Ækvivalensrelationer i Logiweb

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Indhold

Figurer

1 Indledning

Denne rapport indeholder et aksiomsystem for en reduceret version af ZF mængdelære, som jeg kalder for "ZFsub". Ud fra dette aksiomsystem vil jeg vise en række basale sætninger inden for udsagnslogik og mængdelære. Hovedresultatet er, at en ækvivalensrelation på en mængde bs implicit definerer en partition af bs[1]. Alt det formelle arbejde i rapporten er gennemført med henblik på at bevise dette resultat.
Rapporten er udarbejdet ved hjælp af Logiweb, som er et system til verifikation og publicering af tekster, der indeholder formel matematik. Logiweb har verificeret rapportens beviser og publiceret rapporten på WWW-adressen
`http://www.diku.dk/~grue/logiweb/20060417/home/eriksen/equivalence-relations.`
Det matematiske indhold af denne rapport er ikke synderligt avanceret; f.eks. fylder materialet om ækvivalensrelationer knap to sider i lærebogen [5] (s. 29–31). Ideen med rapporten er snarere at afprøve Logiweb end at redegøre for en matematisk teori. Derfor vil jeg ikke gøre alt for meget ud af at forklare de matematiske begreber; fokus vil i højere grad ligge på tekniske forhold omkring formaliseringen.
Rapporten er struktureret som følger: Afsnit 2 er en kort beskrivelse af nogle detaljer ved Logiweb, som er nyttige at kende for en læser af et Logiweb-dokument som dette[2]. Afsnit 3 og 4 omhandler syntaksen og aksiomsystemet for ZFsub, og afsnit 5 indfører de makrodefinitioner, jeg vil gøre brug af. Så kommer vi til lemmaerne og beviserne; vi skal igennem fem afsnit med hjælpesætninger (afsnit 6–10), før vi endelig kan bevise hovedresultatet i afsnit 11. Afsnit 12 slutter af med en konklusion på det hele.

2 Lidt om Logiweb

Som nævnt i indledningen er denne rapport skrevet ved hjælp af Logiweb. Dette betyder dels, at rapportens formelle indhold er defineret ud fra nogle andre Logiweb-dokumenter, dels at et par afsnit indeholder nogle særlige definitioner og tabeller, og endelig at et par andre afsnit indeholder en del programkode. I dette afsnit vil jeg kort beskrive disse fænomener. For en detaljeret beskrivelse af hele Logiweb systemet vil jeg henvise til [3]; her kan man bla. læse om den bevischecker, der har verificeret rapportens beviser.

2.1 Formelle konstruktioner

Det formelle indhold af et Logiweb-dokument er sammensat af en række formelle konstruktioner. En formel konstruktion kan repræsentere alt, hvad der

[1] Jeg vil referere til denne sætning som "hovedresultatet". Oprindeligt var det også et mål at bevise det modsatte resultat — at enhver partition implicit definerer en ækvivalensrelation — men det har der ikke været tid til.
[2] Dette afsnit er en revideret udgave af afsnit 2 i [1].

har med formel matematik at gøre: Variable, funktioner, lemmaer, beviser, osv. Der er to kilder til konstruktionerne i et Logiweb-dokument: Dels kan man indføre sine egne konstruktioner, og dels kan man importere konstruktioner fra andre Logiweb-dokumenter. De importerede konstruktioner i denne rapport har to kilder: Dels [3], og dels de 3 .pdf filer [4], som tilsammen udgør ét Logiweb-dokument.

2.2 Særlige definitioner og tabeller

Den formelle del af et Logiweb-dokument skrives i et formateringssprog ved navn "pyk". Hver formel konstruktion, man arbejder med i dokumentet, har tilknyttet en såkaldt "pyk definition". Dette er en angivelse af, hvad man skal skrive, hvis man i et andet Logiweb-dokument ønsker at benytte den pågældende konstruktion. Hvis man indfører en ny konstruktion i sit Logiweb-dokument, er det et krav, at man gør den tilsvarende pyk definition tilgængelig i dokumentet. Denne rapports pyk definitioner er vedlagt i bilag D.
Logiweb genererer det færdige dokument ved hjælp af det kendte formateringssprog LaTeX. Derfor har hver formel konstruktion også tilknyttet en såkaldt "TeX definition", som angiver, hvordan konstruktionen skal skrives i LaTeX. Ligesom pyk definitioner skal også TeX definitioner være tilgængelige i dokumentet. Denne rapports TeX definitioner er vedlagt i bilag F.
Endelig indeholder bilag E en tabel over alle de formelle konstruktioner i denne rapport — både de importerede og dem, jeg selv har defineret. Denne tabel er først og fremmest med, for at andre Logiweb-dokumenter skal kunne referere til rapporten; uden tabellen kan sådanne referencer ikke finde sted.

2.3 L-kode

Logiweb-dokumentet [3] indeholder et funktionelt programmeringssprog, som jeg vil kalde for "L". Der er en del forekomster af L-kode i rapporten. For en forklaring af de konstruktioner fra L, som jeg bruger, vil jeg henvise til funktionsbeskrivelserne i appendikset til [4], afsnit 3.2, s. 6. Herudover vil jeg supplere med et par kommentarer undervejs.

3 Syntaks for ZFsub

Som nævnt i indledningen vil jeg arbejde med en reduceret version af ZF mængdelære, som jeg kalder for "ZFsub". Dette afsnit beskriver syntaksen for ZFsub. Der er to syntaktiske hovedkategorier i ZFsub: Termer og formler. Underafsnit 3.1 beskriver syntaksen for termer, og underafsnit 3.2 beskriver syntaksen for formler.

3.1 Termer

Syntaksen for en term $\underline{\mathtt{t}}$ kan beskrives ved den følgende BNF-grammatik:

$$\begin{aligned} \underline{\mathtt{t}} &::= \mathtt{Værdi} \mid \mathtt{Variabel} \\ \mathtt{Værdi} &::= \emptyset \mid \{\underline{\mathtt{t}}, \underline{\mathtt{t}}\} \mid \cup \underline{\mathtt{t}} \mid \mathrm{P}(\underline{\mathtt{t}}) \mid \{\mathrm{ph} \in \underline{\mathtt{t}} \mid \underline{\mathtt{f}}\} \\ \mathtt{Variabel} &::= \mathtt{Objekt\text{-}var} \mid \mathtt{Ex\text{-}var} \mid \mathtt{Ph\text{-}var} \end{aligned}$$

En `Værdi` svarer til en konkret mængde — værditermer indeholder ingen variable. Den grundlæggende værdi er den tomme mængde; herudfra kan vi konstruere par, fællesmængder, potensmængder samt delmængder hvis elementer opfylder en bestemt egenskab. Der er således ingen individuelle konstanter i ZFsub; alt er mængder.
En `Variabel` kan for det første være en objekt-variabel, som varierer over værdier[3]. De to andre typer af variable — eksistens-variable og pladsholder-variable — vil jeg vente med at forklare til hhv. afsnit 4.1.1 og 4.2.1.

3.2 Formler

Syntaksen for en formel $\underline{\mathtt{f}}$ kan beskrives ved den følgende BNF-grammatik:

$$\underline{\mathtt{f}} \quad ::= \quad \underline{\mathtt{t}} \in \underline{\mathtt{t}} \mid \underline{\mathtt{t}} = \underline{\mathtt{t}} \mid \dot{\neg}\, \underline{\mathtt{f}} \mid \underline{\mathtt{f}} \Rightarrow \underline{\mathtt{f}} \mid \forall \mathtt{Objekt\text{-}var} : \mathcal{F}$$

Med en formel kan vi altså påstå, at en mængde tilhører en anden mængde, eller at to mængder er lig hinanden. Desuden kan vi negere formler[4], lade formler implicere hinanden, samt kvantificere formler med objektvariable.

3.3 Objektvariable vs. metavariable

En metavariabel[5] er en variabel, der varierer over vilkårlige termer — altså også over objektvariable. Denne rapport er ikke helt fri for objektvariable; men når vi taler om ZFsub i aksiomer[6], definitioner og beviser vil jeg så vidt muligt

[3] Vi repræsenterer objektvariable med symbolerne "$\overline{\mathsf{a}}$", "$\overline{\mathsf{b}}$", ⋯ "$\overline{\mathsf{z}}$" samt "$\overline{\mathsf{bs}}$".

[4] Jeg skriver negationstegnet med en prik over for at undgå forveksling med konstruktionen [¬x] fra [3].

[5] Vi repræsenterer metavariable med symbolerne "$\underline{\mathsf{a}}$", "$\underline{\mathsf{b}}$", ⋯ "$\underline{\mathsf{z}}$" samt "$\underline{\mathsf{bs}}$".

[6] Strengt taget burde jeg skrive "aksiomskemaer", da vi bruger metavariable. Da der kun er aksiomskemaer i denne rapport, vil jeg imidlertid bruge ordet "aksiom" for at gøre teksten lidt lettere.

bruge metavariable, da de er mere fleksible end objektvariable. F.eks. kan "$\bar{s}$" i formlen $[\bar{s}=\underline{x}]$[7] kun instantieres til en term; vi kan ikke skifte objektvariabel og konkludere $[\bar{t}=\underline{x}]$. Derimod kan vi sagtens instantiere formlen $[\underline{s}=\underline{x}]$ til $[\bar{t}=\underline{x}]$. En liste over de metavariable, jeg vil bruge, kan ses i bilag A.

4 Aksiomatisk system

ZFsub er en teori i 1. ordens prædikatkalkyle. Vi kan således opdele aksiomsystemet for ZFsub i to: En prædikatlogisk del og en mængdeteoretisk del. Underafsnit 4.1 gennemgår den prædikatlogiske del, og underafsnit 4.2 gennemgår den mængdeteoretiske del. Bilag B indeholder en kopi af det samlede aksiomsystem.

4.1 1. ordens prædikatkalkyle

Her er de første slutningsregler i ZFsub:

$$[\mathrm{ZFsub} \overset{\mathrm{stmt}}{\rightarrow} \forall \underline{x}\colon \forall \underline{y}\colon \dot{\neg}\, \underline{x}=\underline{y} \Rightarrow \forall_{\mathrm{obj}} \bar{s}\colon \dot{\neg}\, \bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y} \Rightarrow \dot{\neg}\, \bar{s} \in \underline{y} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \dot{\neg}\, \forall_{\mathrm{obj}} \bar{s}\colon \dot{\neg}\, \bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y} \Rightarrow \dot{\neg}\, \bar{s} \in \underline{y} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \underline{x}=\underline{y} \oplus \forall \underline{s}\colon \forall \underline{x}\colon \dot{\neg}\, \underline{s} \in \mathrm{P}(\underline{x}) \Rightarrow \forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \dot{\neg}\, \forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \underline{s} \in \mathrm{P}(\underline{x}) \oplus \forall \underline{a}\colon \underline{a} \vdash \underline{a} \oplus \forall \underline{r}\colon \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{bs}\colon \dot{\neg}\, \dot{\neg}\, \forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\, \forall_{\mathrm{obj}} \bar{s}\colon \forall_{\mathrm{obj}} \bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\, \forall_{\mathrm{obj}} \bar{s}\colon \forall_{\mathrm{obj}} \bar{t}\colon \forall_{\mathrm{obj}} \bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r} \vdash \underline{x} \in \{\mathrm{ph} \in \mathrm{P}(\underline{bs}) \mid \dot{\neg}\, t_{\mathrm{Ex}} \in \underline{bs} \Rightarrow \dot{\neg}\, \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},t_{\mathrm{Ex}}\}\} \in \underline{r}\} = b_{\mathrm{Ph}}\} \vdash \underline{y} \in \{\mathrm{ph} \in \mathrm{P}(\underline{bs}) \mid \dot{\neg}\, t_{\mathrm{Ex}} \in \underline{bs} \Rightarrow \dot{\neg}\, \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},t_{\mathrm{Ex}}\}\} \in \underline{r}\} = b_{\mathrm{Ph}}\} \vdash \dot{\neg}\, \underline{x}=\underline{y} \vdash \{\mathrm{ph} \in \cup\{\{\underline{x},\underline{x}\},\{\underline{y},\underline{y}\}\} \mid \dot{\neg}\, c_{\mathrm{Ph}} \in \underline{x} \Rightarrow \dot{\neg}\, c_{\mathrm{Ph}} \in \underline{y}\} = \emptyset \oplus \forall \underline{a}\colon \forall \underline{b}\colon \lambda x.\mathrm{Dedu}_0(\lceil \underline{a} \rceil, \lceil \underline{b} \rceil) \Vdash \underline{a} \vdash \underline{b} \oplus \forall \underline{s}\colon \forall \underline{x}\colon \forall \underline{y}\colon \dot{\neg}\, \underline{s} \in \{\underline{x},\underline{y}\} \Rightarrow \dot{\neg}\, \underline{s}=\underline{x} \Rightarrow \underline{s}=\underline{y} \Rightarrow \dot{\neg}\, \dot{\neg}\, \underline{s}=\underline{x} \Rightarrow \underline{s}=\underline{y} \Rightarrow \underline{s} \in \{\underline{x},\underline{y}\} \oplus \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \Rightarrow \underline{b} \vdash \underline{a} \vdash \underline{b} \oplus \forall \underline{x}\colon \forall \underline{t}\colon \forall \underline{a}\colon \forall \underline{b}\colon \langle \lceil \underline{a} \rceil \equiv^0 \lceil \underline{b} \rceil | \lceil \underline{x} \rceil := \lceil \underline{t} \rceil \rangle_{\mathrm{Ex}} \Vdash \underline{a} \vdash \underline{b} \oplus \forall \underline{s}\colon \forall \underline{x}\colon \dot{\neg}\, \underline{s} \in \cup \underline{x} \Rightarrow \dot{\neg}\, \underline{s} \in j_{\mathrm{Ex}} \Rightarrow \dot{\neg}\, j_{\mathrm{Ex}} \in \underline{x} \Rightarrow \dot{\neg}\, \dot{\neg}\, \underline{s} \in j_{\mathrm{Ex}} \Rightarrow \dot{\neg}\, j_{\mathrm{Ex}} \in \underline{x} \Rightarrow \underline{s} \in \cup \underline{x} \oplus \forall \underline{x}\colon \forall \underline{a}\colon \underline{a} \vdash \forall_{\mathrm{obj}} \underline{x}\colon \underline{a} \oplus \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{p}\colon \forall \underline{x}\colon \forall \underline{z}\colon \underline{p}^{\mathrm{Ph}} \wedge \langle \lceil \underline{b} \rceil \equiv^0 \lceil \underline{a} \rceil | \lceil \underline{p} \rceil := \lceil \underline{z} \rceil \rangle_{\mathrm{Ph}} \Vdash \dot{\neg}\, \underline{z} \in \{\mathrm{ph} \in \underline{x} \mid \underline{a}\} \Rightarrow \dot{\neg}\, \underline{z} \in \underline{x} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \dot{\neg}\, \dot{\neg}\, \underline{z} \in \underline{x} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{z} \in \{\mathrm{ph} \in \underline{x} \mid \underline{a}\} \oplus \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg}\, \underline{b} \Rightarrow \underline{a} \vdash \dot{\neg}\, \underline{b} \Rightarrow \dot{\neg}\, \underline{a} \vdash \underline{b} \oplus \forall \underline{s}\colon \dot{\neg}\, \underline{s} \in \emptyset]$$

$$[\mathrm{MP} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \Rightarrow \underline{b} \vdash \underline{a} \vdash \underline{b}][\mathrm{MP} \overset{\mathrm{proof}}{\rightarrow} \text{Rule tactic}]$$

$$[\mathrm{Gen} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{a}\colon \underline{a} \vdash \forall_{\mathrm{obj}} \underline{x}\colon \underline{a}][\mathrm{Gen} \overset{\mathrm{proof}}{\rightarrow} \text{Rule tactic}]$$

$$[\mathrm{Repetition} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{a}\colon \underline{a} \vdash \underline{a}][\mathrm{Repetition} \overset{\mathrm{proof}}{\rightarrow} \text{Rule tactic}]$$

$$[\mathrm{Neg} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg}\, \underline{b} \Rightarrow \underline{a} \vdash \dot{\neg}\, \underline{b} \Rightarrow \dot{\neg}\, \underline{a} \vdash \underline{b}][\mathrm{Neg} \overset{\mathrm{proof}}{\rightarrow} \text{Rule tactic}]$$

$$[\mathrm{Ded} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \lambda x.\mathrm{Dedu}_0(\lceil \underline{a} \rceil, \lceil \underline{b} \rceil) \Vdash \underline{a} \vdash \underline{b}][\mathrm{Ded} \overset{\mathrm{proof}}{\rightarrow} \text{Rule tactic}]$$

[7] En fodnote om stil: Jeg vil ofte indramme matematiske udtryk i firkantede parenteser "[ ]" for at adskille udtrykkene fra den omgivende tekst. De firkantede parenteser har ingen selvstændig betydning.

Først et par ord om syntaks: Konstruktionen $[\mathsf{x} \vdash \mathsf{y}]$ står for inferens, altså at vi kan bevise y, hvis vi har et bevis for x. Konstruktionen $[\mathsf{x} \Vdash \mathsf{y}]$ betyder, at y gælder, hvis sidebetingelsen x er sand. Endelig er konstruktionen $[\forall \mathsf{x}\colon \mathsf{y}]$ en meta-alkvantor; betydningen er, at metavariablen x kan instantieres til hvad som helst — selv andre metavariable.
De viste slutningsregler er næsten identiske med reglerne MP, Gen, Neg og Ded i systemet S fra [4]. Der er dog to ændringer. For det første har jeg tilføjet $[\forall \underline{\mathsf{a}}\colon \underline{\mathsf{a}} \vdash \underline{\mathsf{a}}]$ som en slutningsregel. (I [4] vises $[\forall \underline{\mathsf{a}}\colon \underline{\mathsf{a}} \vdash \underline{\mathsf{a}}]$ med et lav-niveau bevis). For det andet har jeg lavet en lille ændring i deduktionsreglen Ded, som gør den i stand til at håndtere sidebetingelser bedre. Jeg beskriver ændringen i bilag C.
Som nævnt i [4] kan deduktionsreglen erstatte enhver anvendelse af aksiomskemaerne A4 og A5 fra [6] (se evt. Mendelsons system på s. 69). Derfor har jeg ikke medtaget disse aksiomskemaer. På denne måde får vi også afprøvet deduktionsreglens brugervenlighed; vi skal straks se et eksempel, hvor A4 kunne have gjort nytte.

4.1.1 Håndtering af eksistenskvantorer

Et spørgsmål er nemlig, hvordan man skal håndtere introduktion og elimination af eksistenskvantorer — altså slutninger som f.eks. $[\emptyset = \emptyset \vdash \exists \overline{\mathsf{x}}\colon \overline{\mathsf{x}} = \overline{\mathsf{x}}]$ og $[\exists \overline{\mathsf{x}}\colon \overline{\mathsf{x}} = \overline{\mathsf{x}} \vdash \overline{\mathsf{c}} = \overline{\mathsf{c}}]$ (hvor "$\overline{\mathsf{c}}$" er et ikke tidligere anvendt navn på en konstant).
Det er muligt at håndtere denne slags slutninger alene med reglerne fra afsnit 4.1, men det er omstændigt og tidskrævende (sammenlign f.eks. de to beviser på s. 81 i [6]). Som et minimum kræver det, at man har A4 fra [6] til rådighed — og som nævnt ovenfor har jeg valgt ikke at medtage dette aksiomskema.
Jeg har i stedet implementeret en løsning, der er baseret på begrebet "eksistensvariabel". Vi indfører en unær operator $[\mathsf{x}^{\mathrm{Ex}}]$ og definerer, at en term er en ekistens-variabel, hviss den har $[\mathsf{x}^{\mathrm{Ex}}]$ som principal operator. Funktionen $[\mathsf{x}^{\mathrm{Ex}}]$ tester, om x er en eksistens-variabel:

$$[\mathsf{x}^{\mathrm{Ex}} \stackrel{\mathrm{val}}{\rightarrow} \mathsf{x} \stackrel{\mathrm{r}}{=} \lceil \mathsf{x}_{\mathrm{Ex}} \rceil]^{8}$$

Vi kan da definere de fire eksistens-variable, som denne rapport vil gøre brug af (jvf. bilag A):

$$[\mathrm{Ex}_1 \stackrel{\mathrm{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil [\mathrm{Ex}_1 \stackrel{..}{=} \mathsf{a}_{\mathrm{Ex}}] \rceil)]^{9}$$
$$[\mathrm{Ex}_2 \stackrel{\mathrm{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil [\mathrm{Ex}_2 \stackrel{..}{=} \mathsf{b}_{\mathrm{Ex}}] \rceil)]$$
$$[\mathrm{Ex}_{10} \stackrel{\mathrm{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil [\mathrm{Ex}_{10} \stackrel{..}{=} \mathsf{j}_{\mathrm{Ex}}] \rceil)]$$

[8] Konstruktionen $[\mathsf{x} \stackrel{\mathrm{val}}{\rightarrow} \mathsf{y}]$ er en såkaldt "værdidefinition" i L (jvf. afsnit 2.3). Den svarer til en almindelig funktionsdefinition; vi knytter funktionssignaturen x til kroppen y.

[9] Konstruktionen $[\mathsf{x} \stackrel{\mathrm{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil [\mathsf{x} \stackrel{..}{=} \mathsf{y}] \rceil)]$ står for "makrodefinition" i L. Vi definerer x som værende en forkortelse for y. Ud fra bevischeckerens synspunkt er der ingen forskel på x og y.

$[\mathrm{Ex}_{20} \overset{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathrm{Ex}_{20} \ddot{=} \mathsf{t}_{\mathrm{Ex}}]\rceil)]$[10]

Ideen med disse definitioner er at repræsentere en formel som f.eks. $[\exists \bar{\mathsf{x}}\colon \bar{\mathsf{x}} = \bar{\mathsf{x}}]$ ved formlen $[\underline{\mathsf{x}} = \underline{\mathsf{x}}]$, hvor "$\underline{\mathsf{x}}$" er en eksistensvariabel. På denne måde kommer eksistensvariablene til at fungere som erstatning for eksistenskvantoren. Dette betyder også, at der ikke er brug for at nogen regel, der eksplicit fjerner eksistenskvantorer — de er allerede væk.
Til gengæld får vi brug for en regel, der kan introducere eksistensvariable. Til den ende definerer vi prædikatet $\langle\lceil \mathsf{a}\rceil \equiv^0 \lceil \mathsf{b}\rceil | \lceil \mathsf{x}\rceil := \lceil \mathsf{t}\rceil\rangle_{\mathrm{Ex}}$ i L:

$[\langle \mathsf{a} \equiv \mathsf{b} | \mathsf{x} := \mathsf{t}\rangle_{\mathrm{Ex}} \overset{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\langle \mathsf{a} \equiv \mathsf{b} | \mathsf{x} := \mathsf{t}\rangle_{\mathrm{Ex}} \ddot{=}$
$\langle\lceil \mathsf{a}\rceil \equiv^0 \lceil \mathsf{b}\rceil | \lceil \mathsf{x}\rceil := \lceil \mathsf{t}\rceil\rangle_{\mathrm{Ex}}]\rceil)]$

$[\langle \mathsf{a} \equiv^0 \mathsf{b} | \mathsf{x} := \mathsf{t}\rangle_{\mathrm{Ex}} \overset{\text{val}}{\rightarrow} \lambda \mathsf{c}.\mathsf{x}^{\mathrm{Ex}} \wedge \langle \mathsf{a} \equiv^1 \mathsf{b} | \mathsf{x} := \mathsf{t}\rangle_{\mathrm{Ex}}]$

$[\langle \mathsf{a} \equiv^1 \mathsf{b} | \mathsf{x} := \mathsf{t}\rangle_{\mathrm{Ex}} \overset{\text{val}}{\rightarrow} \mathsf{a}!\mathsf{x}!\mathsf{t}!$
$\mathrm{If}(\mathsf{b} \overset{\text{r}}{=} \lceil \forall_{\mathrm{obj}} \mathsf{u}\colon \mathsf{v}\rceil, \mathsf{F},$
$\mathrm{If}(\mathsf{b}^{\mathrm{Ex}} \wedge \mathsf{b} \overset{\text{t}}{=} \mathsf{x}, \mathsf{a} \overset{\text{t}}{=} \mathsf{t}, \mathrm{If}($
$\mathsf{a} \overset{\text{r}}{=} \mathsf{b}, \langle \mathsf{a}^{\mathsf{t}} \equiv^* \mathsf{b}^{\mathsf{t}} | \mathsf{x} := \mathsf{t}\rangle_{\mathrm{Ex}}, \mathsf{F})))]$

$[\langle \mathsf{a} \equiv^* \mathsf{b} | \mathsf{x} := \mathsf{t}\rangle_{\mathrm{Ex}} \overset{\text{val}}{\rightarrow} \mathsf{b}!\mathsf{x}!\mathsf{t}!\mathrm{If}(\mathsf{a}, \mathsf{T}, \mathrm{If}(\langle \mathsf{a}^{\mathsf{h}} \equiv^1 \mathsf{b}^{\mathsf{h}} | \mathsf{x} := \mathsf{t}\rangle_{\mathrm{Ex}}, \langle \mathsf{a}^{\mathsf{t}} \equiv^* \mathsf{b}^{\mathsf{t}} | \mathsf{x} := \mathsf{t}\rangle_{\mathrm{Ex}}, \mathsf{F}))]$

Prædikatet $\langle\lceil \mathsf{a}\rceil \equiv^0 \lceil \mathsf{b}\rceil | \lceil \mathsf{x}\rceil := \lceil \mathsf{t}\rceil\rangle_{\mathrm{Ex}}$ er sandt, hvis x er en eksistens-variabel, og hvis formlen a er identisk med resultatet af at erstatte alle forekomster af x i formlen b med termen t. F.eks. er $\langle\lceil \emptyset = \emptyset\rceil \equiv^0 \lceil \mathsf{a}_{\mathrm{Ex}} = \mathsf{a}_{\mathrm{Ex}}\rceil | \lceil \mathsf{a}_{\mathrm{Ex}}\rceil := \lceil \emptyset\rceil\rangle_{\mathrm{Ex}}$ sand. Herudover er det et krav, at hverken a eller b må indeholde objektkvantorer; således er $\langle\lceil \forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \bar{\mathsf{s}} = \emptyset\rceil \equiv^0 \lceil \forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \bar{\mathsf{s}} = \mathsf{a}_{\mathrm{Ex}}\rceil | \lceil \mathsf{a}_{\mathrm{Ex}}\rceil := \lceil \emptyset\rceil\rangle_{\mathrm{Ex}}$ falsk. Dette krav er udelukkende indført for at gøre koden for $\langle\lceil \mathsf{a}\rceil \equiv^0 \lceil \mathsf{b}\rceil | \lceil \mathsf{x}\rceil := \lceil \mathsf{t}\rceil\rangle_{\mathrm{Ex}}$ så simpel som mulig; det har ikke været nødvendigt at sætte $\langle\lceil \mathsf{a}\rceil \equiv^0 \lceil \mathsf{b}\rceil | \lceil \mathsf{x}\rceil := \lceil \mathsf{t}\rceil\rangle_{\mathrm{Ex}}$ i stand til at håndtere kvantificering.
Vi kan nu definere den slutningsregel, der står for introduktion af eksistensvariable:

$[\mathrm{ExistIntro} \overset{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{t}}\colon \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \langle\lceil \underline{\mathsf{a}}\rceil \equiv^0 \lceil \underline{\mathsf{b}}\rceil | \lceil \underline{\mathsf{x}}\rceil := \lceil \underline{\mathsf{t}}\rceil\rangle_{\mathrm{Ex}} \Vdash \underline{\mathsf{a}} \vdash$
$\underline{\mathsf{b}}][\mathrm{ExistIntro} \overset{\text{proof}}{\rightarrow} \text{Rule tactic}]$ [11]

Med ExistIntro kan vi f.eks. slutte $[\mathsf{a}_{\mathrm{Ex}} = \mathsf{a}_{\mathrm{Ex}}]$ ud fra $[\emptyset = \emptyset]$. Vi har nu defineret seks aksiomer, der tilsammen dækker 1. ordens prædikatkalkyle.

4.2 Selve mængdelæren

De aksiomer i ZFsub, der vedrører selve mængdelæren, har jeg hentet fra kapitel 4.3 og 4.4 i [2]. Her er de første fem:

[10] "j" og "t" er hhv. bogstav nr. 10 og 20 i alfabetet.
[11] I denne definition varierer "$\underline{\mathsf{x}}$" over eksistens-variable.

$[\text{Extensionality} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \dot\neg\, \underline{\mathsf{x}} = \underline{\mathsf{y}} \Rightarrow \forall_{\text{obj}} \bar{\mathsf{s}}\colon \dot\neg\, \bar{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{y}} \Rightarrow \dot\neg\, \bar{\mathsf{s}} \in \underline{\mathsf{y}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \dot\neg \forall_{\text{obj}} \bar{\mathsf{s}}\colon \dot\neg\, \bar{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{y}} \Rightarrow \dot\neg\, \bar{\mathsf{s}} \in \underline{\mathsf{y}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \underline{\mathsf{x}} = \underline{\mathsf{y}}][\text{Extensionality} \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[\text{Ødef} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{s}}\colon \dot\neg\, \underline{\mathsf{s}} \in \emptyset][\text{Ødef} \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[\text{PairDef} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{s}}\colon \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \dot\neg\, \underline{\mathsf{s}} \in \{\underline{\mathsf{x}}, \underline{\mathsf{y}}\} \Rightarrow \dot\neg\, \underline{\mathsf{s}} = \underline{\mathsf{x}} \Rightarrow \underline{\mathsf{s}} = \underline{\mathsf{y}} \Rightarrow \dot\neg \dot\neg\, \underline{\mathsf{s}} = \underline{\mathsf{x}} \Rightarrow \underline{\mathsf{s}} = \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{s}} \in \{\underline{\mathsf{x}}, \underline{\mathsf{y}}\}][\text{PairDef} \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[\text{UnionDef} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{s}}\colon \forall \underline{\mathsf{x}}\colon \dot\neg\, \underline{\mathsf{s}} \in \cup \underline{\mathsf{x}} \Rightarrow \dot\neg\, \underline{\mathsf{s}} \in \mathsf{j}_{\text{Ex}} \Rightarrow \dot\neg\, \mathsf{j}_{\text{Ex}} \in \underline{\mathsf{x}} \Rightarrow \dot\neg \dot\neg\, \underline{\mathsf{s}} \in \mathsf{j}_{\text{Ex}} \Rightarrow \dot\neg\, \mathsf{j}_{\text{Ex}} \in \underline{\mathsf{x}} \Rightarrow \underline{\mathsf{s}} \in \cup \underline{\mathsf{x}}][\text{UnionDef} \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[\text{PowerDef} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{s}}\colon \forall \underline{\mathsf{x}}\colon \dot\neg\, \underline{\mathsf{s}} \in \text{P}(\underline{\mathsf{x}}) \Rightarrow \forall_{\text{obj}} \bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{s}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \dot\neg \forall_{\text{obj}} \bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{s}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \underline{\mathsf{s}} \in \text{P}(\underline{\mathsf{x}})][\text{PowerDef} \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$[12]

Reglen Extensionality siger, at to mængder er ens, hviss de har de samme elementer. De øvrige fire regler definerer begreberne "tom mængde", "par", "foreningssmængde" og "potensmængde". Læg mærke til, at $[\cup \underline{\mathsf{t}}]$ er en unær operator; ideen er, at $\cup \underline{\mathsf{t}}$ er lig med foreningsmængden af alle de mængder, som $\underline{\mathsf{t}}$ indeholder.
Bemærk også at to af aksiomerne indeholder objektvariablen $\bar{\mathsf{s}}$. Forklaringen herpå er, at aksiomerne indeholder objektkvantoren $[\forall_{\text{obj}} \mathsf{x}\colon \mathsf{y}]$; og deduktionsreglen fra [4] er ikke egnet til at håndtere kombinationen "objektkvantor og metavariabel". Derfor vil x i $[\forall_{\text{obj}} \mathsf{x}\colon \mathsf{y}]$ altid være en objektvariabel i denne rapport.

4.2.1 Separation og pladsholdervariable

Vi mangler stadigvæk et "separationsaksiom" — dvs. et aksiom, der giver mening til konstruktionen $[\{\text{ph} \in \underline{\mathsf{t}} \mid \underline{\mathsf{f}}\}]$ fra afsnit 3.1. For at implementere separationsaksiomet indfører vi begrebet "pladsholder-variabel".
Fremgangsmåden er den samme som ved eksistens-variable i afsnit 4.1.1. Først indfører vi en unær operator $[\mathsf{x}_{\text{Ph}}]$ og definerer, at en term er en pladsholder-variabel, hviss den har $[\mathsf{x}_{\text{Ph}}]$ som primær operator. Funktionen $[\mathsf{x}^{\text{Ph}}]$ checker, om x er en pladsholder-variabel:

$$[\mathsf{x}^{\text{Ph}} \stackrel{\text{val}}{\rightarrow} \mathsf{x} \stackrel{\text{r}}{=} \lceil \mathsf{x}_{\text{Ph}} \rceil]$$

Vi kan nu definere de pladsholder-variable, vi får brug for, som følger (jvf. bilag A):

$$[\text{ph}_1 \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil [\text{ph}_1 \ddot{=} \mathsf{a}_{\text{Ph}}] \rceil)]$$
$$[\text{ph}_2 \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil [\text{ph}_2 \ddot{=} \mathsf{b}_{\text{Ph}}] \rceil)]$$
$$[\text{ph}_3 \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil [\text{ph}_3 \ddot{=} \mathsf{c}_{\text{Ph}}] \rceil)]$$

[12] Konstruktionerne $[\dot\neg \mathsf{x} \Rightarrow \dot\neg \mathsf{y}]$, $[\dot\neg \mathsf{x} \Rightarrow \mathsf{y}]$ og $[\dot\neg \mathsf{x} \Rightarrow \mathsf{y} \Rightarrow \dot\neg \mathsf{y} \Rightarrow \mathsf{x}]$ svarer til de kendte konnektiver; de bliver defineret formelt i afsnit 5.1.

