1 + 1))n)n)n \gg \dot{\vdash} (0 \leq (1 + 1) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = (1 + 1))n)n)n \rceil, p_0, c)]$

$[0 < 1/2 \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \dot{\vdash} (0 \leq \text{rec}(1 + 1) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \text{rec}(1 + 1))n)n)n]$   
 $[0 < 1/2 \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash 0 < 2 \gg \dot{\vdash} (0 \leq (1 + 1) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = (1 + 1))n)n)n; \text{FirstConjunct} \triangleright \dot{\vdash} (0 \leq (1 + 1) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = (1 + 1))n)n)n \gg 0 \leq (1 + 1); \text{SecondConjunct} \triangleright \dot{\vdash} (0 \leq (1 + 1) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = (1 + 1))n)n)n \gg \dot{\vdash} (0 = (1 + 1))n; \text{NeqSymmetry} \triangleright \dot{\vdash} (0 = (1 + 1))n \gg \dot{\vdash} ((1 + 1) = 0)n; 0 < 1 \gg \dot{\vdash} (0 \leq 1 \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = 1)n)n)n; x * 0 = 0 \gg ((1 + 1) * 0) = 0; x * y = z \text{Backwards} \triangleright ((1 + 1) * 0) = 0 \gg 0 = (0 * (1 + 1)); \text{SubLessLeft} \triangleright 0 = (0 * (1 + 1)) \triangleright \dot{\vdash} (0 \leq 1 \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = 1)n)n)n \gg \dot{\vdash} ((0 * (1 + 1)) \leq 1 \Rightarrow \dot{\vdash} (\dot{\vdash} ((0 * (1 + 1)) = 1)n)n)n; \text{Reciprocal} \triangleright \dot{\vdash} ((1 + 1) = 0)n \gg ((1 + 1) * \text{rec}(1 + 1)) = 1; x * y = z \text{Backwards} \triangleright ((1 + 1) * \text{rec}(1 + 1)) = 1 \gg 1 = (\text{rec}(1 + 1) * (1 + 1)); \text{SubLessRight} \triangleright 1 = (\text{rec}(1 + 1) * (1 + 1)) \triangleright \dot{\vdash} ((0 * (1 + 1)) \leq 1 \Rightarrow \dot{\vdash} (\dot{\vdash} ((0 * (1 + 1)) = 1)n)n)n \gg \dot{\vdash} (0 * (1 + 1)) \leq (\text{rec}(1 + 1) * (1 + 1)) \Rightarrow \dot{\vdash} (\dot{\vdash} ((0 * (1 + 1)) = (\text{rec}(1 + 1) * (1 + 1)))n)n)n; \text{LessDivision} \triangleright 0 \leq (1 + 1) \triangleright \dot{\vdash} ((0 * (1 + 1)) \leq (\text{rec}(1 + 1) * (1 + 1)) \Rightarrow \dot{\vdash} (\dot{\vdash} ((0 * (1 + 1)) = (\text{rec}(1 + 1) * (1 + 1)))n)n)n \gg \dot{\vdash} (0 \leq \text{rec}(1 + 1) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \text{rec}(1 + 1))n)n)n \rceil, p_0, c)]$

$[\text{PositiveHalved} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}. \dot{\vdash} (0 \leq \underline{x} \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \underline{x})n)n)n \vdash \dot{\vdash} (0 \leq (\text{rec}(1 + 1) * \underline{x}) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = (\text{rec}(1 + 1) * \underline{x}))n)n)n]$   
 $[\text{PositiveHalved} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}. \dot{\vdash} (0 \leq \underline{x} \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \underline{x})n)n)n \vdash 0 < 1/2 \gg \dot{\vdash} (0 \leq \text{rec}(1 + 1) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \text{rec}(1 + 1))n)n)n; \text{LessMultiplicationLeft} \triangleright \dot{\vdash} (0 \leq \text{rec}(1 + 1) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \text{rec}(1 + 1))n)n)n \triangleright \dot{\vdash} (0 \leq \underline{x} \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \underline{x})n)n)n \gg \dot{\vdash} ((\text{rec}(1 + 1) * 0) \leq (\text{rec}(1 + 1) * \underline{x}) \Rightarrow \dot{\vdash} (\dot{\vdash} ((\text{rec}(1 + 1) * 0) = (\text{rec}(1 + 1) * \underline{x}))n)n)n; x * 0 = 0 \gg (\text{rec}(1 + 1) * 0) = 0; \text{SubLessLeft} \triangleright (\text{rec}(1 + 1) * 0) = 0 \triangleright \dot{\vdash} ((\text{rec}(1 + 1) * 0) \leq (\text{rec}(1 + 1) * \underline{x}) \Rightarrow \dot{\vdash} (\dot{\vdash} ((\text{rec}(1 + 1) * 0) = (\text{rec}(1 + 1) * \underline{x}))n)n)n \gg \dot{\vdash} (0 \leq (\text{rec}(1 + 1) * \underline{x}) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = (\text{rec}(1 + 1) * \underline{x}))n)n)n \rceil, p_0, c)]$

$[\text{FromNot} \ll \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall (\underline{fx}): \forall (\underline{fy}): \dot{\vdash} (\dot{\vdash} (\forall \text{obj} \overline{\epsilon}): \dot{\vdash} (\dot{\vdash} (\forall \text{obj} \overline{n}: \dot{\vdash} (\forall \text{obj} \overline{m}: \dot{\vdash} (\dot{\vdash} (0 \leq \overline{\epsilon}) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \overline{\epsilon}))n)n)n \Rightarrow \dot{\vdash} (\overline{n} \leq \overline{m} \Rightarrow \underline{fx}[\overline{m}] \leq ((\underline{fy})[\overline{m}] + (-u(\overline{\epsilon})))n)n)n)n)n) \vdash \dot{\vdash} (\dot{\vdash} (\forall \text{obj} \overline{\epsilon}): \dot{\vdash} (\dot{\vdash} (\forall \text{obj} \overline{n}: \dot{\vdash} (\forall \text{obj} \overline{m}: \dot{\vdash} (\dot{\vdash} (0 \leq \overline{\epsilon}) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \overline{\epsilon}))n)n)n \Rightarrow \dot{\vdash} (\overline{n} \leq \overline{m} \Rightarrow \underline{fx}[\overline{m}] \leq ((\underline{fy})[\overline{m}] + (-u(\overline{\epsilon})))n)n)n)n)n)]$

$[\text{FromNot} \ll \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall (\underline{fx}): \forall (\underline{fy}): \text{AutoImPLY} \gg \dot{\vdash} (\forall \text{obj} \overline{\epsilon}): \dot{\vdash} (\dot{\vdash} (\forall \text{obj} \overline{n}: \dot{\vdash} (\forall \text{obj} \overline{m}: \dot{\vdash} (\dot{\vdash} (0 \leq \overline{\epsilon}) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \overline{\epsilon}))n)n)n \Rightarrow \dot{\vdash} (\overline{n} \leq \overline{m} \Rightarrow \underline{fx}[\overline{m}] \leq ((\underline{fy})[\overline{m}] + (-u(\overline{\epsilon})))n)n)n)n) \Rightarrow \dot{\vdash} (\forall \text{obj} \overline{\epsilon}): \dot{\vdash} (\dot{\vdash} (\forall \text{obj} \overline{n}: \dot{\vdash} (\forall \text{obj} \overline{m}: \dot{\vdash} (\dot{\vdash} (0 \leq \overline{\epsilon}) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \overline{\epsilon}))n)n)n \Rightarrow$





$0; \text{eqSymmetry} \triangleright (-u0) = 0 \gg 0 = (-u0); \text{EqNegated} \triangleright 0 = \underline{x} \gg (-u0) = (-\underline{ux}); \text{eqTransitivity5} \triangleright |\underline{x}| = \underline{x} \triangleright \underline{x} = 0 \triangleright 0 = (-u0) \triangleright (-u0) = (-\underline{ux}) \gg |\underline{x}| = (-\underline{ux}); \forall \underline{x}: \text{Ded} \triangleright \forall \underline{x}: \dot{\neg}(\underline{x} <= 0 \Rightarrow \dot{\neg}(\dot{\neg}(\underline{x} = 0)n)n) \vdash |\underline{x}| = (-\underline{ux}) \gg \dot{\neg}(\underline{x} <= 0 \Rightarrow \dot{\neg}(\dot{\neg}(\underline{x} = 0)n)n) \Rightarrow |\underline{x}| = (-\underline{ux}); \text{Ded} \triangleright \forall \underline{x}: \underline{x} = 0 \vdash |\underline{x}| = (-\underline{ux}) \gg \underline{x} = 0 \Rightarrow |\underline{x}| = (-\underline{ux}); \underline{x} <= 0 \vdash \text{LeqLessEq} \triangleright \underline{x} <= 0 \gg \dot{\neg}(\dot{\neg}(\underline{x} <= 0 \Rightarrow \dot{\neg}(\dot{\neg}(\underline{x} = 0)n)n) \Rightarrow \underline{x} = 0); \text{FromDisjuncts} \triangleright \dot{\neg}(\dot{\neg}(\underline{x} <= 0 \Rightarrow \dot{\neg}(\dot{\neg}(\underline{x} = 0)n)n) \Rightarrow \underline{x} = 0) \triangleright \dot{\neg}(\underline{x} <= 0 \Rightarrow \dot{\neg}(\dot{\neg}(\underline{x} = 0)n)n) \Rightarrow |\underline{x}| = (-\underline{ux}) \triangleright \underline{x} = 0 \Rightarrow |\underline{x}| = (-\underline{ux}) \gg |\underline{x}| = (-\underline{ux}), p_0, c]$

$[|0| = 0 \xrightarrow{\text{stmt}} \text{SystemQ} \vdash |0| = 0]$

$[|0| = 0 \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \text{leqReflexivity} \gg 0 <= 0; \text{NonnegativeNumerical} \triangleright 0 <= 0 \gg |0| = 0 \rceil, p_0, c)]$

$[0 <= |\underline{x}| \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: 0 <= |\underline{x}|]$

$[0 <= |\underline{x}| \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}: 0 <= \underline{x} \vdash \text{NonnegativeNumerical} \triangleright 0 <= \underline{x} \gg |\underline{x}| = \underline{x}; \text{eqSymmetry} \triangleright |\underline{x}| = \underline{x} \gg \underline{x} = |\underline{x}|; \text{subLeqRight} \triangleright \underline{x} = |\underline{x}| \triangleright 0 <= \underline{x} \gg 0 <= |\underline{x}|; \forall \underline{x}: \dot{\neg}(0 <= \underline{x}) \vdash \text{ToLess} \triangleright \dot{\neg}(0 <= \underline{x}) \gg \dot{\neg}(\underline{x} <= 0 \Rightarrow \dot{\neg}(\dot{\neg}(\underline{x} = 0)n)n); \text{NegativeNumerical} \triangleright \dot{\neg}(\underline{x} <= 0 \Rightarrow \dot{\neg}(\dot{\neg}(\underline{x} = 0)n)n) \gg |\underline{x}| = (-\underline{ux}); \text{eqSymmetry} \triangleright |\underline{x}| = (-\underline{ux}) \gg (-\underline{ux}) = |\underline{x}|; \text{NegativeNegated} \triangleright \dot{\neg}(\underline{x} <= 0 \Rightarrow \dot{\neg}(\dot{\neg}(\underline{x} = 0)n)n) \gg \dot{\neg}(0 <= (-\underline{ux}) \Rightarrow \dot{\neg}(\dot{\neg}(0 = (-\underline{ux})n)n)); \text{LessLeq} \triangleright \dot{\neg}(0 <= (-\underline{ux}) \Rightarrow \dot{\neg}(\dot{\neg}(0 = (-\underline{ux})n)n) \gg 0 <= (-\underline{ux}); \text{subLeqRight} \triangleright (-\underline{ux}) = |\underline{x}| \triangleright 0 <= (-\underline{ux}) \gg 0 <= |\underline{x}|; \forall \underline{x}: \text{Ded} \triangleright \forall \underline{x}: 0 <= \underline{x} \vdash 0 <= |\underline{x}| \gg 0 <= \underline{x} \Rightarrow 0 <= |\underline{x}|; \text{Ded} \triangleright \forall \underline{x}: \dot{\neg}(0 <= \underline{x}) \vdash 0 <= |\underline{x}| \gg \dot{\neg}(0 <= \underline{x}) \Rightarrow 0 <= |\underline{x}|; \text{FromNegations} \triangleright 0 <= \underline{x} \Rightarrow 0 <= |\underline{x}| \triangleright \dot{\neg}(0 <= \underline{x}) \Rightarrow 0 <= |\underline{x}| \gg 0 <= |\underline{x}| \rceil, p_0, c)]$

$[\text{SameNumerical} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \underline{x} = \underline{y} \vdash |\underline{x}| = |\underline{y}|]$

$[\text{SameNumerical} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: 0 <= \underline{x} \vdash \underline{x} = \underline{y} \vdash \text{NonnegativeNumerical} \triangleright 0 <= \underline{x} \gg |\underline{x}| = \underline{x}; \text{subLeqRight} \triangleright \underline{x} = \underline{y} \triangleright 0 <= \underline{x} \gg 0 <= \underline{y}; \text{NonnegativeNumerical} \triangleright 0 <= \underline{y} \gg |\underline{y}| = \underline{y}; \text{eqSymmetry} \triangleright |\underline{y}| = \underline{y} \gg \underline{y} = |\underline{y}|; \text{eqTransitivity4} \triangleright |\underline{x}| = \underline{x} \triangleright \underline{x} = \underline{y} \triangleright \underline{y} = |\underline{y}| \gg |\underline{x}| = |\underline{y}|; \forall \underline{x}: \forall \underline{y}: \dot{\neg}(0 <= \underline{x}) \vdash \underline{x} = \underline{y} \vdash \text{ToLess} \triangleright \dot{\neg}(0 <= \underline{x}) \gg \dot{\neg}(\underline{x} <= 0 \Rightarrow \dot{\neg}(\dot{\neg}(\underline{x} = 0)n)n); \text{NegativeNumerical} \triangleright \dot{\neg}(\underline{x} <= 0 \Rightarrow \dot{\neg}(\dot{\neg}(\underline{x} = 0)n)n) \gg |\underline{x}| = (-\underline{ux}); \text{SubLessLeft} \triangleright \underline{x} = \underline{y} \triangleright \dot{\neg}(\underline{x} <= 0 \Rightarrow \dot{\neg}(\dot{\neg}(\underline{x} = 0)n)n) \gg \dot{\neg}(\underline{y} <= 0 \Rightarrow \dot{\neg}(\dot{\neg}(\underline{y} = 0)n)n); \text{NegativeNumerical} \triangleright \dot{\neg}(\underline{y} <= 0 \Rightarrow \dot{\neg}(\dot{\neg}(\underline{y} = 0)n)n) \gg |\underline{y}| = (-\underline{uy}); \text{eqSymmetry} \triangleright |\underline{y}| = (-\underline{uy}) \gg (-\underline{uy}) = |\underline{y}|; \text{EqNegated} \triangleright \underline{x} = \underline{y} \gg (-\underline{ux}) = (-\underline{uy}); \text{eqTransitivity4} \triangleright |\underline{x}| = (-\underline{ux}) \triangleright (-\underline{ux}) = (-\underline{uy}) \triangleright (-\underline{uy}) = |\underline{y}| \gg |\underline{x}| = |\underline{y}|; \forall \underline{x}: \forall \underline{y}: \underline{x} = \underline{y} \vdash \text{Ded} \triangleright \forall \underline{x}: \forall \underline{y}: 0 <= \underline{x} \vdash \underline{x} = \underline{y} \vdash |\underline{x}| = |\underline{y}| \gg 0 <= \underline{x} \Rightarrow \underline{x} = \underline{y} \Rightarrow |\underline{x}| = |\underline{y}|; \text{Ded} \triangleright \forall \underline{x}: \forall \underline{y}: \dot{\neg}(0 <= \underline{x}) \vdash \underline{x} = \underline{y} \vdash |\underline{x}| = |\underline{y}| \gg \dot{\neg}(0 <= \underline{x}) \Rightarrow \underline{x} = \underline{y} \Rightarrow |\underline{x}| = |\underline{y}|; \text{FromNegations} \triangleright 0 <= \underline{x} \Rightarrow \underline{x} = \underline{y} \Rightarrow |\underline{x}| = |\underline{y}| \triangleright \dot{\neg}(0 <= \underline{x}) \Rightarrow \underline{x} = \underline{y} \Rightarrow |\underline{x}| = |\underline{y}| \gg \underline{x} = \underline{y} \Rightarrow |\underline{x}| = |\underline{y}|; \text{MP} \triangleright \underline{x} = \underline{y} \Rightarrow |\underline{x}| = |\underline{y}| \triangleright \underline{x} = \underline{y} \gg |\underline{x}| = |\underline{y}| \rceil, p_0, c)]$

$[\text{SameSeries}(\text{Gen}) \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{m}: \forall \underline{n}: \forall (\underline{fx}): \forall (\underline{fy}): \forall (\underline{sz}): \underline{m} \in \mathbb{N} \vdash \underline{n} \in \mathbb{N} \vdash \dot{\neg}(\dot{\neg}(\forall_{\text{obj}}(\underline{r1}): (\underline{r1}) \in (\underline{fx}) \Rightarrow \dot{\neg}(\forall_{\text{obj}}(\underline{op1}): \dot{\neg}(\dot{\neg}(\forall_{\text{obj}}(\underline{op2}): \dot{\neg}(\dot{\neg}(\dot{\neg}(\underline{op1}) \in \mathbb{N} \Rightarrow \dot{\neg}(\underline{op2}) \in (\underline{sz}))n)n) \Rightarrow \dot{\neg}(\underline{r1}) =$

$\{\{\underline{op1}\}, \underline{op1}\}, \{\{\underline{op1}\}, \underline{op2}\})n)n)n) \Rightarrow \dot{\neg}(\forall_{\text{obj}}(\underline{f1}): \forall_{\text{obj}}(\underline{f2}): \forall_{\text{obj}}(\underline{f3}): \forall_{\text{obj}}(\underline{f4}): \{\{\underline{f1}\}, \underline{f1}\}, \{\underline{f1}\}, \underline{f2}) \in (\underline{fx}) \Rightarrow$







$\underline{x}; \text{eqSymmetry} \triangleright |(-\underline{ux})| = \underline{x} \gg \underline{x} = |(-\underline{ux})|; \text{eqTransitivity} \triangleright |\underline{x}| = \underline{x} \triangleright \underline{x} = |(-\underline{ux})| \gg |\underline{x}| = |(-\underline{ux})|, p_0, c]$   
 $[\text{SignNumerical} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: |\underline{x}| = |(-\underline{ux})|]$   
 $[\text{SignNumerical} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \forall \underline{x}: \dot{\vdash} (\underline{x} \leq 0 \Rightarrow \dot{\vdash} (\dot{\vdash} (\underline{x} = 0)n)n)n \vdash \text{NegativeNegated} \triangleright \dot{\vdash} (\underline{x} \leq 0 \Rightarrow \dot{\vdash} (\dot{\vdash} (\underline{x} = 0)n)n)n \gg \dot{\vdash} (0 \leq (-\underline{ux}) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = (-\underline{ux})n)n)n); \text{SignNumerical}(+) \triangleright \dot{\vdash} (0 \leq (-\underline{ux}) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = (-\underline{ux})n)n)n \gg |(-\underline{ux})| = |(-u(-\underline{ux}))|; \text{DoubleMinus} \gg (-u(-\underline{ux})) = \underline{x}; \text{SameNumerical} \triangleright (-u(-\underline{ux})) = \underline{x} \gg |(-u(-\underline{ux}))| = |\underline{x}|; \text{eqTransitivity} \triangleright |(-\underline{ux})| = |(-u(-\underline{ux}))| \triangleright |(-u(-\underline{ux}))| = |\underline{x}| \gg |(-\underline{ux})| = |\underline{x}|; \text{eqSymmetry} \triangleright |(-\underline{ux})| = |\underline{x}| \gg |\underline{x}| = |(-\underline{ux})|; \forall \underline{x}: \underline{x} = 0 \vdash \text{EqNegated} \triangleright \underline{x} = 0 \gg (-\underline{ux}) = (-u0); -0 = 0 \gg (-u0) = 0; \text{eqSymmetry} \triangleright \underline{x} = 0 \gg 0 = \underline{x}; \text{eqTransitivity4} \triangleright (-\underline{ux}) = (-u0) \triangleright (-u0) = 0 \triangleright 0 = \underline{x} \gg (-\underline{ux}) = \underline{x}; \text{eqSymmetry} \triangleright (-\underline{ux}) = \underline{x} \gg \underline{x} = (-\underline{ux}); \text{SameNumerical} \triangleright \underline{x} = (-\underline{ux}) \gg |\underline{x}| = |(-\underline{ux})|; \forall \underline{x}: \dot{\vdash} (0 \leq \underline{x} \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \underline{x})n)n)n \vdash \text{SignNumerical}(+) \triangleright \dot{\vdash} (0 \leq \underline{x} \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \underline{x})n)n)n \gg |\underline{x}| = |(-\underline{ux})|; \forall \underline{x}: \text{Ded} \triangleright \forall \underline{x}: \dot{\vdash} (\underline{x} \leq 0 \Rightarrow \dot{\vdash} (\dot{\vdash} (\underline{x} = 0)n)n)n \vdash |\underline{x}| = |(-\underline{ux})| \gg \dot{\vdash} (\underline{x} \leq 0 \Rightarrow \dot{\vdash} (\dot{\vdash} (\underline{x} = 0)n)n)n \Rightarrow |\underline{x}| = |(-\underline{ux})|; \text{Ded} \triangleright \forall \underline{x}: \underline{x} = 0 \vdash |\underline{x}| = |(-\underline{ux})| \gg \underline{x} = 0 \Rightarrow |\underline{x}| = |(-\underline{ux})|; \text{Ded} \triangleright \forall \underline{x}: \dot{\vdash} (0 \leq \underline{x} \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \underline{x})n)n)n \vdash |\underline{x}| = |(-\underline{ux})| \gg \dot{\vdash} (0 \leq \underline{x} \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \underline{x})n)n)n \Rightarrow |\underline{x}| = |(-\underline{ux})|; \text{LessTotality} \gg \dot{\vdash} (\dot{\vdash} (\underline{x} \leq 0 \Rightarrow \dot{\vdash} (\dot{\vdash} (\underline{x} = 0)n)n)n) \Rightarrow \dot{\vdash} (\underline{x} = 0)n \Rightarrow \dot{\vdash} (0 \leq \underline{x} \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \underline{x})n)n)n); \text{From3Disjuncts} \triangleright \dot{\vdash} (\dot{\vdash} (\underline{x} \leq 0 \Rightarrow \dot{\vdash} (\dot{\vdash} (\underline{x} = 0)n)n)n) \Rightarrow \dot{\vdash} (\underline{x} = 0)n \Rightarrow \dot{\vdash} (0 \leq \underline{x} \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \underline{x})n)n)n \triangleright \dot{\vdash} (\underline{x} \leq 0 \Rightarrow \dot{\vdash} (\dot{\vdash} (\underline{x} = 0)n)n)n \Rightarrow |\underline{x}| = |(-\underline{ux})| \triangleright \underline{x} = 0 \Rightarrow |\underline{x}| = |(-\underline{ux})| \triangleright \dot{\vdash} (0 \leq \underline{x} \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \underline{x})n)n)n \Rightarrow |\underline{x}| = |(-\underline{ux})| \gg |\underline{x}| = |(-\underline{ux})|, p_0, c]$   
 $[\text{Times}(-1) \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: (\underline{x} * (-u1)) = (-\underline{ux})]$   
 $[\text{Times}(-1) \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \forall \underline{x}: \text{Negative} \gg (1 + (-u1)) = 0; \text{plusCommutativity} \gg ((-u1) + 1) = (1 + (-u1)); \text{eqTransitivity} \triangleright ((-u1) + 1) = (1 + (-u1)) \triangleright (1 + (-u1)) = 0 \gg ((-u1) + 1) = 0; \text{EqMultiplicationLeft} \triangleright ((-u1) + 1) = 0 \gg (\underline{x} * ((-u1) + 1)) = (\underline{x} * 0); \underline{x} * 0 = 0 \gg (\underline{x} * 0) = 0; \text{eqTransitivity} \triangleright (\underline{x} * ((-u1) + 1)) = (\underline{x} * 0) \triangleright (\underline{x} * 0) = 0 \gg (\underline{x} * ((-u1) + 1)) = 0; \text{Distribution} \gg (\underline{x} * ((-u1) + 1)) = ((\underline{x} * (-u1)) + (\underline{x} * 1)); \text{eqSymmetry} \triangleright (\underline{x} * ((-u1) + 1)) = ((\underline{x} * (-u1)) + (\underline{x} * 1)) \gg ((\underline{x} * (-u1)) + (\underline{x} * 1)) = (\underline{x} * ((-u1) + 1)); \text{eqTransitivity} \triangleright ((\underline{x} * (-u1)) + (\underline{x} * 1)) = (\underline{x} * ((-u1) + 1)) \triangleright (\underline{x} * ((-u1) + 1)) = 0 \gg ((\underline{x} * (-u1)) + (\underline{x} * 1)) = 0; \text{PositiveToRight}(\text{Eq}) \triangleright ((\underline{x} * (-u1)) + (\underline{x} * 1)) = 0 \gg (\underline{x} * (-u1)) = (0 + (-u(\underline{x} * 1))); \text{plus0Left} \gg (0 + (-u(\underline{x} * 1))) = (-u(\underline{x} * 1)); \text{eqTransitivity} \triangleright (\underline{x} * (-u1)) = (0 + (-u(\underline{x} * 1))) \triangleright (0 + (-u(\underline{x} * 1))) = (-u(\underline{x} * 1)) \gg (\underline{x} * (-u1)) = (-u(\underline{x} * 1)); \text{times1} \gg (\underline{x} * 1) = \underline{x}; \text{EqNegated} \triangleright (\underline{x} * 1) = \underline{x} \gg (-u(\underline{x} * 1)) = (-\underline{ux}); \text{eqTransitivity} \triangleright (\underline{x} * (-u1)) = (-u(\underline{x} * 1)) \triangleright (-u(\underline{x} * 1)) = (-\underline{ux}) \gg (\underline{x} * (-u1)) = (-\underline{ux})], p_0, c]$   
 $[\text{Times}(-1)\text{Left} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: ((-u1) * \underline{x}) = (-\underline{ux})]$   
 $[\text{Times}(-1)\text{Left} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \forall \underline{x}: \text{Times}(-1) \gg (\underline{x} * (-u1)) = (-\underline{ux}); \text{timesCommutativity} \gg ((-u1) * \underline{x}) = (\underline{x} * (-u1)); \text{eqTransitivity} \triangleright ((-u1) * \underline{x}) = (\underline{x} * (-u1)) \triangleright (\underline{x} * (-u1)) = (-\underline{ux}) \gg ((-u1) * \underline{x}) = (-\underline{ux})], p_0, c)]$

$$\begin{aligned}
& [-x - y = -(x + y) \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: ((-\underline{ux}) + (-\underline{uy})) = (-\underline{u(x + y)})] \\
& [-x - y = -(x + y) \xrightarrow{\text{proof}} \lambda c. \lambda x. \lambda y. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \text{Times}(-1)\text{Left} \gg \\
& ((-u1) * \underline{x}) = (-\underline{ux}); \text{Times}(-1)\text{Left} \gg ((-u1) * \underline{y}) = (-\underline{uy}); \text{AddEquations} \triangleright \\
& ((-u1) * \underline{x}) = (-\underline{ux}) \triangleright ((-u1) * \underline{y}) = (-\underline{uy}) \gg (((-u1) * \underline{x}) + ((-u1) * \underline{y})) = \\
& ((-\underline{ux}) + (-\underline{uy})); \text{eqSymmetry} \triangleright (((-u1) * \underline{x}) + ((-u1) * \underline{y})) = ((-\underline{ux}) + (-\underline{uy})) \gg \\
& ((-\underline{ux}) + (-\underline{uy})) = (((-u1) * \underline{x}) + ((-u1) * \underline{y})); \text{DistributionOut} \gg \\
& (((-u1) * \underline{x}) + ((-u1) * \underline{y})) = (-u1) * (\underline{x} + \underline{y}); \text{Times}(-1)\text{Left} \gg \\
& (-u1) * (\underline{x} + \underline{y}) = (-\underline{u(x + y)}); \text{eqTransitivity4} \triangleright ((-\underline{ux}) + (-\underline{uy})) = \\
& (((-u1) * \underline{x}) + ((-u1) * \underline{y})) \triangleright (((-u1) * \underline{x}) + ((-u1) * \underline{y})) = (-u1) * (\underline{x} + \underline{y}) \triangleright \\
& (-u1) * (\underline{x} + \underline{y}) = (-\underline{u(x + y)}) \gg ((-\underline{ux}) + (-\underline{uy})) = (-\underline{u(x + y)}), p_0, c \rceil]
\end{aligned}$$

$$[\text{MinusNegated} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: (-\underline{u(x + (-\underline{uy}))}) = (\underline{y} + (-\underline{ux}))]$$

$$\begin{aligned}
& [\text{MinusNegated} \xrightarrow{\text{proof}} \lambda c. \lambda x. \lambda y. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \text{DoubleMinus} \gg (-\underline{u(-\underline{uy}))} = \\
& \underline{y}; \text{eqAddition} \triangleright (-\underline{u(-\underline{uy}))} = \underline{y} \gg ((-\underline{u(-\underline{uy}))} + (-\underline{ux})) = \\
& (\underline{y} + (-\underline{ux})); \text{eqSymmetry} \triangleright ((-\underline{u(-\underline{uy}))} + (-\underline{ux})) = (\underline{y} + (-\underline{ux})) \gg \\
& (\underline{y} + (-\underline{ux})) = ((-\underline{u(-\underline{uy}))} + (-\underline{ux})); -x - y = -(x + y) \gg \\
& ((-\underline{u(-\underline{uy}))} + (-\underline{ux})) = (-\underline{u((- \underline{uy}) + \underline{x})); \text{plusCommutativity} \gg ((-\underline{uy}) + \underline{x}) = \\
& (\underline{x} + (-\underline{uy})); \text{EqNegated} \triangleright ((-\underline{uy}) + \underline{x}) = (\underline{x} + (-\underline{uy})) \gg (-\underline{u((- \underline{uy}) + \underline{x})) = \\
& (-\underline{u(\underline{x} + (-\underline{uy}))); \text{eqTransitivity4} \triangleright (\underline{y} + (-\underline{ux})) = ((-\underline{u(-\underline{uy}))} + (-\underline{ux})) \triangleright \\
& ((-\underline{u(-\underline{uy}))} + (-\underline{ux})) = (-\underline{u((- \underline{uy}) + \underline{x})) \triangleright (-\underline{u((- \underline{uy}) + \underline{x}))} = \\
& (-\underline{u(\underline{x} + (-\underline{uy}))) \gg (\underline{y} + (-\underline{ux})) = (-\underline{u(\underline{x} + (-\underline{uy}))); \text{eqSymmetry} \triangleright (\underline{y} + (-\underline{ux})) = \\
& (-\underline{u(\underline{x} + (-\underline{uy}))) \gg (-\underline{u(\underline{x} + (-\underline{uy})))} = (\underline{y} + (-\underline{ux})), p_0, c \rceil]
\end{aligned}$$

$$[\text{NumericalDifference} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: |(\underline{x} + (-\underline{uy}))| = |(\underline{y} + (-\underline{ux}))|]$$

$$\begin{aligned}
& [\text{NumericalDifference} \xrightarrow{\text{proof}} \lambda c. \lambda x. \lambda y. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \text{SignNumerical} \gg \\
& |(\underline{x} + (-\underline{uy}))| = |(-\underline{u(\underline{x} + (-\underline{uy})))|; \text{MinusNegated} \gg (-\underline{u(\underline{x} + (-\underline{uy})))} = \\
& (\underline{y} + (-\underline{ux})); \text{SameNumerical} \triangleright (-\underline{u(\underline{x} + (-\underline{uy})))} = (\underline{y} + (-\underline{ux})) \gg \\
& |(-\underline{u(\underline{x} + (-\underline{uy})))| = |(\underline{y} + (-\underline{ux}))|; \text{eqTransitivity} \triangleright |(\underline{x} + (-\underline{uy}))| = \\
& |(-\underline{u(\underline{x} + (-\underline{uy})))| \triangleright |(-\underline{u(\underline{x} + (-\underline{uy})))| = |(\underline{y} + (-\underline{ux}))| \gg |(\underline{x} + (-\underline{uy}))| = \\
& |(\underline{y} + (-\underline{ux}))|, p_0, c \rceil]
\end{aligned}$$

$$[\text{SplitNumericalSumHelper} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: |((-\underline{ux}) + (-\underline{uy}))| \leq (|(-\underline{ux})| + |(-\underline{uy})|) \vdash |(\underline{x} + \underline{y})| \leq (|\underline{x}| + |\underline{y}|)]$$

$$\begin{aligned}
& [\text{SplitNumericalSumHelper} \xrightarrow{\text{proof}} \lambda c. \lambda x. \lambda y. \mathcal{P}(\lceil \text{SystemQ} \vdash \\
& \forall \underline{x}: \forall \underline{y}: |((-\underline{ux}) + (-\underline{uy}))| \leq (|(-\underline{ux})| + |(-\underline{uy})|) \vdash \text{SignNumerical} \gg |\underline{x}| = \\
& |(-\underline{ux})|; \text{SignNumerical} \gg |\underline{y}| = |(-\underline{uy})|; \text{AddEquations} \triangleright |\underline{x}| = |(-\underline{ux})| \triangleright |\underline{y}| = \\
& |(-\underline{uy})| \gg (|\underline{x}| + |\underline{y}|) = (|(-\underline{ux})| + |(-\underline{uy})|); \text{eqSymmetry} \triangleright (|\underline{x}| + |\underline{y}|) = \\
& (|(-\underline{ux})| + |(-\underline{uy})|) \gg (|(-\underline{ux})| + |(-\underline{uy})|) = (|\underline{x}| + |\underline{y}|); -x - y = -(x + y) \gg \\
& ((-\underline{ux}) + (-\underline{uy})) = (-\underline{u(x + y)}); \text{SameNumerical} \triangleright ((-\underline{ux}) + (-\underline{uy})) = \\
& (-\underline{u(\underline{x} + \underline{y})}) \gg |((-\underline{ux}) + (-\underline{uy}))| = |(-\underline{u(\underline{x} + \underline{y})})|; \text{SignNumerical} \gg |(\underline{x} + \underline{y})| = \\
& |(-\underline{u(\underline{x} + \underline{y})})|; \text{eqSymmetry} \triangleright |(\underline{x} + \underline{y})| = |(-\underline{u(\underline{x} + \underline{y})})| \gg |(-\underline{u(\underline{x} + \underline{y})})| = \\
& |(\underline{x} + \underline{y})|; \text{eqTransitivity} \triangleright |((-\underline{ux}) + (-\underline{uy}))| = |(-\underline{u(\underline{x} + \underline{y})})| \triangleright |(-\underline{u(\underline{x} + \underline{y})})| = \\
& |(\underline{x} + \underline{y})| \gg |((-\underline{ux}) + (-\underline{uy}))| = |(\underline{x} + \underline{y})|; \text{subLeqRight} \triangleright (|(-\underline{ux})| + |(-\underline{uy})|) = \\
& (|\underline{x}| + |\underline{y}|) \triangleright |((-\underline{ux}) + (-\underline{uy}))| \leq (|(-\underline{ux})| + |(-\underline{uy})|) \gg |((-\underline{ux}) + (-\underline{uy}))| \leq \\
& (|\underline{x}| + |\underline{y}|); \text{subLeqLeft} \triangleright |((-\underline{ux}) + (-\underline{uy}))| = |(\underline{x} + \underline{y})| \triangleright |((-\underline{ux}) + (-\underline{uy}))| \leq \\
& (|\underline{x}| + |\underline{y}|) \gg |(\underline{x} + \underline{y})| \leq (|\underline{x}| + |\underline{y}|), p_0, c \rceil]
\end{aligned}$$

$[\text{splitNumericalSum}(++) \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: 0 \leq \underline{x} \vdash 0 \leq \underline{y} \vdash$   
 $|\underline{x} + \underline{y}| \leq (|\underline{x}| + |\underline{y}|)]$

$[\text{splitNumericalSum}(++) \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: 0 \leq \underline{x} \vdash 0 \leq \underline{y} \vdash$   
 $\text{AddEquations}(\text{Leq}) \triangleright 0 \leq \underline{x} \triangleright 0 \leq \underline{y} \gg (0 + 0) \leq (\underline{x} + \underline{y}); \text{plus0} \gg$   
 $(0 + 0) = 0; \text{subLeqLeft} \triangleright (0 + 0) = 0 \triangleright (0 + 0) \leq (\underline{x} + \underline{y}) \gg 0 \leq$   
 $(\underline{x} + \underline{y}); \text{NonnegativeNumerical} \triangleright 0 \leq (\underline{x} + \underline{y}) \gg |\underline{x} + \underline{y}| =$   
 $(\underline{x} + \underline{y}); \text{NonnegativeNumerical} \triangleright 0 \leq \underline{x} \gg |\underline{x}| =$   
 $\underline{x}; \text{NonnegativeNumerical} \triangleright 0 \leq \underline{y} \gg |\underline{y}| = \underline{y}; \text{AddEquations} \triangleright |\underline{x}| = \underline{x} \triangleright |\underline{y}| =$   
 $\underline{y} \gg (|\underline{x}| + |\underline{y}|) = (\underline{x} + \underline{y}); \text{eqSymmetry} \triangleright (|\underline{x}| + |\underline{y}|) = (\underline{x} + \underline{y}) \gg (\underline{x} + \underline{y}) =$   
 $(|\underline{x}| + |\underline{y}|); \text{eqTransitivity} \triangleright |(\underline{x} + \underline{y})| = (\underline{x} + \underline{y}) \triangleright (\underline{x} + \underline{y}) = (|\underline{x}| + |\underline{y}|) \gg |(\underline{x} + \underline{y})| =$   
 $(|\underline{x}| + |\underline{y}|); \text{eqLeq} \triangleright |(\underline{x} + \underline{y})| = (|\underline{x}| + |\underline{y}|) \gg |(\underline{x} + \underline{y})| \leq (|\underline{x}| + |\underline{y}|), \text{Po}, c)]$

$[\text{splitNumericalSum}(--) \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \underline{x} \leq 0 \vdash \underline{y} \leq 0 \vdash$   
 $|\underline{x} + \underline{y}| \leq (|\underline{x}| + |\underline{y}|)]$

$[\text{splitNumericalSum}(--) \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \underline{x} \leq 0 \vdash \underline{y} \leq 0 \vdash$   
 $\text{NonpositiveNegated} \triangleright \underline{x} \leq 0 \gg 0 \leq (-\underline{u}\underline{x}); \text{NonpositiveNegated} \triangleright \underline{y} \leq 0 \gg$   
 $0 \leq (-\underline{u}\underline{y}); \text{splitNumericalSum}(++) \triangleright 0 \leq (-\underline{u}\underline{x}) \triangleright 0 \leq (-\underline{u}\underline{y}) \gg$   
 $|((-\underline{u}\underline{x}) + (-\underline{u}\underline{y}))| \leq (|(-\underline{u}\underline{x})| + |(-\underline{u}\underline{y})|); \text{SplitNumericalSumHelper} \triangleright$   
 $|((-\underline{u}\underline{x}) + (-\underline{u}\underline{y}))| \leq (|(-\underline{u}\underline{x})| + |(-\underline{u}\underline{y})|) \gg |(\underline{x} + \underline{y})| \leq (|\underline{x}| + |\underline{y}|), \text{Po}, c)]$

$[\text{splitNumericalSum}(+ - \text{small}) \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: 0 \leq \underline{x} \vdash \underline{y} \leq 0 \vdash$   
 $|\underline{y}| \leq |\underline{x}| \vdash |(\underline{x} + \underline{y})| \leq |\underline{x}|]$

$[\text{splitNumericalSum}(+ - \text{small}) \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: 0 \leq \underline{x} \vdash$   
 $\underline{y} \leq 0 \vdash |\underline{y}| \leq |\underline{x}| \vdash \text{LeqAdditionLeft} \triangleright \underline{y} \leq 0 \gg (\underline{x} + \underline{y}) \leq (\underline{x} + 0); \text{plus0} \gg$   
 $(\underline{x} + 0) = \underline{x}; \text{subLeqRight} \triangleright (\underline{x} + 0) = \underline{x} \triangleright (\underline{x} + \underline{y}) \leq (\underline{x} + 0) \gg (\underline{x} + \underline{y}) \leq$   
 $\underline{x}; \text{PositiveToRight}(\text{Leq})(1\text{term}) \triangleright |\underline{y}| \leq |\underline{x}| \gg 0 \leq$   
 $(|\underline{x}| + (-\underline{u}|\underline{y}|)); \text{lemma nonpositiveNumerical} \triangleright \underline{y} \leq 0 \gg |\underline{y}| =$   
 $(-\underline{u}\underline{y}); \text{EqNegated} \triangleright |\underline{y}| = (-\underline{u}\underline{y}) \gg (-\underline{u}|\underline{y}|) = (-\underline{u}(-\underline{u}\underline{y})); \text{DoubleMinus} \gg$   
 $(-\underline{u}(-\underline{u}\underline{y})) = \underline{y}; \text{eqTransitivity} \triangleright (-\underline{u}|\underline{y}|) = (-\underline{u}(-\underline{u}\underline{y})) \triangleright (-\underline{u}(-\underline{u}\underline{y})) = \underline{y} \gg$   
 $(-\underline{u}|\underline{y}|) = \underline{y}; \text{NonnegativeNumerical} \triangleright 0 \leq \underline{x} \gg |\underline{x}| = \underline{x}; \text{AddEquations} \triangleright |\underline{x}| =$   
 $\underline{x} \triangleright (-\underline{u}|\underline{y}|) = \underline{y} \gg (|\underline{x}| + (-\underline{u}|\underline{y}|)) = (\underline{x} + \underline{y}); \text{subLeqRight} \triangleright (|\underline{x}| + (-\underline{u}|\underline{y}|)) =$   
 $(\underline{x} + \underline{y}) \triangleright 0 \leq (|\underline{x}| + (-\underline{u}|\underline{y}|)) \gg 0 \leq (\underline{x} + \underline{y}); \text{NonnegativeNumerical} \triangleright 0 \leq$   
 $(\underline{x} + \underline{y}) \gg |(\underline{x} + \underline{y})| = (\underline{x} + \underline{y}); \text{eqSymmetry} \triangleright |(\underline{x} + \underline{y})| = (\underline{x} + \underline{y}) \gg (\underline{x} + \underline{y}) =$   
 $|(\underline{x} + \underline{y})|; \text{eqSymmetry} \triangleright |\underline{x}| = \underline{x} \gg \underline{x} = |\underline{x}|; \text{subLeqLeft} \triangleright (\underline{x} + \underline{y}) =$   
 $|(\underline{x} + \underline{y})| \triangleright (\underline{x} + \underline{y}) \leq \underline{x} \gg |(\underline{x} + \underline{y})| \leq \underline{x}; \text{subLeqRight} \triangleright \underline{x} = |\underline{x}| \triangleright |(\underline{x} + \underline{y})| \leq$   
 $\underline{x} \gg |(\underline{x} + \underline{y})| \leq |\underline{x}|, \text{Po}, c)]$

$[\text{splitNumericalSum}(+ - \text{big}) \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: 0 \leq \underline{x} \vdash \underline{y} \leq 0 \vdash$   
 $\dot{\neg}(|\underline{x}| \leq |\underline{y}| \Rightarrow \dot{\neg}(\dot{\neg}(|\underline{x}| = |\underline{y}|)n)n)n \vdash |(\underline{x} + \underline{y})| \leq |\underline{y}|]$

$[\text{splitNumericalSum}(+ - \text{big}) \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: 0 \leq \underline{x} \vdash \underline{y} \leq$   
 $0 \vdash \dot{\neg}(|\underline{x}| \leq |\underline{y}| \Rightarrow \dot{\neg}(\dot{\neg}(|\underline{x}| = |\underline{y}|)n)n)n \vdash \text{NonnegativeNegated} \triangleright 0 \leq \underline{x} \gg$   
 $(-\underline{u}\underline{x}) \leq 0; \text{NonpositiveNegated} \triangleright \underline{y} \leq 0 \gg 0 \leq (-\underline{u}\underline{y}); \text{SignNumerical} \gg$   
 $|\underline{x}| = |(-\underline{u}\underline{x})|; \text{SubLessLeft} \triangleright |\underline{x}| = |(-\underline{u}\underline{x})| \triangleright \dot{\neg}(|\underline{x}| \leq |\underline{y}| \Rightarrow \dot{\neg}(\dot{\neg}(|\underline{x}| =$   
 $|\underline{y}|)n)n)n \gg \dot{\neg}(|(-\underline{u}\underline{x})| \leq |\underline{y}| \Rightarrow \dot{\neg}(\dot{\neg}(|(-\underline{u}\underline{x})| = |\underline{y}|)n)n)n; \text{SignNumerical} \gg$   
 $|\underline{y}| = |(-\underline{u}\underline{y})|; \text{SubLessRight} \triangleright |\underline{y}| = |(-\underline{u}\underline{y})| \triangleright \dot{\neg}(|(-\underline{u}\underline{x})| \leq |\underline{y}| \Rightarrow$   
 $\dot{\neg}(\dot{\neg}(|(-\underline{u}\underline{x})| = |\underline{y}|)n)n)n \gg \dot{\neg}(|(-\underline{u}\underline{x})| \leq |(-\underline{u}\underline{y})| \Rightarrow \dot{\neg}(\dot{\neg}(|(-\underline{u}\underline{x})| =$



$|(-\underline{uy})|n)n)n$ ; LessLeq  $\triangleright \dot{\vdash} (|(-\underline{ux})| \leq |(-\underline{uy})| \Rightarrow \dot{\vdash} (\dot{\vdash} (|(-\underline{ux})| = |(-\underline{uy})|n)n)n \gg |(-\underline{ux})| \leq |(-\underline{uy})|$ ; splitNumericalSum(+ - small)  $\triangleright 0 <= (-\underline{uy}) \triangleright (-\underline{ux}) \leq 0 \triangleright |(-\underline{ux})| \leq |(-\underline{uy})| \gg |((- \underline{uy}) + (-\underline{ux}))| \leq |(-\underline{uy})|$ ; SignNumerical  $\gg |(\underline{x} + \underline{y})| = |(-\underline{u}(\underline{x} + \underline{y}))|$ ;  $-\underline{x} - \underline{y} = -(\underline{x} + \underline{y}) \gg ((-\underline{ux}) + (-\underline{uy})) = (-\underline{u}(\underline{x} + \underline{y}))$ ; plusCommutativity  $\gg ((-\underline{ux}) + (-\underline{uy})) = ((-\underline{uy}) + (-\underline{ux}))$ ; Equality  $\triangleright ((-\underline{ux}) + (-\underline{uy})) = (-\underline{u}(\underline{x} + \underline{y})) \triangleright ((-\underline{ux}) + (-\underline{uy})) = ((-\underline{uy}) + (-\underline{ux})) \gg (-\underline{u}(\underline{x} + \underline{y})) = ((-\underline{uy}) + (-\underline{ux}))$ ; SameNumerical  $\triangleright (-\underline{u}(\underline{x} + \underline{y})) = ((-\underline{uy}) + (-\underline{ux})) \gg |(-\underline{u}(\underline{x} + \underline{y}))| = |((- \underline{uy}) + (-\underline{ux}))|$ ; eqTransitivity  $\triangleright |(\underline{x} + \underline{y})| = |(-\underline{u}(\underline{x} + \underline{y}))| \triangleright |(-\underline{u}(\underline{x} + \underline{y}))| = |((- \underline{uy}) + (-\underline{ux}))| \gg |(\underline{x} + \underline{y})| = |((- \underline{uy}) + (-\underline{ux}))|$ ; eqSymmetry  $\triangleright |(\underline{x} + \underline{y})| = |((- \underline{uy}) + (-\underline{ux}))| \gg |((- \underline{uy}) + (-\underline{ux}))| = |(\underline{x} + \underline{y})|$ ; eqSymmetry  $\triangleright |\underline{y}| = |(-\underline{uy})| \gg |(-\underline{uy})| = |\underline{y}|$ ; subLeqLeft  $\triangleright |((- \underline{uy}) + (-\underline{ux}))| = |(\underline{x} + \underline{y})| \triangleright |((- \underline{uy}) + (-\underline{ux}))| \leq |(-\underline{uy})| \gg |(\underline{x} + \underline{y})| \leq |(-\underline{uy})|$ ; subLeqRight  $\triangleright |(-\underline{uy})| = |\underline{y}| \triangleright |(\underline{x} + \underline{y})| \leq |(-\underline{uy})| \gg |(\underline{x} + \underline{y})| \leq |\underline{y}|$ , po, c)

$[\text{splitNumericalSum}(+-) \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: 0 <= \underline{x} \vdash \underline{y} <= 0 \vdash |(\underline{x} + \underline{y})| \leq (|\underline{x}| + |\underline{y}|)]$

$[\text{splitNumericalSum}(+-) \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: |\underline{y}| \leq |\underline{x}| \vdash 0 <= \underline{x} \vdash \underline{y} <= 0 \vdash \text{splitNumericalSum}(+ - \text{small}) \triangleright 0 <= \underline{x} \triangleright \underline{y} <= 0 \triangleright |\underline{y}| \leq |\underline{x}| \gg |(\underline{x} + \underline{y})| \leq |\underline{x}|; 0 <= |\underline{x}| \gg 0 <= |\underline{y}|$ ; LeqAdditionLeft  $\triangleright 0 <= |\underline{y}| \gg (|\underline{x}| + 0) \leq (|\underline{x}| + |\underline{y}|)$ ; plus0  $\gg (|\underline{x}| + 0) = |\underline{x}|$ ; subLeqLeft  $\triangleright (|\underline{x}| + 0) = |\underline{x}| \triangleright (|\underline{x}| + 0) \leq (|\underline{x}| + |\underline{y}|) \gg |\underline{x}| \leq (|\underline{x}| + |\underline{y}|)$ ; leqTransitivity  $\triangleright |(\underline{x} + \underline{y})| \leq |\underline{x}| \triangleright |\underline{x}| \leq (|\underline{x}| + |\underline{y}|) \gg |(\underline{x} + \underline{y})| \leq (|\underline{x}| + |\underline{y}|)$ ;  $\forall \underline{x}: \forall \underline{y}: \dot{\vdash} (|\underline{y}| \leq |\underline{x}|)n \vdash 0 <= \underline{x} \vdash \underline{y} <= 0 \vdash \text{ToLess} \triangleright \dot{\vdash} (|\underline{y}| \leq |\underline{x}|)n \gg \dot{\vdash} (|\underline{x}| \leq |\underline{y}|) \Rightarrow \dot{\vdash} (\dot{\vdash} (|\underline{x}| = |\underline{y}|)n)n)n$ ; splitNumericalSum(+ - big)  $\triangleright 0 <= \underline{x} \triangleright \underline{y} <= 0 \triangleright \dot{\vdash} (|\underline{x}| \leq |\underline{y}|) \Rightarrow \dot{\vdash} (\dot{\vdash} (|\underline{x}| = |\underline{y}|)n)n)n \gg |(\underline{x} + \underline{y})| \leq |\underline{y}|$ ;  $0 <= |\underline{x}| \gg 0 <= |\underline{x}|$ ; leqAddition  $\triangleright 0 <= |\underline{x}| \gg (0 + |\underline{y}|) \leq (|\underline{x}| + |\underline{y}|)$ ; plus0Left  $\gg (0 + |\underline{y}|) = |\underline{y}|$ ; subLeqLeft  $\triangleright (0 + |\underline{y}|) = |\underline{y}| \triangleright (0 + |\underline{y}|) \leq (|\underline{x}| + |\underline{y}|) \gg |\underline{y}| \leq (|\underline{x}| + |\underline{y}|)$ ; leqTransitivity  $\triangleright |(\underline{x} + \underline{y})| \leq |\underline{y}| \triangleright |\underline{y}| \leq (|\underline{x}| + |\underline{y}|) \gg |(\underline{x} + \underline{y})| \leq (|\underline{x}| + |\underline{y}|)$ ;  $\forall \underline{x}: \forall \underline{y}: \text{Ded} \triangleright \forall \underline{x}: \forall \underline{y}: |\underline{y}| \leq |\underline{x}| \vdash 0 <= \underline{x} \vdash \underline{y} <= 0 \vdash |(\underline{x} + \underline{y})| \leq (|\underline{x}| + |\underline{y}|) \gg |\underline{y}| \leq |\underline{x}| \Rightarrow 0 <= \underline{x} \Rightarrow \underline{y} <= 0 \Rightarrow |(\underline{x} + \underline{y})| \leq (|\underline{x}| + |\underline{y}|)$ ; Ded  $\triangleright \forall \underline{x}: \forall \underline{y}: \dot{\vdash} (|\underline{y}| \leq |\underline{x}|)n \vdash 0 <= \underline{x} \vdash \underline{y} <= 0 \vdash |(\underline{x} + \underline{y})| \leq (|\underline{x}| + |\underline{y}|) \gg \dot{\vdash} (|\underline{y}| \leq |\underline{x}|)n \Rightarrow 0 <= \underline{x} \Rightarrow \underline{y} <= 0 \Rightarrow |(\underline{x} + \underline{y})| \leq (|\underline{x}| + |\underline{y}|)$ ;  $0 <= \underline{x} \vdash \underline{y} <= 0 \vdash \text{FromNegations} \triangleright |\underline{y}| \leq |\underline{x}| \Rightarrow 0 <= \underline{x} \Rightarrow \underline{y} <= 0 \Rightarrow |(\underline{x} + \underline{y})| \leq (|\underline{x}| + |\underline{y}|) \triangleright \dot{\vdash} (|\underline{y}| \leq |\underline{x}|)n \Rightarrow 0 <= \underline{x} \Rightarrow \underline{y} <= 0 \Rightarrow |(\underline{x} + \underline{y})| \leq (|\underline{x}| + |\underline{y}|) \gg 0 <= \underline{x} \Rightarrow \underline{y} <= 0 \Rightarrow |(\underline{x} + \underline{y})| \leq (|\underline{x}| + |\underline{y}|)$ ; MP2  $\triangleright 0 <= \underline{x} \Rightarrow \underline{y} <= 0 \Rightarrow |(\underline{x} + \underline{y})| \leq (|\underline{x}| + |\underline{y}|) \triangleright 0 <= \underline{x} \triangleright \underline{y} <= 0 \gg |(\underline{x} + \underline{y})| \leq (|\underline{x}| + |\underline{y}|)$ , po, c)

$[\text{splitNumericalSum}(+-) \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \underline{x} \leq 0 \vdash 0 <= \underline{y} \vdash |(\underline{x} + \underline{y})| \leq (|\underline{x}| + |\underline{y}|)]$

$[\text{splitNumericalSum}(+-) \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \underline{x} \leq 0 \vdash 0 <= \underline{y} \vdash \text{NonpositiveNegated} \triangleright \underline{x} \leq 0 \gg 0 <= (-\underline{ux})$ ; NonnegativeNegated  $\triangleright 0 <= \underline{y} \gg (-\underline{uy}) \leq 0$ ; splitNumericalSum(+ -)  $\triangleright 0 <= (-\underline{ux}) \triangleright (-\underline{uy}) \leq 0 \gg |((- \underline{ux}) + (-\underline{uy}))| \leq (|(-\underline{ux})| + |(-\underline{uy})|)$ ; SplitNumericalSumHelper  $\triangleright |((- \underline{ux}) + (-\underline{uy}))| \leq (|(-\underline{ux})| + |(-\underline{uy})|) \gg |(\underline{x} + \underline{y})| \leq (|\underline{x}| + |\underline{y}|)$ , po, c)

$[\text{splitNumericalSum} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: |(\underline{x} + \underline{y})| \leq (|\underline{x}| + |\underline{y}|)]$

$$\begin{aligned}
& \text{[splitNumericalSum} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\ulcorner \text{SystemQ} \vdash \forall \underline{x}. \forall \underline{y}. 0 \leq \underline{x} \vdash 0 \leq \underline{y} \vdash \\
& \text{splitNumericalSum}(++) \triangleright 0 \leq \underline{x} \triangleright 0 \leq \underline{y} \gg |\underline{x} + \underline{y}| \leq \\
& (|\underline{x}| + |\underline{y}|); \forall \underline{x}. \forall \underline{y}. 0 \leq \underline{x} \vdash \underline{y} \leq 0 \vdash \text{splitNumericalSum}(+-) \triangleright 0 \leq \underline{x} \triangleright \underline{y} \leq \\
& 0 \gg |\underline{x} + \underline{y}| \leq (|\underline{x}| + |\underline{y}|); \forall \underline{x}. \forall \underline{y}. \underline{x} \leq 0 \vdash 0 \leq \underline{y} \vdash \\
& \text{splitNumericalSum}(-+) \triangleright \underline{x} \leq 0 \triangleright 0 \leq \underline{y} \gg |\underline{x} + \underline{y}| \leq \\
& (|\underline{x}| + |\underline{y}|); \forall \underline{x}. \forall \underline{y}. \underline{x} \leq 0 \vdash \underline{y} \leq 0 \vdash \text{splitNumericalSum}(- -) \triangleright \underline{x} \leq 0 \triangleright \underline{y} \leq \\
& 0 \gg |\underline{x} + \underline{y}| \leq (|\underline{x}| + |\underline{y}|); \forall \underline{x}. \forall \underline{y}. \text{Ded} \triangleright \forall \underline{x}. \forall \underline{y}. 0 \leq \underline{x} \vdash 0 \leq \underline{y} \vdash |\underline{x} + \underline{y}| \leq \\
& (|\underline{x}| + |\underline{y}|) \gg 0 \leq \underline{x} \Rightarrow 0 \leq \underline{y} \Rightarrow |\underline{x} + \underline{y}| \leq (|\underline{x}| + |\underline{y}|); \text{Ded} \triangleright \forall \underline{x}. \forall \underline{y}. 0 \leq \\
& \underline{x} \vdash \underline{y} \leq 0 \vdash |\underline{x} + \underline{y}| \leq (|\underline{x}| + |\underline{y}|) \gg 0 \leq \underline{x} \Rightarrow \underline{y} \leq 0 \Rightarrow |\underline{x} + \underline{y}| \leq \\
& (|\underline{x}| + |\underline{y}|); \text{Ded} \triangleright \forall \underline{x}. \forall \underline{y}. \underline{x} \leq 0 \vdash 0 \leq \underline{y} \vdash |\underline{x} + \underline{y}| \leq (|\underline{x}| + |\underline{y}|) \gg \underline{x} \leq 0 \Rightarrow \\
& 0 \leq \underline{y} \Rightarrow |\underline{x} + \underline{y}| \leq (|\underline{x}| + |\underline{y}|); \text{Ded} \triangleright \forall \underline{x}. \forall \underline{y}. \underline{x} \leq 0 \vdash \underline{y} \leq 0 \vdash |\underline{x} + \underline{y}| \leq \\
& (|\underline{x}| + |\underline{y}|) \gg \underline{x} \leq 0 \Rightarrow \underline{y} \leq 0 \Rightarrow |\underline{x} + \underline{y}| \leq (|\underline{x}| + |\underline{y}|); \text{FromLeqGeq} \triangleright 0 \leq \\
& \underline{x} \Rightarrow 0 \leq \underline{y} \Rightarrow |\underline{x} + \underline{y}| \leq (|\underline{x}| + |\underline{y}|) \triangleright \underline{x} \leq 0 \Rightarrow 0 \leq \underline{y} \Rightarrow |\underline{x} + \underline{y}| \leq \\
& (|\underline{x}| + |\underline{y}|) \gg 0 \leq \underline{y} \Rightarrow |\underline{x} + \underline{y}| \leq (|\underline{x}| + |\underline{y}|); \text{FromLeqGeq} \triangleright 0 \leq \underline{x} \Rightarrow \underline{y} \leq \\
& 0 \Rightarrow |\underline{x} + \underline{y}| \leq (|\underline{x}| + |\underline{y}|) \triangleright \underline{x} \leq 0 \Rightarrow \underline{y} \leq 0 \Rightarrow |\underline{x} + \underline{y}| \leq (|\underline{x}| + |\underline{y}|) \gg \\
& \underline{y} \leq 0 \Rightarrow |\underline{x} + \underline{y}| \leq (|\underline{x}| + |\underline{y}|); \text{FromLeqGeq} \triangleright 0 \leq \underline{y} \Rightarrow |\underline{x} + \underline{y}| \leq \\
& (|\underline{x}| + |\underline{y}|) \triangleright \underline{y} \leq 0 \Rightarrow |\underline{x} + \underline{y}| \leq (|\underline{x}| + |\underline{y}|) \gg |\underline{x} + \underline{y}| \leq (|\underline{x}| + |\underline{y}|), p_0, c)]
\end{aligned}$$

$$\begin{aligned}
& \text{[insertMiddleTerm(Sum)} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}. \forall \underline{y}. \forall \underline{z}. (\underline{x} + \underline{y}) = \\
& (\underline{x} + (-\underline{uz})) + (\underline{z} + \underline{y})]
\end{aligned}$$

$$\begin{aligned}
& \text{[insertMiddleTerm(Sum)} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\ulcorner \text{SystemQ} \vdash \forall \underline{x}. \forall \underline{y}. \forall \underline{z}. x = x + y - y \gg \\
& x = (\underline{x} + \underline{z}) + (-\underline{uz}); \text{Three2threeTerms} \gg ((\underline{x} + \underline{z}) + (-\underline{uz})) = \\
& ((\underline{x} + (-\underline{uz})) + \underline{z}); \text{eqTransitivity} \triangleright x = ((\underline{x} + \underline{z}) + (-\underline{uz})) \triangleright ((\underline{x} + \underline{z}) + (-\underline{uz})) = \\
& ((\underline{x} + (-\underline{uz})) + \underline{z}) \gg x = ((\underline{x} + (-\underline{uz})) + \underline{z}); \text{eqAddition} \triangleright x = ((\underline{x} + (-\underline{uz})) + \underline{z}) \gg \\
& (\underline{x} + \underline{y}) = (((\underline{x} + (-\underline{uz})) + \underline{z}) + \underline{y}); \text{plusAssociativity} \gg (((\underline{x} + (-\underline{uz})) + \underline{z}) + \underline{y}) = \\
& ((\underline{x} + (-\underline{uz})) + (\underline{z} + \underline{y})); \text{eqTransitivity} \triangleright (\underline{x} + \underline{y}) = (((\underline{x} + (-\underline{uz})) + \underline{z}) + \underline{y}) \triangleright (((\underline{x} + \\
& (-\underline{uz})) + \underline{z}) + \underline{y}) = ((\underline{x} + (-\underline{uz})) + (\underline{z} + \underline{y})) \gg (\underline{x} + \underline{y}) = ((\underline{x} + (-\underline{uz})) + (\underline{z} + \underline{y})), p_0, c)]
\end{aligned}$$

$$\begin{aligned}
& \text{[insertMiddleTerm(Numerical)} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}. \forall \underline{y}. \forall \underline{z}. |\underline{x} + \underline{y}| \leq \\
& (|\underline{x} + (-\underline{uz})| + |\underline{z} + \underline{y}|)]
\end{aligned}$$

$$\begin{aligned}
& \text{[insertMiddleTerm(Numerical)} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\ulcorner \text{SystemQ} \vdash \\
& \forall \underline{x}. \forall \underline{y}. \forall \underline{z}. \text{splitNumericalSum} \gg (|\underline{x} + (-\underline{uz})| + |\underline{z} + \underline{y}|) \leq \\
& (|\underline{x} + (-\underline{uz})| + |\underline{z} + \underline{y}|); \text{insertMiddleTerm(Sum)} \gg (\underline{x} + \underline{y}) = \\
& ((\underline{x} + (-\underline{uz})) + (\underline{z} + \underline{y})); \text{SameNumerical} \triangleright (\underline{x} + \underline{y}) = ((\underline{x} + (-\underline{uz})) + (\underline{z} + \underline{y})) \gg \\
& |\underline{x} + \underline{y}| = (|((\underline{x} + (-\underline{uz})) + (\underline{z} + \underline{y}))|); \text{eqSymmetry} \triangleright |\underline{x} + \underline{y}| = \\
& |((\underline{x} + (-\underline{uz})) + (\underline{z} + \underline{y}))| \gg |((\underline{x} + (-\underline{uz})) + (\underline{z} + \underline{y}))| = \\
& |\underline{x} + \underline{y}|); \text{subLeqLeft} \triangleright |((\underline{x} + (-\underline{uz})) + (\underline{z} + \underline{y}))| = |\underline{x} + \underline{y}| \triangleright |((\underline{x} + (-\underline{uz})) + (\underline{z} + \underline{y}))| \leq \\
& (|\underline{x} + (-\underline{uz})| + |\underline{z} + \underline{y}|) \gg |\underline{x} + \underline{y}| \leq (|\underline{x} + (-\underline{uz})| + |\underline{z} + \underline{y}|), p_0, c)] \\
& (***) \text{ REGNESTYKKER} (***)
\end{aligned}$$

$$\begin{aligned}
& \text{[insertMiddleTerm(Difference)} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}. \forall \underline{y}. \forall \underline{z}. (\underline{x} + (-\underline{uy})) = \\
& (\underline{x} + \underline{z}) + (-\underline{u}(\underline{y} + \underline{z}))]
\end{aligned}$$

$$\begin{aligned}
& \text{[insertMiddleTerm(Difference)} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\ulcorner \text{SystemQ} \vdash \\
& \forall \underline{x}. \forall \underline{y}. \forall \underline{z}. \text{insertMiddleTerm(Sum)} \gg (\underline{x} + (-\underline{uy})) = \\
& ((\underline{x} + (-\underline{u}(-\underline{uz}))) + ((-\underline{uz}) + (-\underline{uy}))); \text{DoubleMinus} \gg (-\underline{u}(-\underline{uz})) = \\
& \underline{z}; \text{EqAdditionLeft} \triangleright (-\underline{u}(-\underline{uz})) = \underline{z} \gg (\underline{x} + (-\underline{u}(-\underline{uz}))) =
\end{aligned}$$

$$\begin{aligned}
& (\underline{x} + \underline{z}); \text{plusCommutativity} \gg ((-\underline{u}\underline{z}) + (-\underline{u}\underline{y})) = ((-\underline{u}\underline{y}) + (-\underline{u}\underline{z})); -x - y = \\
& -(x + y) \gg ((-\underline{u}\underline{y}) + (-\underline{u}\underline{z})) = (-\underline{u}(\underline{y} + \underline{z})); \text{eqTransitivity} \triangleright ((-\underline{u}\underline{z}) + (-\underline{u}\underline{y})) = \\
& ((-\underline{u}\underline{y}) + (-\underline{u}\underline{z})) \triangleright ((-\underline{u}\underline{y}) + (-\underline{u}\underline{z})) = (-\underline{u}(\underline{y} + \underline{z})) \gg ((-\underline{u}\underline{z}) + (-\underline{u}\underline{y})) = \\
& (-\underline{u}(\underline{y} + \underline{z})); \text{AddEquations} \triangleright (\underline{x} + (-\underline{u}(-\underline{u}\underline{z}))) = (\underline{x} + \underline{z}) \triangleright ((-\underline{u}\underline{z}) + (-\underline{u}\underline{y})) = \\
& (-\underline{u}(\underline{y} + \underline{z})) \gg ((\underline{x} + (-\underline{u}(-\underline{u}\underline{z}))) + ((-\underline{u}\underline{z}) + (-\underline{u}\underline{y}))) = \\
& ((\underline{x} + \underline{z}) + (-\underline{u}(\underline{y} + \underline{z}))); \text{eqTransitivity} \triangleright (\underline{x} + (-\underline{u}\underline{y})) = \\
& ((\underline{x} + (-\underline{u}(-\underline{u}\underline{z}))) + ((-\underline{u}\underline{z}) + (-\underline{u}\underline{y}))) \triangleright ((\underline{x} + (-\underline{u}(-\underline{u}\underline{z}))) + ((-\underline{u}\underline{z}) + (-\underline{u}\underline{y}))) = \\
& ((\underline{x} + \underline{z}) + (-\underline{u}(\underline{y} + \underline{z}))) \gg (\underline{x} + (-\underline{u}\underline{y})) = ((\underline{x} + \underline{z}) + (-\underline{u}(\underline{y} + \underline{z}))), p_0, c]
\end{aligned}$$

$$[\text{DistributionOutLeft} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}. \forall \underline{y}. \forall \underline{z}. ((\underline{y} * \underline{x}) + (\underline{z} * \underline{x})) = (\underline{x} * (\underline{y} + \underline{z}))]$$

$$\begin{aligned}
& [\text{DistributionOutLeft} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \\
& \forall \underline{x}. \forall \underline{y}. \forall \underline{z}. \text{timesCommutativity} \gg (\underline{y} * \underline{x}) = (\underline{x} * \underline{y}); \text{timesCommutativity} \gg \\
& (\underline{z} * \underline{x}) = (\underline{x} * \underline{z}); \text{AddEquations} \triangleright (\underline{y} * \underline{x}) = (\underline{x} * \underline{y}) \triangleright (\underline{z} * \underline{x}) = (\underline{x} * \underline{z}) \gg \\
& ((\underline{y} * \underline{x}) + (\underline{z} * \underline{x})) = ((\underline{x} * \underline{y}) + (\underline{x} * \underline{z})); \text{DistributionOut} \gg ((\underline{x} * \underline{y}) + (\underline{x} * \underline{z})) = \\
& (\underline{x} * (\underline{y} + \underline{z})); \text{eqTransitivity} \triangleright ((\underline{y} * \underline{x}) + (\underline{z} * \underline{x})) = ((\underline{x} * \underline{y}) + (\underline{x} * \underline{z})) \triangleright ((\underline{x} * \underline{y}) + (\underline{x} * \underline{z})) = \\
& (\underline{x} * (\underline{y} + \underline{z})) \gg ((\underline{y} * \underline{x}) + (\underline{z} * \underline{x})) = (\underline{x} * (\underline{y} + \underline{z})), p_0, c)]
\end{aligned}$$

$$[\text{TwoWholes} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}. (\underline{x} + \underline{x}) = ((1 + 1) * \underline{x})]$$

$$\begin{aligned}
& [\text{TwoWholes} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \forall \underline{x}. \text{times1} \gg (\underline{x} * 1) = \underline{x}; \text{eqSymmetry} \gg \\
& \underline{x} = (\underline{x} * 1); \text{EqAdditionLeft} \triangleright \underline{x} = (\underline{x} * 1) \gg (\underline{x} + \underline{x}) = (\underline{x} + (\underline{x} * 1)); \text{eqAddition} \triangleright \underline{x} = \\
& (\underline{x} * 1) \gg (\underline{x} + (\underline{x} * 1)) = ((\underline{x} * 1) + (\underline{x} * 1)); \text{eqTransitivity} \triangleright (\underline{x} + \underline{x}) = (\underline{x} + (\underline{x} * 1)) \triangleright \\
& (\underline{x} + (\underline{x} * 1)) = ((\underline{x} * 1) + (\underline{x} * 1)) \gg (\underline{x} + \underline{x}) = ((\underline{x} * 1) + (\underline{x} * 1)); \text{DistributionOut} \gg \\
& ((\underline{x} * 1) + (\underline{x} * 1)) = (\underline{x} * (1 + 1)); \text{Repetition} \triangleright ((\underline{x} * 1) + (\underline{x} * 1)) = (\underline{x} * (1 + 1)) \gg \\
& ((\underline{x} * 1) + (\underline{x} * 1)) = (\underline{x} * (1 + 1)); \text{timesCommutativity} \gg (\underline{x} * (1 + 1)) = \\
& ((1 + 1) * \underline{x}); \text{eqTransitivity4} \triangleright (\underline{x} + \underline{x}) = ((\underline{x} * 1) + (\underline{x} * 1)) \triangleright ((\underline{x} * 1) + (\underline{x} * 1)) = \\
& (\underline{x} * (1 + 1)) \triangleright (\underline{x} * (1 + 1)) = ((1 + 1) * \underline{x}) \gg (\underline{x} + \underline{x}) = ((1 + 1) * \underline{x}), p_0, c)]
\end{aligned}$$

$$[\text{TwoHalves} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}. ((\text{rec}(1 + 1) * \underline{x}) + (\text{rec}(1 + 1) * \underline{x})) = \underline{x}]$$

$$\begin{aligned}
& [\text{TwoHalves} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \forall \underline{x}. 0 < 2 \gg \dot{\vdash} (0 <= (1 + 1)) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \\
& (1 + 1))n)n)n; \text{LessNeq} \triangleright \dot{\vdash} (0 <= (1 + 1)) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = (1 + 1))n)n)n \gg \dot{\vdash} (0 = \\
& (1 + 1))n; \text{NeqSymmetry} \triangleright \dot{\vdash} (0 = (1 + 1))n \gg \dot{\vdash} ((1 + 1) = 0)n; \text{TwoWholes} \gg \\
& ((\text{rec}(1 + 1) * \underline{x}) + (\text{rec}(1 + 1) * \underline{x})) = ((1 + 1) * (\text{rec}(1 + 1) * \underline{x})); \text{timesAssociativity} \gg \\
& (((1 + 1) * \text{rec}(1 + 1)) * \underline{x}) = ((1 + 1) * (\text{rec}(1 + 1) * \underline{x})); \text{eqSymmetry} \triangleright (((1 + 1) * \\
& \text{rec}(1 + 1)) * \underline{x}) = ((1 + 1) * (\text{rec}(1 + 1) * \underline{x})) \gg ((1 + 1) * (\text{rec}(1 + 1) * \underline{x})) = \\
& (((1 + 1) * \text{rec}(1 + 1)) * \underline{x}); \text{Reciprocal} \triangleright \dot{\vdash} ((1 + 1) = 0)n \gg ((1 + 1) * \text{rec}(1 + 1)) = \\
& 1; \text{eqMultiplication} \triangleright ((1 + 1) * \text{rec}(1 + 1)) = 1 \gg (((1 + 1) * \text{rec}(1 + 1)) * \underline{x}) = \\
& (1 * \underline{x}); \text{times1Left} \gg (1 * \underline{x}) = \underline{x}; \text{eqTransitivity5} \triangleright ((\text{rec}(1 + 1) * \underline{x}) + (\text{rec}(1 + 1) * \underline{x})) = \\
& ((1 + 1) * (\text{rec}(1 + 1) * \underline{x})) \triangleright ((1 + 1) * (\text{rec}(1 + 1) * \underline{x})) = \\
& (((1 + 1) * \text{rec}(1 + 1)) * \underline{x}) \triangleright (((1 + 1) * \text{rec}(1 + 1)) * \underline{x}) = (1 * \underline{x}) \triangleright (1 * \underline{x}) = \underline{x} \gg \\
& ((\text{rec}(1 + 1) * \underline{x}) + (\text{rec}(1 + 1) * \underline{x})) = \underline{x}], p_0, c)]
\end{aligned}$$

$$[\text{ThreeWholes} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}. ((\underline{x} + \underline{x}) + \underline{x}) = (((1 + 1) + 1) * \underline{x})]$$

$$\begin{aligned}
& [\text{ThreeWholes} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \forall \underline{x}. \text{TwoWholes} \gg (\underline{x} + \underline{x}) = \\
& ((1 + 1) * \underline{x}); \text{times1Left} \gg (1 * \underline{x}) = \underline{x}; \text{eqSymmetry} \triangleright (1 * \underline{x}) = \underline{x} \gg \underline{x} = \\
& (1 * \underline{x}); \text{AddEquations} \triangleright (\underline{x} + \underline{x}) = ((1 + 1) * \underline{x}) \triangleright \underline{x} = (1 * \underline{x}) \gg ((\underline{x} + \underline{x}) + \underline{x}) = \\
& (((1 + 1) * \underline{x}) + (1 * \underline{x})); \text{DistributionOutLeft} \gg (((1 + 1) * \underline{x}) + (1 * \underline{x})) = \\
& (\underline{x} * ((1 + 1) + 1)); \text{timesCommutativity} \gg (\underline{x} * ((1 + 1) + 1)) =
\end{aligned}$$

$((1 + 1) + 1) * \underline{x}$ ; eqTransitivity4  $\triangleright ((\underline{x} + \underline{x}) + \underline{x}) =$   
 $((1+1)*\underline{x}) + (1*\underline{x}) \triangleright (((1+1)*\underline{x}) + (1*\underline{x})) = (\underline{x} * ((1+1)+1)) \triangleright (\underline{x} * ((1+1)+1)) =$   
 $((1+1)+1) * \underline{x} \gg ((\underline{x} + \underline{x}) + \underline{x}) = (((1+1)+1) * \underline{x})$ ; Repetition  $\triangleright ((\underline{x} + \underline{x}) + \underline{x}) =$   
 $((1+1)+1) * \underline{x} \gg ((\underline{x} + \underline{x}) + \underline{x}) = (((1+1)+1) * \underline{x}]$ ,  $p_0, c$ ]  
 $[0 < 3 \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \dot{\neg}(0 \leq ((1+1)+1) \Rightarrow \dot{\neg}(\dot{\neg}(0 = ((1+1)+1))n)n)n]$   
 $[0 < 3 \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash 0 < 2 \gg \dot{\neg}(0 \leq (1+1) \Rightarrow \dot{\neg}(\dot{\neg}(0 =$   
 $(1+1))n)n)n$ ; LessLeq  $\triangleright \dot{\neg}(0 \leq (1+1) \Rightarrow \dot{\neg}(\dot{\neg}(0 = (1+1))n)n)n \gg 0 < =$   
 $(1+1)$ ; Leq + 1  $\triangleright 0 \leq (1+1) \gg \dot{\neg}(0 \leq ((1+1)+1) \Rightarrow \dot{\neg}(\dot{\neg}(0 =$   
 $((1+1)+1))n)n)n$ ; Repetition  $\triangleright \dot{\neg}(0 \leq ((1+1)+1) \Rightarrow \dot{\neg}(\dot{\neg}(0 =$   
 $((1+1)+1))n)n)n \gg \dot{\neg}(0 \leq ((1+1)+1) \Rightarrow \dot{\neg}(\dot{\neg}(0 = ((1+1)+1))n)n)n]$ ,  $p_0, c$ ]  
 $[\text{ThreeThirds} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash$   
 $\forall \underline{x}: (((\text{rec}((1+1)+1) * \underline{x}) + (\text{rec}((1+1)+1) * \underline{x})) + (\text{rec}((1+1)+1) * \underline{x})) = \underline{x}]$   
 $[\text{ThreeThirds} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}: 0 < 3 \gg \dot{\neg}(0 \leq ((1+1)+1) \Rightarrow$   
 $\dot{\neg}(\dot{\neg}(0 = ((1+1)+1))n)n)n$ ; PositiveNonzero  $\triangleright \dot{\neg}(0 \leq ((1+1)+1) \Rightarrow$   
 $\dot{\neg}(\dot{\neg}(0 = ((1+1)+1))n)n)n \gg \dot{\neg}(((1+1)+1) = 0)n$ ; ThreeWholes  $\gg$   
 $((\text{rec}((1+1)+1) * \underline{x}) + (\text{rec}((1+1)+1) * \underline{x})) + (\text{rec}((1+1)+1) * \underline{x})) = (((1+1)+1) *$   
 $(\text{rec}((1+1)+1) * \underline{x}))$ ; timesAssociativity  $\gg (((1+1)+1) * \text{rec}((1+1)+1) * \underline{x}) =$   
 $((1+1)+1) * (\text{rec}((1+1)+1) * \underline{x})$ ; eqSymmetry  $\triangleright (((1+1)+1) * \text{rec}((1+1)+1) * \underline{x}) =$   
 $((1+1)+1) * (\text{rec}((1+1)+1) * \underline{x}) \gg (((1+1)+1) * (\text{rec}((1+1)+1) * \underline{x})) =$   
 $((1+1)+1) * \text{rec}((1+1)+1) * \underline{x}$ ; Reciprocal  $\triangleright \dot{\neg}(((1+1)+1) = 0)n \gg (((1+1)+1) * \text{rec}((1+1)+1) = 1$ ; eqMultiplication  $\triangleright (((1+1)+1) * \text{rec}((1+1)+1) = 1 \gg (((1+1)+1) * \text{rec}((1+1)+1) * \underline{x}) = (1 * \underline{x})$ ; timesLeft  $\gg (1 * \underline{x}) = \underline{x}$ ; eqTransitivity5  $\triangleright (((\text{rec}((1+1)+1) * \underline{x}) + (\text{rec}((1+1)+1) * \underline{x})) + (\text{rec}((1+1)+1) * \underline{x})) = (((1+1)+1) * (\text{rec}((1+1)+1) * \underline{x})) \triangleright (((1+1)+1) * (\text{rec}((1+1)+1) * \underline{x})) = (((1+1)+1) * \text{rec}((1+1)+1) * \underline{x}) \triangleright (((1+1)+1) * \text{rec}((1+1)+1) * \underline{x}) = (1 * \underline{x}) \triangleright (1 * \underline{x}) = \underline{x} \gg$   
 $((\text{rec}((1+1)+1) * \underline{x}) + (\text{rec}((1+1)+1) * \underline{x})) + (\text{rec}((1+1)+1) * \underline{x})) = \underline{x}]$ ,  $p_0, c$ ]  
XX 0<sub>i</sub>1/2 er et specialtilfaelde [ $\text{PositiveInverted} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: \dot{\neg}(0 \leq \underline{x} \Rightarrow \dot{\neg}(\dot{\neg}(0 = \underline{x})n)n)n \vdash \dot{\neg}(0 \leq \text{rec} \underline{x} \Rightarrow \dot{\neg}(\dot{\neg}(0 = \text{rec} \underline{x})n)n)n]$   
 $[\text{PositiveInverted} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}: \dot{\neg}(0 \leq \underline{x} \Rightarrow \dot{\neg}(\dot{\neg}(0 = \underline{x})n)n)n \vdash \text{FirstConjunct} \triangleright \dot{\neg}(0 \leq \underline{x} \Rightarrow \dot{\neg}(\dot{\neg}(0 = \underline{x})n)n)n \gg 0 \leq \underline{x}$ ; SecondConjunct  $\triangleright \dot{\neg}(0 \leq \underline{x} \Rightarrow \dot{\neg}(\dot{\neg}(0 = \underline{x})n)n)n \gg \dot{\neg}(0 = \underline{x})n$ ; NeqSymmetry  $\triangleright \dot{\neg}(0 = \underline{x})n \gg \dot{\neg}(\underline{x} = 0)n$ ;  $0 < 1 \gg \dot{\neg}(0 \leq 1 \Rightarrow \dot{\neg}(\dot{\neg}(0 = 1)n)n)n$ ;  $x * 0 = 0 \gg (\underline{x} * 0) = 0$ ;  $x * y = z$ Backwards  $\triangleright (\underline{x} * 0) = 0 \gg 0 = (0 * \underline{x})$ ; SubLessLeft  $\triangleright 0 = (0 * \underline{x}) \triangleright \dot{\neg}(0 \leq 1 \Rightarrow \dot{\neg}(\dot{\neg}(0 = 1)n)n)n \gg \dot{\neg}((0 * \underline{x}) \leq 1 \Rightarrow \dot{\neg}(\dot{\neg}((0 * \underline{x}) = 1)n)n)n$ ; Reciprocal  $\triangleright \dot{\neg}(\underline{x} = 0)n \gg (\underline{x} * \text{rec} \underline{x}) = 1$ ;  $x * y = z$ Backwards  $\triangleright (\underline{x} * \text{rec} \underline{x}) = 1 \gg 1 = (\text{rec} \underline{x} * \underline{x})$ ; SubLessRight  $\triangleright 1 = (\text{rec} \underline{x} * \underline{x}) \triangleright \dot{\neg}((0 * \underline{x}) \leq 1 \Rightarrow \dot{\neg}(\dot{\neg}((0 * \underline{x}) = 1)n)n)n \gg \dot{\neg}((0 * \underline{x}) \leq (\text{rec} \underline{x} * \underline{x}) \Rightarrow \dot{\neg}(\dot{\neg}((0 * \underline{x}) = (\text{rec} \underline{x} * \underline{x}))n)n)n$ ; LessDivision  $\triangleright 0 \leq \underline{x} \triangleright \dot{\neg}((0 * \underline{x}) \leq (\text{rec} \underline{x} * \underline{x}) \Rightarrow \dot{\neg}(\dot{\neg}((0 * \underline{x}) = (\text{rec} \underline{x} * \underline{x}))n)n)n \gg \dot{\neg}(0 \leq \text{rec} \underline{x} \Rightarrow \dot{\neg}(\dot{\neg}(0 = \text{rec} \underline{x})n)n)n]$ ,  $p_0, c$ ]  
 $[0 < 1/3 \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \dot{\neg}(0 \leq \text{rec}((1+1)+1) \Rightarrow \dot{\neg}(\dot{\neg}(0 = \text{rec}((1+1)+1))n)n)n]$   
 $[0 < 1/3 \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash 0 < 3 \gg \dot{\neg}(0 \leq ((1+1)+1) \Rightarrow \dot{\neg}(\dot{\neg}(0 = ((1+1)+1))n)n)n$ ; PositiveInverted  $\triangleright \dot{\neg}(0 \leq ((1+1)+1) \Rightarrow \dot{\neg}(\dot{\neg}(0 = ((1+1)+1))n)n)n]$

$(1)n)n)n \gg \dot{\vdash} (0 \leq \text{rec}((1+1)+1) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \text{rec}((1+1)+1))n)n)n], p_0, c)$   
 (\*\*\*) KVANTI (\*\*\*)

$[\text{Nat}(x) \xrightarrow{\text{macro}} \lambda t. \lambda s. \lambda c. \tilde{\mathcal{M}}_4(t, s, c, [[\text{Nat}(x) \doteq \lambda c. [x] \in_t ([V_{2n}] :: [\mathcal{M}] :: [\mathcal{N}] :: T)])]]]$

$\langle [a \equiv b | x := t]_{\text{Me}} \xrightarrow{\text{macro}} \lambda t. \lambda s. \lambda c. \tilde{\mathcal{M}}_4(t, s, c, [[\langle [a \equiv b | x := t]_{\text{Me}} \doteq \langle [a] \equiv [b] | [x] := [t] \rangle_{\text{Me}}]]] \rangle$

$\langle [a \equiv^1 b | x := t]_{\text{Me}} \xrightarrow{\text{val}} a!x!t!$   
 $\text{If}(\text{If}(b \stackrel{r}{=} [\forall_{\text{obj}} u: v], b^1 \stackrel{t}{=} x, F), a \stackrel{t}{=} b,$   
 $\text{If}(b \stackrel{t}{=} x, a \stackrel{t}{=} t, \text{If}($   
 $a \stackrel{r}{=} b, \langle a^t \equiv^* b^t | x := t \rangle_{\text{Me}}, F))]]]$

$\langle [a \equiv^* b | x := t]_{\text{Me}} \xrightarrow{\text{val}} b!x!t! \text{If}(a, T, \text{If}(\langle a^h \equiv^1 b^h | x := t \rangle_{\text{Me}}, \langle a^t \equiv^* b^t | x := t \rangle_{\text{Me}}, F))]]]$

$[\text{FromNotSameF}(\text{Weak})(\text{Helper}) \xrightarrow{\text{stmt}} \text{SystemQ} \vdash$

$\forall \underline{x}: \forall \underline{y}: \forall \underline{z}: \dot{\vdash} (\dot{\vdash} (|\underline{x} + (-\underline{uy})| \leq \underline{z} \Rightarrow \dot{\vdash} (\dot{\vdash} (|\underline{x} + (-\underline{uy})| = \underline{z})n)n)n) \vdash$   
 $\dot{\vdash} (\underline{x} \leq (\underline{y} + (-\underline{uz})))n \Rightarrow \underline{y} \leq (\underline{x} + (-\underline{uz}))]$

$[\text{FromNotSameF}(\text{Weak})(\text{Helper}) \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \forall \underline{z}: \underline{z} \leq$   
 $|\underline{x} + (-\underline{uy})| \vdash 0 \leq (\underline{x} + (-\underline{uy})) \vdash \text{NonnegativeNumerical} \triangleright 0 \leq (\underline{x} + (-\underline{uy})) \gg$   
 $|\underline{x} + (-\underline{uy})| = (\underline{x} + (-\underline{uy})); \text{subLeqRight} \triangleright |\underline{x} + (-\underline{uy})| = (\underline{x} + (-\underline{uy})) \triangleright \underline{z} \leq$   
 $|\underline{x} + (-\underline{uy})| \gg \underline{z} \leq (\underline{x} + (-\underline{uy})); \text{negativeToLeft}(\text{Leq}) \triangleright \underline{z} \leq (\underline{x} + (-\underline{uy})) \gg$   
 $(\underline{z} + \underline{y}) \leq \underline{x}; \text{plusCommutativity} \gg (\underline{z} + \underline{y}) = (\underline{y} + \underline{z}); \text{subLeqLeft} \triangleright (\underline{z} + \underline{y}) =$   
 $(\underline{y} + \underline{z}) \triangleright (\underline{z} + \underline{y}) \leq \underline{x} \gg (\underline{y} + \underline{z}) \leq \underline{x}; \text{PositiveToRight}(\text{Leq}) \triangleright (\underline{y} + \underline{z}) \leq \underline{x} \gg$   
 $\underline{y} \leq (\underline{x} + (-\underline{uz})); \text{WeakenOr1} \triangleright \underline{y} \leq (\underline{x} + (-\underline{uz})) \gg \dot{\vdash} (\underline{x} \leq (\underline{y} + (-\underline{uz})))n \Rightarrow$   
 $\underline{y} \leq (\underline{x} + (-\underline{uz})); \forall \underline{x}: \forall \underline{y}: \forall \underline{z}: \underline{z} \leq |\underline{x} + (-\underline{uy})| \vdash \dot{\vdash} (0 \leq (\underline{x} + (-\underline{uy})))n \vdash$   
 $\text{ToLess} \triangleright \dot{\vdash} (0 \leq (\underline{x} + (-\underline{uy})))n \gg \dot{\vdash} ((\underline{x} + (-\underline{uy})) \leq 0 \Rightarrow \dot{\vdash} (\dot{\vdash} ((\underline{x} + (-\underline{uy})) =$   
 $0)n)n)n; \text{NegativeNumerical} \gg |(\underline{x} + (-\underline{uy}))| =$   
 $(-\underline{u}(\underline{x} + (-\underline{uy}))); \text{MinusNegated} \gg (-\underline{u}(\underline{x} + (-\underline{uy}))) =$   
 $(\underline{y} + (-\underline{ux})); \text{eqTransitivity} \triangleright |(\underline{x} + (-\underline{uy}))| = (-\underline{u}(\underline{x} + (-\underline{uy}))) \triangleright (-\underline{u}(\underline{x} + (-\underline{uy}))) =$   
 $(\underline{y} + (-\underline{ux})) \gg |(\underline{x} + (-\underline{uy}))| = (\underline{y} + (-\underline{ux})); \text{subLeqRight} \triangleright |(\underline{x} + (-\underline{uy}))| =$   
 $(\underline{y} + (-\underline{ux})) \triangleright \underline{z} \leq |(\underline{x} + (-\underline{uy}))| \gg \underline{z} \leq (\underline{y} + (-\underline{ux})); \text{negativeToLeft}(\text{Leq}) \triangleright \underline{z} \leq$   
 $(\underline{y} + (-\underline{ux})) \gg (\underline{z} + \underline{x}) \leq \underline{y}; \text{plusCommutativity} \gg (\underline{z} + \underline{x}) =$   
 $(\underline{x} + \underline{z}); \text{subLeqLeft} \triangleright (\underline{z} + \underline{x}) = (\underline{x} + \underline{z}) \triangleright (\underline{z} + \underline{x}) \leq \underline{y} \gg (\underline{x} + \underline{z}) \leq$   
 $\underline{y}; \text{PositiveToRight}(\text{Leq}) \triangleright (\underline{x} + \underline{z}) \leq \underline{y} \gg \underline{x} \leq (\underline{y} + (-\underline{uz})); \text{WeakenOr2} \triangleright \underline{x} \leq$   
 $(\underline{y} + (-\underline{uz})) \gg \dot{\vdash} (\underline{x} \leq (\underline{y} + (-\underline{uz})))n \Rightarrow \underline{y} \leq (\underline{x} + (-\underline{uz})); \forall \underline{x}: \forall \underline{y}: \forall \underline{z}: \text{Ded} \triangleright$   
 $\forall \underline{x}: \forall \underline{y}: \forall \underline{z}: \underline{z} \leq |(\underline{x} + (-\underline{uy}))| \vdash 0 \leq (\underline{x} + (-\underline{uy})) \vdash \dot{\vdash} (\underline{x} \leq (\underline{y} + (-\underline{uz})))n \Rightarrow$   
 $\underline{y} \leq (\underline{x} + (-\underline{uz})) \gg \underline{z} \leq |(\underline{x} + (-\underline{uy}))| \Rightarrow 0 \leq (\underline{x} + (-\underline{uy})) \Rightarrow \dot{\vdash} (\underline{x} \leq$   
 $(\underline{y} + (-\underline{uz})))n \Rightarrow \underline{y} \leq (\underline{x} + (-\underline{uz})); \text{Ded} \triangleright \forall \underline{x}: \forall \underline{y}: \forall \underline{z}: \underline{z} \leq |(\underline{x} + (-\underline{uy}))| \vdash$   
 $\dot{\vdash} (0 \leq (\underline{x} + (-\underline{uy})))n \vdash \dot{\vdash} (\underline{x} \leq (\underline{y} + (-\underline{uz})))n \Rightarrow \underline{y} \leq (\underline{x} + (-\underline{uz})) \gg \underline{z} \leq$   
 $|(\underline{x} + (-\underline{uy}))| \Rightarrow \dot{\vdash} (0 \leq (\underline{x} + (-\underline{uy})))n \Rightarrow \dot{\vdash} (\underline{x} \leq (\underline{y} + (-\underline{uz})))n \Rightarrow \underline{y} \leq$   
 $(\underline{x} + (-\underline{uz})); \dot{\vdash} (\dot{\vdash} (|\underline{x} + (-\underline{uy})| \leq \underline{z} \Rightarrow \dot{\vdash} (\dot{\vdash} (|\underline{x} + (-\underline{uy})| = \underline{z})n)n)n) \vdash$   
 $\text{fromNotLess} \triangleright \dot{\vdash} (\dot{\vdash} (|\underline{x} + (-\underline{uy})| \leq \underline{z} \Rightarrow \dot{\vdash} (\dot{\vdash} (|\underline{x} + (-\underline{uy})| = \underline{z})n)n)n) \gg$   
 $\underline{z} \leq |(\underline{x} + (-\underline{uy}))|]; \text{MP} \triangleright \underline{z} \leq |(\underline{x} + (-\underline{uy}))| \Rightarrow 0 \leq (\underline{x} + (-\underline{uy})) \Rightarrow \dot{\vdash} (\underline{x} \leq$











$$\begin{aligned}
& [\text{MinusTimesMinus} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \text{DoubleMinus} \gg \\
& (-u(-uy)) = \underline{y}; \text{Times}(-1)\text{Left} \gg ((-u1) * (-uy)) = \\
& (-u(-uy)); \text{eqTransitivity} \triangleright ((-u1) * (-uy)) = (-u(-uy)) \triangleright (-u(-uy)) = \underline{y} \gg \\
& ((-u1) * (-uy)) = \underline{y}; \text{EqMultiplicationLeft} \triangleright ((-u1) * (-uy)) = \underline{y} \gg \\
& (\underline{x} * ((-u1) * (-uy))) = (\underline{x} * \underline{y}); \text{Times}(-1) \gg (\underline{x} * (-u1)) = \\
& (-u\underline{x}); \text{eqMultiplication} \triangleright (\underline{x} * (-u1)) = (-u\underline{x}) \gg ((\underline{x} * (-u1)) * (-uy)) = \\
& ((-u\underline{x}) * (-uy)); \text{timesAssociativity} \gg ((\underline{x} * (-u1)) * (-uy)) = \\
& (\underline{x} * ((-u1) * (-uy))); \text{Equality} \triangleright ((\underline{x} * (-u1)) * (-uy)) = \\
& ((-u\underline{x}) * (-uy)) \triangleright ((\underline{x} * (-u1)) * (-uy)) = (\underline{x} * ((-u1) * (-uy))) \gg ((-u\underline{x}) * (-uy)) = \\
& (\underline{x} * ((-u1) * (-uy))); \text{eqTransitivity} \triangleright ((-u\underline{x}) * (-uy)) = (\underline{x} * ((-u1) * \\
& (-uy))) \triangleright (\underline{x} * ((-u1) * (-uy))) = (\underline{x} * \underline{y}) \gg ((-u\underline{x}) * (-uy)) = (\underline{x} * \underline{y}) \rceil, p_0, c) ]
\end{aligned}$$

$$[\text{PlusTimesMinus} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: (\underline{x} * (-uy)) = (-u(\underline{x} * \underline{y}))]$$

$$\begin{aligned}
& [\text{PlusTimesMinus} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \text{Times}(-1)\text{Left} \gg \\
& ((-u1) * \underline{y}) = (-uy); \text{EqMultiplicationLeft} \triangleright ((-u1) * \underline{y}) = (-uy) \gg \\
& (\underline{x} * ((-u1) * \underline{y})) = (\underline{x} * (-uy)); \text{timesAssociativity} \gg ((\underline{x} * (-u1)) * \underline{y}) = \\
& (\underline{x} * ((-u1) * \underline{y})); \text{timesCommutativity} \gg (\underline{x} * (-u1)) = \\
& ((-u1) * \underline{x}); \text{eqMultiplication} \triangleright (\underline{x} * (-u1)) = ((-u1) * \underline{x}) \gg ((\underline{x} * (-u1)) * \underline{y}) = \\
& (((-u1) * \underline{x}) * \underline{y}); \text{timesAssociativity} \gg (((-u1) * \underline{x}) * \underline{y}) = \\
& ((-u1) * (\underline{x} * \underline{y})); \text{Times}(-1)\text{Left} \gg ((-u1) * (\underline{x} * \underline{y})) = \\
& (-u(\underline{x} * \underline{y})); \text{eqTransitivity4} \triangleright ((\underline{x} * (-u1)) * \underline{y}) = \\
& (((-u1) * \underline{x}) * \underline{y}) \triangleright (((-u1) * \underline{x}) * \underline{y}) = ((-u1) * (\underline{x} * \underline{y})) \triangleright ((-u1) * (\underline{x} * \underline{y})) = \\
& (-u(\underline{x} * \underline{y})) \gg ((\underline{x} * (-u1)) * \underline{y}) = (-u(\underline{x} * \underline{y})); \text{Equality} \triangleright ((\underline{x} * (-u1)) * \underline{y}) = \\
& (-u(\underline{x} * \underline{y})) \triangleright ((\underline{x} * (-u1)) * \underline{y}) = (\underline{x} * ((-u1) * \underline{y})) \gg (-u(\underline{x} * \underline{y})) = \\
& (\underline{x} * ((-u1) * \underline{y})); \text{eqTransitivity} \triangleright (-u(\underline{x} * \underline{y})) = (\underline{x} * ((-u1) * \underline{y})) \triangleright (\underline{x} * ((-u1) * \underline{y})) = \\
& (\underline{x} * (-uy)) \gg (-u(\underline{x} * \underline{y})) = (\underline{x} * (-uy)); \text{eqSymmetry} \triangleright (-u(\underline{x} * \underline{y})) = \\
& (\underline{x} * (-uy)) \gg (\underline{x} * (-uy)) = (-u(\underline{x} * \underline{y})) \rceil, p_0, c) ]
\end{aligned}$$

$$[\text{SplitNumericalProduct}(++) \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: 0 \leq \underline{x} \vdash 0 \leq \underline{y} \vdash |(\underline{x} * \underline{y})| = (|\underline{x}| * |\underline{y}|)]$$

$$\begin{aligned}
& [\text{SplitNumericalProduct}(++) \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: 0 \leq \underline{x} \vdash 0 \leq \\
& \underline{y} \vdash \text{NonnegativeFactors} \triangleright 0 \leq \underline{x} \triangleright 0 \leq \underline{y} \gg 0 \leq \\
& (\underline{x} * \underline{y}); \text{NonnegativeNumerical} \triangleright 0 \leq (\underline{x} * \underline{y}) \gg |(\underline{x} * \underline{y})| = \\
& (\underline{x} * \underline{y}); \text{NonnegativeNumerical} \triangleright 0 \leq \underline{x} \gg |\underline{x}| = \underline{x}; \text{NonnegativeNumerical} \triangleright 0 \leq \\
& \underline{y} \gg |\underline{y}| = \underline{y}; \text{MultiplyEquations} \triangleright |\underline{x}| = \underline{x} \triangleright |\underline{y}| = \underline{y} \gg (|\underline{x}| * |\underline{y}|) = \\
& (\underline{x} * \underline{y}); \text{eqSymmetry} \triangleright (|\underline{x}| * |\underline{y}|) = (\underline{x} * \underline{y}) \gg (\underline{x} * \underline{y}) = (|\underline{x}| * |\underline{y}|); \text{eqTransitivity} \triangleright \\
& |(\underline{x} * \underline{y})| = (\underline{x} * \underline{y}) \triangleright (\underline{x} * \underline{y}) = (|\underline{x}| * |\underline{y}|) \gg |(\underline{x} * \underline{y})| = (|\underline{x}| * |\underline{y}|) \rceil, p_0, c) ]
\end{aligned}$$

$$[\text{SplitNumericalProduct}(+-) \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: 0 \leq \underline{x} \vdash \underline{y} \leq 0 \vdash |(\underline{x} * \underline{y})| = (|\underline{x}| * |\underline{y}|)]$$

$$\begin{aligned}
& [\text{SplitNumericalProduct}(+-) \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: 0 \leq \underline{x} \vdash \underline{y} \leq \\
& 0 \vdash \text{SignNumerical} \gg |(\underline{x} * \underline{y})| = |(-u(\underline{x} * \underline{y}))|; \text{eqSymmetry} \triangleright |(\underline{x} * \underline{y})| = \\
& |(-u(\underline{x} * \underline{y}))| \gg |(-u(\underline{x} * \underline{y}))| = |(\underline{x} * \underline{y})|; \text{PlusTimesMinus} \gg (\underline{x} * (-uy)) = \\
& (-u(\underline{x} * \underline{y})); \text{SameNumerical} \triangleright (\underline{x} * (-uy)) = (-u(\underline{x} * \underline{y})) \gg |(\underline{x} * (-uy))| = \\
& |(-u(\underline{x} * \underline{y}))|; \text{eqTransitivity} \triangleright |(\underline{x} * (-uy))| = |(-u(\underline{x} * \underline{y}))| \triangleright |(-u(\underline{x} * \underline{y}))| = |(\underline{x} * \underline{y})| \gg \\
& |(\underline{x} * (-uy))| = |(\underline{x} * \underline{y})|; \text{SignNumerical} \gg |\underline{y}| = |(-uy)|; \text{eqSymmetry} \triangleright |\underline{y}| =
\end{aligned}$$