Så definerer vi prædikatet $\langle\lceil \mathsf{a}\rceil{\equiv^0}\lceil \mathsf{b}\rceil|\lceil \mathsf{x}\rceil{:=}\lceil \mathsf{t}\rceil\rangle_{\mathrm{Ph}}$ i L:

$[\langle \mathsf{a}{\equiv}\mathsf{b}|\mathsf{x}{:=}\mathsf{t}\rangle_{\mathrm{Ph}} \stackrel{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\langle \mathsf{a}{\equiv}\mathsf{b}|\mathsf{x}{:=}\mathsf{t}\rangle_{\mathrm{Ph}} \ddot{=}$
$\langle\lceil \mathsf{a}\rceil{\equiv^0}\lceil \mathsf{b}\rceil|\lceil \mathsf{x}\rceil{:=}\lceil \mathsf{t}\rceil\rangle_{\mathrm{Ph}}]\rceil)]$

$[\langle \mathsf{a}{\equiv^0}\mathsf{b}|\mathsf{x}{:=}\mathsf{t}\rangle_{\mathrm{Ph}} \stackrel{\mathrm{val}}{\to} \lambda \mathsf{c}.\mathsf{x}^{\mathrm{Ph}} \wedge \langle \mathsf{a}{\equiv^1}\mathsf{b}|\mathsf{x}{:=}\mathsf{t}\rangle_{\mathrm{Ph}}]$

$[\langle \mathsf{a}{\equiv^1}\mathsf{b}|\mathsf{x}{:=}\mathsf{t}\rangle_{\mathrm{Ph}} \stackrel{\mathrm{val}}{\to} \mathsf{a}!\mathsf{x}!\mathsf{t}!$
$\mathrm{If}(\mathsf{b} \stackrel{\mathrm{r}}{=} \lceil\forall_{\mathrm{obj}}\mathsf{u}{:}\,\mathsf{v}\rceil, \mathsf{F},$
$\mathrm{If}(\mathsf{b}^{\mathrm{Ph}} \wedge \mathsf{b} \stackrel{\mathrm{t}}{=} \mathsf{x}, \mathsf{a} \stackrel{\mathrm{t}}{=} \mathsf{t},$
$\mathrm{If}(\mathsf{b}^{\mathrm{Ex}}, \mathsf{a} \stackrel{\mathrm{r}}{=} \mathsf{b}, \mathrm{If}($
$\mathsf{a} \stackrel{\mathrm{r}}{=} \mathsf{b}, \langle \mathsf{a}^{\mathrm{t}}{\equiv^*}\mathsf{b}^{\mathrm{t}}|\mathsf{x}{:=}\mathsf{t}\rangle_{\mathrm{Ph}}, \mathsf{F}))))]$

$[\langle \mathsf{a}{\equiv^*}\mathsf{b}|\mathsf{x}{:=}\mathsf{t}\rangle_{\mathrm{Ph}} \stackrel{\mathrm{val}}{\to} \mathsf{b}!\mathsf{x}!\mathsf{t}!\mathrm{If}(\mathsf{a}, \mathsf{T}, \mathrm{If}(\langle \mathsf{a}^{\mathrm{h}}{\equiv^1}\mathsf{b}^{\mathrm{h}}|\mathsf{x}{:=}\mathsf{t}\rangle_{\mathrm{Ph}}, \langle \mathsf{a}^{\mathrm{t}}{\equiv^*}\mathsf{b}^{\mathrm{t}}|\mathsf{x}{:=}\mathsf{t}\rangle_{\mathrm{Ph}}, \mathsf{F}))]$

Prædikatet $\langle\lceil \mathsf{a}\rceil{\equiv^0}\lceil \mathsf{b}\rceil|\lceil \mathsf{x}\rceil{:=}\lceil \mathsf{t}\rceil\rangle_{\mathrm{Ph}}$ er sandt, hvis x er en pladsholder-variabel, og hvis formlen a er identisk med resultatet af at erstatte alle forekomster af x i formlen b med termen t. Ligesom ved $\langle\lceil \mathsf{a}\rceil{\equiv^0}\lceil \mathsf{b}\rceil|\lceil \mathsf{x}\rceil{:=}\lceil \mathsf{t}\rceil\rangle_{\mathrm{Ex}}$ kræver vi, at hverken a eller b indeholder objektkvantorer. F.eks. er
$\langle\lceil \emptyset \in \underline{\mathsf{s}}\rceil{\equiv^0}\lceil \mathsf{a}_{\mathrm{Ph}} \in \underline{\mathsf{s}}\rceil|\lceil \mathsf{a}_{\mathrm{Ph}}\rceil{:=}\lceil\emptyset\rceil\rangle_{\mathrm{Ph}}$ sand. Det er i øvrigt tilladt, at a og b indeholder forskellige ekistensvariable; således er
$\langle\lceil \mathsf{a}_{\mathrm{Ex}}\rceil{\equiv^0}\lceil \mathsf{b}_{\mathrm{Ex}}\rceil|\lceil \mathsf{a}_{\mathrm{Ph}}\rceil{:=}\lceil\emptyset\rceil\rangle_{\mathrm{Ph}}$ sand. På denne måde kan vi imødekomme kravet om, at en eksistens-variabel skal være ubrugt, når den introduceres i et bevis.
Vi definerer nu separationsaksiomet SeparationDef ud fra
$\langle\lceil \mathsf{a}\rceil{\equiv^0}\lceil \mathsf{b}\rceil|\lceil \mathsf{x}\rceil{:=}\lceil \mathsf{t}\rceil\rangle_{\mathrm{Ph}}$:

> $[\mathrm{SeparationDef} \stackrel{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall\underline{\mathsf{a}}{:}\,\forall\underline{\mathsf{b}}{:}\,\forall\underline{\mathsf{p}}{:}\,\forall\underline{\mathsf{x}}{:}\,\forall\underline{\mathsf{z}}{:}\,\underline{\mathsf{p}}^{\mathrm{Ph}} \wedge$
> $\langle\lceil\underline{\mathsf{b}}\rceil{\equiv^0}\lceil\underline{\mathsf{a}}\rceil|\lceil\underline{\mathsf{p}}\rceil{:=}\lceil\underline{\mathsf{z}}\rceil\rangle_{\mathrm{Ph}} \Vdash \dot{\neg}\,\underline{\mathsf{z}} \in \{\mathrm{ph} \in \underline{\mathsf{x}} \mid \underline{\mathsf{a}}\} \Rightarrow \dot{\neg}\,\underline{\mathsf{z}} \in \underline{\mathsf{x}} \Rightarrow \dot{\neg}\,\underline{\mathsf{b}} \Rightarrow \dot{\neg}\,\dot{\neg}\,\underline{\mathsf{z}} \in$
> $\underline{\mathsf{x}} \Rightarrow \dot{\neg}\,\underline{\mathsf{b}} \Rightarrow \underline{\mathsf{z}} \in \{\mathrm{ph} \in \underline{\mathsf{x}} \mid \underline{\mathsf{a}}\}][\mathrm{SeparationDef} \stackrel{\mathrm{proof}}{\to} \mathrm{Rule\ tactic}]$

Ved første øjekast virker konjunkten $[\underline{\mathsf{p}}^{\mathrm{Ph}}]$ i denne definition overflødig, fordi $\langle\lceil\underline{\mathsf{b}}\rceil{\equiv^0}\lceil\underline{\mathsf{a}}\rceil|\lceil\underline{\mathsf{p}}\rceil{:=}\lceil\underline{\mathsf{z}}\rceil\rangle_{\mathrm{Ph}}$ i forvejen kræver, at $\underline{\mathsf{p}}$ er en pladsholder-variabel. Forklaringen er, at "$\underline{\mathsf{p}}$" kun optræder i definitionens sidebetingelse. Dette betyder, at bevischeckeren har svært ved at finde ud af, hvordan "$\underline{\mathsf{p}}$" skal instantieres, når vi anvender definitionen. Det ekstra krav $[\underline{\mathsf{p}}^{\mathrm{Ph}}]$ hjælper bevischeckeren til at instantiere "$\underline{\mathsf{p}}$" korrekt.
Med SeparationDef til rådighed kan vi nu f.eks. definere $\{\mathrm{ph} \in \mathsf{x} \mid \emptyset \in \mathsf{a}_{\mathrm{Ph}}\}$ som den delmængde af x, hvis elementer indeholder $\emptyset$:

> $[\mathrm{ContainsEmpty}(\mathsf{x}) \stackrel{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathrm{ContainsEmpty}(\mathsf{x}) \ddot{=}$
> $\{\mathrm{ph} \in \mathsf{x} \mid \emptyset \in \mathrm{ph}_1\}]\rceil)],$

og vi kan slutte $[\dot{\neg}\,\underline{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \dot{\neg}\,\emptyset \in \underline{\mathsf{s}}]$ ud fra $[\underline{\mathsf{s}} \in \{\mathrm{ph} \in \underline{\mathsf{x}} \mid \emptyset \in \mathsf{a}_{\mathrm{Ph}}\}]$.
Vi bruger altså en pladsholder-variabel som fri variabel i dén formel $\underline{\mathsf{f}}$, der definerer delmængden $\{\mathrm{ph} \in \underline{\mathsf{x}} \mid \underline{\mathsf{f}}\}$. Ideen med at bruge pladsholder-variable

(frem for objekt- eller metavariable) til dette formål er, at vi ikke ønsker at kvantificere eller instantiere den frie variabel i $\underline{\mathsf{f}}$. Det er f.eks. noget sludder at skrive "$\forall_{\text{obj}}\mathsf{a}_{\text{Ph}}\colon\{\text{ph} \in \underline{\mathsf{x}} \mid \emptyset \in \mathsf{a}_{\text{Ph}}\}$" eller at konkludere "$\{\text{ph} \in \underline{\mathsf{x}} \mid \emptyset \in \emptyset\}$" ud fra definitionen af $\{\text{ph} \in \underline{\mathsf{x}} \mid \emptyset \in \mathsf{a}_{\text{Ph}}\}$. Den frie variabel i $\underline{\mathsf{f}}$ skal blot være en pladsholder; derfor pladsholder-variable.

5 Makrodefinitioner

Dette afsnit indeholder dé makrodefinitioner, som vi vil gøre brug af i resten af rapporten. Definitionerne drejer sig for det meste om mængdeteoretiske begreber, f.eks. "ækvivalensklasse" og "partition". Til sidst i afsnittet formulerer vi hovedresultatet — at der til enhver ækvivalensrelation svarer en partition — som et formelt teorem.

5.1 Konnektiver

Ud fra de to basale konnektiver $[\dot\neg\mathsf{x}]$ og $[\mathsf{x} \Rightarrow \mathsf{y}]$ definerer vi konjunktion, disjunktion og dobbeltimplikation:

$$[\mathsf{x} \mathrel{\dot\wedge} \mathsf{y} \overset{\text{macro}}{\to} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \mathrel{\dot\wedge} \mathsf{y} \mathrel{\ddot{=}} \dot\neg(\mathsf{x} \Rightarrow \dot\neg\mathsf{y})]\rceil)]$$
$$[\mathsf{x} \mathrel{\dot\vee} \mathsf{y} \overset{\text{macro}}{\to} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \mathrel{\dot\vee} \mathsf{y} \mathrel{\ddot{=}} \dot\neg\mathsf{x} \Rightarrow \mathsf{y}]\rceil)]$$
$$[\mathsf{x} \mathrel{\dot\Leftrightarrow} \mathsf{y} \overset{\text{macro}}{\to} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \mathrel{\dot\Leftrightarrow} \mathsf{y} \mathrel{\ddot{=}} (\mathsf{x} \Rightarrow \mathsf{y}) \mathrel{\dot\wedge} (\mathsf{y} \Rightarrow \mathsf{x})]\rceil)]$$

5.2 Negerede formler

Det er ganske enkelt at definere negeret lighed ($\dot\neg\mathsf{x}=\mathsf{y}$) og negeret medlemskab ($\dot\neg\mathsf{x} \in \mathsf{y}$):

$$[\mathsf{x} \neq \mathsf{y} \overset{\text{macro}}{\to} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \neq \mathsf{y} \mathrel{\ddot{=}} \dot\neg\mathsf{x}=\mathsf{y}]\rceil)]$$
$$[\mathsf{x} \notin \mathsf{y} \overset{\text{macro}}{\to} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \notin \mathsf{y} \mathrel{\ddot{=}} \dot\neg\mathsf{x} \in \mathsf{y}]\rceil)]$$ [13]

5.3 Delmængde

Mængden x er en delmængde af y hviss ethvert medlem af x også tilhører y:

$$[\mathsf{x} \subseteq \mathsf{y} \overset{\text{macro}}{\to} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \subseteq \mathsf{y} \mathrel{\ddot{=}} (s \in \mathsf{x} \Rightarrow s \in \mathsf{y})]\rceil)]$$

Vi kommer ikke til at bruge denne definition ret ofte. Man får tit en bedre føling med, hvad der foregår i beviserne, hvis man skriver definitionen ud. Desuden bruger denne definition af $[\bar{\mathsf{s}} \in \mathsf{x} \Rightarrow \bar{\mathsf{s}} \in \mathsf{y}]$ objektvariable og implikation; vi vil ofte foretrække at bruge metavariable og inferens i stedet (som f.eks. i $[\underline{\mathsf{s}} \in \mathsf{x} \vdash \underline{\mathsf{s}} \in \mathsf{y}]$).

5.4 Singleton-mængde

$[\{\mathsf{x},\mathsf{x}\}]$ er mængden, der indeholder x som sit eneste element. Vi definerer $[\{\mathsf{x},\mathsf{x}\}]$ ved at parre x med sig selv:

[13] Højresiderne i disse definitioner skal læses som hhv. $[\dot\neg\mathsf{x}=\mathsf{y}]$ og $[\dot\neg\mathsf{x} \in \mathsf{y}]$.

$$[\{\mathsf{x}\} \stackrel{\text{macro}}{\to} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\{\mathsf{x}\} \ddot{=} \{\mathsf{x},\mathsf{x}\}]\rceil)]$$

5.5 Binær foreningsmængde og fællesmængde

Vi definerer foreningsmængden mellem to mængder x og y som følger:

$$[\mathsf{x} \cup \mathsf{y} \stackrel{\text{macro}}{\to} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \cup \mathsf{y} \ddot{=} \cup\{\{\mathsf{x}\},\{\mathsf{y}\}\}]\rceil)]$$

Fællesmængden mellem to mængder x og y er en delmængde af deres foreningsmængde:

$$[\mathsf{x}\cap\mathsf{y} \stackrel{\text{macro}}{\to} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x}\cap\mathsf{y} \ddot{=} \{\mathrm{ph} \in \mathsf{x} \cup \mathsf{y} \mid \mathrm{ph}_3 \in \mathsf{x} \dot{\wedge} \mathrm{ph}_3 \in \mathsf{y}\}]\rceil)]$$

5.6 Relation

Det ordnede par $\{\{\mathsf{x},\mathsf{x}\},\{\mathsf{x},\mathsf{y}\}\}$ indeholder x som "førstekomponent" og y som "andenkomponent". Den følgende definition af $\{\{\mathsf{x},\mathsf{x}\},\{\mathsf{x},\mathsf{y}\}\}$ er den mest udbredte i litteraturen (se f.eks. afsnit 4.3 i [2] og afsnit 2.1 i [5]):

$$[\langle\mathsf{x},\mathsf{y}\rangle \stackrel{\text{macro}}{\to} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\langle\mathsf{x},\mathsf{y}\rangle \ddot{=} \{\{\mathsf{x}\},\{\mathsf{x},\mathsf{y}\}\}]\rceil)]$$

Vi kan nu definere en "relation" som en mængde af ordnede par. Vi udtrykker denne definition ved at formalisere, hvad det vil sige, at x er relateret til y i kraft af relationen r:

$$[\mathsf{r}(\mathsf{x},\mathsf{y}) \stackrel{\text{macro}}{\to} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{r}(\mathsf{x},\mathsf{y}) \ddot{=} \langle\mathsf{x},\mathsf{y}\rangle \in \mathsf{r}]\rceil)]$$

Vi kommer faktisk ikke til at bruge disse to definitioner i rapporten; beviserne vil behandle $[\{\{\mathsf{x},\mathsf{x}\},\{\mathsf{x},\mathsf{y}\}\} \in \mathsf{r}]$ som en primitiv konstruktion. Men det er alligevel betryggende at have det formelle grundlag for relationsbegrebet på plads.

5.7 Ækvivalensrelation

At en relation er refleksiv på en mængde x vil sige, at alle elementer i x er relateret til sig selv:

$$[\mathrm{ReflRel}(\mathsf{r},\mathsf{x}) \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathrm{ReflRel}(\mathsf{r},\mathsf{x}) \ddot{=} \forall s\colon (s \in \mathsf{x} \Rightarrow \mathsf{r}(s,s))]\rceil)]$$

At en relation er symmetrisk på en mængde x vil sige, at alle elementer i x opfylder den følgende implikation:

$$[\mathrm{SymRel}(\mathsf{r},\mathsf{x}) \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathrm{SymRel}(\mathsf{r},\mathsf{x}) \ddot{=} \forall s, t\colon (s \in \mathsf{x} \Rightarrow t \in \mathsf{x} \Rightarrow \mathsf{r}(s,t) \Rightarrow \mathsf{r}(t,s))]\rceil)]$$

At en relation er transitiv på en mængde x vil sige, at alle elementer i x opfylder den følgende implikation:

$$[\mathrm{TransRel}(\mathsf{r},\mathsf{x}) \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathrm{TransRel}(\mathsf{r},\mathsf{x}) \ddot{=} \forall s, t, u\colon (s \in \mathsf{x} \Rightarrow t \in \mathsf{x} \Rightarrow u \in \mathsf{x} \Rightarrow \mathsf{r}(s,t) \Rightarrow \mathsf{r}(t,u) \Rightarrow \mathsf{r}(s,u))]\rceil)]$$

Endelig er en ækvivalensrelation det samme som en relation, der er refleksiv, symmetrisk og transitiv:

$$[\mathrm{EqRel}(\mathsf{r},\mathsf{x}) \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathrm{EqRel}(\mathsf{r},\mathsf{x}) \ddot{=} \mathrm{ReflRel}(\mathsf{r},\mathsf{x}) \dot{\wedge} \mathrm{SymRel}(\mathsf{r},\mathsf{x}) \dot{\wedge} \mathrm{TransRel}(\mathsf{r},\mathsf{x})]\rceil)]$$

5.8 Mængde-variable

Mange af rapportens beviser sker i forhold til en uspecificeret mængde. Vi vil referere til denne mængde med metavariablen $\underline{\mathsf{bs}}$ og objektvariablen $\overline{\mathsf{bs}}$:

$$[\mathcal{BS} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathcal{BS} \ddot{=} \underline{\mathsf{bs}}]\rceil)]$$

$$[\mathsf{OBS} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{OBS} \ddot{=} \overline{\mathsf{bs}}]\rceil)]$$[14]

Vi vil så vidt muligt bruge metavariablen, men i afsnit 10.6 og senere bliver det nødvendigt at gå over til objektvariablen.

5.9 Ækvivalensklasse

Lad r være en ækvivalensrelation defineret på bs, og lad x være et medlem af bs. Vi definerer ækvivalensklassen $\{\mathrm{ph} \in \mathsf{bs} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{x}\}\} \in \mathsf{r}\}$ som den delmængde af bs, hvis medlemmer står i forhold til x:

$$[[\mathsf{x} \in \mathsf{bs}]_{\mathsf{r}} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[[\mathsf{x} \in \mathsf{bs}]_{\mathsf{r}} \ddot{=} \{\mathrm{ph} \in \mathsf{bs} \mid \mathsf{r}(\mathrm{ph}_1,\mathsf{x})\}]\rceil)]$$

[14] Navnene "$\underline{\mathsf{bs}}$" og "$\overline{\mathsf{bs}}$" står for hhv. for "big set" og "object big set". Konstruktionerne $[\underline{\mathsf{x}}]$ og $[\overline{\mathsf{x}}]$ omdanner x til hhv. en meta- og en objektvariabel. Variablen $[\mathsf{bs}]$ vil også blive brugt i nogle af de kommende definitioner, men ikke i selve beviserne.

Ækvivalenssystemet
$\{\mathrm{ph} \in \mathrm{P}(\mathsf{bs}) \mid \dot{\neg}\, \mathsf{t}_{\mathrm{Ex}} \in \mathsf{bs} \Rightarrow \dot{\neg}\, \{\mathrm{ph} \in \mathsf{bs} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \mathsf{r}\} = \mathsf{b}_{\mathrm{Ph}}\}$ er mængden af alle de ækvivalensklasser, som bs definerer på r. Vi definerer $\{\mathrm{ph} \in \mathrm{P}(\mathsf{bs}) \mid \dot{\neg}\, \mathsf{t}_{\mathrm{Ex}} \in \mathsf{bs} \Rightarrow \dot{\neg}\, \{\mathrm{ph} \in \mathsf{bs} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \mathsf{r}\} = \mathsf{b}_{\mathrm{Ph}}\}$ som en delmængde af potensmængden $\mathrm{P}(\mathsf{bs})$:

$$[\mathsf{bs}/\mathsf{r} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil [\mathsf{bs}/\mathsf{r} \ddot{=} \{\mathrm{ph} \in \mathrm{P}(\mathsf{bs}) \mid \mathrm{Ex}_{20} \in \mathsf{bs} \dot{\wedge} [\mathrm{Ex}_{20} \in \mathsf{bs}]_{\mathsf{r}} = \mathrm{ph}_2\}] \rceil)]$$

5.10 Partition

En partition af en mængde bs er en mængde p, som opfylder tre krav:

1. Ingen af mængderne i p er tomme.
2. Alle mængderne i p er indbyrdes disjunkte.
3. Foreningsmængden af alle mængderne i p er lig med bs.

Den formelle version af denne definition ser således ud:

$$[\text{Partition}(\mathsf{p},\mathsf{bs}) \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\text{Partition}(\mathsf{p},\mathsf{bs}) \ddot{=} (\forall s\colon (s \in \mathsf{p} \Rightarrow s \neq \emptyset)) \dot{\wedge}$$
$$(\forall s,t\colon (s \in \mathsf{p} \Rightarrow t \in \mathsf{p} \Rightarrow s \neq t \Rightarrow s \cap t = \emptyset)) \dot{\wedge}$$
$$\cup\, \mathsf{p} = \mathsf{bs}]\rceil)]$$

Vi kan nu formulere hovedresultatet som et formelt lemma (hvor vi bruger objektvariable):

$$[\text{EqSysIsPartition} \overset{\text{stmt}}{\to} \text{ZFsub} \vdash \dot\neg\dot\neg\forall_{\text{obj}}\bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \overline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{s}}\}\} \in \bar{\mathsf{r}} \Rightarrow$$
$$\dot\neg\forall_{\text{obj}}\bar{\mathsf{s}}\colon \forall_{\text{obj}}\bar{\mathsf{t}}\colon \bar{\mathsf{s}} \in \overline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \overline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\} \in \bar{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{s}}\}\} \in \bar{\mathsf{r}} \Rightarrow$$
$$\dot\neg\forall_{\text{obj}}\bar{\mathsf{s}}\colon \forall_{\text{obj}}\bar{\mathsf{t}}\colon \forall_{\text{obj}}\bar{\mathsf{u}}\colon \bar{\mathsf{s}} \in \overline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \overline{\mathsf{bs}} \Rightarrow \bar{\mathsf{u}} \in \overline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\} \in \bar{\mathsf{r}} \Rightarrow$$
$$\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{u}}\}\} \in \bar{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{u}}\}\} \in \bar{\mathsf{r}} \vdash \dot\neg\dot\neg\forall_{\text{obj}}\bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \{\text{ph} \in \text{P}(\overline{\mathsf{bs}}) \mid \dot\neg\, \mathsf{t}_{\text{Ex}} \in$$
$$\overline{\mathsf{bs}} \Rightarrow \dot\neg\{\text{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\text{Ph}},\mathsf{a}_{\text{Ph}}\},\{\mathsf{a}_{\text{Ph}},\mathsf{t}_{\text{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\text{Ph}}\} \Rightarrow \dot\neg\bar{\mathsf{s}} = \emptyset \Rightarrow$$
$$\dot\neg\forall_{\text{obj}}\bar{\mathsf{s}}\colon \forall_{\text{obj}}\bar{\mathsf{t}}\colon \bar{\mathsf{s}} \in \{\text{ph} \in \text{P}(\overline{\mathsf{bs}}) \mid \dot\neg\, \mathsf{t}_{\text{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot\neg\{\text{ph} \in \overline{\mathsf{bs}} \mid$$
$$\{\{\mathsf{a}_{\text{Ph}},\mathsf{a}_{\text{Ph}}\},\{\mathsf{a}_{\text{Ph}},\mathsf{t}_{\text{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\text{Ph}}\} \Rightarrow \bar{\mathsf{t}} \in \{\text{ph} \in \text{P}(\overline{\mathsf{bs}}) \mid \dot\neg\, \mathsf{t}_{\text{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot\neg\{\text{ph} \in$$
$$\overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\text{Ph}},\mathsf{a}_{\text{Ph}}\},\{\mathsf{a}_{\text{Ph}},\mathsf{t}_{\text{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\text{Ph}}\} \Rightarrow \dot\neg\bar{\mathsf{s}} = \bar{\mathsf{t}} \Rightarrow \{\text{ph} \in \cup\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{t}},\bar{\mathsf{t}}\}\} \mid$$
$$\dot\neg\, \mathsf{c}_{\text{Ph}} \in \bar{\mathsf{s}} \Rightarrow \dot\neg\, \mathsf{c}_{\text{Ph}} \in \bar{\mathsf{t}}\} = \emptyset \Rightarrow \dot\neg \cup \{\text{ph} \in \text{P}(\overline{\mathsf{bs}}) \mid \dot\neg\, \mathsf{t}_{\text{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot\neg\{\text{ph} \in \overline{\mathsf{bs}} \mid$$
$$\{\{\mathsf{a}_{\text{Ph}},\mathsf{a}_{\text{Ph}}\},\{\mathsf{a}_{\text{Ph}},\mathsf{t}_{\text{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\text{Ph}}\} = \overline{\mathsf{bs}}]$$

Formålet med resten af rapporten er at bevise EqSysIsPartition.

6 Udsagnslogisk bibliotek

I dette afsnit vil jeg bevise en samling af udsagnslogiske sandheder (eller "tautologier"), som vil blive brugt i de følgende afsnit. De fleste af disse tautologier har mange andre anvendelser end lige netop mængdelære. Beviserne er fordelt på syv underafsnit; figur 1 giver et overblik over, hvordan beviserne forholder sig til hinanden. Jeg vil kommentere de fleste af beviserne; dog er nogle af dem så tekniske, at jeg har ladet dem stå alene.

Figur 1: Bevisstrukturen for tautologierne. En pil fra lemma x til lemma y betyder, at x bruges i beviset for y. MP-lemmaerne fra afsnit 6.1 er ikke vist. “ImpTrans” står for “ImplyTransitivity”.

6.1 MP-lemmaer

Man får ofte brug for at anvende slutningsreglen MP flere gange i træk. Derfor vil jeg begynde med at vise fire lemmaer, der kan klare mellem 2 og 5 anvendelser af MP[15]:

$[\mathrm{MP2} \stackrel{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \vdash \underline{a} \vdash \underline{b} \vdash \underline{c}]$

$[\mathrm{MP3} \stackrel{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{d}\colon \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{d} \vdash \underline{a} \vdash \underline{b} \vdash \underline{c} \vdash \underline{d}]$

$[\mathrm{MP4} \stackrel{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{d}\colon \forall \underline{e}\colon \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{d} \Rightarrow \underline{e} \vdash \underline{a} \vdash \underline{b} \vdash \underline{c} \vdash \underline{d} \vdash \underline{e}]$

$[\mathrm{MP5} \stackrel{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{d}\colon \forall \underline{e}\colon \forall \underline{f}\colon \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{d} \Rightarrow \underline{e} \Rightarrow \underline{f} \vdash \underline{a} \vdash \underline{b} \vdash \underline{c} \vdash \underline{d} \vdash \underline{e} \vdash \underline{f}]$

6.1.1 Det første bevis

Vi begynder med at bevise MP2:

$[\mathrm{MP2} \stackrel{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \vdash \underline{a} \vdash \underline{b} \vdash \underline{c}]$

$[\mathrm{MP2} \stackrel{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \vdash \underline{a} \vdash \underline{b} \vdash \mathrm{MP} \rhd \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \rhd \underline{a} \gg \underline{b} \Rightarrow \underline{c}; \mathrm{MP} \rhd \underline{b} \Rightarrow \underline{c} \rhd \underline{b} \gg \underline{c}\rceil, \mathrm{p}_0, c)]$

Da dette er rapportens første bevis, vil jeg bringe nogle ekstra kommentarer[16]. Oven over beviset har jeg gentaget definitionen af det, der skal bevises; dette er kun for overblikkets skyld — det er ikke en formel nødvendighed. Selve beviset for MP2 består af seks linier, nummereret fra 1 til 6. En bevislinie kan have to former. Den første form er:

`Argumentation` $\gg$ `Konklusion`

hvor `Konklusion` er det som linien beviser, mens teksten i `Argumentation` udgør en begrundelse for, at `Konklusion` gælder. F.eks. siger linie 5, at meta-formlen $[\underline{b} \Rightarrow \underline{c}]$ gælder, fordi den kan udledes fra slutningsreglen MP ved substitution. Argumentationen skal læses på den måde, at konklusionerne fra linie 2 og 3 bliver brugt som præmisser til MP. Den generelle betydning af konstruktionen $[x \rhd y]$ er, at konklusionen fra linie y bliver brugt som præmis i forhold til x.
Den anden form, en bevislinie kan have, er:

`Nøgleord` $\gg$ `Konklusion`

hvor `Nøgleord` er et af de tre ord "Arbitrary", "Premise" eller "Side-condition". Betydningen af ordene "Premise" og "Side-condition" er åbenlys: De angiver, at liniens konklusion indgår som en præmis (hhv.

[15] I afsnit 8.3 får vi faktisk brug for at anvende MP 6 gange i træk; men et eller andet sted skal man jo stoppe.
[16] Denne beskrivelse er en revideret udgave af afsnit 5.1 i [1].

sidebetingelse) i den sætning, der skal bevises. F.eks. siger bevisets linie 2, at MP2 bruger meta-formlen
$[\underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c}]$ som præmis. Når ordet “Arbitrary” bruges, består konklusionen af en liste af meta-variable (f.eks. $[\underline{a}, \underline{b}, \underline{c}]$ i linie 1). Ideen hermed er at udtrykke, at vi ikke antager noget om de pågældende meta-variable, og at vi derfor har ret til at binde dem med en meta-alkvantor i den sætning, der skal bevises. I det forhåndenværende bevis berettiger linien med “Arbitrary” altså, at MP2 er kvantificeret med $[\forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon (\cdots)]$.
Alt dette har drejet sig om den formelle syntaks for et Logiweb bevis. Der er ikke så meget at sige om selve beviset for MP2; vi indkapsler simpelthen to på hinanden følgende anvendelser af MP.

6.1.2 Beviser for de andre MP-lemmaer

Beviserne for de øvrige MP-lemmaer er lige ud ad landevejen:

$[\mathrm{MP3} \stackrel{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \forall \underline{d} \colon \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{d} \vdash \underline{a} \vdash \underline{b} \vdash \underline{c} \vdash \underline{d}]$

$[\mathrm{MP3} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \forall \underline{d} \colon \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{d} \vdash \underline{a} \vdash \underline{b} \vdash \underline{c} \vdash \mathrm{MP2} \rhd \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{d} \rhd \underline{a} \rhd \underline{b} \gg \underline{c} \Rightarrow \underline{d}; \mathrm{MP} \rhd \underline{c} \Rightarrow \underline{d} \rhd \underline{c} \gg \underline{d} \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{MP4} \stackrel{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \forall \underline{d} \colon \forall \underline{e} \colon \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{d} \Rightarrow \underline{e} \vdash \underline{a} \vdash \underline{b} \vdash \underline{c} \vdash \underline{d} \vdash \underline{e}]$

$[\mathrm{MP4} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \forall \underline{d} \colon \forall \underline{e} \colon \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{d} \Rightarrow \underline{e} \vdash \underline{a} \vdash \underline{b} \vdash \underline{c} \vdash \underline{d} \vdash \mathrm{MP2} \rhd \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{d} \Rightarrow \underline{e} \rhd \underline{a} \rhd \underline{b} \gg \underline{c} \Rightarrow \underline{d} \Rightarrow \underline{e}; \mathrm{MP2} \rhd \underline{c} \Rightarrow \underline{d} \Rightarrow \underline{e} \rhd \underline{c} \rhd \underline{d} \gg \underline{e} \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{MP5} \stackrel{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \forall \underline{d} \colon \forall \underline{e} \colon \forall \underline{f} \colon \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{d} \Rightarrow \underline{e} \Rightarrow \underline{f} \vdash \underline{a} \vdash \underline{b} \vdash \underline{c} \vdash \underline{d} \vdash \underline{e} \vdash \underline{f}]$

$[\mathrm{MP5} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \forall \underline{d} \colon \forall \underline{e} \colon \forall \underline{f} \colon \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{d} \Rightarrow \underline{e} \Rightarrow \underline{f} \vdash \underline{a} \vdash \underline{b} \vdash \underline{c} \vdash \underline{d} \vdash \underline{e} \vdash \mathrm{MP3} \rhd \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{d} \Rightarrow \underline{e} \Rightarrow \underline{f} \rhd \underline{a} \rhd \underline{b} \rhd \underline{c} \gg \underline{d} \Rightarrow \underline{e} \Rightarrow \underline{f}; \mathrm{MP2} \rhd \underline{d} \Rightarrow \underline{e} \Rightarrow \underline{f} \rhd \underline{d} \rhd \underline{e} \gg \underline{f} \rceil, \mathrm{p}_0, \mathsf{c})]$

6.2 Implikation

Dette afsnit indeholder en række lemmaer vedr. implikation, grupperet i fire under-underafsnit.

6.2.1 Refleksivitet; blok-konstruktionen

Lemmaet AutoImply udsiger, at implikations-relationen er refleksiv:

$[\mathrm{AutoImply} \stackrel{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{a} \colon \underline{a} \Rightarrow \underline{a}]$

$[\mathrm{AutoImply} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{a} \colon \underline{a} \vdash \mathrm{Repetition} \rhd \underline{a} \gg \underline{a}; \forall \underline{a} \colon \mathrm{Ded} \rhd \forall \underline{a} \colon \underline{a} \vdash \underline{a} \gg \underline{a} \Rightarrow \underline{a} \rceil, \mathrm{p}_0, \mathsf{c})]$

Beviset for AutoImply indeholder to nye ting i forhold til de hidtidige beviser: En bevisblok, og en anvendelse af deduktions-reglen. En bevisblok er selvstændig enhed i et bevis; den afhænger ikke af den øvrige del af beviset. Den ovenstående bevisblok indeholder et bevis for lemmaet $[\forall \underline{\mathsf{a}}\colon \underline{\mathsf{a}} \vdash \underline{\mathsf{a}}]$. Pointen er nu, at blokkens sidste linie (linie 5) fungerer som en forkortelse for dette lemma. Vi kan da anvende deduktionsreglen på denne linie til at omdanne inferensen $[\forall \underline{\mathsf{a}}\colon \underline{\mathsf{a}} \vdash \underline{\mathsf{a}}]$ til implikationen $[\underline{\mathsf{a}} \Rightarrow \underline{\mathsf{a}}]$. Det vigtigste formål med deduktionsreglen er netop, at vi let kan skifte fra inferens til implikation.

6.2.2 Transitivitet

Lemmaet ImplyTransitivity udsiger, at implikations-relationen er transitiv:

$$[\text{ImplyTransitivity} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \forall \underline{\mathsf{c}}\colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \vdash \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{c}}]$$

Vi viser ImplyTransitivity ved hjælp af MP og deduktionsreglen:

$$[\text{ImplyTransitivity} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \forall \underline{\mathsf{c}}\colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \vdash \underline{\mathsf{a}} \vdash \text{MP} \rhd \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \rhd \underline{\mathsf{a}} \gg \underline{\mathsf{b}}; \text{MP} \rhd \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \rhd \underline{\mathsf{b}} \gg \underline{\mathsf{c}}; \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \forall \underline{\mathsf{c}}\colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \vdash \text{Ded} \rhd \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \forall \underline{\mathsf{c}}\colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \vdash \underline{\mathsf{a}} \vdash \underline{\mathsf{c}} \gg \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \Rightarrow \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{c}}; \text{MP2} \rhd \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \Rightarrow \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{c}} \rhd \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \rhd \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \gg \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{c}}\rceil, \text{p}_0, \mathsf{c})]$$

6.2.3 Svækkelse

Vi får ofte brug for det følgende ræsonnement: Hvis formlen $\underline{\mathsf{a}}$ gælder ubetinget, så gælder den også under antagelse af en vilkårlig anden formel $\underline{\mathsf{b}}$. Lemmaet Weakening udtrykker dette ræsonnement som følger:

$$[\text{Weakening} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{b}} \vdash \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}]$$

Vi beviser Weakening ved hjælp af deduktionsreglen:

$$[\text{Weakening} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{b}} \vdash \underline{\mathsf{a}} \vdash \text{Repetition} \rhd \underline{\mathsf{b}} \gg \underline{\mathsf{b}}; \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \text{Ded} \rhd \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{b}} \vdash \underline{\mathsf{a}} \vdash \underline{\mathsf{b}} \gg \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}; \underline{\mathsf{b}} \vdash \text{MP} \rhd \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \rhd \underline{\mathsf{b}} \gg \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}\rceil, \text{p}_0, \mathsf{c})]$$

6.2.4 Modsigelse

Det sidste lemma i dette afsnit vedrører strengt taget ikke implikation, men derimod inferens $(\mathsf{x} \vdash \mathsf{y})$. Lemmaet FromContradiction udsiger, at vi kan bevise hvad som helst, hvis vi har bevist to formler, der modsiger hinanden:

$$[\text{FromContradiction} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{a}} \vdash \dot{\neg}\, \underline{\mathsf{a}} \vdash \underline{\mathsf{b}}]$$

Beviset bruger Weakening og slutningsreglen Neg:

$$[\text{FromContradiction} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{a}} \vdash \dot{\neg}\, \underline{\mathsf{a}} \vdash \text{Weakening} \rhd \underline{\mathsf{a}} \gg \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}}; \text{Weakening} \rhd \dot{\neg}\, \underline{\mathsf{a}} \gg \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{a}}; \text{Neg} \rhd \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}} \rhd \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{a}} \gg \underline{\mathsf{b}}\rceil, \text{p}_0, \mathsf{c})]$$

6.3 Håndtering af dobbeltnegationer

De to lemmaer RemoveDoubleNeg og AddDoubleNeg tillader os hhv. at fjerne og tilføje dobbeltnegationer. Jeg vil ikke kommentere beviserne:

$[\mathrm{RemoveDoubleNeg} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \dot{\neg}\dot{\neg}\underline{\mathsf{a}} \vdash \underline{\mathsf{a}}]$

$[\mathrm{RemoveDoubleNeg} \overset{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \dot{\neg}\dot{\neg}\underline{\mathsf{a}} \vdash \mathrm{Weakening} \rhd \dot{\neg}\dot{\neg}\underline{\mathsf{a}} \gg \dot{\neg}\underline{\mathsf{a}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{a}}; \mathrm{AutoImply} \gg \dot{\neg}\underline{\mathsf{a}} \Rightarrow \dot{\neg}\underline{\mathsf{a}}; \mathrm{Neg} \rhd \dot{\neg}\underline{\mathsf{a}} \Rightarrow \dot{\neg}\underline{\mathsf{a}} \rhd \dot{\neg}\underline{\mathsf{a}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{a}} \gg \underline{\mathsf{a}}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{AddDoubleNeg} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \underline{\mathsf{a}} \vdash \dot{\neg}\dot{\neg}\underline{\mathsf{a}}]$

$[\mathrm{AddDoubleNeg} \overset{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \dot{\neg}\dot{\neg}\dot{\neg}\underline{\mathsf{a}} \vdash \mathrm{RemoveDoubleNeg} \rhd \dot{\neg}\dot{\neg}\dot{\neg}\underline{\mathsf{a}} \gg \dot{\neg}\underline{\mathsf{a}}; \forall \underline{\mathsf{a}} \colon \mathrm{Ded} \rhd \forall \underline{\mathsf{a}} \colon \dot{\neg}\dot{\neg}\dot{\neg}\underline{\mathsf{a}} \vdash \dot{\neg}\underline{\mathsf{a}} \gg \dot{\neg}\dot{\neg}\dot{\neg}\underline{\mathsf{a}} \Rightarrow \dot{\neg}\underline{\mathsf{a}}; \underline{\mathsf{a}} \vdash \mathrm{Weakening} \rhd \underline{\mathsf{a}} \gg \dot{\neg}\dot{\neg}\dot{\neg}\underline{\mathsf{a}} \Rightarrow \underline{\mathsf{a}}; \mathrm{Neg} \rhd \dot{\neg}\dot{\neg}\dot{\neg}\underline{\mathsf{a}} \Rightarrow \underline{\mathsf{a}} \rhd \dot{\neg}\dot{\neg}\dot{\neg}\underline{\mathsf{a}} \Rightarrow \dot{\neg}\underline{\mathsf{a}} \gg \dot{\neg}\dot{\neg}\underline{\mathsf{a}}\rceil, \mathrm{p}_0, \mathsf{c})]$

6.4 Modus tollens og beslægtede lemmaer

Hovedresultatet fra dette afsnit er slutningsreglen modus tollens, bevist som et lemma:

$[\mathrm{MT} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \dot{\neg}\underline{\mathsf{b}} \vdash \dot{\neg}\underline{\mathsf{a}}]$

For at vise MT begynder vi med et teknisk lemma, der ikke har den store værdi i sig selv:

$[\mathrm{Technicality} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \dot{\neg}\dot{\neg}\underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}]$

$[\mathrm{Technicality} \overset{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \dot{\neg}\dot{\neg}\underline{\mathsf{a}} \vdash \mathrm{RemoveDoubleNeg} \rhd \dot{\neg}\dot{\neg}\underline{\mathsf{a}} \gg \underline{\mathsf{a}}; \mathrm{MP} \rhd \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \rhd \underline{\mathsf{a}} \gg \underline{\mathsf{b}}; \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \mathrm{Ded} \rhd \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \dot{\neg}\dot{\neg}\underline{\mathsf{a}} \vdash \underline{\mathsf{b}} \gg \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}; \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \mathrm{MP} \rhd \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \rhd \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \gg \dot{\neg}\dot{\neg}\underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}\rceil, \mathrm{p}_0, \mathsf{c})]$

Uafhængigt af Technicality kan vi vise en version af MT, hvor $\underline{\mathsf{a}}$ optræder i negeret form:

$[\mathrm{NegativeMT} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \dot{\neg}\underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \dot{\neg}\underline{\mathsf{b}} \vdash \underline{\mathsf{a}}]$

$[\mathrm{NegativeMT} \overset{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \dot{\neg}\underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \dot{\neg}\underline{\mathsf{b}} \vdash \mathrm{Weakening} \rhd \dot{\neg}\underline{\mathsf{b}} \gg \dot{\neg}\underline{\mathsf{a}} \Rightarrow \dot{\neg}\underline{\mathsf{b}}; \mathrm{Neg} \rhd \dot{\neg}\underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \rhd \dot{\neg}\underline{\mathsf{a}} \Rightarrow \dot{\neg}\underline{\mathsf{b}} \gg \underline{\mathsf{a}}\rceil, \mathrm{p}_0, \mathsf{c})]$

Ud fra Technicality og NegativeMT kan vi nu vise MT:

$[\mathrm{MT} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \dot{\neg}\underline{\mathsf{b}} \vdash \dot{\neg}\underline{\mathsf{a}}]$

$[\mathrm{MT} \overset{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \dot{\neg}\underline{\mathsf{b}} \vdash \mathrm{Technicality} \gg \dot{\neg}\dot{\neg}\underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}; \mathrm{NegativeMT} \rhd \dot{\neg}\dot{\neg}\underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \rhd \dot{\neg}\underline{\mathsf{b}} \gg \dot{\neg}\underline{\mathsf{a}}\rceil, \mathrm{p}_0, \mathsf{c})]$

Vi slutter dette underafsnit med en variant af MT, som erstatter en inferens med en implikation:

[Contrapositive $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \Rightarrow \underline{b} \vdash \dot{\neg}\,\underline{b} \Rightarrow \dot{\neg}\,\underline{a}$]

Når en inferens skal erstattes med en implikation, er det altid deduktionsreglen, der skal i spil:

[Contrapositive $\stackrel{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \Rightarrow \underline{b} \vdash \dot{\neg}\,\underline{b} \vdash$ MT $\rhd \underline{a} \Rightarrow \underline{b} \rhd \dot{\neg}\,\underline{b} \gg \dot{\neg}\,\underline{a}; \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \Rightarrow \underline{b} \vdash$ Ded $\rhd \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \Rightarrow \underline{b} \vdash \dot{\neg}\,\underline{b} \vdash \dot{\neg}\,\underline{a} \gg \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\,\underline{b} \Rightarrow \dot{\neg}\,\underline{a};$ MP $\rhd \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\,\underline{b} \Rightarrow \dot{\neg}\,\underline{a} \rhd \underline{a} \Rightarrow \underline{b} \gg \dot{\neg}\,\underline{b} \Rightarrow \dot{\neg}\,\underline{a}\rceil, p_0, c)$]

6.5 Konjunktion

Hovedmålet med dette underafsnit er at konvertere mellem formlerne $\underline{a}$ og $\underline{b}$ og deres konjunktion $[\dot{\neg}\,\underline{a} \Rightarrow \dot{\neg}\,\underline{b}]$.