[NonzeroFactors  $\xrightarrow{\text{stnt}}$  SystemQ  $\vdash \forall \underline{x}: \forall \underline{y}: \dot{\vdash} (\underline{x} = 0)n \vdash \dot{\vdash} (\underline{y} = 0)n \vdash \dot{\vdash} ((\underline{x} * \underline{y}) = 0)n]$

[NonzeroFactors  $\xrightarrow{\text{proof}}$   $\lambda c. \lambda x. \mathcal{P}(\ulcorner \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \dot{\vdash} (\underline{x} = 0)n \vdash \dot{\vdash} (\underline{y} = 0)n \vdash \text{NeqMultiplication} \triangleright \dot{\vdash} (\underline{y} = 0)n \triangleright \dot{\vdash} (\underline{x} = 0)n \gg \dot{\vdash} ((\underline{x} * \underline{y}) = (0 * \underline{y}))n; \text{timesCommutativity} \gg (0 * \underline{y}) = (\underline{y} * 0); x * 0 = 0 \gg (\underline{y} * 0) = 0; \text{eqTransitivity} \triangleright (0 * \underline{y}) = (\underline{y} * 0) \triangleright (\underline{y} * 0) = 0 \gg (0 * \underline{y}) = 0; \text{SubNeqRight} \triangleright (0 * \underline{y}) = 0 \triangleright \dot{\vdash} ((\underline{x} * \underline{y}) = (0 * \underline{y}))n \gg \dot{\vdash} ((\underline{x} * \underline{y}) = 0)n \urcorner, p_0, c)]$

[PositiveFactors  $\xrightarrow{\text{stnt}}$  SystemQ  $\vdash \forall \underline{x}: \forall \underline{y}: \dot{\vdash} (0 \leq \underline{x} \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \underline{x})n)n) \vdash \dot{\vdash} (0 \leq \underline{y} \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \underline{y})n)n) \vdash \dot{\vdash} (0 \leq (\underline{x} * \underline{y}) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = (\underline{x} * \underline{y}))n)n)]$

[PositiveFactors  $\xrightarrow{\text{proof}}$   $\lambda c. \lambda x. \mathcal{P}(\ulcorner \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \dot{\vdash} (0 \leq \underline{x} \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \underline{x})n)n) \vdash \dot{\vdash} (0 \leq \underline{y} \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \underline{y})n)n) \vdash \text{Repetition} \triangleright \dot{\vdash} (0 \leq \underline{x} \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \underline{x})n)n) \gg \dot{\vdash} (0 \leq \underline{x} \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \underline{x})n)n); \text{FirstConjunct} \triangleright \dot{\vdash} (0 \leq \underline{x} \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \underline{x})n)n) \gg 0 \leq \underline{x}; \text{SecondConjunct} \triangleright \dot{\vdash} (0 \leq \underline{x} \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \underline{x})n)n) \gg \dot{\vdash} (0 = \underline{x})n; \text{NeqSymmetry} \triangleright \dot{\vdash} (0 = \underline{x})n \gg \dot{\vdash} (\underline{x} = 0)n; \text{Repetition} \triangleright \dot{\vdash} (0 \leq \underline{y} \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \underline{y})n)n) \gg \dot{\vdash} (0 \leq \underline{y} \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \underline{y})n)n); \text{FirstConjunct} \triangleright \dot{\vdash} (0 \leq \underline{y} \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \underline{y})n)n) \gg 0 \leq \underline{y}; \text{SecondConjunct} \triangleright \dot{\vdash} (0 \leq \underline{y} \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \underline{y})n)n) \gg \dot{\vdash} (0 = \underline{y})n; \text{NeqSymmetry} \triangleright \dot{\vdash} (0 = \underline{y})n \gg \dot{\vdash} (\underline{y} = 0)n; \text{NonnegativeFactors} \triangleright 0 \leq \underline{x} \triangleright 0 \leq \underline{y} \gg 0 \leq (\underline{x} * \underline{y}); \text{NonzeroFactors} \triangleright \dot{\vdash} (\underline{x} = 0)n \triangleright \dot{\vdash} (\underline{y} = 0)n \gg \dot{\vdash} ((\underline{x} * \underline{y}) = 0)n; \text{NeqSymmetry} \triangleright \dot{\vdash} ((\underline{x} * \underline{y}) = 0)n \gg \dot{\vdash} (0 = (\underline{x} * \underline{y}))n; \text{JoinConjuncts} \triangleright 0 \leq (\underline{x} * \underline{y}) \triangleright \dot{\vdash} (0 = (\underline{x} * \underline{y}))n \gg \dot{\vdash} (0 \leq (\underline{x} * \underline{y}) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = (\underline{x} * \underline{y}))n)n); \text{Repetition} \triangleright \dot{\vdash} (0 \leq (\underline{x} * \underline{y}) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = (\underline{x} * \underline{y}))n)n) \gg \dot{\vdash} (0 \leq (\underline{x} * \underline{y}) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = (\underline{x} * \underline{y}))n)n) \urcorner, p_0, c)]$

[SeriesSubsetCP  $\xrightarrow{\text{stnt}}$  SystemQ  $\vdash \forall (\underline{fx}): \forall (\underline{sy}): \dot{\vdash} (\dot{\vdash} (\forall \text{obj} \overline{(r1)}: \overline{(r1)} \in \underline{(fx)} \Rightarrow \dot{\vdash} (\forall \text{obj} \overline{(op1)}: \dot{\vdash} (\dot{\vdash} (\forall \text{obj} \overline{(op2)}: \dot{\vdash} (\dot{\vdash} (\overline{(op1)} \in \mathbb{N} \Rightarrow \dot{\vdash} (\overline{(op2)} \in \underline{(sy)})n)n) \Rightarrow \dot{\vdash} (\overline{(r1)} = \{\{\overline{(op1)}, \overline{(op1)}\}, \{\overline{(op1)}, \overline{(op2)}\}\})n)n)n) \Rightarrow \dot{\vdash} (\forall \text{obj} \overline{(f1)}: \forall \text{obj} \overline{(f2)}: \forall \text{obj} \overline{(f3)}: \forall \text{obj} \overline{(f4)}: \{\{\overline{(f1)}, \overline{(f1)}\}, \{\overline{(f1)}, \overline{(f2)}\}\} \in \underline{(fx)} \Rightarrow \{\{\overline{(f3)}, \overline{(f3)}\}, \{\overline{(f3)}, \overline{(f4)}\}\} \in \underline{(fx)} \Rightarrow \overline{(f1)} = \overline{(f3)} \Rightarrow \overline{(f2)} = \overline{(f4)})n) \Rightarrow \dot{\vdash} (\forall \text{obj} \overline{(s1)}: \overline{(s1)} \in \mathbb{N} \Rightarrow \dot{\vdash} (\forall \text{obj} \overline{(s2)}: \dot{\vdash} (\{\{\overline{(s1)}, \overline{(s1)}\}, \{\overline{(s1)}, \overline{(s2)}\}\} \in \underline{(fx)})n)n) \vdash \forall \text{obj} \overline{(s1)}: \overline{(s1)} \in \underline{(fx)} \Rightarrow \overline{(s1)} \in \{\text{ph} \in \text{P}(\text{P}(\text{Union}(\{\mathbb{N}, \underline{(sy)}\}))))) \mid \dot{\vdash} (\forall \text{obj} \overline{(op1)}: \dot{\vdash} (\dot{\vdash} (\forall \text{obj} \overline{(op2)}: \dot{\vdash} (\dot{\vdash} (\overline{(op1)} \in \mathbb{N} \Rightarrow \dot{\vdash} (\overline{(op2)} \in \underline{(sy)})n)n) \Rightarrow \dot{\vdash} (\text{aPh} = \{\{\{\overline{(op1)}, \overline{(op1)}\}, \{\overline{(op1)}, \overline{(op2)}\}\})n)n)n) \urcorner]$

[SeriesSubsetCP  $\xrightarrow{\text{proof}}$   $\lambda c. \lambda x. \mathcal{P}(\ulcorner \text{SystemQ} \vdash \forall (\underline{fx}): \forall (\underline{sy}): \dot{\vdash} (\dot{\vdash} (\forall \text{obj} \overline{(r1)}: \overline{(r1)} \in \underline{(fx)} \Rightarrow \dot{\vdash} (\forall \text{obj} \overline{(op1)}: \dot{\vdash} (\dot{\vdash} (\forall \text{obj} \overline{(op2)}: \dot{\vdash} (\dot{\vdash} (\overline{(op1)} \in \mathbb{N} \Rightarrow \dot{\vdash} (\overline{(op2)} \in \underline{(sy)})n)n) \Rightarrow \dot{\vdash} (\overline{(r1)} = \{\{\overline{(op1)}, \overline{(op1)}\}, \{\overline{(op1)}, \overline{(op2)}\}\})n)n)n) \Rightarrow \dot{\vdash} (\forall \text{obj} \overline{(f1)}: \forall \text{obj} \overline{(f2)}: \forall \text{obj} \overline{(f3)}: \forall \text{obj} \overline{(f4)}: \{\{\overline{(f1)}, \overline{(f1)}\}, \{\overline{(f1)}, \overline{(f2)}\}\} \in \underline{(fx)} \Rightarrow \{\{\overline{(f3)}, \overline{(f3)}\}, \{\overline{(f3)}, \overline{(f4)}\}\} \in \underline{(fx)} \Rightarrow \overline{(f1)} = \overline{(f3)} \Rightarrow \overline{(f2)} = \overline{(f4)})n) \Rightarrow \dot{\vdash} (\forall \text{obj} \overline{(s1)}: \overline{(s1)} \in \mathbb{N} \Rightarrow \dot{\vdash} (\forall \text{obj} \overline{(s2)}: \dot{\vdash} (\{\{\overline{(s1)}, \overline{(s1)}\}, \{\overline{(s1)}, \overline{(s2)}\}\} \in \underline{(fx)})n)n) \vdash \overline{(s1)} \in \underline{(fx)} \vdash \text{FromSeries} \triangleright \dot{\vdash} (\dot{\vdash} (\forall \text{obj} \overline{(r1)}: \overline{(r1)} \in \underline{(fx)} \Rightarrow \dot{\vdash} (\forall \text{obj} \overline{(op1)}: \dot{\vdash} (\dot{\vdash} (\forall \text{obj} \overline{(op2)}: \dot{\vdash} (\dot{\vdash} (\overline{(op1)} \in \mathbb{N} \Rightarrow \dot{\vdash} (\overline{(op2)} \in \underline{(sy)})n)n) \Rightarrow$





















































$$\begin{aligned}
& \dot{\vdash} (\forall_{\text{obj}} \overline{f1}): \forall_{\text{obj}} \overline{f2}): \forall_{\text{obj}} \overline{f3}): \forall_{\text{obj}} \overline{f4}): \{ \{ \overline{f1}, \overline{f1} \}, \{ \overline{f1}, \overline{f2} \} \} \in \overline{fx} \Rightarrow \\
& \{ \{ \overline{f3}, \overline{f3} \}, \{ \overline{f3}, \overline{f4} \} \} \in \overline{fx} \Rightarrow \overline{f1} = \overline{f3} \Rightarrow \overline{f2} = \overline{f4} \text{)n)n} \Rightarrow \\
& \dot{\vdash} (\forall_{\text{obj}} \overline{s1}): \overline{s1} \in \mathbb{N} \Rightarrow \dot{\vdash} (\forall_{\text{obj}} \overline{s2}): \dot{\vdash} (\{ \{ \overline{s1}, \overline{s1} \}, \{ \overline{s1}, \overline{s2} \} \} \in \\
& \overline{fx}) \text{n)n)n)n} \triangleright \dot{\vdash} (\dot{\vdash} (\forall_{\text{obj}} \overline{r1}): \overline{r1} \in \overline{fy}) \Rightarrow \\
& \dot{\vdash} (\forall_{\text{obj}} \overline{op1}): \dot{\vdash} (\dot{\vdash} (\forall_{\text{obj}} \overline{op2}): \dot{\vdash} (\dot{\vdash} (\overline{op1} \in \mathbb{N} \Rightarrow \dot{\vdash} (\overline{op2} \in \mathbb{Q}) \text{n)n} \Rightarrow \\
& \dot{\vdash} (\overline{r1} = \{ \{ \overline{op1}, \overline{op1} \}, \{ \overline{op1}, \overline{op2} \} \} \text{n)n)n) \text{n)n)n)n} \Rightarrow \\
& \dot{\vdash} (\forall_{\text{obj}} \overline{f1}): \forall_{\text{obj}} \overline{f2}): \forall_{\text{obj}} \overline{f3}): \forall_{\text{obj}} \overline{f4}): \{ \{ \overline{f1}, \overline{f1} \}, \{ \overline{f1}, \overline{f2} \} \} \in \overline{fy} \Rightarrow \\
& \{ \{ \overline{f3}, \overline{f3} \}, \{ \overline{f3}, \overline{f4} \} \} \in \overline{fy} \Rightarrow \overline{f1} = \overline{f3} \Rightarrow \overline{f2} = \overline{f4} \text{)n)n} \Rightarrow \\
& \dot{\vdash} (\forall_{\text{obj}} \overline{s1}): \overline{s1} \in \mathbb{N} \Rightarrow \dot{\vdash} (\forall_{\text{obj}} \overline{s2}): \dot{\vdash} (\{ \{ \overline{s1}, \overline{s1} \}, \{ \overline{s1}, \overline{s2} \} \} \in \\
& \overline{fy}) \text{n)n)n)n)n} \triangleright \forall_{\text{obj}} \underline{m}: \overline{fx}[\underline{m}] = \overline{fy}[\underline{m}] \gg \overline{s1} \in \overline{fx} \Rightarrow \overline{s1} \in \overline{fy}; \text{Gen} \triangleright \overline{s1} \in \\
& \overline{fx} \Rightarrow \overline{s1} \in \overline{fy} \gg \forall_{\text{obj}} \overline{s1}): \overline{s1} \in \overline{fx} \Rightarrow \overline{s1} \in \overline{fy}; \text{Repetition} \triangleright \\
& \forall_{\text{obj}} \overline{s1}): \overline{s1} \in \overline{fx} \Rightarrow \overline{s1} \in \overline{fy} \gg \forall_{\text{obj}} \overline{s1}): \overline{s1} \in \overline{fx} \Rightarrow \overline{s1} \in \overline{fy}], p_0, c) \\
& [\text{To} = f \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{m}: \forall \underline{fx}: \forall \underline{fy}: \lambda c. \text{Typeseries0}(\ulcorner \underline{fx} \urcorner, \ulcorner \underline{Q} \urcorner) \Vdash \\
& \lambda c. \text{Typeseries0}(\ulcorner \underline{fy} \urcorner, \ulcorner \underline{Q} \urcorner) \Vdash \forall_{\text{obj}} \underline{m}: \overline{fx}[\underline{m}] = \overline{fy}[\underline{m}] \vdash \overline{fx} = \overline{fy}] \\
& [\text{To} = f \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\ulcorner \text{SystemQ} \vdash \\
& \forall \underline{m}: \forall \underline{fx}: \forall \underline{fy}: \lambda c. \text{Typeseries0}(\ulcorner \underline{fx} \urcorner, \ulcorner \underline{Q} \urcorner) \Vdash \lambda c. \text{Typeseries0}(\ulcorner \underline{fy} \urcorner, \ulcorner \underline{Q} \urcorner) \Vdash \\
& \forall_{\text{obj}} \underline{m}: \overline{fx}[\underline{m}] = \overline{fy}[\underline{m}] \vdash \text{A4} @ \underline{m} \triangleright \forall_{\text{obj}} \underline{m}: \overline{fx}[\underline{m}] = \overline{fy}[\underline{m}] \gg \overline{fx}[\underline{m}] = \\
& \overline{fy}[\underline{m}]; \text{eqSymmetry} \triangleright \overline{fx}[\underline{m}] = \overline{fy}[\underline{m}] \gg \overline{fy}[\underline{m}] = \overline{fx}[\underline{m}]; \text{Gen} \triangleright \overline{fy}[\underline{m}] = \\
& \overline{fx}[\underline{m}] \gg \forall_{\text{obj}} \underline{m}: \overline{fy}[\underline{m}] = \overline{fx}[\underline{m}]; \text{To} = f(\text{Subset}) \triangleright \\
& \lambda c. \text{Typeseries0}(\ulcorner \underline{fx} \urcorner, \ulcorner \underline{Q} \urcorner) \triangleright \lambda c. \text{Typeseries0}(\ulcorner \underline{fy} \urcorner, \ulcorner \underline{Q} \urcorner) \triangleright \forall_{\text{obj}} \underline{m}: \overline{fx}[\underline{m}] = \\
& \overline{fy}[\underline{m}] \gg \forall_{\text{obj}} \overline{s1}): \overline{s1} \in \overline{fx} \Rightarrow \overline{s1} \in \overline{fy}; \text{To} = f(\text{Subset}) \triangleright \\
& \lambda c. \text{Typeseries0}(\ulcorner \underline{fy} \urcorner, \ulcorner \underline{Q} \urcorner) \triangleright \lambda c. \text{Typeseries0}(\ulcorner \underline{fx} \urcorner, \ulcorner \underline{Q} \urcorner) \triangleright \forall_{\text{obj}} \underline{m}: \overline{fy}[\underline{m}] = \\
& \overline{fx}[\underline{m}] \gg \forall_{\text{obj}} \overline{s1}): \overline{s1} \in \overline{fy} \Rightarrow \overline{s1} \in \overline{fx}; \text{ToSetEquality} \triangleright \forall_{\text{obj}} \overline{s1}): \overline{s1} \in \\
& \overline{fx} \Rightarrow \overline{s1} \in \overline{fy} \triangleright \forall_{\text{obj}} \overline{s1}): \overline{s1} \in \overline{fy} \Rightarrow \overline{s1} \in \overline{fx} \gg \overline{fx} = \overline{fy}], p_0, c) \\
& [\text{Tester1} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{m}: \forall \underline{fx}: \forall \underline{fy}: \forall_{\text{obj}} \underline{m}: \overline{fx}[\underline{m}] = \overline{fy}[\underline{m}] \vdash \overline{fx} = \overline{fy}] \\
& [\text{Tester1} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\ulcorner \text{SystemQ} \vdash \forall \underline{m}: \forall \underline{fx}: \forall \underline{fy}: \forall_{\text{obj}} \underline{m}: \overline{fx}[\underline{m}] = \overline{fy}[\underline{m}] \vdash \\
& \text{To} = f \triangleright \forall_{\text{obj}} \underline{m}: \overline{fx}[\underline{m}] = \overline{fy}[\underline{m}] \gg \overline{fx} = \overline{fy}], p_0, c) \\
& [\text{reciprocalToLeft(Less)} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \forall \underline{z}: \dot{\vdash} (0 \leq \underline{z} \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \\
& \underline{z}) \text{n)n) \vdash \dot{\vdash} (\underline{x} \leq (\underline{y} * \text{rec} \underline{z})) \Rightarrow \dot{\vdash} (\dot{\vdash} (\underline{x} = (\underline{y} * \text{rec} \underline{z})) \text{n)n) \vdash \dot{\vdash} ((\underline{x} * \underline{z}) \leq \underline{y} \Rightarrow \\
& \dot{\vdash} (\dot{\vdash} ((\underline{x} * \underline{z}) = \underline{y}) \text{n)n) \\
& [\text{reciprocalToLeft(Less)} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\ulcorner \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \forall \underline{z}: \dot{\vdash} (0 \leq \underline{z} \Rightarrow \\
& \dot{\vdash} (\dot{\vdash} (0 = \underline{z}) \text{n)n) \vdash \dot{\vdash} (\underline{x} \leq (\underline{y} * \text{rec} \underline{z})) \Rightarrow \dot{\vdash} (\dot{\vdash} (\underline{x} = (\underline{y} * \text{rec} \underline{z})) \text{n)n) \vdash \\
& \text{LessMultiplication} \triangleright \dot{\vdash} (0 \leq \underline{z} \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \underline{z}) \text{n)n) \triangleright \dot{\vdash} (\underline{x} \leq (\underline{y} * \text{rec} \underline{z})) \Rightarrow \\
& \dot{\vdash} (\dot{\vdash} (\underline{x} = (\underline{y} * \text{rec} \underline{z})) \text{n)n) \gg \dot{\vdash} ((\underline{x} * \underline{z}) \leq ((\underline{y} * \text{rec} \underline{z}) * \underline{z})) \Rightarrow \dot{\vdash} (\dot{\vdash} ((\underline{x} * \underline{z}) = \\
& ((\underline{y} * \text{rec} \underline{z}) * \underline{z})) \text{n)n) \text{n); Three2threeFactors} \gg ((\underline{y} * \text{rec} \underline{z}) * \underline{z}) = \\
& ((\underline{y} * \underline{z}) * \text{rec} \underline{z}); \text{PositiveNonzero} \triangleright \dot{\vdash} (0 \leq \underline{z} \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \underline{z}) \text{n)n) \gg \dot{\vdash} (\underline{z} = \\
& 0) \text{n); } \underline{x} = \underline{x} * \underline{y} * (1/\underline{y}) \triangleright \dot{\vdash} (\underline{z} = 0) \text{n} \gg \underline{y} = ((\underline{y} * \underline{z}) * \text{rec} \underline{z}); \text{eqSymmetry} \triangleright \underline{y} = \\
& ((\underline{y} * \underline{z}) * \text{rec} \underline{z}) \gg ((\underline{y} * \underline{z}) * \text{rec} \underline{z}) = \underline{y}; \text{eqTransitivity} \triangleright ((\underline{y} * \text{rec} \underline{z}) * \underline{z}) = \\
& ((\underline{y} * \underline{z}) * \text{rec} \underline{z}) \triangleright ((\underline{y} * \underline{z}) * \text{rec} \underline{z}) = \underline{y} \gg ((\underline{y} * \text{rec} \underline{z}) * \underline{z}) = \\
& \underline{y}; \text{SubLessRight} \triangleright ((\underline{y} * \text{rec} \underline{z}) * \underline{z}) = \underline{y} \triangleright \dot{\vdash} ((\underline{x} * \underline{z}) \leq ((\underline{y} * \text{rec} \underline{z}) * \underline{z})) \Rightarrow \dot{\vdash} (\dot{\vdash} ((\underline{x} * \underline{z}) =
\end{aligned}$$























$(\underline{y} + (-\underline{uz})) \Rightarrow \dot{\vdash}(\dot{\vdash}(\underline{x} = (\underline{y} + (-\underline{uz})))n)n \gg \dot{\vdash}((\underline{x} + \underline{z}) <= ((\underline{y} + (-\underline{uz})) + \underline{z})) \Rightarrow$   
 $\dot{\vdash}(\dot{\vdash}((\underline{x} + \underline{z}) = ((\underline{y} + (-\underline{uz})) + \underline{z}))n)n); \text{Three2threeTerms} \gg ((\underline{y} + (-\underline{uz})) + \underline{z}) =$   
 $((\underline{y} + \underline{z}) + (-\underline{uz})); \underline{x} = \underline{x} + \underline{y} - \underline{y} \gg \underline{y} = ((\underline{y} + \underline{z}) + (-\underline{uz})); \text{eqSymmetry} \triangleright \underline{y} =$   
 $((\underline{y} + \underline{z}) + (-\underline{uz})) \gg ((\underline{y} + \underline{z}) + (-\underline{uz})) = \underline{y}; \text{eqTransitivity} \triangleright ((\underline{y} + (-\underline{uz})) + \underline{z}) =$   
 $((\underline{y} + \underline{z}) + (-\underline{uz})) \triangleright ((\underline{y} + \underline{z}) + (-\underline{uz})) = \underline{y} \gg ((\underline{y} + (-\underline{uz})) + \underline{z}) = \underline{y}; \text{SubLessRight} \triangleright$   
 $((\underline{y} + (-\underline{uz})) + \underline{z}) = \underline{y} \triangleright \dot{\vdash}(\dot{\vdash}((\underline{x} + \underline{z}) <= ((\underline{y} + (-\underline{uz})) + \underline{z})) \Rightarrow \dot{\vdash}(\dot{\vdash}((\underline{x} + \underline{z}) =$   
 $((\underline{y} + (-\underline{uz})) + \underline{z}))n)n \gg \dot{\vdash}((\underline{x} + \underline{z}) <= \underline{y}) \Rightarrow \dot{\vdash}(\dot{\vdash}((\underline{x} + \underline{z}) = \underline{y})n)n]$

$[\text{PositiveTripled} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}. \dot{\vdash}(0 <= (\text{rec}((1 + 1) + 1) * \underline{x})) \Rightarrow \dot{\vdash}(\dot{\vdash}(0 =$   
 $(\text{rec}((1 + 1) + 1) * \underline{x}))n)n] \vdash \dot{\vdash}(0 <= \underline{x} \Rightarrow \dot{\vdash}(\dot{\vdash}(0 = \underline{x})n)n]$

$[\text{PositiveTripled} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}. \dot{\vdash}(0 <= (\text{rec}((1 + 1) + 1) * \underline{x})) \Rightarrow$   
 $\dot{\vdash}(\dot{\vdash}(0 = (\text{rec}((1 + 1) + 1) * \underline{x}))n)n] \vdash 0 < 3 \gg \dot{\vdash}(0 <= ((1 + 1) + 1) \Rightarrow$   
 $\dot{\vdash}(\dot{\vdash}(0 = ((1 + 1) + 1))n)n); \text{PositiveFactors} \triangleright \dot{\vdash}(0 <= ((1 + 1) + 1) \Rightarrow$   
 $\dot{\vdash}(\dot{\vdash}(0 = ((1 + 1) + 1))n)n] \triangleright \dot{\vdash}(0 <= (\text{rec}((1 + 1) + 1) * \underline{x})) \Rightarrow \dot{\vdash}(\dot{\vdash}(0 =$   
 $(\text{rec}((1 + 1) + 1) * \underline{x}))n)n \gg \dot{\vdash}(0 <= (((1 + 1) + 1) * (\text{rec}((1 + 1) + 1) * \underline{x})) \Rightarrow$   
 $\dot{\vdash}(\dot{\vdash}(0 = (((1 + 1) + 1) * (\text{rec}((1 + 1) + 1) * \underline{x})))n)n); \text{timesAssociativity} \gg$   
 $((((1 + 1) + 1) * \text{rec}((1 + 1) + 1)) * \underline{x}) = (((1 + 1) + 1) * (\text{rec}((1 + 1) + 1) * \underline{x}))$   
 $\underline{x}); \text{eqSymmetry} \triangleright (((1 + 1) + 1) * \text{rec}((1 + 1) + 1)) * \underline{x} =$   
 $((1 + 1) + 1) * (\text{rec}((1 + 1) + 1) * \underline{x}) \gg (((1 + 1) + 1) * (\text{rec}((1 + 1) + 1) * \underline{x})) =$   
 $((1 + 1) + 1) * (\text{rec}((1 + 1) + 1) * \underline{x}); \text{PositiveNonzero} \triangleright \dot{\vdash}(0 <=$   
 $((1 + 1) + 1) \Rightarrow \dot{\vdash}(\dot{\vdash}(0 = ((1 + 1) + 1))n)n \gg \dot{\vdash}(((1 + 1) + 1) =$   
 $0)n); \text{Reciprocal} \triangleright \dot{\vdash}(((1 + 1) + 1) = 0)n \gg (((1 + 1) + 1) * \text{rec}((1 + 1) + 1)) =$   
 $1; \text{eqMultiplication} \triangleright (((1 + 1) + 1) * \text{rec}((1 + 1) + 1)) = 1 \gg$   
 $((1 + 1) + 1) * \text{rec}((1 + 1) + 1) * \underline{x} = (1 * \underline{x}); \text{times1Left} \gg (1 * \underline{x}) =$   
 $\underline{x}; \text{eqTransitivity4} \triangleright (((1 + 1) + 1) * (\text{rec}((1 + 1) + 1) * \underline{x})) =$   
 $((1 + 1) + 1) * (\text{rec}((1 + 1) + 1) * \underline{x}) \triangleright (((1 + 1) + 1) * \text{rec}((1 + 1) + 1)) * \underline{x} =$   
 $(1 * \underline{x}) \triangleright (1 * \underline{x}) = \underline{x} \gg (((1 + 1) + 1) * (\text{rec}((1 + 1) + 1) * \underline{x})) =$   
 $\underline{x}; \text{SubLessRight} \triangleright (((1 + 1) + 1) * (\text{rec}((1 + 1) + 1) * \underline{x})) = \underline{x} \triangleright \dot{\vdash}(0 <=$   
 $((1 + 1) + 1) * (\text{rec}((1 + 1) + 1) * \underline{x})) \Rightarrow \dot{\vdash}(\dot{\vdash}(0 = (((1 + 1) + 1) * (\text{rec}((1 + 1) +$   
 $1) * \underline{x})))n)n \gg \dot{\vdash}(0 <= \underline{x} \Rightarrow \dot{\vdash}(\dot{\vdash}(0 = \underline{x})n)n]$

$[\text{PositiveDividedBy3} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}. \dot{\vdash}(0 <= \underline{x} \Rightarrow \dot{\vdash}(\dot{\vdash}(0 = \underline{x})n)n] \vdash$   
 $\dot{\vdash}(0 <= (\text{rec}((1 + 1) + 1) * \underline{x})) \Rightarrow \dot{\vdash}(\dot{\vdash}(0 = (\text{rec}((1 + 1) + 1) * \underline{x}))n)n]$

$[\text{PositiveDividedBy3} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}. \dot{\vdash}(0 <= \underline{x} \Rightarrow \dot{\vdash}(\dot{\vdash}(0 =$   
 $\underline{x})n)n] \vdash 0 < 3 \gg \dot{\vdash}(0 <= ((1 + 1) + 1) \Rightarrow \dot{\vdash}(\dot{\vdash}(0 =$   
 $((1 + 1) + 1))n)n); \text{PositiveInverted} \triangleright \dot{\vdash}(0 <= ((1 + 1) + 1) \Rightarrow \dot{\vdash}(\dot{\vdash}(0 =$   
 $((1 + 1) + 1))n)n \gg \dot{\vdash}(0 <= \text{rec}((1 + 1) + 1) \Rightarrow \dot{\vdash}(\dot{\vdash}(0 =$   
 $\text{rec}((1 + 1) + 1))n)n); \text{PositiveFactors} \triangleright \dot{\vdash}(0 <= \text{rec}((1 + 1) + 1) \Rightarrow \dot{\vdash}(\dot{\vdash}(0 =$   
 $\text{rec}((1 + 1) + 1))n)n] \triangleright \dot{\vdash}(0 <= \underline{x} \Rightarrow \dot{\vdash}(\dot{\vdash}(0 = \underline{x})n)n \gg \dot{\vdash}(0 <=$   
 $(\text{rec}((1 + 1) + 1) * \underline{x})) \Rightarrow \dot{\vdash}(\dot{\vdash}(0 = (\text{rec}((1 + 1) + 1) * \underline{x}))n)n]$

$[\underline{x} - \underline{x}] = 0 \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}. |(\underline{x} + (-\underline{ux}))| = 0]$

$[\underline{x} - \underline{x}] = 0 \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}. \text{eqReflexivity} \gg \underline{x} =$   
 $\underline{x}; \text{PositiveToLeft}(\text{Eq})(1\text{term}) \triangleright \underline{x} = \underline{x} \gg (\underline{x} + (-\underline{ux})) =$   
 $0; \text{SameNumerical} \triangleright (\underline{x} + (-\underline{ux})) = 0 \gg |(\underline{x} + (-\underline{ux}))| = |0|; |0| = 0 \gg |0| =$   
 $0; \text{eqTransitivity} \triangleright |(\underline{x} + (-\underline{ux}))| = |0| \triangleright |0| = 0 \gg |(\underline{x} + (-\underline{ux}))| = 0]$

$[1 < 2 \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \dot{\vdash}(1 <= (1 + 1) \Rightarrow \dot{\vdash}(\dot{\vdash}(1 = (1 + 1))n)n)]$

$[1 < 2 \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash 0 < 1 \gg \dot{\vdash} (0 <= 1 \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = 1) \text{n}) \text{n}) \text{n}; \text{LessAddition} \triangleright \dot{\vdash} (0 <= 1 \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = 1) \text{n}) \text{n}) \text{n} \gg \dot{\vdash} ((0 + 1) <= (1 + 1) \Rightarrow \dot{\vdash} (\dot{\vdash} ((0 + 1) = (1 + 1)) \text{n}) \text{n}) \text{n}; \text{plus0Left} \gg (0 + 1) = 1; \text{SubLessLeft} \triangleright (0 + 1) = 1 \triangleright \dot{\vdash} ((0 + 1) <= (1 + 1) \Rightarrow \dot{\vdash} (\dot{\vdash} ((0 + 1) = (1 + 1)) \text{n}) \text{n}) \text{n} \gg \dot{\vdash} (1 <= (1 + 1) \Rightarrow \dot{\vdash} (\dot{\vdash} (1 = (1 + 1)) \text{n}) \text{n}) \text{n}; \text{Repetition} \triangleright \dot{\vdash} (1 <= (1 + 1) \Rightarrow \dot{\vdash} (\dot{\vdash} (1 = (1 + 1)) \text{n}) \text{n}) \text{n} \gg \dot{\vdash} (1 <= (1 + 1) \Rightarrow \dot{\vdash} (\dot{\vdash} (1 = (1 + 1)) \text{n}) \text{n}) \text{n}] , p_0, c]$

$[1/3 < 2/3 \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \dot{\vdash} (\text{rec}((1 + 1) + 1) <= ((1 + 1) * \text{rec}((1 + 1) + 1))) \Rightarrow \dot{\vdash} (\dot{\vdash} (\text{rec}((1 + 1) + 1) = ((1 + 1) * \text{rec}((1 + 1) + 1))) \text{n}) \text{n}]$

$[1/3 < 2/3 \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash 1 < 2 \gg \dot{\vdash} (1 <= (1 + 1) \Rightarrow \dot{\vdash} (\dot{\vdash} (1 = (1 + 1)) \text{n}) \text{n}) \text{n}; 0 < 1/3 \gg \dot{\vdash} (0 <= \text{rec}((1 + 1) + 1) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \text{rec}((1 + 1) + 1)) \text{n}) \text{n}) \text{n}; \text{LessMultiplication} \triangleright \dot{\vdash} (0 <= \text{rec}((1 + 1) + 1) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = \text{rec}((1 + 1) + 1)) \text{n}) \text{n}) \text{n} \triangleright \dot{\vdash} (1 <= (1 + 1) \Rightarrow \dot{\vdash} (\dot{\vdash} (1 = (1 + 1)) \text{n}) \text{n}) \text{n} \gg \dot{\vdash} ((1 * \text{rec}((1 + 1) + 1)) <= ((1 + 1) * \text{rec}((1 + 1) + 1))) \Rightarrow \dot{\vdash} (\dot{\vdash} ((1 * \text{rec}((1 + 1) + 1)) = ((1 + 1) * \text{rec}((1 + 1) + 1))) \text{n}) \text{n}) \text{n}; \text{times1Left} \gg (1 * \text{rec}((1 + 1) + 1)) = \text{rec}((1 + 1) + 1); \text{SubLessLeft} \triangleright (1 * \text{rec}((1 + 1) + 1)) = \text{rec}((1 + 1) + 1) \triangleright \dot{\vdash} ((1 * \text{rec}((1 + 1) + 1)) <= ((1 + 1) * \text{rec}((1 + 1) + 1))) \Rightarrow \dot{\vdash} (\dot{\vdash} ((1 * \text{rec}((1 + 1) + 1)) = ((1 + 1) * \text{rec}((1 + 1) + 1))) \text{n}) \text{n}) \text{n} \gg \dot{\vdash} (\text{rec}((1 + 1) + 1) <= ((1 + 1) * \text{rec}((1 + 1) + 1))) \Rightarrow \dot{\vdash} (\dot{\vdash} (\text{rec}((1 + 1) + 1) = ((1 + 1) * \text{rec}((1 + 1) + 1))) \text{n}) \text{n}) \text{n}; \text{Repetition} \triangleright \dot{\vdash} (\text{rec}((1 + 1) + 1) <= ((1 + 1) * \text{rec}((1 + 1) + 1))) \Rightarrow \dot{\vdash} (\dot{\vdash} (\text{rec}((1 + 1) + 1) = ((1 + 1) * \text{rec}((1 + 1) + 1))) \text{n}) \text{n}) \text{n} \gg \dot{\vdash} (\text{rec}((1 + 1) + 1) <= ((1 + 1) * \text{rec}((1 + 1) + 1))) \Rightarrow \dot{\vdash} (\dot{\vdash} (\text{rec}((1 + 1) + 1) = ((1 + 1) * \text{rec}((1 + 1) + 1))) \text{n}) \text{n}) \text{n}] , p_0, c]$

$[(1/3)x + (1/3)x = (2/3)x \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}. ((\text{rec}((1 + 1) + 1) * \underline{x}) + (\text{rec}((1 + 1) + 1) * \underline{x})) = (((1 + 1) * \text{rec}((1 + 1) + 1)) * \underline{x})]$

$[(1/3)x + (1/3)x = (2/3)x \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}. \text{TwoWholes} \gg ((\text{rec}((1 + 1) + 1) * \underline{x}) + (\text{rec}((1 + 1) + 1) * \underline{x})) = ((1 + 1) * (\text{rec}((1 + 1) + 1) * \underline{x})); \text{timesAssociativity} \gg (((1 + 1) * \text{rec}((1 + 1) + 1)) * \underline{x}) = ((1 + 1) * (\text{rec}((1 + 1) + 1) * \underline{x})); \text{eqSymmetry} \triangleright (((1 + 1) * \text{rec}((1 + 1) + 1)) * \underline{x}) = ((1 + 1) * (\text{rec}((1 + 1) + 1) * \underline{x})) \gg ((1 + 1) * (\text{rec}((1 + 1) + 1) * \underline{x})) = (((1 + 1) * \text{rec}((1 + 1) + 1)) * \underline{x}); \text{eqTransitivity} \triangleright ((\text{rec}((1 + 1) + 1) * \underline{x}) + (\text{rec}((1 + 1) + 1) * \underline{x})) = ((1 + 1) * (\text{rec}((1 + 1) + 1) * \underline{x})) \triangleright ((1 + 1) * (\text{rec}((1 + 1) + 1) * \underline{x})) = (((1 + 1) * \text{rec}((1 + 1) + 1)) * \underline{x}) \gg ((\text{rec}((1 + 1) + 1) * \underline{x}) + (\text{rec}((1 + 1) + 1) * \underline{x})) = (((1 + 1) * \text{rec}((1 + 1) + 1)) * \underline{x}); \text{Repetition} \triangleright ((\text{rec}((1 + 1) + 1) * \underline{x}) + (\text{rec}((1 + 1) + 1) * \underline{x})) = (((1 + 1) * \text{rec}((1 + 1) + 1)) * \underline{x}) \gg ((\text{rec}((1 + 1) + 1) * \underline{x}) + (\text{rec}((1 + 1) + 1) * \underline{x})) = (((1 + 1) * \text{rec}((1 + 1) + 1)) * \underline{x}) , p_0, c]$

$[-(1/3)x - (1/3)x = -(2/3)x \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}. ((-u(\text{rec}((1 + 1) + 1) * \underline{x})) + (-u(\text{rec}((1 + 1) + 1) * \underline{x}))) = (-u(((1 + 1) * \text{rec}((1 + 1) + 1)) * \underline{x}))]$

$[-(1/3)x - (1/3)x = -(2/3)x \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}. (1/3)x + (1/3)x = (2/3)x \gg ((\text{rec}((1 + 1) + 1) * \underline{x}) + (\text{rec}((1 + 1) + 1) * \underline{x})) = (((1 + 1) * \text{rec}((1 + 1) + 1)) * \underline{x}); \text{EqNegated} \triangleright ((\text{rec}((1 + 1) + 1) * \underline{x}) + (\text{rec}((1 + 1) + 1) * \underline{x})) = (((1 + 1) * \text{rec}((1 + 1) + 1)) * \underline{x}) \gg (-u((\text{rec}((1 + 1) + 1) * \underline{x}) + (\text{rec}((1 + 1) + 1) * \underline{x}))) = (-u(((1 + 1) * \text{rec}((1 + 1) + 1)) * \underline{x})); -x - y = -(x + y) \gg ((-u(\text{rec}((1 + 1) + 1) * \underline{x})) + (-u(\text{rec}((1 + 1) + 1) * \underline{x}))) = (-u((\text{rec}((1 + 1) + 1) * \underline{x}) + (\text{rec}((1 + 1) + 1) * \underline{x}))) ; \text{eqTransitivity} \triangleright ((-u(\text{rec}((1 + 1) + 1) * \underline{x})) + (-u(\text{rec}((1 + 1) + 1) * \underline{x}))) =$



$(-u(\text{rec}((1+1)+1) * \underline{x}) + (\text{rec}((1+1)+1) * \underline{x})) \triangleright (-u(\text{rec}((1+1)+1) * \underline{x}) + (\text{rec}((1+1)+1) * \underline{x})) = (-u(((1+1) * \text{rec}((1+1)+1)) * \underline{x})) \gg ((-u(\text{rec}((1+1)+1) * \underline{x})) + (-u(\text{rec}((1+1)+1) * \underline{x}))) = (-u(((1+1) * \text{rec}((1+1)+1)) * \underline{x})), p_0, c]$   
 $[(2/3)x + (1/3)x = x \xrightarrow{\text{stmt}} \text{SystemQ} \vdash$   
 $\forall \underline{x}: (((1+1) * \text{rec}((1+1)+1)) * \underline{x}) + (\text{rec}((1+1)+1) * \underline{x}) = \underline{x}]$   
 $[(2/3)x + (1/3)x = x \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}: \text{TwoWholes} \gg$   
 $((\text{rec}((1+1)+1) * \underline{x}) + (\text{rec}((1+1)+1) * \underline{x})) =$   
 $((1+1) * (\text{rec}((1+1)+1) * \underline{x})); \text{timesAssociativity} \gg (((1+1) * \text{rec}((1+1)+1)) * \underline{x}) =$   
 $((1+1) * (\text{rec}((1+1)+1) * \underline{x})); \text{eqSymmetry} \triangleright (((1+1) * \text{rec}((1+1)+1)) * \underline{x}) =$   
 $((1+1) * (\text{rec}((1+1)+1) * \underline{x})) \gg ((1+1) * (\text{rec}((1+1)+1) * \underline{x})) =$   
 $((1+1) * \text{rec}((1+1)+1) * \underline{x}); \text{eqTransitivity} \triangleright ((\text{rec}((1+1)+1) * \underline{x}) + (\text{rec}((1+1)+1) * \underline{x})) =$   
 $((1+1) * (\text{rec}((1+1)+1) * \underline{x})) \triangleright ((1+1) * (\text{rec}((1+1)+1) * \underline{x})) =$   
 $((1+1) * \text{rec}((1+1)+1) * \underline{x}) \gg ((\text{rec}((1+1)+1) * \underline{x}) + (\text{rec}((1+1)+1) * \underline{x})) =$   
 $((1+1) * \text{rec}((1+1)+1) * \underline{x}); \text{eqAddition} \triangleright ((\text{rec}((1+1)+1) * \underline{x}) + (\text{rec}((1+1)+1) * \underline{x})) =$   
 $((1+1) * \text{rec}((1+1)+1) * \underline{x}) \gg$   
 $((\text{rec}((1+1)+1) * \underline{x}) + (\text{rec}((1+1)+1) * \underline{x})) + (\text{rec}((1+1)+1) * \underline{x}) =$   
 $((1+1) * \text{rec}((1+1)+1) * \underline{x}) + (\text{rec}((1+1)+1) * \underline{x}); \text{ThreeThirds} \gg$   
 $((\text{rec}((1+1)+1) * \underline{x}) + (\text{rec}((1+1)+1) * \underline{x})) + (\text{rec}((1+1)+1) * \underline{x}) =$   
 $\underline{x}; \text{Equality} \triangleright (((\text{rec}((1+1)+1) * \underline{x}) + (\text{rec}((1+1)+1) * \underline{x})) + (\text{rec}((1+1)+1) * \underline{x})) =$   
 $((1+1) * \text{rec}((1+1)+1) * \underline{x}) + (\text{rec}((1+1)+1) * \underline{x}) \triangleright (((\text{rec}((1+1)+1) * \underline{x}) + (\text{rec}((1+1)+1) * \underline{x})) + (\text{rec}((1+1)+1) * \underline{x})) = \underline{x} \gg$   
 $((1+1) * \text{rec}((1+1)+1) * \underline{x}) + (\text{rec}((1+1)+1) * \underline{x}) = \underline{x}], p_0, c]$   
 $[-x + (1/3)x = -(2/3)x \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: ((-\underline{ux}) + (\text{rec}((1+1)+1) * \underline{x})) =$   
 $(-u(((1+1) * \text{rec}((1+1)+1)) * \underline{x}))]$   
 $[-x + (1/3)x = -(2/3)x \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}: (2/3)x + (1/3)x = x \gg$   
 $((1+1) * \text{rec}((1+1)+1) * \underline{x}) + (\text{rec}((1+1)+1) * \underline{x}) =$   
 $\underline{x}; \text{PositiveToRight}(\text{Eq}) \triangleright (((1+1) * \text{rec}((1+1)+1)) * \underline{x}) + (\text{rec}((1+1)+1) * \underline{x}) =$   
 $\underline{x} \gg ((1+1) * \text{rec}((1+1)+1) * \underline{x}) =$   
 $(\underline{x} + (-u(\text{rec}((1+1)+1) * \underline{x}))); \text{EqNegated} \triangleright (((1+1) * \text{rec}((1+1)+1)) * \underline{x}) =$   
 $(\underline{x} + (-u(\text{rec}((1+1)+1) * \underline{x}))) \gg (-u(((1+1) * \text{rec}((1+1)+1)) * \underline{x})) =$   
 $(-u(\underline{x} + (-u(\text{rec}((1+1)+1) * \underline{x}))))); \text{MinusNegated} \gg (-u(\underline{x} + (-u(\text{rec}((1+1)+1) * \underline{x})))) =$   
 $((\text{rec}((1+1)+1) * \underline{x}) + (-\underline{ux})); \text{plusCommutativity} \gg$   
 $((\text{rec}((1+1)+1) * \underline{x}) + (-\underline{ux})) =$   
 $((-\underline{ux}) + (\text{rec}((1+1)+1) * \underline{x})); \text{eqTransitivity4} \triangleright (-u(((1+1) * \text{rec}((1+1)+1)) * \underline{x})) =$   
 $(-u(\underline{x} + (-u(\text{rec}((1+1)+1) * \underline{x})))) \triangleright (-u(\underline{x} + (-u(\text{rec}((1+1)+1) * \underline{x})))) =$   
 $((\text{rec}((1+1)+1) * \underline{x}) + (-\underline{ux})) \triangleright ((\text{rec}((1+1)+1) * \underline{x}) + (-\underline{ux})) =$   
 $((-\underline{ux}) + (\text{rec}((1+1)+1) * \underline{x})) \gg (-u(((1+1) * \text{rec}((1+1)+1)) * \underline{x})) =$   
 $((-\underline{ux}) + (\text{rec}((1+1)+1) * \underline{x})); \text{eqSymmetry} \triangleright (-u(((1+1) * \text{rec}((1+1)+1)) * \underline{x})) =$   
 $((-\underline{ux}) + (\text{rec}((1+1)+1) * \underline{x})) \gg ((-\underline{ux}) + (\text{rec}((1+1)+1) * \underline{x})) =$   
 $(-u(((1+1) * \text{rec}((1+1)+1)) * \underline{x})), p_0, c]$   
line ell y because lemma  $(2/3)x+(1/3)x=x$  indeed  $2/3 * \text{meta } x + 1/3 * \text{meta } x = \text{meta } x$  end line  
line ell a because lemma  $\text{eqTransitivity modus ponens}$  ell  
x modus ponens ell y indeed  $1/3 * \text{meta } x + 2/3 * \text{meta } x = \text{meta } x$  end line  
 $[\text{PreserveLessGreater} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall (\underline{x1}): \forall (\underline{x2}): \forall (\underline{y1}): \forall (\underline{y2}): \forall \underline{z}: (\underline{x1}) < =$



$$\begin{aligned}
& \dot{\vdash}(((y1) + (-u(\text{rec}((1+1)+1)*z))) <= (y2) \Rightarrow \dot{\vdash}(\dot{\vdash}(((y1) + (-u(\text{rec}((1+1)+1)*z))) = (y2))n)n)n; \text{LessAddition} \triangleright \dot{\vdash}(((y1) + (-u(\text{rec}((1+1)+1)*z))) <= (y2) \Rightarrow \\
& \dot{\vdash}(\dot{\vdash}(((y1) + (-u(\text{rec}((1+1)+1)*z))) = (y2))n)n)n \gg \dot{\vdash}(((y1) + (-u(\text{rec}((1+1)+1)*z))) + (-u(\text{rec}((1+1)+1)*z))) <= ((y2) + (-u(\text{rec}((1+1)+1)*z))) \Rightarrow \\
& \dot{\vdash}(\dot{\vdash}(((y1) + (-u(\text{rec}((1+1)+1)*z))) + (-u(\text{rec}((1+1)+1)*z))) = ((y2) + (-u(\text{rec}((1+1)+1)*z))))n)n)n; \text{plusAssociativity} \gg \\
& (((y1) + (-u(\text{rec}((1+1)+1)*z))) + (-u(\text{rec}((1+1)+1)*z))) = \\
& ((y1) + ((-u(\text{rec}((1+1)+1)*z)) + (-u(\text{rec}((1+1)+1)*z))))); - (1/3)x - (1/3)x = \\
& - (2/3)x \gg ((-u(\text{rec}((1+1)+1)*z)) + (-u(\text{rec}((1+1)+1)*z))) = \\
& (-u(((1+1)*\text{rec}((1+1)+1))*z)); \text{EqAdditionLeft} \triangleright ((-u(\text{rec}((1+1)+1)*z)) + (-u(\text{rec}((1+1)+1)*z))) = (-u(((1+1)*\text{rec}((1+1)+1))*z)) \gg \\
& ((y1) + ((-u(\text{rec}((1+1)+1)*z)) + (-u(\text{rec}((1+1)+1)*z)))) = \\
& ((y1) + (-u(((1+1)*\text{rec}((1+1)+1))*z))); \text{eqTransitivity} \triangleright (((y1) + \\
& (-u(\text{rec}((1+1)+1)*z))) + (-u(\text{rec}((1+1)+1)*z))) = ((y1) + ((-u(\text{rec}((1+1)+1)*z)) + \\
& (-u(\text{rec}((1+1)+1)*z)))) = ((y1) + (-u(((1+1)*\text{rec}((1+1)+1))*z))) \gg \\
& (((y1) + (-u(\text{rec}((1+1)+1)*z))) + (-u(\text{rec}((1+1)+1)*z))) = \\
& ((y1) + (-u(((1+1)*\text{rec}((1+1)+1))*z))); \text{SubLessLeft} \triangleright (((y1) + (-u(\text{rec}((1+1)+1)*z))) + (-u(\text{rec}((1+1)+1)*z))) = ((y1) + (-u(((1+1)*\text{rec}((1+1)+1))*z))) \triangleright \dot{\vdash}(((y1) + (-u(\text{rec}((1+1)+1)*z))) + (-u(\text{rec}((1+1)+1)*z))) <= \\
& ((y2) + (-u(\text{rec}((1+1)+1)*z))) \Rightarrow \dot{\vdash}(\dot{\vdash}(((y1) + (-u(\text{rec}((1+1)+1)*z))) + (-u(\text{rec}((1+1)+1)*z))) + \\
& (-u(\text{rec}((1+1)+1)*z))) = ((y2) + (-u(\text{rec}((1+1)+1)*z))))n)n)n \gg \\
& \dot{\vdash}(((y1) + (-u(((1+1)*\text{rec}((1+1)+1))*z))) <= ((y2) + (-u(\text{rec}((1+1)+1)*z))) \\
& \Rightarrow \dot{\vdash}(\dot{\vdash}(((y1) + (-u(((1+1)*\text{rec}((1+1)+1))*z))) <= ((y2) + (-u(\text{rec}((1+1)+1)*z))))n)n)n; \text{LessLeqTransitivity} \triangleright \dot{\vdash}((x2) <= \\
& ((x1) + (\text{rec}((1+1)+1)*z))) \Rightarrow \dot{\vdash}(\dot{\vdash}((x2) = ((x1) + (\text{rec}((1+1)+1)*z))))n)n)n \triangleright \\
& ((x1) + (\text{rec}((1+1)+1)*z)) <= ((y1) + (-u(((1+1)*\text{rec}((1+1)+1))*z))) \gg \\
& \dot{\vdash}((x2) <= ((y1) + (-u(((1+1)*\text{rec}((1+1)+1))*z)))) \Rightarrow \dot{\vdash}(\dot{\vdash}((x2) = \\
& ((y1) + (-u(((1+1)*\text{rec}((1+1)+1))*z))))n)n)n; \text{LessTransitivity} \triangleright \dot{\vdash}((x2) <= \\
& ((y1) + (-u(((1+1)*\text{rec}((1+1)+1))*z)))) \Rightarrow \dot{\vdash}(\dot{\vdash}((x2) = \\
& ((y1) + (-u(((1+1)*\text{rec}((1+1)+1))*z))))n)n)n \triangleright \dot{\vdash}(((y1) + (-u(((1+1)*\text{rec}((1+1)+1))*z))) <= ((y2) + (-u(\text{rec}((1+1)+1)*z)))) \Rightarrow \\
& \dot{\vdash}(\dot{\vdash}(((y1) + (-u(((1+1)*\text{rec}((1+1)+1))*z))) = ((y2) + (-u(\text{rec}((1+1)+1)*z))))n)n)n \gg \dot{\vdash}((x2) <= ((y2) + (-u(\text{rec}((1+1)+1)*z)))) \Rightarrow \dot{\vdash}(\dot{\vdash}((x2) = \\
& ((y2) + (-u(\text{rec}((1+1)+1)*z))))n)n)n; \text{LessLeq} \triangleright \dot{\vdash}((x2) <= \\
& ((y2) + (-u(\text{rec}((1+1)+1)*z)))) \Rightarrow \dot{\vdash}(\dot{\vdash}((x2) = ((y2) + (-u(\text{rec}((1+1)+1)*z))))n)n)n \gg \underline{x2} <= ((y2) + (-u(\text{rec}((1+1)+1)*z))), \text{Po, c}]
\end{aligned}$$

$$\begin{aligned}
& [\text{ClosetolessIsLess} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall(x1): \forall(x2): \forall y: \forall z: (x1) <= (y + (-uz)) \vdash \\
& \dot{\vdash}(|((x1) + (-u(x2)))| <= (\text{rec}((1+1)+1)*z) \Rightarrow \dot{\vdash}(\dot{\vdash}(|((x1) + (-u(x2)))| = \\
& (\text{rec}((1+1)+1)*z))n)n)n \vdash \underline{x2} <= (y + (-u(\text{rec}((1+1)+1)*z)))]
\end{aligned}$$

$$\begin{aligned}
& [\text{ClosetolessIsLess} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\vdash \text{SystemQ} \vdash \forall(x1): \forall(x2): \forall y: \forall z: (x1) <= \\
& (y + (-uz)) \vdash \dot{\vdash}(|((x1) + (-u(x2)))| <= (\text{rec}((1+1)+1)*z) \Rightarrow \\
& \dot{\vdash}(\dot{\vdash}(|((x1) + (-u(x2)))| = (\text{rec}((1+1)+1)*z))n)n)n \vdash 0 <= |x| \gg 0 <= \\
& |((x1) + (-u(x2)))|); \text{leqLessTransitivity} \triangleright 0 <= \\
& |((x1) + (-u(x2)))| \triangleright \dot{\vdash}(|((x1) + (-u(x2)))| <= (\text{rec}((1+1)+1)*z) \Rightarrow
\end{aligned}$$











































































































































































































































































$$\begin{aligned}
& \{\{\overline{(f3)}, \overline{(f3)}\}, \{\overline{(f3)}, \overline{(f4)}\}\} \in f_{Ph} \Rightarrow \overline{(f1)} = \overline{(f3)} \Rightarrow \overline{(f2)} = \overline{(f4)} \} n \Rightarrow \\
& \dot{\neg} (\forall_{obj} \overline{(s1)} : (s1) \in N \Rightarrow \dot{\neg} (\forall_{obj} \overline{(s2)} : \dot{\neg} (\{\{\overline{(s1)}, \overline{(s1)}\}, \{\overline{(s1)}, \overline{(s2)}\}\} \in \\
& f_{Ph} \} n) n) n) \} | \forall_{obj} \overline{(\epsilon)} : \dot{\neg} (\forall_{obj} \overline{n} : \dot{\neg} (\forall_{obj} \overline{m} : \dot{\neg} (0 \leq \overline{(\epsilon)} \Rightarrow \dot{\neg} (\dot{\neg} (0 = \overline{(\epsilon)}) n) n) n) \Rightarrow \\
& \overline{n} \leq \overline{m} \Rightarrow \dot{\neg} (|\{\{ph \in \{ph \in P(P(\text{Union}(\{N, Q\}))\})\}| \\
& \dot{\neg} (\forall_{obj} \overline{(op1)} : \dot{\neg} (\dot{\neg} (\forall_{obj} \overline{(op2)} : \dot{\neg} (\dot{\neg} (\dot{\neg} (\overline{(op1)} \in N \Rightarrow \dot{\neg} (\overline{(op2)} \in Q) n) n) \Rightarrow \\
& \dot{\neg} (a_{Ph} = \{\{\overline{(op1)}, \overline{(op1)}\}, \{\overline{(op1)}, \overline{(op2)}\}\} n) n) n) n) n) \} | \dot{\neg} (\forall_{obj} \overline{m} : \dot{\neg} (e_{Ph} = \\
& \{\{\overline{m}, \overline{m}\}, \{\overline{m}, ((fx)[\overline{m}] * (fz)[\overline{m}])\}\} n) n) \} [\overline{m}] + (-ud_{Ph}[\overline{m}])) | \leq \overline{(\epsilon)} \Rightarrow \\
& \dot{\neg} (\dot{\neg} (|\{\{ph \in \{ph \in P(P(\text{Union}(\{N, Q\}))\})\}| \\
& \dot{\neg} (\forall_{obj} \overline{(op1)} : \dot{\neg} (\dot{\neg} (\forall_{obj} \overline{(op2)} : \dot{\neg} (\dot{\neg} (\dot{\neg} (\overline{(op1)} \in N \Rightarrow \dot{\neg} (\overline{(op2)} \in Q) n) n) \Rightarrow \\
& \dot{\neg} (a_{Ph} = \{\{\overline{(op1)}, \overline{(op1)}\}, \{\overline{(op1)}, \overline{(op2)}\}\} n) n) n) n) n) \} | \dot{\neg} (\dot{\neg} (\forall_{obj} \overline{(r1)} : \overline{(r1)} \in \\
& f_{Ph} \Rightarrow \dot{\neg} (\forall_{obj} \overline{(op1)} : \dot{\neg} (\dot{\neg} (\forall_{obj} \overline{(op2)} : \dot{\neg} (\dot{\neg} (\dot{\neg} (\overline{(op1)} \in N \Rightarrow \dot{\neg} (\overline{(op2)} \in Q) n) n) \Rightarrow \\
& \dot{\neg} (\overline{(r1)} = \{\{\overline{(op1)}, \overline{(op1)}\}, \{\overline{(op1)}, \overline{(op2)}\}\} n) n) n) n) n) \Rightarrow \\
& \dot{\neg} (\forall_{obj} \overline{(f1)} : \forall_{obj} \overline{(f2)} : \forall_{obj} \overline{(f3)} : \forall_{obj} \overline{(f4)} : \{\{\overline{(f1)}, \overline{(f1)}\}, \{\overline{(f1)}, \overline{(f2)}\}\} \in f_{Ph} \Rightarrow \\
& \{\{\overline{(f3)}, \overline{(f3)}\}, \{\overline{(f3)}, \overline{(f4)}\}\} \in f_{Ph} \Rightarrow \overline{(f1)} = \overline{(f3)} \Rightarrow \overline{(f2)} = \overline{(f4)} \} n \Rightarrow \\
& \dot{\neg} (\forall_{obj} \overline{(s1)} : (s1) \in N \Rightarrow \dot{\neg} (\forall_{obj} \overline{(s2)} : \dot{\neg} (\{\{\overline{(s1)}, \overline{(s1)}\}, \{\overline{(s1)}, \overline{(s2)}\}\} \in \\
& f_{Ph} \} n) n) n) \} | \forall_{obj} \overline{(\epsilon)} : \dot{\neg} (\forall_{obj} \overline{n} : \dot{\neg} (\forall_{obj} \overline{m} : \dot{\neg} (0 \leq \overline{(\epsilon)} \Rightarrow \dot{\neg} (\dot{\neg} (0 = \overline{(\epsilon)}) n) n) n) \Rightarrow \\
& \overline{n} \leq \overline{m} \Rightarrow \dot{\neg} (|\{\{ph \in \{ph \in P(P(\text{Union}(\{N, Q\}))\})\}| \\
& \dot{\neg} (\forall_{obj} \overline{(op1)} : \dot{\neg} (\dot{\neg} (\forall_{obj} \overline{(op2)} : \dot{\neg} (\dot{\neg} (\dot{\neg} (\overline{(op1)} \in N \Rightarrow \dot{\neg} (\overline{(op2)} \in Q) n) n) \Rightarrow \\
& \dot{\neg} (a_{Ph} = \{\{\overline{(op1)}, \overline{(op1)}\}, \{\overline{(op1)}, \overline{(op2)}\}\} n) n) n) n) n) \} | \dot{\neg} (\forall_{obj} \overline{m} : \dot{\neg} (e_{Ph} = \\
& \{\{\overline{m}, \overline{m}\}, \{\overline{m}, ((fy)[\overline{m}] * (fz)[\overline{m}])\}\} n) n) \} [\overline{m}] + (-ud_{Ph}[\overline{m}])) | \leq \overline{(\epsilon)} \Rightarrow \\
& \dot{\neg} (\dot{\neg} (|\{\{ph \in \{ph \in P(P(\text{Union}(\{N, Q\}))\})\}| \\
& \dot{\neg} (\forall_{obj} \overline{(op1)} : \dot{\neg} (\dot{\neg} (\forall_{obj} \overline{(op2)} : \dot{\neg} (\dot{\neg} (\dot{\neg} (\overline{(op1)} \in N \Rightarrow \dot{\neg} (\overline{(op2)} \in Q) n) n) \Rightarrow \\
& \dot{\neg} (a_{Ph} = \{\{\overline{(op1)}, \overline{(op1)}\}, \{\overline{(op1)}, \overline{(op2)}\}\} n) n) n) n) n) \} | \dot{\neg} (\forall_{obj} \overline{m} : \dot{\neg} (e_{Ph} = \\
& \{\{\overline{m}, \overline{m}\}, \{\overline{m}, ((fy)[\overline{m}] * (fz)[\overline{m}])\}\} n) n) \} [\overline{m}] + (-ud_{Ph}[\overline{m}])) | = \\
& \overline{(\epsilon)} \} n) n) n) n) \}, p_0, c)
\end{aligned}$$

[Cauchy(2)(Helper)  $\xrightarrow{stmt}$  SystemQ]  $\vdash$

$$\begin{aligned}
& \forall \overline{(v1)} : \forall \overline{(v2)} : \forall \overline{(n1)} : \forall \overline{(n2)} : \forall \overline{(\epsilon)} : \forall \overline{(fx)} : \forall \overline{(fy)} : \forall_{obj} \overline{(v1)} : \forall_{obj} \overline{(v2)} : \dot{\neg} (0 \leq \overline{(\epsilon)} \Rightarrow \\
& \dot{\neg} (\dot{\neg} (0 = \overline{(\epsilon)}) n) n) n) \Rightarrow \overline{(n1)} \leq \overline{(v1)} \Rightarrow \overline{(n1)} \leq \overline{(v2)} \Rightarrow \\
& \dot{\neg} (|\{((fx)[\overline{(v1)}] + (-u(fx)[\overline{(v2)}]))\} | \leq \overline{(\epsilon)} \Rightarrow \dot{\neg} (\dot{\neg} (|\{((fx)[\overline{(v1)}] + (-u(fx)[\overline{(v2)}]))\} | = \\
& \overline{(\epsilon)}) n) n) n) \Rightarrow \forall_{obj} \overline{(v1)} : \forall_{obj} \overline{(v2)} : \dot{\neg} (0 \leq \overline{(\epsilon)} \Rightarrow \dot{\neg} (\dot{\neg} (0 = \overline{(\epsilon)}) n) n) n) \Rightarrow \overline{(n2)} \leq \\
& \overline{(v1)} \Rightarrow \overline{(n2)} \leq \overline{(v2)} \Rightarrow \dot{\neg} (|\{((fy)[\overline{(v1)}] + (-u(fy)[\overline{(v2)}]))\} | \leq \overline{(\epsilon)} \Rightarrow \\
& \dot{\neg} (\dot{\neg} (|\{((fy)[\overline{(v1)}] + (-u(fy)[\overline{(v2)}]))\} | = \overline{(\epsilon)}) n) n) n) \Rightarrow \dot{\neg} (0 \leq \overline{(\epsilon)} \Rightarrow \dot{\neg} (\dot{\neg} (0 = \\
& \overline{(\epsilon)}) n) n) n) \Rightarrow \text{if}(\overline{(n2)} \leq \overline{(n1)}, \overline{(n1)}, \overline{(n2)}) \leq \overline{(v1)} \Rightarrow \text{if}(\overline{(n2)} \leq \\
& \overline{(n1)}, \overline{(n1)}, \overline{(n2)}) \leq \overline{(v2)} \Rightarrow \dot{\neg} (\dot{\neg} (|\{((fx)[\overline{(v1)}] + (-u(fx)[\overline{(v2)}]))\} | \leq \overline{(\epsilon)} \Rightarrow \\
& \dot{\neg} (\dot{\neg} (|\{((fx)[\overline{(v1)}] + (-u(fx)[\overline{(v2)}]))\} | = \overline{(\epsilon)}) n) n) n) \Rightarrow \dot{\neg} (\dot{\neg} (|\{((fy)[\overline{(v1)}] + \\
& (-u(fy)[\overline{(v2)}]))\} | \leq \overline{(\epsilon)} \Rightarrow \dot{\neg} (\dot{\neg} (|\{((fy)[\overline{(v1)}] + (-u(fy)[\overline{(v2)}]))\} | = \overline{(\epsilon)}) n) n) n) n)
\end{aligned}$$

[Cauchy(2)(Helper)  $\xrightarrow{proof}$   $\lambda c. \lambda x. \mathcal{P}(\ulcorner \text{SystemQ} \urcorner \vdash$

$$\begin{aligned}
& \forall \overline{(v1)} : \forall \overline{(v2)} : \forall \overline{(n1)} : \forall \overline{(n2)} : \forall \overline{(\epsilon)} : \forall \overline{(fx)} : \forall \overline{(fy)} : \forall_{obj} \overline{(v1)} : \forall_{obj} \overline{(v2)} : \dot{\neg} (0 \leq \overline{(\epsilon)} \Rightarrow \\
& \dot{\neg} (\dot{\neg} (0 = \overline{(\epsilon)}) n) n) n) \Rightarrow \overline{(n1)} \leq \overline{(v1)} \Rightarrow \overline{(n1)} \leq \overline{(v2)} \Rightarrow
\end{aligned}$$









$$\begin{aligned}
& (|(x + (-uz))| + |(z + (-uu))| + |(u + y)|) \gg |(x + y)| \leq = \\
& (|(x + (-uz))| + |(z + (-uu))| + |(u + y)|); \text{plusAssociativity} \gg \\
& ((|(x + (-uz))| + |(z + (-uu))| + |(u + y)|) = (|(x + (-uz))| + |(z + (-uu))| + |(u + y)|)); \text{eqSymmetry} \triangleright ((|(x + (-uz))| + |(z + (-uu))| + |(u + y)|) = (|(x + (-uz))| + |(z + (-uu))| + |(u + y)|)); \text{subLeqRight} \triangleright ((|(x + (-uz))| + |(z + (-uu))| + |(u + y)|) \triangleright |(x + y)|) \leq = \\
& (|(x + (-uz))| + |(z + (-uu))| + |(u + y)|) \gg |(x + y)| \leq = \\
& ((|(x + (-uz))| + |(z + (-uu))| + |(u + y)|), p_0, c)
\end{aligned}$$

$$[\text{FromPositiveNumerical} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall x: \dot{\vdash} (0 \leq |x| \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = |x|)n)n) \vdash \dot{\vdash} (x = 0)n]$$

$$\begin{aligned}
& [\text{FromPositiveNumerical} \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall x: 0 \leq x \vdash \dot{\vdash} (0 \leq |x| \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = |x|)n)n) \vdash \text{NonnegativeNumerical} \triangleright 0 \leq x \gg |x| = \\
& x; \text{SubLessRight} \triangleright |x| = x \triangleright \dot{\vdash} (0 \leq |x| \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = |x|)n)n) \gg \dot{\vdash} (0 \leq x \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = x)n)n); \text{LessNeq} \triangleright \dot{\vdash} (0 \leq x \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = x)n)n) \gg \dot{\vdash} (0 = x)n; \text{NeqSymmetry} \triangleright \dot{\vdash} (0 = x)n \gg \dot{\vdash} (x = 0)n; \forall x: x \leq 0 \vdash \dot{\vdash} (0 \leq |x| \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = |x|)n)n) \vdash \text{lemma nonpositiveNumerical} \triangleright x \leq 0 \gg |x| = (-ux); \text{SubLessRight} \triangleright |x| = (-ux) \triangleright \dot{\vdash} (0 \leq |x| \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = |x|)n)n) \gg \dot{\vdash} (0 \leq (-ux) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = (-ux))n)n); \text{PositiveNegated} \triangleright \dot{\vdash} (0 \leq (-ux) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = (-ux))n)n) \gg \dot{\vdash} ((-u(-ux)) \leq 0 \Rightarrow \dot{\vdash} (\dot{\vdash} ((-u(-ux)) = 0)n)n); \text{DoubleMinus} \gg (-u(-ux)) = x; \text{SubLessLeft} \triangleright (-u(-ux)) = x \triangleright \dot{\vdash} ((-u(-ux)) \leq 0 \Rightarrow \dot{\vdash} (\dot{\vdash} ((-u(-ux)) = 0)n)n) \gg \dot{\vdash} (x \leq 0 \Rightarrow \dot{\vdash} (\dot{\vdash} (x = 0)n)n); \text{LessNeq} \triangleright \dot{\vdash} (x \leq 0 \Rightarrow \dot{\vdash} (\dot{\vdash} (x = 0)n)n) \gg \dot{\vdash} (x = 0)n; \forall x: \text{Ded} \triangleright \forall x: 0 \leq x \vdash \dot{\vdash} (0 \leq |x| \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = |x|)n)n) \vdash \dot{\vdash} (x = 0)n \gg 0 \leq x \Rightarrow \dot{\vdash} (0 \leq |x| \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = |x|)n)n) \Rightarrow \dot{\vdash} (x = 0)n; \text{Ded} \triangleright \forall x: x \leq 0 \vdash \dot{\vdash} (0 \leq |x| \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = |x|)n)n) \vdash \dot{\vdash} (x = 0)n \gg x \leq 0 \Rightarrow \dot{\vdash} (0 \leq |x| \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = |x|)n)n) \Rightarrow \dot{\vdash} (x = 0)n; \dot{\vdash} (0 \leq |x| \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = |x|)n)n) \vdash \text{FromLeqGeq} \triangleright 0 \leq x \Rightarrow \dot{\vdash} (0 \leq |x| \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = |x|)n)n) \Rightarrow \dot{\vdash} (x = 0)n \triangleright x \leq 0 \Rightarrow \dot{\vdash} (0 \leq |x| \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = |x|)n)n) \Rightarrow \dot{\vdash} (x = 0)n \gg \dot{\vdash} (0 \leq |x| \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = |x|)n)n) \Rightarrow \dot{\vdash} (x = 0)n; \text{MP} \triangleright \dot{\vdash} (0 \leq |x| \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = |x|)n)n) \Rightarrow \dot{\vdash} (x = 0)n \triangleright \dot{\vdash} (0 \leq |x| \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = |x|)n)n) \gg \dot{\vdash} (x = 0)n \rceil, p_0, c)
\end{aligned}$$

$$[\text{NegativeToRight}(\text{Neq})(1\text{term}) \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall x: \forall y: \dot{\vdash} ((x + (-uy)) = 0)n \vdash \dot{\vdash} (x = y)n]$$

$$\begin{aligned}
& [\text{NegativeToRight}(\text{Neq})(1\text{term}) \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall x: \forall y: x = y \vdash \text{PositiveToLeft}(\text{Eq})(1\text{term}) \triangleright x = y \gg (x + (-uy)) = 0; \forall x: \forall y: \text{Ded} \triangleright \forall x: \forall y: x = y \vdash (x + (-uy)) = 0 \gg x = y \Rightarrow (x + (-uy)) = 0; \dot{\vdash} ((x + (-uy)) = 0)n \vdash \text{MT} \triangleright x = y \Rightarrow (x + (-uy)) = 0 \triangleright \dot{\vdash} ((x + (-uy)) = 0)n \gg \dot{\vdash} (x = y)n \rceil, p_0, c)
\end{aligned}$$

$$[\text{NonzeroProduct}(2) \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall x: \forall y: \dot{\vdash} ((x * y) = 0)n \vdash \dot{\vdash} (y = 0)n]$$

$$\begin{aligned}
& [\text{NonzeroProduct}(2) \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall x: \forall y: y = 0 \vdash \text{EqMultiplicationLeft} \triangleright y = 0 \gg (x * y) = (x * 0); x * 0 = 0 \gg (x * 0) = 0; \text{eqTransitivity} \triangleright (x * y) = (x * 0) \triangleright (x * 0) = 0 \gg (x * y) = 0; \forall x: \forall y: \text{Ded} \triangleright \forall x: \forall y: y = 0 \vdash (x * y) = 0 \gg y = 0 \Rightarrow (x * y) = 0; \dot{\vdash} ((x * y) = 0)n \vdash \text{MT} \triangleright y = 0 \Rightarrow (x * y) = 0 \triangleright \dot{\vdash} ((x * y) = 0)n \gg \dot{\vdash} (y = 0)n \rceil, p_0, c)
\end{aligned}$$

[NonreciprocalToRight(Eq)(1term)  $\xrightarrow{\text{stmt}}$  SystemQ  $\vdash \forall \underline{x}: \forall \underline{y}: (\underline{x} * \underline{y}) = 1 \vdash \underline{x} = \text{recy}$ ]

[NonreciprocalToRight(Eq)(1term)  $\xrightarrow{\text{proof}}$   $\lambda c. \lambda x. \lambda \mathcal{P}(\ulcorner \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: (\underline{x} * \underline{y}) = 1 \vdash \text{eqMultiplication} \triangleright (\underline{x} * \underline{y}) = 1 \gg ((\underline{x} * \underline{y}) * \text{recy}) = (1 * \text{recy}); 0 < 1 \gg \dot{\vdash} (0 < = 1 \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = 1)n)n)n; \text{PositiveNonzero} \triangleright \dot{\vdash} (0 < = 1 \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = 1)n)n)n \gg \dot{\vdash} (1 = 0)n; \text{eqSymmetry} \triangleright (\underline{x} * \underline{y}) = 1 \gg 1 = (\underline{x} * \underline{y}); \text{SubNeqLeft} \triangleright 1 = (\underline{x} * \underline{y}) \triangleright \dot{\vdash} (1 = 0)n \gg \dot{\vdash} ((\underline{x} * \underline{y}) = 0)n; \text{NonzeroProduct}(2) \triangleright \dot{\vdash} ((\underline{x} * \underline{y}) = 0)n \gg \dot{\vdash} (\underline{y} = 0)n; \underline{x} = \underline{x} * \underline{y} * (1/\underline{y}) \triangleright \dot{\vdash} (\underline{y} = 0)n \gg \underline{x} = ((\underline{x} * \underline{y}) * \text{recy}); \text{times1Left} \gg (1 * \text{recy}) = \text{recy}; \text{eqTransitivity4} \triangleright \underline{x} = ((\underline{x} * \underline{y}) * \text{recy}) \triangleright ((\underline{x} * \underline{y}) * \text{recy}) = (1 * \text{recy}) \triangleright (1 * \text{recy}) = \text{recy} \gg \underline{x} = \text{recy} \rceil, p_0, c)$ ]

[NonreciprocalToRight(Eq)  $\xrightarrow{\text{stmt}}$  SystemQ  $\vdash \forall \underline{x}: \forall \underline{y}: \forall \underline{z}: \dot{\vdash} (\underline{y} = 0)n \vdash (\underline{x} * \underline{y}) = \underline{z} \vdash \underline{x} = (\underline{z} * \text{recy})$ ]

[NonreciprocalToRight(Eq)  $\xrightarrow{\text{proof}}$   $\lambda c. \lambda x. \lambda \mathcal{P}(\ulcorner \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \forall \underline{z}: \dot{\vdash} (\underline{y} = 0)n \vdash (\underline{x} * \underline{y}) = \underline{z} \vdash \text{eqMultiplication} \triangleright (\underline{x} * \underline{y}) = \underline{z} \gg ((\underline{x} * \underline{y}) * \text{recy}) = (\underline{z} * \text{recy}); \underline{x} = \underline{x} * \underline{y} * (1/\underline{y}) \triangleright \dot{\vdash} (\underline{y} = 0)n \gg \underline{x} = ((\underline{x} * \underline{y}) * \text{recy}); \text{eqTransitivity} \triangleright \underline{x} = ((\underline{x} * \underline{y}) * \text{recy}) \triangleright ((\underline{x} * \underline{y}) * \text{recy}) = (\underline{z} * \text{recy}) \gg \underline{x} = (\underline{z} * \text{recy}) \rceil, p_0, c)$ ]

[NonreciprocalToLeft(Eq)(1term)  $\xrightarrow{\text{stmt}}$  SystemQ  $\vdash \forall \underline{x}: \forall \underline{y}: 1 = (\underline{x} * \underline{y}) \vdash \text{recy} = \underline{x}$ ]

[NonreciprocalToLeft(Eq)(1term)  $\xrightarrow{\text{proof}}$   $\lambda c. \lambda x. \lambda \mathcal{P}(\ulcorner \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: 1 = (\underline{x} * \underline{y}) \vdash \text{eqSymmetry} \triangleright 1 = (\underline{x} * \underline{y}) \gg (\underline{x} * \underline{y}) = 1; \text{NonreciprocalToRight(Eq)(1term)} \triangleright (\underline{x} * \underline{y}) = 1 \gg \underline{x} = \text{recy}; \text{eqSymmetry} \triangleright \underline{x} = \text{recy} \gg \text{recy} = \underline{x} \rceil, p_0, c)$ ]

[SameReciprocal  $\xrightarrow{\text{stmt}}$  SystemQ  $\vdash \forall \underline{x}: \forall \underline{y}: \dot{\vdash} (\underline{x} = 0)n \vdash \underline{x} = \underline{y} \vdash \text{recx} = \text{recy}$ ]

[SameReciprocal  $\xrightarrow{\text{proof}}$   $\lambda c. \lambda x. \lambda \mathcal{P}(\ulcorner \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \dot{\vdash} (\underline{x} = 0)n \vdash \underline{x} = \underline{y} \vdash \text{times1Left} \gg (1 * \underline{x}) = \underline{x}; \text{eqTransitivity} \triangleright (1 * \underline{x}) = \underline{x} \triangleright \underline{x} = \underline{y} \gg (1 * \underline{x}) = \underline{y}; \text{NonreciprocalToRight(Eq)} \triangleright \dot{\vdash} (\underline{x} = 0)n \triangleright (1 * \underline{x}) = \underline{y} \gg 1 = (\underline{y} * \text{recx}); \text{timesCommutativity} \gg (\underline{y} * \text{recx}) = (\text{recx} * \underline{y}); \text{eqTransitivity} \triangleright 1 = (\underline{y} * \text{recx}) \triangleright (\underline{y} * \text{recx}) = (\text{recx} * \underline{y}) \gg 1 = (\text{recx} * \underline{y}); \text{NonreciprocalToLeft(Eq)(1term)} \triangleright 1 = (\text{recx} * \underline{y}) \gg \text{recy} = \text{recx}; \text{eqSymmetry} \triangleright \text{recy} = \text{recx} \gg \text{recx} = \text{recy} \rceil, p_0, c)$ ]

[OrderedPairEquality  $\xrightarrow{\text{stmt}}$  SystemQ  $\vdash$

$\forall (\underline{sx}): \forall (\underline{sx1}): \forall (\underline{sy}): \forall (\underline{sy1}): \forall (\underline{sz}): \forall (\underline{sz1}): \forall (\underline{su}): \forall (\underline{su1}): \{ \{ (\underline{sx}), (\underline{sx}) \}, \{ (\underline{sx}), (\underline{sx1}) \} \} = \{ \{ (\underline{sy}), (\underline{sy}) \}, \{ (\underline{sy}), (\underline{sy1}) \} \} \vdash \{ \{ (\underline{sz}), (\underline{sz}) \}, \{ (\underline{sz}), (\underline{sz1}) \} \} = \{ \{ (\underline{su}), (\underline{su}) \}, \{ (\underline{su}), (\underline{su1}) \} \} \vdash (\underline{sx}) = (\underline{sz}) \vdash (\underline{sy}) = (\underline{su})$

[OrderedPairEquality  $\xrightarrow{\text{proof}}$   $\lambda c. \lambda x. \lambda \mathcal{P}(\ulcorner \text{SystemQ} \vdash$

$\forall (\underline{sx}): \forall (\underline{sx1}): \forall (\underline{sy}): \forall (\underline{sy1}): \forall (\underline{sz}): \forall (\underline{sz1}): \forall (\underline{su}): \forall (\underline{su1}): \{ \{ (\underline{sx}), (\underline{sx}) \}, \{ (\underline{sx}), (\underline{sx1}) \} \} = \{ \{ (\underline{sy}), (\underline{sy}) \}, \{ (\underline{sy}), (\underline{sy1}) \} \} \vdash \{ \{ (\underline{sz}), (\underline{sz}) \}, \{ (\underline{sz}), (\underline{sz1}) \} \} = \{ \{ (\underline{su}), (\underline{su}) \}, \{ (\underline{su}), (\underline{su1}) \} \} \vdash (\underline{sx}) = (\underline{sz}) \vdash$

$\text{FromOrderedPair}(1) \triangleright \{ \{ (\underline{sx}), (\underline{sx}) \}, \{ (\underline{sx}), (\underline{sx1}) \} \} = \{ \{ (\underline{sy}), (\underline{sy}) \}, \{ (\underline{sy}), (\underline{sy1}) \} \} \gg (\underline{sx}) = (\underline{sy}); \text{eqSymmetry} \triangleright (\underline{sx}) = (\underline{sy}) \gg (\underline{sy}) = (\underline{sx}); \text{FromOrderedPair}(1) \triangleright \{ \{ (\underline{sz}), (\underline{sz}) \}, \{ (\underline{sz}), (\underline{sz1}) \} \} = \{ \{ (\underline{su}), (\underline{su}) \}, \{ (\underline{su}), (\underline{su1}) \} \} \gg (\underline{sz}) = (\underline{su}); \text{eqTransitivity4} \triangleright (\underline{sy}) = (\underline{sx}) \triangleright (\underline{sx}) = (\underline{sz}) \triangleright (\underline{sz}) = (\underline{su}) \gg (\underline{sy}) = (\underline{su}) \rceil, p_0, c)$

[ReciprocalIsFunction  $\xrightarrow{\text{stmt}}$  SystemQ  $\vdash$

$\forall \underline{m}: \forall \underline{m}: \forall (\underline{fx}): \forall_{\text{obj}}(\underline{f1}): \forall_{\text{obj}}(\underline{f2}): \forall_{\text{obj}}(\underline{f3}): \forall_{\text{obj}}(\underline{f4}): \{ \{ (\underline{f1}), (\underline{f1}) \}, \{ (\underline{f1}), (\underline{f2}) \} \} \in$

















































$(\underline{y} + \underline{z})n)n \gg \dot{\vdash}((\underline{x} + (-\underline{uz})) <= \underline{y} \Rightarrow \dot{\vdash}(\dot{\vdash}((\underline{x} + (-\underline{uz})) = \underline{y})n)n)n], p_0, c)]$

$[\text{SwitchTerms}(x < y - z) \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \forall \underline{z}: \dot{\vdash}(\underline{x} <= (\underline{y} + (-\underline{uz})) \Rightarrow \dot{\vdash}(\dot{\vdash}(\underline{x} = (\underline{y} + (-\underline{uz})))n)n)n \vdash \dot{\vdash}(\underline{z} <= (\underline{y} + (-\underline{ux})) \Rightarrow \dot{\vdash}(\dot{\vdash}(\underline{z} = (\underline{y} + (-\underline{ux})))n)n)n]$

$[\text{SwitchTerms}(x < y - z) \xrightarrow{\text{proof}} \lambda c. \lambda x. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \forall \underline{z}: \dot{\vdash}(\underline{x} <= (\underline{y} + (-\underline{uz})) \Rightarrow \dot{\vdash}(\dot{\vdash}(\underline{x} = (\underline{y} + (-\underline{uz})))n)n)n \vdash \text{NegativeToLeft}(\text{Less}) \triangleright \dot{\vdash}(\underline{x} <= (\underline{y} + (-\underline{uz})) \Rightarrow \dot{\vdash}(\dot{\vdash}(\underline{x} = (\underline{y} + (-\underline{uz})))n)n)n \gg \dot{\vdash}((\underline{x} + \underline{z}) <= \underline{y} \Rightarrow \dot{\vdash}(\dot{\vdash}((\underline{x} + \underline{z}) = \underline{y})n)n)n; \text{plusCommutativity} \gg (\underline{x} + \underline{z}) = (\underline{z} + \underline{x}); \text{SubLessLeft} \triangleright (\underline{x} + \underline{z}) = (\underline{z} + \underline{x}) \triangleright \dot{\vdash}((\underline{x} + \underline{z}) <= \underline{y} \Rightarrow \dot{\vdash}(\dot{\vdash}((\underline{x} + \underline{z}) = \underline{y})n)n)n \gg \dot{\vdash}((\underline{z} + \underline{x}) <= \underline{y} \Rightarrow \dot{\vdash}(\dot{\vdash}((\underline{z} + \underline{x}) = \underline{y})n)n)n; \text{PositiveToRight}(\text{Less}) \triangleright \dot{\vdash}(\underline{z} + \underline{x}) <= \underline{y} \Rightarrow \dot{\vdash}(\dot{\vdash}((\underline{z} + \underline{x}) = \underline{y})n)n)n \gg \dot{\vdash}(\underline{z} <= (\underline{y} + (-\underline{ux})) \Rightarrow \dot{\vdash}(\dot{\vdash}(\underline{z} = (\underline{y} + (-\underline{ux})))n)n)n], p_0, c)]$

$[\text{insertTwoMiddleTerms}(\text{Sum}) \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \forall \underline{z}: \forall \underline{u}: (\underline{x} + \underline{y}) = ((\underline{x} + (-\underline{uz})) + (\underline{z} + (-\underline{uu}))) + (\underline{u} + \underline{y})]$

$[\text{insertTwoMiddleTerms}(\text{Sum}) \xrightarrow{\text{proof}} \lambda c. \lambda x. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \forall \underline{z}: \forall \underline{u}: \text{insertMiddleTerm}(\text{Sum}) \gg (\underline{x} + \underline{y}) = ((\underline{x} + (-\underline{uz})) + (\underline{z} + \underline{y})); \text{insertMiddleTerm}(\text{Sum}) \gg (\underline{z} + \underline{y}) = ((\underline{z} + (-\underline{uu})) + (\underline{u} + \underline{y})); \text{EqAdditionLeft} \triangleright (\underline{z} + \underline{y}) = ((\underline{z} + (-\underline{uu})) + (\underline{u} + \underline{y})) \gg ((\underline{x} + (-\underline{uz})) + (\underline{z} + \underline{y})) = ((\underline{x} + (-\underline{uz})) + ((\underline{z} + (-\underline{uu})) + (\underline{u} + \underline{y}))); \text{plusAssociativity} \gg (((\underline{x} + (-\underline{uz})) + (\underline{z} + (-\underline{uu}))) + (\underline{u} + \underline{y})) = ((\underline{x} + (-\underline{uz})) + ((\underline{z} + (-\underline{uu})) + (\underline{u} + \underline{y}))); \text{eqSymmetry} \triangleright (((\underline{x} + (-\underline{uz})) + (\underline{z} + (-\underline{uu}))) + (\underline{u} + \underline{y})) = ((\underline{x} + (-\underline{uz})) + ((\underline{z} + (-\underline{uu})) + (\underline{u} + \underline{y}))) \gg (((\underline{x} + (-\underline{uz})) + (\underline{z} + (-\underline{uu}))) + (\underline{u} + \underline{y})) = (((\underline{x} + (-\underline{uz})) + (\underline{z} + (-\underline{uu}))) + (\underline{u} + \underline{y})); \text{eqTransitivity4} \triangleright (\underline{x} + \underline{y}) = ((\underline{x} + (-\underline{uz})) + (\underline{z} + \underline{y})) \triangleright ((\underline{x} + (-\underline{uz})) + (\underline{z} + \underline{y})) = ((\underline{x} + (-\underline{uz})) + ((\underline{z} + (-\underline{uu})) + (\underline{u} + \underline{y}))) \triangleright ((\underline{x} + (-\underline{uz})) + ((\underline{z} + (-\underline{uu})) + (\underline{u} + \underline{y}))) = (((\underline{x} + (-\underline{uz})) + (\underline{z} + (-\underline{uu}))) + (\underline{u} + \underline{y})) \gg (\underline{x} + \underline{y}) = (((\underline{x} + (-\underline{uz})) + (\underline{z} + (-\underline{uu}))) + (\underline{u} + \underline{y})), p_0, c)]$