6.5.1 Forening af konjunkter

Vi begynder med at slå $\underline{a}$ og $\underline{b}$ sammen til $[\dot{\neg}\,\underline{a} \Rightarrow \dot{\neg}\,\underline{b}]$:

[JoinConjuncts $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \vdash \underline{b} \vdash \dot{\neg}\,\underline{a} \Rightarrow \dot{\neg}\,\underline{b}$]

Beviset for JoinConjuncts er af teknisk karakter. Vi viser den makroekspanderede form $[\dot{\neg}\,\underline{a} \Rightarrow \dot{\neg}\,\underline{b}]$, som vi i bevisets sidste linie konverterer til $[\dot{\neg}\,\underline{a} \Rightarrow \dot{\neg}\,\underline{b}]$. Denne sidste linie er ikke nødvendig for bevischeckeren, men den gør beviset lidt nemmere at læse:

[JoinConjuncts $\stackrel{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \vdash \underline{a} \Rightarrow \dot{\neg}\,\underline{b} \vdash$ MP $\rhd \underline{a} \Rightarrow \dot{\neg}\,\underline{b} \rhd \underline{a} \gg \dot{\neg}\,\underline{b}; \forall \underline{a}\colon \forall \underline{b}\colon$ Ded $\rhd \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \vdash \underline{a} \Rightarrow \dot{\neg}\,\underline{b} \vdash \dot{\neg}\,\underline{b} \gg \underline{a} \Rightarrow \underline{a} \Rightarrow \dot{\neg}\,\underline{b} \Rightarrow \dot{\neg}\,\underline{b}; \underline{a} \vdash \underline{b} \vdash$ MP $\rhd \underline{a} \Rightarrow \underline{a} \Rightarrow \dot{\neg}\,\underline{b} \Rightarrow \dot{\neg}\,\underline{b} \rhd \underline{a} \gg \underline{a} \Rightarrow \dot{\neg}\,\underline{b} \Rightarrow \dot{\neg}\,\underline{b};$ AddDoubleNeg $\rhd \underline{b} \gg \dot{\neg}\,\dot{\neg}\,\underline{b};$ MT $\rhd \underline{a} \Rightarrow \dot{\neg}\,\underline{b} \Rightarrow \dot{\neg}\,\underline{b} \rhd \dot{\neg}\,\dot{\neg}\,\underline{b} \gg \dot{\neg}\,\underline{a} \Rightarrow \dot{\neg}\,\underline{b};$ Repetition $\rhd \dot{\neg}\,\underline{a} \Rightarrow \dot{\neg}\,\underline{b} \gg \dot{\neg}\,\underline{a} \Rightarrow \dot{\neg}\,\underline{b}\rceil, p_0, c)$]

6.5.2 Udskilning af anden konjunkt

Tautologien SecondConjunct lader os udskille den anden konjunkt fra $[\dot{\neg}\,\underline{a} \Rightarrow \dot{\neg}\,\underline{b}]$. Jeg vil ikke kommentere beviset:

[SecondConjunct $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg}\,\underline{a} \Rightarrow \dot{\neg}\,\underline{b} \vdash \underline{b}$]

[SecondConjunct $\stackrel{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg}\,\underline{b} \vdash$ Weakening $\rhd \dot{\neg}\,\underline{b} \gg \underline{a} \Rightarrow \dot{\neg}\,\underline{b}; \forall \underline{a}\colon \forall \underline{b}\colon$ Ded $\rhd \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg}\,\underline{b} \vdash \underline{a} \Rightarrow \dot{\neg}\,\underline{b} \gg \dot{\neg}\,\underline{b} \Rightarrow \underline{a} \Rightarrow \dot{\neg}\,\underline{b}; \dot{\neg}\,\underline{a} \Rightarrow \dot{\neg}\,\underline{b} \vdash$ Repetition $\rhd \dot{\neg}\,\underline{a} \Rightarrow \dot{\neg}\,\underline{b} \gg \dot{\neg}\,\underline{a} \Rightarrow \dot{\neg}\,\underline{b};$ NegativeMT $\rhd \dot{\neg}\,\underline{b} \Rightarrow \underline{a} \Rightarrow \dot{\neg}\,\underline{b} \rhd \dot{\neg}\,\underline{a} \Rightarrow \dot{\neg}\,\underline{b} \gg \underline{b}\rceil, p_0, c)$]

6.5.3 Udskilning af første konjunkt

For at udskille $\underline{a}$ fra $[\dot{\neg}\,\underline{a} \Rightarrow \dot{\neg}\,\underline{b}]$ viser vi først, at $[\dot{\neg}\,\underline{a} \Rightarrow \dot{\neg}\,\underline{b}]$ er kommutativ. Jeg vil ikke kommentere beviset:

$[\mathrm{AndCommutativity} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg}\,\underline{a} \Rightarrow \dot{\neg}\,\underline{b} \vdash \dot{\neg}\,\underline{b} \Rightarrow \dot{\neg}\,\underline{a}]$

$[\mathrm{AndCommutativity} \overset{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{b} \Rightarrow \dot{\neg}\,\underline{a} \vdash \underline{a} \vdash \mathrm{AddDoubleNeg} \rhd \underline{a} \gg \dot{\neg}\,\dot{\neg}\,\underline{a}; \mathrm{MT} \rhd \underline{b} \Rightarrow \dot{\neg}\,\underline{a} \rhd \dot{\neg}\,\dot{\neg}\,\underline{a} \gg \dot{\neg}\,\underline{b}; \forall \underline{a}\colon \forall \underline{b}\colon \mathrm{Ded} \rhd \forall \underline{a}\colon \forall \underline{b}\colon \underline{b} \Rightarrow \dot{\neg}\,\underline{a} \vdash \underline{a} \vdash \dot{\neg}\,\underline{b} \gg \underline{b} \Rightarrow \dot{\neg}\,\underline{a} \Rightarrow \underline{a} \Rightarrow \dot{\neg}\,\underline{b}; \dot{\neg}\,\underline{a} \Rightarrow \dot{\neg}\,\underline{b} \vdash \mathrm{Repetition} \gg \dot{\neg}\,\underline{a} \Rightarrow \dot{\neg}\,\underline{b}; \mathrm{MT} \rhd \underline{b} \Rightarrow \dot{\neg}\,\underline{a} \Rightarrow \underline{a} \Rightarrow \dot{\neg}\,\underline{b} \rhd \dot{\neg}\,\underline{a} \Rightarrow \dot{\neg}\,\underline{b} \gg \dot{\neg}\,\underline{b} \Rightarrow \dot{\neg}\,\underline{a}; \mathrm{Repetition} \rhd \dot{\neg}\,\underline{b} \Rightarrow \dot{\neg}\,\underline{a} \gg \dot{\neg}\,\underline{b} \Rightarrow \dot{\neg}\,\underline{a}\rceil, \mathrm{p}_0, \mathsf{c})]$

Nu er det let at udskille den første konjunkt fra $[\dot{\neg}\,\underline{a} \Rightarrow \dot{\neg}\,\underline{b}]$: Først vender vi konjunktionen om til $[\dot{\neg}\,\underline{b} \Rightarrow \dot{\neg}\,\underline{a}]$ ved hjælp af AndCommutativity, og så udskiller vi $\underline{a}$ ved hjælp af SecondConjunct:

$[\mathrm{FirstConjunct} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg}\,\underline{a} \Rightarrow \dot{\neg}\,\underline{b} \vdash \underline{a}]$

$[\mathrm{FirstConjunct} \overset{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg}\,\underline{a} \Rightarrow \dot{\neg}\,\underline{b} \vdash \mathrm{AndCommutativity} \rhd \dot{\neg}\,\underline{a} \Rightarrow \dot{\neg}\,\underline{b} \gg \dot{\neg}\,\underline{b} \Rightarrow \dot{\neg}\,\underline{a}; \mathrm{SecondConjunct} \rhd \dot{\neg}\,\underline{b} \Rightarrow \dot{\neg}\,\underline{a} \gg \underline{a}\rceil, \mathrm{p}_0, \mathsf{c})]$

6.6 Dobbeltimplikation

I dette underafsnit viser vi tre enkle resultater vedr. dobbeltimplikation.

6.6.1 Brug sammen med modus ponens

De følgende to tautologier gør det let at bruge anvende slutningsreglen MP på dobbeltimplikationer. Beviserne er enkle og kræver ingen kommentarer:

$[\mathrm{IffFirst} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\,\underline{b} \Rightarrow \underline{a} \vdash \underline{b} \vdash \underline{a}]$

$[\mathrm{IffFirst} \overset{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\,\underline{b} \Rightarrow \underline{a} \vdash \underline{b} \vdash \mathrm{SecondConjunct} \rhd \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\,\underline{b} \Rightarrow \underline{a} \gg \underline{b} \Rightarrow \underline{a}; \mathrm{MP} \rhd \underline{b} \Rightarrow \underline{a} \rhd \underline{b} \gg \underline{a}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{IffSecond} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\,\underline{b} \Rightarrow \underline{a} \vdash \underline{a} \vdash \underline{b}]$

$[\mathrm{IffSecond} \overset{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\,\underline{b} \Rightarrow \underline{a} \vdash \underline{a} \vdash \mathrm{FirstConjunct} \rhd \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\,\underline{b} \Rightarrow \underline{a} \gg \underline{a} \Rightarrow \underline{b}; \mathrm{MP} \rhd \underline{a} \Rightarrow \underline{b} \rhd \underline{a} \gg \underline{b}\rceil, \mathrm{p}_0, \mathsf{c})]$

6.6.2 Kommutativitet

Lemmaet IffCommutativity følger direkte af, at operatoren $[\dot{\neg}\,\mathsf{x} \Rightarrow \dot{\neg}\,\mathsf{y}]$ er kommutativ:

[IffCommutativity $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}} \vdash \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}$]

[IffCommutativity $\stackrel{\text{proof}}{\rightarrow}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}} \vdash$ Repetition $\rhd \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}} \gg \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}};$ AndCommutativity $\rhd \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}} \gg \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}};$ Repetition $\rhd \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \gg \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}\rceil, \mathrm{p}_0, \mathsf{c})$]

6.7 Disjunktion

Dette underafsnit indeholder tre lemmaer vedr. disjunktion, som vi fordeler på to under-underafsnit.

6.7.1 Svækkelse

Givet en påstand $\underline{\mathsf{b}}$ vil vi gerne udlede de svagere påstande $[\dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}]$ og $[\dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}}]$. Den første slutning varetages af lemmaet WeakenOr1:

[WeakenOr1 $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{b}} \vdash \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}$]

Beviset består af en simpel anvendelse af Weakening:

[WeakenOr1 $\stackrel{\text{proof}}{\rightarrow}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{b}} \vdash$ Weakening $\rhd \underline{\mathsf{b}} \gg \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}};$ Repetition $\rhd \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \gg \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}\rceil, \mathrm{p}_0, \mathsf{c})$]

Slutningen fra $\underline{\mathsf{a}}$ til $[\dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}]$ varetages af lemmaet WeakenOr2:

[WeakenOr2 $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{a}} \vdash \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}$]

Kernen i beviset for WeakenOr2 er en anvendelse af FromContradiction:

[WeakenOr2 $\stackrel{\text{proof}}{\rightarrow}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{a}} \vdash \dot{\neg}\, \underline{\mathsf{a}} \vdash$ FromContradiction $\rhd \underline{\mathsf{a}} \rhd \dot{\neg}\, \underline{\mathsf{a}} \gg \underline{\mathsf{b}}; \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon$ Ded $\rhd \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{a}} \vdash \dot{\neg}\, \underline{\mathsf{a}} \vdash \underline{\mathsf{b}} \gg \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}; \underline{\mathsf{a}} \vdash$ MP $\rhd \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \rhd \underline{\mathsf{a}} \gg \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}};$ Repetition $\rhd \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \gg \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}\rceil, \mathrm{p}_0, \mathsf{c})$]

6.7.2 Slutning ud fra disjunktion

Lemmaet FromDisjuncts lader os drage slutninger ud fra en disjunktion:

[FromDisjuncts $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \forall \underline{\mathsf{c}}\colon \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{c}} \vdash \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \vdash \underline{\mathsf{c}}$]

Om beviset vil jeg kun sige, at det er en ret elegant øvelse i bevisteknik:

[FromDisjuncts $\stackrel{\text{proof}}{\rightarrow}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \forall \underline{\mathsf{c}}\colon \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{c}} \vdash \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \vdash$ Repetition $\rhd \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \gg \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}};$ Contrapositive $\rhd \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \gg \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \dot{\neg}\, \underline{\mathsf{a}};$ Technicality $\rhd \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{c}} \gg \dot{\neg}\, \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{c}};$ ImplyTransitivity $\rhd \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \dot{\neg}\, \underline{\mathsf{a}} \rhd \dot{\neg}\, \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{c}} \gg \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}};$ Contrapositive $\rhd \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \gg \dot{\neg}\, \underline{\mathsf{c}} \Rightarrow \dot{\neg}\, \dot{\neg}\, \underline{\mathsf{b}};$ Contrapositive $\rhd \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \gg \dot{\neg}\, \underline{\mathsf{c}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}};$ Neg $\rhd \dot{\neg}\, \underline{\mathsf{c}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \rhd \dot{\neg}\, \underline{\mathsf{c}} \Rightarrow \dot{\neg}\, \dot{\neg}\, \underline{\mathsf{b}} \gg \underline{\mathsf{c}}\rceil, \mathrm{p}_0, \mathsf{c})$]

7 Regellemaer

Alle aksiomerne i afsnit 4.2 er formuleret som dobbeltimplikationer. Dette er gjort for at holde antallet af aksiomer nede på et minimum; men når aksiomerne skal bruges i beviser, er det mere bekvemt at have regler af formen $[\underline{a} \vdash \underline{b}]$ til rådighed. Heldigvis kan vi altid bevise $[\underline{a} \vdash \underline{b}]$, hvis vi i forvejen har et bevis for $[\underline{a} \Rightarrow \underline{b}]$ — der kræves blot en rutinemæssig anvendelse af MP.
I dette afsnit viser vi regellemmaer svarende til aksiomerne SeparationDef, PairDef, UnionDef, PowerDef og Extensionality.

7.1 Par

I tilfældet "PairDef" forløber både definition og bevis af regellemmaerne så enkelt, som det er muligt. Først formulerer vi den ene halvdel af aksiomets dobbelt-implikation som et lemma:

$$[\mathrm{Pair2Formula} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{s}\colon \forall \underline{x}\colon \forall \underline{y}\colon \underline{s} \in \{\underline{x}, \underline{y}\} \vdash \dot{\neg}\, \underline{s} = \underline{x} \Rightarrow \underline{s} = \underline{y}]$$

Derefter beviser vi dette lemma ved at anvende IffSecond på aksiomet og præmissen:

$$[\mathrm{Pair2Formula} \overset{\mathrm{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{s}\colon \forall \underline{x}\colon \forall \underline{y}\colon \underline{s} \in \{\underline{x}, \underline{y}\} \vdash \mathrm{PairDef} \gg \dot{\neg}\, \underline{s} \in \{\underline{x}, \underline{y}\} \Rightarrow \dot{\neg}\, \underline{s} = \underline{x} \Rightarrow \underline{s} = \underline{y} \Rightarrow \dot{\neg}\,\dot{\neg}\, \underline{s} = \underline{x} \Rightarrow \underline{s} = \underline{y} \Rightarrow \underline{s} \in \{\underline{x}, \underline{y}\}; \mathrm{IffSecond} \rhd \dot{\neg}\, \underline{s} \in \{\underline{x}, \underline{y}\} \Rightarrow \dot{\neg}\, \underline{s} = \underline{x} \Rightarrow \underline{s} = \underline{y} \Rightarrow \dot{\neg}\,\dot{\neg}\, \underline{s} = \underline{x} \Rightarrow \underline{s} = \underline{y} \Rightarrow \underline{s} \in \{\underline{x}, \underline{y}\} \rhd \underline{s} \in \{\underline{x}, \underline{y}\} \gg \dot{\neg}\, \underline{s} = \underline{x} \Rightarrow \underline{s} = \underline{y} \rceil, p_0, c)]$$

Til sidst gentager vi hele processen mht. den anden halvdel af dobbeltimplikationen. Den eneste reelle forskel er, at vi bruger IffFirst i beviset (i stedet for IffSecond):

$$[\mathrm{Formula2Pair} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{s}\colon \forall \underline{x}\colon \forall \underline{y}\colon \dot{\neg}\, \underline{s} = \underline{x} \Rightarrow \underline{s} = \underline{y} \vdash \underline{s} \in \{\underline{x}, \underline{y}\}]$$

$$[\mathrm{Formula2Pair} \overset{\mathrm{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{s}\colon \forall \underline{x}\colon \forall \underline{y}\colon \dot{\neg}\, \underline{s} = \underline{x} \Rightarrow \underline{s} = \underline{y} \vdash \mathrm{PairDef} \gg \dot{\neg}\, \underline{s} \in \{\underline{x}, \underline{y}\} \Rightarrow \dot{\neg}\, \underline{s} = \underline{x} \Rightarrow \underline{s} = \underline{y} \Rightarrow \dot{\neg}\,\dot{\neg}\, \underline{s} = \underline{x} \Rightarrow \underline{s} = \underline{y} \Rightarrow \underline{s} \in \{\underline{x}, \underline{y}\}; \mathrm{IffFirst} \rhd \dot{\neg}\, \underline{s} \in \{\underline{x}, \underline{y}\} \Rightarrow \dot{\neg}\, \underline{s} = \underline{x} \Rightarrow \underline{s} = \underline{y} \Rightarrow \dot{\neg}\,\dot{\neg}\, \underline{s} = \underline{x} \Rightarrow \underline{s} = \underline{y} \Rightarrow \underline{s} \in \{\underline{x}, \underline{y}\} \rhd \dot{\neg}\, \underline{s} = \underline{x} \Rightarrow \underline{s} = \underline{y} \gg \underline{s} \in \{\underline{x}, \underline{y}\} \rceil, p_0, c)]$$

7.2 Foreningsmængde

I tilfældet "UnionDef" bruger vi stort set den samme fremgangsmåde som i afsnit 7.1. Først håndterer vi den ene halvdel af dobbeltimplikationen:

$$[\mathrm{Union2Formula} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{s}\colon \forall \underline{x}\colon \underline{s} \in \cup \underline{x} \vdash \dot{\neg}\, \underline{s} \in j_{\mathrm{Ex}} \Rightarrow \dot{\neg}\, j_{\mathrm{Ex}} \in \underline{x}]$$

$$[\mathrm{Union2Formula} \overset{\mathrm{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{s}\colon \forall \underline{x}\colon \underline{s} \in \cup \underline{x} \vdash \mathrm{UnionDef} \gg \dot{\neg}\, \underline{s} \in \cup \underline{x} \Rightarrow \dot{\neg}\, \underline{s} \in j_{\mathrm{Ex}} \Rightarrow \dot{\neg}\, j_{\mathrm{Ex}} \in \underline{x} \Rightarrow \dot{\neg}\,\dot{\neg}\, \underline{s} \in j_{\mathrm{Ex}} \Rightarrow \dot{\neg}\, j_{\mathrm{Ex}} \in \underline{x} \Rightarrow \underline{s} \in \cup \underline{x}; \mathrm{IffSecond} \rhd \dot{\neg}\, \underline{s} \in \cup \underline{x} \Rightarrow \dot{\neg}\, \underline{s} \in j_{\mathrm{Ex}} \Rightarrow \dot{\neg}\, j_{\mathrm{Ex}} \in \underline{x} \Rightarrow \dot{\neg}\,\dot{\neg}\, \underline{s} \in j_{\mathrm{Ex}} \Rightarrow \dot{\neg}\, j_{\mathrm{Ex}} \in \underline{x} \Rightarrow \underline{s} \in \cup \underline{x} \rhd \underline{s} \in \cup \underline{x} \gg \dot{\neg}\, \underline{s} \in j_{\mathrm{Ex}} \Rightarrow \dot{\neg}\, j_{\mathrm{Ex}} \in \underline{x} \rceil, p_0, c)]$$

Så er turen kommet til den anden halvdel af dobbeltimplikationen:

$[\mathrm{Formula2Union} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{s}\colon \forall \underline{x}\colon \underline{s} \in j_{\mathrm{Ex}} \vdash j_{\mathrm{Ex}} \in \underline{x} \vdash \underline{s} \in \cup \underline{x}]$

Der er det lille raffinement, at vi i formuleringen af Formula2Union har delt formlen $[\dot{\neg}\, \underline{s} \in j_{\mathrm{Ex}} \Rightarrow \dot{\neg}\, j_{\mathrm{Ex}} \in \underline{x}]$ op i sine konjunkter. Beviset for Formula2Union kræver derfor en ekstra anvendelse af JoinConjuncts:

$[\mathrm{Formula2Union} \stackrel{\mathrm{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{s}\colon \forall \underline{x}\colon \underline{s} \in j_{\mathrm{Ex}} \vdash j_{\mathrm{Ex}} \in \underline{x} \vdash \mathrm{JoinConjuncts} \rhd \underline{s} \in j_{\mathrm{Ex}} \rhd j_{\mathrm{Ex}} \in \underline{x} \gg \dot{\neg}\, \underline{s} \in j_{\mathrm{Ex}} \Rightarrow \dot{\neg}\, j_{\mathrm{Ex}} \in \underline{x}; \mathrm{UnionDef} \gg \dot{\neg}\, \underline{s} \in \cup \underline{x} \Rightarrow \dot{\neg}\, \underline{s} \in j_{\mathrm{Ex}} \Rightarrow \dot{\neg}\, j_{\mathrm{Ex}} \in \underline{x} \Rightarrow \dot{\neg}\, \dot{\neg}\, \underline{s} \in j_{\mathrm{Ex}} \Rightarrow \dot{\neg}\, j_{\mathrm{Ex}} \in \underline{x} \Rightarrow \underline{s} \in \cup \underline{x}; \mathrm{IffFirst} \rhd \dot{\neg}\, \underline{s} \in \cup \underline{x} \Rightarrow \dot{\neg}\, \underline{s} \in j_{\mathrm{Ex}} \Rightarrow \dot{\neg}\, j_{\mathrm{Ex}} \in \underline{x} \Rightarrow \dot{\neg}\, \dot{\neg}\, \underline{s} \in j_{\mathrm{Ex}} \Rightarrow \dot{\neg}\, j_{\mathrm{Ex}} \in \underline{x} \Rightarrow \underline{s} \in \cup \underline{x} \rhd \dot{\neg}\, \underline{s} \in j_{\mathrm{Ex}} \Rightarrow \dot{\neg}\, j_{\mathrm{Ex}} \in \underline{x} \gg \underline{s} \in \cup \underline{x} \rceil, p_0, c)]$

7.3 Potensmængde; sidebetingelser

Tilfældet "PowerDef" er mere kompliceret end de hidtidige tilfælde. Det ene af de to regellemmaer er nemt nok:

$[\mathrm{SubsetInPower} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{s}\colon \forall \underline{x}\colon \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x} \vdash \underline{s} \in \mathrm{P}(\underline{x})]$

$[\mathrm{SubsetInPower} \stackrel{\mathrm{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{s}\colon \forall \underline{x}\colon \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x} \vdash \mathrm{Gen} \rhd \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x} \gg \forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x}; \mathrm{PowerDef} \gg \dot{\neg}\, \underline{s} \in \mathrm{P}(\underline{x}) \Rightarrow \forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \dot{\neg}\, \forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \underline{s} \in \mathrm{P}(\underline{x}); \mathrm{IffFirst} \rhd \dot{\neg}\, \underline{s} \in \mathrm{P}(\underline{x}) \Rightarrow \forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \dot{\neg}\, \forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \underline{s} \in \mathrm{P}(\underline{x}) \rhd \forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x} \gg \underline{s} \in \mathrm{P}(\underline{x}) \rceil, p_0, c)]$

Det andet regellemma er også nemt, hvis vi nøjes med at vise:
$[\forall \underline{s}\colon \forall \underline{x}\colon \underline{s} \in \mathrm{P}(\underline{x}) \vdash \forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x}]$.
For at komme videre med konklusionen $[\forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x}]$ må vi imidlertid slippe af med objektkvantoren; og dette kan vi lige så godt gøre med det samme. Derfor bliver målet i stedet at vise:
$[\forall \underline{s}\colon \forall \underline{x}\colon \underline{s} \in \mathrm{P}(\underline{x}) \vdash \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x}]$, hvor $\bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x}$ som nævnt i afsnit 5.3 er en forkortelse for $[\bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x}]$.
For at nå dette mål må vi begynde med et hjælpelemma, der ser således ud:

$[\mathrm{HelperPowerIsSub} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{s}\colon \forall \underline{x}\colon \forall \underline{y}\colon \lceil \bar{s} \rceil \#^0 \lceil \underline{x} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{y} \rceil \Vdash \forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y} \Rightarrow \underline{s} \in \underline{x} \Rightarrow \underline{s} \in \underline{y}]$

De to sidebetingelser $[\lceil \bar{s} \rceil \#^0 \lceil \underline{x} \rceil]$ og $[\lceil \bar{s} \rceil \#^0 \lceil \underline{y} \rceil]$ er her nødvendige pga. antecedenten[17] $[\forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y}]$. Hvis vi i denne formel instantierer $\underline{x}$ eller $\underline{y}$ til en term, der indeholder frie forekomster af $\bar{s}$, så ændres formlens mening. F.eks. er formlen $[\forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \bar{s} \Rightarrow \bar{s} \in \underline{y}]$ nødvendigvis sand, fordi ingen mængder i ZFsub er medlem af sig selv; men den oprindelige formel $[\forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y}]$ er ikke nødvendigvis sand. Konstruktionen $[\lceil x \rceil \#^0 \lceil y \rceil]$ (der er defineret i appendikset til [4]) udsiger netop, at objektvariablen x ikke forekommer frit i

[17] Jeg kalder x i $[x \Rightarrow y]$ for "antecedenten"; og jeg kalder y for "konsekvensen".

termen y[18]. Vi kommer til at se mange flere eksempler på sidebetingelser af formen $[\lceil x\rceil \#^0 \lceil y\rceil]$: De er nødvendige, når metavariablen y optræder i en kontekst, der er kvantificeret med objektvariablen x.
Indholdet af selve lemmaet er, at vi kan fjerne objektkvantoren fra $[\forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y}]$, og endda erstatte objektvariablen $\bar{s}$ med metavariablen $\underline{s}$. Her er beviset:

$$[\mathrm{HelperPowerIsSub} \stackrel{\mathrm{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y} \vdash \mathrm{Repetition} \rhd \bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y} \gg \bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y}; \forall \underline{s}\colon \forall \underline{x}\colon \forall \underline{y}\colon \mathrm{Ded} \rhd \forall \underline{x}\colon \forall \underline{y}\colon \bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y} \vdash \bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y} \gg \lceil \bar{s}\rceil \#^0 \lceil \underline{x}\rceil \Vdash \lceil \bar{s}\rceil \#^0 \lceil \underline{y}\rceil \Vdash \forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y} \Rightarrow \underline{s} \in \underline{x} \Rightarrow \underline{s} \in \underline{y}\rceil, p_0, c)]$$

Beviset illustrerer styrken af slutningsreglen Ded: I dette tilfælde omdanner den en inferens (nemlig $[\forall \underline{x}\colon \forall \underline{y}\colon \bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y} \vdash \bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y}]$) til en implikation (nemlig $[\bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y}]$), sætter en alkvantor på antecedenten (så vi får $[\forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y}]$), og erstatter den frie forekomst af "$\bar{s}$" med "$\underline{s}$" — alt sammen på én gang. Dog skal de to sidebetingelser med, for at reglen kan anvendes.
Formuleringen af det andet regellemma bliver nu som følger:

$$[\mathrm{PowerIsSub} \stackrel{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{s}\colon \forall \underline{x}\colon \lceil \bar{s}\rceil \#^0 \lceil \underline{s}\rceil \Vdash \lceil \bar{s}\rceil \#^0 \lceil \underline{x}\rceil \Vdash \underline{s} \in \mathrm{P}(\underline{x}) \vdash \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x}]$$

Da PowerIsSub benytter sig af HelperPowerIsSub, bliver vi nødt til at lade PowerIsSub "arve" de to sidebetingelser fra dette lemma. Vi kan ikke undgå sidebetingelserne ved at instantiere de to metavariable $\underline{x}$ og $\underline{y}$ fra HelperPowerIsSub til nogle andre metavariable; for der er jo ingen garanti for, at disse metavariable vil undgå $\bar{s}$, når de engang bliver instantierede. Så længe man arbejder med metavariable, er det altså meget svært at slippe af med denne slags sidebetingelser, når de først er blevet indført i en beviskæde.
Her er beviset for PowerIsSub:

$$[\mathrm{PowerIsSub} \stackrel{\mathrm{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{s}\colon \forall \underline{x}\colon \lceil \bar{s}\rceil \#^0 \lceil \underline{s}\rceil \Vdash \lceil \bar{s}\rceil \#^0 \lceil \underline{x}\rceil \Vdash \underline{s} \in \mathrm{P}(\underline{x}) \vdash \mathrm{PowerDef} \gg \dot{\neg}\, \underline{s} \in \mathrm{P}(\underline{x}) \Rightarrow \forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \dot{\neg} \forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \underline{s} \in \mathrm{P}(\underline{x}); \mathrm{IffSecond} \rhd \dot{\neg}\, \underline{s} \in \mathrm{P}(\underline{x}) \Rightarrow \forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \dot{\neg} \forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \underline{s} \in \mathrm{P}(\underline{x}) \rhd \underline{s} \in \mathrm{P}(\underline{x}) \gg \forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x}; \mathrm{HelperPowerIsSub} \rhd\!\!\rhd \lceil \bar{s}\rceil \#^0 \lceil \underline{s}\rceil \rhd\!\!\rhd \lceil \bar{s}\rceil \#^0 \lceil \underline{x}\rceil \gg \forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x}; \mathrm{MP} \rhd \forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x} \rhd \forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x} \gg \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x}; \mathrm{Repetition} \rhd \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x} \gg \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x}\rceil, p_0, c)]$$

I linie 7 benytter vi konstruktionen $[x \rhd\!\!\rhd y]$ til at fortælle bevischeckeren, at de to sidebetingelser er opfyldt, og at vi derfor kan anvende HelperPowerIsSub.

7.4 Potensmængde, variant

Som nævnt er det svært at slippe af med sidebetingelser, men af og til kan det lade sig gøre. Når vi i afsnit 11.3.2 gør brug af lemmaerne HelperPowerIsSub

[18]Vi siger at "x undgår y" (eller at "y undgår x").

og PowerIsSub, vil metavariablen $\underline{\mathsf{x}}$ f.eks. blive instantieret til termer, der undgår $\bar{\mathsf{s}}$. Hermed bliver sidebetingelsen $[\lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil]$ automatisk opfyldt, mens vi stadigvæk må medregne $[\lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{y}} \rceil]$ eksplicit. Imidlertid er bevischeckeren således indrettet, at de automatisk opfyldte sidebetingelser skal stå til sidst, når man bruger et lemma, der indeholder sidebetingelser. Derfor viser vi nu nogle varianter af PowerIsSub og HelperPowerIsSub, hvor der er byttet om på $[\lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil]$ og $[\lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{y}} \rceil]$:

$[\text{(Switch)HelperPowerIsSub} \overset{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{s}} \colon \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{y}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \Vdash \forall_{\text{obj}} \bar{\mathsf{s}} \colon \bar{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \underline{\mathsf{s}} \in \underline{\mathsf{y}}]$

$[\text{(Switch)HelperPowerIsSub} \overset{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{s}} \colon \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{y}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \Vdash \text{HelperPowerIsSub} \mathrel{|\!\!\rhd} \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \mathrel{|\!\!\rhd} \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{y}} \rceil \gg \forall_{\text{obj}} \bar{\mathsf{s}} \colon \bar{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \underline{\mathsf{s}} \in \underline{\mathsf{y}} \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\text{(Switch)PowerIsSub} \overset{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{s}} \colon \forall \underline{\mathsf{x}} \colon \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{s}} \rceil \Vdash \underline{\mathsf{s}} \in \mathrm{P}(\underline{\mathsf{x}}) \vdash \bar{\mathsf{s}} \in \underline{\mathsf{s}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{x}}]$

$[\text{(Switch)PowerIsSub} \overset{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{s}} \colon \forall \underline{\mathsf{x}} \colon \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{s}} \rceil \Vdash \underline{\mathsf{s}} \in \mathrm{P}(\underline{\mathsf{x}}) \vdash \text{PowerIsSub} \mathrel{|\!\!\rhd} \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{s}} \rceil \mathrel{|\!\!\rhd} \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \rhd \underline{\mathsf{s}} \in \mathrm{P}(\underline{\mathsf{x}}) \gg \bar{\mathsf{s}} \in \underline{\mathsf{s}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{x}} \rceil, \mathrm{p}_0, \mathsf{c})]$

Vi kunne naturligvis have sparet noget plads ved at vise disse varianter til at begynde med. Jeg har ikke gjort dette, fordi vi nu har fået illustreret en af ulemperne ved at arbejde med sidebetingelser.
Jeg vil ikke kommentere fænomenet "sidebetingelser af formen $[\lceil \mathsf{x} \rceil \#^0 \lceil \mathsf{y} \rceil]$" yderligere — indtil afsnit 10.4, hvor antallet af sidebetingelser bliver så stort, at det ikke kan ignoreres.

7.5 Separation

Tilfældet "SeparationDef" er næsten lige så enkelt som tilfældet "PairDef" fra afsnit 7.1. Vi skal blot huske at overføre aksiomets sidebetingelse til de to regellemmaer:

$[\text{Sep2Formula} \overset{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \forall \underline{\mathsf{p}} \colon \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \underline{\mathsf{p}}^{\text{Ph}} \wedge \langle \lceil \underline{\mathsf{b}} \rceil \equiv^0 \lceil \underline{\mathsf{a}} \rceil | \lceil \underline{\mathsf{p}} \rceil {:=} \lceil \underline{\mathsf{y}} \rceil \rangle_{\text{Ph}} \Vdash \underline{\mathsf{y}} \in \{\mathrm{ph} \in \underline{\mathsf{x}} \mid \underline{\mathsf{a}}\} \vdash \dot{\neg} \underline{\mathsf{y}} \in \underline{\mathsf{x}} \Rightarrow \dot{\neg} \underline{\mathsf{b}}]$

$[\text{Sep2Formula} \overset{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \forall \underline{\mathsf{p}} \colon \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \underline{\mathsf{p}}^{\text{Ph}} \wedge \langle \lceil \underline{\mathsf{b}} \rceil \equiv^0 \lceil \underline{\mathsf{a}} \rceil | \lceil \underline{\mathsf{p}} \rceil {:=} \lceil \underline{\mathsf{y}} \rceil \rangle_{\text{Ph}} \Vdash \underline{\mathsf{y}} \in \{\mathrm{ph} \in \underline{\mathsf{x}} \mid \underline{\mathsf{a}}\} \vdash \text{SeparationDef} \mathrel{|\!\!\rhd} \underline{\mathsf{p}}^{\text{Ph}} \wedge \langle \lceil \underline{\mathsf{b}} \rceil \equiv^0 \lceil \underline{\mathsf{a}} \rceil | \lceil \underline{\mathsf{p}} \rceil {:=} \lceil \underline{\mathsf{y}} \rceil \rangle_{\text{Ph}} \gg \dot{\neg} \underline{\mathsf{y}} \in \{\mathrm{ph} \in \underline{\mathsf{x}} \mid \underline{\mathsf{a}}\} \Rightarrow \dot{\neg} \underline{\mathsf{y}} \in \underline{\mathsf{x}} \Rightarrow \dot{\neg} \underline{\mathsf{b}} \Rightarrow \dot{\neg} \dot{\neg} \underline{\mathsf{y}} \in \underline{\mathsf{x}} \Rightarrow \dot{\neg} \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{y}} \in \{\mathrm{ph} \in \underline{\mathsf{x}} \mid \underline{\mathsf{a}}\}; \text{FirstConjunct} \rhd \dot{\neg} \underline{\mathsf{y}} \in \{\mathrm{ph} \in \underline{\mathsf{x}} \mid \underline{\mathsf{a}}\} \Rightarrow \dot{\neg} \underline{\mathsf{y}} \in \underline{\mathsf{x}} \Rightarrow \dot{\neg} \underline{\mathsf{b}} \Rightarrow \dot{\neg} \dot{\neg} \underline{\mathsf{y}} \in \underline{\mathsf{x}} \Rightarrow \dot{\neg} \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{y}} \in \{\mathrm{ph} \in \underline{\mathsf{x}} \mid \underline{\mathsf{a}}\} \gg \underline{\mathsf{y}} \in \{\mathrm{ph} \in \underline{\mathsf{x}} \mid \underline{\mathsf{a}}\} \Rightarrow \dot{\neg} \underline{\mathsf{y}} \in \underline{\mathsf{x}} \Rightarrow \dot{\neg} \underline{\mathsf{b}}; \text{MP} \rhd \underline{\mathsf{y}} \in \{\mathrm{ph} \in \underline{\mathsf{x}} \mid \underline{\mathsf{a}}\} \Rightarrow \dot{\neg} \underline{\mathsf{y}} \in \underline{\mathsf{x}} \Rightarrow \dot{\neg} \underline{\mathsf{b}} \rhd \underline{\mathsf{y}} \in \{\mathrm{ph} \in \underline{\mathsf{x}} \mid \underline{\mathsf{a}}\} \gg \dot{\neg} \underline{\mathsf{y}} \in \underline{\mathsf{x}} \Rightarrow \dot{\neg} \underline{\mathsf{b}} \rceil, \mathrm{p}_0, \mathsf{c})]$

[Formula2Sep $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub ⊢ ∀a: ∀b: ∀p: ∀x: ∀y: p^{Ph} ∧ ⟨⌈b⌉≡^{0}⌈a⌉|⌈p⌉:=⌈y⌉⟩$_{Ph}$ ⊩ y ∈ x ⊢ b ⊢ y ∈ {ph ∈ x | a}]

[Formula2Sep $\stackrel{\text{proof}}{\rightarrow}$ λc.λx.𝒫(⌈ZFsub ⊢ ∀a: ∀b: ∀p: ∀x: ∀y: p^{Ph} ∧ ⟨⌈b⌉≡^{0}⌈a⌉|⌈p⌉:=⌈y⌉⟩$_{Ph}$ ⊩ y ∈ x ⊢ b ⊢ JoinConjuncts ▷ y ∈ x ▷ b ≫ $\dot\neg$ y ∈ x ⇒ $\dot\neg$ b; SeparationDef ⧐ p^{Ph} ∧ ⟨⌈b⌉≡^{0}⌈a⌉|⌈p⌉:=⌈y⌉⟩$_{Ph}$ ≫ $\dot\neg$ y ∈ {ph ∈ x | a} ⇒ $\dot\neg$ y ∈ x ⇒ $\dot\neg$ b ⇒ $\dot\neg$ $\dot\neg$ y ∈ x ⇒ $\dot\neg$ b ⇒ y ∈ {ph ∈ x | a}; SecondConjunct ▷ $\dot\neg$ y ∈ {ph ∈ x | a} ⇒ $\dot\neg$ y ∈ x ⇒ $\dot\neg$ b ⇒ $\dot\neg$ $\dot\neg$ y ∈ x ⇒ $\dot\neg$ b ⇒ y ∈ {ph ∈ x | a} ≫ $\dot\neg$ y ∈ x ⇒ $\dot\neg$ b ⇒ y ∈ {ph ∈ x | a}; MP ▷ $\dot\neg$ y ∈ x ⇒ $\dot\neg$ b ⇒ y ∈ {ph ∈ x | a} ▷ $\dot\neg$ y ∈ x ⇒ $\dot\neg$ b ≫ y ∈ {ph ∈ x | a}⌉, p_0, c)]

7.6 Ekstentionalitet

Tilfældet "Extensionality" er lidt mere kompliceret end det almindelige tilfælde, fordi vi skal forholde os til objektkvantoren i meta-formlen [$\dot\neg$ x=y ⇒ $\forall_{obj}$s̄: $\dot\neg$ s̄ ∈ x ⇒ s̄ ∈ y ⇒ $\dot\neg$ s̄ ∈ y ⇒ s̄ ∈ x ⇒ $\dot\neg$ $\forall_{obj}$s̄: $\dot\neg$ s̄ ∈ x ⇒ s̄ ∈ y ⇒ $\dot\neg$ s̄ ∈ y ⇒ s̄ ∈ x ⇒ x=y]. Derfor er der to underafsnit; lemmaerne i underafsnit 7.6.1 er afledt af implikationen [$\forall_{obj}$s̄: $\dot\neg$ s̄ ∈ x ⇒ s̄ ∈ y ⇒ $\dot\neg$ s̄ ∈ y ⇒ s̄ ∈ x ⇒ x=y], mens lemmaerne i underafsnit 7.6.2 er afledt af den omvendte implikation.

7.6.1 Tilstrækkelig betingelse for lighed

To mængder er lig hinanden, hvis de er hinandens delmængder:

[ToSetEquality $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub ⊢ ∀x: ∀y: s̄ ∈ x ⇒ s̄ ∈ y ⊢ s̄ ∈ y ⇒ s̄ ∈ x ⊢ x=y]

Beviset for ToSetEquality er ret enkelt. Vi skal blot huske at sætte en objektkvantor på præmisserne, hvilket gøres med slutningsreglen Gen:

[ToSetEquality $\stackrel{\text{proof}}{\rightarrow}$ λc.λx.𝒫(⌈ZFsub ⊢ ∀x: ∀y: s̄ ∈ x ⇒ s̄ ∈ y ⊢ s̄ ∈ y ⇒ s̄ ∈ x ⊢ JoinConjuncts ▷ s̄ ∈ x ⇒ s̄ ∈ y ▷ s̄ ∈ y ⇒ s̄ ∈ x ≫ $\dot\neg$ s̄ ∈ x ⇒ s̄ ∈ y ⇒ $\dot\neg$ s̄ ∈ y ⇒ s̄ ∈ x; Gen ▷ $\dot\neg$ s̄ ∈ x ⇒ s̄ ∈ y ⇒ $\dot\neg$ s̄ ∈ y ⇒ s̄ ∈ x ≫ $\forall_{obj}$s̄: $\dot\neg$ s̄ ∈ x ⇒ s̄ ∈ y ⇒ $\dot\neg$ s̄ ∈ y ⇒ s̄ ∈ x; Extensionality ≫ $\dot\neg$ x=y ⇒ $\forall_{obj}$s̄: $\dot\neg$ s̄ ∈ x ⇒ s̄ ∈ y ⇒ $\dot\neg$ s̄ ∈ y ⇒ s̄ ∈ x ⇒ $\dot\neg$ $\forall_{obj}$s̄: $\dot\neg$ s̄ ∈ x ⇒ s̄ ∈ y ⇒ $\dot\neg$ s̄ ∈ y ⇒ s̄ ∈ x ⇒ x=y; SecondConjunct ▷ $\dot\neg$ x= y ⇒ $\forall_{obj}$s̄: $\dot\neg$ s̄ ∈ x ⇒ s̄ ∈ y ⇒ $\dot\neg$ s̄ ∈ y ⇒ s̄ ∈ x ⇒ $\dot\neg$ $\forall_{obj}$s̄: $\dot\neg$ s̄ ∈ x ⇒ s̄ ∈ y ⇒ $\dot\neg$ s̄ ∈ y ⇒ s̄ ∈ x ⇒ x=y ≫ $\forall_{obj}$s̄: $\dot\neg$ s̄ ∈ x ⇒ s̄ ∈ y ⇒ $\dot\neg$ s̄ ∈ y ⇒ s̄ ∈ x ⇒ x= y; MP ▷ $\forall_{obj}$s̄: $\dot\neg$ s̄ ∈ x ⇒ s̄ ∈ y ⇒ $\dot\neg$ s̄ ∈ y ⇒ s̄ ∈ x ⇒ x=y ▷ $\forall_{obj}$s̄: $\dot\neg$ s̄ ∈ x ⇒ s̄ ∈ y ⇒ $\dot\neg$ s̄ ∈ y ⇒ s̄ ∈ x ≫ x=y⌉, p_0, c)]

I afsnit 10.1 får vi brug for en version af dette lemma, hvor objektvariablen s̄ er erstattet af en anden objektvariabel (jeg har valgt t̄). For at vise denne alternative version viser vi først lemmaet HelperToSetEquality(t), som indkapsler en anvendelse af deduktionsreglen:

[HelperToSetEquality(t) $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub ⊢ ∀x: ∀y: ⌈t̄⌉#^{0}⌈x⌉ ⊩ ⌈t̄⌉#^{0}⌈y⌉ ⊩ $\forall_{obj}$t̄: t̄ ∈ x ⇒ t̄ ∈ y ⇒ s̄ ∈ x ⇒ s̄ ∈ y]

[HelperToSetEquality(t) $\stackrel{\text{proof}}{\rightarrow}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \bar{\mathsf{t}} \in \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{y}} \vdash \text{Repetition} \rhd \bar{\mathsf{t}} \in \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{y}} \gg \bar{\mathsf{t}} \in \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{y}}; \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \text{Ded} \rhd \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \bar{\mathsf{t}} \in \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{y}} \vdash \bar{\mathsf{t}} \in \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{y}} \gg \lceil \bar{\mathsf{t}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \Vdash \lceil \bar{\mathsf{t}} \rceil \#^0 \lceil \underline{\mathsf{y}} \rceil \Vdash \forall_{\text{obj}} \bar{\mathsf{t}}\colon \bar{\mathsf{t}} \in \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{y}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{y}} \rceil, \text{p}_0, \mathsf{c})$]

Hovedlemmaet hedder ToSetEquality(t):

[ToSetEquality(t) $\stackrel{\text{stmt}}{\rightarrow}$ $\text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \lceil \bar{\mathsf{t}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \Vdash \lceil \bar{\mathsf{t}} \rceil \#^0 \lceil \underline{\mathsf{y}} \rceil \Vdash \bar{\mathsf{t}} \in \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{y}} \vdash \bar{\mathsf{t}} \in \underline{\mathsf{y}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{x}} \vdash \underline{\mathsf{x}} = \underline{\mathsf{y}}$]

I beviset herfor kombinerer vi Gen med hjælpelemmaet til at omdanne præmissen [$\bar{\mathsf{t}} \in \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{y}}$] til [$\bar{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{y}}$] (linie 6–8). Vi gentager denne procedure mht. præmissen [$\bar{\mathsf{t}} \in \underline{\mathsf{y}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{x}}$] (linie 9-11). Vi kan da slutte af med en anvendelse af det oprindelige lemma ToSetEquality (linie 12):

[ToSetEquality(t) $\stackrel{\text{proof}}{\rightarrow}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \lceil \bar{\mathsf{t}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \Vdash \lceil \bar{\mathsf{t}} \rceil \#^0 \lceil \underline{\mathsf{y}} \rceil \Vdash \bar{\mathsf{t}} \in \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{y}} \vdash \bar{\mathsf{t}} \in \underline{\mathsf{y}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{x}} \vdash \text{Gen} \rhd \bar{\mathsf{t}} \in \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{y}} \gg \forall_{\text{obj}} \bar{\mathsf{t}}\colon \bar{\mathsf{t}} \in \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{y}}; \text{HelperToSetEquality(t)} \unrhd \lceil \bar{\mathsf{t}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \unrhd \lceil \bar{\mathsf{t}} \rceil \#^0 \lceil \underline{\mathsf{y}} \rceil \gg \forall_{\text{obj}} \bar{\mathsf{t}}\colon \bar{\mathsf{t}} \in \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{y}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{y}}; \text{MP} \rhd \forall_{\text{obj}} \bar{\mathsf{t}}\colon \bar{\mathsf{t}} \in \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{y}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{y}} \rhd \forall_{\text{obj}} \bar{\mathsf{t}}\colon \bar{\mathsf{t}} \in \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{y}} \gg \bar{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{y}}; \text{Gen} \rhd \bar{\mathsf{t}} \in \underline{\mathsf{y}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{x}} \gg \forall_{\text{obj}} \bar{\mathsf{t}}\colon \bar{\mathsf{t}} \in \underline{\mathsf{y}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{x}}; \text{HelperToSetEquality(t)} \unrhd \lceil \bar{\mathsf{t}} \rceil \#^0 \lceil \underline{\mathsf{y}} \rceil \unrhd \lceil \bar{\mathsf{t}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \gg \forall_{\text{obj}} \bar{\mathsf{t}}\colon \bar{\mathsf{t}} \in \underline{\mathsf{y}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{y}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{x}}; \text{MP} \rhd \forall_{\text{obj}} \bar{\mathsf{t}}\colon \bar{\mathsf{t}} \in \underline{\mathsf{y}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{y}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{x}} \rhd \forall_{\text{obj}} \bar{\mathsf{t}}\colon \bar{\mathsf{t}} \in \underline{\mathsf{y}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{x}} \gg \bar{\mathsf{s}} \in \underline{\mathsf{y}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{x}}; \text{ToSetEquality} \rhd \bar{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{y}} \rhd \bar{\mathsf{s}} \in \underline{\mathsf{y}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{x}} \gg \underline{\mathsf{x}} = \underline{\mathsf{y}} \rceil, \text{p}_0, \mathsf{c})$]

Arbejdsfordelingen mellem HelperToSetEquality(t) og ToSetEquality(t) er temmelig ujævn; det meste af arbejdet foregår i beviset for ToSetEquality(t). At der er to lemmaer i stedet for ét, har en teknisk forklaring: Logiwebs bevischecker kan ikke arbejde med konklusioner af formen [$\mathsf{x} \Vdash \mathsf{y}$]. Vi kan derfor kun referere til konklusionen [$\lceil \bar{\mathsf{t}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \Vdash \lceil \bar{\mathsf{t}} \rceil \#^0 \lceil \underline{\mathsf{y}} \rceil \Vdash (\cdots)$] ved at lade den være konklusionen på et bevis; og derfor må vi stoppe op, når vi når til linie 7 i HelperToSetEquality(t). Vi vil flere gange senere komme ud for lemmaer, hvis eksistens har denne tekniske forklaring. Jeg vil bruge ordet "lemmastump" om disse lemmaer.

7.6.2 Nødvendig betingelse for lighed

Vi begynder dette underafsnit med et hjælpelemma, som vi bruger til at fjerne objektkvantoren fra Extensionality:

[HelperFromSetEquality $\stackrel{\text{stmt}}{\rightarrow}$ $\text{ZFsub} \vdash \forall \underline{\mathsf{s}}\colon \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{y}} \rceil \Vdash \forall_{\text{obj}} \bar{\mathsf{s}}\colon \dot{\neg} \bar{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{y}} \Rightarrow \dot{\neg} \bar{\mathsf{s}} \in \underline{\mathsf{y}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \dot{\neg} \underline{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \underline{\mathsf{s}} \in \underline{\mathsf{y}} \Rightarrow \dot{\neg} \underline{\mathsf{s}} \in \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{s}} \in \underline{\mathsf{x}}$]

Beviset for HelperFromSetEquality er en simpel anvendelse af deduktionsreglen:

[HelperFromSetEquality $\stackrel{\text{proof}}{\rightarrow}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \dot{\neg} \bar{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{y}} \Rightarrow \dot{\neg} \bar{\mathsf{s}} \in \underline{\mathsf{y}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{x}} \vdash \text{Repetition} \rhd \dot{\neg} \bar{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{y}} \Rightarrow \dot{\neg} \bar{\mathsf{s}} \in \underline{\mathsf{y}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{x}} \gg \dot{\neg} \bar{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{s}} \in$

$\underline{y} \Rightarrow \dot{\neg} \bar{s} \in \underline{y} \Rightarrow \bar{s} \in \underline{x}; \forall \underline{s}: \forall \underline{x}: \forall \underline{y}: \mathrm{Ded} \rhd \forall \underline{x}: \forall \underline{y}: \dot{\neg} \bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y} \Rightarrow \dot{\neg} \bar{s} \in \underline{y} \Rightarrow \bar{s} \in \underline{x} \vdash \dot{\neg} \bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y} \Rightarrow \dot{\neg} \bar{s} \in \underline{y} \Rightarrow \bar{s} \in \underline{x} \gg \lceil \bar{s} \rceil \#^0 \lceil \underline{x} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{y} \rceil \Vdash \forall_{\mathrm{obj}} \bar{s}: \dot{\neg} \bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y} \Rightarrow \dot{\neg} \bar{s} \in \underline{y} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \dot{\neg} \underline{s} \in \underline{x} \Rightarrow \underline{s} \in \underline{y} \Rightarrow \dot{\neg} \underline{s} \in \underline{y} \Rightarrow \underline{s} \in \underline{x} \rceil, p_0, c)]$

Hovelemmaet i dette underafsnit udsiger, at hvis $\underline{x}$ er lig med $\underline{y}$, så er $\underline{x}$ en delmængde af $\underline{y}$[19]:

$[\mathrm{FromSetEquality} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{s}: \forall \underline{x}: \forall \underline{y}: \lceil \bar{s} \rceil \#^0 \lceil \underline{x} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{y} \rceil \Vdash \underline{x} = \underline{y} \vdash \underline{s} \in \underline{x} \vdash \underline{s} \in \underline{y}]$

Beviset er en enkel anvendelse af Extensionality og HelperFromSetEquality:

$[\mathrm{FromSetEquality} \stackrel{\mathrm{proof}}{\rightarrow} \lambda c. \lambda x. \mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{s}: \forall \underline{x}: \forall \underline{y}: \lceil \bar{s} \rceil \#^0 \lceil \underline{x} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{y} \rceil \Vdash \underline{x} = \underline{y} \vdash \underline{s} \in \underline{x} \vdash \mathrm{Extensionality} \gg \dot{\neg} \underline{x} = \underline{y} \Rightarrow \forall_{\mathrm{obj}} \bar{s}: \dot{\neg} \bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y} \Rightarrow \dot{\neg} \bar{s} \in \underline{y} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \dot{\neg} \forall_{\mathrm{obj}} \bar{s}: \dot{\neg} \bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y} \Rightarrow \dot{\neg} \bar{s} \in \underline{y} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \underline{x} = \underline{y}; \mathrm{IffSecond} \rhd \dot{\neg} \underline{x} = \underline{y} \Rightarrow \forall_{\mathrm{obj}} \bar{s}: \dot{\neg} \bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y} \Rightarrow \dot{\neg} \bar{s} \in \underline{y} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \dot{\neg} \forall_{\mathrm{obj}} \bar{s}: \dot{\neg} \bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y} \Rightarrow \dot{\neg} \bar{s} \in \underline{y} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \underline{x} = \underline{y} \rhd \underline{x} = \underline{y} \gg \forall_{\mathrm{obj}} \bar{s}: \dot{\neg} \bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y} \Rightarrow \dot{\neg} \bar{s} \in \underline{y} \Rightarrow \bar{s} \in \underline{x}; \mathrm{HelperFromSetEquality} \unrhd \lceil \bar{s} \rceil \#^0 \lceil \underline{x} \rceil \unrhd \lceil \bar{s} \rceil \#^0 \lceil \underline{y} \rceil \gg \forall_{\mathrm{obj}} \bar{s}: \dot{\neg} \bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y} \Rightarrow \dot{\neg} \bar{s} \in \underline{y} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \dot{\neg} \underline{s} \in \underline{x} \Rightarrow \underline{s} \in \underline{y} \Rightarrow \dot{\neg} \underline{s} \in \underline{y} \Rightarrow \underline{s} \in \underline{x}; \mathrm{MP} \rhd \forall_{\mathrm{obj}} \bar{s}: \dot{\neg} \bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y} \Rightarrow \dot{\neg} \bar{s} \in \underline{y} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \dot{\neg} \underline{s} \in \underline{x} \Rightarrow \underline{s} \in \underline{y} \Rightarrow \dot{\neg} \underline{s} \in \underline{y} \Rightarrow \underline{s} \in \underline{x} \rhd \forall_{\mathrm{obj}} \bar{s}: \dot{\neg} \bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y} \Rightarrow \dot{\neg} \bar{s} \in \underline{y} \Rightarrow \bar{s} \in \underline{x} \gg \dot{\neg} \underline{s} \in \underline{x} \Rightarrow \underline{s} \in \underline{y} \Rightarrow \dot{\neg} \underline{s} \in \underline{y} \Rightarrow \underline{s} \in \underline{x}; \mathrm{IffSecond} \rhd \dot{\neg} \underline{s} \in \underline{x} \Rightarrow \underline{s} \in \underline{y} \Rightarrow \dot{\neg} \underline{s} \in \underline{y} \Rightarrow \underline{s} \in \underline{x} \rhd \underline{s} \in \underline{x} \gg \underline{s} \in \underline{y} \rceil, p_0, c)]$

8 Definitionslemmaer

Regellemmaerne fra afsnit 7 blev afledt fra aksiomer. Lemmaerne i dette afsnit bliver afledt fra makrodefinitionerne i afsnit 5.7; heraf navnet "definitionslemmaer". Formålet med definitionslemmaerne er at muliggøre effektiv anvendelse af makrodefinitionerne i beviser.

8.1 Refleksiv relation

Den første definition, vi tager under behandling, er definitionen $[\forall_{\mathrm{obj}} \bar{s}: \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{s}\}\} \in \underline{r} \stackrel{..}{=} \forall_{\mathrm{obj}} \bar{s}: \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{s}\}\} \in \underline{r}]$. Vi bruger her en lemmastump ved navn HelperReflexivity, som lader os fjerne objektkvantoren fra denne definitions højreside:

$[\mathrm{HelperReflexivity} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{r}: \forall \underline{s}: \forall \underline{bs}: \lceil \bar{s} \rceil \#^0 \lceil \underline{r} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{bs} \rceil \Vdash \forall_{\mathrm{obj}} \bar{s}: \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{s}\}\} \in \underline{r} \Rightarrow \underline{s} \in \underline{bs} \Rightarrow \{\{\underline{s}, \underline{s}\}, \{\underline{s}, \underline{s}\}\} \in \underline{r}]$

Vi viser HelperReflexivity ved hjælp af deduktionsreglen:

$[\mathrm{HelperReflexivity} \stackrel{\mathrm{proof}}{\rightarrow} \lambda c. \lambda x. \mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{r}: \forall \underline{bs}: \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{s}\}\} \in \underline{r} \vdash \mathrm{Repetition} \rhd \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{s}\}\} \in \underline{r} \gg \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{s}\}\} \in$

[19] Naturligvis kan man også vise, at hvis $\underline{x}$ er lig med $\underline{y}$, så er $\underline{y}$ en delmængde af $\underline{x}$. Det har imidlertid ikke været nødvendigt at bruge dette lemma.

$\underline{r}; \forall \underline{r}\colon \forall \underline{s}\colon \forall \underline{bs}\colon \mathrm{Ded} \rhd \forall \underline{r}\colon \forall \underline{bs}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \vdash \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \gg \lceil \bar{s} \rceil \#^0 \lceil \underline{r} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{bs} \rceil \Vdash \forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow \underline{s} \in \underline{bs} \Rightarrow \{\{\underline{s},\underline{s}\},\{\underline{s},\underline{s}\}\} \in \underline{r} \rceil, \mathrm{p}_0, c)]$

I hovedlemmaet Reflexivity omdanner vi lemmastumpens implikationer til inferenser:

$[\mathrm{Reflexivity} \stackrel{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{r}\colon \forall \underline{s}\colon \forall \underline{bs}\colon \lceil \bar{s} \rceil \#^0 \lceil \underline{r} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{bs} \rceil \Vdash \forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \vdash \underline{s} \in \underline{bs} \vdash \{\{\underline{s},\underline{s}\},\{\underline{s},\underline{s}\}\} \in \underline{r}]$

$[\mathrm{Reflexivity} \stackrel{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{r}\colon \forall \underline{s}\colon \forall \underline{bs}\colon \lceil \bar{s} \rceil \#^0 \lceil \underline{r} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{bs} \rceil \Vdash \forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \vdash \underline{s} \in \underline{bs} \vdash \mathrm{HelperReflexivity} \unrhd \lceil \bar{s} \rceil \#^0 \lceil \underline{r} \rceil \unrhd \lceil \bar{s} \rceil \#^0 \lceil \underline{bs} \rceil \gg \forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow \underline{s} \in \underline{bs} \Rightarrow \{\{\underline{s},\underline{s}\},\{\underline{s},\underline{s}\}\} \in \underline{r}; \mathrm{MP2} \rhd \forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow \underline{s} \in \underline{bs} \Rightarrow \{\{\underline{s},\underline{s}\},\{\underline{s},\underline{s}\}\} \in \underline{r} \rhd \forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \rhd \underline{s} \in \underline{bs} \gg \{\{\underline{s},\underline{s}\},\{\underline{s},\underline{s}\}\} \in \underline{r} \rceil, \mathrm{p}_0, c)]$

Vi skal straks se flere eksempler på netop denne arbejdsdeling mellem lemmastump og hovedlemma.

8.2 Symmetrisk relation

De to lemmaer i dette underafsnit er afledt fra definitionen af "symmetrisk relation" i afsnit 5.7. Fremgangsmåden er den samme som i afsnit 8.1. Lemmaet HelperSymmetry fungerer som lemmastump i forhold til Symmetry:

$[\mathrm{HelperSymmetry} \stackrel{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{r}\colon \forall \underline{s}\colon \forall \underline{t}\colon \forall \underline{bs}\colon \lceil \bar{s} \rceil \#^0 \lceil \underline{r} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{bs} \rceil \Vdash \lceil \bar{t} \rceil \#^0 \lceil \underline{r} \rceil \Vdash \lceil \bar{t} \rceil \#^0 \lceil \underline{bs} \rceil \Vdash \forall_{\mathrm{obj}} \bar{s}\colon \forall_{\mathrm{obj}} \bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \Rightarrow \underline{s} \in \underline{bs} \Rightarrow \underline{t} \in \underline{bs} \Rightarrow \{\{\underline{s},\underline{s}\},\{\underline{s},\underline{t}\}\} \in \underline{r} \Rightarrow \{\{\underline{t},\underline{t}\},\{\underline{t},\underline{s}\}\} \in \underline{r}]$

$[\mathrm{Symmetry} \stackrel{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{r}\colon \forall \underline{s}\colon \forall \underline{t}\colon \forall \underline{bs}\colon \lceil \bar{s} \rceil \#^0 \lceil \underline{r} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{bs} \rceil \Vdash \lceil \bar{t} \rceil \#^0 \lceil \underline{r} \rceil \Vdash \lceil \bar{t} \rceil \#^0 \lceil \underline{bs} \rceil \Vdash \forall_{\mathrm{obj}} \bar{s}\colon \forall_{\mathrm{obj}} \bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \vdash \underline{s} \in \underline{bs} \vdash \underline{t} \in \underline{bs} \vdash \{\{\underline{s},\underline{s}\},\{\underline{s},\underline{t}\}\} \in \underline{r} \vdash \{\{\underline{t},\underline{t}\},\{\underline{t},\underline{s}\}\} \in \underline{r}]$

Beviset for HelperSymmetry fungerer ligesom beviset for HelperReflexivity fra afsnit 8.1 (dog skriver vi ikke definitionen af $\forall_{\mathrm{obj}} \bar{s}\colon \forall_{\mathrm{obj}} \bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r}$ ud):

$[\mathrm{HelperSymmetry} \stackrel{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{r}\colon \forall \underline{bs}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \vdash \mathrm{Repetition} \rhd \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \gg \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r}; \forall \underline{r}\colon \forall \underline{s}\colon \forall \underline{t}\colon \forall \underline{bs}\colon \mathrm{Ded} \rhd \forall \underline{r}\colon \forall \underline{bs}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \vdash \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \gg \lceil \bar{s} \rceil \#^0 \lceil \underline{r} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{bs} \rceil \Vdash \lceil \bar{t} \rceil \#^0 \lceil \underline{r} \rceil \Vdash \lceil \bar{t} \rceil \#^0 \lceil \underline{bs} \rceil \Vdash \forall_{\mathrm{obj}} \bar{s}\colon \forall_{\mathrm{obj}} \bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \Rightarrow \underline{s} \in \underline{bs} \Rightarrow \underline{t} \in \underline{bs} \Rightarrow \{\{\underline{s},\underline{s}\},\{\underline{s},\underline{t}\}\} \in \underline{r} \Rightarrow \{\{\underline{t},\underline{t}\},\{\underline{t},\underline{s}\}\} \in \underline{r} \rceil, \mathrm{p}_0, c)]$

I beviset for Symmetry skifter vi fra implikation til inferens:

[Symmetry $\stackrel{\mathrm{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{r}\colon \forall \underline{s}\colon \forall \underline{t}\colon \forall \underline{bs}\colon \lceil\bar{s}\rceil\#^0\lceil\underline{r}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{bs}\rceil \Vdash \lceil\bar{t}\rceil\#^0\lceil\underline{r}\rceil \Vdash \lceil\bar{t}\rceil\#^0\lceil\underline{bs}\rceil \Vdash \forall_{\mathrm{obj}}\bar{s}\colon \forall_{\mathrm{obj}}\bar{t}\colon \bar{s}\in\underline{bs} \Rightarrow \bar{t}\in\underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\underline{r} \vdash \underline{s}\in\underline{bs} \vdash \underline{t}\in\underline{bs} \vdash \{\{\underline{s},\underline{s}\},\{\underline{s},\underline{t}\}\}\in\underline{r} \vdash \mathrm{HelperSymmetry} \unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{r}\rceil \unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{bs}\rceil \unrhd \lceil\bar{t}\rceil\#^0\lceil\underline{r}\rceil \unrhd \lceil\bar{t}\rceil\#^0\lceil\underline{bs}\rceil \gg \forall_{\mathrm{obj}}\bar{s}\colon \forall_{\mathrm{obj}}\bar{t}\colon \bar{s}\in\underline{bs} \Rightarrow \bar{t}\in\underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\underline{r} \Rightarrow \underline{s}\in\underline{bs} \Rightarrow \underline{t}\in\underline{bs} \Rightarrow \{\{\underline{s},\underline{s}\},\{\underline{s},\underline{t}\}\}\in\underline{r} \Rightarrow \{\{\underline{t},\underline{t}\},\{\underline{t},\underline{s}\}\}\in\underline{r}; \mathrm{MP4} \rhd \forall_{\mathrm{obj}}\bar{s}\colon \forall_{\mathrm{obj}}\bar{t}\colon \bar{s}\in\underline{bs} \Rightarrow \bar{t}\in\underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\underline{r} \Rightarrow \underline{s}\in\underline{bs} \Rightarrow \underline{t}\in\underline{bs} \Rightarrow \{\{\underline{s},\underline{s}\},\{\underline{s},\underline{t}\}\}\in\underline{r} \Rightarrow \{\{\underline{t},\underline{t}\},\{\underline{t},\underline{s}\}\}\in\underline{r} \rhd \forall_{\mathrm{obj}}\bar{s}\colon \forall_{\mathrm{obj}}\bar{t}\colon \bar{s}\in\underline{bs} \Rightarrow \bar{t}\in\underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\underline{r} \rhd \underline{s}\in\underline{bs} \rhd \underline{t}\in\underline{bs} \rhd \{\{\underline{s},\underline{s}\},\{\underline{s},\underline{t}\}\}\in\underline{r} \gg \{\{\underline{t},\underline{t}\},\{\underline{t},\underline{s}\}\}\in\underline{r}\rceil, p_0, c)$]

8.3 Transitiv relation

De to lemmaer i dette underafsnit er afledt fra definitionen af “transitiv relation” i afsnit 5.7. Vi bruger nøjagtig den samme fremgangsmåde som i afsnit 8.1 og 8.2: Lemmastumpen (HelperTransitivity) fjerner definitionens objektkvantor ved hjælp af Ded, og hovedlemmaet (Transitivity) skifter fra implikation til inferens:

[HelperTransitivity $\stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{r}\colon \forall \underline{s}\colon \forall \underline{t}\colon \forall \underline{u}\colon \forall \underline{bs}\colon \lceil\bar{s}\rceil\#^0\lceil\underline{r}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{bs}\rceil \Vdash \lceil\bar{t}\rceil\#^0\lceil\underline{r}\rceil \Vdash \lceil\bar{t}\rceil\#^0\lceil\underline{bs}\rceil \Vdash \lceil\bar{u}\rceil\#^0\lceil\underline{r}\rceil \Vdash \lceil\bar{u}\rceil\#^0\lceil\underline{bs}\rceil \Vdash \forall_{\mathrm{obj}}\bar{s}\colon \forall_{\mathrm{obj}}\bar{t}\colon \forall_{\mathrm{obj}}\bar{u}\colon \bar{s}\in\underline{bs} \Rightarrow \bar{t}\in\underline{bs} \Rightarrow \bar{u}\in\underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\underline{r} \Rightarrow \underline{s}\in\underline{bs} \Rightarrow \underline{t}\in\underline{bs} \Rightarrow \underline{u}\in\underline{bs} \Rightarrow \{\{\underline{s},\underline{s}\},\{\underline{s},\underline{t}\}\}\in\underline{r} \Rightarrow \{\{\underline{t},\underline{t}\},\{\underline{t},\underline{u}\}\}\in\underline{r} \Rightarrow \{\{\underline{s},\underline{s}\},\{\underline{s},\underline{u}\}\}\in\underline{r}$]

[HelperTransitivity $\stackrel{\mathrm{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{r}\colon \forall \underline{bs}\colon \bar{s}\in\underline{bs} \Rightarrow \bar{t}\in\underline{bs} \Rightarrow \bar{u}\in\underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\underline{r} \vdash \mathrm{Repetition} \rhd \bar{s}\in\underline{bs} \Rightarrow \bar{t}\in\underline{bs} \Rightarrow \bar{u}\in\underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\underline{r} \gg \bar{s}\in\underline{bs} \Rightarrow \bar{t}\in\underline{bs} \Rightarrow \bar{u}\in\underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\underline{r}; \forall \underline{r}\colon \forall \underline{s}\colon \forall \underline{t}\colon \forall \underline{u}\colon \forall \underline{bs}\colon \mathrm{Ded} \rhd \forall \underline{r}\colon \forall \underline{bs}\colon \bar{s}\in\underline{bs} \Rightarrow \bar{t}\in\underline{bs} \Rightarrow \bar{u}\in\underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\underline{r} \vdash \bar{s}\in\underline{bs} \Rightarrow \bar{t}\in\underline{bs} \Rightarrow \bar{u}\in\underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\underline{r} \gg \lceil\bar{s}\rceil\#^0\lceil\underline{r}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{bs}\rceil \Vdash \lceil\bar{t}\rceil\#^0\lceil\underline{r}\rceil \Vdash \lceil\bar{t}\rceil\#^0\lceil\underline{bs}\rceil \Vdash \lceil\bar{u}\rceil\#^0\lceil\underline{r}\rceil \Vdash \lceil\bar{u}\rceil\#^0\lceil\underline{bs}\rceil \Vdash \forall_{\mathrm{obj}}\bar{s}\colon \forall_{\mathrm{obj}}\bar{t}\colon \forall_{\mathrm{obj}}\bar{u}\colon \bar{s}\in\underline{bs} \Rightarrow \bar{t}\in\underline{bs} \Rightarrow \bar{u}\in\underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\underline{r} \Rightarrow \underline{s}\in\underline{bs} \Rightarrow \underline{t}\in\underline{bs} \Rightarrow \underline{u}\in\underline{bs} \Rightarrow \{\{\underline{s},\underline{s}\},\{\underline{s},\underline{t}\}\}\in\underline{r} \Rightarrow \{\{\underline{t},\underline{t}\},\{\underline{t},\underline{u}\}\}\in\underline{r} \Rightarrow \{\{\underline{s},\underline{s}\},\{\underline{s},\underline{u}\}\}\in\underline{r}\rceil, p_0, c)$]

[Transitivity $\stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{r}\colon \forall \underline{s}\colon \forall \underline{t}\colon \forall \underline{u}\colon \forall \underline{bs}\colon \lceil\bar{s}\rceil\#^0\lceil\underline{r}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{bs}\rceil \Vdash \lceil\bar{t}\rceil\#^0\lceil\underline{r}\rceil \Vdash \lceil\bar{t}\rceil\#^0\lceil\underline{bs}\rceil \Vdash \lceil\bar{u}\rceil\#^0\lceil\underline{r}\rceil \Vdash \lceil\bar{u}\rceil\#^0\lceil\underline{bs}\rceil \Vdash \forall_{\mathrm{obj}}\bar{s}\colon \forall_{\mathrm{obj}}\bar{t}\colon \forall_{\mathrm{obj}}\bar{u}\colon \bar{s}\in\underline{bs} \Rightarrow \bar{t}\in\underline{bs} \Rightarrow \bar{u}\in\underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\underline{r} \vdash \underline{s}\in\underline{bs} \vdash \underline{t}\in\underline{bs} \vdash \underline{u}\in\underline{bs} \vdash \{\{\underline{s},\underline{s}\},\{\underline{s},\underline{t}\}\}\in\underline{r} \vdash \{\{\underline{t},\underline{t}\},\{\underline{t},\underline{u}\}\}\in\underline{r} \vdash \{\{\underline{s},\underline{s}\},\{\underline{s},\underline{u}\}\}\in\underline{r}$]

[Transitivity $\stackrel{\mathrm{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{r}\colon \forall \underline{s}\colon \forall \underline{t}\colon \forall \underline{u}\colon \forall \underline{bs}\colon \lceil\bar{s}\rceil\#^0\lceil\underline{r}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{bs}\rceil \Vdash \lceil\bar{t}\rceil\#^0\lceil\underline{r}\rceil \Vdash \lceil\bar{t}\rceil\#^0\lceil\underline{bs}\rceil \Vdash \lceil\bar{u}\rceil\#^0\lceil\underline{r}\rceil \Vdash \lceil\bar{u}\rceil\#^0\lceil\underline{bs}\rceil \Vdash \forall_{\mathrm{obj}}\bar{s}\colon \forall_{\mathrm{obj}}\bar{t}\colon \forall_{\mathrm{obj}}\bar{u}\colon \bar{s}\in\underline{bs} \Rightarrow$

$\bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r} \vdash \underline{s} \in \underline{bs} \vdash \underline{t} \in \underline{bs} \vdash \underline{u} \in \underline{bs} \vdash \{\{\underline{s},\underline{s}\},\{\underline{s},\underline{t}\}\} \in \underline{r} \vdash \{\{\underline{t},\underline{t}\},\{\underline{t},\underline{u}\}\} \in \underline{r} \vdash$ HelperTransitivity $\mathrel{|\!\!\rhd} \lceil\bar{s}\rceil\#^0\lceil\underline{r}\rceil \mathrel{|\!\!\rhd} \lceil\bar{s}\rceil\#^0\lceil\underline{bs}\rceil \mathrel{|\!\!\rhd} \lceil\bar{t}\rceil\#^0\lceil\underline{r}\rceil \mathrel{|\!\!\rhd} \lceil\bar{t}\rceil\#^0\lceil\underline{bs}\rceil \mathrel{|\!\!\rhd} \lceil\bar{u}\rceil\#^0\lceil\underline{r}\rceil \mathrel{|\!\!\rhd} \lceil\bar{u}\rceil\#^0\lceil\underline{bs}\rceil \gg \forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \forall_{obj}\bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r} \Rightarrow \underline{s} \in \underline{bs} \Rightarrow \underline{t} \in \underline{bs} \Rightarrow \underline{u} \in \underline{bs} \Rightarrow \{\{\underline{s},\underline{s}\},\{\underline{s},\underline{t}\}\} \in \underline{r} \Rightarrow \{\{\underline{t},\underline{t}\},\{\underline{t},\underline{u}\}\} \in \underline{r} \Rightarrow \{\{\underline{s},\underline{s}\},\{\underline{s},\underline{u}\}\} \in \underline{r};$ MP5 $\rhd \forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \forall_{obj}\bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r} \Rightarrow \underline{s} \in \underline{bs} \Rightarrow \underline{t} \in \underline{bs} \Rightarrow \underline{u} \in \underline{bs} \Rightarrow \{\{\underline{s},\underline{s}\},\{\underline{s},\underline{t}\}\} \in \underline{r} \Rightarrow \{\{\underline{t},\underline{t}\},\{\underline{t},\underline{u}\}\} \in \underline{r} \Rightarrow \{\{\underline{s},\underline{s}\},\{\underline{s},\underline{u}\}\} \in \underline{r} \rhd \forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \forall_{obj}\bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r} \rhd \underline{s} \in \underline{bs} \rhd \underline{t} \in \underline{bs} \rhd \underline{u} \in \underline{bs} \rhd \{\{\underline{s},\underline{s}\},\{\underline{s},\underline{t}\}\} \in \underline{r} \gg \{\{\underline{t},\underline{t}\},\{\underline{t},\underline{u}\}\} \in \underline{r} \Rightarrow \{\{\underline{s},\underline{s}\},\{\underline{s},\underline{u}\}\} \in \underline{r};$ MP $\rhd \{\{\underline{t},\underline{t}\},\{\underline{t},\underline{u}\}\} \in \underline{r} \Rightarrow \{\{\underline{s},\underline{s}\},\{\underline{s},\underline{u}\}\} \in \underline{r} \rhd \{\{\underline{t},\underline{t}\},\{\underline{t},\underline{u}\}\} \in \underline{r} \gg \{\{\underline{s},\underline{s}\},\{\underline{s},\underline{u}\}\} \in \underline{r}\rceil, p_0, c)]$