$[\text{FromNumericalGreater} \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \dot{\vdash}(\underline{x} <= |\underline{y}| \Rightarrow \dot{\vdash}(\dot{\vdash}(\underline{x} = |\underline{y}|)n)n)n \vdash \dot{\vdash}(\dot{\vdash}(\underline{y} <= (-\underline{ux}) \Rightarrow \dot{\vdash}(\dot{\vdash}(\underline{y} = (-\underline{ux}))n)n)n)n \Rightarrow \dot{\vdash}(\underline{x} <= \underline{y} \Rightarrow \dot{\vdash}(\dot{\vdash}(\underline{x} = \underline{y})n)n)n]$

$[\text{FromNumericalGreater} \xrightarrow{\text{proof}} \lambda c. \lambda x. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \dot{\vdash}(\underline{x} <= |\underline{y}| \Rightarrow \dot{\vdash}(\dot{\vdash}(\underline{x} = |\underline{y}|)n)n)n \vdash 0 <= \underline{y} \vdash \text{NonnegativeNumerical} \triangleright 0 <= \underline{y} \gg |\underline{y}| = \underline{y}; \text{SubLessRight} \triangleright |\underline{y}| = \underline{y} \triangleright \dot{\vdash}(\underline{x} <= |\underline{y}| \Rightarrow \dot{\vdash}(\dot{\vdash}(\underline{x} = |\underline{y}|)n)n)n \gg \dot{\vdash}(\underline{x} <= \underline{y} \Rightarrow \dot{\vdash}(\dot{\vdash}(\underline{x} = \underline{y})n)n)n; \text{WeakenOr1} \triangleright \dot{\vdash}(\underline{x} <= \underline{y} \Rightarrow \dot{\vdash}(\dot{\vdash}(\underline{x} = \underline{y})n)n)n \gg \dot{\vdash}(\dot{\vdash}(\underline{y} <= (-\underline{ux}) \Rightarrow \dot{\vdash}(\dot{\vdash}(\underline{y} = (-\underline{ux}))n)n)n)n \Rightarrow \dot{\vdash}(\underline{x} <= \underline{y} \Rightarrow \dot{\vdash}(\dot{\vdash}(\underline{x} = \underline{y})n)n)n); \forall \underline{x}: \forall \underline{y}: \dot{\vdash}(\underline{x} <= |\underline{y}| \Rightarrow \dot{\vdash}(\dot{\vdash}(\underline{x} = |\underline{y}|)n)n)n \vdash \underline{y} <= 0 \vdash \text{lemma nonpositiveNumerical} \triangleright \underline{y} <= 0 \gg |\underline{y}| = (-\underline{uy}); \text{SubLessRight} \triangleright |\underline{y}| = (-\underline{uy}) \triangleright \dot{\vdash}(\underline{x} <= |\underline{y}| \Rightarrow \dot{\vdash}(\dot{\vdash}(\underline{x} = |\underline{y}|)n)n)n \gg \dot{\vdash}(\underline{x} <= (-\underline{uy}) \Rightarrow \dot{\vdash}(\dot{\vdash}(\underline{x} = (-\underline{uy}))n)n)n; \text{LessNegated} \triangleright \dot{\vdash}(\underline{x} <= (-\underline{uy}) \Rightarrow \dot{\vdash}(\dot{\vdash}(\underline{x} = (-\underline{uy}))n)n)n \gg \dot{\vdash}((-u(-\underline{uy})) <= (-\underline{ux}) \Rightarrow \dot{\vdash}(\dot{\vdash}((-u(-\underline{uy})) = (-\underline{ux}))n)n)n); \text{DoubleMinus} \gg (-u(-\underline{uy})) = \underline{y}; \text{SubLessLeft} \triangleright (-u(-\underline{uy})) = \underline{y} \triangleright \dot{\vdash}((-u(-\underline{uy})) <= (-\underline{ux}) \Rightarrow \dot{\vdash}(\dot{\vdash}((-u(-\underline{uy})) = (-\underline{ux}))n)n)n \gg \dot{\vdash}(\underline{y} <= (-\underline{ux}) \Rightarrow \dot{\vdash}(\dot{\vdash}(\underline{y} = (-\underline{ux}))n)n)n; \text{WeakenOr2} \triangleright \dot{\vdash}(\underline{y} <= (-\underline{ux}) \Rightarrow \dot{\vdash}(\dot{\vdash}(\underline{y} = (-\underline{ux}))n)n)n \gg \dot{\vdash}(\dot{\vdash}(\underline{y} <= (-\underline{ux}) \Rightarrow \dot{\vdash}(\dot{\vdash}(\underline{y} = (-\underline{ux}))n)n)n)n \Rightarrow \dot{\vdash}(\underline{x} <= \underline{y} \Rightarrow \dot{\vdash}(\dot{\vdash}(\underline{x} = \underline{y})n)n)n]$











$$\begin{aligned}
& \dot{\vdash} (\dot{\vdash} (0 = (\text{rec}((1+1)+1) * (\epsilon)))n)n \Rightarrow \dot{\vdash} ((n1) <= (n2)) \Rightarrow \\
& \dot{\vdash} (\dot{\vdash} (((\text{fy})[(n2)] + (-u(\text{rec}((1+1)+1) * (\epsilon)))) <= (\text{fx})[(n2)]) \Rightarrow \dot{\vdash} (\dot{\vdash} (((\text{fy})[(n2)] + \\
& (-u(\text{rec}((1+1)+1) * (\epsilon)))) = (\text{fx})[(n2)])n)n)n \Rightarrow \forall_{\text{obj}}(v1): \forall_{\text{obj}}(v2): \dot{\vdash} (0 <= \\
& (\text{rec}((1+1)+1) * (\epsilon)) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = (\text{rec}((1+1)+1) * (\epsilon)))n)n \Rightarrow (n1) <= (v1)) \Rightarrow \\
& (n1) <= (v2)) \Rightarrow \dot{\vdash} (\dot{\vdash} (|((\text{fx})[(v1)] + (-u(\text{fx})[(v2)]))| <= (\text{rec}((1+1)+1) * (\epsilon)) \Rightarrow \\
& \dot{\vdash} (\dot{\vdash} (|((\text{fx})[(v1)] + (-u(\text{fx})[(v2)]))| = (\text{rec}((1+1)+1) * (\epsilon)))n)n \Rightarrow \\
& \dot{\vdash} (\dot{\vdash} (|((\text{fy})[(v1)] + (-u(\text{fy})[(v2)]))| <= (\text{rec}((1+1)+1) * (\epsilon)) \Rightarrow \\
& \dot{\vdash} (\dot{\vdash} (|((\text{fy})[(v1)] + (-u(\text{fy})[(v2)]))| = (\text{rec}((1+1)+1) * (\epsilon)))n)n)n \Rightarrow \\
& \dot{\vdash} (0 <= (\epsilon)) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = (\epsilon))n)n \Rightarrow (n2) <= \underline{m} \Rightarrow \\
& \dot{\vdash} (((\text{fy})[\underline{m}] + (-u(\text{fx})[\underline{m}])) <= (\epsilon)) \Rightarrow \dot{\vdash} (\dot{\vdash} (((\text{fy})[\underline{m}] + (-u(\text{fx})[\underline{m}])) = (\epsilon))n)n)n] \\
& [\text{FromNot} < \text{f}(\text{Strong})(\text{Helper}) \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \\
& \forall(v1): \forall(v2): \forall m: \forall(n1): \forall(n2): \forall(\epsilon): \forall(\text{fx}): \forall(\text{fy}): \dot{\vdash} (0 <= (\text{rec}((1+1)+1) * (\epsilon)) \Rightarrow \\
& \dot{\vdash} (\dot{\vdash} (0 = (\text{rec}((1+1)+1) * (\epsilon)))n)n \Rightarrow \dot{\vdash} ((n1) <= (n2)) \Rightarrow \\
& \dot{\vdash} (\dot{\vdash} (((\text{fy})[(n2)] + (-u(\text{rec}((1+1)+1) * (\epsilon)))) <= (\text{fx})[(n2)]) \Rightarrow \dot{\vdash} (\dot{\vdash} (((\text{fy})[(n2)] + \\
& (-u(\text{rec}((1+1)+1) * (\epsilon)))) = (\text{fx})[(n2)])n)n)n \vdash \forall_{\text{obj}}(v1): \forall_{\text{obj}}(v2): \dot{\vdash} (0 <= \\
& (\text{rec}((1+1)+1) * (\epsilon)) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = (\text{rec}((1+1)+1) * (\epsilon)))n)n \Rightarrow (n1) <= (v1)) \Rightarrow \\
& (n1) <= (v2)) \Rightarrow \dot{\vdash} (\dot{\vdash} (|((\text{fx})[(v1)] + (-u(\text{fx})[(v2)]))| <= (\text{rec}((1+1)+1) * (\epsilon)) \Rightarrow \\
& \dot{\vdash} (\dot{\vdash} (|((\text{fx})[(v1)] + (-u(\text{fx})[(v2)]))| = (\text{rec}((1+1)+1) * (\epsilon)))n)n \Rightarrow \\
& \dot{\vdash} (\dot{\vdash} (|((\text{fy})[(v1)] + (-u(\text{fy})[(v2)]))| <= (\text{rec}((1+1)+1) * (\epsilon)) \Rightarrow \\
& \dot{\vdash} (\dot{\vdash} (|((\text{fy})[(v1)] + (-u(\text{fy})[(v2)]))| = (\text{rec}((1+1)+1) * (\epsilon)))n)n)n \vdash \\
& \dot{\vdash} (0 <= (\epsilon)) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = (\epsilon))n)n \vdash (n2) <= \underline{m} \vdash 0 < 3 \gg \dot{\vdash} (0 <= \\
& ((1+1)+1) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = ((1+1)+1))n)n \text{; PositiveInverted} \triangleright \dot{\vdash} (0 <= \\
& ((1+1)+1) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = ((1+1)+1))n)n \gg \dot{\vdash} (0 <= \text{rec}((1+1)+1) \Rightarrow \\
& \dot{\vdash} (\dot{\vdash} (0 = \text{rec}((1+1)+1))n)n \text{; PositiveFactors} \triangleright \dot{\vdash} (0 <= \text{rec}((1+1)+1) \Rightarrow \\
& \dot{\vdash} (\dot{\vdash} (0 = \text{rec}((1+1)+1))n)n \triangleright \dot{\vdash} (0 <= (\epsilon)) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = (\epsilon))n)n \gg \dot{\vdash} (0 <= \\
& (\text{rec}((1+1)+1) * (\epsilon)) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = (\text{rec}((1+1)+1) * (\epsilon)))n)n \text{; MP} \triangleright \dot{\vdash} (0 <= \\
& (\text{rec}((1+1)+1) * (\epsilon)) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = (\text{rec}((1+1)+1) * (\epsilon)))n)n \Rightarrow \dot{\vdash} ((n1) <= \\
& (n2)) \Rightarrow \dot{\vdash} (\dot{\vdash} (((\text{fy})[(n2)] + (-u(\text{rec}((1+1)+1) * (\epsilon)))) <= (\text{fx})[(n2)]) \Rightarrow \\
& \dot{\vdash} (\dot{\vdash} (((\text{fy})[(n2)] + (-u(\text{rec}((1+1)+1) * (\epsilon)))) = (\text{fx})[(n2)])n)n \triangleright \dot{\vdash} (0 <= \\
& (\text{rec}((1+1)+1) * (\epsilon)) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = (\text{rec}((1+1)+1) * (\epsilon)))n)n \gg \dot{\vdash} ((n1) <= \\
& (n2)) \Rightarrow \dot{\vdash} (\dot{\vdash} (((\text{fy})[(n2)] + (-u(\text{rec}((1+1)+1) * (\epsilon)))) <= (\text{fx})[(n2)]) \Rightarrow \\
& \dot{\vdash} (\dot{\vdash} (((\text{fy})[(n2)] + (-u(\text{rec}((1+1)+1) * (\epsilon)))) = \\
& (\text{fx})[(n2)])n)n \text{; FirstConjunct} \triangleright \dot{\vdash} ((n1) <= (n2)) \Rightarrow \\
& \dot{\vdash} (\dot{\vdash} (((\text{fy})[(n2)] + (-u(\text{rec}((1+1)+1) * (\epsilon)))) <= (\text{fx})[(n2)]) \Rightarrow \\
& \dot{\vdash} (\dot{\vdash} (((\text{fy})[(n2)] + (-u(\text{rec}((1+1)+1) * (\epsilon)))) = (\text{fx})[(n2)])n)n \gg (n1) <= \\
& (n2) \text{; SecondConjunct} \triangleright \dot{\vdash} ((n1) <= (n2)) \Rightarrow \dot{\vdash} (\dot{\vdash} (((\text{fy})[(n2)] + (-u(\text{rec}((1+1)+ \\
& 1) * (\epsilon)))) <= (\text{fx})[(n2)]) \Rightarrow \dot{\vdash} (\dot{\vdash} (((\text{fy})[(n2)] + (-u(\text{rec}((1+1)+1) * (\epsilon)))) = \\
& (\text{fx})[(n2)])n)n \gg \dot{\vdash} (((\text{fy})[(n2)] + (-u(\text{rec}((1+1)+1) * (\epsilon)))) <= \\
& (\text{fx})[(n2)]) \Rightarrow \dot{\vdash} (\dot{\vdash} (((\text{fy})[(n2)] + (-u(\text{rec}((1+1)+1) * (\epsilon)))) = \\
& (\text{fx})[(n2)])n)n \text{; SwitchTerms}(x - y < z) \triangleright \dot{\vdash} (((\text{fy})[(n2)] + (-u(\text{rec}((1+1)+ \\
& 1) * (\epsilon)))) <= (\text{fx})[(n2)]) \Rightarrow \dot{\vdash} (\dot{\vdash} (((\text{fy})[(n2)] + (-u(\text{rec}((1+1)+1) * (\epsilon)))) = \\
& (\text{fx})[(n2)])n)n \gg \dot{\vdash} (((\text{fy})[(n2)] + (-u(\text{fx})[(n2)])) <= (\text{rec}((1+1)+1) * (\epsilon)) \Rightarrow \\
& \dot{\vdash} (\dot{\vdash} (((\text{fy})[(n2)] + (-u(\text{fx})[(n2)])) = (\text{rec}((1+1)+1) * (\epsilon)))n)n \text{; leqTransitivity} \triangleright \\
& (n1) <= (n2) \triangleright (n2) <= \underline{m} \gg (n1) <= \underline{m} \text{; A4} @ \underline{m} \triangleright \forall_{\text{obj}}(v1): \forall_{\text{obj}}(v2): \dot{\vdash} (0 <= \\
& (\text{rec}((1+1)+1) * (\epsilon)) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = (\text{rec}((1+1)+1) * (\epsilon)))n)n \Rightarrow (n1) <= (v1)) \Rightarrow
\end{aligned}$$



$$\begin{aligned}
& 1) * (\epsilon) \Rightarrow \dot{\neg}(\dot{\neg}(((fy)[(n2)] + (-u(fx)[(n2)]))) = (\text{rec}((1+1)+1) * (\epsilon))n)n \gg \\
& \dot{\neg}(((fy)[m] + (-u(fx)[m]))) <= (\epsilon) \Rightarrow \dot{\neg}(\dot{\neg}(((fy)[m] + (-u(fx)[m]))) = \\
& (\epsilon)n)n)n; \forall(v1): \forall(v2): \forall m: \forall(n1): \forall(n2): \forall(\epsilon): \forall(fx): \forall(fy): \text{Ded} \triangleright \\
& \forall(v1): \forall(v2): \forall m: \forall(n1): \forall(n2): \forall(\epsilon): \forall(fx): \forall(fy): \dot{\neg}(0 <= (\text{rec}((1+1)+1) * (\epsilon)) \Rightarrow \\
& \dot{\neg}(\dot{\neg}(0 = (\text{rec}((1+1)+1) * (\epsilon))n)n)n \Rightarrow \dot{\neg}((n1) <= (n2)) \Rightarrow \\
& \dot{\neg}(\dot{\neg}(((fy)[(n2)] + (-u(\text{rec}((1+1)+1) * (\epsilon)))) <= (fx)[(n2)]) \Rightarrow \dot{\neg}(\dot{\neg}(((fy)[(n2)] + \\
& (-u(\text{rec}((1+1)+1) * (\epsilon)))) = (fx)[(n2)]n)n)n)n \vdash \forall_{\text{obj}}(v1): \forall_{\text{obj}}(v2): \dot{\neg}(0 <= \\
& (\text{rec}((1+1)+1) * (\epsilon)) \Rightarrow \dot{\neg}(\dot{\neg}(0 = (\text{rec}((1+1)+1) * (\epsilon))n)n)n \Rightarrow (n1) <= (v1)) \Rightarrow \\
& (n1) <= (v2)) \Rightarrow \dot{\neg}(\dot{\neg}(|((fx)[(v1)] + (-u(fx)[(v2)])))| <= (\text{rec}((1+1)+1) * (\epsilon)) \Rightarrow \\
& \dot{\neg}(\dot{\neg}(|((fx)[(v1)] + (-u(fx)[(v2)])))| = (\text{rec}((1+1)+1) * (\epsilon))n)n)n \Rightarrow \\
& \dot{\neg}(\dot{\neg}(|((fy)[(v1)] + (-u(fy)[(v2)])))| <= (\text{rec}((1+1)+1) * (\epsilon)) \Rightarrow \\
& \dot{\neg}(\dot{\neg}(|((fy)[(v1)] + (-u(fy)[(v2)])))| = (\text{rec}((1+1)+1) * (\epsilon))n)n)n)n \vdash \dot{\neg}(0 <= \\
& (\epsilon) \Rightarrow \dot{\neg}(\dot{\neg}(0 = (\epsilon)n)n)n \vdash (n2) <= m \vdash \dot{\neg}(((fy)[m] + (-u(fx)[m]))) <= (\epsilon) \Rightarrow \\
& \dot{\neg}(\dot{\neg}(((fy)[m] + (-u(fx)[m]))) = (\epsilon)n)n)n \gg \dot{\neg}(0 <= (\text{rec}((1+1)+1) * (\epsilon)) \Rightarrow \\
& \dot{\neg}(\dot{\neg}(0 = (\text{rec}((1+1)+1) * (\epsilon))n)n)n \Rightarrow \dot{\neg}((n1) <= (n2)) \Rightarrow \\
& \dot{\neg}(\dot{\neg}(((fy)[(n2)] + (-u(\text{rec}((1+1)+1) * (\epsilon)))) <= (fx)[(n2)]) \Rightarrow \dot{\neg}(\dot{\neg}(((fy)[(n2)] + \\
& (-u(\text{rec}((1+1)+1) * (\epsilon)))) = (fx)[(n2)]n)n)n)n \Rightarrow \forall_{\text{obj}}(v1): \forall_{\text{obj}}(v2): \dot{\neg}(0 <= \\
& (\text{rec}((1+1)+1) * (\epsilon)) \Rightarrow \dot{\neg}(\dot{\neg}(0 = (\text{rec}((1+1)+1) * (\epsilon))n)n)n \Rightarrow (n1) <= (v1)) \Rightarrow \\
& (n1) <= (v2)) \Rightarrow \dot{\neg}(\dot{\neg}(|((fx)[(v1)] + (-u(fx)[(v2)])))| <= (\text{rec}((1+1)+1) * (\epsilon)) \Rightarrow \\
& \dot{\neg}(\dot{\neg}(|((fx)[(v1)] + (-u(fx)[(v2)])))| = (\text{rec}((1+1)+1) * (\epsilon))n)n)n \Rightarrow \\
& \dot{\neg}(\dot{\neg}(|((fy)[(v1)] + (-u(fy)[(v2)])))| <= (\text{rec}((1+1)+1) * (\epsilon)) \Rightarrow \\
& \dot{\neg}(\dot{\neg}(|((fy)[(v1)] + (-u(fy)[(v2)])))| = (\text{rec}((1+1)+1) * (\epsilon))n)n)n)n \Rightarrow \dot{\neg}(0 <= \\
& (\epsilon) \Rightarrow \dot{\neg}(\dot{\neg}(0 = (\epsilon)n)n)n \Rightarrow (n2) <= m \Rightarrow \dot{\neg}(((fy)[m] + (-u(fx)[m]))) <= \\
& (\epsilon) \Rightarrow \dot{\neg}(\dot{\neg}(((fy)[m] + (-u(fx)[m]))) = (\epsilon)n)n)n, p_0, c]
\end{aligned}$$

[FromNot < f(Strong)  $\xrightarrow{\text{stmt}}$  SystemQ  $\vdash$

$$\begin{aligned}
& \forall(v1): \forall(v2): \forall m: \forall(n2): \forall(\epsilon): \forall(fx): \forall(fy): \dot{\neg}(\dot{\neg}(\forall_{\text{obj}}(\overline{\epsilon}): \dot{\neg}(\dot{\neg}(\forall_{\text{obj}}\overline{n}: \dot{\neg}(\forall_{\text{obj}}\overline{m}: \dot{\neg}(\dot{\neg}(0 \\
& (\epsilon) \Rightarrow \dot{\neg}(\dot{\neg}(0 = (\epsilon)n)n)n \Rightarrow \dot{\neg}(\overline{n} <= \overline{m} \Rightarrow (fx)[\overline{m}] <= ((fy)[\overline{m}] + \\
& (-u(\overline{\epsilon})))n)n)n)n)n \vdash \dot{\neg}(0 <= (\epsilon) \Rightarrow \dot{\neg}(\dot{\neg}(0 = (\epsilon)n)n)n \Rightarrow (n2) <= m \Rightarrow \\
& \dot{\neg}(((fy)[m] + (-u(fx)[m]))) <= (\epsilon) \Rightarrow \dot{\neg}(\dot{\neg}(((fy)[m] + (-u(fx)[m]))) = (\epsilon)n)n)n]
\end{aligned}$$

[FromNot < f(Strong)  $\xrightarrow{\text{proof}}$   $\lambda c. \lambda x. \mathcal{P}(\ulcorner \text{SystemQ} \vdash$

$$\begin{aligned}
& \forall(v1): \forall(v2): \forall m: \forall(n2): \forall(\epsilon): \forall(fx): \forall(fy): \dot{\neg}(\dot{\neg}(\forall_{\text{obj}}(\overline{\epsilon}): \dot{\neg}(\dot{\neg}(\forall_{\text{obj}}\overline{n}: \dot{\neg}(\forall_{\text{obj}}\overline{m}: \dot{\neg}(\dot{\neg}(0 \\
& (\epsilon) \Rightarrow \dot{\neg}(\dot{\neg}(0 = (\overline{\epsilon})n)n)n \Rightarrow \dot{\neg}(\overline{n} <= \overline{m} \Rightarrow (fx)[\overline{m}] <= \\
& ((fy)[\overline{m}] + (-u(\overline{\epsilon})))n)n)n)n)n \vdash \text{FromNot} <
\end{aligned}$$

$$\begin{aligned}
& \text{f(Weak)} \triangleright \dot{\neg}(\dot{\neg}(\forall_{\text{obj}}(\overline{\epsilon}): \dot{\neg}(\dot{\neg}(\forall_{\text{obj}}\overline{n}: \dot{\neg}(\forall_{\text{obj}}\overline{m}: \dot{\neg}(\dot{\neg}(0 <= \overline{\epsilon})) \Rightarrow \dot{\neg}(\dot{\neg}(0 = \\
& (\overline{\epsilon})n)n)n \Rightarrow \dot{\neg}(\overline{n} <= \overline{m} \Rightarrow (fx)[\overline{m}] <= ((fy)[\overline{m}] + (-u(\overline{\epsilon})))n)n)n)n)n \gg \\
& \forall_{\text{obj}}(\overline{\epsilon}): \forall_{\text{obj}}\overline{n}: \dot{\neg}(\forall_{\text{obj}}\overline{n2}): \dot{\neg}(\dot{\neg}(0 <= (\overline{\epsilon}) \Rightarrow \dot{\neg}(\dot{\neg}(0 = (\overline{\epsilon})n)n)n \Rightarrow \dot{\neg}(\overline{n} <= \\
& (\overline{n2}) \Rightarrow \dot{\neg}(\dot{\neg}(((fy)[(\overline{n2})] + (-u(\overline{\epsilon}))) <= (fx)[(\overline{n2})]) \Rightarrow \\
& \dot{\neg}(\dot{\neg}(((fy)[(\overline{n2})] + (-u(\overline{\epsilon}))) = (fx)[(\overline{n2})]n)n)n)n)n; A4 @(\text{rec}((1+1)+1) * \\
& (\overline{\epsilon})) \triangleright \forall_{\text{obj}}(\overline{\epsilon}): \forall_{\text{obj}}\overline{n}: \dot{\neg}(\forall_{\text{obj}}\overline{n2}): \dot{\neg}(\dot{\neg}(0 <= (\overline{\epsilon}) \Rightarrow \dot{\neg}(\dot{\neg}(0 = (\overline{\epsilon})n)n)n \Rightarrow \\
& \dot{\neg}(\overline{n} <= (\overline{n2}) \Rightarrow \dot{\neg}(\dot{\neg}(((fy)[(\overline{n2})] + (-u(\overline{\epsilon}))) <= (fx)[(\overline{n2})]) \Rightarrow \dot{\neg}(\dot{\neg}(((fy)[(\overline{n2})] + \\
& (-u(\overline{\epsilon}))) = (fx)[(\overline{n2})]n)n)n)n)n \gg \forall_{\text{obj}}\overline{n}: \dot{\neg}(\forall_{\text{obj}}\overline{n2}): \dot{\neg}(\dot{\neg}(0 <= \\
& (\text{rec}((1+1)+1) * (\overline{\epsilon})) \Rightarrow \dot{\neg}(\dot{\neg}(0 = (\text{rec}((1+1)+1) * (\overline{\epsilon}))n)n)n \Rightarrow \dot{\neg}(\overline{n} <= \\
& (\overline{n2}) \Rightarrow \dot{\neg}(\dot{\neg}(((fy)[(\overline{n2})] + (-u(\text{rec}((1+1)+1) * (\overline{\epsilon})))) <= (fx)[(\overline{n2})]) \Rightarrow
\end{aligned}$$



$$\begin{aligned}
& (\epsilon) \Rightarrow \dot{\vdash} (\dot{\vdash} (((\underline{fy})[\underline{m}] + (-u(\underline{fx})[\underline{m}])) = (\underline{\epsilon}))n)n) \triangleright \dot{\vdash} (\forall_{\text{obj}}(\underline{n2}): \dot{\vdash} (\dot{\vdash} (0 <= \\
& \underline{\text{rec}}((1+1)+1) * (\underline{\epsilon})) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = (\underline{\text{rec}}((1+1)+1) * (\underline{\epsilon}))n)n) \Rightarrow \dot{\vdash} (\underline{(n1)} <= \\
& \underline{(n2)}) \Rightarrow \dot{\vdash} (\dot{\vdash} (((\underline{fy})[\underline{(n2)}] + (-u(\underline{\text{rec}}((1+1)+1) * (\underline{\epsilon})))) <= (\underline{fx})[\underline{(n2)}]) \Rightarrow \\
& \dot{\vdash} (\dot{\vdash} (((\underline{fy})[\underline{(n2)}] + (-u(\underline{\text{rec}}((1+1)+1) * (\underline{\epsilon})))) = \\
& (\underline{fx})[\underline{(n2)}])n)n)n)n) \triangleright \dot{\vdash} (\forall_{\text{obj}}(\underline{n1}): \dot{\vdash} (\forall_{\text{obj}}(\underline{v1}): \forall_{\text{obj}}(\underline{v2}): \dot{\vdash} (0 <= \\
& \underline{\text{rec}}((1+1)+1) * (\underline{\epsilon})) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = (\underline{\text{rec}}((1+1)+1) * (\underline{\epsilon}))n)n) \Rightarrow (\underline{n1}) <= (\underline{v1}) \Rightarrow \\
& (\underline{n1}) <= (\underline{v2}) \Rightarrow \dot{\vdash} (\dot{\vdash} (|((\underline{fx})[\underline{(v1)}] + (-u(\underline{fx})[\underline{(v2)}]))| <= (\underline{\text{rec}}((1+1)+1) * (\underline{\epsilon})) \Rightarrow \\
& \dot{\vdash} (\dot{\vdash} (|((\underline{fx})[\underline{(v1)}] + (-u(\underline{fx})[\underline{(v2)}]))| = (\underline{\text{rec}}((1+1)+1) * (\underline{\epsilon}))n)n) \Rightarrow \\
& \dot{\vdash} (\dot{\vdash} (|((\underline{fy})[\underline{(v1)}] + (-u(\underline{fy})[\underline{(v2)}]))| <= (\underline{\text{rec}}((1+1)+1) * (\underline{\epsilon})) \Rightarrow \\
& \dot{\vdash} (\dot{\vdash} (|((\underline{fy})[\underline{(v1)}] + (-u(\underline{fy})[\underline{(v2)}]))| = (\underline{\text{rec}}((1+1)+1) * (\underline{\epsilon}))n)n)n)n) \triangleright \\
& \dot{\vdash} (0 <= (\underline{\epsilon}) \Rightarrow \dot{\vdash} (\dot{\vdash} (0 = (\underline{\epsilon}))n)n) \Rightarrow (\underline{n2}) <= \underline{m} \Rightarrow \dot{\vdash} (((\underline{fy})[\underline{m}] + \\
& (-u(\underline{fx})[\underline{m}])) <= (\underline{\epsilon}) \Rightarrow \dot{\vdash} (\dot{\vdash} (((\underline{fy})[\underline{m}] + (-u(\underline{fx})[\underline{m}])) = (\underline{\epsilon}))n)n) \mid, p_0, c)
\end{aligned}$$

$$\begin{aligned}
& [\text{fromNotSameF}(\text{Strongest})(\text{Helper2}) \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \\
& \forall \underline{x}: \forall \underline{y}: \forall \underline{z}: \forall \underline{u}: \forall \underline{v}: \dot{\vdash} (|(\underline{x} + (-\underline{uy}))| <= (\underline{\text{rec}}((1+1)+1) * \underline{v}) \Rightarrow \\
& \dot{\vdash} (\dot{\vdash} (|(\underline{x} + (-\underline{uy}))| = (\underline{\text{rec}}((1+1)+1) * \underline{v}))n)n) \vdash \dot{\vdash} (|(\underline{z} + (-\underline{uu}))| <= \\
& (\underline{\text{rec}}((1+1)+1) * \underline{v}) \Rightarrow \dot{\vdash} (\dot{\vdash} (|(\underline{z} + (-\underline{uu}))| = (\underline{\text{rec}}((1+1)+1) * \underline{v}))n)n) \vdash \\
& \underline{v} <= |(\underline{y} + (-\underline{uu}))| \vdash \dot{\vdash} ((\underline{\text{rec}}((1+1)+1) * \underline{v}) <= |(\underline{x} + (-\underline{uz}))| \Rightarrow \\
& \dot{\vdash} (\dot{\vdash} ((\underline{\text{rec}}((1+1)+1) * \underline{v}) = |(\underline{x} + (-\underline{uz}))|)n)n) \mid
\end{aligned}$$

$$\begin{aligned}
& [\text{fromNotSameF}(\text{Strongest})(\text{Helper2}) \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}([\text{SystemQ} \vdash \\
& \forall \underline{x}: \forall \underline{y}: \forall \underline{z}: \forall \underline{u}: \forall \underline{v}: \dot{\vdash} (|(\underline{x} + (-\underline{uy}))| <= (\underline{\text{rec}}((1+1)+1) * \underline{v}) \Rightarrow \\
& \dot{\vdash} (\dot{\vdash} (|(\underline{x} + (-\underline{uy}))| = (\underline{\text{rec}}((1+1)+1) * \underline{v}))n)n) \vdash \dot{\vdash} (|(\underline{z} + (-\underline{uu}))| <= \\
& (\underline{\text{rec}}((1+1)+1) * \underline{v}) \Rightarrow \dot{\vdash} (\dot{\vdash} (|(\underline{z} + (-\underline{uu}))| = (\underline{\text{rec}}((1+1)+1) * \underline{v}))n)n) \vdash \\
& \underline{v} <= |(\underline{y} + (-\underline{uu}))| \vdash \text{NumericalDifference} \gg |(\underline{x} + (-\underline{uy}))| = \\
& |(\underline{y} + (-\underline{ux}))|; \text{SubLessLeft} \triangleright |(\underline{x} + (-\underline{uy}))| = |(\underline{y} + (-\underline{ux}))| \triangleright \dot{\vdash} (|(\underline{x} + (-\underline{uy}))| <= \\
& (\underline{\text{rec}}((1+1)+1) * \underline{v}) \Rightarrow \dot{\vdash} (\dot{\vdash} (|(\underline{x} + (-\underline{uy}))| = (\underline{\text{rec}}((1+1)+1) * \underline{v}))n)n) \gg \\
& \dot{\vdash} (|(\underline{y} + (-\underline{ux}))| <= (\underline{\text{rec}}((1+1)+1) * \underline{v}) \Rightarrow \dot{\vdash} (\dot{\vdash} (|(\underline{y} + (-\underline{ux}))| = (\underline{\text{rec}}((1+1)+ \\
& 1) * \underline{v}))n)n); \text{LessNegated} \triangleright \dot{\vdash} (|(\underline{y} + (-\underline{ux}))| <= (\underline{\text{rec}}((1+1)+1) * \underline{v}) \Rightarrow \\
& \dot{\vdash} (\dot{\vdash} (|(\underline{y} + (-\underline{ux}))| = (\underline{\text{rec}}((1+1)+1) * \underline{v}))n)n) \gg \dot{\vdash} ((-u(\underline{\text{rec}}((1+1)+1) * \underline{v})) <= \\
& (-u|(\underline{y} + (-\underline{ux}))|)) \Rightarrow \dot{\vdash} (\dot{\vdash} ((-u(\underline{\text{rec}}((1+1)+1) * \underline{v})) = \\
& (-u|(\underline{y} + (-\underline{ux}))|))n)n); \text{LessNegated} \triangleright \dot{\vdash} (|(\underline{z} + (-\underline{uu}))| <= (\underline{\text{rec}}((1+1)+1) * \underline{v}) \Rightarrow \\
& \dot{\vdash} (\dot{\vdash} (|(\underline{z} + (-\underline{uu}))| = (\underline{\text{rec}}((1+1)+1) * \underline{v}))n)n) \gg \dot{\vdash} ((-u(\underline{\text{rec}}((1+1)+1) * \underline{v})) <= \\
& (-u|(\underline{z} + (-\underline{uu}))|)) \Rightarrow \dot{\vdash} (\dot{\vdash} ((-u(\underline{\text{rec}}((1+1)+1) * \underline{v})) = \\
& (-u|(\underline{z} + (-\underline{uu}))|))n)n); \text{AddEquations}(\text{LeqLess}) \triangleright \underline{v} <= \\
& |(\underline{y} + (-\underline{uu}))| \triangleright \dot{\vdash} ((-u(\underline{\text{rec}}((1+1)+1) * \underline{v})) <= (-u|(\underline{y} + (-\underline{ux}))|)) \Rightarrow \\
& \dot{\vdash} (\dot{\vdash} ((-u(\underline{\text{rec}}((1+1)+1) * \underline{v})) = (-u|(\underline{y} + (-\underline{ux}))|))n)n) \gg \\
& \dot{\vdash} ((\underline{v} + (-u(\underline{\text{rec}}((1+1)+1) * \underline{v}))) <= (|(\underline{y} + (-\underline{uu}))| + (-u|(\underline{y} + (-\underline{ux}))|))) \Rightarrow \\
& \dot{\vdash} (\dot{\vdash} ((\underline{v} + (-u(\underline{\text{rec}}((1+1)+1) * \underline{v}))) = (|(\underline{y} + (-\underline{uu}))| + (-u|(\underline{y} + \\
& (-\underline{ux}))|))n)n); \text{AddEquations}(\text{Less}) \triangleright \dot{\vdash} ((\underline{v} + (-u(\underline{\text{rec}}((1+1)+1) * \underline{v}))) <= \\
& (|(\underline{y} + (-\underline{uu}))| + (-u|(\underline{y} + (-\underline{ux}))|))) \Rightarrow \dot{\vdash} (\dot{\vdash} ((\underline{v} + (-u(\underline{\text{rec}}((1+1)+1) * \underline{v}))) = \\
& (|(\underline{y} + (-\underline{uu}))| + (-u|(\underline{y} + (-\underline{ux}))|))n)n) \triangleright \dot{\vdash} ((-u(\underline{\text{rec}}((1+1)+1) * \underline{v})) <= \\
& (-u|(\underline{z} + (-\underline{uu}))|)) \Rightarrow \dot{\vdash} (\dot{\vdash} ((-u(\underline{\text{rec}}((1+1)+1) * \underline{v})) = (-u|(\underline{z} + (-\underline{uu}))|))n)n) \gg \\
& \dot{\vdash} (((\underline{v} + (-u(\underline{\text{rec}}((1+1)+1) * \underline{v}))) + (-u(\underline{\text{rec}}((1+1)+1) * \underline{v}))) <= \\
& ((|(\underline{y} + (-\underline{uu}))| + (-u|(\underline{y} + (-\underline{ux}))|)) + (-u|(\underline{z} + (-\underline{uu}))|))) \Rightarrow \\
& \dot{\vdash} (\dot{\vdash} (((\underline{v} + (-u(\underline{\text{rec}}((1+1)+1) * \underline{v}))) + (-u(\underline{\text{rec}}((1+1)+1) * \underline{v}))) = \\
& ((|(\underline{y} + (-\underline{uu}))| + (-u|(\underline{y} + (-\underline{ux}))|)) + (-u|(\underline{z} +
\end{aligned}$$

$(-u\underline{u}))\underline{))n)n)n$ ; insertTwoMiddleTerms(Numerical)  $\gg |(\underline{y} + (-u\underline{u}))| <=$   
 $(|(\underline{y} + (-u\underline{x}))| + |(\underline{x} + (-u\underline{z}))|) + |(\underline{z} + (-u\underline{u}))|)$ ; plusAssociativity  $\gg$   
 $(|(\underline{y} + (-u\underline{x}))| + |(\underline{x} + (-u\underline{z}))|) + |(\underline{z} + (-u\underline{u}))|) =$   
 $(|(\underline{y} + (-u\underline{x}))| + (|(\underline{x} + (-u\underline{z}))| + |(\underline{z} + (-u\underline{u}))|))$ ; plusCommutativity  $\gg$   
 $(|(\underline{y} + (-u\underline{x}))| + (|(\underline{x} + (-u\underline{z}))| + |(\underline{z} + (-u\underline{u}))|)) = (|(\underline{x} + (-u\underline{z}))| + |(\underline{z} + (-u\underline{u}))|) +$   
 $|(\underline{y} + (-u\underline{x}))|)$ ; eqTransitivity  $\triangleright (|(\underline{y} + (-u\underline{x}))| + |(\underline{x} + (-u\underline{z}))|) + |(\underline{z} + (-u\underline{u}))|) =$   
 $(|(\underline{y} + (-u\underline{x}))| + (|(\underline{x} + (-u\underline{z}))| + |(\underline{z} + (-u\underline{u}))|)) \triangleright (|(\underline{y} + (-u\underline{x}))| + (|(\underline{x} +$   
 $(-u\underline{z}))| + |(\underline{z} + (-u\underline{u}))|)) = (|(\underline{x} + (-u\underline{z}))| + |(\underline{z} + (-u\underline{u}))|) + |(\underline{y} + (-u\underline{x}))|) \gg$   
 $(|(\underline{y} + (-u\underline{x}))| + |(\underline{x} + (-u\underline{z}))|) + |(\underline{z} + (-u\underline{u}))|) = (|(\underline{x} + (-u\underline{z}))| + |(\underline{z} + (-u\underline{u}))|) +$   
 $|(\underline{y} + (-u\underline{x}))|)$ ; subLeqRight  $\triangleright (|(\underline{y} + (-u\underline{x}))| + |(\underline{x} + (-u\underline{z}))|) + |(\underline{z} + (-u\underline{u}))|) =$   
 $(|(\underline{x} + (-u\underline{z}))| + |(\underline{z} + (-u\underline{u}))|) + |(\underline{y} + (-u\underline{x}))|) \triangleright |(\underline{y} + (-u\underline{u}))| <=$   
 $(|(\underline{y} + (-u\underline{x}))| + |(\underline{x} + (-u\underline{z}))|) + |(\underline{z} + (-u\underline{u}))|) \gg |(\underline{y} + (-u\underline{u}))| <=$   
 $(|(\underline{x} + (-u\underline{z}))| + |(\underline{z} + (-u\underline{u}))|) + |(\underline{y} + (-u\underline{x}))|)$ ; PositiveToLeft(Leq)  $\triangleright |(\underline{y} +$   
 $(-u\underline{u}))| <= (|(\underline{x} + (-u\underline{z}))| + |(\underline{z} + (-u\underline{u}))|) + |(\underline{y} + (-u\underline{x}))|) \gg (|(\underline{y} + (-u\underline{u}))| +$   
 $(-u|(\underline{y} + (-u\underline{x}))|)) <= (|(\underline{x} + (-u\underline{z}))| + |(\underline{z} + (-u\underline{u}))|)$ ; PositiveToLeft(Leq)  $\triangleright$   
 $(|(\underline{y} + (-u\underline{u}))| + (-u|(\underline{y} + (-u\underline{x}))|)) <= (|(\underline{x} + (-u\underline{z}))| + |(\underline{z} + (-u\underline{u}))|) \gg$   
 $(|(\underline{y} + (-u\underline{u}))| + (-u|(\underline{y} + (-u\underline{x}))|)) + (-u|(\underline{z} + (-u\underline{u}))|)) <= |(\underline{x} +$   
 $(-u\underline{z}))|$ ; LessLeqTransitivity  $\triangleright \dot{\dot{}}((\underline{v} + (-u(\text{rec}((1+1)+1) * \underline{v}))) + (-u(\text{rec}((1+1)+1) * \underline{v}))) <=$   
 $(|(\underline{y} + (-u\underline{u}))| + (-u|(\underline{y} + (-u\underline{x}))|)) + (-u|(\underline{z} + (-u\underline{u}))|)) \Rightarrow \dot{\dot{}}(\dot{\dot{}}((\underline{v} + (-u(\text{rec}((1+1)+1) * \underline{v}))) + (-u(\text{rec}((1+1)+1) * \underline{v}))) =$   
 $(|(\underline{y} + (-u\underline{u}))| + (-u|(\underline{y} + (-u\underline{x}))|)) + (-u|(\underline{z} + (-u\underline{u}))|))n)n)n \triangleright (|(\underline{y} +$   
 $(-u\underline{u}))| + (-u|(\underline{y} + (-u\underline{x}))|)) + (-u|(\underline{z} + (-u\underline{u}))|)) <= |(\underline{x} + (-u\underline{z}))| \gg$   
 $\dot{\dot{}}((\underline{v} + (-u(\text{rec}((1+1)+1) * \underline{v}))) + (-u(\text{rec}((1+1)+1) * \underline{v}))) <= |(\underline{x} + (-u\underline{z}))| \Rightarrow$   
 $\dot{\dot{}}(\dot{\dot{}}((\underline{v} + (-u(\text{rec}((1+1)+1) * \underline{v}))) + (-u(\text{rec}((1+1)+1) * \underline{v}))) =$   
 $|(\underline{x} + (-u\underline{z}))|)n)n)n$ ; ThreeThirds  $\gg$   
 $((\text{rec}((1+1)+1) * \underline{v}) + (\text{rec}((1+1)+1) * \underline{v})) + (\text{rec}((1+1)+1) * \underline{v})) =$   
 $\underline{v}$ ; PositiveToRight(Eq)  $\triangleright ((\text{rec}((1+1)+1) * \underline{v}) + (\text{rec}((1+1)+1) * \underline{v})) + (\text{rec}((1+1)+1) * \underline{v})) = \underline{v} \gg$   
 $((\text{rec}((1+1)+1) * \underline{v}) + (\text{rec}((1+1)+1) * \underline{v})) = (\underline{v} + (-u(\text{rec}((1+1)+1) * \underline{v})))$ ; PositiveToRight(Eq)  $\triangleright ((\text{rec}((1+1)+1) * \underline{v}) + (\text{rec}((1+1)+1) * \underline{v})) =$   
 $(\underline{v} + (-u(\text{rec}((1+1)+1) * \underline{v}))) \gg (\text{rec}((1+1)+1) * \underline{v}) = ((\underline{v} + (-u(\text{rec}((1+1)+1) * \underline{v}))) + (-u(\text{rec}((1+1)+1) * \underline{v})))$ ; eqSymmetry  $\triangleright (\text{rec}((1+1)+1) * \underline{v}) =$   
 $((\underline{v} + (-u(\text{rec}((1+1)+1) * \underline{v}))) + (-u(\text{rec}((1+1)+1) * \underline{v}))) \gg$   
 $((\underline{v} + (-u(\text{rec}((1+1)+1) * \underline{v}))) + (-u(\text{rec}((1+1)+1) * \underline{v}))) = (\text{rec}((1+1)+1) * \underline{v})$ ; SubLessLeft  $\triangleright ((\underline{v} + (-u(\text{rec}((1+1)+1) * \underline{v}))) + (-u(\text{rec}((1+1)+1) * \underline{v}))) =$   
 $(\text{rec}((1+1)+1) * \underline{v}) \triangleright \dot{\dot{}}((\underline{v} + (-u(\text{rec}((1+1)+1) * \underline{v}))) + (-u(\text{rec}((1+1)+1) * \underline{v}))) <= |(\underline{x} + (-u\underline{z}))| \Rightarrow \dot{\dot{}}(\dot{\dot{}}((\underline{v} + (-u(\text{rec}((1+1)+1) * \underline{v}))) +$   
 $(-u(\text{rec}((1+1)+1) * \underline{v}))) = |(\underline{x} + (-u\underline{z}))|)n)n)n \gg \dot{\dot{}}((\text{rec}((1+1)+1) * \underline{v}) <=$   
 $|(\underline{x} + (-u\underline{z}))| \Rightarrow \dot{\dot{}}(\dot{\dot{}}((\text{rec}((1+1)+1) * \underline{v}) = |(\underline{x} + (-u\underline{z}))|)n)n)n]$ , po, c)]  
 $[fromNotSameF(Strongest)(Helper) \xrightarrow{\text{stmt}} \text{SystemQ} \vdash$   
 $\forall(\underline{v1}): \forall(\underline{v2}): \forall \underline{m}: \forall(\underline{n1}): \forall(\underline{n2}): \forall(\epsilon): \forall(\underline{fx}): \forall(\underline{fy}): \dot{\dot{}}(\dot{\dot{}}(0 <= (\epsilon) \Rightarrow \dot{\dot{}}(\dot{\dot{}}(0 =$   
 $(\epsilon))n)n)n \Rightarrow (\underline{n1}) <= (\underline{n2}) \Rightarrow \dot{\dot{}}(|((\underline{fx})[(\underline{n2})] + (-u(\underline{fy})[(\underline{n2})]))| <= (\epsilon) \Rightarrow$   
 $\dot{\dot{}}(\dot{\dot{}}(|((\underline{fx})[(\underline{n2})] + (-u(\underline{fy})[(\underline{n2})]))| = (\epsilon))n)n)n \Rightarrow \forall_{\text{obj}}(\underline{v1}): \forall_{\text{obj}}(\underline{v2}): \dot{\dot{}}(0 <=$   
 $(\text{rec}((1+1)+1) * (\epsilon)) \Rightarrow \dot{\dot{}}(\dot{\dot{}}(0 = (\text{rec}((1+1)+1) * (\epsilon))n)n)n \Rightarrow (\underline{n1}) <= (\underline{v1}) \Rightarrow$   
 $(\underline{n1}) <= (\underline{v2}) \Rightarrow \dot{\dot{}}(\dot{\dot{}}(|((\underline{fx})[(\underline{v1})] + (-u(\underline{fx})[(\underline{v2})]))| <= (\text{rec}((1+1)+1) * (\epsilon)) \Rightarrow$   
 $\dot{\dot{}}(\dot{\dot{}}(|((\underline{fx})[(\underline{v1})] + (-u(\underline{fx})[(\underline{v2})]))| = (\text{rec}((1+1)+1) * (\epsilon))n)n)n \Rightarrow$





























