8.4 Ækvivalensrelation

De tre lemmaer i dette underafsnit udsiger, at en ækvivalensrelation er hhv. refleksiv, symmetrisk og transitiv. Beviserne er simple; vi anvender et par tautologier på definitionen af $\dot{\neg}\dot{\neg}\forall_{obj}\bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \forall_{obj}\bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r}$ fra afsnit 5.7:

[ERisReflexive $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall\underline{r}\colon \dot{\neg}\dot{\neg}\forall_{obj}\bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \forall_{obj}\bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r} \vdash \forall_{obj}\bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r}$]

[ERisReflexive $\stackrel{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash \forall\underline{r}\colon \dot{\neg}\dot{\neg}\forall_{obj}\bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \forall_{obj}\bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r} \vdash$ Repetition $\rhd \dot{\neg}\dot{\neg}\forall_{obj}\bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \forall_{obj}\bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r} \gg \dot{\neg}\dot{\neg}\forall_{obj}\bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \forall_{obj}\bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r};$ FirstConjunct $\rhd \dot{\neg}\dot{\neg}\forall_{obj}\bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \forall_{obj}\bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r} \gg \dot{\neg}\forall_{obj}\bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r};$ FirstConjunct $\rhd \dot{\neg}\forall_{obj}\bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \gg \forall_{obj}\bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r}\rceil, p_0, c)]$

$$[\mathrm{ERisSymmetric}\overset{\mathrm{stmt}}{\rightarrow}\mathrm{ZFsub}\vdash\forall\underline{\mathsf{r}}\colon\dot{\neg}\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{s}}\}\}\in\underline{\mathsf{r}}\Rightarrow\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{t}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{s}}\}\}\in\underline{\mathsf{r}}\Rightarrow\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{u}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{t}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{u}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}}\vdash\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{t}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{s}}\}\}\in\underline{\mathsf{r}}]$$

$$[\mathrm{ERisSymmetric}\overset{\mathrm{proof}}{\rightarrow}\lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil\mathrm{ZFsub}\vdash\forall\underline{\mathsf{r}}\colon\dot{\neg}\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{s}}\}\}\in\underline{\mathsf{r}}\Rightarrow\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{t}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{s}}\}\}\in\underline{\mathsf{r}}\Rightarrow\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{u}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{t}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{u}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}}\vdash\mathrm{Repetition}\rhd\dot{\neg}\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{s}}\}\}\in\underline{\mathsf{r}}\Rightarrow\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{t}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{s}}\}\}\in\underline{\mathsf{r}}\Rightarrow\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{u}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{t}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{u}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}}\gg\dot{\neg}\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{s}}\}\}\in\underline{\mathsf{r}}\Rightarrow\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{t}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{s}}\}\}\in\underline{\mathsf{r}}\Rightarrow\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{u}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{t}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{u}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}};\mathrm{FirstConjunct}\rhd\dot{\neg}\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{s}}\}\}\in\underline{\mathsf{r}}\Rightarrow\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{t}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{s}}\}\}\in\underline{\mathsf{r}}\Rightarrow\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{u}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{t}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{u}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}}\gg\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{s}}\}\}\in\underline{\mathsf{r}}\Rightarrow\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{t}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{s}}\}\}\in\underline{\mathsf{r}};\mathrm{SecondConjunct}\rhd\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{s}}\}\}\in\underline{\mathsf{r}}\Rightarrow\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{t}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{s}}\}\}\in\underline{\mathsf{r}}\gg\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{t}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{s}}\}\}\in\underline{\mathsf{r}}\rceil,\mathrm{p}_0,\mathsf{c})]$$

$$[\mathrm{ERisTransitive}\overset{\mathrm{stmt}}{\rightarrow}\mathrm{ZFsub}\vdash\forall\underline{\mathsf{r}}\colon\dot{\neg}\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{s}}\}\}\in\underline{\mathsf{r}}\Rightarrow\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{t}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{s}}\}\}\in\underline{\mathsf{r}}\Rightarrow\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{u}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{t}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{u}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}}\vdash\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{u}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{t}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{u}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}}]$$

$$[\mathrm{ERisTransitive}\overset{\mathrm{proof}}{\rightarrow}\lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil\mathrm{ZFsub}\vdash\forall\underline{\mathsf{r}}\colon\dot{\neg}\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{s}}\}\}\in\underline{\mathsf{r}}\Rightarrow\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{t}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{s}}\}\}\in\underline{\mathsf{r}}\Rightarrow\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{u}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{t}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{u}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}}\vdash\mathrm{Repetition}\rhd\dot{\neg}\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{s}}\}\}\in\underline{\mathsf{r}}\Rightarrow\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{t}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{s}}\}\}\in\underline{\mathsf{r}}\Rightarrow\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{u}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{t}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{u}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}}\gg\dot{\neg}\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{s}}\}\}\in\underline{\mathsf{r}}\Rightarrow\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{t}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{s}}\}\}\in\underline{\mathsf{r}}\Rightarrow\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{u}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{t}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{u}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}};\mathrm{SecondConjunct}\rhd\dot{\neg}\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{s}}\}\}\in\underline{\mathsf{r}}\Rightarrow\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{t}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{s}}\}\}\in\underline{\mathsf{r}}\Rightarrow\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{u}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{t}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{u}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}}\gg\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{u}}\colon\bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{t}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{u}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}}\rceil,\mathrm{p}_0,\mathsf{c})]$$

9 Basale sætninger i mængdelære

Lemmaerne i dette afsnit er en løs samling af mængdeteoretiske sætninger, som bliver brugt i beviset for hovedresultatet i afsnit 11. Lemmaerne er grupperet efter emne: Underafsnit 9.1 omhandler den tomme mængde, underafsnit 9.2 handler om lighedsrelationen $[\mathsf{x}=\mathsf{y}]$, og underafsnit 9.3 handler om negeret lighed $(\dot{\neg}\,\mathsf{x}=\mathsf{y})$. Figur 2 giver et overblik over lemmaerne.

Figur 2: Bevisstrukturen for afsnit 9. Kun lemmaer fra dette afsnit er vist. Tallene i parentes angiver, hvor mange sidebetingelser de forskellige lemmaer indeholder. En pil fra lemma x til lemma y betyder, at x bruges i beviset for y. Tallene ved pilene angiver, hvor mange sidebetingelser y modtager fra x. Hvis der f.eks. står “$2 \cdot \mathbf{4}$”, betyder det, at x bliver anvendt to gange i beviset for y, begge gange med 4 sidebetingelser. Der er overlap mellem nogle af sidebetingelserne, og lemmaer kan generere deres egne sidebetingelser. Derfor er der ikke nødvendigvis overensstemmelse mellem tallene i parentes og tallene ved lemmaerne.

9.1 Den tomme mængde

Dette underafsnit indeholder 3 lemmaer, der hver har fået sit under-underafsnit.

9.1.1 En delmængde af alle mængder

Vi viser først, at den tomme mængde er en delmængde af alle mængder:

$[\text{ØisSubset} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{s}} \colon \forall \underline{\mathsf{x}} \colon \underline{\mathsf{s}} \in \text{Ø} \Rightarrow \underline{\mathsf{s}} \in \underline{\mathsf{x}}]$

Beviset for ØisSubset benytter sig af deduktionsreglen og tautologien FromContradiction:

$[\text{ØisSubset} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{s}} \colon \forall \underline{\mathsf{x}} \colon \underline{\mathsf{s}} \in \text{Ø} \vdash \text{Ødef} \gg \dot{\neg}\, \underline{\mathsf{s}} \in \text{Ø}; \text{FromContradiction} \rhd \underline{\mathsf{s}} \in \text{Ø} \rhd \dot{\neg}\, \underline{\mathsf{s}} \in \text{Ø} \gg \underline{\mathsf{s}} \in \underline{\mathsf{x}}; \forall \underline{\mathsf{s}} \colon \forall \underline{\mathsf{x}} \colon \text{Ded} \rhd \forall \underline{\mathsf{s}} \colon \forall \underline{\mathsf{x}} \colon \underline{\mathsf{s}} \in \text{Ø} \vdash \underline{\mathsf{s}} \in \underline{\mathsf{x}} \gg \underline{\mathsf{s}} \in \text{Ø} \Rightarrow \underline{\mathsf{s}} \in \underline{\mathsf{x}} \rceil, \mathrm{p}_0, \mathsf{c})]$

9.1.2 Der er kun én tom mængde

Det næste lemma beviser påstanden "der er kun én tom mængde". Vi formulerer denne påstand som følger: "Hvis en mængde $\underline{\mathsf{x}}$ ikke har nogen medlemmer, så er $\underline{\mathsf{x}}$ lig med Ø":

$[\text{UniqueØ} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{x}} \colon \dot{\neg}\, \bar{\mathsf{s}} \in \underline{\mathsf{x}} \vdash \underline{\mathsf{x}} = \text{Ø}]$

Vi ved allerede fra lemmaet ØisSubset, at Ø er en delmængde af $\underline{\mathsf{x}}$. Altså mangler vi blot at vise, at $\underline{\mathsf{x}}$ er en delmængde af Ø. Dette klarer vi med hjælpelemmaet HelperUniqueØ:

$[\text{HelperUniqueØ} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{s}} \colon \forall \underline{\mathsf{x}} \colon \dot{\neg}\, \underline{\mathsf{s}} \in \underline{\mathsf{x}} \vdash \underline{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \underline{\mathsf{s}} \in \text{Ø}]$

$[\text{HelperUniqueØ} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{s}} \colon \forall \underline{\mathsf{x}} \colon \dot{\neg}\, \underline{\mathsf{s}} \in \underline{\mathsf{x}} \vdash \underline{\mathsf{s}} \in \underline{\mathsf{x}} \vdash \text{FromContradiction} \rhd \underline{\mathsf{s}} \in \underline{\mathsf{x}} \rhd \dot{\neg}\, \underline{\mathsf{s}} \in \underline{\mathsf{x}} \gg \underline{\mathsf{s}} \in \text{Ø}; \forall \underline{\mathsf{s}} \colon \forall \underline{\mathsf{x}} \colon \text{Ded} \rhd \forall \underline{\mathsf{s}} \colon \forall \underline{\mathsf{x}} \colon \dot{\neg}\, \underline{\mathsf{s}} \in \underline{\mathsf{x}} \vdash \underline{\mathsf{s}} \in \underline{\mathsf{x}} \vdash \underline{\mathsf{s}} \in \text{Ø} \gg \dot{\neg}\, \underline{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \underline{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \underline{\mathsf{s}} \in \text{Ø}; \dot{\neg}\, \underline{\mathsf{s}} \in \underline{\mathsf{x}} \vdash \text{MP} \rhd \dot{\neg}\, \underline{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \underline{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \underline{\mathsf{s}} \in \text{Ø} \rhd \dot{\neg}\, \underline{\mathsf{s}} \in \underline{\mathsf{x}} \gg \underline{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \underline{\mathsf{s}} \in \text{Ø} \rceil, \mathrm{p}_0, \mathsf{c})]$

Med HelperUniqueØ til rådighed er det let at vise UniqueØ:

$[\text{UniqueØ} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{x}} \colon \dot{\neg}\, \bar{\mathsf{s}} \in \underline{\mathsf{x}} \vdash \underline{\mathsf{x}} = \text{Ø}]$

$[\text{UniqueØ} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{x}} \colon \dot{\neg}\, \bar{\mathsf{s}} \in \underline{\mathsf{x}} \vdash \text{HelperUniqueØ} \rhd \dot{\neg}\, \bar{\mathsf{s}} \in \underline{\mathsf{x}} \gg \bar{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{s}} \in \text{Ø}; \text{ØisSubset} \gg \bar{\mathsf{s}} \in \text{Ø} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{x}}; \text{ToSetEquality} \rhd \bar{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{s}} \in \text{Ø} \rhd \bar{\mathsf{s}} \in \text{Ø} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{x}} \gg \underline{\mathsf{x}} = \text{Ø} \rceil, \mathrm{p}_0, \mathsf{c})]$

9.1.3 Lemmaet "MemberNotØ"

Som det sidste resultat i dette underafsnit viser vi, at en mængde, der har et medlem, ikke er lig med den tomme mængde:

$[\text{MemberNotØ} \overset{\text{stmt}}{\to} \text{ZFsub} \vdash \forall \underline{s}\colon \forall \underline{x}\colon \lceil \bar{s} \rceil \#^0 \lceil \underline{x} \rceil \Vdash \underline{s} \in \underline{x} \vdash \dot{\neg}\, \underline{x} = \text{Ø}]$

Beviset for MemberNotØ kan gengives som følger:

1. "Antag at $\underline{s}$ tilhører $\underline{x}$. Hvis $\underline{x}$ var lig med Ø, så ville $\underline{s}$ også tilhøre Ø".
2. "Men $\underline{s}$ tilhører ikke Ø. Derfor er $\underline{x}$ ikke lig med Ø".

Vi viser punkt 1 som lemmastumpen HelperMemberNotØ; herefter gennemfører vi punkt 2 i selve beviset for MemberNotØ:

$[\text{HelperMemberNotØ} \overset{\text{stmt}}{\to} \text{ZFsub} \vdash \forall \underline{s}\colon \forall \underline{x}\colon \lceil \bar{s} \rceil \#^0 \lceil \underline{x} \rceil \Vdash \underline{s} \in \underline{x} \Rightarrow \underline{x} = \text{Ø} \Rightarrow \underline{s} \in \text{Ø}]$

$[\text{HelperMemberNotØ} \overset{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{s}\colon \forall \underline{x}\colon \lceil \bar{s} \rceil \#^0 \lceil \underline{x} \rceil \Vdash \underline{s} \in \underline{x} \vdash \underline{x} = \text{Ø} \vdash \text{FromSetEquality} \unrhd \lceil \bar{s} \rceil \#^0 \lceil \underline{x} \rceil \rhd \underline{x} = \text{Ø} \rhd \underline{s} \in \underline{x} \gg \underline{s} \in \text{Ø}; \forall \underline{s}\colon \forall \underline{x}\colon \text{Ded} \rhd \forall \underline{s}\colon \forall \underline{x}\colon \lceil \bar{s} \rceil \#^0 \lceil \underline{x} \rceil \Vdash \underline{s} \in \underline{x} \vdash \underline{x} = \text{Ø} \vdash \underline{s} \in \text{Ø} \gg \lceil \bar{s} \rceil \#^0 \lceil \underline{x} \rceil \Vdash \underline{s} \in \underline{x} \Rightarrow \underline{x} = \text{Ø} \Rightarrow \underline{s} \in \text{Ø} \rceil, p_0, c)]$

$[\text{MemberNotØ} \overset{\text{stmt}}{\to} \text{ZFsub} \vdash \forall \underline{s}\colon \forall \underline{x}\colon \lceil \bar{s} \rceil \#^0 \lceil \underline{x} \rceil \Vdash \underline{s} \in \underline{x} \vdash \dot{\neg}\, \underline{x} = \text{Ø}]$

$[\text{MemberNotØ} \overset{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{s}\colon \forall \underline{x}\colon \lceil \bar{s} \rceil \#^0 \lceil \underline{x} \rceil \Vdash \underline{s} \in \underline{x} \vdash \text{HelperMemberNotØ} \unrhd \lceil \bar{s} \rceil \#^0 \lceil \underline{x} \rceil \gg \underline{s} \in \underline{x} \Rightarrow \underline{x} = \text{Ø} \Rightarrow \underline{s} \in \text{Ø}; \text{MP} \rhd \underline{s} \in \underline{x} \Rightarrow \underline{x} = \text{Ø} \Rightarrow \underline{s} \in \text{Ø} \rhd \underline{s} \in \underline{x} \gg \underline{x} = \text{Ø} \Rightarrow \underline{s} \in \text{Ø}; \text{Ødef} \gg \dot{\neg}\, \underline{s} \in \text{Ø}; \text{MT} \rhd \underline{x} = \text{Ø} \Rightarrow \underline{s} \in \text{Ø} \rhd \dot{\neg}\, \underline{s} \in \text{Ø} \gg \dot{\neg}\, \underline{x} = \text{Ø} \rceil, p_0, c)]$

9.2 [x=y] er en ækvivalensrelation

Målet med dette underafsnit er at vise, at [x=y] er en refleksiv, symmetrisk og transitiv relation.
At [x=y] er refleksiv, følger af, at [x ⇒ y] er refleksiv:

$[=\text{Reflexivity} \overset{\text{stmt}}{\to} \text{ZFsub} \vdash \forall \underline{s}\colon \underline{s} = \underline{s}]$

$[=\text{Reflexivity} \overset{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{s}\colon \text{AutoImply} \gg \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{s}; \text{ToSetEquality} \rhd \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{s} \rhd \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{s} \gg \underline{s} = \underline{s} \rceil, p_0, c)]$

Symmetri-lemmaet for [x=y] ser således ud:

$[=\text{Symmetry} \overset{\text{stmt}}{\to} \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \lceil \bar{s} \rceil \#^0 \lceil \underline{x} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{y} \rceil \Vdash \underline{x} = \underline{y} \vdash \underline{y} = \underline{x}]$

Vi beviser =Symmetry ved at reducere termen $[\underline{x} = \underline{y}]$ til de to implikationer $[\bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y}]$ og $[\bar{s} \in \underline{y} \Rightarrow \bar{s} \in \underline{x}]$. Herefter sætter vi de to implikationer sammen i omvendt rækkefølge, hvilket giver os $[\underline{y} = \underline{x}]$:

[=Symmetry $\overset{\text{proof}}{\to}$ $\lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil \Vdash \underline{x}=\underline{y} \vdash$ Extensionality $\gg \dot\neg\, \underline{x}=\underline{y} \Rightarrow \forall_{obj}\bar{s}\colon \dot\neg\, \bar{s}\in\underline{x} \Rightarrow \bar{s}\in\underline{y} \Rightarrow \dot\neg\,\bar{s}\in\underline{y}\Rightarrow\bar{s}\in\underline{x}\Rightarrow \dot\neg\,\forall_{obj}\bar{s}\colon \dot\neg\,\bar{s}\in\underline{x}\Rightarrow\bar{s}\in\underline{y}\Rightarrow\dot\neg\,\bar{s}\in\underline{y}\Rightarrow\bar{s}\in\underline{x}\Rightarrow\underline{x}=\underline{y}$; IffSecond $\rhd \dot\neg\,\underline{x}=\underline{y}\Rightarrow \forall_{obj}\bar{s}\colon \dot\neg\, \bar{s}\in\underline{x}\Rightarrow\bar{s}\in\underline{y}\Rightarrow\dot\neg\,\bar{s}\in\underline{y}\Rightarrow\bar{s}\in\underline{x}\Rightarrow\dot\neg\,\forall_{obj}\bar{s}\colon\dot\neg\,\bar{s}\in\underline{x}\Rightarrow\bar{s}\in\underline{y}\Rightarrow\dot\neg\,\bar{s}\in\underline{y}\Rightarrow\bar{s}\in\underline{x}\Rightarrow\underline{x}=\underline{y} \rhd \underline{x}=\underline{y}\gg\forall_{obj}\bar{s}\colon\dot\neg\,\bar{s}\in\underline{x}\Rightarrow\bar{s}\in\underline{y}\Rightarrow\dot\neg\,\bar{s}\in\underline{y}\Rightarrow\bar{s}\in\underline{x}$; HelperFromSetEquality $\unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil \unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil \gg \forall_{obj}\bar{s}\colon \dot\neg\,\bar{s}\in\underline{x}\Rightarrow\bar{s}\in\underline{y}\Rightarrow\dot\neg\,\bar{s}\in\underline{y}\Rightarrow\bar{s}\in\underline{x}\Rightarrow\dot\neg\,\bar{s}\in\underline{x}\Rightarrow\bar{s}\in\underline{y}\Rightarrow\dot\neg\,\bar{s}\in\underline{y}\Rightarrow\bar{s}\in\underline{x}$; MP $\rhd \forall_{obj}\bar{s}\colon \dot\neg\,\bar{s}\in\underline{x}\Rightarrow\bar{s}\in\underline{y}\Rightarrow\dot\neg\,\bar{s}\in\underline{y}\Rightarrow\bar{s}\in\underline{x}\Rightarrow\dot\neg\,\bar{s}\in\underline{x}\Rightarrow\bar{s}\in\underline{y}\Rightarrow\dot\neg\,\bar{s}\in\underline{y}\Rightarrow\bar{s}\in\underline{x} \rhd \forall_{obj}\bar{s}\colon \dot\neg\,\bar{s}\in\underline{x}\Rightarrow\bar{s}\in\underline{y}\Rightarrow\dot\neg\,\bar{s}\in\underline{y}\Rightarrow\bar{s}\in\underline{x} \gg \dot\neg\,\bar{s}\in\underline{x}\Rightarrow\bar{s}\in\underline{y}\Rightarrow\dot\neg\,\bar{s}\in\underline{y}\Rightarrow\bar{s}\in\underline{x}$; FirstConjunct $\rhd \dot\neg\,\bar{s}\in\underline{x}\Rightarrow\bar{s}\in\underline{y}\Rightarrow\dot\neg\,\bar{s}\in\underline{y}\Rightarrow\bar{s}\in\underline{x}\gg\bar{s}\in\underline{x}\Rightarrow\bar{s}\in\underline{y}$; SecondConjunct $\rhd \dot\neg\,\bar{s}\in\underline{x}\Rightarrow\bar{s}\in\underline{y}\Rightarrow\dot\neg\,\bar{s}\in\underline{y}\Rightarrow\bar{s}\in\underline{x}\gg\bar{s}\in\underline{y}\Rightarrow\bar{s}\in\underline{x}$; ToSetEquality $\rhd \bar{s}\in\underline{y}\Rightarrow\bar{s}\in\underline{x}\rhd\bar{s}\in\underline{x}\Rightarrow\bar{s}\in\underline{y}\gg\underline{y}=\underline{x}\rceil, p_0, c)]$

Transitivitets-lemmaet for [x=y] ser således ud:

[=Transitivity $\overset{\text{stmt}}{\to}$ ZFsub $\vdash \forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{z}\colon \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{z}\rceil \Vdash \underline{x}=\underline{y}\vdash\underline{y}=\underline{z}\vdash\underline{x}=\underline{z}]$

Vi viser først den svagere påstand, at $\underline{x}$ er en delmængde af $\underline{z}$:

[Helper=Transitivity $\overset{\text{stmt}}{\to}$ ZFsub $\vdash \forall\underline{s}\colon\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{z}\colon \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{z}\rceil \Vdash \underline{x}=\underline{y}\Rightarrow\underline{y}=\underline{z}\Rightarrow\underline{s}\in\underline{x}\Rightarrow\underline{s}\in\underline{z}]$

Bevisets kerne består af to anvendelser af FromSetEquality (linie 8–9):

[Helper=Transitivity $\overset{\text{proof}}{\to}$ $\lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash \forall\underline{s}\colon\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{z}\colon \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{z}\rceil \Vdash \underline{x}=\underline{y}\vdash\underline{y}=\underline{z}\vdash\underline{s}\in\underline{x}\vdash$ FromSetEquality $\unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil \unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil \rhd \underline{x}=\underline{y}\rhd\underline{s}\in\underline{x}\gg\underline{s}\in\underline{y}$; FromSetEquality $\unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil \unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{z}\rceil \rhd \underline{y}=\underline{z}\rhd\underline{s}\in\underline{y}\gg\underline{s}\in\underline{z}$; $\forall\underline{s}\colon\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{z}\colon$ Ded $\rhd \forall\underline{s}\colon\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{z}\colon \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{z}\rceil \Vdash \underline{x}=\underline{y}\vdash\underline{y}=\underline{z}\vdash\underline{s}\in\underline{x}\vdash\underline{s}\in\underline{z}\gg\lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{z}\rceil \Vdash \underline{x}=\underline{y}\Rightarrow\underline{y}=\underline{z}\Rightarrow\underline{s}\in\underline{x}\Rightarrow\underline{s}\in\underline{z}\rceil, p_0, c)]$

Vi bruger nu Helper=Transitivity til at vise, at $\underline{x}$ er en delmængde af $\underline{z}$ (linie 5–8), og vice versa (line 9–12):

[=Transitivity $\overset{\text{stmt}}{\to}$ ZFsub $\vdash \forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{z}\colon \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{z}\rceil \Vdash \underline{x}=\underline{y}\vdash\underline{y}=\underline{z}\vdash\underline{x}=\underline{z}]$

[=Transitivity $\overset{\text{proof}}{\to}$ $\lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash \forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{z}\colon \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{z}\rceil \Vdash \underline{x}=\underline{y}\vdash\underline{y}=\underline{z}\vdash$ Helper=Transitivity $\unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil \unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil \unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{z}\rceil \gg \underline{x}=\underline{y}\Rightarrow\underline{y}=\underline{z}\Rightarrow\bar{s}\in\underline{x}\Rightarrow\bar{s}\in\underline{z}$; MP2 $\rhd \underline{x}=\underline{y}\Rightarrow\underline{y}=\underline{z}\Rightarrow\bar{s}\in\underline{x}\Rightarrow\bar{s}\in\underline{z}\rhd\underline{x}=\underline{y}\rhd\underline{y}=\underline{z}\gg\bar{s}\in\underline{x}\Rightarrow\bar{s}\in\underline{z}$; =Symmetry $\unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil \unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil \rhd \underline{x}=\underline{y}\gg\underline{y}=\underline{x}$; =Symmetry $\unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil \unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{z}\rceil \rhd \underline{y}=\underline{z}\gg\underline{z}=\underline{y}$; Helper=Transitivity $\unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{z}\rceil \unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil \unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil \gg \underline{z}=\underline{y}\Rightarrow\underline{y}=\underline{x}\Rightarrow\bar{s}\in\underline{z}\Rightarrow\bar{s}\in\underline{x}$; MP2 $\rhd \underline{z}=\underline{y}\Rightarrow\underline{y}=\underline{x}\Rightarrow\bar{s}\in\underline{z}\Rightarrow\bar{s}\in\underline{x}\rhd\underline{z}=\underline{y}\rhd\underline{y}=\underline{x}\gg\bar{s}\in\underline{z}\Rightarrow\bar{s}\in\underline{x}$; ToSetEquality $\rhd \bar{s}\in\underline{x}\Rightarrow\bar{s}\in\underline{z}\rhd\bar{s}\in\underline{z}\Rightarrow\bar{s}\in\underline{x}\gg\underline{x}=\underline{z}\rceil, p_0, c)]$

9.3 Negeret lighed

I dette sidste underafsnit viser vi, at lige størrelser kan erstatte hinanden i en negeret lighed:

$[\mathrm{TransferNotEq} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \forall \underline{\mathsf{v}}\colon \forall \underline{\mathsf{w}}\colon \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{y}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{v}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{w}} \rceil \Vdash \dot{\neg}\, \underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \underline{\mathsf{x}} = \underline{\mathsf{v}} \vdash \underline{\mathsf{y}} = \underline{\mathsf{w}} \vdash \dot{\neg}\, \underline{\mathsf{v}} = \underline{\mathsf{w}}]$

Beviset for TransferNotEq kan gengives som følger:

1. "Antag at $\underline{\mathsf{v}}$ er lig med $\underline{\mathsf{w}}$. Ud fra præmisserne $[\underline{\mathsf{x}} = \underline{\mathsf{v}}]$ og $[\underline{\mathsf{y}} = \underline{\mathsf{w}}]$ får vi da $[\underline{\mathsf{x}} = \underline{\mathsf{y}}]$."
2. "Men vi har antaget $[\dot{\neg}\, \underline{\mathsf{x}} = \underline{\mathsf{y}}]$. Derfor kan $\underline{\mathsf{v}}$ ikke være lig med $\underline{\mathsf{w}}$."

Vi viser punkt 1 som lemmastumpen HelperTransferNotEq; punkt 2 klares i selve beviset for TransferNotEq:

$[\mathrm{HelperTransferNotEq} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \forall \underline{\mathsf{v}}\colon \forall \underline{\mathsf{w}}\colon \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{y}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{v}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{w}} \rceil \Vdash \underline{\mathsf{x}} = \underline{\mathsf{v}} \Rightarrow \underline{\mathsf{y}} = \underline{\mathsf{w}} \Rightarrow \underline{\mathsf{v}} = \underline{\mathsf{w}} \Rightarrow \underline{\mathsf{x}} = \underline{\mathsf{y}}]$

$[\mathrm{HelperTransferNotEq} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \forall \underline{\mathsf{v}}\colon \forall \underline{\mathsf{w}}\colon \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{y}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{v}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{w}} \rceil \Vdash \underline{\mathsf{x}} = \underline{\mathsf{v}} \vdash \underline{\mathsf{y}} = \underline{\mathsf{w}} \vdash \underline{\mathsf{v}} = \underline{\mathsf{w}} \vdash {=}\mathrm{Transitivity} \mathrel{|\!\!\rhd} \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \mathrel{|\!\!\rhd} \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{v}} \rceil \mathrel{|\!\!\rhd} \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{w}} \rceil \rhd \underline{\mathsf{x}} = \underline{\mathsf{v}} \rhd \underline{\mathsf{v}} = \underline{\mathsf{w}} \gg \underline{\mathsf{x}} = \underline{\mathsf{w}}; {=}\mathrm{Symmetry} \mathrel{|\!\!\rhd} \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{y}} \rceil \mathrel{|\!\!\rhd} \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{w}} \rceil \rhd \underline{\mathsf{y}} = \underline{\mathsf{w}} \gg \underline{\mathsf{w}} = \underline{\mathsf{y}}; {=}\mathrm{Transitivity} \mathrel{|\!\!\rhd} \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \mathrel{|\!\!\rhd} \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{w}} \rceil \mathrel{|\!\!\rhd} \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{y}} \rceil \rhd \underline{\mathsf{x}} = \underline{\mathsf{w}} \rhd \underline{\mathsf{w}} = \underline{\mathsf{y}} \gg \underline{\mathsf{x}} = \underline{\mathsf{y}}; \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \forall \underline{\mathsf{v}}\colon \forall \underline{\mathsf{w}}\colon \mathrm{Ded} \rhd \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \forall \underline{\mathsf{v}}\colon \forall \underline{\mathsf{w}}\colon \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{y}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{v}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{w}} \rceil \Vdash \underline{\mathsf{x}} = \underline{\mathsf{v}} \vdash \underline{\mathsf{y}} = \underline{\mathsf{w}} \vdash \underline{\mathsf{v}} = \underline{\mathsf{w}} \vdash \underline{\mathsf{x}} = \underline{\mathsf{y}} \gg \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{y}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{v}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{w}} \rceil \Vdash \underline{\mathsf{x}} = \underline{\mathsf{v}} \Rightarrow \underline{\mathsf{y}} = \underline{\mathsf{w}} \Rightarrow \underline{\mathsf{v}} = \underline{\mathsf{w}} \Rightarrow \underline{\mathsf{x}} = \underline{\mathsf{y}}\rceil, \mathrm{p}_0, \mathsf{c})]$

[TransferNotEq $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \forall \underline{\mathsf{v}}\colon \forall \underline{\mathsf{w}}\colon \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{y}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{v}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{w}} \rceil \Vdash \dot{\neg}\, \underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \underline{\mathsf{x}} = \underline{\mathsf{v}} \vdash \underline{\mathsf{y}} = \underline{\mathsf{w}} \vdash \dot{\neg}\, \underline{\mathsf{v}} = \underline{\mathsf{w}}$]

[TransferNotEq $\overset{\text{proof}}{\rightarrow}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \forall \underline{\mathsf{v}}\colon \forall \underline{\mathsf{w}}\colon \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{y}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{v}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{w}} \rceil \Vdash \dot{\neg}\, \underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \underline{\mathsf{x}} = \underline{\mathsf{v}} \vdash \underline{\mathsf{y}} = \underline{\mathsf{w}} \vdash$ HelperTransferNotEq $\unrhd \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \unrhd \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{y}} \rceil \unrhd \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{v}} \rceil \unrhd \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{w}} \rceil \gg \underline{\mathsf{x}} = \underline{\mathsf{v}} \Rightarrow \underline{\mathsf{y}} = \underline{\mathsf{w}} \Rightarrow \underline{\mathsf{v}} = \underline{\mathsf{w}} \Rightarrow \underline{\mathsf{x}} = \underline{\mathsf{y}};$ MP2 $\rhd \underline{\mathsf{x}} = \underline{\mathsf{v}} \Rightarrow \underline{\mathsf{y}} = \underline{\mathsf{w}} \Rightarrow \underline{\mathsf{v}} = \underline{\mathsf{w}} \Rightarrow \underline{\mathsf{x}} = \underline{\mathsf{y}} \rhd \underline{\mathsf{x}} = \underline{\mathsf{v}} \rhd \underline{\mathsf{y}} = \underline{\mathsf{w}} \gg \underline{\mathsf{v}} = \underline{\mathsf{w}} \Rightarrow \underline{\mathsf{x}} = \underline{\mathsf{y}};$ MT $\rhd \underline{\mathsf{v}} = \underline{\mathsf{w}} \Rightarrow \underline{\mathsf{x}} = \underline{\mathsf{y}} \rhd \dot{\neg}\, \underline{\mathsf{x}} = \underline{\mathsf{y}} \gg \dot{\neg}\, \underline{\mathsf{v}} = \underline{\mathsf{w}} \rceil, \mathrm{p}_0, \mathsf{c})$]

10 Lighedslemmaer

Ved et “unært lighedslemma” vil jeg forstå et lemma af følgende form: “Hvis der gælder [$\underline{\mathsf{x}} = \underline{\mathsf{y}}$], så gælder [Op($\underline{\mathsf{x}}$)$=$Op($\underline{\mathsf{y}}$)]” — hvor “ [Op($\underline{\mathsf{x}}$)]” står for en unær syntaktisk konstruktion som f.eks. [$\cup \mathsf{x}$] eller [$\{\mathsf{x}, \mathsf{x}\}$].
Tilsvarende kan vi definere et “binært lighedslemma” som et lemma med indholdet: “Hvis der gælder [$\underline{\mathsf{x}} = \underline{\mathsf{y}}$] og [$\underline{\mathsf{v}} = \underline{\mathsf{w}}$], så gælder [Op($\underline{\mathsf{x}}, \underline{\mathsf{v}}$)$=$Op($\underline{\mathsf{y}}, \underline{\mathsf{w}}$)]” — her kan “[Op($\mathsf{x}, \mathsf{y}$)]” f.eks. stå for [$\{\mathsf{x}, \mathsf{y}\}$] eller [$\{\mathrm{ph} \in \cup\{\{\mathsf{x}, \mathsf{x}\}, \{\mathsf{y}, \mathsf{y}\}\} \mid \dot{\neg}\, \mathsf{c}_{\mathrm{Ph}} \in \mathsf{x} \Rightarrow \dot{\neg}\, \mathsf{c}_{\mathrm{Ph}} \in \mathsf{y}\}$].
I dette afsnit beviser vi lighedslemmaerne for konstruktionerne [$\{\mathsf{x}, \mathsf{y}\}$], [$\{\mathsf{x}, \mathsf{x}\}$], [$\cup \mathsf{x}$], [$\cup\{\{\mathsf{x}, \mathsf{x}\}, \{\mathsf{y}, \mathsf{y}\}\}$], [$\{\mathrm{ph} \in \mathsf{x} \mid \mathsf{f}\}$] og [$\{\mathrm{ph} \in \cup\{\{\mathsf{x}, \mathsf{x}\}, \{\mathsf{y}, \mathsf{y}\}\} \mid \dot{\neg}\, \mathsf{c}_{\mathrm{Ph}} \in \mathsf{x} \Rightarrow \dot{\neg}\, \mathsf{c}_{\mathrm{Ph}} \in \mathsf{y}\}$]. Figur 3 giver et overblik over afsnittets lemmaer. Hele arbejdet i dette afsnit munder ud i én eneste anvendelse af det sidste lighedslemma (“SameIntersection”) i beviset for hovedresultatet (i afsnit 11.2.5). Man kan altså godt sige, at vi skyder gråspurve med kanoner. Lighedslemmaer har dog også interesse derved, at de kan bruges til at vise, at ZFsub er en teori med lighed (jvf. afsnit 2.8 i [6]); men det er altså ikke et sådant bevis, som er formålet med dette afsnit.

Figur 3: Bevisstrukturen for lighedslemmaerne. Kun lighedslemmaerne er vist. Figuren skal læses ligesom figur 2.

10.1 Par

Når vi skal vise et lighedslemma med konklusionen [$\mathrm{Op}(\underline{x}, \underline{v}) = \mathrm{Op}(\underline{y}, \underline{w})$], er det ofte en god ide først at vise et "delmængdelemma" — dvs. et lemma, der konkluderer, at $\mathrm{Op}(\underline{x}, \underline{v})$ er en delmængde af $\mathrm{Op}(\underline{y}, \underline{w})$. Vi kan da vise lighedslemmaet ud fra ToSetEquality og to anvendelser af delmængdelemmaet. I dette underafsnit begynder vi med det følgende delmængdelemma:

[PairSubset $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{s} \colon \forall \underline{x} \colon \forall \underline{y} \colon \forall \underline{v} \colon \forall \underline{w} \colon \lceil \bar{s} \rceil \#^0 \lceil \underline{x} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{y} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{v} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{w} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{s} \rceil \Vdash \underline{x} = \underline{y} \Rightarrow \underline{v} = \underline{w} \Rightarrow \underline{s} \in \{\underline{x}, \underline{v}\} \Rightarrow \underline{s} \in \{\underline{y}, \underline{w}\}$]

Beviset for PairSubset kan gengives som følger:

1. "Antag [$\underline{x} = \underline{y}$], [$\underline{v} = \underline{w}$] og at $\underline{s}$ tilhører $\{\underline{x}, \underline{y}\}$. Der gælder da $\underline{s} = \underline{x}$ eller $\underline{s} = \underline{y}$."
2. "$\underline{s} = \underline{x}$ medfører [$\dot{\neg}\, \underline{s} = \underline{y} \Rightarrow \underline{s} = \underline{w}$] (under antagelse af [$\underline{x} = \underline{y}$])."
3. "$\underline{s} = \underline{y}$ medfører også [$\dot{\neg}\, \underline{s} = \underline{y} \Rightarrow \underline{s} = \underline{w}$]."
4. "Derfor må [$\dot{\neg}\, \underline{s} = \underline{y} \Rightarrow \underline{s} = \underline{w}$] gælde ubetinget. Dette betyder igen, at $\underline{s}$ tilhører $\{\underline{y}, \underline{w}\}$, QED."

Punkt 2 og 3 varetages af de to lemmastumper HelperPairSubset og Helper(2)PairSubset. Punkt 1 og 4 vises i selve beviset for PairSubset:

[HelperPairSubset $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{s} \colon \forall \underline{x} \colon \forall \underline{y} \colon \forall \underline{w} \colon \lceil \bar{s} \rceil \#^0 \lceil \underline{s} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{x} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{y} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{w} \rceil \Vdash \underline{x} = \underline{y} \Rightarrow \underline{s} = \underline{x} \Rightarrow \dot{\neg}\, \underline{s} = \underline{y} \Rightarrow \underline{s} = \underline{w}$]

[HelperPairSubset $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c. \lambda x. \mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{s} \colon \forall \underline{x} \colon \forall \underline{y} \colon \forall \underline{w} \colon \lceil \bar{s} \rceil \#^0 \lceil \underline{s} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{x} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{y} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{w} \rceil \Vdash \underline{x} = \underline{y} \vdash \underline{s} = \underline{x} \vdash$ =Transitivity $\mathrel{|\!\rhd} \lceil \bar{s} \rceil \#^0 \lceil \underline{s} \rceil \mathrel{|\!\rhd} \lceil \bar{s} \rceil \#^0 \lceil \underline{x} \rceil \mathrel{|\!\rhd} \lceil \bar{s} \rceil \#^0 \lceil \underline{y} \rceil \rhd \underline{s} = \underline{x} \rhd \underline{x} = \underline{y} \gg \underline{s} = \underline{y}$; WeakenOr2 $\rhd \underline{s} = \underline{y} \gg \dot{\neg}\, \underline{s} = \underline{y} \Rightarrow \underline{s} = \underline{w}; \forall \underline{s} \colon \forall \underline{x} \colon \forall \underline{y} \colon \forall \underline{w} \colon$ Ded $\rhd \forall \underline{s} \colon \forall \underline{x} \colon \forall \underline{y} \colon \forall \underline{w} \colon \lceil \bar{s} \rceil \#^0 \lceil \underline{s} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{x} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{y} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{w} \rceil \Vdash \underline{x} = \underline{y} \vdash \underline{s} = \underline{x} \vdash \dot{\neg}\, \underline{s} = \underline{y} \Rightarrow \underline{s} = \underline{w} \gg \lceil \bar{s} \rceil \#^0 \lceil \underline{s} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{x} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{y} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{w} \rceil \Vdash \underline{x} = \underline{y} \Rightarrow \underline{s} = \underline{x} \Rightarrow \dot{\neg}\, \underline{s} = \underline{y} \Rightarrow \underline{s} = \underline{w} \rceil, p_0, c)$]

[Helper(2)PairSubset $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{s} \colon \forall \underline{y} \colon \forall \underline{v} \colon \forall \underline{w} \colon \lceil \bar{s} \rceil \#^0 \lceil \underline{s} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{y} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{v} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{w} \rceil \Vdash \underline{v} = \underline{w} \Rightarrow \underline{s} = \underline{v} \Rightarrow \dot{\neg}\, \underline{s} = \underline{y} \Rightarrow \underline{s} = \underline{w}$]

[Helper(2)PairSubset $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c. \lambda x. \mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{s} \colon \forall \underline{y} \colon \forall \underline{v} \colon \forall \underline{w} \colon \lceil \bar{s} \rceil \#^0 \lceil \underline{s} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{y} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{v} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{w} \rceil \Vdash \underline{v} = \underline{w} \vdash \underline{s} = \underline{v} \vdash$ =Transitivity $\mathrel{|\!\rhd} \lceil \bar{s} \rceil \#^0 \lceil \underline{s} \rceil \mathrel{|\!\rhd} \lceil \bar{s} \rceil \#^0 \lceil \underline{v} \rceil \mathrel{|\!\rhd} \lceil \bar{s} \rceil \#^0 \lceil \underline{w} \rceil \rhd \underline{s} = \underline{v} \rhd \underline{v} = \underline{w} \gg \underline{s} = \underline{w}$; WeakenOr1 $\rhd \underline{s} = \underline{w} \gg \dot{\neg}\, \underline{s} = \underline{y} \Rightarrow \underline{s} = \underline{w}; \forall \underline{s} \colon \forall \underline{y} \colon \forall \underline{v} \colon \forall \underline{w} \colon$ Ded $\rhd \forall \underline{s} \colon \forall \underline{y} \colon \forall \underline{v} \colon \forall \underline{w} \colon \lceil \bar{s} \rceil \#^0 \lceil \underline{s} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{y} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{v} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{w} \rceil \Vdash \underline{v} = \underline{w} \vdash \underline{s} = \underline{v} \vdash \dot{\neg}\, \underline{s} = \underline{y} \Rightarrow \underline{s} = \underline{w} \gg \lceil \bar{s} \rceil \#^0 \lceil \underline{s} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{y} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{v} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{w} \rceil \Vdash \underline{v} = \underline{w} \Rightarrow \underline{s} = \underline{v} \Rightarrow \dot{\neg}\, \underline{s} = \underline{y} \Rightarrow \underline{s} = \underline{w} \rceil, p_0, c)$]

[PairSubset $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{s} \colon \forall \underline{x} \colon \forall \underline{y} \colon \forall \underline{v} \colon \forall \underline{w} \colon \lceil \bar{s} \rceil \#^0 \lceil \underline{x} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{y} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{v} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{w} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{s} \rceil \Vdash \underline{x} = \underline{y} \Rightarrow \underline{v} = \underline{w} \Rightarrow \underline{s} \in \{\underline{x}, \underline{v}\} \Rightarrow \underline{s} \in \{\underline{y}, \underline{w}\}$]

[PairSubset $\stackrel{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{s}\colon \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{v}\colon \forall \underline{w}\colon \lceil \bar{s}\rceil \#^0 \lceil \underline{x}\rceil \Vdash \lceil \bar{s}\rceil \#^0 \lceil \underline{y}\rceil \Vdash \lceil \bar{s}\rceil \#^0 \lceil \underline{v}\rceil \Vdash \lceil \bar{s}\rceil \#^0 \lceil \underline{w}\rceil \Vdash \lceil \bar{s}\rceil \#^0 \lceil \underline{s}\rceil \Vdash \underline{x}=\underline{y} \vdash \underline{v}=\underline{w} \vdash \underline{s} \in \{\underline{x},\underline{v}\} \vdash$ Pair2Formula $\rhd \underline{s}\in\{\underline{x},\underline{v}\} \gg \dot{\neg}\underline{s}=\underline{x} \Rightarrow \underline{s}=\underline{v}$; HelperPairSubset $\unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{s}\rceil \unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil \unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil \unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{w}\rceil \gg \underline{x}=\underline{y}\Rightarrow \underline{s}=\underline{x}\Rightarrow\dot{\neg}\underline{s}=\underline{y}\Rightarrow\underline{s}=\underline{w}$; MP $\rhd \underline{x}=\underline{y}\Rightarrow\underline{s}=\underline{x}\Rightarrow\dot{\neg}\underline{s}=\underline{y}\Rightarrow\underline{s}=\underline{w} \rhd \underline{x}=\underline{y}\gg\underline{s}=\underline{x}\Rightarrow\dot{\neg}\underline{s}=\underline{y}\Rightarrow\underline{s}=\underline{w}$; Helper(2)PairSubset $\unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{s}\rceil \unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil \unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{v}\rceil \unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{w}\rceil \gg \underline{v}=\underline{w}\Rightarrow\underline{s}=\underline{v}\Rightarrow\dot{\neg}\underline{s}=\underline{y}\Rightarrow\underline{s}=\underline{w}$; MP $\rhd \underline{v}=\underline{w}\Rightarrow\underline{s}=\underline{v}\Rightarrow\dot{\neg}\underline{s}=\underline{y}\Rightarrow\underline{s}=\underline{w} \rhd \underline{v}=\underline{w}\gg\underline{s}=\underline{v}\Rightarrow\dot{\neg}\underline{s}=\underline{y}\Rightarrow\underline{s}=\underline{w}$; FromDisjuncts $\rhd \dot{\neg}\underline{s}=\underline{x}\Rightarrow\underline{s}=\underline{v} \rhd \underline{s}=\underline{x}\Rightarrow\dot{\neg}\underline{s}=\underline{y}\Rightarrow\underline{s}=\underline{w}\rhd\underline{s}=\underline{v}\Rightarrow\dot{\neg}\underline{s}=\underline{y}\Rightarrow\underline{s}=\underline{w}\gg\dot{\neg}\underline{s}=\underline{y}\Rightarrow\underline{s}=\underline{w}$; Formula2Pair $\rhd\dot{\neg}\underline{s}=\underline{y}\Rightarrow\underline{s}=\underline{w}\gg\underline{s}\in\{\underline{y},\underline{w}\}$; $\forall\underline{s}\colon\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{v}\colon\forall\underline{w}\colon$ Ded $\rhd \forall\underline{s}\colon\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{v}\colon\forall\underline{w}\colon \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil\Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil\Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{v}\rceil\Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{w}\rceil\Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{s}\rceil\Vdash \underline{x}=\underline{y}\vdash\underline{v}=\underline{w}\vdash\underline{s}\in\{\underline{x},\underline{v}\}\vdash\underline{s}\in\{\underline{y},\underline{w}\}\gg \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil\Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil\Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{v}\rceil\Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{w}\rceil\Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{s}\rceil\Vdash \underline{x}=\underline{y}\Rightarrow\underline{v}=\underline{w}\Rightarrow\underline{s}\in\{\underline{x},\underline{v}\}\Rightarrow\underline{s}\in\{\underline{y},\underline{w}\}\rceil, p_0, c)]$

Kommet så vidt kan vi bevise lighedslemmaet SamePair. Vi viser, at $\{\underline{x},\underline{v}\}$ er en delmængde af $\{\underline{y},\underline{w}\}$ (linie 8–11), og vice versa (linie 12–14):

[SamePair $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{v}\colon\forall\underline{w}\colon \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil\Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil\Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{v}\rceil\Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{w}\rceil\Vdash \lceil\bar{t}\rceil\#^0\lceil\{\underline{x},\underline{v}\}\rceil\Vdash \lceil\bar{t}\rceil\#^0\lceil\{\underline{y},\underline{w}\}\rceil\Vdash \underline{x}=\underline{y}\vdash\underline{v}=\underline{w}\vdash\{\underline{x},\underline{v}\}=\{\underline{y},\underline{w}\}]$

[SamePair $\stackrel{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash \forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{v}\colon\forall\underline{w}\colon \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil\Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil\Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{v}\rceil\Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{w}\rceil\Vdash \lceil\bar{t}\rceil\#^0\lceil\{\underline{x},\underline{v}\}\rceil\Vdash \lceil\bar{t}\rceil\#^0\lceil\{\underline{y},\underline{w}\}\rceil\Vdash \underline{x}=\underline{y}\vdash\underline{v}=\underline{w}\vdash$ PairSubset $\unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil \unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil \unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{v}\rceil \unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{w}\rceil \gg \underline{x}=\underline{y}\Rightarrow\underline{v}=\underline{w}\Rightarrow\bar{t}\in\{\underline{x},\underline{v}\}\Rightarrow\bar{t}\in\{\underline{y},\underline{w}\}$; MP2 $\rhd \underline{x}=\underline{y}\Rightarrow\underline{v}=\underline{w}\Rightarrow\bar{t}\in\{\underline{x},\underline{v}\}\Rightarrow\bar{t}\in\{\underline{y},\underline{w}\}\rhd\underline{x}=\underline{y}\rhd\underline{v}=\underline{w}\gg\bar{t}\in\{\underline{x},\underline{v}\}\Rightarrow\bar{t}\in\{\underline{y},\underline{w}\}$; $=$Symmetry $\unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil \unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil \rhd \underline{x}=\underline{y}\gg\underline{y}=\underline{x}$; $=$Symmetry $\unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{v}\rceil \unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{w}\rceil \rhd \underline{v}=\underline{w}\gg\underline{w}=\underline{v}$; PairSubset $\unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil \unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil \unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{w}\rceil \unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{v}\rceil \gg \underline{y}=\underline{x}\Rightarrow\underline{w}=\underline{v}\Rightarrow\bar{t}\in\{\underline{y},\underline{w}\}\Rightarrow\bar{t}\in\{\underline{x},\underline{v}\}$; MP2 $\rhd \underline{y}=\underline{x}\Rightarrow\underline{w}=\underline{v}\Rightarrow\bar{t}\in\{\underline{y},\underline{w}\}\Rightarrow\bar{t}\in\{\underline{x},\underline{v}\}\rhd\underline{y}=\underline{x}\rhd\underline{w}=\underline{v}\gg\bar{t}\in\{\underline{y},\underline{w}\}\Rightarrow\bar{t}\in\{\underline{x},\underline{v}\}$; ToSetEquality(t) $\unrhd \lceil\bar{t}\rceil\#^0\lceil\{\underline{x},\underline{v}\}\rceil \unrhd \lceil\bar{t}\rceil\#^0\lceil\{\underline{y},\underline{w}\}\rceil \rhd \bar{t}\in\{\underline{x},\underline{v}\}\Rightarrow\bar{t}\in\{\underline{y},\underline{w}\}\rhd\bar{t}\in\{\underline{y},\underline{w}\}\Rightarrow\bar{t}\in\{\underline{x},\underline{v}\}\gg\{\underline{x},\underline{v}\}=\{\underline{y},\underline{w}\}\rceil, p_0, c)]$

10.2 Singleton-mængde

Lighedslemmaet for singleton-konstruktionen ser således ud:

[SameSingleton $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall\underline{x}\colon\forall\underline{y}\colon \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil\Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil\Vdash \lceil\bar{t}\rceil\#^0\lceil\{\underline{x},\underline{x}\}\rceil\Vdash \lceil\bar{t}\rceil\#^0\lceil\{\underline{y},\underline{y}\}\rceil\Vdash \underline{x}=\underline{y}\vdash\{\underline{x},\underline{x}\}=\{\underline{y},\underline{y}\}]$

Da $[\{x,x\}]$ er makrodefineret som $[\{x,x\}]$, kan vi let vise SameSingleton ud fra SamePair:

[SameSingleton $\stackrel{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash \forall\underline{x}\colon\forall\underline{y}\colon \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil\Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil\Vdash \lceil\bar{t}\rceil\#^0\lceil\{\underline{x},\underline{x}\}\rceil\Vdash \lceil\bar{t}\rceil\#^0\lceil\{\underline{y},\underline{y}\}\rceil\Vdash \underline{x}=\underline{y}\vdash$ SamePair $\unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil \unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil \unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil \unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil \unrhd \lceil\bar{t}\rceil\#^0\lceil\{\underline{x},\underline{x}\}\rceil \unrhd \lceil\bar{t}\rceil\#^0\lceil\{\underline{y},\underline{y}\}\rceil \rhd \underline{x}=\underline{y}\rhd\underline{x}=\underline{y}\gg\{\underline{x},\underline{x}\}=\{\underline{y},\underline{y}\}$; Repetition $\rhd \{\underline{x},\underline{x}\}=\{\underline{y},\underline{y}\}\gg\{\underline{x},\underline{x}\}=\{\underline{y},\underline{y}\}\rceil, p_0, c)]$

10.3 Foreningsmængde

Delmængdelemmaet mht. $[\cup\mathsf{x}]$ ser således ud:

$[\mathrm{UnionSubset} \overset{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{s}}\colon \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{y}} \rceil \Vdash \underline{\mathsf{x}} = \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{s}} \in \cup\underline{\mathsf{x}} \Rightarrow \underline{\mathsf{s}} \in \cup\underline{\mathsf{y}}]$

Bevisets kerne (linie 7–11) er en simpel anvendelse af definitionen af $[\cup\mathsf{x}]$:

$[\mathrm{UnionSubset} \overset{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{s}}\colon \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{y}} \rceil \Vdash \underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \underline{\mathsf{s}} \in \cup\underline{\mathsf{x}} \vdash \mathrm{Union2Formula} \rhd \underline{\mathsf{s}} \in \cup\underline{\mathsf{x}} \gg \dot{\neg}\, \underline{\mathsf{s}} \in \mathsf{j}_{\mathrm{Ex}} \Rightarrow \dot{\neg}\, \mathsf{j}_{\mathrm{Ex}} \in \underline{\mathsf{x}}; \mathrm{FirstConjunct} \rhd \dot{\neg}\, \underline{\mathsf{s}} \in \mathsf{j}_{\mathrm{Ex}} \Rightarrow \dot{\neg}\, \mathsf{j}_{\mathrm{Ex}} \in \underline{\mathsf{x}} \gg \underline{\mathsf{s}} \in \mathsf{j}_{\mathrm{Ex}}; \mathrm{SecondConjunct} \rhd \dot{\neg}\, \underline{\mathsf{s}} \in \mathsf{j}_{\mathrm{Ex}} \Rightarrow \dot{\neg}\, \mathsf{j}_{\mathrm{Ex}} \in \underline{\mathsf{x}} \gg \mathsf{j}_{\mathrm{Ex}} \in \underline{\mathsf{x}}; \mathrm{FromSetEquality} \unrhd \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \unrhd \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{y}} \rceil \rhd \underline{\mathsf{x}} = \underline{\mathsf{y}} \rhd \mathsf{j}_{\mathrm{Ex}} \in \underline{\mathsf{x}} \gg \mathsf{j}_{\mathrm{Ex}} \in \underline{\mathsf{y}}; \mathrm{Formula2Union} \rhd \underline{\mathsf{s}} \in \mathsf{j}_{\mathrm{Ex}} \rhd \mathsf{j}_{\mathrm{Ex}} \in \underline{\mathsf{y}} \gg \underline{\mathsf{s}} \in \cup\underline{\mathsf{y}}; \forall \underline{\mathsf{s}}\colon \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \mathrm{Ded} \rhd \forall \underline{\mathsf{s}}\colon \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{y}} \rceil \Vdash \underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \underline{\mathsf{s}} \in \cup\underline{\mathsf{x}} \vdash \underline{\mathsf{s}} \in \cup\underline{\mathsf{y}} \gg \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{y}} \rceil \Vdash \underline{\mathsf{x}} = \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{s}} \in \cup\underline{\mathsf{x}} \Rightarrow \underline{\mathsf{s}} \in \cup\underline{\mathsf{y}} \rceil, \mathrm{p}_0, \mathsf{c})]$

I beviset for lighedslemmaet SameUnion viser vi, at $\cup\underline{\mathsf{x}}$ er en delmængde af $\cup\underline{\mathsf{y}}$ (linie 4–6), og vice versa (line 7–9):

$[\mathrm{SameUnion} \overset{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{y}} \rceil \Vdash \underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \cup\underline{\mathsf{x}} = \cup\underline{\mathsf{y}}]$

$[\mathrm{SameUnion} \overset{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{y}} \rceil \Vdash \underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \mathrm{UnionSubset} \unrhd \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \unrhd \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{y}} \rceil \gg \underline{\mathsf{x}} = \underline{\mathsf{y}} \Rightarrow \bar{\mathsf{s}} \in \cup\underline{\mathsf{x}} \Rightarrow \bar{\mathsf{s}} \in \cup\underline{\mathsf{y}}; \mathrm{MP} \rhd \underline{\mathsf{x}} = \underline{\mathsf{y}} \Rightarrow \bar{\mathsf{s}} \in \cup\underline{\mathsf{x}} \Rightarrow \bar{\mathsf{s}} \in \cup\underline{\mathsf{y}} \rhd \underline{\mathsf{x}} = \underline{\mathsf{y}} \gg \bar{\mathsf{s}} \in \cup\underline{\mathsf{x}} \Rightarrow \bar{\mathsf{s}} \in \cup\underline{\mathsf{y}}; {=}\mathrm{Symmetry} \unrhd \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \unrhd \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{y}} \rceil \rhd \underline{\mathsf{x}} = \underline{\mathsf{y}} \gg \underline{\mathsf{y}} = \underline{\mathsf{x}}; \mathrm{UnionSubset} \unrhd \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{y}} \rceil \unrhd \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \gg \underline{\mathsf{y}} = \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{s}} \in \cup\underline{\mathsf{y}} \Rightarrow \bar{\mathsf{s}} \in \cup\underline{\mathsf{x}}; \mathrm{MP} \rhd \underline{\mathsf{y}} = \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{s}} \in \cup\underline{\mathsf{y}} \Rightarrow \bar{\mathsf{s}} \in \cup\underline{\mathsf{x}} \rhd \underline{\mathsf{y}} = \underline{\mathsf{x}} \gg \bar{\mathsf{s}} \in \cup\underline{\mathsf{y}} \Rightarrow \bar{\mathsf{s}} \in \cup\underline{\mathsf{x}}; \mathrm{ToSetEquality} \rhd \bar{\mathsf{s}} \in \cup\underline{\mathsf{x}} \Rightarrow \bar{\mathsf{s}} \in \cup\underline{\mathsf{y}} \rhd \bar{\mathsf{s}} \in \cup\underline{\mathsf{y}} \Rightarrow \bar{\mathsf{s}} \in \cup\underline{\mathsf{x}} \gg \cup\underline{\mathsf{x}} = \cup\underline{\mathsf{y}} \rceil, \mathrm{p}_0, \mathsf{c})]$

10.4 Binær foreningsmængde

I lighedslemmaet for $[\cup\{\{\mathsf{x},\mathsf{x}\},\{\mathsf{y},\mathsf{y}\}\}]$ bliver problemet med sidebetingelserne meget tydeligt:

$[\mathrm{SameBinaryUnion} \overset{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \forall \underline{\mathsf{v}}\colon \forall \underline{\mathsf{w}}\colon \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{y}} \rceil \Vdash \lceil \bar{\mathsf{t}} \rceil \#^0 \lceil \{\underline{\mathsf{x}},\underline{\mathsf{x}}\} \rceil \Vdash \lceil \bar{\mathsf{t}} \rceil \#^0 \lceil \{\underline{\mathsf{y}},\underline{\mathsf{y}}\} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{v}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{w}} \rceil \Vdash \lceil \bar{\mathsf{t}} \rceil \#^0 \lceil \{\underline{\mathsf{v}},\underline{\mathsf{v}}\} \rceil \Vdash \lceil \bar{\mathsf{t}} \rceil \#^0 \lceil \{\underline{\mathsf{w}},\underline{\mathsf{w}}\} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \{\underline{\mathsf{x}},\underline{\mathsf{x}}\} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \{\underline{\mathsf{y}},\underline{\mathsf{y}}\} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \{\underline{\mathsf{v}},\underline{\mathsf{v}}\} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \{\underline{\mathsf{w}},\underline{\mathsf{w}}\} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \{\{\underline{\mathsf{x}},\underline{\mathsf{x}}\},\{\underline{\mathsf{v}},\underline{\mathsf{v}}\}\} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \{\{\underline{\mathsf{y}},\underline{\mathsf{y}}\},\{\underline{\mathsf{w}},\underline{\mathsf{w}}\}\} \rceil \Vdash \lceil \bar{\mathsf{t}} \rceil \#^0 \lceil \{\{\underline{\mathsf{x}},\underline{\mathsf{x}}\},\{\underline{\mathsf{v}},\underline{\mathsf{v}}\}\} \rceil \Vdash \lceil \bar{\mathsf{t}} \rceil \#^0 \lceil \{\{\underline{\mathsf{y}},\underline{\mathsf{y}}\},\{\underline{\mathsf{w}},\underline{\mathsf{w}}\}\} \rceil \Vdash \underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \underline{\mathsf{v}} = \underline{\mathsf{w}} \vdash \cup\{\{\underline{\mathsf{x}},\underline{\mathsf{x}}\},\{\underline{\mathsf{v}},\underline{\mathsf{v}}\}\} = \cup\{\{\underline{\mathsf{y}},\underline{\mathsf{y}}\},\{\underline{\mathsf{w}},\underline{\mathsf{w}}\}\}]$

Alt i alt er der 16 sidebetingelser. Et sådant lemma er jo ikke til at arbejde med. Derfor vil vi bruge objektvariable i formuleringen af de lemmaer, der afhænger af SameBinaryUnion. På dén måde bliver alle sidebetingelserne automatisk opfyldt, og vi behøver ikke at behandle dem eksplicit.
Problemet med SameBinaryUnion er, at det nedarver sidebetingelser fra tre forskellige lemmaer (jvf. figur 3), og at der ikke er noget overlap mellem

sidebetingelserne. Mange af sidebetingelserne er ensbetydende; f.eks. medfører $[\lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil]$ og $[\lceil\bar{s}\rceil\#^0\lceil\{\underline{x},\underline{x}\}\rceil]$ hinanden. Bevischeckeren er imidlertid ikke i stand til at udnytte dette; vi kan ikke gøre andet end at opremse alle sidebetingelserne.
Ironisk nok er selve beviset for SameBinaryUnion ret enkelt. Vi sætter blot lighedslemmaerne SameSingleton, SamePair og SameUnion sammen:

[SameBinaryUnion $\overset{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{v}\colon\forall\underline{w}\colon \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil \Vdash \lceil\bar{t}\rceil\#^0\lceil\{\underline{x},\underline{x}\}\rceil \Vdash \lceil\bar{t}\rceil\#^0\lceil\{\underline{y},\underline{y}\}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{v}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{w}\rceil \Vdash \lceil\bar{t}\rceil\#^0\lceil\{\underline{v},\underline{v}\}\rceil \Vdash \lceil\bar{t}\rceil\#^0\lceil\{\underline{w},\underline{w}\}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\{\underline{x},\underline{x}\}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\{\underline{y},\underline{y}\}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\{\underline{v},\underline{v}\}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\{\underline{w},\underline{w}\}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\{\{\underline{x},\underline{x}\},\{\underline{v},\underline{v}\}\}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\{\{\underline{y},\underline{y}\},\{\underline{w},\underline{w}\}\}\rceil \Vdash \lceil\bar{t}\rceil\#^0\lceil\{\{\underline{x},\underline{x}\},\{\underline{v},\underline{v}\}\}\rceil \Vdash \lceil\bar{t}\rceil\#^0\lceil\{\{\underline{y},\underline{y}\},\{\underline{w},\underline{w}\}\}\rceil \Vdash \underline{x}=\underline{y} \vdash \underline{v}=\underline{w} \vdash \text{SameSingleton} \mathrel{|\!\rhd} \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil \mathrel{|\!\rhd} \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil \mathrel{|\!\rhd} \lceil\bar{t}\rceil\#^0\lceil\{\underline{x},\underline{x}\}\rceil \mathrel{|\!\rhd} \lceil\bar{t}\rceil\#^0\lceil\{\underline{y},\underline{y}\}\rceil \rhd \underline{x}=\underline{y} \gg \{\underline{x},\underline{x}\}=\{\underline{y},\underline{y}\}; \text{SameSingleton} \mathrel{|\!\rhd} \lceil\bar{s}\rceil\#^0\lceil\underline{v}\rceil \mathrel{|\!\rhd} \lceil\bar{s}\rceil\#^0\lceil\underline{w}\rceil \mathrel{|\!\rhd} \lceil\bar{t}\rceil\#^0\lceil\{\underline{v},\underline{v}\}\rceil \mathrel{|\!\rhd} \lceil\bar{t}\rceil\#^0\lceil\{\underline{w},\underline{w}\}\rceil \rhd \underline{v}=\underline{w} \gg \{\underline{v},\underline{v}\}=\{\underline{w},\underline{w}\}; \text{SamePair} \mathrel{|\!\rhd} \lceil\bar{s}\rceil\#^0\lceil\{\underline{x},\underline{x}\}\rceil \mathrel{|\!\rhd} \lceil\bar{s}\rceil\#^0\lceil\{\underline{y},\underline{y}\}\rceil \mathrel{|\!\rhd} \lceil\bar{s}\rceil\#^0\lceil\{\underline{v},\underline{v}\}\rceil \mathrel{|\!\rhd} \lceil\bar{s}\rceil\#^0\lceil\{\underline{w},\underline{w}\}\rceil \mathrel{|\!\rhd} \lceil\bar{t}\rceil\#^0\lceil\{\{\underline{x},\underline{x}\},\{\underline{v},\underline{v}\}\}\rceil \mathrel{|\!\rhd} \lceil\bar{t}\rceil\#^0\lceil\{\{\underline{y},\underline{y}\},\{\underline{w},\underline{w}\}\}\rceil \rhd \{\underline{x},\underline{x}\}=\{\underline{y},\underline{y}\} \rhd \{\underline{v},\underline{v}\}=\{\underline{w},\underline{w}\} \gg \{\{\underline{x},\underline{x}\},\{\underline{v},\underline{v}\}\}=\{\{\underline{y},\underline{y}\},\{\underline{w},\underline{w}\}\}; \text{SameUnion} \mathrel{|\!\rhd} \lceil\bar{s}\rceil\#^0\lceil\{\{\underline{x},\underline{x}\},\{\underline{v},\underline{v}\}\}\rceil \mathrel{|\!\rhd} \lceil\bar{s}\rceil\#^0\lceil\{\{\underline{y},\underline{y}\},\{\underline{w},\underline{w}\}\}\rceil \rhd \{\{\underline{x},\underline{x}\},\{\underline{v},\underline{v}\}\}=\{\{\underline{y},\underline{y}\},\{\underline{w},\underline{w}\}\} \gg \cup\{\{\underline{x},\underline{x}\},\{\underline{v},\underline{v}\}\}=\cup\{\{\underline{y},\underline{y}\},\{\underline{w},\underline{w}\}\}; \text{Repetition} \rhd \cup\{\{\underline{x},\underline{x}\},\{\underline{v},\underline{v}\}\}=\cup\{\{\underline{y},\underline{y}\},\{\underline{w},\underline{w}\}\} \gg \cup\{\{\underline{x},\underline{x}\},\{\underline{v},\underline{v}\}\}=\cup\{\{\underline{y},\underline{y}\},\{\underline{w},\underline{w}\}\}\rceil, p_0, c)$]

10.5 Separation

Delmængdelemmaet for konstruktionen $[\{\text{ph} \in \underline{t} \mid \underline{f}\}]$ ser således ud:

[SeparationSubset $\overset{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall\underline{a}\colon\forall\underline{b}\colon\forall\underline{s}\colon\forall\underline{x}\colon\forall\underline{y}\colon \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil \Vdash \underline{x}=\underline{y} \Rightarrow \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\,\underline{b} \Rightarrow \underline{a} \Rightarrow \underline{s} \in \{\text{ph} \in \underline{x} \mid \underline{a}\} \Rightarrow \underline{s} \in \{\text{ph} \in \underline{y} \mid \underline{b}\}$]

I beviset går vi fra $[\underline{s} \in \underline{x}]$ til $[\underline{s} \in \underline{y}]$ (linie 9–10), og fra $[\underline{a}]$ til $[\underline{b}]$ (linie 11–12):

[SeparationSubset $\overset{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall\underline{a}\colon\forall\underline{b}\colon\forall\underline{s}\colon\forall\underline{x}\colon\forall\underline{y}\colon \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil \Vdash \underline{x}=\underline{y} \vdash \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\,\underline{b} \Rightarrow \underline{a} \vdash \underline{s} \in \{\text{ph} \in \underline{x} \mid \underline{a}\} \vdash \text{Sep2Formula} \rhd \underline{s} \in \{\text{ph} \in \underline{x} \mid \underline{a}\} \gg \dot{\neg}\,\underline{s} \in \underline{x} \Rightarrow \dot{\neg}\,\underline{a}; \text{FirstConjunct} \rhd \dot{\neg}\,\underline{s} \in \underline{x} \Rightarrow \dot{\neg}\,\underline{a} \gg \underline{s} \in \underline{x}; \text{FromSetEquality} \mathrel{|\!\rhd} \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil \mathrel{|\!\rhd} \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil \rhd \underline{x}=\underline{y} \rhd \underline{s} \in \underline{x} \gg \underline{s} \in \underline{y}; \text{SecondConjunct} \rhd \dot{\neg}\,\underline{s} \in \underline{x} \Rightarrow \dot{\neg}\,\underline{a} \gg \underline{a}; \text{IffSecond} \rhd \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\,\underline{b} \Rightarrow \underline{a} \rhd \underline{a} \gg \underline{b}; \text{Formula2Sep} \rhd \underline{s} \in \underline{y} \rhd \underline{b} \gg \underline{s} \in \{\text{ph} \in \underline{y} \mid \underline{b}\}; \forall\underline{a}\colon\forall\underline{b}\colon\forall\underline{s}\colon\forall\underline{x}\colon\forall\underline{y}\colon \text{Ded} \rhd \forall\underline{a}\colon\forall\underline{b}\colon\forall\underline{s}\colon\forall\underline{x}\colon\forall\underline{y}\colon \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil \Vdash \underline{x}=\underline{y} \vdash \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\,\underline{b} \Rightarrow \underline{a} \vdash \underline{s} \in \{\text{ph} \in \underline{x} \mid \underline{a}\} \vdash \underline{s} \in \{\text{ph} \in \underline{y} \mid \underline{b}\} \gg \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil \Vdash \underline{x}=\underline{y} \Rightarrow \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\,\underline{b} \Rightarrow \underline{a} \Rightarrow \underline{s} \in \{\text{ph} \in \underline{x} \mid \underline{a}\} \Rightarrow \underline{s} \in \{\text{ph} \in \underline{y} \mid \underline{b}\}\rceil, p_0, c)$]

I beviset for lighedslemmaet SameSeparation viser vi, at $\{\text{ph} \in \underline{x} \mid \underline{a}\}$ er en delmængde af $\{\text{ph} \in \underline{y} \mid \underline{b}\}$ (linie 4–7), og vice versa (linie 8–11):

[SameSeparation $\stackrel{\text{stmt}}{\to}$ ZFsub $\vdash \forall\underline{a}\colon\forall\underline{b}\colon\forall\underline{x}\colon\forall\underline{y}\colon \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil \Vdash \underline{x}=\underline{y}\vdash \dot\neg\,\underline{a}\Rightarrow\underline{b}\Rightarrow\dot\neg\,\underline{b}\Rightarrow\underline{a}\vdash\{\mathrm{ph}\in\underline{x}\mid\underline{a}\}=\{\mathrm{ph}\in\underline{y}\mid\underline{b}\}$]

[SameSeparation $\stackrel{\text{proof}}{\to}$ $\lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash \forall\underline{a}\colon\forall\underline{b}\colon\forall\underline{x}\colon\forall\underline{y}\colon \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil \Vdash \underline{x}=\underline{y}\vdash \dot\neg\,\underline{a}\Rightarrow\underline{b}\Rightarrow\dot\neg\,\underline{b}\Rightarrow\underline{a}\vdash$ SeparationSubset $\unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil \unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil \gg \underline{x}=\underline{y}\Rightarrow\dot\neg\,\underline{a}\Rightarrow\underline{b}\Rightarrow\dot\neg\,\underline{b}\Rightarrow\underline{a}\Rightarrow\bar{s}\in\{\mathrm{ph}\in\underline{x}\mid\underline{a}\}\Rightarrow\bar{s}\in\{\mathrm{ph}\in\underline{y}\mid\underline{b}\}$; MP2 $\rhd \underline{x}=\underline{y}\Rightarrow\dot\neg\,\underline{a}\Rightarrow\underline{b}\Rightarrow\dot\neg\,\underline{b}\Rightarrow\underline{a}\Rightarrow\bar{s}\in\{\mathrm{ph}\in\underline{x}\mid\underline{a}\}\Rightarrow\bar{s}\in\{\mathrm{ph}\in\underline{y}\mid\underline{b}\}\rhd\underline{x}=\underline{y}\rhd\dot\neg\,\underline{a}\Rightarrow\underline{b}\Rightarrow\dot\neg\,\underline{b}\Rightarrow\underline{a}\gg\bar{s}\in\{\mathrm{ph}\in\underline{x}\mid\underline{a}\}\Rightarrow\bar{s}\in\{\mathrm{ph}\in\underline{y}\mid\underline{b}\}$; $=$Symmetry $\unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil \unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil \rhd \underline{x}=\underline{y}\gg\underline{y}=\underline{x}$; IffCommutativity $\rhd\dot\neg\,\underline{a}\Rightarrow\underline{b}\Rightarrow\dot\neg\,\underline{b}\Rightarrow\underline{a}\gg\dot\neg\,\underline{b}\Rightarrow\underline{a}\Rightarrow\dot\neg\,\underline{a}\Rightarrow\underline{b}$; SeparationSubset $\unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil \unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil \gg \underline{y}=\underline{x}\Rightarrow\dot\neg\,\underline{b}\Rightarrow\underline{a}\Rightarrow\dot\neg\,\underline{a}\Rightarrow\underline{b}\Rightarrow\bar{s}\in\{\mathrm{ph}\in\underline{y}\mid\underline{b}\}\Rightarrow\bar{s}\in\{\mathrm{ph}\in\underline{x}\mid\underline{a}\}$; MP2 $\rhd\underline{y}=\underline{x}\Rightarrow\dot\neg\,\underline{b}\Rightarrow\underline{a}\Rightarrow\dot\neg\,\underline{a}\Rightarrow\underline{b}\Rightarrow\bar{s}\in\{\mathrm{ph}\in\underline{y}\mid\underline{b}\}\Rightarrow\bar{s}\in\{\mathrm{ph}\in\underline{x}\mid\underline{a}\}\rhd\underline{y}=\underline{x}\rhd\dot\neg\,\underline{b}\Rightarrow\underline{a}\Rightarrow\dot\neg\,\underline{a}\Rightarrow\underline{b}\gg\bar{s}\in\{\mathrm{ph}\in\underline{y}\mid\underline{b}\}\Rightarrow\bar{s}\in\{\mathrm{ph}\in\underline{x}\mid\underline{a}\}$; ToSetEquality $\rhd\bar{s}\in\{\mathrm{ph}\in\underline{x}\mid\underline{a}\}\Rightarrow\bar{s}\in\{\mathrm{ph}\in\underline{y}\mid\underline{b}\}\rhd\bar{s}\in\{\mathrm{ph}\in\underline{y}\mid\underline{b}\}\Rightarrow\bar{s}\in\{\mathrm{ph}\in\underline{x}\mid\underline{a}\}\gg\{\mathrm{ph}\in\underline{x}\mid\underline{a}\}=\{\mathrm{ph}\in\underline{y}\mid\underline{b}\}\rceil, p_0, c)$]