$$\begin{aligned}
& \dot{\neg}(\dot{\neg}(\overline{((fy)[\bar{m}] + (-ud_{Ph}[\bar{m}])))} = \overline{(\epsilon)})n)n)n)n} = \{ph \in P(\{ph \in P(\{ph \in P(P(\text{Union}\{N, Q\})) \mid \dot{\neg}(\forall_{obj}(\overline{op1}): \dot{\neg}(\forall_{obj}(\overline{op2}): \dot{\neg}(\dot{\neg}(\overline{op1}) \in N \Rightarrow \dot{\neg}(\overline{op2}) \in Q)n)n} \Rightarrow \dot{\neg}(\underline{a_{Ph}} = \{\{\overline{op1}, \overline{op1}\}, \{\overline{op1}, \overline{op2}\}\})n)n)n)n)n) \mid \dot{\neg}(\dot{\neg}(\forall_{obj}(\underline{r1}): (\underline{r1}) \in \underline{f_{Ph}} \Rightarrow \dot{\neg}(\forall_{obj}(\overline{op1}): \dot{\neg}(\dot{\neg}(\forall_{obj}(\overline{op2}): \dot{\neg}(\dot{\neg}(\overline{op1}) \in N \Rightarrow \dot{\neg}(\overline{op2}) \in Q)n)n} \Rightarrow \dot{\neg}(\underline{r1}) = \{\{\overline{op1}, \overline{op1}\}, \{\overline{op1}, \overline{op2}\}\})n)n)n)n)n) \Rightarrow \dot{\neg}(\forall_{obj}(\underline{f1}): \forall_{obj}(\underline{f2}): \forall_{obj}(\underline{f3}): \forall_{obj}(\underline{f4}): \{\{\overline{f1}, \overline{f1}\}, \{\overline{f1}, \overline{f2}\}\} \in \underline{f_{Ph}} \Rightarrow \{\{\overline{f3}, \overline{f3}\}, \{\overline{f3}, \overline{f4}\}\} \in \underline{f_{Ph}} \Rightarrow \underline{f1} = \underline{f3} \Rightarrow \underline{f2} = \underline{f4})n)n} \Rightarrow \dot{\neg}(\forall_{obj}(\underline{s1}): (\underline{s1}) \in N \Rightarrow \dot{\neg}(\forall_{obj}(\underline{s2}): \dot{\neg}(\{\{\overline{s1}, \overline{s1}\}, \{\overline{s1}, \overline{s2}\}\} \in \underline{f_{Ph}})n)n)n) \mid \forall_{obj}(\underline{\epsilon}): \dot{\neg}(\forall_{obj}\bar{n}: \dot{\neg}(\forall_{obj}\bar{m}: \dot{\neg}(0 \leq \underline{\epsilon}) \Rightarrow \dot{\neg}(\dot{\neg}(0 = \underline{\epsilon})n)n) \Rightarrow \bar{n} \leq \bar{m} \Rightarrow \dot{\neg}(\overline{((fx)[\bar{m}] + (-ud_{Ph}[\bar{m}])))} \leq \underline{\epsilon}) \Rightarrow \dot{\neg}(\dot{\neg}(\overline{((fx)[\bar{m}] + (-ud_{Ph}[\bar{m}])))} = \underline{\epsilon})n)n)n)n} \mid p_0, c\}
\end{aligned}$$

$$\begin{aligned}
& [\text{FromNotSameF}(\text{Strong})(\text{Helper2}) \xrightarrow{\text{stmt}} \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \forall \underline{z}: \forall \underline{u}: \forall \underline{v}: \underline{v} \leq = |(\underline{x} + (-\underline{uz}))| \vdash \dot{\neg}(|(\underline{x} + (-\underline{uy}))| \leq (\text{rec}(1+1) * \underline{v}) \Rightarrow \dot{\neg}(\dot{\neg}(|(\underline{x} + (-\underline{uy}))| = (\text{rec}(1+1) * \underline{v})n)n) \vdash \dot{\neg}(|(\underline{z} + (-\underline{uu}))| \leq (\text{rec}(1+1) * \underline{v}) \Rightarrow \dot{\neg}(\dot{\neg}(|(\underline{z} + (-\underline{uu}))| = (\text{rec}(1+1) * \underline{v})n)n) \vdash \dot{\neg}(\underline{y} = \underline{u})n] \\
& [\text{FromNotSameF}(\text{Strong})(\text{Helper2}) \xrightarrow{\text{proof}} \lambda c. \lambda x. \mathcal{P}(\lceil \text{SystemQ} \vdash \forall \underline{x}: \forall \underline{y}: \forall \underline{z}: \forall \underline{u}: \forall \underline{v}: \underline{v} \leq = |(\underline{x} + (-\underline{uz}))| \vdash \dot{\neg}(|(\underline{x} + (-\underline{uy}))| \leq (\text{rec}(1+1) * \underline{v}) \Rightarrow \dot{\neg}(\dot{\neg}(|(\underline{x} + (-\underline{uy}))| = (\text{rec}(1+1) * \underline{v})n)n) \vdash \dot{\neg}(|(\underline{z} + (-\underline{uu}))| \leq (\text{rec}(1+1) * \underline{v}) \Rightarrow \dot{\neg}(\dot{\neg}(|(\underline{z} + (-\underline{uu}))| = (\text{rec}(1+1) * \underline{v})n)n) \vdash \text{LessNegated} \triangleright \dot{\neg}(|(\underline{x} + (-\underline{uy}))| \leq (\text{rec}(1+1) * \underline{v}) \Rightarrow \dot{\neg}(\dot{\neg}(|(\underline{x} + (-\underline{uy}))| = (\text{rec}(1+1) * \underline{v})n)n) \gg \dot{\neg}((-u(\text{rec}(1+1) * \underline{v})) \leq (-u|(\underline{x} + (-\underline{uy}))|) \Rightarrow \dot{\neg}(\dot{\neg}((-u(\text{rec}(1+1) * \underline{v})) = (-u|(\underline{x} + (-\underline{uy}))|)n)n); \text{NumericalDifference} \gg |(\underline{z} + (-\underline{uu}))| = |(\underline{u} + (-\underline{uz}))|; \text{SubLessLeft} \triangleright |(\underline{z} + (-\underline{uu}))| = |(\underline{u} + (-\underline{uz}))| \triangleright \dot{\neg}(|(\underline{z} + (-\underline{uu}))| \leq (\text{rec}(1+1) * \underline{v}) \Rightarrow \dot{\neg}(\dot{\neg}(|(\underline{z} + (-\underline{uu}))| = (\text{rec}(1+1) * \underline{v})n)n) \gg \dot{\neg}(|(\underline{u} + (-\underline{uz}))| \leq (\text{rec}(1+1) * \underline{v}) \Rightarrow \dot{\neg}(\dot{\neg}(|(\underline{u} + (-\underline{uz}))| = (\text{rec}(1+1) * \underline{v})n)n); \text{LessNegated} \triangleright \dot{\neg}(|(\underline{u} + (-\underline{uz}))| \leq (\text{rec}(1+1) * \underline{v}) \Rightarrow \dot{\neg}(\dot{\neg}(|(\underline{u} + (-\underline{uz}))| = (\text{rec}(1+1) * \underline{v})n)n) \gg \dot{\neg}((-u(\text{rec}(1+1) * \underline{v})) \leq (-u|(\underline{u} + (-\underline{uz}))|) \Rightarrow \dot{\neg}(\dot{\neg}((-u(\text{rec}(1+1) * \underline{v})) = (-u|(\underline{u} + (-\underline{uz}))|)n)n); \text{AddEquations(Less)} \triangleright \dot{\neg}((-u(\text{rec}(1+1) * \underline{v})) \leq (-u|(\underline{x} + (-\underline{uy}))|) \Rightarrow \dot{\neg}(\dot{\neg}((-u(\text{rec}(1+1) * \underline{v})) = (-u|(\underline{x} + (-\underline{uy}))|)n)n) \triangleright \dot{\neg}((-u(\text{rec}(1+1) * \underline{v})) \leq (-u|(\underline{u} + (-\underline{uz}))|) \Rightarrow \dot{\neg}(\dot{\neg}((-u(\text{rec}(1+1) * \underline{v})) = (-u|(\underline{u} + (-\underline{uz}))|)n)n) \gg \dot{\neg}(((\text{rec}(1+1) * \underline{v}) + (-u(\text{rec}(1+1) * \underline{v}))) \leq ((-u|(\underline{x} + (-\underline{uy}))|) + (-u|(\underline{u} + (-\underline{uz}))|)) \Rightarrow \dot{\neg}(\dot{\neg}(((\text{rec}(1+1) * \underline{v}) + (-u(\text{rec}(1+1) * \underline{v}))) = ((-u|(\underline{x} + (-\underline{uy}))|) + (-u|(\underline{u} + (-\underline{uz}))|))n)n); \text{TwoHalves} \gg ((\text{rec}(1+1) * \underline{v}) + (\text{rec}(1+1) * \underline{v})) = \underline{v}; \text{EqNegated} \triangleright ((\text{rec}(1+1) * \underline{v}) + (\text{rec}(1+1) * \underline{v})) = \underline{v} \gg (-u((\text{rec}(1+1) * \underline{v}) + (\text{rec}(1+1) * \underline{v}))) = (-\underline{uv}); -x - y = -(x + y) \gg ((-u(\text{rec}(1+1) * \underline{v})) + (-u(\text{rec}(1+1) * \underline{v}))) = (-u((\text{rec}(1+1) * \underline{v}) + (\text{rec}(1+1) * \underline{v}))) \triangleright ((-u(\text{rec}(1+1) * \underline{v}) + (-u(\text{rec}(1+1) * \underline{v}))) = (-u((\text{rec}(1+1) * \underline{v}) + (\text{rec}(1+1) * \underline{v}))) \triangleright (-u((\text{rec}(1+1) * \underline{v}) + (\text{rec}(1+1) * \underline{v}))) = (-\underline{uv}) \gg ((-u(\text{rec}(1+1) * \underline{v}) + (-u(\text{rec}(1+1) * \underline{v}))) = (-\underline{uv}); \text{SubLessLeft} \triangleright ((-u(\text{rec}(1+1) * \underline{v}) + (-u(\text{rec}(1+1) * \underline{v}))) = (-\underline{uv}) \triangleright \dot{\neg}(((\text{rec}(1+1) * \underline{v}) + (-u(\text{rec}(1+1) * \underline{v}))) \leq ((-u|(\underline{x} + (-\underline{uy}))|) + (-u|(\underline{u} + (-\underline{uz}))|)) \Rightarrow \dot{\neg}(\dot{\neg}(((\text{rec}(1+1) * \underline{v}) + (-u(\text{rec}(1+1) * \underline{v}))) = ((-u|(\underline{x} + (-\underline{uy}))|) + (-u|(\underline{u} + (-\underline{uz}))|))n)n) \gg
\end{aligned}$$





























































$\{\{\underline{m}, \underline{m}\}, \{\underline{m}, (\overline{(\underline{f}x)[\underline{m}]} * \{ph \in \{ph \in P(P(\text{Union}(\{N, Q\})))\}) \mid$   
 $\dot{\neg}(\forall_{\text{obj}}\overline{(\text{op1})}) : \dot{\neg}(\dot{\neg}(\forall_{\text{obj}}\overline{(\text{op2})}) : \dot{\neg}(\dot{\neg}(\overline{(\text{op1})} \in N \Rightarrow \dot{\neg}(\overline{(\text{op2})} \in Q))n)n \Rightarrow$   
 $\dot{\neg}(\underline{a}_{Ph} = \{\{\overline{(\text{op1})}, \overline{(\text{op1})}\}, \{\overline{(\text{op1})}, \overline{(\text{op2})}\})n)n)n)n)n\} \mid$   
 $\dot{\neg}(\forall_{\text{obj}}\underline{m} : \dot{\neg}(\dot{\neg}(\dot{\neg}(\overline{(\underline{f}x)[\underline{m}]} = 0))n \Rightarrow \dot{\neg}(\underline{f}_{Ph} =$   
 $\{\{\underline{m}, \underline{m}\}, \{\underline{m}, \text{rec}(\underline{f}x)[\underline{m}]\})n)n)n \Rightarrow \dot{\neg}(\overline{(\underline{f}x)[\underline{m}]} = 0 \Rightarrow \dot{\neg}(\underline{f}_{Ph} =$   
 $\{\{\underline{m}, \underline{m}\}, \{\underline{m}, 0\})n)n)n)n\}[\underline{m}]\})n\}[\overline{m}] + (-\text{ud}_{Ph}[\overline{m}])\} | <= \overline{(\epsilon)} \Rightarrow$   
 $\dot{\neg}(\dot{\neg}(\{ \{ph \in \{ph \in P(P(\text{Union}(\{N, Q\}))) \} \mid$   
 $\dot{\neg}(\forall_{\text{obj}}\overline{(\text{op1})}) : \dot{\neg}(\dot{\neg}(\forall_{\text{obj}}\overline{(\text{op2})}) : \dot{\neg}(\dot{\neg}(\overline{(\text{op1})} \in N \Rightarrow \dot{\neg}(\overline{(\text{op2})} \in Q))n)n \Rightarrow$   
 $\dot{\neg}(\underline{a}_{Ph} = \{\{\overline{(\text{op1})}, \overline{(\text{op1})}\}, \{\overline{(\text{op1})}, \overline{(\text{op2})}\})n)n)n)n)n\} \mid \dot{\neg}(\forall_{\text{obj}}\underline{m} : \dot{\neg}(\underline{e}_{Ph} =$   
 $\{\{\underline{m}, \underline{m}\}, \{\underline{m}, (\overline{(\underline{f}x)[\underline{m}]} * \{ph \in \{ph \in P(P(\text{Union}(\{N, Q\})))\}) \mid$   
 $\dot{\neg}(\forall_{\text{obj}}\overline{(\text{op1})}) : \dot{\neg}(\dot{\neg}(\forall_{\text{obj}}\overline{(\text{op2})}) : \dot{\neg}(\dot{\neg}(\overline{(\text{op1})} \in N \Rightarrow \dot{\neg}(\overline{(\text{op2})} \in Q))n)n \Rightarrow$   
 $\dot{\neg}(\underline{a}_{Ph} = \{\{\overline{(\text{op1})}, \overline{(\text{op1})}\}, \{\overline{(\text{op1})}, \overline{(\text{op2})}\})n)n)n)n)n\} \mid$   
 $\dot{\neg}(\forall_{\text{obj}}\underline{m} : \dot{\neg}(\dot{\neg}(\dot{\neg}(\overline{(\underline{f}x)[\underline{m}]} = 0))n \Rightarrow \dot{\neg}(\underline{f}_{Ph} =$   
 $\{\{\underline{m}, \underline{m}\}, \{\underline{m}, \text{rec}(\underline{f}x)[\underline{m}]\})n)n)n \Rightarrow \dot{\neg}(\overline{(\underline{f}x)[\underline{m}]} = 0 \Rightarrow \dot{\neg}(\underline{f}_{Ph} =$   
 $\{\{\underline{m}, \underline{m}\}, \{\underline{m}, 0\})n)n)n)n\}[\underline{m}]\})n\}[\overline{m}] + (-\text{ud}_{Ph}[\overline{m}])\} | = \overline{(\epsilon)}n)n)n)n\} =$   
 $\{ph \in P(\{ph \in P(\{ph \in P(P(\text{Union}(\{N, Q\})))\}) \mid$   
 $\dot{\neg}(\forall_{\text{obj}}\overline{(\text{op1})}) : \dot{\neg}(\dot{\neg}(\forall_{\text{obj}}\overline{(\text{op2})}) : \dot{\neg}(\dot{\neg}(\overline{(\text{op1})} \in N \Rightarrow \dot{\neg}(\overline{(\text{op2})} \in Q))n)n \Rightarrow$   
 $\dot{\neg}(\underline{a}_{Ph} = \{\{\overline{(\text{op1})}, \overline{(\text{op1})}\}, \{\overline{(\text{op1})}, \overline{(\text{op2})}\})n)n)n)n)n\} \mid \dot{\neg}(\dot{\neg}(\forall_{\text{obj}}\overline{(\underline{r1})}) : \overline{(\underline{r1})} \in$   
 $\underline{f}_{Ph} \Rightarrow \dot{\neg}(\forall_{\text{obj}}\overline{(\text{op1})}) : \dot{\neg}(\dot{\neg}(\forall_{\text{obj}}\overline{(\text{op2})}) : \dot{\neg}(\dot{\neg}(\overline{(\text{op1})} \in N \Rightarrow \dot{\neg}(\overline{(\text{op2})} \in Q))n)n \Rightarrow$   
 $\dot{\neg}(\overline{(\underline{r1})} = \{\{\overline{(\text{op1})}, \overline{(\text{op1})}\}, \{\overline{(\text{op1})}, \overline{(\text{op2})}\})n)n)n)n)n\} \Rightarrow$   
 $\dot{\neg}(\underline{v}_{\text{obj}}(\underline{f1}) : \underline{v}_{\text{obj}}(\underline{f2}) : \underline{v}_{\text{obj}}(\underline{f3}) : \underline{v}_{\text{obj}}(\underline{f4}) : \{\{\overline{(\underline{f1})}, \overline{(\underline{f1})}\}, \{\overline{(\underline{f1})}, \overline{(\underline{f2})}\}) \in \underline{f}_{Ph} \Rightarrow$   
 $\{\{\overline{(\underline{f3})}, \overline{(\underline{f3})}\}, \{\overline{(\underline{f3})}, \overline{(\underline{f4})}\}) \in \underline{f}_{Ph} \Rightarrow \overline{(\underline{f1})} = \overline{(\underline{f3})} \Rightarrow \overline{(\underline{f2})} = \overline{(\underline{f4})})n)n \Rightarrow$   
 $\dot{\neg}(\forall_{\text{obj}}(\underline{s1}) : (\underline{s1}) \in N \Rightarrow \dot{\neg}(\forall_{\text{obj}}(\underline{s2}) : \dot{\neg}(\{\{\overline{(\underline{s1})}, \overline{(\underline{s1})}\}, \{\overline{(\underline{s1})}, \overline{(\underline{s2})}\}) \in$   
 $\underline{f}_{Ph})n)n)n)n\} \mid \forall_{\text{obj}}(\overline{(\underline{\epsilon})}) : \dot{\neg}(\forall_{\text{obj}}\overline{(\underline{n})}) : \dot{\neg}(\forall_{\text{obj}}\overline{(\underline{m})}) : \dot{\neg}(0 <= \overline{(\underline{\epsilon})} \Rightarrow \dot{\neg}(\dot{\neg}(0 = \overline{(\underline{\epsilon})})n)n)n \Rightarrow$   
 $\overline{(\underline{n})} <= \overline{(\underline{m})} \Rightarrow \dot{\neg}(\{ \{ph \in \{ph \in P(P(\text{Union}(\{N, Q\})))\}) \mid$   
 $\dot{\neg}(\forall_{\text{obj}}\overline{(\text{op1})}) : \dot{\neg}(\dot{\neg}(\forall_{\text{obj}}\overline{(\text{op2})}) : \dot{\neg}(\dot{\neg}(\overline{(\text{op1})} \in N \Rightarrow \dot{\neg}(\overline{(\text{op2})} \in Q))n)n \Rightarrow$   
 $\dot{\neg}(\underline{a}_{Ph} = \{\{\overline{(\text{op1})}, \overline{(\text{op1})}\}, \{\overline{(\text{op1})}, \overline{(\text{op2})}\})n)n)n)n)n\} \mid \dot{\neg}(\forall_{\text{obj}}\overline{(\underline{crs1})}) : \dot{\neg}(\underline{c}_{Ph} =$   
 $\{\{\overline{(\underline{crs1})}, \overline{(\underline{crs1})}\}, \{\overline{(\underline{crs1})}, \underline{1}\})n)n\}[\underline{m}] + (-\text{ud}_{Ph}[\underline{m}])\} | <= \overline{(\epsilon)} \Rightarrow \dot{\neg}(\dot{\neg}(\{ \{ph \in$   
 $\{ph \in P(P(\text{Union}(\{N, Q\})))\}) \mid \dot{\neg}(\forall_{\text{obj}}\overline{(\text{op1})}) : \dot{\neg}(\dot{\neg}(\forall_{\text{obj}}\overline{(\text{op2})}) : \dot{\neg}(\dot{\neg}(\overline{(\text{op1})} \in$   
 $N \Rightarrow \dot{\neg}(\overline{(\text{op2})} \in Q))n)n \Rightarrow \dot{\neg}(\underline{a}_{Ph} =$   
 $\{\{\overline{(\text{op1})}, \overline{(\text{op1})}\}, \{\overline{(\text{op1})}, \overline{(\text{op2})}\})n)n)n)n)n\} \mid \dot{\neg}(\forall_{\text{obj}}\overline{(\underline{crs1})}) : \dot{\neg}(\underline{c}_{Ph} =$   
 $\{\{\overline{(\underline{crs1})}, \overline{(\underline{crs1})}\}, \{\overline{(\underline{crs1})}, \underline{1}\})n)n\}[\underline{m}] + (-\text{ud}_{Ph}[\underline{m}])\} | = \overline{(\epsilon)}n)n)n)n\}, p_0, c]$

$[\text{sup} \xrightarrow{\text{prio}}$

## Preassociative

$[\text{sup}], [\text{base}], [\text{bracket} * \text{end bracket}], [\text{big bracket} * \text{end bracket}], [ * * * ],$

$[\text{flush left} *], [x], [y], [z], [ * \times * ], [ * \rightarrow * ], [\text{pyk}], [\text{tex}], [\text{name}], [\text{prio}], [ * ], [T],$

$[\text{if}(*, *, *)], [ * \xrightarrow{*} * ], [\text{val}], [\text{claim}], [\perp], [f(*)], [(*)^1], [F], [0], [1], [2], [3], [4], [5], [6],$

$[7], [8], [9], [0], [1], [2], [3], [4], [5], [6], [7], [8], [9], [a], [b], [c], [d], [e], [f], [g], [h], [i], [j],$

$[k], [l], [m], [n], [o], [p], [q], [r], [s], [t], [u], [v], [w], [(*)^M], [If(*, *, *)],$

$[\text{array} \{ * \} * \text{end array}], [l], [c], [r], [\text{empty}], [ * | * := * ], [\mathcal{M}(*)], [\tilde{\mathcal{U}}(*)], [\mathcal{U}(*)],$

$[\mathcal{U}^M(*)], [\text{apply}(*, *)], [\text{apply}_1(*, *)], [\text{identifier}(*)], [\text{identifier}_1(*, *)], [\text{array-$

plus(\*, \*), [array-remove(\*, \*, \*)], [array-put(\*, \*, \*, \*)], [array-add(\*, \*, \*, \*, \*)],  
 [bit(\*, \*)], [bit<sub>1</sub>(\*, \*)], [rack], ["vector"], ["bibliography"], ["dictionary"],  
 ["body"], ["codex"], ["expansion"], ["code"], ["cache"], ["diagnose"], ["pyk"],  
 ["tex"], ["texname"], ["value"], ["message"], ["macro"], ["definition"],  
 ["unpack"], ["claim"], ["priority"], ["lambda"], ["apply"], ["true"], ["if"],  
 ["quote"], ["proclaim"], ["define"], ["introduce"], ["hide"], ["pre"], ["post"],  
 [ $\mathcal{E}(*, *, *)$ ], [ $\mathcal{E}_2(*, *, *, *, *)$ ], [ $\mathcal{E}_3(*, *, *, *, *)$ ], [ $\mathcal{E}_4(*, *, *, *, *)$ ], **lookup**(\*, \*, \*),  
**abstract**(\*, \*, \*, \*), [ $\{*\}$ ], [ $\mathcal{M}(*, *, *)$ ], [ $\mathcal{M}_2(*, *, *, *)$ ], [ $\mathcal{M}^*(*, *, *)$ ], [macro],  
 [ $s_0$ ], [**zip**(\*, \*)], [**assoc**<sub>1</sub>(\*, \*, \*)], [ $(*)^P$ ], [self], [ $[* \doteq *]$ ], [ $[* \dot{=} *]$ ], [ $[* \dot{=} *]$ ],  
 [ $[* \stackrel{\text{pyk}}{=} *]$ ], [ $[* \stackrel{\text{tex}}{=} *]$ ], [ $[* \stackrel{\text{name}}{=} *]$ ], [**Priority table**[\*]], [ $\tilde{\mathcal{M}}_1$ ], [ $\tilde{\mathcal{M}}_2(*)$ ], [ $\tilde{\mathcal{M}}_3(*)$ ],  
 [ $\tilde{\mathcal{M}}_4(*, *, *, *)$ ], [ $\mathcal{M}(*, *, *)$ ], [ $\tilde{\mathcal{Q}}(*, *, *)$ ], [ $\tilde{\mathcal{Q}}_2(*, *, *)$ ], [ $\tilde{\mathcal{Q}}_3(*, *, *, *)$ ], [ $\tilde{\mathcal{Q}}^*(*, *, *)$ ],  
 [ $(*)$ ], [ $(*)$ ], [display(\*)], [statement(\*)], [ $[*]$ ], [ $[*]^-$ ], [**aspect**(\*, \*)],  
**aspect**(\*, \*, \*), [ $\langle * \rangle$ ], [**tuple**<sub>1</sub>(\*)], [**tuple**<sub>2</sub>(\*)], [let<sub>2</sub>(\*, \*)], [let<sub>1</sub>(\*, \*)],  
 [ $[* \stackrel{\text{claim}}{=} *]$ ], [checker], [**check**(\*, \*)], [**check**<sub>2</sub>(\*, \*, \*)], [**check**<sub>3</sub>(\*, \*, \*)],  
**check**<sup>\*</sup>(\*, \*), [**check**<sub>2</sub><sup>\*</sup>(\*, \*, \*)], [ $[*]$ ], [ $[*]^-$ ], [ $[*]^\circ$ ], [msg], [ $[* \stackrel{\text{msg}}{=} *]$ ], [ $\langle \text{stmt} \rangle$ ],  
 [stmt], [ $[* \stackrel{\text{stmt}}{=} *]$ ], [HeadNil'], [HeadPair'], [Transitivity'], [ $\perp$ ], [Contra'], [T<sub>E</sub>],  
 [L<sub>1</sub>], [ $\underline{*}$ ], [A], [B], [C], [D], [E], [F], [G], [H], [I], [J], [K], [L], [M], [N], [O], [P], [Q],  
 [R], [S], [T], [U], [V], [W], [X], [Y], [Z], [ $[* * := *]$ ], [ $[* * := *]$ ], [ $\emptyset$ ], [Remainder],  
 [ $(*)^\vee$ ], [intro(\*, \*, \*, \*)], [intro(\*, \*, \*)], [error(\*, \*)], [error<sub>2</sub>(\*, \*)], [proof(\*, \*, \*)],  
 [proof<sub>2</sub>(\*, \*)], [S(\*, \*)], [S<sup>I</sup>(\*, \*)], [S<sup>D</sup>(\*, \*)], [S<sup>D</sup>(\*, \*, \*)], [S<sup>E</sup>(\*, \*)], [S<sup>E</sup>(\*, \*, \*)],  
 [S<sup>+</sup>(\*, \*)], [S<sup>+</sup>(\*, \*, \*)], [S<sup>-</sup>(\*, \*)], [S<sup>-</sup>(\*, \*, \*)], [S<sup>\*</sup>(\*, \*)], [S<sup>\*</sup>(\*, \*, \*)],  
 [S<sub>2</sub><sup>\*</sup>(\*, \*, \*, \*)], [S<sup>@</sup>(\*, \*)], [S<sup>@</sup>(\*, \*, \*)], [S<sup>+</sup>(\*, \*)], [S<sub>1</sub><sup>+</sup>(\*, \*, \*, \*)], [S<sup>+</sup>(\*, \*)],  
 [S<sub>1</sub><sup>+</sup>(\*, \*, \*, \*)], [S<sup>i.e.</sup>(\*, \*)], [S<sup>i.e.</sup>(\*, \*, \*, \*)], [S<sub>2</sub><sup>i.e.</sup>(\*, \*, \*, \*, \*)], [S<sup>V</sup>(\*, \*)],  
 [S<sub>1</sub><sup>V</sup>(\*, \*, \*, \*)], [S<sup>i</sup>(\*, \*)], [S<sub>1</sub><sup>i</sup>(\*, \*, \*)], [S<sub>2</sub><sup>i</sup>(\*, \*, \*, \*)], [T(\*)], [claims(\*, \*, \*)],  
 [claims<sub>2</sub>(\*, \*, \*)], [ $\langle \text{proof} \rangle$ ], [proof], [**Lemma** \*: \*], [**Proof of** \*: \*],  
 [**\* lemma** \*: \*], [**\* antilemma** \*: \*], [**\* rule** \*: \*], [**\* antirule** \*: \*],  
 [verifier], [V<sub>1</sub>(\*)], [V<sub>2</sub>(\*, \*)], [V<sub>3</sub>(\*, \*, \*, \*)], [V<sub>4</sub>(\*, \*)], [V<sub>5</sub>(\*, \*, \*, \*)], [V<sub>6</sub>(\*, \*, \*, \*)],  
 [V<sub>7</sub>(\*, \*, \*, \*)], [Cut(\*, \*)], [Head<sub>⊕</sub>(\*)], [Tail<sub>⊕</sub>(\*)], [rule<sub>1</sub>(\*, \*)], [rule(\*, \*)],  
 [Rule tactic], [Plus(\*, \*)], [**Theory** \*], [theory<sub>2</sub>(\*, \*)], [theory<sub>3</sub>(\*, \*)],  
 [theory<sub>4</sub>(\*, \*, \*)], [HeadNil''], [HeadPair''], [Transitivity''], [Contra''], [HeadNil],  
 [HeadPair], [Transitivity], [Contra], [T<sub>E</sub>], [ragged right],  
 [ragged right expansion ], [parm(\*, \*, \*)], [parm<sup>\*</sup>(\*, \*, \*)], [inst(\*, \*)],  
 [inst<sup>\*</sup>(\*, \*)], [occur(\*, \*, \*)], [occur<sup>\*</sup>(\*, \*, \*)], [unify(\* = \*, \*)], [unify<sup>\*</sup>(\* = \*, \*)],  
 [unify<sub>2</sub>(\* = \*, \*)], [L<sub>a</sub>], [L<sub>b</sub>], [L<sub>c</sub>], [L<sub>d</sub>], [L<sub>e</sub>], [L<sub>f</sub>], [L<sub>g</sub>], [L<sub>h</sub>], [L<sub>i</sub>], [L<sub>j</sub>], [L<sub>k</sub>], [L<sub>l</sub>], [L<sub>m</sub>],  
 [L<sub>n</sub>], [L<sub>o</sub>], [L<sub>p</sub>], [L<sub>q</sub>], [L<sub>r</sub>], [L<sub>s</sub>], [L<sub>t</sub>], [L<sub>u</sub>], [L<sub>v</sub>], [L<sub>w</sub>], [L<sub>x</sub>], [L<sub>y</sub>], [L<sub>z</sub>], [L<sub>A</sub>], [L<sub>B</sub>], [L<sub>C</sub>],  
 [L<sub>D</sub>], [L<sub>E</sub>], [L<sub>F</sub>], [L<sub>G</sub>], [L<sub>H</sub>], [L<sub>I</sub>], [L<sub>J</sub>], [L<sub>K</sub>], [L<sub>L</sub>], [L<sub>M</sub>], [L<sub>N</sub>], [L<sub>O</sub>], [L<sub>P</sub>], [L<sub>Q</sub>], [L<sub>R</sub>],  
 [L<sub>S</sub>], [L<sub>T</sub>], [L<sub>U</sub>], [L<sub>V</sub>], [L<sub>W</sub>], [L<sub>X</sub>], [L<sub>Y</sub>], [L<sub>Z</sub>], [L<sub>?</sub>], [Reflexivity], [Reflexivity<sub>1</sub>],  
 [Commutativity], [Commutativity<sub>1</sub>], [ $\langle \text{tactic} \rangle$ ], [tactic], [ $[* \stackrel{\text{tactic}}{=} *]$ ], [P(\*, \*, \*)],  
 [P<sup>\*</sup>(\*, \*, \*)], [p<sub>0</sub>], [conclude<sub>1</sub>(\*, \*)], [conclude<sub>2</sub>(\*, \*, \*)], [conclude<sub>3</sub>(\*, \*, \*, \*)],  
 [conclude<sub>4</sub>(\*, \*)], [check], [ $[* \stackrel{\circ}{=} *]$ ], [RootVisible(\*)], [A], [R], [C], [T], [L], [ $\{*\}$ ], [ $\bar{*}$ ],  
 [a], [b], [c], [d], [e], [f], [g], [h], [i], [j], [k], [l], [m], [n], [o], [p], [q], [r], [s], [t], [u], [v],  
 [w], [x], [y], [z], [ $[* \equiv * | * := *]$ ], [ $[* \equiv^0 * | * := *]$ ], [ $[* \equiv^1 * | * := *]$ ], [ $[* \equiv^* * | * := *]$ ],  
 [Ded(\*, \*)], [Ded<sub>0</sub>(\*, \*)], [Ded<sub>1</sub>(\*, \*, \*)], [Ded<sub>2</sub>(\*, \*, \*)], [Ded<sub>3</sub>(\*, \*, \*, \*)],  
 [Ded<sub>4</sub>(\*, \*, \*, \*)], [Ded<sub>4</sub><sup>\*</sup>(\*, \*, \*, \*)], [Ded<sub>5</sub>(\*, \*, \*)], [Ded<sub>6</sub>(\*, \*, \*, \*)],



[Ded<sub>6</sub><sup>\*</sup>(\*, \*, \*, \*)], [Ded<sub>7</sub>(\*)], [Ded<sub>8</sub>(\*, \*)], [Ded<sub>8</sub><sup>\*</sup>(\*, \*)], [S], [Neg], [MP], [Gen],  
 [Ded], [S1], [S2], [S3], [S4], [S5], [S6], [S7], [S8], [S9], [Repetition], [A1'], [A2'], [A4'],  
 [A5'], [Prop 3.2a], [Prop 3.2b], [Prop 3.2c], [Prop 3.2d], [Prop 3.2e<sub>1</sub>], [Prop 3.2e<sub>2</sub>],  
 [Prop 3.2e], [Prop 3.2f<sub>1</sub>], [Prop 3.2f<sub>2</sub>], [Prop 3.2f], [Prop 3.2g<sub>1</sub>], [Prop 3.2g<sub>2</sub>],  
 [Prop 3.2g], [Prop 3.2h<sub>1</sub>], [Prop 3.2h<sub>2</sub>], [Prop 3.2h], [Block<sub>1</sub>(\*, \*, \*)], [Block<sub>2</sub>(\*)],  
 [kvanti], [UniqueMember], [UniqueMember(Type)], [SameSeries], [A4],  
 [SameMember], [Qclosed(Addition)], [Qclosed(Multiplication)],  
 [FromCartProd(1)], [1rule fromCartProd(2)], [constantRationalSeries(\*)],  
 [cartProd(\*)], [Power(\*)], [binaryUnion(\*, \*)], [SetOfRationalSeries],  
 [IsSubset(\*, \*)], [(p\*, \*)], [(s\*), [(· · ·)], [Objekt-var], [Ex-var], [Ph-var], [Værdi],  
 [Variabel], [Op(\*)], [Op(\*, \*)], [\* ::= \*], [ContainsEmpty(\*)], [Nat(\*)],  
 [Dedu(\*, \*)], [Dedu<sub>0</sub>(\*, \*)], [Dedu<sub>s</sub>(\*, \*, \*)], [Dedu<sub>1</sub>(\*, \*, \*)], [Dedu<sub>2</sub>(\*, \*, \*)],  
 [Dedu<sub>3</sub>(\*, \*, \*, \*)], [Dedu<sub>4</sub>(\*, \*, \*, \*)], [Dedu<sub>4</sub><sup>\*</sup>(\*, \*, \*, \*)], [Dedu<sub>5</sub>(\*, \*, \*)],  
 [Dedu<sub>6</sub>(\*, \*, \*, \*)], [Dedu<sub>6</sub><sup>\*</sup>(\*, \*, \*, \*)], [Dedu<sub>7</sub>(\*)], [Dedu<sub>8</sub>(\*, \*)], [Dedu<sub>8</sub><sup>\*</sup>(\*, \*)],  
 [Ex<sub>1</sub>], [Ex<sub>2</sub>], [Ex<sub>3</sub>], [Ex<sub>10</sub>], [Ex<sub>20</sub>], [\*<sub>Ex</sub>], [\*<sup>Ex</sup>], [( $\equiv$  \* | \* ::= \*)<sub>Ex</sub>],  
 [( $\equiv$ <sup>0</sup> \* | \* ::= \*)<sub>Ex</sub>], [( $\equiv$ <sup>1</sup> \* | \* ::= \*)<sub>Ex</sub>], [( $\equiv$ <sup>\*</sup> \* | \* ::= \*)<sub>Ex</sub>], [ph<sub>1</sub>], [ph<sub>2</sub>], [ph<sub>3</sub>],  
 [\*<sub>Ph</sub>], [\*<sup>Ph</sup>], [( $\equiv$  \* | \* ::= \*)<sub>Ph</sub>], [( $\equiv$ <sup>0</sup> \* | \* ::= \*)<sub>Ph</sub>], [( $\equiv$ <sup>1</sup> \* | \* ::= \*)<sub>Ph</sub>],  
 [( $\equiv$ <sup>\*</sup> \* | \* ::= \*)<sub>Ph</sub>], [( $\equiv$  \* | \* ::= \*)<sub>Me</sub>], [( $\equiv$ <sup>1</sup> \* | \* ::= \*)<sub>Me</sub>],  
 [( $\equiv$ <sup>\*</sup> \* | \* ::= \*)<sub>Me</sub>], [bs], [OBS], [BS], [Ø], [SystemQ], [MP], [Gen], [Repetition],  
 [Neg], [Ded], [ExistIntro], [Extensionality], [Ødef], [PairDef], [UnionDef],  
 [PowerDef], [SeparationDef], [AddDoubleNeg], [RemoveDoubleNeg],  
 [AndCommutativity], [AutoImPLY], [Contrapositive], [FirstConjunct],  
 [SecondConjunct], [FromContradiction], [FromDisjuncts], [IffCommutativity],  
 [IffFirst], [IffSecond], [ImPLYTransitivity], [JoinConjuncts], [MP2], [MP3], [MP4],  
 [MP5], [MT], [NegativeMT], [Technicality], [Weakening], [WeakenOr1],  
 [WeakenOr2], [Formula2Pair], [Pair2Formula], [Formula2Union],  
 [Union2Formula], [Formula2Sep], [Sep2Formula], [Formula2Power],  
 [SubsetInPower], [HelperPowerIsSub], [PowerIsSub],  
 [(Switch)HelperPowerIsSub], [(Switch)PowerIsSub], [ToSetEquality],  
 [HelperToSetEquality(t)], [ToSetEquality(t)], [HelperFromSetEquality],  
 [FromSetEquality], [HelperReflexivity], [Reflexivity], [HelperSymmetry],  
 [Symmetry], [HelperTransitivity], [Transitivity], [ERisReflexive],  
 [ERisSymmetric], [ERisTransitive], [ØisSubset], [HelperMemberNotØ],  
 [MemberNotØ], [HelperUniqueØ], [UniqueØ], [= Reflexivity], [= Symmetry],  
 [Helper == Transitivity], [= Transitivity], [HelperTransferNotEq],  
 [TransferNotEq], [HelperPairSubset], [Helper(2)PairSubset], [PairSubset],  
 [SamePair], [SameSingleton], [UnionSubset], [SameUnion], [SeparationSubset],  
 [SameSeparation], [SameBinaryUnion], [IntersectionSubset], [SameIntersection],  
 [AutoMember], [HelperEqSysNotØ], [EqSysNotØ], [HelperEqSubset],  
 [EqSubset], [HelperEqNecessary], [EqNecessary], [HelperNoneEqNecessary],  
 [Helper(2)NoneEqNecessary], [NoneEqNecessary], [EqClassIsSubset],  
 [EqClassesAreDisjoint], [AllDisjoint], [AllDisjointImPLY], [BSsubset],  
 [Union(BS/R)subset], [UnionIdentity], [EqSysIsPartition], [(x1)], [(x2)], [(y1)],  
 [(y2)], [(v1)], [(v2)], [(v3)], [(v4)], [(v2n)], [(m1)], [(m2)], [(n1)], [(n2)], [(n3)], [(ε)],  
 [(ε)<sub>1</sub>], [(ε)<sub>2</sub>], [(fep)], [(fx)], [(fy)], [(fz)], [(fu)], [(fv)], [(fw)], [(rx)], [(ry)], [(rz)],  
 [(ru)], [(sx)], [(sx1)], [(sy)], [(sy1)], [(sz)], [(sz1)], [(su)], [(su1)], [(fxs)], [(fys)],

[(crs1)], [(f1)], [(f2)], [(f3)], [(f4)], [(op1)], [(op2)], [(r1)], [(s1)], [(s2)], [X<sub>1</sub>], [X<sub>2</sub>], [Y<sub>1</sub>], [Y<sub>2</sub>], [V<sub>1</sub>], [V<sub>2</sub>], [V<sub>3</sub>], [V<sub>4</sub>], [V<sub>2n</sub>], [M<sub>1</sub>], [M<sub>2</sub>], [N<sub>1</sub>], [N<sub>2</sub>], [N<sub>3</sub>], [ε], [ε<sub>1</sub>], [ε<sub>2</sub>], [FX], [FY], [FZ], [FU], [FV], [FW], [FEP], [RX], [RY], [RZ], [RU], [(SX)], [(SX1)], [(SY)], [(SY1)], [(SZ)], [(SZ1)], [(SU)], [(SU1)], [FXS], [FYS], [(F1)], [(F2)], [(F3)], [(F4)], [(OP1)], [(OP2)], [(R1)], [(S1)], [(S2)], [(EPob)], [(CRS1ob)], [(F1ob)], [(F2ob)], [(F3ob)], [(F4ob)], [(N1ob)], [(N2ob)], [(OP1ob)], [(OP2ob)], [(R1ob)], [(S1ob)], [(S2ob)], [ph<sub>4</sub>], [ph<sub>5</sub>], [ph<sub>6</sub>], [NAT], [RATIONALSERIES], [SERIES], [SetOfReals], [SetOfFxs], [N], [Q], [X], [xs], [xaF], [ysF], [us], [usFoelge], [0], [1], [(-1)], [2], [3], [1/2], [1/3], [2/3], [0f], [1f], [00], [01], [(- - 01)], [02], [01/02], [PlusAssociativity(R)], [PlusAssociativity(R)XX], [Plus0(R)], [Negative(R)], [Times1(R)], [lessAddition(R)], [PlusCommutativity(R)], [LeqAntisymmetry(R)], [LeqTransitivity(R)], [leqAddition(R)], [Distribution(R)], [A4(Axiom)], [InductionAxiom], [EqualityAxiom], [EqLeqAxiom], [EqAdditionAxiom], [EqMultiplicationAxiom], [QisClosed(Reciprocal)(Implied)], [QisClosed(Reciprocal)], [QisClosed(Negative)(Implied)], [QisClosed(Negative)], [leqReflexivity], [leqAntisymmetryAxiom], [leqTransitivityAxiom], [leqTotality], [leqAdditionAxiom], [leqMultiplicationAxiom], [plusAssociativity], [plusCommutativity], [Negative], [plus0], [timesAssociativity], [timesCommutativity], [ReciprocalAxiom], [times1], [Distribution], [0not1], [lemma eqLeq(R)], [TimesAssociativity(R)], [TimesCommutativity(R)], [(Adgic)SameR], [Separation2formula(1)], [Separation2formula(2)], [Cauchy], [PlusF], [ReciprocalF], [From ==], [To ==], [FromInR], [PlusR(Sym)], [ReciprocalR(Axiom)], [LessMinus1(N)], [Nonnegative(N)], [US0], [NextXS(UpperBound)], [NextXS(NoUpperBound)], [NextUS(UpperBound)], [NextUS(NoUpperBound)], [ExpZero], [ExpPositive], [ExpZero(R)], [ExpPositive(R)], [BSzero], [BSpositive], [USteleScope(Zero)], [USteleScope(Positive)], [EqAddition(R)], [FromLimit], [ToUpperBound], [FromUpperBound], [USisUpperBound], [0not1(R)], [ExpUnbounded(R)], [FromLeq(Advanced)(N)], [FromLeastUpperBound], [ToLeastUpperBound], [XSisNotUpperBound], [ysFGreater], [ysFLess], [SmallInverse], [NatType], [RationalType], [SeriesType], [Max], [Numerical], [NumericalF], [MemberOfSeries(Implied)], [JoinConjuncts(2conditions)], [prop lemma imply negation], [TND], [FromNegatedImplied], [ToNegatedImplied], [FromNegated(2 \* Implied)], [FromNegatedAnd], [FromNegatedOr], [ToNegatedOr], [FromNegations], [From3Disjuncts], [From2 \* 2Disjuncts], [NegateDisjunct1], [NegateDisjunct2], [ExpandDisjuncts], [SENC1], [SENC2], [LessLeq(R)], [MemberOfSeries], [memberOfSeries(Type)], [ToNegatedAnd(1)], [UniqueNegative], [DoubleMinus], [MinusNegated], [eqReflexivity], [eqSymmetry], [eqTransitivity], [eqTransitivity4], [eqTransitivity5], [eqTransitivity6], [AddEquations], [SubtractEquations], [SubtractEquationsLeft], [MultiplyEquations], [EqNegated], [PositiveToRight(Eq)], [PositiveToLeft(Eq)(1term)], [NegativeToLeft(Eq)], [NonreciprocalToRight(Eq)(1term)], [PlusAssociativity(4terms)], [LessNeq], [NeqSymmetry], [NeqNegated], [SubNeqRight], [SubNeqLeft], [NegativeToRight(Neq)(1term)], [NeqAddition], [NeqMultiplication],

[NonzeroProduct(2)], [UStelescope(+1)], [TelescopeBound(Base)],  
 [TelescopeBound(Indu)], [TelescopeBound], [IntervalSize(Base)],  
 [IntervalSize(Indu)], [IntervalSize], [XS < US], [lemma USdecreasing(+1)],  
 [CloseUS], [CloseUS(n + 1)], [AllNegated(ImPLY)], [ExistNegated(ImPLY)],  
 [IntroExist(Helper)], [IntroExist], [ExistMP], [ExistMP2], [TwiceExistMP],  
 [TwiceExistMP2], [EAE - MP], [AddAll], [AddExist(Helper1)],  
 [AddExist(Helper2)], [AddExist], [AddExist(SimpleAnt)], [AddExist(Simple)],  
 [AddEAE], [AEA - negated], [EEA - negated], [Induction], [leqAntisymmetry],  
 [leqTransitivity], [leqAddition], [leqMultiplication], [Reciprocal], [Equality],  
 [eqLeq], [eqAddition], [eqMultiplication], [LeqMultiplicationLeft], [LeqLessEq],  
 [LessLeq], [FromLeqGeq], [subLeqRight], [subLeqLeft], [Leq + 1],  
 [PositiveToRight(Leq)], [PositiveToRight(Leq)(1term)],  
 [lemma negativeToRight(Leq)], [PositiveToLeft(Leq)], [negativeToLeft(Leq)],  
 [negativeToLeft(Leq)(1term)], [LeqAdditionLeft], [leqSubtraction],  
 [leqSubtractionLeft], [thirdGeq], [LeqNegated], [AddEquations(Leq)],  
 [MultiplyEquations(Leq)], [ThirdGeqSeries], [LeqNeqLess], [FromLess], [ToLess],  
 [fromNotLess], [toNotLess], [NegativeLessPositive], [leqLessTransitivity],  
 [LessLeqTransitivity], [LessTransitivity], [LessTotality], [SubLessRight],  
 [SubLessLeft], [SwitchTerms(x < y - z)], [SwitchTerms(x - y < z)],  
 [LessAddition], [LessAdditionLeft], [LessMultiplication],  
 [LessMultiplicationLeft], [LessDivision], [PositiveToRight(Less)],  
 [PositiveToLeft(Less)], [NegativeToLeft(Less)], [NegativeToRight(Less)],  
 [AddEquations(Less)], [AddEquations(LeqLess)], [reciprocalToLeft(Less)],  
 [LessNegated], [PositiveNonzero], [PositiveNegated], [NonpositiveNegated],  
 [NegativeNegated], [NonnegativeNegated], [PositiveHalved], [PositiveInverted],  
 [NonnegativeNumerical], [NegativeNumerical], [PositiveNumerical],  
 [lemma nonpositiveNumerical], [|0| = 0], [0 <= |x|], [x <= |x|],  
 [FromPositiveNumerical], [SameNumerical], [SignNumerical(+)],  
 [SignNumerical], [ToNumericalLess], [FromNumericalGreater],  
 [NumericalDifference], [NumericalDifferenceLess(Helper)],  
 [NumericalDifferenceLess], [SplitNumericalSumHelper],  
 [splitNumericalSum(++)], [splitNumericalSum(--)],  
 [splitNumericalSum(+ - small)], [splitNumericalSum(+ - big)],  
 [splitNumericalSum(++-)], [splitNumericalSum(-+)], [splitNumericalSum],  
 [SplitNumericalProduct(++)], [SplitNumericalProduct(++-)],  
 [SplitNumericalProduct], [insertMiddleTerm(Numerical)],  
 [insertTwoMiddleTerms(Numerical)], [Three2twoTerms], [Three2threeTerms],  
 [Three2twoFactors], [Three2threeFactors], [Times(-1)], [Times(-1)Left],  
 [MaxLeq(1)], [MaxLeq(2)], [LessThanMax], [x + y = zBackwards],  
 [x \* y = zBackwards], [x = x + (y - y)], [x = x + y - y], [x = x \* y \* (1/y)],  
 [insertMiddleTerm(Sum)], [insertTwoMiddleTerms(Sum)],  
 [insertMiddleTerm(Difference)], [x \* 0 + x = x], [x \* 0 = 0], [NonnegativeFactors],  
 [NonzeroFactors], [PositiveFactors], [PlusTimesMinus], [MinusTimesMinus],  
 [(-1) \* (-1) + (-1) \* 1 = 0], [(-1) \* (-1) = 1], [0 < 1Helper], [0 < 1], [0 < 2],  
 [0 < 3], [0 < 1/2], [0 < 1/3], [TwoWholes], [ThreeWholes], [TwoHalves],  
 [ThreeThirds], [-x - y = -(x + y)], [-x \* y = -(x \* y)], [-0 = 0],

[SFsymmetry], [SFtransitivity], [f2R(Plus)], [f2R(Times)],  
 [<< TransitivityHelper(Q)], [<< Transitivity], [<<== Reflexivity],  
 [<<== AntisymmetryHelper(Q)], [FromNot < f(Weak)(Helper)],  
 [FromNot < f(Weak)], [FromNot < f(Strong)(Helper2)],  
 [FromNot < f(Strong)(Helper)], [FromNot < f(Strong)],  
 [fromNotSameF(Strongest)(Helper2)], [fromNotSameF(Strongest)(Helper)],  
 [fromNotSameF(Strongest)], [ToLess(F)(Helper)], [ToLess(F)], [FromNot <<],  
 [ToLess(R)], [FromNotSameF(Weak)(Helper)], [FromNotSameF(Weak)],  
 [FromNotLess(F)], [== Addition], [== AdditionLeft],  
 [Fpart - Bounded(Base)], [Fpart - Bounded(InduHelper)],  
 [Fpart - Bounded(Indu)], [Fpart - Bounded], [F - Bounded(Helper)],  
 [F - Bounded], [SameFmultiplication(Helper)], [SameFmultiplication],  
 [EqMultiplication(R)], [EqMultiplicationLeft(R)], [ $x * 0 = 0(F)$ ], [ $x * 0 = 0(R)$ ],  
 [LessMultiplication(F)(Helper2)], [LessMultiplication(F)(Helper)],  
 [LessMultiplication(F)], [LessMultiplication(R)], [LeqMultiplication(R)],  
 [PlusAssociativity(F)], [Plus0(F)], [PlusCommutativity(F)],  
 [TimesAssociativity(F)], [Times1f], [Cauchy(2)(Helper)], [Cauchy(2)],  
 [ReciprocalFnonzero], [(Eventually = f)2sameF(Helper)],  
 [(Eventually = f)2sameF], [FromNotSameF(Strong)(Helper2)],  
 [FromNotSameF(Strong)(Helper)], [FromNotSameF(Strong)],  
 [SameFreciprocal(Helper)], [SameFreciprocal], [From!! ==], [Reciprocal(R)],  
 [TimesCommutativity(F)], [Distribution(F)], [FromMax(1)], [FromMax(2)],  
 [ToNegatedAnd], [DistributionOut], [DistributionOutLeft], [DistributionLeft],  
 [FromNotLess(R)], [CartProdIsRelation], [FromSubset], [SubsetIsRelation],  
 [ToSeries], [FromSeries], [SeriesSubsetCP], [ValueType], [RemoveOr],  
 [FromSingleton], [InPair(1)], [InPair(2)], [SameMember(2)], [ToBinaryUnion(1)],  
 [ToBinaryUnion(2)], [FromOrderedPair(TwoLevels)], [ToCartProd(Helper)],  
 [ToCartProd], [NonreciprocalToRight(Eq)], [NonreciprocalToLeft(Eq)(1term)],  
 [SameReciprocal], [CPseparationIsRelation], [OrderedPairEquality],  
 [ReciprocalIsFunction], [ReciprocalIsTotal], [ReciprocalIsRationalSeries],  
 [CrsIsRelation], [CrsIsFunction], [CrsIsTotal], [CrsIsSeries], [CrsLookup], [0f],  
 [1f], [ToSingleton], [FromSameSingleton], [SingletonmembersEqual],  
 [UnequalsNotInSingleton], [NonsingletonmembersUnequal], [FromOrderedPair],  
 [FromOrderedPair(1)], [FromOrderedPair(2)], [FromCartProd],  
 [FromCartProd(1)], [FromCartProd(2)], [sameOrderedPair], [InSeriesHelper],  
 [InSeries], [To = f(Subset)(Helper)], [To = f(Subset)], [To = f],  
 [productIsFunction], [productIsTotal], [ProductIsRationalSeries], [TimesF],  
 [ $-x + (1/2)x = -(1/2)x$ ], [PositiveTripled], [PositiveDividedBy3], [ $|x - x| = 0$ ],  
 [ $1 < 2$ ], [ $1/3 < 2/3$ ], [ $(1/3)x + (1/3)x = (2/3)x$ ], [ $(2/3)x + (1/3)x = x$ ],  
 [ $-x + (2/3)x = -(1/3)x$ ], [ $-(1/3)x - (1/3)x = -(2/3)x$ ],  
 [ $-x + (1/3)x = -(2/3)x$ ], [PreserveLessGreater], [ClosetolessIsLess],  
 [SubLessLeft(F)], [SubLessLeft(R)], [ClosetogreaterIsGreater],  
 [SubLessRight(F)], [SubLessRight(R)], [plus0Left], [times1Left],  
 [EqAdditionLeft], [EqMultiplicationLeft], [PlusF(Sym)], [TimesF(Sym)],  
 [SameSeries(Gen)], [EqualsSameF], [LeqReflexivity(R)];

**Preassociative**

[Tester1], [Tester2], [Tester3], [Tester4], [Tester5], [Tester6];

### Preassociative

[\*\_{}], [\*/indexintro(\*, \*, \*, \*)], [\*/intro(\*, \*, \*)], [\*/bothintro(\*, \*, \*, \*, \*)],  
[\*/nameintro(\*, \*, \*, \*)], [\*/], [\*[ \* ]], [\*[ \* → \* ]], [\*[ \* ⇒ \* ]], [\*\_0], [\*\_1], [0b], [\*\_color(\*)],  
[\*\_color\* (\*)], [\*\_H], [\*\_T], [\*\_U], [\*\_h], [\*\_t], [\*\_s], [\*\_c], [\*\_d], [\*\_a], [\*\_C], [\*\_M], [\*\_B], [\*\_r], [\*\_i],  
[\*\_d], [\*\_R], [\*\_0], [\*\_1], [\*\_2], [\*\_3], [\*\_4], [\*\_5], [\*\_6], [\*\_7], [\*\_8], [\*\_9], [\*\_E], [\*\_V], [\*\_C], [\*\_C\*],  
[\*\_hide];

### Preassociative

[“ \* ”], [], [(\*)^t], [string(\*) + \*], [string(\*) ++ \*], [  
\*, [ \* ], [! \*], [\" \*], [# \*], [\$ \*], [% \*], [& \*], [’ \*], [( \* ), ( \* )], [\*\*], [+ \*], [ \* ], [- \*], [ \* ], [ / \* ],  
[0 \* ], [1 \* ], [2 \* ], [3 \* ], [4 \* ], [5 \* ], [6 \* ], [7 \* ], [8 \* ], [9 \* ], [ : \* ], [ ; \* ], [ < \* ], [= \* ], [ > \* ], [ ? \* ],  
[@ \* ], [A \* ], [B \* ], [C \* ], [D \* ], [E \* ], [F \* ], [G \* ], [H \* ], [I \* ], [J \* ], [K \* ], [L \* ], [M \* ], [N \* ],  
[O \* ], [P \* ], [Q \* ], [R \* ], [S \* ], [T \* ], [U \* ], [V \* ], [W \* ], [X \* ], [Y \* ], [Z \* ], [[ \* ], [ \ \* ], [ ] \* ], [ ^ \* ],  
[ \_ \* ], [ ‘ \* ], [ a \* ], [ b \* ], [ c \* ], [ d \* ], [ e \* ], [ f \* ], [ g \* ], [ h \* ], [ i \* ], [ j \* ], [ k \* ], [ l \* ], [ m \* ], [ n \* ], [ o \* ],  
[ p \* ], [ q \* ], [ r \* ], [ s \* ], [ t \* ], [ u \* ], [ v \* ], [ w \* ], [ x \* ], [ y \* ], [ z \* ], [ { \* }, [ | \* ], [ } \* ], [ ~ \* ],  
[Preassociative \* ; \*], [Postassociative \* ; \*], [[ \* ], \*], [priority \* end],  
[newline \*], [macro newline \*], [MacroIndent(\*)];

### Preassociative

[\* ’ \*], [\* ‘ \*];

### Preassociative

[\*(exp)\*];

### Preassociative

[\*’], [R(\*)], [— R(\*)], [rec\*];

### Preassociative

[\*/ \*], [ \* ∩ \* ], [ \* [ \* ]];

### Preassociative

[∪ \*], [ \* ∪ \* ], [P(\*)];

### Preassociative

[{ \* }], [StateExpand(\*, \*, \*)], [extractSeries(\*)], [SetOfSeries(\*)], [— Macro(\*)],  
[ExpandList(\*, \*, \*)], [ \* Macro(\*)], [ + + Macro(\*)], [ < < Macro(\*)],  
[| Macro(\*)], [01 // Macro(\*)], [UB(\*, \*)], [LUB(\*, \*)], [BS(\*, \*)],  
[UStescope(\*, \*)], [(\*)], [| f \* |], [| r \* |], [Limit(\*, \*)], [Union(\*)],  
[IsOrderedPair(\*, \*, \*)], [IsRelation(\*, \*, \*)], [isFunction(\*, \*, \*)], [IsSeries(\*, \*)],  
[IsNatural(\*, \*)], [OrderedPair(\*, \*)], [TypeNat(\*)], [TypeNat0(\*)],  
[TypeRational(\*)], [TypeRational0(\*)], [TypeSeries(\*, \*)], [Typeseries0(\*, \*)];

### Preassociative

[{\* \* }], [ < \* , \* ], [(-u\*)], [-\_f \*], [(- - \*)], [1f / \*], [01 // temp\*];

### Preassociative

[\*( \* , \* )], [RefRel(\*, \*)], [SymRel(\*, \*)], [TransRel(\*, \*)], [EqRel(\*, \*)], [[ \* ∈ \* ] \*],  
[Partition(\*, \*)];

### Preassociative

[ \* · \* ], [ \* ·\_0 \* ], [( \* \* \* )], [ \* \*\_f \* ], [ \* \* \* \* ];

### Preassociative

[ \* + \* ], [ \* +\_0 \* ], [ \* +\_1 \* ], [ \* - \* ], [ \* -\_0 \* ], [ \* -\_1 \* ], [( \* + \* )], [( \* - \* )], [ \* +\_f \* ],  
[ \* -\_f \* ], [ \* + + \* ], [R(\*) - R(\*)];

### Preassociative

$[* \in *]$ ;

### Preassociative

$[[* \mid], [\text{if}(*, *, *)], [\text{Max}(*, *)], [\text{Max}(*, *)]$ ;

### Preassociative

$[* = *], [* \neq *], [* < = *], [* < *], [* <_f *], [* \leq_f *], [\text{SF}(*, *)], [* == *],$   
 $[*!! == *], [* << *], [* << == *]$ ;

### Preassociative

$[* \cup \{*\}], [* \cup *], [* \setminus \{*\}]$ ;

### Postassociative

$[* \dot{:} *], [* \dot{:} *], [* \dot{:} : *], [* \underline{+2*} *], [* :: *], [* +2* *]$ ;

### Postassociative

$[*, *]$ ;

### Preassociative

$[* \overset{B}{\approx} *], [* \overset{D}{\approx} *], [* \overset{C}{\approx} *], [* \overset{P}{\approx} *], [* \approx *], [* = *], [* \dashv *], [* \overset{t}{=} *], [* \overset{t^*}{=} *], [* \overset{r}{=} *],$   
 $[* \in_t *], [* \subseteq_T *], [* \overset{T}{=} *], [* \overset{s}{=} *], [* \text{free in } *], [* \text{free in}^* *], [* \text{free for } * \text{ in } *],$   
 $[* \text{free for}^* * \text{ in } *], [* \in_c *], [* < *], [* <' *], [* \leq' *], [* = *], [* \neq *], [*^{\text{var}}],$   
 $[* \#^0 *], [* \#^1 *], [* \#^* *], [* == *], [* \subseteq *]$ ;

### Preassociative

$[\neg *], [\dot{\neg} (*n)], [* \notin *], [* \neq *]$ ;

### Preassociative

$[* \wedge *], [* \overset{\sim}{\wedge} *], [* \overset{\sim}{\wedge} *], [* \wedge_c *], [* \overset{\dot{\wedge}}{\wedge} *]$ ;

### Preassociative

$[* \vee *], [* \parallel *], [* \overset{\ddot{\vee}}{\vee} *]$ ;

### Postassociative

$[* \overset{\dot{\vee}}{\vee} *]$ ;

### Preassociative

$[\exists *: *], [\forall *: *], [\forall_{\text{obj}} *: *], [\exists *: *]$ ;

### Postassociative

$[* \overset{\dot{\Rightarrow}}{\Rightarrow} *], [* \Rightarrow *], [* \Leftrightarrow *], [* \overset{\dot{\Leftrightarrow}}{\Leftrightarrow} *]$ ;

### Preassociative

$[\{\text{ph} \in * \mid *\}]$ ;

### Postassociative

$[* : *], [* \text{spy} *], [*! *]$ ;

### Preassociative

$[* \left\{ \begin{array}{c} * \\ * \end{array} \right. ]$ ;

### Preassociative

$[\lambda * . *], [\Lambda * . *], [\Lambda *], [\text{if } * \text{ then } * \text{ else } *], [\text{let } * = * \text{ in } *], [\text{let } * \doteq * \text{ in } *]$ ;

### Preassociative

$[* \# *]$ ;

### Preassociative

$[*^I], [*^\triangleright], [*^V], [*^+], [*^-], [*^*]$ ;

### Preassociative

$[* @ *], [* \triangleright *], [* \blacktriangleright *], [* \gg *], [* \trianglerighteq *]$ ;

### Postassociative

[\* ⊢ \*], [\* ⊨ \*], [\* i.e. \*];

**Preassociative**

[∀\*: \*], [Π\*: \*];

**Postassociative**

[\* ⊕ \*];

**Postassociative**

[\*, \*];

**Preassociative**

[\* proves \*];

**Preassociative**

[\* **proof of** \* : \*], [Line \* : \* >> \*; \*], [Last line \* >> \* □],

[Line \* : Premise >> \*; \*], [Line \* : Side-condition >> \*; \*], [Arbitrary >> \*; \*],

[Local >> \* = \*; \*], [Begin \*; \* : End; \*], [Last block line \* >> \*; \*],

[Arbitrary >> \*; \*];

**Postassociative**

[\* | \*];

**Postassociative**

[\* , \*], [\* [ \* ]\*];

**Preassociative**

[\*&\*];

**Preassociative**

[\*\\\*, [\* linebreak[4] \*], [\*\\\*];]

## A Pyk definitioner

[ToNegatedAnd(1)  $\xrightarrow{\text{pyk}}$  “prop lemma to negated and(1)”]

[UniqueNegative  $\xrightarrow{\text{pyk}}$  “lemma uniqueNegative”]

[DoubleMinus  $\xrightarrow{\text{pyk}}$  “lemma doubleMinus”]

[MinusNegated  $\xrightarrow{\text{pyk}}$  “lemma minusNegated”]

[eqReflexivity  $\xrightarrow{\text{pyk}}$  “lemma eqReflexivity”]

[eqSymmetry  $\xrightarrow{\text{pyk}}$  “lemma eqSymmetry”]

[eqTransitivity  $\xrightarrow{\text{pyk}}$  “lemma eqTransitivity”]

[eqTransitivity4  $\xrightarrow{\text{pyk}}$  “lemma eqTransitivity4”]

[eqTransitivity5  $\xrightarrow{\text{pyk}}$  “lemma eqTransitivity5”]

[eqTransitivity6  $\xrightarrow{\text{pyk}}$  “lemma eqTransitivity6”]

[AddEquations  $\xrightarrow{\text{pyk}}$  “lemma addEquations”]

[SubtractEquations  $\xrightarrow{\text{pyk}}$  “lemma subtractEquations”]

[SubtractEquationsLeft  $\xrightarrow{\text{pyk}}$  “lemma subtractEquationsLeft”]

[MultiplyEquations  $\xrightarrow{\text{pyk}}$  “lemma multiplyEquations”]

[EqNegated  $\xrightarrow{\text{pyk}}$  “lemma eqNegated”]

[PositiveToRight(Eq)  $\xrightarrow{\text{pyk}}$  “lemma positiveToRight(Eq)”]  
 [PositiveToLeft(Eq)(1term)  $\xrightarrow{\text{pyk}}$  “lemma positiveToLeft(Eq)(1 term)”]  
 [NegativeToLeft(Eq)  $\xrightarrow{\text{pyk}}$  “lemma negativeToLeft(Eq)”]  
 [NonreciprocalToRight(Eq)(1term)  $\xrightarrow{\text{pyk}}$  “lemma nonreciprocalToRight(Eq)(1 term)”]  
 [PlusAssociativity(4terms)  $\xrightarrow{\text{pyk}}$  “lemma plusAssociativity(4 terms)”]  
 [LessNeq  $\xrightarrow{\text{pyk}}$  “lemma lessNeq”]  
 [NeqSymmetry  $\xrightarrow{\text{pyk}}$  “lemma neqSymmetry”]  
 [NeqNegated  $\xrightarrow{\text{pyk}}$  “lemma neqNegated”]  
 [SubNeqRight  $\xrightarrow{\text{pyk}}$  “lemma subNeqRight”]  
 [SubNeqLeft  $\xrightarrow{\text{pyk}}$  “lemma subNeqLeft”]  
 [NegativeToRight(Neq)(1term)  $\xrightarrow{\text{pyk}}$  “lemma negativeToRight(Neq)(1 term)”]  
 [NeqAddition  $\xrightarrow{\text{pyk}}$  “lemma neqAddition”]  
 [NeqMultiplication  $\xrightarrow{\text{pyk}}$  “lemma neqMultiplication”]  
 [NonzeroProduct(2)  $\xrightarrow{\text{pyk}}$  “lemma nonzeroProduct(2)”]  
 [UStelescope(+1)  $\xrightarrow{\text{pyk}}$  “lemma UStelescope(+1)”]  
 [TelescopeBound(Base)  $\xrightarrow{\text{pyk}}$  “lemma telescopeBound base”]  
 [TelescopeBound(Indu)  $\xrightarrow{\text{pyk}}$  “lemma telescopeBound indu”]  
 [TelescopeBound  $\xrightarrow{\text{pyk}}$  “lemma telescopeBound”]  
 [IntervalSize(Base)  $\xrightarrow{\text{pyk}}$  “lemma intervalSize base”]  
 [IntervalSize(Indu)  $\xrightarrow{\text{pyk}}$  “lemma intervalSize indu”]  
 [IntervalSize  $\xrightarrow{\text{pyk}}$  “lemma intervalSize”]  
 [XS < US  $\xrightarrow{\text{pyk}}$  “lemma XSlessUS”]  
 [lemma USdecreasing(+1)  $\xrightarrow{\text{pyk}}$  “lemma USdecreasing(+1)”]  
 [CloseUS  $\xrightarrow{\text{pyk}}$  “lemma closeUS”]  
 [CloseUS(n + 1)  $\xrightarrow{\text{pyk}}$  “lemma closeUS(n+1)”]  
 [AllNegated(ImPLY)  $\xrightarrow{\text{pyk}}$  “pred lemma allNegated(ImPLY)”]  
 [ExistNegated(ImPLY)  $\xrightarrow{\text{pyk}}$  “pred lemma existNegated(ImPLY)”]  
 [IntroExist(Helper)  $\xrightarrow{\text{pyk}}$  “pred lemma intro exist helper”]  
 [IntroExist  $\xrightarrow{\text{pyk}}$  “pred lemma intro exist”]  
 [ExistMP  $\xrightarrow{\text{pyk}}$  “pred lemma exist mp”]  
 [ExistMP2  $\xrightarrow{\text{pyk}}$  “pred lemma exist mp2”]  
 [TwiceExistMP  $\xrightarrow{\text{pyk}}$  “pred lemma 2exist mp”]  
 [TwiceExistMP2  $\xrightarrow{\text{pyk}}$  “pred lemma 2exist mp2”]  
 [EAE – MP  $\xrightarrow{\text{pyk}}$  “pred lemma EAE mp”]



[AddAll  $\xrightarrow{\text{pyk}}$  “pred lemma addAll”]  
 [AddExist(Helper1)  $\xrightarrow{\text{pyk}}$  “pred lemma addExist helper1”]  
 [AddExist(Helper2)  $\xrightarrow{\text{pyk}}$  “pred lemma addExist helper2”]  
 [AddExist  $\xrightarrow{\text{pyk}}$  “pred lemma addExist”]  
 [AddExist(SimpleAnt)  $\xrightarrow{\text{pyk}}$  “pred lemma addExist(SimpleAnt)”]  
 [AddExist(Simple)  $\xrightarrow{\text{pyk}}$  “pred lemma addExist(Simple)”]  
 [AddEAE  $\xrightarrow{\text{pyk}}$  “pred lemma addEAE”]  
 [AEA – negated  $\xrightarrow{\text{pyk}}$  “pred lemma AEAnegated”]  
 [EEA – negated  $\xrightarrow{\text{pyk}}$  “pred lemma EEAnegated”]  
 [Induction  $\xrightarrow{\text{pyk}}$  “lemma induction”]  
 [leqAntisymmetry  $\xrightarrow{\text{pyk}}$  “lemma leqAntisymmetry”]  
 [leqTransitivity  $\xrightarrow{\text{pyk}}$  “lemma leqTransitivity”]  
 [leqAddition  $\xrightarrow{\text{pyk}}$  “lemma leqAddition”]  
 [leqMultiplication  $\xrightarrow{\text{pyk}}$  “lemma leqMultiplication”]  
 [Reciprocal  $\xrightarrow{\text{pyk}}$  “lemma reciprocal”]  
 [Equality  $\xrightarrow{\text{pyk}}$  “lemma equality”]  
 [eqLeq  $\xrightarrow{\text{pyk}}$  “lemma eqLeq”]  
 [eqAddition  $\xrightarrow{\text{pyk}}$  “lemma eqAddition”]  
 [eqMultiplication  $\xrightarrow{\text{pyk}}$  “lemma eqMultiplication”]  
 [LeqMultiplicationLeft  $\xrightarrow{\text{pyk}}$  “lemma leqMultiplicationLeft”]  
 [LeqLessEq  $\xrightarrow{\text{pyk}}$  “lemma leqLessEq”]  
 [LessLeq  $\xrightarrow{\text{pyk}}$  “lemma lessLeq”]  
 [FromLeqGeq  $\xrightarrow{\text{pyk}}$  “lemma from leqGeq”]  
 [subLeqRight  $\xrightarrow{\text{pyk}}$  “lemma subLeqRight”]  
 [subLeqLeft  $\xrightarrow{\text{pyk}}$  “lemma subLeqLeft”]  
 [Leq + 1  $\xrightarrow{\text{pyk}}$  “lemma leqPlus1”]  
 [PositiveToRight(Leq)  $\xrightarrow{\text{pyk}}$  “lemma positiveToRight(Leq)”]  
 [PositiveToRight(Leq)(1term)  $\xrightarrow{\text{pyk}}$  “lemma positiveToRight(Leq)(1 term)”]  
 [lemma negativeToRight(Leq)  $\xrightarrow{\text{pyk}}$  “lemma negativeToRight(Leq)”]  
 [PositiveToLeft(Leq)  $\xrightarrow{\text{pyk}}$  “lemma positiveToLeft(Leq)”]  
 [negativeToLeft(Leq)  $\xrightarrow{\text{pyk}}$  “lemma negativeToLeft(Leq)”]  
 [negativeToLeft(Leq)(1term)  $\xrightarrow{\text{pyk}}$  “lemma negativeToLeft(Leq)(1 term)”]  
 [LeqAdditionLeft  $\xrightarrow{\text{pyk}}$  “lemma leqAdditionLeft”]  
 [leqSubtraction  $\xrightarrow{\text{pyk}}$  “lemma leqSubtraction”]  
 [leqSubtractionLeft  $\xrightarrow{\text{pyk}}$  “lemma leqSubtractionLeft”]

$[\text{thirdGeq} \xrightarrow{\text{pyk}} \text{"lemma thirdGeq"}]$   
 $[\text{LeqNegated} \xrightarrow{\text{pyk}} \text{"lemma leqNegated"}]$   
 $[\text{AddEquations}(\text{Leq}) \xrightarrow{\text{pyk}} \text{"lemma addEquations}(\text{Leq})"]]$   
 $[\text{MultiplyEquations}(\text{Leq}) \xrightarrow{\text{pyk}} \text{"lemma multiplyEquations}(\text{Leq})"]]$   
 $[\text{ThirdGeqSeries} \xrightarrow{\text{pyk}} \text{"lemma thirdGeqSeries"}]$   
 $[\text{LeqNeqLess} \xrightarrow{\text{pyk}} \text{"lemma leqNeqLess"}]$   
 $[\text{FromLess} \xrightarrow{\text{pyk}} \text{"lemma fromLess"}]$   
 $[\text{ToLess} \xrightarrow{\text{pyk}} \text{"lemma toLess"}]$   
 $[\text{fromNotLess} \xrightarrow{\text{pyk}} \text{"lemma fromNotLess"}]$   
 $[\text{toNotLess} \xrightarrow{\text{pyk}} \text{"lemma toNotLess"}]$   
 $[\text{NegativeLessPositive} \xrightarrow{\text{pyk}} \text{"lemma negativeLessPositive"}]$   
 $[\text{leqLessTransitivity} \xrightarrow{\text{pyk}} \text{"lemma leqLessTransitivity"}]$   
 $[\text{LessLeqTransitivity} \xrightarrow{\text{pyk}} \text{"lemma lessLeqTransitivity"}]$   
 $[\text{LessTransitivity} \xrightarrow{\text{pyk}} \text{"lemma lessTransitivity"}]$   
 $[\text{LessTotality} \xrightarrow{\text{pyk}} \text{"lemma lessTotality"}]$   
 $[\text{SubLessRight} \xrightarrow{\text{pyk}} \text{"lemma subLessRight"}]$   
 $[\text{SubLessLeft} \xrightarrow{\text{pyk}} \text{"lemma subLessLeft"}]$   
 $[\text{SwitchTerms}(x < y - z) \xrightarrow{\text{pyk}} \text{"lemma switchTerms}(x < y - z)"]]$   
 $[\text{SwitchTerms}(x - y < z) \xrightarrow{\text{pyk}} \text{"lemma switchTerms}(x - y < z)"]]$   
 $[\text{LessAddition} \xrightarrow{\text{pyk}} \text{"lemma lessAddition"}]$   
 $[\text{LessAdditionLeft} \xrightarrow{\text{pyk}} \text{"lemma lessAdditionLeft"}]$   
 $[\text{LessMultiplication} \xrightarrow{\text{pyk}} \text{"lemma lessMultiplication"}]$   
 $[\text{LessMultiplicationLeft} \xrightarrow{\text{pyk}} \text{"lemma lessMultiplicationLeft"}]$   
 $[\text{LessDivision} \xrightarrow{\text{pyk}} \text{"lemma lessDivision"}]$   
 $[\text{PositiveToRight}(\text{Less}) \xrightarrow{\text{pyk}} \text{"lemma positiveToRight}(\text{Less})"]]$   
 $[\text{PositiveToLeft}(\text{Less}) \xrightarrow{\text{pyk}} \text{"lemma positiveToLeft}(\text{Less})"]]$   
 $[\text{NegativeToLeft}(\text{Less}) \xrightarrow{\text{pyk}} \text{"lemma negativeToLeft}(\text{Less})"]]$   
 $[\text{NegativeToRight}(\text{Less}) \xrightarrow{\text{pyk}} \text{"lemma negativeToRight}(\text{Less})"]]$   
 $[\text{AddEquations}(\text{Less}) \xrightarrow{\text{pyk}} \text{"lemma addEquations}(\text{Less})"]]$   
 $[\text{AddEquations}(\text{LeqLess}) \xrightarrow{\text{pyk}} \text{"lemma addEquations}(\text{LeqLess})"]]$   
 $[\text{reciprocalToLeft}(\text{Less}) \xrightarrow{\text{pyk}} \text{"lemma reciprocalToLeft}(\text{Less})"]]$   
 $[\text{LessNegated} \xrightarrow{\text{pyk}} \text{"lemma lessNegated"}]$   
 $[\text{PositiveNonzero} \xrightarrow{\text{pyk}} \text{"lemma positiveNonzero"}]$   
 $[\text{PositiveNegated} \xrightarrow{\text{pyk}} \text{"lemma positiveNegated"}]$   
 $[\text{NonpositiveNegated} \xrightarrow{\text{pyk}} \text{"lemma nonpositiveNegated"}]$

[NegativeNegated  $\xrightarrow{\text{pyk}}$  “lemma negativeNegated”]  
 [NonnegativeNegated  $\xrightarrow{\text{pyk}}$  “lemma nonnegativeNegated”]  
 [PositiveHalved  $\xrightarrow{\text{pyk}}$  “lemma positiveHalved”]  
 [PositiveInverted  $\xrightarrow{\text{pyk}}$  “lemma positiveInverted”]  
 [NonnegativeNumerical  $\xrightarrow{\text{pyk}}$  “lemma nonnegativeNumerical”]  
 [NegativeNumerical  $\xrightarrow{\text{pyk}}$  “lemma negativeNumerical”]  
 [PositiveNumerical  $\xrightarrow{\text{pyk}}$  “lemma positiveNumerical”]  
 [lemma nonpositiveNumerical  $\xrightarrow{\text{pyk}}$  “lemma nonpositiveNumerical”]  
 [|0| = 0  $\xrightarrow{\text{pyk}}$  “lemma |0|=0”]  
 [0 <= |x|  $\xrightarrow{\text{pyk}}$  “lemma 0<=|x|”]  
 [x <= |x|  $\xrightarrow{\text{pyk}}$  “lemma x<=|x|”]  
 [FromPositiveNumerical  $\xrightarrow{\text{pyk}}$  “lemma fromPositiveNumerical”]  
 [SameNumerical  $\xrightarrow{\text{pyk}}$  “lemma sameNumerical”]  
 [SignNumerical(+)  $\xrightarrow{\text{pyk}}$  “lemma signNumerical(+)”]  
 [SignNumerical  $\xrightarrow{\text{pyk}}$  “lemma signNumerical”]  
 [ToNumericalLess  $\xrightarrow{\text{pyk}}$  “lemma toNumericalLess”]  
 [FromNumericalGreater  $\xrightarrow{\text{pyk}}$  “lemma fromNumericalGreater”]  
 [NumericalDifference  $\xrightarrow{\text{pyk}}$  “lemma numericalDifference”]  
 [NumericalDifferenceLess(Helper)  $\xrightarrow{\text{pyk}}$  “lemma numericalDifferenceLess helper”]  
 [NumericalDifferenceLess  $\xrightarrow{\text{pyk}}$  “lemma numericalDifferenceLess”]  
 [SplitNumericalSumHelper  $\xrightarrow{\text{pyk}}$  “lemma splitNumericalSumHelper”]  
 [splitNumericalSum(++)  $\xrightarrow{\text{pyk}}$  “lemma splitNumericalSum(++)”]  
 [splitNumericalSum(--)  $\xrightarrow{\text{pyk}}$  “lemma splitNumericalSum(--)”]  
 [splitNumericalSum(+ - small)  $\xrightarrow{\text{pyk}}$  “lemma splitNumericalSum(+-,  
 smallNegative)”]  
 [splitNumericalSum(+ - big)  $\xrightarrow{\text{pyk}}$  “lemma splitNumericalSum(+-,  
 bigNegative)”]  
 [splitNumericalSum(+-)  $\xrightarrow{\text{pyk}}$  “lemma splitNumericalSum(+-)”]  
 [splitNumericalSum(-+)  $\xrightarrow{\text{pyk}}$  “lemma splitNumericalSum(-+)”]  
 [splitNumericalSum  $\xrightarrow{\text{pyk}}$  “lemma splitNumericalSum”]  
 [SplitNumericalProduct(+++)  $\xrightarrow{\text{pyk}}$  “lemma splitNumericalProduct(+++)”]  
 [SplitNumericalProduct(+-)  $\xrightarrow{\text{pyk}}$  “lemma splitNumericalProduct(+-)”]  
 [SplitNumericalProduct  $\xrightarrow{\text{pyk}}$  “lemma splitNumericalProduct”]  
 [insertMiddleTerm(Numerical)  $\xrightarrow{\text{pyk}}$  “lemma insertMiddleTerm(Numerical)”]  
 [insertTwoMiddleTerms(Numerical)  $\xrightarrow{\text{pyk}}$  “lemma  
 insertTwoMiddleTerms(Numerical)”]

$[Three2twoTerms \xrightarrow{pyk} \text{“lemma three2twoTerms”}]$   
 $[Three2threeTerms \xrightarrow{pyk} \text{“lemma three2threeTerms”}]$   
 $[Three2twoFactors \xrightarrow{pyk} \text{“lemma three2twoFactors”}]$   
 $[Three2threeFactors \xrightarrow{pyk} \text{“lemma three2threeFactors”}]$   
 $[Times(-1) \xrightarrow{pyk} \text{“lemma times(-1)”}]$   
 $[Times(-1)Left \xrightarrow{pyk} \text{“lemma times(-1)Left”}]$   
 $[MaxLeq(1) \xrightarrow{pyk} \text{“lemma leqMax1”}]$   
 $[MaxLeq(2) \xrightarrow{pyk} \text{“lemma leqMax2”}]$   
 $[LessThanMax \xrightarrow{pyk} \text{“lemma lessThanMax”}]$   
 $[x + y = zBackwards \xrightarrow{pyk} \text{“lemma x+y=zBackwards”}]$   
 $[x * y = zBackwards \xrightarrow{pyk} \text{“lemma x*y=zBackwards”}]$   
 $[x = x + (y - y) \xrightarrow{pyk} \text{“lemma x=x+(y-y)”}]$   
 $[x = x + y - y \xrightarrow{pyk} \text{“lemma x=x+y-y”}]$   
 $[x = x * y * (1/y) \xrightarrow{pyk} \text{“lemma x=x*y*(1/y)”}]$   
 $[insertMiddleTerm(Sum) \xrightarrow{pyk} \text{“lemma insertMiddleTerm(Sum)”}]$   
 $[insertTwoMiddleTerms(Sum) \xrightarrow{pyk} \text{“lemma insertTwoMiddleTerms(Sum)”}]$   
 $[insertMiddleTerm(Difference) \xrightarrow{pyk} \text{“lemma insertMiddleTerm(Difference)”}]$   
 $[x * 0 + x = x \xrightarrow{pyk} \text{“lemma x*0+x=x”}]$   
 $[x * 0 = 0 \xrightarrow{pyk} \text{“lemma x*0=0”}]$   
 $[NonnegativeFactors \xrightarrow{pyk} \text{“lemma nonnegativeFactors”}]$   
 $[NonzeroFactors \xrightarrow{pyk} \text{“lemma nonzeroFactors”}]$   
 $[PositiveFactors \xrightarrow{pyk} \text{“lemma positiveFactors”}]$   
 $[PlusTimesMinus \xrightarrow{pyk} \text{“lemma plusTimesMinus”}]$   
 $[MinusTimesMinus \xrightarrow{pyk} \text{“lemma minusTimesMinus”}]$   
 $[(-1) * (-1) + (-1) * 1 = 0 \xrightarrow{pyk} \text{“lemma (-1)*(-1)+(-1)*1=0”}]$   
 $[(-1) * (-1) = 1 \xrightarrow{pyk} \text{“lemma (-1)*(-1)=1”}]$   
 $[0 < 1Helper \xrightarrow{pyk} \text{“lemma 0<1Helper”}]$   
 $[0 < 1 \xrightarrow{pyk} \text{“lemma 0<1”}]$   
 $[0 < 2 \xrightarrow{pyk} \text{“lemma 0<2”}]$   
 $[0 < 3 \xrightarrow{pyk} \text{“lemma 0<3”}]$   
 $[0 < 1/2 \xrightarrow{pyk} \text{“lemma 0<1/2”}]$   
 $[0 < 1/3 \xrightarrow{pyk} \text{“lemma 0<1/3”}]$   
 $[TwoWholes \xrightarrow{pyk} \text{“lemma x+x=2*x”}]$   
 $[ThreeWholes \xrightarrow{pyk} \text{“lemma x+x+x=3*x”}]$   
 $[TwoHalves \xrightarrow{pyk} \text{“lemma (1/2)x+(1/2)x=x”}]$

$[ThreeThirds \xrightarrow{pyk} \text{“lemma } (1/3)x+(1/3)x+(1/3)x=x\text{”}]$   
 $[-x - y = -(x + y) \xrightarrow{pyk} \text{“lemma } -x-y=-(x+y)\text{”}]$   
 $[-x * y = -(x * y) \xrightarrow{pyk} \text{“lemma } -x*y=-(x*y)\text{”}]$   
 $[-0 = 0 \xrightarrow{pyk} \text{“lemma } -0=0\text{”}]$   
 $[SFsymmetry \xrightarrow{pyk} \text{“lemma sameFSymmetry”}]$   
 $[SFtransitivity \xrightarrow{pyk} \text{“lemma sameFtransitivity”}]$   
 $[f2R(Plus) \xrightarrow{pyk} \text{“lemma f2R(Plus)”}]$   
 $[f2R(Times) \xrightarrow{pyk} \text{“lemma f2R(Times)”}]$   
 $[<< TransitivityHelper(Q) \xrightarrow{pyk} \text{“lemma } <<TransitivityHelper(Q)\text{”}]$   
 $[<< Transitivity \xrightarrow{pyk} \text{“lemma } <<Transitivity\text{”}]$   
 $[<<== Reflexivity \xrightarrow{pyk} \text{“lemma } <<==Reflexivity\text{”}]$   
 $[<<== AntisymmetryHelper(Q) \xrightarrow{pyk} \text{“lemma } <<==AntisymmetryHelper(Q)\text{”}]$   
 $[FromNot < f(Weak)(Helper) \xrightarrow{pyk} \text{“lemma fromNot<f(Weak) helper”}]$   
 $[FromNot < f(Weak) \xrightarrow{pyk} \text{“lemma fromNot<f(Weak)”}]$   
 $[FromNot < f(Strong)(Helper2) \xrightarrow{pyk} \text{“lemma fromNot<f(Strong) helper2”}]$   
 $[FromNot < f(Strong)(Helper) \xrightarrow{pyk} \text{“lemma fromNot<f(Strong) helper”}]$   
 $[FromNot < f(Strong) \xrightarrow{pyk} \text{“lemma fromNot<f(Strong)”}]$   
 $[fromNotSameF(Strongest)(Helper2) \xrightarrow{pyk} \text{“lemma fromNotSameF(Strongest) helper2”}]$   
 $[fromNotSameF(Strongest)(Helper) \xrightarrow{pyk} \text{“lemma fromNotSameF(Strongest) helper”}]$   
 $[fromNotSameF(Strongest) \xrightarrow{pyk} \text{“lemma fromNotSameF(Strongest)”}]$   
 $[ToLess(F)(Helper) \xrightarrow{pyk} \text{“lemma toLess(F) helper”}]$   
 $[ToLess(F) \xrightarrow{pyk} \text{“lemma toLess(F)”}]$   
 $[FromNot << \xrightarrow{pyk} \text{“lemma fromNot<<”}]$   
 $[ToLess(R) \xrightarrow{pyk} \text{“lemma toLess(R)”}]$   
 $[FromNotSameF(Weak)(Helper) \xrightarrow{pyk} \text{“lemma fromNotSameF(Weak)(Helper)”}]$   
 $[FromNotSameF(Weak) \xrightarrow{pyk} \text{“lemma fromNotSameF(Weak)”}]$   
 $[FromNotLess(F) \xrightarrow{pyk} \text{“lemma fromNotLess(F)”}]$   
 $[== Addition \xrightarrow{pyk} \text{“lemma ==Addition”}]$   
 $[== AdditionLeft \xrightarrow{pyk} \text{“lemma ==AdditionLeft”}]$   
 $[Fpart - Bounded(Base) \xrightarrow{pyk} \text{“lemma fpart-Bounded base”}]$   
 $[Fpart - Bounded(InduHelper) \xrightarrow{pyk} \text{“lemma fpart-Bounded indu helper”}]$   
 $[Fpart - Bounded(Indu) \xrightarrow{pyk} \text{“lemma fpart-Bounded indu”}]$   
 $[Fpart - Bounded \xrightarrow{pyk} \text{“lemma fpart-Bounded”}]$

$[F - \text{Bounded}(\text{Helper}) \xrightarrow{\text{pyk}} \text{"lemma f-Bounded helper"}]$   
 $[F - \text{Bounded} \xrightarrow{\text{pyk}} \text{"lemma f-Bounded"}]$   
 $[\text{SameFmultiplication}(\text{Helper}) \xrightarrow{\text{pyk}} \text{"lemma sameFmultiplication helper"}]$   
 $[\text{SameFmultiplication} \xrightarrow{\text{pyk}} \text{"lemma sameFmultiplication"}]$   
 $[\text{EqMultiplication}(\text{R}) \xrightarrow{\text{pyk}} \text{"lemma eqMultiplication(R)"}]$   
 $[\text{EqMultiplicationLeft}(\text{R}) \xrightarrow{\text{pyk}} \text{"lemma eqMultiplicationLeft(R)"}]$   
 $[x * 0 = 0(\text{F}) \xrightarrow{\text{pyk}} \text{"lemma x*0=0(F)"}]$   
 $[x * 0 = 0(\text{R}) \xrightarrow{\text{pyk}} \text{"lemma x*0=0(R)"}]$   
 $[\text{LessMultiplication}(\text{F})(\text{Helper2}) \xrightarrow{\text{pyk}} \text{"lemma lessMultiplication(F) helper2"}]$   
 $[\text{LessMultiplication}(\text{F})(\text{Helper}) \xrightarrow{\text{pyk}} \text{"lemma lessMultiplication(F) helper"}]$   
 $[\text{LessMultiplication}(\text{F}) \xrightarrow{\text{pyk}} \text{"lemma lessMultiplication(F)"}]$   
 $[\text{LessMultiplication}(\text{R}) \xrightarrow{\text{pyk}} \text{"lemma lessMultiplication(R)"}]$   
 $[\text{LeqMultiplication}(\text{R}) \xrightarrow{\text{pyk}} \text{"lemma leqMultiplication(R)"}]$   
 $[\text{PlusAssociativity}(\text{F}) \xrightarrow{\text{pyk}} \text{"lemma plusAssociativity(F)"}]$   
 $[\text{Plus0}(\text{F}) \xrightarrow{\text{pyk}} \text{"lemma plus0(F)"}]$   
 $[\text{PlusCommutativity}(\text{F}) \xrightarrow{\text{pyk}} \text{"lemma plusCommutativity(F)"}]$   
 $[\text{TimesAssociativity}(\text{F}) \xrightarrow{\text{pyk}} \text{"lemma timesAssociativity(F)"}]$   
 $[\text{Times1f} \xrightarrow{\text{pyk}} \text{"lemma times1f"}]$   
 $[\text{Cauchy}(2)(\text{Helper}) \xrightarrow{\text{pyk}} \text{"lemma 2cauchy helper"}]$   
 $[\text{Cauchy}(2) \xrightarrow{\text{pyk}} \text{"lemma 2cauchy"}]$   
 $[\text{ReciprocalFnonzero} \xrightarrow{\text{pyk}} \text{"lemma reciprocalF nonzero"}]$   
 $[(\text{Eventually} = f)2\text{sameF}(\text{Helper}) \xrightarrow{\text{pyk}} \text{"lemma eventually=f to sameF helper"}]$   
 $[(\text{Eventually} = f)2\text{sameF} \xrightarrow{\text{pyk}} \text{"lemma eventually=f to sameF"}]$   
 $[\text{FromNotSameF}(\text{Strong})(\text{Helper2}) \xrightarrow{\text{pyk}} \text{"lemma fromNotSameF(Strong) helper2"}]$   
 $[\text{FromNotSameF}(\text{Strong})(\text{Helper}) \xrightarrow{\text{pyk}} \text{"lemma fromNotSameF(Strong) helper"}]$   
 $[\text{FromNotSameF}(\text{Strong}) \xrightarrow{\text{pyk}} \text{"lemma fromNotSameF(Strong)"}]$   
 $[\text{SameFreciprocal}(\text{Helper}) \xrightarrow{\text{pyk}} \text{"lemma sameFreciprocal helper"}]$   
 $[\text{SameFreciprocal} \xrightarrow{\text{pyk}} \text{"lemma sameFreciprocal"}]$   
 $[\text{From!!} == \xrightarrow{\text{pyk}} \text{"lemma from!!=="}]$   
 $[\text{Reciprocal}(\text{R}) \xrightarrow{\text{pyk}} \text{"lemma reciprocal(R)"}]$   
 $[\text{TimesCommutativity}(\text{F}) \xrightarrow{\text{pyk}} \text{"lemma timesCommutativity(F)"}]$   
 $[\text{Distribution}(\text{F}) \xrightarrow{\text{pyk}} \text{"lemma distribution(F)"}]$   
 $[\text{FromMax}(1) \xrightarrow{\text{pyk}} \text{"lemma fromMax(1)"}]$   
 $[\text{FromMax}(2) \xrightarrow{\text{pyk}} \text{"lemma fromMax(2)"}]$

[ToNegatedAnd  $\xrightarrow{\text{pyk}}$  “prop lemma to negated and”]  
 [DistributionOut  $\xrightarrow{\text{pyk}}$  “lemma distributionOut”]  
 [DistributionOutLeft  $\xrightarrow{\text{pyk}}$  “lemma distributionOutLeft”]  
 [DistributionLeft  $\xrightarrow{\text{pyk}}$  “lemma distributionLeft”]  
 [FromNotLess(R)  $\xrightarrow{\text{pyk}}$  “lemma fromNotLess(R)”]  
 [CartProdIsRelation  $\xrightarrow{\text{pyk}}$  “lemma cartProdIsRelation”]  
 [FromSubset  $\xrightarrow{\text{pyk}}$  “lemma fromSubset”]  
 [SubsetIsRelation  $\xrightarrow{\text{pyk}}$  “lemma subsetIsRelation”]  
 [ToSeries  $\xrightarrow{\text{pyk}}$  “lemma toSeries”]  
 [FromSeries  $\xrightarrow{\text{pyk}}$  “lemma fromSeries”]  
 [SeriesSubsetCP  $\xrightarrow{\text{pyk}}$  “lemma seriesSubsetCP”]  
 [ValueType  $\xrightarrow{\text{pyk}}$  “lemma valueType”]  
 [RemoveOr  $\xrightarrow{\text{pyk}}$  “prop lemma remove or”]  
 [FromSingleton  $\xrightarrow{\text{pyk}}$  “lemma fromSingleton”]  
 [InPair(1)  $\xrightarrow{\text{pyk}}$  “lemma inPair(1)”]  
 [InPair(2)  $\xrightarrow{\text{pyk}}$  “lemma inPair(2)”]  
 [SameMember(2)  $\xrightarrow{\text{pyk}}$  “lemma sameMember(2)”]  
 [ToBinaryUnion(1)  $\xrightarrow{\text{pyk}}$  “lemma toBinaryUnion(1)”]  
 [ToBinaryUnion(2)  $\xrightarrow{\text{pyk}}$  “lemma toBinaryUnion(2)”]  
 [FromOrderedPair(TwoLevels)  $\xrightarrow{\text{pyk}}$  “lemma fromOrderedPair(twoLevels)”]  
 [ToCartProd(Helper)  $\xrightarrow{\text{pyk}}$  “lemma toCartProd helper”]  
 [ToCartProd  $\xrightarrow{\text{pyk}}$  “lemma toCartProd”]  
 [NonreciprocalToRight(Eq)  $\xrightarrow{\text{pyk}}$  “lemma nonreciprocalToRight(Eq)”]  
 [NonreciprocalToLeft(Eq)(1term)  $\xrightarrow{\text{pyk}}$  “lemma nonreciprocalToLeft(Eq)(1 term)”]  
 [SameReciprocal  $\xrightarrow{\text{pyk}}$  “lemma sameReciprocal”]  
 [CPseparationIsRelation  $\xrightarrow{\text{pyk}}$  “lemma CPseparationIsRelation”]  
 [OrderedPairEquality  $\xrightarrow{\text{pyk}}$  “lemma orderedPairEquality”]  
 [ReciprocalIsFunction  $\xrightarrow{\text{pyk}}$  “lemma reciprocalIsFunction”]  
 [ReciprocalIsTotal  $\xrightarrow{\text{pyk}}$  “lemma reciprocalIsTotal”]  
 [ReciprocalIsRationalSeries  $\xrightarrow{\text{pyk}}$  “lemma reciprocalIsRationalSeries”]  
 [CrsIsRelation  $\xrightarrow{\text{pyk}}$  “lemma crsIsRelation”]  
 [CrsIsFunction  $\xrightarrow{\text{pyk}}$  “lemma crsIsFunction”]  
 [CrsIsTotal  $\xrightarrow{\text{pyk}}$  “lemma crsIsTotal”]  
 [CrsIsSeries  $\xrightarrow{\text{pyk}}$  “lemma crsIsSeries”]

$[CrsLookup \xrightarrow{pyk} \text{“lemma crsLookup”}]$   
 $[Of \xrightarrow{pyk} \text{“lemma Of”}]$   
 $[If \xrightarrow{pyk} \text{“lemma If”}]$   
 $[ToSingleton \xrightarrow{pyk} \text{“lemma toSingleton”}]$   
 $[FromSameSingleton \xrightarrow{pyk} \text{“lemma fromSameSingleton”}]$   
 $[SingletonmembersEqual \xrightarrow{pyk} \text{“lemma singletonmembersEqual”}]$   
 $[UnequalsNotInSingleton \xrightarrow{pyk} \text{“lemma unequalsNotInSingleton”}]$   
 $[NonsingletonmembersUnequal \xrightarrow{pyk} \text{“lemma nonsingletonmembersUnequal”}]$   
 $[FromOrderedPair \xrightarrow{pyk} \text{“lemma fromOrderedPair”}]$   
 $[FromOrderedPair(1) \xrightarrow{pyk} \text{“lemma fromOrderedPair(1)”}]$   
 $[FromOrderedPair(2) \xrightarrow{pyk} \text{“lemma fromOrderedPair(2)”}]$   
 $[FromCartProd \xrightarrow{pyk} \text{“lemma fromCartProd”}]$   
 $[FromCartProd(1) \xrightarrow{pyk} \text{“lemma fromCartProd(1)”}]$   
 $[FromCartProd(2) \xrightarrow{pyk} \text{“lemma fromCartProd(2)”}]$   
 $[sameOrderedPair \xrightarrow{pyk} \text{“lemma sameOrderedPair”}]$   
 $[InSeriesHelper \xrightarrow{pyk} \text{“lemma inSeries helper”}]$   
 $[InSeries \xrightarrow{pyk} \text{“lemma inSeries”}]$   
 $[To = f(Subset)(Helper) \xrightarrow{pyk} \text{“lemma to=f subset helper”}]$   
 $[To = f(Subset) \xrightarrow{pyk} \text{“lemma to=f subset”}]$   
 $[To = f \xrightarrow{pyk} \text{“lemma to=f”}]$   
 $[productIsFunction \xrightarrow{pyk} \text{“lemma productIsFunction”}]$   
 $[productIsTotal \xrightarrow{pyk} \text{“lemma productIsTotal”}]$   
 $[ProductIsRationalSeries \xrightarrow{pyk} \text{“lemma productIsRationalSeries”}]$   
 $[TimesF \xrightarrow{pyk} \text{“lemma timesF”}]$   
 $[-x + (1/2)x = -(1/2)x \xrightarrow{pyk} \text{“lemma -x+(1/2)x=-(1/2)x”}]$   
 $[PositiveTripled \xrightarrow{pyk} \text{“lemma positiveTripled”}]$   
 $[PositiveDividedBy3 \xrightarrow{pyk} \text{“lemma positiveDividedBy3”}]$   
 $[|x - x| = 0 \xrightarrow{pyk} \text{“lemma |x-x|=0”}]$   
 $[1 < 2 \xrightarrow{pyk} \text{“lemma 1<2”}]$   
 $[1/3 < 2/3 \xrightarrow{pyk} \text{“lemma 1/3<2/3”}]$   
 $[(1/3)x + (1/3)x = (2/3)x \xrightarrow{pyk} \text{“lemma (1/3)x+(1/3)x=(2/3)x”}]$   
 $[(2/3)x + (1/3)x = x \xrightarrow{pyk} \text{“lemma (2/3)x+(1/3)x=x”}]$   
 $[-x + (2/3)x = -(1/3)x \xrightarrow{pyk} \text{“lemma -x+(2/3)x=-(1/3)x”}]$   
 $[-(1/3)x - (1/3)x = -(2/3)x \xrightarrow{pyk} \text{“lemma -(1/3)x-(1/3)x=-(2/3)x”}]$   
 $[-x + (1/3)x = -(2/3)x \xrightarrow{pyk} \text{“lemma -x+(1/3)x=-(2/3)x”}]$



[PreserveLessGreater  $\xrightarrow{\text{pyk}}$  “lemma preserveLessGreater”]  
 [ClosetolessIsLess  $\xrightarrow{\text{pyk}}$  “lemma closetolessIsLess”]  
 [SubLessLeft(F)  $\xrightarrow{\text{pyk}}$  “lemma subLessLeft(F)”]  
 [SubLessLeft(R)  $\xrightarrow{\text{pyk}}$  “lemma subLessLeft(R)”]  
 [ClosetogreaterIsGreater  $\xrightarrow{\text{pyk}}$  “lemma closetogreaterIsGreater”]  
 [SubLessRight(F)  $\xrightarrow{\text{pyk}}$  “lemma subLessRight(F)”]  
 [SubLessRight(R)  $\xrightarrow{\text{pyk}}$  “lemma subLessRight(R)”]  
 [plus0Left  $\xrightarrow{\text{pyk}}$  “lemma plus0Left”]  
 [times1Left  $\xrightarrow{\text{pyk}}$  “lemma times1Left”]  
 [EqAdditionLeft  $\xrightarrow{\text{pyk}}$  “lemma eqAdditionLeft”]  
 [EqMultiplicationLeft  $\xrightarrow{\text{pyk}}$  “lemma eqMultiplicationLeft”]  
 [PlusF(Sym)  $\xrightarrow{\text{pyk}}$  “lemma plusF(Sym)”]  
 [TimesF(Sym)  $\xrightarrow{\text{pyk}}$  “lemma timesF(Sym)”]  
 [SameSeries(Gen)  $\xrightarrow{\text{pyk}}$  “lemma sameSeries(Gen)”]  
 [EqualsSameF  $\xrightarrow{\text{pyk}}$  “lemma equalsSameF”]  
 [LeqReflexivity(R)  $\xrightarrow{\text{pyk}}$  “lemma leqReflexivity(R)”]  
 [Tester1  $\xrightarrow{\text{pyk}}$  “tester1”]  
 [Tester2  $\xrightarrow{\text{pyk}}$  “tester2”]  
 [Tester3  $\xrightarrow{\text{pyk}}$  “tester3”]  
 [Tester4  $\xrightarrow{\text{pyk}}$  “tester4”]  
 [Tester5  $\xrightarrow{\text{pyk}}$  “tester5”]  
 [Tester6  $\xrightarrow{\text{pyk}}$  “tester6”]  
 [sup  $\xrightarrow{\text{pyk}}$  “sup”]

[sup  $\xrightarrow{\text{tex}}$  “sup”]

[x(exp)y  $\xrightarrow{\text{tex}}$  “(#1.  
(expARGH!) #2.  
)”]

[NonreciprocalToRight(Eq)(1term)  $\xrightarrow{\text{tex}}$  “NonreciprocalToRight(Eq)(1 term)”]

[PlusAssociativity(4terms)  $\xrightarrow{\text{tex}}$  “PlusAssociativity(4 terms)”]

[NonzeroProduct(2)  $\xrightarrow{\text{tex}}$  “NonzeroProduct(2)”]

[lemma eqLeq(R)  $\xrightarrow{\text{tex}}$  “eqLeq(R)”]

[ThirdGeqSeries  $\xrightarrow{\text{tex}}$  “ThirdGeqSeries”]

[negativeToLeft(Leq)  $\xrightarrow{\text{tex}}$  “negativeToLeft(Leq)”]

[negativeToLeft(Leq)(1term)  $\xrightarrow{\text{tex}}$  “negativeToLeft(Leq)(1 term)”]

[UStelescope(+1)  $\xrightarrow{\text{tex}}$  “UStelescope(+1)”]

[TelescopeBound(Base)  $\xrightarrow{\text{tex}}$  “TelescopeBound(Base)”]

[TelescopeBound(Indu)  $\xrightarrow{\text{tex}}$  “TelescopeBound(Indu)”]

[TelescopeBound  $\xrightarrow{\text{tex}}$  “TelescopeBound”]

[IntervalSize(Base)  $\xrightarrow{\text{tex}}$  “IntervalSize(Base)”]

[IntervalSize(Indu)  $\xrightarrow{\text{tex}}$  “IntervalSize(Indu)”]

[IntervalSize  $\xrightarrow{\text{tex}}$  “IntervalSize”]

[XS < US  $\xrightarrow{\text{tex}}$  “XS<US”]

[CloseUS  $\xrightarrow{\text{tex}}$  “CloseUS”]

[CloseUS(n + 1)  $\xrightarrow{\text{tex}}$  “CloseUS(n+1)”]

[Induction  $\xrightarrow{\text{tex}}$  “Induction”]

[leqAntisymmetry  $\xrightarrow{\text{tex}}$  “leqAntisymmetry”]

[leqTransitivity  $\xrightarrow{\text{tex}}$  “leqTransitivity”]

[leqAddition  $\xrightarrow{\text{tex}}$  “leqAddition”]

[Reciprocal  $\xrightarrow{\text{tex}}$  “Reciprocal”]

[Equality  $\xrightarrow{\text{tex}}$  “Equality”]

[eqLeq  $\xrightarrow{\text{tex}}$  “eqLeq”]

[eqAddition  $\xrightarrow{\text{tex}}$  “eqAddition”]

[eqMultiplication  $\xrightarrow{\text{tex}}$  “eqMultiplication”]

[eqReflexivity  $\xrightarrow{\text{tex}}$  “eqReflexivity”]

[eqSymmetry  $\xrightarrow{\text{tex}}$  “eqSymmetry”]

[eqTransitivity  $\xrightarrow{\text{tex}}$  “eqTransitivity”]

[eqTransitivity4  $\xrightarrow{\text{tex}}$  “eqTransitivity4”]

[eqTransitivity5  $\xrightarrow{\text{tex}}$  “eqTransitivity5”]

[eqTransitivity6  $\xrightarrow{\text{tex}}$  “eqTransitivity6”]

[plus0Left  $\xrightarrow{\text{tex}}$  “plus0Left”]

[times1Left  $\xrightarrow{\text{tex}}$  “times1Left”]

[EqMultiplicationLeft  $\xrightarrow{\text{tex}}$  “EqMultiplicationLeft”]

[DistributionLeft  $\xrightarrow{\text{tex}}$  “DistributionLeft”]

[DistributionOut  $\xrightarrow{\text{tex}}$  “DistributionOut”]

[DistributionOutLeft  $\xrightarrow{\text{tex}}$  “DistributionOutLeft”]

[Three2twoTerms  $\xrightarrow{\text{tex}}$  “Three2twoTerms”]

[Three2threeTerms  $\xrightarrow{\text{tex}}$  “Three2threeTerms”]

[Three2twoFactors  $\xrightarrow{\text{tex}}$  “Three2twoFactors”]

[Three2threeFactors  $\xrightarrow{\text{tex}}$  “Three2threeFactors”]

[AddEquations  $\xrightarrow{\text{tex}}$  “AddEquations”]

[SubtractEquations  $\xrightarrow{\text{tex}}$  “SubtractEquations”]

[SubtractEquationsLeft  $\xrightarrow{\text{tex}}$  “SubtractEquationsLeft”]

[MultiplyEquations  $\xrightarrow{\text{tex}}$  “MultiplyEquations”]

[EqNegated  $\xrightarrow{\text{tex}}$  “EqNegated”]

[PositiveToRight(Eq)  $\xrightarrow{\text{tex}}$  “PositiveToRight(Eq)”]

[PositiveToLeft(Eq)(1term)  $\xrightarrow{\text{tex}}$  “PositiveToLeft(Eq)(1 term)”]

[NegativeToLeft(Eq)  $\xrightarrow{\text{tex}}$  “NegativeToLeft(Eq)”]

[reciprocalToLeft(Less)  $\xrightarrow{\text{tex}}$  “reciprocalToLeft(Less)”]

[LessNeq  $\xrightarrow{\text{tex}}$  “LessNeq”]

[NeqSymmetry  $\xrightarrow{\text{tex}}$  “NeqSymmetry”]

[NeqNegated  $\xrightarrow{\text{tex}}$  “NeqNegated”]

[SubNeqRight  $\xrightarrow{\text{tex}}$  “SubNeqRight”]

[SubNeqLeft  $\xrightarrow{\text{tex}}$  “SubNeqLeft”]

[NegativeToRight(Neq)(1term)  $\xrightarrow{\text{tex}}$  “NegativeToRight(Neq)(1 term)”]

[NeqAddition  $\xrightarrow{\text{tex}}$  “NeqAddition”]

[NeqMultiplication  $\xrightarrow{\text{tex}}$  “NeqMultiplication”]

[UniqueNegative  $\xrightarrow{\text{tex}}$  “UniqueNegative”]

[DoubleMinus  $\xrightarrow{\text{tex}}$  “DoubleMinus”]

[LeqMultiplicationLeft  $\xrightarrow{\text{tex}}$  “LeqMultiplicationLeft ”]

[LeqLessEq  $\xrightarrow{\text{tex}}$  “LeqLessEq”]

[LessLeq  $\xrightarrow{\text{tex}}$  “LessLeq”]

[FromLeqGeq  $\xrightarrow{\text{tex}}$  “FromLeqGeq”]

[subLeqRight  $\xrightarrow{\text{tex}}$  “subLeqRight”]

[subLeqLeft  $\xrightarrow{\text{tex}}$  “subLeqLeft”]

[Leq + 1  $\xrightarrow{\text{tex}}$  “Leq+1”]

[PositiveToRight(Leq)  $\xrightarrow{\text{tex}}$  “PositiveToRight(Leq)”]

[PositiveToRight(Leq)(1term)  $\xrightarrow{\text{tex}}$  “PositiveToRight(Leq)(1 term)”]

[PositiveToLeft(Leq)  $\xrightarrow{\text{tex}}$  “PositiveToLeft(Leq)”]

$[\text{LeqAdditionLeft} \xrightarrow{\text{tex}} \text{"LeqAdditionLeft"}]$   
 $[\text{leqSubtraction} \xrightarrow{\text{tex}} \text{"leqSubtraction"}]$   
 $[\text{leqSubtractionLeft} \xrightarrow{\text{tex}} \text{"leqSubtractionLeft"}]$   
 $[\text{leqMultiplication} \xrightarrow{\text{tex}} \text{"leqMultiplication"}]$   
 $[\text{thirdGeq} \xrightarrow{\text{tex}} \text{"thirdGeq"}]$   
 $[\text{LeqNegated} \xrightarrow{\text{tex}} \text{"LeqNegated"}]$   
 $[\text{AddEquations(Leq)} \xrightarrow{\text{tex}} \text{"AddEquations(Leq)"}]$   
 $[\text{MultiplyEquations(Leq)} \xrightarrow{\text{tex}} \text{"MultiplyEquations(Leq)"}]$   
 $[\text{LeqNeqLess} \xrightarrow{\text{tex}} \text{"LeqNeqLess"}]$   
 $[\text{FromLess} \xrightarrow{\text{tex}} \text{"FromLess"}]$   
 $[\text{ToLess} \xrightarrow{\text{tex}} \text{"ToLess"}]$   
 $[\text{fromNotLess} \xrightarrow{\text{tex}} \text{"fromNotLess"}]$   
 $[\text{toNotLess} \xrightarrow{\text{tex}} \text{"toNotLess"}]$   
 $[\text{LessAddition} \xrightarrow{\text{tex}} \text{"LessAddition"}]$   
 $[\text{LessAdditionLeft} \xrightarrow{\text{tex}} \text{"LessAdditionLeft"}]$   
 $[\text{LessMultiplication} \xrightarrow{\text{tex}} \text{"LessMultiplication"}]$   
 $[\text{LessMultiplicationLeft} \xrightarrow{\text{tex}} \text{"LessMultiplicationLeft"}]$   
 $[\text{LessDivision} \xrightarrow{\text{tex}} \text{"LessDivision"}]$   
 $[\text{PositiveToRight(Less)} \xrightarrow{\text{tex}} \text{"PositiveToRight(Less)"}]$   
 $[\text{PositiveToLeft(Less)} \xrightarrow{\text{tex}} \text{"PositiveToLeft(Less)"}]$   
 $[\text{NegativeToLeft(Less)} \xrightarrow{\text{tex}} \text{"NegativeToLeft(Less)"}]$   
 $[\text{NegativeToRight(Less)} \xrightarrow{\text{tex}} \text{"NegativeToRight(Less)"}]$   
 $[\text{AddEquations(Less)} \xrightarrow{\text{tex}} \text{"AddEquations(Less)"}]$   
 $[\text{AddEquations(LeqLess)} \xrightarrow{\text{tex}} \text{"AddEquations(LeqLess)"}]$   
 $[\text{NegativeLessPositive} \xrightarrow{\text{tex}} \text{"NegativeLessPositive"}]$

$[\text{leqLessTransitivity} \xrightarrow{\text{tex}} \text{"leqLessTransitivity"}]$   
 $[\text{LessLeqTransitivity} \xrightarrow{\text{tex}} \text{"LessLeqTransitivity"}]$   
 $[\text{LessTransitivity} \xrightarrow{\text{tex}} \text{"LessTransitivity"}]$   
 $[\text{LessTotality} \xrightarrow{\text{tex}} \text{"LessTotality"}]$   
 $[\text{SubLessRight} \xrightarrow{\text{tex}} \text{"SubLessRight"}]$   
 $[\text{SubLessLeft} \xrightarrow{\text{tex}} \text{"SubLessLeft"}]$   
 $[\text{SwitchTerms}(x < y - z) \xrightarrow{\text{tex}} \text{"SwitchTerms}(x < y - z)"]]$   
 $[\text{SwitchTerms}(x - y < z) \xrightarrow{\text{tex}} \text{"SwitchTerms}(x - y < z)"]]$   
 $[\text{LessNegated} \xrightarrow{\text{tex}} \text{"LessNegated"}]$   
 $[\text{PositiveNonzero} \xrightarrow{\text{tex}} \text{"PositiveNonzero"}]$   
 $[\text{PositiveNegated} \xrightarrow{\text{tex}} \text{"PositiveNegated"}]$   
 $[\text{NonpositiveNegated} \xrightarrow{\text{tex}} \text{"NonpositiveNegated"}]$   
 $[\text{NegativeNegated} \xrightarrow{\text{tex}} \text{"NegativeNegated"}]$   
 $[\text{NonnegativeNegated} \xrightarrow{\text{tex}} \text{"NonnegativeNegated"}]$   
 $[\text{PositiveInverted} \xrightarrow{\text{tex}} \text{"PositiveInverted"}]$   
 $[\text{PositiveHalved} \xrightarrow{\text{tex}} \text{"PositiveHalved"}]$   
 $[\text{NonnegativeNumerical} \xrightarrow{\text{tex}} \text{"NonnegativeNumerical"}]$   
 $[\text{NegativeNumerical} \xrightarrow{\text{tex}} \text{"NegativeNumerical"}]$   
 $[\text{PositiveNumerical} \xrightarrow{\text{tex}} \text{"PositiveNumerical"}]$   
 $[|0| = 0 \xrightarrow{\text{tex}} \text{"|0|=0"}]$   
 $[0 <= |x| \xrightarrow{\text{tex}} \text{"0 <= |x|"}]$   
 $[x <= |x| \xrightarrow{\text{tex}} \text{"x <= |x|"}]$   
 $[\text{FromPositiveNumerical} \xrightarrow{\text{tex}} \text{"FromPositiveNumerical"}]$   
 $[\text{SameNumerical} \xrightarrow{\text{tex}} \text{"SameNumerical"}]$   
 $[\text{SignNumerical}(+) \xrightarrow{\text{tex}} \text{"SignNumerical}(+)"]]$

[SignNumerical  $\xrightarrow{\text{tex}}$  “SignNumerical”]

[ToNumericalLess  $\xrightarrow{\text{tex}}$  “ToNumericalLess”]

[FromNumericalGreater  $\xrightarrow{\text{tex}}$  “FromNumericalGreater”]

[NumericalDifference  $\xrightarrow{\text{tex}}$  “NumericalDifference”]

[NumericalDifferenceLess(Helper)  $\xrightarrow{\text{tex}}$  “NumericalDifferenceLess(Helper)”]

[NumericalDifferenceLess  $\xrightarrow{\text{tex}}$  “NumericalDifferenceLess”]

[SplitNumericalSumHelper  $\xrightarrow{\text{tex}}$  “SplitNumericalSumHelper”]

[splitNumericalSum(++)  $\xrightarrow{\text{tex}}$  “splitNumericalSum(++)”]

[splitNumericalSum(--)  $\xrightarrow{\text{tex}}$  “splitNumericalSum(--)”]

[splitNumericalSum(+ - small)  $\xrightarrow{\text{tex}}$  “splitNumericalSum(+ -small)”]

[splitNumericalSum(+ - big)  $\xrightarrow{\text{tex}}$  “splitNumericalSum(+ -big)”]

[splitNumericalSum(+ -)  $\xrightarrow{\text{tex}}$  “splitNumericalSum(+ -)”]

[splitNumericalSum(- +)  $\xrightarrow{\text{tex}}$  “splitNumericalSum(- +)”]

[splitNumericalSum  $\xrightarrow{\text{tex}}$  “splitNumericalSum”]

[SplitNumericalProduct(++)  $\xrightarrow{\text{tex}}$  “SplitNumericalProduct(++)”]

[SplitNumericalProduct(+ -)  $\xrightarrow{\text{tex}}$  “SplitNumericalProduct(+ -)”]

[SplitNumericalProduct  $\xrightarrow{\text{tex}}$  “SplitNumericalProduct”]

[insertMiddleTerm(Numerical)  $\xrightarrow{\text{tex}}$  “insertMiddleTerm(Numerical)”]

[insertTwoMiddleTerms(Numerical)  $\xrightarrow{\text{tex}}$  “insertTwoMiddleTerms(Numerical)”]

[MaxLeq(1)  $\xrightarrow{\text{tex}}$  “MaxLeq(1)”]

[MaxLeq(2)  $\xrightarrow{\text{tex}}$  “MaxLeq(2)”]

[LessThanMax  $\xrightarrow{\text{tex}}$  “LessThanMax”]

[x + y = zBackwards  $\xrightarrow{\text{tex}}$  “x+y=zBackwards”]

[x \* y = zBackwards  $\xrightarrow{\text{tex}}$  “x\*y=zBackwards”]

[x = x + (y - y)  $\xrightarrow{\text{tex}}$  “x=x+(y-y)”]

$$[x = x + y - y \xrightarrow{\text{tex}} \text{“}x=x+y-y\text{”}]$$

$$[x = x * y * (1/y) \xrightarrow{\text{tex}} \text{“}x=x*y*(1/y)\text{”}]$$

$$[\text{insertMiddleTerm(Sum)} \xrightarrow{\text{tex}} \text{“insertMiddleTerm(Sum)”}]$$

$$[\text{insertTwoMiddleTerms(Sum)} \xrightarrow{\text{tex}} \text{“insertTwoMiddleTerms(Sum)”}]$$

$$[\text{insertMiddleTerm(Difference)} \xrightarrow{\text{tex}} \text{“insertMiddleTerm(Difference)”}]$$

$$[x * 0 + x = x \xrightarrow{\text{tex}} \text{“}x*0+x=x\text{”}]$$

$$[\text{NonnegativeFactors} \xrightarrow{\text{tex}} \text{“NonnegativeFactors”}]$$

$$[\text{NonzeroFactors} \xrightarrow{\text{tex}} \text{“NonzeroFactors”}]$$

$$[\text{PositiveFactors} \xrightarrow{\text{tex}} \text{“PositiveFactors”}]$$

$$[\text{PlusTimesMinus} \xrightarrow{\text{tex}} \text{“PlusTimesMinus”}]$$

$$[\text{MinusTimesMinus} \xrightarrow{\text{tex}} \text{“MinusTimesMinus”}]$$

$$[x * 0 = 0 \xrightarrow{\text{tex}} \text{“}x*0=0\text{”}]$$

$$[(-1) * (-1) + (-1) * 1 = 0 \xrightarrow{\text{tex}} \text{“}(-1)*(-1)+(-1)*1=0\text{”}]$$

$$[(-1) * (-1) = 1 \xrightarrow{\text{tex}} \text{“}(-1)*(-1)=1\text{”}]$$

$$[0 < 1\text{Helper} \xrightarrow{\text{tex}} \text{“}0<1\text{Helper”}]$$

$$[0 < 1 \xrightarrow{\text{tex}} \text{“}0<1\text{”}]$$

$$[0 < 2 \xrightarrow{\text{tex}} \text{“}0<2\text{”}]$$

$$[0 < 3 \xrightarrow{\text{tex}} \text{“}0<3\text{”}]$$

$$[0 < 1/2 \xrightarrow{\text{tex}} \text{“}0<1/2\text{”}]$$

$$[0 < 1/3 \xrightarrow{\text{tex}} \text{“}0<1/3\text{”}]$$

$$[\text{TwoWholes} \xrightarrow{\text{tex}} \text{“TwoWholes”}]$$

$$[\text{ThreeWholes} \xrightarrow{\text{tex}} \text{“ThreeWholes”}]$$

$$[\text{TwoHalves} \xrightarrow{\text{tex}} \text{“TwoHalves”}]$$

$$[\text{ThreeThirds} \xrightarrow{\text{tex}} \text{“ThreeThirds”}]$$

$$[-x - y = -(x + y) \xrightarrow{\text{tex}} \text{“}-x-y=-(x+y)\text{”}]$$



$[-x * y = -(x * y) \xrightarrow{\text{tex}} \text{"-x*y=-(x*y)}"]$

$[\text{MinusNegated} \xrightarrow{\text{tex}} \text{"MinusNegated"}]$

$[\text{Times}(-1) \xrightarrow{\text{tex}} \text{"Times(-1)}"]$

$[\text{Times}(-1)\text{Left} \xrightarrow{\text{tex}} \text{"Times(-1)Left"}]$

$[-0 = 0 \xrightarrow{\text{tex}} \text{"-0=0"}]$

$[\text{AllNegated}(\text{Imply}) \xrightarrow{\text{tex}} \text{"AllNegated}(\text{Imply)}"]$

$[\text{ExistNegated}(\text{Imply}) \xrightarrow{\text{tex}} \text{"ExistNegated}(\text{Imply)}"]$

$[\text{IntroExist}(\text{Helper}) \xrightarrow{\text{tex}} \text{"IntroExist}(\text{Helper)}"]$

$[\text{IntroExist} \xrightarrow{\text{tex}} \text{"IntroExist"}]$

$[\text{ExistMP} \xrightarrow{\text{tex}} \text{"ExistMP"}]$

$[\text{ExistMP2} \xrightarrow{\text{tex}} \text{"ExistMP2"}]$

$[\text{TwiceExistMP} \xrightarrow{\text{tex}} \text{"TwiceExistMP"}]$

$[\text{TwiceExistMP2} \xrightarrow{\text{tex}} \text{"TwiceExistMP2"}]$

$[\text{EAE} - \text{MP} \xrightarrow{\text{tex}} \text{"EAE-MP"}]$

$[\text{AddAll} \xrightarrow{\text{tex}} \text{"AddAll "}]$

$[\text{AddExist}(\text{Helper1}) \xrightarrow{\text{tex}} \text{"AddExist}(\text{Helper1)}"]$

$[\text{AddExist}(\text{Helper2}) \xrightarrow{\text{tex}} \text{"AddExist}(\text{Helper2)}"]$

$[\text{AddExist} \xrightarrow{\text{tex}} \text{"AddExist"}]$

$[\text{AddExist}(\text{SimpleAnt}) \xrightarrow{\text{tex}} \text{"AddExist}(\text{SimpleAnt)}"]$

$[\text{AddExist}(\text{Simple}) \xrightarrow{\text{tex}} \text{"AddExist}(\text{Simple)}"]$

$[\text{AddEAE} \xrightarrow{\text{tex}} \text{"AddEAE"}]$

$[\text{AEA} - \text{negated} \xrightarrow{\text{tex}} \text{"AEA-negated"}]$

$[\text{EEA} - \text{negated} \xrightarrow{\text{tex}} \text{"EEA-negated"}]$

$[\text{ToNegatedAnd} \xrightarrow{\text{tex}} \text{"ToNegatedAnd"}]$

$[\text{eqTransitivity4} \xrightarrow{\text{tex}} \text{"eqTransitivity4"}]$

[SFsymmetry  $\xrightarrow{\text{tex}}$  “SFsymmetry”]

[SFtransitivity  $\xrightarrow{\text{tex}}$  “SFtransitivity”]

[PlusF(Sym)  $\xrightarrow{\text{tex}}$  “PlusF(Sym)”]

[TimesF(Sym)  $\xrightarrow{\text{tex}}$  “TimesF(Sym)”]

[f2R(Plus)  $\xrightarrow{\text{tex}}$  “f2R(Plus)”]

[f2R(Times)  $\xrightarrow{\text{tex}}$  “f2R(Times)”]

[<< TransitivityHelper(Q)  $\xrightarrow{\text{tex}}$  “<<TransitivityHelper(Q)”]

[<< Transitivity  $\xrightarrow{\text{tex}}$  “<<Transitivity”]

[<<== Reflexivity  $\xrightarrow{\text{tex}}$  “<<==Reflexivity”]

[<<== AntisymmetryHelper(Q)  $\xrightarrow{\text{tex}}$  “<<==AntisymmetryHelper(Q)”]

[FromNotSameF(Weak)(Helper)  $\xrightarrow{\text{tex}}$  “FromNotSameF(Weak)(Helper)”]

[FromNotSameF(Weak)  $\xrightarrow{\text{tex}}$  “FromNotSameF(Weak)”]

[FromNotLess(F)  $\xrightarrow{\text{tex}}$  “FromNotLess(F)”]

[Plus0(F)  $\xrightarrow{\text{tex}}$  “Plus0(F)”]

[== Addition  $\xrightarrow{\text{tex}}$  “==Addition”]

[== AdditionLeft  $\xrightarrow{\text{tex}}$  “==AdditionLeft”]

[Fpart – Bounded(Base)  $\xrightarrow{\text{tex}}$  “Fpart-Bounded(Base)”]

[Fpart – Bounded(InduHelper)  $\xrightarrow{\text{tex}}$  “Fpart-Bounded(InduHelper)”]

[Fpart – Bounded(Indu)  $\xrightarrow{\text{tex}}$  “Fpart-Bounded(Indu)”]

[Fpart – Bounded  $\xrightarrow{\text{tex}}$  “Fpart-Bounded”]

[F – Bounded  $\xrightarrow{\text{tex}}$  “F-Bounded”]

[F – Bounded(Helper)  $\xrightarrow{\text{tex}}$  “F-Bounded(Helper)”]

[SameFmultiplication(Helper)  $\xrightarrow{\text{tex}}$  “SameFmultiplication(Helper)”]

[SameFmultiplication  $\xrightarrow{\text{tex}}$  “SameFmultiplication”]

[FromNot < f(Weak)(Helper)  $\xrightarrow{\text{tex}}$  “FromNot<f(Weak)(Helper)”]

[FromNot < f(Weak)  $\xrightarrow{\text{tex}}$  "FromNot<f(Weak)"]

[FromNot < f(Strong)(Helper2)  $\xrightarrow{\text{tex}}$  "FromNot<f(Strong)(Helper2)"]

[FromNot < f(Strong)(Helper)  $\xrightarrow{\text{tex}}$  "FromNot<f(Strong)(Helper)"]

[FromNot < f(Strong)  $\xrightarrow{\text{tex}}$  "FromNot<f(Strong)"]

[fromNotSameF(Strongest)(Helper2)  $\xrightarrow{\text{tex}}$   
"fromNotSameF(Strongest)(Helper2)"]

[fromNotSameF(Strongest)(Helper)  $\xrightarrow{\text{tex}}$  "fromNotSameF(Strongest)(Helper)"]

[fromNotSameF(Strongest)  $\xrightarrow{\text{tex}}$  "fromNotSameF(Strongest)"]

[ToLess(F)(Helper)  $\xrightarrow{\text{tex}}$  "ToLess(F)(Helper)"]

[ToLess(F)  $\xrightarrow{\text{tex}}$  "ToLess(F)"]

[LessMultiplication(F)(Helper2)  $\xrightarrow{\text{tex}}$  "LessMultiplication(F)(Helper2)"]

[LessMultiplication(F)(Helper)  $\xrightarrow{\text{tex}}$  "LessMultiplication(F)(Helper)"]

[LessMultiplication(F)  $\xrightarrow{\text{tex}}$  "LessMultiplication(F)"]

[EqMultiplication(R)  $\xrightarrow{\text{tex}}$  "EqMultiplication(R)"]

[EqMultiplicationLeft(R)  $\xrightarrow{\text{tex}}$  "EqMultiplicationLeft(R)"]

[PlusAssociativity(F)  $\xrightarrow{\text{tex}}$  "PlusAssociativity(F)"]

[FromNot <<  $\xrightarrow{\text{tex}}$  "FromNot<<"]

[ToLess(R)  $\xrightarrow{\text{tex}}$  "ToLess(R)"]

[x \* 0 = 0(F)  $\xrightarrow{\text{tex}}$  "x\*0=0(F)"]

[x \* 0 = 0(R)  $\xrightarrow{\text{tex}}$  "x\*0=0(R)"]

[PlusCommutativity(F)  $\xrightarrow{\text{tex}}$  "PlusCommutativity(F)"]

[Cauchy(2)(Helper)  $\xrightarrow{\text{tex}}$  "Cauchy(2)(Helper)"]

[Cauchy(2)  $\xrightarrow{\text{tex}}$  "Cauchy(2)"]

[TimesAssociativity(F)  $\xrightarrow{\text{tex}}$  "TimesAssociativity(F)"]

[LessMultiplication(R)  $\xrightarrow{\text{tex}}$  "LessMultiplication(R)"]

[LeqMultiplication(R)  $\xrightarrow{\text{tex}}$  “LeqMultiplication(R)”]

[Times1f  $\xrightarrow{\text{tex}}$  “Times1f”]

[ReciprocalFnonzero  $\xrightarrow{\text{tex}}$  “ReciprocalFnonzero”]

[(Eventually = f)2sameF(Helper)  $\xrightarrow{\text{tex}}$  “(Eventually=f)2sameF(Helper)”]

[(Eventually = f)2sameF  $\xrightarrow{\text{tex}}$  “(Eventually=f)2sameF”]

[FromNotSameF(Strong)(Helper2)  $\xrightarrow{\text{tex}}$  “FromNotSameF(Strong)(Helper2)”]

[FromNotSameF(Strong)(Helper)  $\xrightarrow{\text{tex}}$  “FromNotSameF(Strong)(Helper)”]

[FromNotSameF(Strong)  $\xrightarrow{\text{tex}}$  “FromNotSameF(Strong)”]

[SameFreciprocal(Helper)  $\xrightarrow{\text{tex}}$  “SameFreciprocal(Helper)”]

[SameFreciprocal  $\xrightarrow{\text{tex}}$  “SameFreciprocal”]

[From!! ==  $\xrightarrow{\text{tex}}$  “From!!==”]

[Reciprocal(R)  $\xrightarrow{\text{tex}}$  “Reciprocal(R)”]

[TimesCommutativity(F)  $\xrightarrow{\text{tex}}$  “TimesCommutativity(F)”]

[Distribution(F)  $\xrightarrow{\text{tex}}$  “Distribution(F)”]

[FromNotLess(R)  $\xrightarrow{\text{tex}}$  “FromNotLess(R)”]

[ToNegatedAnd(1)  $\xrightarrow{\text{tex}}$  “ToNegatedAnd(1)”]

[FromMax(1)  $\xrightarrow{\text{tex}}$  “FromMax(1)”]

[FromMax(2)  $\xrightarrow{\text{tex}}$  “FromMax(2)”]

[CartProdIsRelation  $\xrightarrow{\text{tex}}$  “CartProdIsRelation”]

[FromSubset  $\xrightarrow{\text{tex}}$  “FromSubset”]

[SubsetIsRelation  $\xrightarrow{\text{tex}}$  “SubsetIsRelation”]

[SeriesSubsetCP  $\xrightarrow{\text{tex}}$  “SeriesSubsetCP”]

[ValueType  $\xrightarrow{\text{tex}}$  “ValueType”]

[ToSeries  $\xrightarrow{\text{tex}}$  “ToSeries”]

[FromSeries  $\xrightarrow{\text{tex}}$  “FromSeries”]

[RemoveOr  $\xrightarrow{\text{tex}}$  “RemoveOr”]

[FromSingleton  $\xrightarrow{\text{tex}}$  “FromSingleton”]

[InPair(1)  $\xrightarrow{\text{tex}}$  “InPair(1)”]

[InPair(2)  $\xrightarrow{\text{tex}}$  “InPair(2)”]

[SameMember(2)  $\xrightarrow{\text{tex}}$  “SameMember(2)”]

[ToBinaryUnion(1)  $\xrightarrow{\text{tex}}$  “ToBinaryUnion(1)”]

[ToBinaryUnion(2)  $\xrightarrow{\text{tex}}$  “ToBinaryUnion(2)”]

[FromOrderedPair(TwoLevels)  $\xrightarrow{\text{tex}}$  “FromOrderedPair(TwoLevels)”]

[ToCartProd(Helper)  $\xrightarrow{\text{tex}}$  “ToCartProd(Helper)”]

[ToCartProd  $\xrightarrow{\text{tex}}$  “ToCartProd”]

[NonreciprocalToRight(Eq)  $\xrightarrow{\text{tex}}$  “NonreciprocalToRight(Eq)”]

[NonreciprocalToLeft(Eq)(1term)  $\xrightarrow{\text{tex}}$  “NonreciprocalToLeft(Eq)(1 term)”]

[SameReciprocal  $\xrightarrow{\text{tex}}$  “SameReciprocal”]

[CPseparationIsRelation  $\xrightarrow{\text{tex}}$  “CPseparationIsRelation”]

[OrderedPairEquality  $\xrightarrow{\text{tex}}$  “OrderedPairEquality”]

[ReciprocalIsFunction  $\xrightarrow{\text{tex}}$  “ReciprocalIsFunction”]

[ReciprocalIsTotal  $\xrightarrow{\text{tex}}$  “ReciprocalIsTotal”]

[ReciprocalIsRationalSeries  $\xrightarrow{\text{tex}}$  “ReciprocalIsRationalSeries”]

[CrsIsRelation  $\xrightarrow{\text{tex}}$  “CrsIsRelation”]

[CrsIsFunction  $\xrightarrow{\text{tex}}$  “CrsIsFunction ”]

[CrsIsTotal  $\xrightarrow{\text{tex}}$  “CrsIsTotal”]

[CrsIsSeries  $\xrightarrow{\text{tex}}$  “CrsIsSeries”]

[CrsLookup  $\xrightarrow{\text{tex}}$  “CrsLookup”]

[0f  $\xrightarrow{\text{tex}}$  “0f”]

[1f  $\xrightarrow{\text{tex}}$  “1f”]

[ToSingleton  $\xrightarrow{\text{tex}}$  "ToSingleton"]

[FromSameSingleton  $\xrightarrow{\text{tex}}$  "FromSameSingleton"]

[SingletonmembersEqual  $\xrightarrow{\text{tex}}$  "SingletonmembersEqual"]

[UnequalsNotInSingleton  $\xrightarrow{\text{tex}}$  "UnequalsNotInSingleton"]

[NonsingletonmembersUnequal  $\xrightarrow{\text{tex}}$  "NonsingletonmembersUnequal"]

[FromOrderedPair  $\xrightarrow{\text{tex}}$  "FromOrderedPair"]

[FromOrderedPair(1)  $\xrightarrow{\text{tex}}$  "FromOrderedPair(1)"]

[FromOrderedPair(2)  $\xrightarrow{\text{tex}}$  "FromOrderedPair(2)"]

[FromCartProd  $\xrightarrow{\text{tex}}$  "FromCartProd"]

[FromCartProd(1)  $\xrightarrow{\text{tex}}$  "FromCartProd(1)"]

[FromCartProd(2)  $\xrightarrow{\text{tex}}$  "FromCartProd(2)"]

[sameOrderedPair  $\xrightarrow{\text{tex}}$  "sameOrderedPair"]

[InSeriesHelper  $\xrightarrow{\text{tex}}$  "InSeriesHelper"]

[InSeries  $\xrightarrow{\text{tex}}$  "InSeries"]

[To = f(Subset)(Helper)  $\xrightarrow{\text{tex}}$  "To=f(Subset)(Helper)"]

[To = f(Subset)  $\xrightarrow{\text{tex}}$  "To=f(Subset)"]

[To = f  $\xrightarrow{\text{tex}}$  "To=f"]

[Tester1  $\xrightarrow{\text{tex}}$  "Tester1"]

[Tester2  $\xrightarrow{\text{tex}}$  "Tester2"]

[Tester3  $\xrightarrow{\text{tex}}$  "Tester3"]

[Tester4  $\xrightarrow{\text{tex}}$  "Tester4"]

[Tester5  $\xrightarrow{\text{tex}}$  "Tester5"]

[Tester6  $\xrightarrow{\text{tex}}$  "Tester6"]

[productIsFunction  $\xrightarrow{\text{tex}}$  "productIsFunction"]

[productIsTotal  $\xrightarrow{\text{tex}}$  "productIsTotal"]

[ProductIsRationalSeries  $\xrightarrow{\text{tex}}$  "ProductIsRationalSeries"]

[TimesF  $\xrightarrow{\text{tex}}$  "TimesF"]

$[-x + (1/2)x = -(1/2)x \xrightarrow{\text{tex}} "-x+(1/2)x=-(1/2)x"]$

[ClosetolessIsLess  $\xrightarrow{\text{tex}}$  "ClosetolessIsLess"]

[SubLessLeft(F)  $\xrightarrow{\text{tex}}$  "SubLessLeft(F)"]

[SubLessLeft(R)  $\xrightarrow{\text{tex}}$  "SubLessLeft(R)"]

[ClosetogreaterIsGreater  $\xrightarrow{\text{tex}}$  "ClosetogreaterIsGreater"]

[SubLessRight(F)  $\xrightarrow{\text{tex}}$  "SubLessRight(F)"]

[SubLessRight(R)  $\xrightarrow{\text{tex}}$  "SubLessRight(R)"]

[PositiveTripled  $\xrightarrow{\text{tex}}$  "PositiveTripled"]

[PositiveDividedBy3  $\xrightarrow{\text{tex}}$  "PositiveDividedBy3"]

$[|x - x| = 0 \xrightarrow{\text{tex}} "|x-x|=0"]$

$[1 < 2 \xrightarrow{\text{tex}} "1<2"]$

$[1/3 < 2/3 \xrightarrow{\text{tex}} "1/3<2/3"]$

$[(1/3)x + (1/3)x = (2/3)x \xrightarrow{\text{tex}} "(1/3)x+(1/3)x=(2/3)x"]$

$[(2/3)x + (1/3)x = x \xrightarrow{\text{tex}} "(2/3)x+(1/3)x=x"]$

$[-x + (2/3)x = -(1/3)x \xrightarrow{\text{tex}} "-x+(2/3)x=-(1/3)x"]$

[PreserveLessGreater  $\xrightarrow{\text{tex}}$  "PreserveLessGreater"]

$[-(1/3)x - (1/3)x = -(2/3)x \xrightarrow{\text{tex}} "-(1/3)x-(1/3)x=-(2/3)x"]$

$[-x + (1/3)x = -(2/3)x \xrightarrow{\text{tex}} "-x+(1/3)x=-(2/3)x"]$

[plus0Left  $\xrightarrow{\text{tex}}$  "plus0Left"]

[times1Left  $\xrightarrow{\text{tex}}$  "times1Left"]

[EqAdditionLeft  $\xrightarrow{\text{tex}}$  "EqAdditionLeft"]

[EqMultiplicationLeft  $\xrightarrow{\text{tex}}$  "EqMultiplicationLeft"]

[PlusF(Sym)  $\xrightarrow{\text{tex}}$  "PlusF(Sym)"]

[TimesF(Sym)  $\xrightarrow{\text{tex}}$  “TimesF(Sym)”]

[SameSeries(Gen)  $\xrightarrow{\text{tex}}$  “SameSeries(Gen)”]

[EqualsSameF  $\xrightarrow{\text{tex}}$  “EqualsSameF”]

[LeqReflexivity(R)  $\xrightarrow{\text{tex}}$  “LeqReflexivity(R)”]