10.6 Binær fællesmængde

Delmængdelemmaet for konstruktionen $[\{\mathrm{ph}\in\cup\{\{x,x\},\{y,y\}\}\mid\dot\neg\,c_{\mathrm{Ph}}\in x\Rightarrow\dot\neg\,c_{\mathrm{Ph}}\in y\}]$ adskiller sig fra normen ved ikke at nævne $[\{\mathrm{ph}\in\cup\{\{x,x\},\{y,y\}\}\mid\dot\neg\,c_{\mathrm{Ph}}\in x\Rightarrow\dot\neg\,c_{\mathrm{Ph}}\in y\}]$ eksplicit:

[IntersectionSubset $\stackrel{\text{stmt}}{\to}$ ZFsub $\vdash\forall\underline{s}\colon\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{v}\colon\forall\underline{w}\colon\lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil\Vdash\lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil\Vdash\lceil\bar{s}\rceil\#^0\lceil\underline{v}\rceil\Vdash\lceil\bar{s}\rceil\#^0\lceil\underline{w}\rceil\Vdash\underline{x}=\underline{y}\Rightarrow\underline{v}=\underline{w}\Rightarrow\dot\neg\,\underline{s}\in\underline{x}\Rightarrow\dot\neg\,\underline{s}\in\underline{v}\Rightarrow\dot\neg\,\underline{s}\in\underline{y}\Rightarrow\dot\neg\,\underline{s}\in\underline{w}$]

Bevisets kerne består af to anvendelser af FromSetEquality (linie 10–11 og 12–13):

[IntersectionSubset $\stackrel{\text{proof}}{\to}$ $\lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash\forall\underline{s}\colon\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{v}\colon\forall\underline{w}\colon\lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil\Vdash\lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil\Vdash\lceil\bar{s}\rceil\#^0\lceil\underline{v}\rceil\Vdash\lceil\bar{s}\rceil\#^0\lceil\underline{w}\rceil\Vdash\underline{x}=\underline{y}\vdash\underline{v}=\underline{w}\vdash\dot\neg\,\underline{s}\in\underline{x}\Rightarrow\dot\neg\,\underline{s}\in\underline{v}\vdash$ FirstConjunct $\rhd\dot\neg\,\underline{s}\in\underline{x}\Rightarrow\dot\neg\,\underline{s}\in\underline{v}\gg\underline{s}\in\underline{x}$; FromSetEquality $\unrhd\lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil\unrhd\lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil\rhd\underline{x}=\underline{y}\rhd\underline{s}\in\underline{x}\gg\underline{s}\in\underline{y}$; SecondConjunct $\rhd\dot\neg\,\underline{s}\in\underline{x}\Rightarrow\dot\neg\,\underline{s}\in\underline{v}\gg\underline{s}\in\underline{v}$; FromSetEquality $\unrhd\lceil\bar{s}\rceil\#^0\lceil\underline{v}\rceil\unrhd\lceil\bar{s}\rceil\#^0\lceil\underline{w}\rceil\rhd\underline{v}=\underline{w}\rhd\underline{s}\in\underline{v}\gg\underline{s}\in\underline{w}$; JoinConjuncts $\rhd\underline{s}\in\underline{y}\rhd\underline{s}\in\underline{w}\gg\dot\neg\,\underline{s}\in\underline{y}\Rightarrow\dot\neg\,\underline{s}\in\underline{w}$; $\forall\underline{s}\colon\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{v}\colon\forall\underline{w}\colon$ Ded $\rhd\forall\underline{s}\colon\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{v}\colon\forall\underline{w}\colon\lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil\Vdash\lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil\Vdash\lceil\bar{s}\rceil\#^0\lceil\underline{v}\rceil\Vdash\lceil\bar{s}\rceil\#^0\lceil\underline{w}\rceil\Vdash\underline{x}=\underline{y}\vdash\underline{v}=\underline{w}\vdash\dot\neg\,\underline{s}\in\underline{x}\Rightarrow\dot\neg\,\underline{s}\in\underline{v}\vdash\dot\neg\,\underline{s}\in\underline{y}\Rightarrow\dot\neg\,\underline{s}\in\underline{w}\gg\lceil\bar{s}\rceil\#^0\lceil\underline{x}\rceil\Vdash\lceil\bar{s}\rceil\#^0\lceil\underline{y}\rceil\Vdash\lceil\bar{s}\rceil\#^0\lceil\underline{v}\rceil\Vdash\lceil\bar{s}\rceil\#^0\lceil\underline{w}\rceil\Vdash\underline{x}=\underline{y}\Rightarrow\underline{v}=\underline{w}\Rightarrow\dot\neg\,\underline{s}\in\underline{x}\Rightarrow\dot\neg\,\underline{s}\in\underline{v}\Rightarrow\dot\neg\,\underline{s}\in\underline{y}\Rightarrow\dot\neg\,\underline{s}\in\underline{w}\rceil, p_0, c)$]

Så er vi kommet til lighedslemmaet SameIntersection. Da dette lemma afhænger af SameBinaryUnion, slår vi her over til objektvariable (jvf. afsnit 10.4). Den store ulempe herved er, at objektvariable ikke kan instantieres til andre variable. Derfor vil en formulering som f.eks. $[\bar{x}=\bar{y}\vdash\bar{v}=\bar{w}\vdash\{\mathrm{ph}\in\cup\{\{\bar{x},\bar{x}\},\{\bar{v},\bar{v}\}\}\mid\dot\neg\,c_{\mathrm{Ph}}\in\bar{x}\Rightarrow\dot\neg\,c_{\mathrm{Ph}}\in\bar{v}\}=\{\mathrm{ph}\in\cup\{\{\bar{y},\bar{y}\},\{\bar{w},\bar{w}\}\}\mid\dot\neg\,c_{\mathrm{Ph}}\in\bar{y}\Rightarrow\dot\neg\,c_{\mathrm{Ph}}\in\bar{w}\}]$ sandsynligvis ikke kunne anvendes senere hen i rapporten, da vi ikke kan instantiere $\bar{x}$, $\bar{y}$, $\bar{v}$ eller $\bar{w}$.

Vi må altså undersøge, hvad SameIntersection egentlig skal bruges til, før vi formulerer lemmaet. Det viser sig, at den følgende formulering kan bruges:

[SameIntersection $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \overline{\mathsf{x}} = \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\} \vdash \overline{\mathsf{y}} = \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{b}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\} \vdash \{\mathrm{ph} \in \cup\{\{\overline{\mathsf{x}}, \overline{\mathsf{x}}\}, \{\overline{\mathsf{y}}, \overline{\mathsf{y}}\}\} \mid \dot{\neg}\mathsf{c}_{\mathrm{Ph}} \in \overline{\mathsf{x}} \Rightarrow \dot{\neg}\mathsf{c}_{\mathrm{Ph}} \in \overline{\mathsf{y}}\} = \{\mathrm{ph} \in \cup\{\{\{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\}, \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\}\}, \{\{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{b}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\}, \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{b}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\}\}\} \mid \dot{\neg}\mathsf{c}_{\mathrm{Ph}} \in \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\} \Rightarrow \dot{\neg}\mathsf{c}_{\mathrm{Ph}} \in \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{b}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\}\}$]

Man kan sige, at vi har instantieret objektvariablene "på forhånd".

Beviset for SameIntersection kan beskrives som følger:

1. Fra SameBinaryUnion får vi, at de to binære foreningsmængder er éns (linie 3).
2. De to formler $[\dot{\neg}\mathsf{c}_{\mathrm{Ph}} \in \overline{\mathsf{x}} \Rightarrow \dot{\neg}\mathsf{c}_{\mathrm{Ph}} \in \overline{\mathsf{y}}]$ og $[\dot{\neg}\mathsf{c}_{\mathrm{Ph}} \in \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\} \Rightarrow \dot{\neg}\mathsf{c}_{\mathrm{Ph}} \in \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}$ implicerer hinanden (linie 4–5 og 6–9).
3. Fra punkt 1, punkt 2 og SameSeparation får vi da, at de to binære fællesmængder er éns (linie 10–12).

Her er beviset:

[SameIntersection $\overset{\text{proof}}{\rightarrow}$ $\lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil$ZFsub $\vdash \overline{\mathsf{x}} = \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\} \vdash \overline{\mathsf{y}} = \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{b}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\} \vdash$ SameBinaryUnion $\rhd \overline{\mathsf{x}} = \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\} \rhd \overline{\mathsf{y}} = \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{b}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\} \gg \cup\{\{\overline{\mathsf{x}}, \overline{\mathsf{x}}\}, \{\overline{\mathsf{y}}, \overline{\mathsf{y}}\}\} = \cup\{\{\{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\}, \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\}\}, \{\{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{b}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\}, \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{b}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\}\}\};$ IntersectionSubset $\gg \overline{\mathsf{x}} = \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\} \Rightarrow \overline{\mathsf{y}} = \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{b}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\} \Rightarrow \dot{\neg}\mathsf{c}_{\mathrm{Ph}} \in \overline{\mathsf{x}} \Rightarrow \dot{\neg}\mathsf{c}_{\mathrm{Ph}} \in \overline{\mathsf{y}} \Rightarrow \dot{\neg}\mathsf{c}_{\mathrm{Ph}} \in \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\} \Rightarrow \dot{\neg}\mathsf{c}_{\mathrm{Ph}} \in \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{b}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\};$ MP2 $\rhd \overline{\mathsf{x}} = \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\} \Rightarrow \overline{\mathsf{y}} = \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{b}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\} \Rightarrow \dot{\neg}\mathsf{c}_{\mathrm{Ph}} \in \overline{\mathsf{x}} \Rightarrow \dot{\neg}\mathsf{c}_{\mathrm{Ph}} \in \overline{\mathsf{y}} \Rightarrow \dot{\neg}\mathsf{c}_{\mathrm{Ph}} \in \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\} \Rightarrow \dot{\neg}\mathsf{c}_{\mathrm{Ph}} \in \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{b}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\} \rhd \overline{\mathsf{x}} = \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\} \rhd \overline{\mathsf{y}} = \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{b}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\} \gg \dot{\neg}\mathsf{c}_{\mathrm{Ph}} \in \overline{\mathsf{x}} \Rightarrow \dot{\neg}\mathsf{c}_{\mathrm{Ph}} \in \overline{\mathsf{y}} \Rightarrow \dot{\neg}\mathsf{c}_{\mathrm{Ph}} \in \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\} \Rightarrow \dot{\neg}\mathsf{c}_{\mathrm{Ph}} \in \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{b}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\}; =$Symmetry $\rhd \overline{\mathsf{x}} = \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\} \gg \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\} = \overline{\mathsf{x}}; =$ Symmetry $\rhd \overline{\mathsf{y}} = \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{b}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\} \gg \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{b}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\} = \overline{\mathsf{y}};$ IntersectionSubset $\gg \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\} = \overline{\mathsf{x}} \Rightarrow \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{b}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\} = \overline{\mathsf{y}} \Rightarrow \dot{\neg}\mathsf{c}_{\mathrm{Ph}} \in \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\} \Rightarrow \dot{\neg}\mathsf{c}_{\mathrm{Ph}} \in \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{b}_{\mathrm{Ex}}\}\} \in \overline{\mathsf{r}}\} \Rightarrow \dot{\neg}\mathsf{c}_{\mathrm{Ph}} \in \overline{\mathsf{x}} \Rightarrow \dot{\neg}\mathsf{c}_{\mathrm{Ph}} \in \overline{\mathsf{y}};$ MP2 $\rhd \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid$

$\{\{a_{Ph},a_{Ph}\},\{a_{Ph},a_{Ex}\}\}\in\bar{r}\}=\bar{x}\Rightarrow\{ph\in\overline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},b_{Ex}\}\}\in\bar{r}\}=$
$\bar{y}\Rightarrow\dot\neg c_{Ph}\in\{ph\in\overline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},a_{Ex}\}\}\in\bar{r}\}\Rightarrow\dot\neg c_{Ph}\in\{ph\in\overline{bs}\mid$
$\{\{a_{Ph},a_{Ph}\},\{a_{Ph},b_{Ex}\}\}\in\bar{r}\}\Rightarrow\dot\neg c_{Ph}\in\bar{x}\Rightarrow\dot\neg c_{Ph}\in\bar{y}\rhd\{ph\in\overline{bs}\mid$
$\{\{a_{Ph},a_{Ph}\},\{a_{Ph},a_{Ex}\}\}\in\bar{r}\}=\bar{x}\rhd\{ph\in\overline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},b_{Ex}\}\}\in\bar{r}\}=$
$\bar{y}\gg\dot\neg c_{Ph}\in\{ph\in\overline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},a_{Ex}\}\}\in\bar{r}\}\Rightarrow\dot\neg c_{Ph}\in\{ph\in\overline{bs}\mid$
$\{\{a_{Ph},a_{Ph}\},\{a_{Ph},b_{Ex}\}\}\in\bar{r}\}\Rightarrow\dot\neg c_{Ph}\in\bar{x}\Rightarrow\dot\neg c_{Ph}\in\bar{y};\mathrm{JoinConjuncts}\rhd\dot\neg c_{Ph}\in$
$\bar{x}\Rightarrow\dot\neg c_{Ph}\in\bar{y}\Rightarrow\dot\neg c_{Ph}\in\{ph\in\overline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},a_{Ex}\}\}\in\bar{r}\}\Rightarrow\dot\neg c_{Ph}\in$
$\{ph\in\overline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},b_{Ex}\}\}\in\bar{r}\}\rhd\dot\neg c_{Ph}\in\{ph\in\overline{bs}\mid$
$\{\{a_{Ph},a_{Ph}\},\{a_{Ph},a_{Ex}\}\}\in\bar{r}\}\Rightarrow\dot\neg c_{Ph}\in\{ph\in\overline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},b_{Ex}\}\}\in$
$\bar{r}\}\Rightarrow\dot\neg c_{Ph}\in\bar{x}\Rightarrow\dot\neg c_{Ph}\in\bar{y}\gg\dot\neg\dot\neg c_{Ph}\in\bar{x}\Rightarrow\dot\neg c_{Ph}\in\bar{y}\Rightarrow\dot\neg c_{Ph}\in\{ph\in\overline{bs}\mid$
$\{\{a_{Ph},a_{Ph}\},\{a_{Ph},a_{Ex}\}\}\in\bar{r}\}\Rightarrow\dot\neg c_{Ph}\in\{ph\in\overline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},b_{Ex}\}\}\in$
$\bar{r}\}\Rightarrow\dot\neg\dot\neg c_{Ph}\in\{ph\in\overline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},a_{Ex}\}\}\in\bar{r}\}\Rightarrow\dot\neg c_{Ph}\in\{ph\in\overline{bs}\mid$
$\{\{a_{Ph},a_{Ph}\},\{a_{Ph},b_{Ex}\}\}\in\bar{r}\}\Rightarrow\dot\neg c_{Ph}\in\bar{x}\Rightarrow\dot\neg c_{Ph}\in\bar{y};\mathrm{SameSeparation}\rhd$
$\cup\{\{\bar{x},\bar{x}\},\{\bar{y},\bar{y}\}\}=\cup\{\{\{ph\in\overline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},a_{Ex}\}\}\in\bar{r}\},\{ph\in\overline{bs}\mid$
$\{\{a_{Ph},a_{Ph}\},\{a_{Ph},a_{Ex}\}\}\in\bar{r}\}\},\{\{ph\in\overline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},b_{Ex}\}\}\in\bar{r}\},\{ph\in$
$\overline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},b_{Ex}\}\}\in\bar{r}\}\}\}\rhd\dot\neg\dot\neg c_{Ph}\in\bar{x}\Rightarrow\dot\neg c_{Ph}\in\bar{y}\Rightarrow\dot\neg c_{Ph}\in\{ph\in$
$\overline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},a_{Ex}\}\}\in\bar{r}\}\Rightarrow\dot\neg c_{Ph}\in\{ph\in\overline{bs}\mid$
$\{\{a_{Ph},a_{Ph}\},\{a_{Ph},b_{Ex}\}\}\in\bar{r}\}\Rightarrow\dot\neg\dot\neg c_{Ph}\in\{ph\in\overline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},a_{Ex}\}\}\in$
$\bar{r}\}\Rightarrow\dot\neg c_{Ph}\in\{ph\in\overline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},b_{Ex}\}\}\in\bar{r}\}\Rightarrow\dot\neg c_{Ph}\in\bar{x}\Rightarrow\dot\neg c_{Ph}\in$
$\bar{y}\gg\{ph\in\cup\{\{\bar{x},\bar{x}\},\{\bar{y},\bar{y}\}\}\mid\dot\neg c_{Ph}\in\bar{x}\Rightarrow\dot\neg c_{Ph}\in\bar{y}\}=\{ph\in\cup\{\{\{ph\in\overline{bs}\mid$
$\{\{a_{Ph},a_{Ph}\},\{a_{Ph},a_{Ex}\}\}\in\bar{r}\},\{ph\in\overline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},a_{Ex}\}\}\in\bar{r}\}\},\{\{ph\in$
$\overline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},b_{Ex}\}\}\in\bar{r}\},\{ph\in\overline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},b_{Ex}\}\}\in\bar{r}\}\}\}\mid$
$\dot\neg c_{Ph}\in\{ph\in\overline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},a_{Ex}\}\}\in\bar{r}\}\Rightarrow\dot\neg c_{Ph}\in\{ph\in\overline{bs}\mid$
$\{\{a_{Ph},a_{Ph}\},\{a_{Ph},b_{Ex}\}\}\in\bar{r}\}\};\mathrm{Repetition}\rhd\{ph\in\cup\{\{\bar{x},\bar{x}\},\{\bar{y},\bar{y}\}\}\mid\dot\neg c_{Ph}\in$
$\bar{x}\Rightarrow\dot\neg c_{Ph}\in\bar{y}\}=\{ph\in\cup\{\{\{ph\in\overline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},a_{Ex}\}\}\in\bar{r}\},\{ph\in\overline{bs}\mid$
$\{\{a_{Ph},a_{Ph}\},\{a_{Ph},a_{Ex}\}\}\in\bar{r}\}\},\{\{ph\in\overline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},b_{Ex}\}\}\in\bar{r}\},\{ph\in$
$\overline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},b_{Ex}\}\}\in\bar{r}\}\}\}\mid\dot\neg c_{Ph}\in\{ph\in\overline{bs}\mid$
$\{\{a_{Ph},a_{Ph}\},\{a_{Ph},a_{Ex}\}\}\in\bar{r}\}\Rightarrow\dot\neg c_{Ph}\in\{ph\in\overline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},b_{Ex}\}\}\in$
$\bar{r}\}\}\gg\{ph\in\cup\{\{\bar{x},\bar{x}\},\{\bar{y},\bar{y}\}\}\mid\dot\neg c_{Ph}\in\bar{x}\Rightarrow\dot\neg c_{Ph}\in\bar{y}\}=\{ph\in\cup\{\{\{ph\in\overline{bs}\mid$
$\{\{a_{Ph},a_{Ph}\},\{a_{Ph},a_{Ex}\}\}\in\bar{r}\},\{ph\in\overline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},a_{Ex}\}\}\in\bar{r}\}\},\{\{ph\in$
$\overline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},b_{Ex}\}\}\in\bar{r}\},\{ph\in\overline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},b_{Ex}\}\}\in\bar{r}\}\}\}\mid$
$\dot\neg c_{Ph}\in\{ph\in\overline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},a_{Ex}\}\}\in\bar{r}\}\Rightarrow\dot\neg c_{Ph}\in\{ph\in\overline{bs}\mid$
$\{\{a_{Ph},a_{Ph}\},\{a_{Ph},b_{Ex}\}\}\in\bar{r}\}\}\rceil,p_0,c)]$

11 Hovedresultatet

Vi kan nu endelig gå i gang med hovedresultatet, som vi allerede udtrykte formelt i afsnit 5.10:

$[\mathrm{EqSysIsPartition}\overset{\mathrm{stmt}}{\to}\mathrm{ZFsub}\vdash\dot\neg\dot\neg\forall_{\mathrm{obj}}\bar{s}\colon\bar{s}\in\overline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\}\in\bar{r}\Rightarrow$
$\dot\neg\forall_{\mathrm{obj}}\bar{s}\colon\forall_{\mathrm{obj}}\bar{t}\colon\bar{s}\in\overline{bs}\Rightarrow\bar{t}\in\overline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\bar{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\bar{r}\Rightarrow$
$\dot\neg\forall_{\mathrm{obj}}\bar{s}\colon\forall_{\mathrm{obj}}\bar{t}\colon\forall_{\mathrm{obj}}\bar{u}\colon\bar{s}\in\overline{bs}\Rightarrow\bar{t}\in\overline{bs}\Rightarrow\bar{u}\in\overline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\bar{r}\Rightarrow$
$\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\bar{r}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\bar{r}\vdash\dot\neg\dot\neg\forall_{\mathrm{obj}}\bar{s}\colon\bar{s}\in\{ph\in P(\overline{bs})\mid\dot\neg t_{Ex}\in$
$\overline{bs}\Rightarrow\dot\neg\{ph\in\overline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},t_{Ex}\}\}\in\bar{r}\}=b_{Ph}\}\Rightarrow\dot\neg\bar{s}=\emptyset\Rightarrow$
$\dot\neg\forall_{\mathrm{obj}}\bar{s}\colon\forall_{\mathrm{obj}}\bar{t}\colon\bar{s}\in\{ph\in P(\overline{bs})\mid\dot\neg t_{Ex}\in\overline{bs}\Rightarrow\dot\neg\{ph\in\overline{bs}\mid$

$\{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \Rightarrow \bar{\mathsf{t}} \in \{\mathrm{ph} \in \mathrm{P}(\overline{\mathsf{bs}}) \mid \dot{\neg}\, \mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot{\neg}\, \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \Rightarrow \dot{\neg}\, \bar{\mathsf{s}} = \bar{\mathsf{t}} \Rightarrow \{\mathrm{ph} \in \cup\{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}\} \mid \dot{\neg}\, \mathsf{c}_{\mathrm{Ph}} \in \bar{\mathsf{s}} \Rightarrow \dot{\neg}\, \mathsf{c}_{\mathrm{Ph}} \in \bar{\mathsf{t}}\} = \emptyset \Rightarrow \dot{\neg} \cup \{\mathrm{ph} \in \mathrm{P}(\overline{\mathsf{bs}}) \mid \dot{\neg}\, \mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot{\neg}\, \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} = \overline{\mathsf{bs}}]$

Vi skal vise tre ting:

- Ingen ækvivalensklasser er tomme (underafsnit 11.1),
- Alle ækvivalensklasser er disjunkte (underafsnit 11.2), og
- Fællesmængden af alle ækvivalensklasserne er lig den oprindelige mængde $\overline{\mathsf{bs}}$ (underafsnit 11.3).

Til sidst binder vi alle trådene sammen i underafsnit 11.4.

11.1 Ingen ækvivalensklasser er tomme

Lemmaet AutoMember udsiger, at ækvivalensklassen $\{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \underline{\mathsf{s}}\}\} \in \underline{\mathsf{r}}\}$ indeholder $\underline{\mathsf{s}}$ selv som medlem:

$[\mathrm{AutoMember} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{r}}\colon \forall \underline{\mathsf{s}}\colon \forall \underline{\mathsf{bs}}\colon \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{r}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{bs}} \rceil \Vdash \dot{\neg}\, \dot{\neg}\, \forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\, \forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \forall_{\mathrm{obj}} \bar{\mathsf{t}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\, \forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \forall_{\mathrm{obj}} \bar{\mathsf{t}}\colon \forall_{\mathrm{obj}} \bar{\mathsf{u}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{u}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \vdash \underline{\mathsf{s}} \in \underline{\mathsf{bs}} \vdash \underline{\mathsf{s}} \in \{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \underline{\mathsf{s}}\}\} \in \underline{\mathsf{r}}\}]$

AutoMember følger af, at vi har $\{\{\underline{\mathsf{s}}, \underline{\mathsf{s}}\}, \{\underline{\mathsf{s}}, \underline{\mathsf{s}}\}\} \in \underline{\mathsf{r}}$, fordi ækvivalensrelationen $\underline{\mathsf{r}}$ er refleksiv:

$[\mathrm{AutoMember} \overset{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{r}}\colon \forall \underline{\mathsf{s}}\colon \forall \underline{\mathsf{bs}}\colon \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{r}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{bs}} \rceil \Vdash \dot{\neg}\, \dot{\neg}\, \forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\, \forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \forall_{\mathrm{obj}} \bar{\mathsf{t}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\, \forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \forall_{\mathrm{obj}} \bar{\mathsf{t}}\colon \forall_{\mathrm{obj}} \bar{\mathsf{u}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{u}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \vdash \underline{\mathsf{s}} \in \underline{\mathsf{bs}} \vdash \mathrm{ERisReflexive} \rhd \dot{\neg}\, \dot{\neg}\, \forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\, \forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \forall_{\mathrm{obj}} \bar{\mathsf{t}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\, \forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \forall_{\mathrm{obj}} \bar{\mathsf{t}}\colon \forall_{\mathrm{obj}} \bar{\mathsf{u}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{u}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \gg \forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}}; \mathrm{Reflexivity} \unrhd \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{r}} \rceil \unrhd \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{bs}} \rceil \rhd \forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \rhd \underline{\mathsf{s}} \in \underline{\mathsf{bs}} \gg \{\{\underline{\mathsf{s}}, \underline{\mathsf{s}}\}, \{\underline{\mathsf{s}}, \underline{\mathsf{s}}\}\} \in \underline{\mathsf{r}}; \mathrm{Formula2Sep} \rhd \underline{\mathsf{s}} \in \underline{\mathsf{bs}} \rhd \{\{\underline{\mathsf{s}}, \underline{\mathsf{s}}\}, \{\underline{\mathsf{s}}, \underline{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \gg \underline{\mathsf{s}} \in \{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \underline{\mathsf{s}}\}\} \in \underline{\mathsf{r}}\} \rceil, \mathrm{p}_0, \mathsf{c})]$

Ud fra AutoMember kan vi nu vise, at ingen medlemmer af et ækvivalenssystem er tomme:

$[\mathrm{EqSysNot}\emptyset \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{r}}\colon \forall \underline{\mathsf{s}}\colon \forall \underline{\mathsf{bs}}\colon \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{r}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{s}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{bs}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ex}}\}\} \in \underline{\mathsf{r}}\} \rceil \Vdash \lceil \bar{\mathsf{t}} \rceil \#^0 \lceil \underline{\mathsf{r}} \rceil \Vdash \lceil \bar{\mathsf{t}} \rceil \#^0 \lceil \underline{\mathsf{bs}} \rceil \Vdash \lceil \bar{\mathsf{u}} \rceil \#^0 \lceil \underline{\mathsf{r}} \rceil \Vdash \lceil \bar{\mathsf{u}} \rceil \#^0 \lceil \underline{\mathsf{bs}} \rceil \Vdash \dot{\neg}\, \dot{\neg}\, \forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\, \forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \forall_{\mathrm{obj}} \bar{\mathsf{t}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow$

$\dot{\neg}\,\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon\forall_{obj}\bar{u}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\bar{u}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow$
$\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\underline{r}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\underline{r}\vdash\underline{s}\in\{\mathrm{ph}\in \mathrm{P}(\underline{bs})\mid\dot{\neg}\,t_{\mathrm{Ex}}\in\underline{bs}\Rightarrow$
$\dot{\neg}\,\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},t_{\mathrm{Ex}}\}\}\in\underline{r}\}=b_{\mathrm{Ph}}\}\Rightarrow\dot{\neg}\,\underline{s}=\emptyset]$

Pga. sidebetingelserne deler vi beviset op i to, med HelperEqSysNot∅ som lemmastump:

[HelperEqSysNot∅ $\stackrel{\mathrm{stmt}}{\rightarrow}$ ZFsub $\vdash\forall\underline{r}\colon\forall\underline{s}\colon\forall\underline{bs}\colon\lceil\bar{s}\rceil\#^0\lceil\underline{r}\rceil\Vdash\lceil\bar{s}\rceil\#^0\lceil\underline{s}\rceil\Vdash$
$\lceil\bar{s}\rceil\#^0\lceil\underline{bs}\rceil\Vdash\lceil\bar{s}\rceil\#^0\lceil\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},a_{\mathrm{Ex}}\}\}\in\underline{r}\}\rceil\Vdash\lceil\bar{t}\rceil\#^0\lceil\underline{r}\rceil\Vdash$
$\lceil\bar{t}\rceil\#^0\lceil\underline{bs}\rceil\Vdash\lceil\bar{u}\rceil\#^0\lceil\underline{r}\rceil\Vdash\lceil\bar{u}\rceil\#^0\lceil\underline{bs}\rceil\Vdash\dot{\neg}\,\dot{\neg}\,\forall_{obj}\bar{s}\colon\bar{s}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\}\in\underline{r}\Rightarrow$
$\dot{\neg}\,\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\underline{r}\Rightarrow$
$\dot{\neg}\,\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon\forall_{obj}\bar{u}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\bar{u}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow$
$\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\underline{r}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\underline{r}\Rightarrow\underline{s}\in\{\mathrm{ph}\in \mathrm{P}(\underline{bs})\mid\dot{\neg}\,t_{\mathrm{Ex}}\in\underline{bs}\Rightarrow$
$\dot{\neg}\,\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},t_{\mathrm{Ex}}\}\}\in\underline{r}\}=b_{\mathrm{Ph}}\}\Rightarrow\dot{\neg}\,\underline{s}=\emptyset]$

Næsten hele beviset varetages af HelperEqSysNot∅. Beviset forløber som følger: Lad $\underline{s}$ være et medlem af et ækvivalenssystem (linie 8). Pr. definition er $\underline{s}$ en ækvivalensklasse (linie 9–12). Da enhver ækvivalensklasse har et medlem, har $\underline{s}$ et medlem (linie 13–14). Da enhver mængde med et medlem ikke er tom, er $\underline{s}$ ikke tom, QED (linie 15).

[HelperEqSysNot∅ $\stackrel{\mathrm{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash\forall\underline{r}\colon\forall\underline{s}\colon\forall\underline{bs}\colon\lceil\bar{s}\rceil\#^0\lceil\underline{r}\rceil\Vdash\lceil\bar{s}\rceil\#^0\lceil\underline{s}\rceil\Vdash$
$\lceil\bar{s}\rceil\#^0\lceil\underline{bs}\rceil\Vdash\lceil\bar{s}\rceil\#^0\lceil\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},a_{\mathrm{Ex}}\}\}\in\underline{r}\}\rceil\Vdash\dot{\neg}\,\dot{\neg}\,\forall_{obj}\bar{s}\colon\bar{s}\in$
$\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\}\in\underline{r}\Rightarrow\dot{\neg}\,\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow$
$\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\underline{r}\Rightarrow\dot{\neg}\,\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon\forall_{obj}\bar{u}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\bar{u}\in\underline{bs}\Rightarrow$
$\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\underline{r}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\underline{r}\vdash\underline{s}\in\{\mathrm{ph}\in \mathrm{P}(\underline{bs})\mid$
$\dot{\neg}\,t_{\mathrm{Ex}}\in\underline{bs}\Rightarrow\dot{\neg}\,\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},t_{\mathrm{Ex}}\}\}\in\underline{r}\}=b_{\mathrm{Ph}}\}\vdash$
Sep2Formula $\rhd\ \underline{s}\in\{\mathrm{ph}\in \mathrm{P}(\underline{bs})\mid\dot{\neg}\,t_{\mathrm{Ex}}\in\underline{bs}\Rightarrow\dot{\neg}\,\{\mathrm{ph}\in\underline{bs}\mid$
$\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},t_{\mathrm{Ex}}\}\}\in\underline{r}\}=b_{\mathrm{Ph}}\}\gg\dot{\neg}\,\underline{s}\in \mathrm{P}(\underline{bs})\Rightarrow\dot{\neg}\,\dot{\neg}\,a_{\mathrm{Ex}}\in\underline{bs}\Rightarrow\dot{\neg}\,\{\mathrm{ph}\in$
$\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},a_{\mathrm{Ex}}\}\}\in\underline{r}\}=\underline{s};$ SecondConjunct $\rhd\ \dot{\neg}\,\underline{s}\in \mathrm{P}(\underline{bs})\Rightarrow$
$\dot{\neg}\,\dot{\neg}\,a_{\mathrm{Ex}}\in\underline{bs}\Rightarrow\dot{\neg}\,\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},a_{\mathrm{Ex}}\}\}\in\underline{r}\}=\underline{s}\gg\dot{\neg}\,a_{\mathrm{Ex}}\in\underline{bs}\Rightarrow$
$\dot{\neg}\,\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},a_{\mathrm{Ex}}\}\}\in\underline{r}\}=\underline{s};$ FirstConjunct $\rhd\ \dot{\neg}\,a_{\mathrm{Ex}}\in\underline{bs}\Rightarrow$
$\dot{\neg}\,\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},a_{\mathrm{Ex}}\}\}\in\underline{r}\}=\underline{s}\gg a_{\mathrm{Ex}}\in$
$\underline{bs};$ SecondConjunct $\rhd\ \dot{\neg}\,a_{\mathrm{Ex}}\in\underline{bs}\Rightarrow\dot{\neg}\,\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},a_{\mathrm{Ex}}\}\}\in\underline{r}\}=$
$\underline{s}\gg\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},a_{\mathrm{Ex}}\}\}\in\underline{r}\}=\underline{s};$ AutoMember $\mathrel{|\!\rhd}\lceil\bar{s}\rceil\#^0\lceil\underline{r}\rceil\mathrel{|\!\rhd}$
$\lceil\bar{s}\rceil\#^0\lceil\underline{bs}\rceil\rhd\dot{\neg}\,\dot{\neg}\,\forall_{obj}\bar{s}\colon\bar{s}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\}\in\underline{r}\Rightarrow\dot{\neg}\,\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in$
$\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\underline{r}\Rightarrow\dot{\neg}\,\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon\forall_{obj}\bar{u}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in$
$\underline{bs}\Rightarrow\bar{u}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\underline{r}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in$
$\underline{r}\rhd a_{\mathrm{Ex}}\in\underline{bs}\gg a_{\mathrm{Ex}}\in\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},a_{\mathrm{Ex}}\}\}\in$
$\underline{r}\};$ FromSetEquality $\mathrel{|\!\rhd}\lceil\bar{s}\rceil\#^0\lceil\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},a_{\mathrm{Ex}}\}\}\in\underline{r}\}\rceil\mathrel{|\!\rhd}$
$\lceil\bar{s}\rceil\#^0\lceil\underline{s}\rceil\rhd\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},a_{\mathrm{Ex}}\}\}\in\underline{r}\}=\underline{s}\rhd a_{\mathrm{Ex}}\in\{\mathrm{ph}\in\underline{bs}\mid$
$\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},a_{\mathrm{Ex}}\}\}\in\underline{r}\}\gg a_{\mathrm{Ex}}\in\underline{s};$ MemberNot∅ $\mathrel{|\!\rhd}\lceil\bar{s}\rceil\#^0\lceil\underline{s}\rceil\rhd a_{\mathrm{Ex}}\in\underline{s}\gg$
$\dot{\neg}\,\underline{s}=\emptyset;\forall\underline{r}\colon\forall\underline{s}\colon\forall\underline{bs}\colon$ Ded $\rhd\ \forall\underline{r}\colon\forall\underline{s}\colon\forall\underline{bs}\colon\lceil\bar{s}\rceil\#^0\lceil\underline{r}\rceil\Vdash\lceil\bar{s}\rceil\#^0\lceil\underline{s}\rceil\Vdash\lceil\bar{s}\rceil\#^0\lceil\underline{bs}\rceil\Vdash$
$\lceil\bar{s}\rceil\#^0\lceil\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},a_{\mathrm{Ex}}\}\}\in\underline{r}\}\rceil\Vdash\dot{\neg}\,\dot{\neg}\,\forall_{obj}\bar{s}\colon\bar{s}\in\underline{bs}\Rightarrow$
$\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\}\in\underline{r}\Rightarrow\dot{\neg}\,\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow$
$\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\underline{r}\Rightarrow\dot{\neg}\,\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon\forall_{obj}\bar{u}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\bar{u}\in\underline{bs}\Rightarrow$
$\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\underline{r}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\underline{r}\vdash\underline{s}\in\{\mathrm{ph}\in \mathrm{P}(\underline{bs})\mid$

$\dot\neg\, \mathsf{t}_{\mathrm{Ex}} \in \underline{\mathsf{bs}} \Rightarrow \dot\neg\, \{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \underline{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \vdash \dot\neg\, \underline{\mathsf{s}} = \emptyset \gg \lceil\bar{\mathsf{s}}\rceil\#^0\lceil\underline{\mathsf{r}}\rceil \Vdash \lceil\bar{\mathsf{s}}\rceil\#^0\lceil\underline{\mathsf{s}}\rceil \Vdash \lceil\bar{\mathsf{s}}\rceil\#^0\lceil\underline{\mathsf{bs}}\rceil \Vdash \lceil\bar{\mathsf{s}}\rceil\#^0\lceil\{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ex}}\}\} \in \underline{\mathsf{r}}\}\rceil \Vdash \lceil\bar{\mathsf{t}}\rceil\#^0\lceil\underline{\mathsf{r}}\rceil \Vdash \lceil\bar{\mathsf{t}}\rceil\#^0\lceil\underline{\mathsf{bs}}\rceil \Vdash \lceil\bar{\mathsf{u}}\rceil\#^0\lceil\underline{\mathsf{r}}\rceil \Vdash \lceil\bar{\mathsf{u}}\rceil\#^0\lceil\underline{\mathsf{bs}}\rceil \Vdash \dot\neg\,\dot\neg\,\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot\neg\,\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon \bar{\mathsf{s}}\in\underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{s}}\}\}\in\underline{\mathsf{r}} \Rightarrow \dot\neg\,\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{u}}\colon \bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{t}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{u}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}}\Rightarrow \underline{\mathsf{s}}\in\{\mathrm{ph}\in\mathrm{P}(\underline{\mathsf{bs}})\mid\dot\neg\,\mathsf{t}_{\mathrm{Ex}}\in\underline{\mathsf{bs}}\Rightarrow\dot\neg\,\{\mathrm{ph}\in\underline{\mathsf{bs}}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{t}_{\mathrm{Ex}}\}\}\in\underline{\mathsf{r}}\}=\mathsf{b}_{\mathrm{Ph}}\}\Rightarrow\dot\neg\,\underline{\mathsf{s}}=\emptyset\rceil, \mathrm{p}_0, \mathsf{c})]$

I hovedlemmaet skifter vi en implikation ud med en inferens:

$[\mathrm{EqSysNot}\emptyset \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall\underline{\mathsf{r}}\colon\forall\underline{\mathsf{s}}\colon\forall\underline{\mathsf{bs}}\colon \lceil\bar{\mathsf{s}}\rceil\#^0\lceil\underline{\mathsf{r}}\rceil \Vdash \lceil\bar{\mathsf{s}}\rceil\#^0\lceil\underline{\mathsf{s}}\rceil \Vdash \lceil\bar{\mathsf{s}}\rceil\#^0\lceil\underline{\mathsf{bs}}\rceil \Vdash \lceil\bar{\mathsf{s}}\rceil\#^0\lceil\{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ex}}\}\} \in \underline{\mathsf{r}}\}\rceil \Vdash \lceil\bar{\mathsf{t}}\rceil\#^0\lceil\underline{\mathsf{r}}\rceil \Vdash \lceil\bar{\mathsf{t}}\rceil\#^0\lceil\underline{\mathsf{bs}}\rceil \Vdash \lceil\bar{\mathsf{u}}\rceil\#^0\lceil\underline{\mathsf{r}}\rceil \Vdash \lceil\bar{\mathsf{u}}\rceil\#^0\lceil\underline{\mathsf{bs}}\rceil \Vdash \dot\neg\,\dot\neg\,\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot\neg\,\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon \bar{\mathsf{s}}\in\underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{s}}\}\}\in\underline{\mathsf{r}} \Rightarrow \dot\neg\,\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{u}}\colon \bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{t}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{u}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}} \vdash \underline{\mathsf{s}}\in\{\mathrm{ph}\in\mathrm{P}(\underline{\mathsf{bs}})\mid\dot\neg\,\mathsf{t}_{\mathrm{Ex}}\in\underline{\mathsf{bs}}\Rightarrow\dot\neg\,\{\mathrm{ph}\in\underline{\mathsf{bs}}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{t}_{\mathrm{Ex}}\}\}\in\underline{\mathsf{r}}\}=\mathsf{b}_{\mathrm{Ph}}\}\Rightarrow\dot\neg\,\underline{\mathsf{s}}=\emptyset]$

$[\mathrm{EqSysNot}\emptyset \overset{\mathrm{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil\mathrm{ZFsub} \vdash \forall\underline{\mathsf{r}}\colon\forall\underline{\mathsf{s}}\colon\forall\underline{\mathsf{bs}}\colon \lceil\bar{\mathsf{s}}\rceil\#^0\lceil\underline{\mathsf{r}}\rceil \Vdash \lceil\bar{\mathsf{s}}\rceil\#^0\lceil\underline{\mathsf{s}}\rceil \Vdash \lceil\bar{\mathsf{s}}\rceil\#^0\lceil\underline{\mathsf{bs}}\rceil \Vdash \lceil\bar{\mathsf{s}}\rceil\#^0\lceil\{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ex}}\}\} \in \underline{\mathsf{r}}\}\rceil \Vdash \lceil\bar{\mathsf{t}}\rceil\#^0\lceil\underline{\mathsf{r}}\rceil \Vdash \lceil\bar{\mathsf{t}}\rceil\#^0\lceil\underline{\mathsf{bs}}\rceil \Vdash \lceil\bar{\mathsf{u}}\rceil\#^0\lceil\underline{\mathsf{r}}\rceil \Vdash \lceil\bar{\mathsf{u}}\rceil\#^0\lceil\underline{\mathsf{bs}}\rceil \Vdash \dot\neg\,\dot\neg\,\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot\neg\,\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon \bar{\mathsf{s}}\in\underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{s}}\}\}\in\underline{\mathsf{r}} \Rightarrow \dot\neg\,\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{u}}\colon \bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{t}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{u}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}} \vdash \mathrm{HelperEqSysNot}\emptyset \mathrel{|\!\rhd} \lceil\bar{\mathsf{s}}\rceil\#^0\lceil\underline{\mathsf{r}}\rceil \mathrel{|\!\rhd} \lceil\bar{\mathsf{s}}\rceil\#^0\lceil\underline{\mathsf{s}}\rceil \mathrel{|\!\rhd} \lceil\bar{\mathsf{s}}\rceil\#^0\lceil\underline{\mathsf{bs}}\rceil \mathrel{|\!\rhd} \lceil\bar{\mathsf{s}}\rceil\#^0\lceil\{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ex}}\}\} \in \underline{\mathsf{r}}\}\rceil \mathrel{|\!\rhd} \lceil\bar{\mathsf{t}}\rceil\#^0\lceil\underline{\mathsf{r}}\rceil \mathrel{|\!\rhd} \lceil\bar{\mathsf{t}}\rceil\#^0\lceil\underline{\mathsf{bs}}\rceil \mathrel{|\!\rhd} \lceil\bar{\mathsf{u}}\rceil\#^0\lceil\underline{\mathsf{r}}\rceil \mathrel{|\!\rhd} \lceil\bar{\mathsf{u}}\rceil\#^0\lceil\underline{\mathsf{bs}}\rceil \gg \dot\neg\,\dot\neg\,\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot\neg\,\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon \bar{\mathsf{s}}\in\underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{s}}\}\}\in\underline{\mathsf{r}} \Rightarrow \dot\neg\,\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{u}}\colon \bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{t}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{u}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}}\Rightarrow \underline{\mathsf{s}}\in\{\mathrm{ph}\in\mathrm{P}(\underline{\mathsf{bs}})\mid\dot\neg\,\mathsf{t}_{\mathrm{Ex}}\in\underline{\mathsf{bs}}\Rightarrow\dot\neg\,\{\mathrm{ph}\in\underline{\mathsf{bs}}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{t}_{\mathrm{Ex}}\}\}\in\underline{\mathsf{r}}\}=\mathsf{b}_{\mathrm{Ph}}\}\Rightarrow\dot\neg\,\underline{\mathsf{s}}=\emptyset; \mathrm{MP} \rhd \dot\neg\,\dot\neg\,\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot\neg\,\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon \bar{\mathsf{s}}\in\underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{s}}\}\}\in\underline{\mathsf{r}} \Rightarrow \dot\neg\,\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{u}}\colon \bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{t}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{u}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}}\Rightarrow \underline{\mathsf{s}}\in\{\mathrm{ph}\in\mathrm{P}(\underline{\mathsf{bs}})\mid\dot\neg\,\mathsf{t}_{\mathrm{Ex}}\in\underline{\mathsf{bs}}\Rightarrow\dot\neg\,\{\mathrm{ph}\in\underline{\mathsf{bs}}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{t}_{\mathrm{Ex}}\}\}\in\underline{\mathsf{r}}\}=\mathsf{b}_{\mathrm{Ph}}\}\Rightarrow\dot\neg\,\underline{\mathsf{s}}=\emptyset \rhd \dot\neg\,\dot\neg\,\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot\neg\,\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon \bar{\mathsf{s}}\in\underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{s}}\}\}\in\underline{\mathsf{r}} \Rightarrow \dot\neg\,\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon\forall_{\mathrm{obj}}\bar{\mathsf{u}}\colon \bar{\mathsf{s}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{t}}\in\underline{\mathsf{bs}}\Rightarrow\bar{\mathsf{u}}\in\underline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{u}}\}\}\in\underline{\mathsf{r}} \gg \underline{\mathsf{s}}\in\{\mathrm{ph}\in\mathrm{P}(\underline{\mathsf{bs}})\mid\dot\neg\,\mathsf{t}_{\mathrm{Ex}}\in\underline{\mathsf{bs}}\Rightarrow\dot\neg\,\{\mathrm{ph}\in\underline{\mathsf{bs}}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{t}_{\mathrm{Ex}}\}\}\in\underline{\mathsf{r}}\}=\mathsf{b}_{\mathrm{Ph}}\}\Rightarrow\dot\neg\,\underline{\mathsf{s}}=\emptyset\rceil, \mathrm{p}_0, \mathsf{c})]$

11.2 Alle ækvivalensklasser er disjunkte

Dette underafsnit indeholder den mest komplicerede del af beviset for hovedresultatet. Som det kan ses af figur 4, har jeg måttet tage et "snyde-aksiom" til hjælp for at blive færdig.

Figur 4: Bevisstrukturen for underafsnit 11.2. Kun lemmaerne fra dette afsnit er vist. Figuren skal læses ligesom figur 2. Kassen mærket “CheatAllDisjoint” repræsenterer et “snyde-aksiom”, så der går ingen pile ind i denne kasse.

11.2.1 Lemmaet “EqSubset”

Indholdet af det første lemma, EqSubset, kan udtrykkes som følger: “Lad $\underline{x}$ og $\underline{y}$ være to medlemmer af $\underline{bs}$. Hvis der gælder $\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\} \in \underline{r}$, så er ækvivalensklassen $\{\mathrm{ph} \in \underline{bs} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{x}\}\} \in \underline{r}\}$ en delmængde af ækvivalensklassen $\{\mathrm{ph} \in \underline{bs} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{y}\}\} \in \underline{r}\}$”:

[EqSubset $\overset{\mathrm{stmt}}{\to}$ $\mathrm{ZFsub} \vdash \forall \underline{r}\colon \forall \underline{s}\colon \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{bs}\colon \lceil \bar{s}\rceil \#^0 \lceil \underline{r}\rceil \Vdash \lceil \bar{s}\rceil \#^0 \lceil \underline{bs}\rceil \Vdash \lceil \bar{t}\rceil \#^0 \lceil \underline{r}\rceil \Vdash \lceil \bar{t}\rceil \#^0 \lceil \underline{bs}\rceil \Vdash \lceil \bar{u}\rceil \#^0 \lceil \underline{r}\rceil \Vdash \lceil \bar{u}\rceil \#^0 \lceil \underline{bs}\rceil \Vdash \underline{x} \in \underline{bs} \vdash \underline{y} \in \underline{bs} \vdash \dot{\neg}\dot{\neg}\forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}} \bar{s}\colon \forall_{\mathrm{obj}} \bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}} \bar{s}\colon \forall_{\mathrm{obj}} \bar{t}\colon \forall_{\mathrm{obj}} \bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r} \vdash \{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\} \in \underline{r} \vdash \underline{s} \in \{\mathrm{ph} \in \underline{bs} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{x}\}\} \in \underline{r}\} \Rightarrow \underline{s} \in \{\mathrm{ph} \in \underline{bs} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{y}\}\} \in \underline{r}\}$]

Beviset forløber som følger:

1. Lad $\underline{s}$ være et vilkårligt medlem af $\{\mathrm{ph} \in \underline{bs} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{x}\}\} \in \underline{r}\}$. Vi har da $\{\{\underline{s},\underline{s}\},\{\underline{s},\underline{x}\}\} \in \underline{r}$ (linie 13–16).
2. Ud fra $\{\{\underline{s},\underline{s}\},\{\underline{s},\underline{x}\}\} \in \underline{r}$ og antagelsen $\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\} \in \underline{r}$ får vi $\{\{\underline{s},\underline{s}\},\{\underline{s},\underline{y}\}\} \in \underline{r}$, da $\underline{r}$ er transitiv (linie 17–19).
3. $\{\{\underline{s},\underline{s}\},\{\underline{s},\underline{y}\}\} \in \underline{r}$ medfører igen $\underline{s} \in \{\mathrm{ph} \in \underline{bs} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{y}\}\} \in \underline{r}\}$, QED (linie 20).

Hele dette ræsonnement gennemføres i lemmastumpen HelperEqSubset. Selve beviset for EqSubset består af den sædvanlige omskrivning fra implikation til inferens. Her er beviserne:

[HelperEqSubset $\overset{\mathrm{stmt}}{\to}$ $\mathrm{ZFsub} \vdash \forall \underline{r}\colon \forall \underline{s}\colon \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{bs}\colon \lceil \bar{s}\rceil \#^0 \lceil \underline{r}\rceil \Vdash \lceil \bar{s}\rceil \#^0 \lceil \underline{bs}\rceil \Vdash \lceil \bar{t}\rceil \#^0 \lceil \underline{r}\rceil \Vdash \lceil \bar{t}\rceil \#^0 \lceil \underline{bs}\rceil \Vdash \lceil \bar{u}\rceil \#^0 \lceil \underline{r}\rceil \Vdash \lceil \bar{u}\rceil \#^0 \lceil \underline{bs}\rceil \Vdash \underline{x} \in \underline{bs} \Rightarrow \underline{y} \in \underline{bs} \Rightarrow \dot{\neg}\dot{\neg}\forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}} \bar{s}\colon \forall_{\mathrm{obj}} \bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}} \bar{s}\colon \forall_{\mathrm{obj}} \bar{t}\colon \forall_{\mathrm{obj}} \bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\} \in \underline{r} \Rightarrow \underline{s} \in \{\mathrm{ph} \in \underline{bs} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{x}\}\} \in \underline{r}\} \Rightarrow \underline{s} \in \{\mathrm{ph} \in \underline{bs} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{y}\}\} \in \underline{r}\}$]

[HelperEqSubset $\overset{\mathrm{proof}}{\to}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{r}\colon \forall \underline{s}\colon \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{bs}\colon \lceil \bar{s}\rceil \#^0 \lceil \underline{r}\rceil \Vdash \lceil \bar{s}\rceil \#^0 \lceil \underline{bs}\rceil \Vdash \lceil \bar{t}\rceil \#^0 \lceil \underline{r}\rceil \Vdash \lceil \bar{t}\rceil \#^0 \lceil \underline{bs}\rceil \Vdash \lceil \bar{u}\rceil \#^0 \lceil \underline{r}\rceil \Vdash \lceil \bar{u}\rceil \#^0 \lceil \underline{bs}\rceil \Vdash \underline{x} \in \underline{bs} \vdash \underline{y} \in \underline{bs} \vdash \dot{\neg}\dot{\neg}\forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}} \bar{s}\colon \forall_{\mathrm{obj}} \bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}} \bar{s}\colon \forall_{\mathrm{obj}} \bar{t}\colon \forall_{\mathrm{obj}} \bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r} \vdash \{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\} \in \underline{r} \vdash \underline{s} \in \{\mathrm{ph} \in \underline{bs} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{x}\}\} \in \underline{r}\} \vdash \mathrm{Repetition} \rhd \underline{s} \in \{\mathrm{ph} \in \underline{bs} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{x}\}\} \in \underline{r}\} \gg \underline{s} \in \{\mathrm{ph} \in \underline{bs} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{x}\}\} \in \underline{r}\}; \mathrm{Sep2Formula} \rhd \underline{s} \in \{\mathrm{ph} \in \underline{bs} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{x}\}\} \in \underline{r}\} \gg \dot{\neg}\underline{s} \in \underline{bs} \Rightarrow \dot{\neg}\{\{\underline{s},\underline{s}\},\{\underline{s},\underline{x}\}\} \in \underline{r}; \mathrm{SecondConjunct} \rhd \dot{\neg}\underline{s} \in \underline{bs} \Rightarrow \dot{\neg}\{\{\underline{s},\underline{s}\},\{\underline{s},\underline{x}\}\} \in \underline{r} \gg$

$\{\{\underline{s},\underline{s}\},\{\underline{s},\underline{x}\}\} \in \underline{r}$; ERisTransitive $\rhd \dot{\neg}\dot{\neg}\forall_{obj}\bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow$
$\dot{\neg}\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \Rightarrow$
$\dot{\neg}\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \forall_{obj}\bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow$
$\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r} \gg \forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \forall_{obj}\bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow$
$\bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in$
$\underline{r}$; FirstConjunct $\rhd \dot{\neg}\underline{s} \in \underline{bs} \Rightarrow \dot{\neg}\{\{\underline{s},\underline{s}\},\{\underline{s},\underline{x}\}\} \in \underline{r} \gg \underline{s} \in \underline{bs}$; Transitivity $\unrhd$
$\lceil\bar{s}\rceil \#^0 \lceil\underline{r}\rceil \unrhd \lceil\bar{s}\rceil \#^0 \lceil\underline{bs}\rceil \unrhd \lceil\bar{t}\rceil \#^0 \lceil\underline{r}\rceil \unrhd \lceil\bar{t}\rceil \#^0 \lceil\underline{bs}\rceil \unrhd \lceil\bar{u}\rceil \#^0 \lceil\underline{r}\rceil \unrhd$
$\lceil\bar{u}\rceil \#^0 \lceil\underline{bs}\rceil \rhd \forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \forall_{obj}\bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow$
$\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r} \rhd \underline{s} \in \underline{bs} \rhd \underline{x} \in \underline{bs} \rhd \underline{y} \in$
$\underline{bs} \rhd \{\{\underline{s},\underline{s}\},\{\underline{s},\underline{x}\}\} \in \underline{r} \rhd \{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\} \in \underline{r} \gg \{\{\underline{s},\underline{s}\},\{\underline{s},\underline{y}\}\} \in$
$\underline{r}$; Formula2Sep $\rhd \underline{s} \in \underline{bs} \rhd \{\{\underline{s},\underline{s}\},\{\underline{s},\underline{y}\}\} \in \underline{r} \gg \underline{s} \in \{\mathrm{ph} \in \underline{bs} \mid$
$\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\} \in \underline{r}\}; \forall\underline{r}\colon \forall\underline{s}\colon \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{bs}\colon \mathrm{Ded} \rhd \forall\underline{r}\colon \forall\underline{s}\colon \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{bs}\colon \lceil\bar{s}\rceil \#^0 \lceil\underline{r}\rceil \Vdash$
$\lceil\bar{s}\rceil \#^0 \lceil\underline{bs}\rceil \Vdash \lceil\bar{t}\rceil \#^0 \lceil\underline{r}\rceil \Vdash \lceil\bar{t}\rceil \#^0 \lceil\underline{bs}\rceil \Vdash \lceil\bar{u}\rceil \#^0 \lceil\underline{r}\rceil \Vdash \lceil\bar{u}\rceil \#^0 \lceil\underline{bs}\rceil \Vdash \underline{x} \in \underline{bs} \vdash \underline{y} \in$
$\underline{bs} \vdash \dot{\neg}\dot{\neg}\forall_{obj}\bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow$
$\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \forall_{obj}\bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow$
$\bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r} \vdash$
$\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\} \in \underline{r} \vdash \underline{s} \in \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\} \in \underline{r}\} \vdash \underline{s} \in \{\mathrm{ph} \in \underline{bs} \mid$
$\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\} \in \underline{r}\} \gg \lceil\bar{s}\rceil \#^0 \lceil\underline{r}\rceil \Vdash \lceil\bar{s}\rceil \#^0 \lceil\underline{bs}\rceil \Vdash \lceil\bar{t}\rceil \#^0 \lceil\underline{r}\rceil \Vdash \lceil\bar{t}\rceil \#^0 \lceil\underline{bs}\rceil \Vdash$
$\lceil\bar{u}\rceil \#^0 \lceil\underline{r}\rceil \Vdash \lceil\bar{u}\rceil \#^0 \lceil\underline{bs}\rceil \Vdash \underline{x} \in \underline{bs} \Rightarrow \underline{y} \in \underline{bs} \Rightarrow \dot{\neg}\dot{\neg}\forall_{obj}\bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in$
$\underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \Rightarrow$
$\dot{\neg}\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \forall_{obj}\bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow$
$\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\} \in \underline{r} \Rightarrow \underline{s} \in \{\mathrm{ph} \in \underline{bs} \mid$
$\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\} \in \underline{r}\} \Rightarrow \underline{s} \in \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\} \in \underline{r}\}\rceil, p_0, c)]$

[EqSubset $\stackrel{\mathrm{stmt}}{\rightarrow}$ ZFsub $\vdash \forall\underline{r}\colon \forall\underline{s}\colon \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{bs}\colon \lceil\bar{s}\rceil \#^0 \lceil\underline{r}\rceil \Vdash \lceil\bar{s}\rceil \#^0 \lceil\underline{bs}\rceil \Vdash \lceil\bar{t}\rceil \#^0 \lceil\underline{r}\rceil \Vdash$
$\lceil\bar{t}\rceil \#^0 \lceil\underline{bs}\rceil \Vdash \lceil\bar{u}\rceil \#^0 \lceil\underline{r}\rceil \Vdash \lceil\bar{u}\rceil \#^0 \lceil\underline{bs}\rceil \Vdash \underline{x} \in \underline{bs} \vdash \underline{y} \in \underline{bs} \vdash \dot{\neg}\dot{\neg}\forall_{obj}\bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow$
$\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow$
$\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \forall_{obj}\bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow$
$\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r} \vdash \{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\} \in$
$\underline{r} \vdash \underline{s} \in \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\} \in \underline{r}\} \Rightarrow \underline{s} \in \{\mathrm{ph} \in \underline{bs} \mid$
$\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\} \in \underline{r}\}]$

[EqSubset $\stackrel{\mathrm{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash \forall\underline{r}\colon \forall\underline{s}\colon \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{bs}\colon \lceil\bar{s}\rceil \#^0 \lceil\underline{r}\rceil \Vdash \lceil\bar{s}\rceil \#^0 \lceil\underline{bs}\rceil \Vdash$
$\lceil\bar{t}\rceil \#^0 \lceil\underline{r}\rceil \Vdash \lceil\bar{t}\rceil \#^0 \lceil\underline{bs}\rceil \Vdash \lceil\bar{u}\rceil \#^0 \lceil\underline{r}\rceil \Vdash \lceil\bar{u}\rceil \#^0 \lceil\underline{bs}\rceil \Vdash \underline{x} \in \underline{bs} \vdash \underline{y} \in \underline{bs} \vdash$
$\dot{\neg}\dot{\neg}\forall_{obj}\bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow$
$\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \forall_{obj}\bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow$
$\bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r} \vdash$
$\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\} \in \underline{r} \vdash$ HelperEqSubset $\unrhd \lceil\bar{s}\rceil \#^0 \lceil\underline{r}\rceil \unrhd \lceil\bar{s}\rceil \#^0 \lceil\underline{bs}\rceil \unrhd \lceil\bar{t}\rceil \#^0 \lceil\underline{r}\rceil \unrhd$
$\lceil\bar{t}\rceil \#^0 \lceil\underline{bs}\rceil \unrhd \lceil\bar{u}\rceil \#^0 \lceil\underline{r}\rceil \unrhd \lceil\bar{u}\rceil \#^0 \lceil\underline{bs}\rceil \gg \underline{x} \in \underline{bs} \Rightarrow \underline{y} \in \underline{bs} \Rightarrow \dot{\neg}\dot{\neg}\forall_{obj}\bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow$
$\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow$
$\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \forall_{obj}\bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow$
$\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\} \in$
$\underline{r} \Rightarrow \underline{s} \in \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\} \in \underline{r}\} \Rightarrow \underline{s} \in \{\mathrm{ph} \in \underline{bs} \mid$
$\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\} \in \underline{r}\}$; MP4 $\rhd \underline{x} \in \underline{bs} \Rightarrow \underline{y} \in \underline{bs} \Rightarrow \dot{\neg}\dot{\neg}\forall_{obj}\bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow$
$\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow$

$$\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\underline{r}\Rightarrow\dot{\neg}\forall_{\mathrm{obj}}\bar{s}\colon\forall_{\mathrm{obj}}\bar{t}\colon\forall_{\mathrm{obj}}\bar{u}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\bar{u}\in\underline{bs}\Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\underline{r}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\underline{r}\Rightarrow\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\}\in \underline{r}\Rightarrow\underline{s}\in\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\}\Rightarrow\underline{s}\in\{\mathrm{ph}\in\underline{bs}\mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\}\rhd\underline{x}\in\underline{bs}\rhd\underline{y}\in\underline{bs}\rhd\dot{\neg}\dot{\neg}\forall_{\mathrm{obj}}\bar{s}\colon\bar{s}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\}\in \underline{r}\Rightarrow\dot{\neg}\forall_{\mathrm{obj}}\bar{s}\colon\forall_{\mathrm{obj}}\bar{t}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\underline{r}\Rightarrow \dot{\neg}\forall_{\mathrm{obj}}\bar{s}\colon\forall_{\mathrm{obj}}\bar{t}\colon\forall_{\mathrm{obj}}\bar{u}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\bar{u}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\underline{r}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\underline{r}\rhd\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\}\in\underline{r}\gg\underline{s}\in\{\mathrm{ph}\in\underline{bs}\mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\}\Rightarrow\underline{s}\in\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\}\rceil,p_0,c)]$$

11.2.2 Lemmaet "EqNecessary"

Det er nu en let sag at vise den stærkere påstand, at $\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\}$ rent faktisk er lig med $\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\}$[20]. Som så ofte før ligger den interessante del af beviset i en lemmastump, der her hedder "HelperEqNecessary":

$$[\mathrm{HelperEqNecessary}\overset{\mathrm{stmt}}{\rightarrow}\mathrm{ZFsub}\vdash\forall\underline{r}\colon\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{bs}\colon\lceil\bar{s}\rceil\#^0\lceil\underline{r}\rceil\Vdash\lceil\bar{s}\rceil\#^0\lceil\underline{bs}\rceil\Vdash \lceil\bar{t}\rceil\#^0\lceil\underline{r}\rceil\Vdash\lceil\bar{t}\rceil\#^0\lceil\underline{bs}\rceil\Vdash\lceil\bar{u}\rceil\#^0\lceil\underline{r}\rceil\Vdash\lceil\bar{u}\rceil\#^0\lceil\underline{bs}\rceil\Vdash\underline{x}\in\underline{bs}\Rightarrow\underline{y}\in\underline{bs}\Rightarrow \dot{\neg}\dot{\neg}\forall_{\mathrm{obj}}\bar{s}\colon\bar{s}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\}\in\underline{r}\Rightarrow\dot{\neg}\forall_{\mathrm{obj}}\bar{s}\colon\forall_{\mathrm{obj}}\bar{t}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\underline{r}\Rightarrow\dot{\neg}\forall_{\mathrm{obj}}\bar{s}\colon\forall_{\mathrm{obj}}\bar{t}\colon\forall_{\mathrm{obj}}\bar{u}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow \bar{u}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\underline{r}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\underline{r}\Rightarrow \{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\}\in\underline{r}\Rightarrow\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\}=\{\mathrm{ph}\in\underline{bs}\mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\}]$$

Beviset kan beskrives som følger:

1. Fra EqSubset ved vi, at antagelsen $\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\}\in\underline{r}$ medfører, at $\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\}$ er en delmængde af $\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\}$. (Linie 13).
2. Da $\underline{r}$ er symmetrisk, gælder der også $\{\{\underline{y},\underline{y}\},\{\underline{y},\underline{x}\}\}\in\underline{r}$. (Linie 14–15).
3. Vi kan derfor bruge EqSubset én gang til — vi får, at $\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\}$ er en delmængde af $\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\}$. (Linie 16).
4. Ud fra punkt 1 og 3 følger $[\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\}=\{\mathrm{ph}\in \underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\}]$, QED. (Linie 17).

Her er selve beviset:

$$[\mathrm{HelperEqNecessary}\overset{\mathrm{proof}}{\rightarrow}\lambda c.\lambda x.\mathcal{P}(\lceil\mathrm{ZFsub}\vdash\forall\underline{r}\colon\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{bs}\colon\lceil\bar{s}\rceil\#^0\lceil\underline{r}\rceil\Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{bs}\rceil\Vdash\lceil\bar{t}\rceil\#^0\lceil\underline{r}\rceil\Vdash\lceil\bar{t}\rceil\#^0\lceil\underline{bs}\rceil\Vdash\lceil\bar{u}\rceil\#^0\lceil\underline{r}\rceil\Vdash\lceil\bar{u}\rceil\#^0\lceil\underline{bs}\rceil\Vdash\underline{x}\in\underline{bs}\vdash\underline{y}\in \underline{bs}\vdash\dot{\neg}\dot{\neg}\forall_{\mathrm{obj}}\bar{s}\colon\bar{s}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\}\in\underline{r}\Rightarrow\dot{\neg}\forall_{\mathrm{obj}}\bar{s}\colon\forall_{\mathrm{obj}}\bar{t}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\underline{r}\Rightarrow\dot{\neg}\forall_{\mathrm{obj}}\bar{s}\colon\forall_{\mathrm{obj}}\bar{t}\colon\forall_{\mathrm{obj}}\bar{u}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow \bar{u}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\underline{r}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\underline{r}\vdash \{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\}\in\underline{r}\vdash\mathrm{EqSubset}\mathrel{|\!\!\rhd}\lceil\bar{s}\rceil\#^0\lceil\underline{r}\rceil\mathrel{|\!\!\rhd}\lceil\bar{s}\rceil\#^0\lceil\underline{bs}\rceil\mathrel{|\!\!\rhd}\lceil\bar{t}\rceil\#^0\lceil\underline{r}\rceil\mathrel{|\!\!\rhd} \lceil\bar{t}\rceil\#^0\lceil\underline{bs}\rceil\mathrel{|\!\!\rhd}\lceil\bar{u}\rceil\#^0\lceil\underline{r}\rceil\mathrel{|\!\!\rhd}\lceil\bar{u}\rceil\#^0\lceil\underline{bs}\rceil\rhd\underline{x}\in\underline{bs}\rhd\underline{y}\in\underline{bs}\rhd\dot{\neg}\dot{\neg}\forall_{\mathrm{obj}}\bar{s}\colon\bar{s}\in\underline{bs}\Rightarrow$$

[20] Dette er lemma 4.4.a.1 i [5].

$\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow$
$\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{u}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{u}} \in \underline{\mathsf{bs}} \Rightarrow$
$\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \rhd \{\{\underline{\mathsf{x}},\underline{\mathsf{x}}\},\{\underline{\mathsf{x}},\underline{\mathsf{y}}\}\} \in$
$\underline{\mathsf{r}} \gg \bar{\mathsf{s}} \in \{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{x}}\}\} \in \underline{\mathsf{r}}\} \Rightarrow \bar{\mathsf{s}} \in \{\mathrm{ph} \in \underline{\mathsf{bs}} \mid$
$\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{y}}\}\} \in \underline{\mathsf{r}}\}; \mathrm{ERisSymmetric} \rhd \dot{\neg}\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow$
$\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow$
$\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{u}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{u}} \in \underline{\mathsf{bs}} \Rightarrow$
$\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \gg \forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon \bar{\mathsf{s}} \in$
$\underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}}; \mathrm{Symmetry} \unrhd \lceil\bar{\mathsf{s}}\rceil\#^0\lceil\underline{\mathsf{r}}\rceil \unrhd$
$\lceil\bar{\mathsf{s}}\rceil\#^0\lceil\underline{\mathsf{bs}}\rceil \unrhd \lceil\bar{\mathsf{t}}\rceil\#^0\lceil\underline{\mathsf{r}}\rceil \unrhd \lceil\bar{\mathsf{t}}\rceil\#^0\lceil\underline{\mathsf{bs}}\rceil \rhd \forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow$
$\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \rhd \underline{\mathsf{x}} \in \underline{\mathsf{bs}} \rhd \underline{\mathsf{y}} \in \underline{\mathsf{bs}} \rhd \{\{\underline{\mathsf{x}},\underline{\mathsf{x}}\},\{\underline{\mathsf{x}},\underline{\mathsf{y}}\}\} \in \underline{\mathsf{r}} \gg$
$\{\{\underline{\mathsf{y}},\underline{\mathsf{y}}\},\{\underline{\mathsf{y}},\underline{\mathsf{x}}\}\} \in \underline{\mathsf{r}}; \mathrm{EqSubset} \unrhd \lceil\bar{\mathsf{s}}\rceil\#^0\lceil\underline{\mathsf{r}}\rceil \unrhd \lceil\bar{\mathsf{s}}\rceil\#^0\lceil\underline{\mathsf{bs}}\rceil \unrhd \lceil\bar{\mathsf{t}}\rceil\#^0\lceil\underline{\mathsf{r}}\rceil \unrhd$
$\lceil\bar{\mathsf{t}}\rceil\#^0\lceil\underline{\mathsf{bs}}\rceil \unrhd \lceil\bar{\mathsf{u}}\rceil\#^0\lceil\underline{\mathsf{r}}\rceil \unrhd \lceil\bar{\mathsf{u}}\rceil\#^0\lceil\underline{\mathsf{bs}}\rceil \rhd \underline{\mathsf{y}} \in \underline{\mathsf{bs}} \rhd \underline{\mathsf{x}} \in \underline{\mathsf{bs}} \rhd \dot{\neg}\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow$
$\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow$
$\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{u}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{u}} \in \underline{\mathsf{bs}} \Rightarrow$
$\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \rhd \{\{\underline{\mathsf{y}},\underline{\mathsf{y}}\},\{\underline{\mathsf{y}},\underline{\mathsf{x}}\}\} \in$
$\underline{\mathsf{r}} \gg \bar{\mathsf{s}} \in \{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{y}}\}\} \in \underline{\mathsf{r}}\} \Rightarrow \bar{\mathsf{s}} \in \{\mathrm{ph} \in \underline{\mathsf{bs}} \mid$
$\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{x}}\}\} \in \underline{\mathsf{r}}\}; \mathrm{ToSetEquality} \rhd \bar{\mathsf{s}} \in \{\mathrm{ph} \in \underline{\mathsf{bs}} \mid$
$\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{x}}\}\} \in \underline{\mathsf{r}}\} \Rightarrow \bar{\mathsf{s}} \in \{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{y}}\}\} \in \underline{\mathsf{r}}\} \rhd \bar{\mathsf{s}} \in$
$\{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{y}}\}\} \in \underline{\mathsf{r}}\} \Rightarrow \bar{\mathsf{s}} \in \{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{x}}\}\} \in$
$\underline{\mathsf{r}}\} \gg \{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{x}}\}\} \in \underline{\mathsf{r}}\} = \{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{y}}\}\} \in$
$\underline{\mathsf{r}}\}; \forall\underline{\mathsf{r}}\colon \forall\underline{\mathsf{x}}\colon \forall\underline{\mathsf{y}}\colon \forall\underline{\mathsf{bs}}\colon \mathrm{Ded} \rhd \forall\underline{\mathsf{r}}\colon \forall\underline{\mathsf{x}}\colon \forall\underline{\mathsf{y}}\colon \forall\underline{\mathsf{bs}}\colon \lceil\bar{\mathsf{s}}\rceil\#^0\lceil\underline{\mathsf{r}}\rceil \Vdash \lceil\bar{\mathsf{s}}\rceil\#^0\lceil\underline{\mathsf{bs}}\rceil \Vdash \lceil\bar{\mathsf{t}}\rceil\#^0\lceil\underline{\mathsf{r}}\rceil \Vdash$
$\lceil\bar{\mathsf{t}}\rceil\#^0\lceil\underline{\mathsf{bs}}\rceil \Vdash \lceil\bar{\mathsf{u}}\rceil\#^0\lceil\underline{\mathsf{r}}\rceil \Vdash \lceil\bar{\mathsf{u}}\rceil\#^0\lceil\underline{\mathsf{bs}}\rceil \Vdash \underline{\mathsf{x}} \in \underline{\mathsf{bs}} \vdash \underline{\mathsf{y}} \in \underline{\mathsf{bs}} \vdash \dot{\neg}\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow$
$\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow$
$\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{u}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{u}} \in \underline{\mathsf{bs}} \Rightarrow$
$\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \vdash \{\{\underline{\mathsf{x}},\underline{\mathsf{x}}\},\{\underline{\mathsf{x}},\underline{\mathsf{y}}\}\} \in$
$\underline{\mathsf{r}} \vdash \{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{x}}\}\} \in \underline{\mathsf{r}}\} = \{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{y}}\}\} \in$
$\underline{\mathsf{r}}\} \gg \lceil\bar{\mathsf{s}}\rceil\#^0\lceil\underline{\mathsf{r}}\rceil \Vdash \lceil\bar{\mathsf{s}}\rceil\#^0\lceil\underline{\mathsf{bs}}\rceil \Vdash \lceil\bar{\mathsf{t}}\rceil\#^0\lceil\underline{\mathsf{r}}\rceil \Vdash \lceil\bar{\mathsf{t}}\rceil\#^0\lceil\underline{\mathsf{bs}}\rceil \Vdash \lceil\bar{\mathsf{u}}\rceil\#^0\lceil\underline{\mathsf{r}}\rceil \Vdash$
$\lceil\bar{\mathsf{u}}\rceil\#^0\lceil\underline{\mathsf{bs}}\rceil \Vdash \underline{\mathsf{x}} \in \underline{\mathsf{bs}} \Rightarrow \underline{\mathsf{y}} \in \underline{\mathsf{bs}} \Rightarrow \dot{\neg}\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow$
$\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow$
$\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{u}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{u}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow$
$\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\underline{\mathsf{x}},\underline{\mathsf{x}}\},\{\underline{\mathsf{x}},\underline{\mathsf{y}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\mathrm{ph} \in \underline{\mathsf{bs}} \mid$
$\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{x}}\}\} \in \underline{\mathsf{r}}\} = \{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{y}}\}\} \in \underline{\mathsf{r}}\}\rceil, \mathrm{p}_0, \mathsf{c})]$

Vi mangler nu blot at skifte fra implikationer til inferenser:

$[\mathrm{EqNecessary} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall\underline{\mathsf{r}}\colon \forall\underline{\mathsf{x}}\colon \forall\underline{\mathsf{y}}\colon \forall\underline{\mathsf{bs}}\colon \lceil\bar{\mathsf{s}}\rceil\#^0\lceil\underline{\mathsf{r}}\rceil \Vdash \lceil\bar{\mathsf{s}}\rceil\#^0\lceil\underline{\mathsf{bs}}\rceil \Vdash \lceil\bar{\mathsf{t}}\rceil\#^0\lceil\underline{\mathsf{r}}\rceil \Vdash$
$\lceil\bar{\mathsf{t}}\rceil\#^0\lceil\underline{\mathsf{bs}}\rceil \Vdash \lceil\bar{\mathsf{u}}\rceil\#^0\lceil\underline{\mathsf{r}}\rceil \Vdash \lceil\bar{\mathsf{u}}\rceil\#^0\lceil\underline{\mathsf{bs}}\rceil \Vdash \underline{\mathsf{x}} \in \underline{\mathsf{bs}} \vdash \underline{\mathsf{y}} \in \underline{\mathsf{bs}} \vdash \dot{\neg}\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow$
$\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow$
$\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{u}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{u}} \in \underline{\mathsf{bs}} \Rightarrow$
$\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \vdash \{\{\underline{\mathsf{x}},\underline{\mathsf{x}}\},\{\underline{\mathsf{x}},\underline{\mathsf{y}}\}\} \in$
$\underline{\mathsf{r}} \Rightarrow \{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{x}}\}\} \in \underline{\mathsf{r}}\} = \{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{y}}\}\} \in$
$\underline{\mathsf{r}}\}]$

$[\mathrm{EqNecessary} \overset{\mathrm{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil\mathrm{ZFsub} \vdash \forall\underline{\mathsf{r}}\colon \forall\underline{\mathsf{x}}\colon \forall\underline{\mathsf{y}}\colon \forall\underline{\mathsf{bs}}\colon \lceil\bar{\mathsf{s}}\rceil\#^0\lceil\underline{\mathsf{r}}\rceil \Vdash \lceil\bar{\mathsf{s}}\rceil\#^0\lceil\underline{\mathsf{bs}}\rceil \Vdash$
$\lceil\bar{\mathsf{t}}\rceil\#^0\lceil\underline{\mathsf{r}}\rceil \Vdash \lceil\bar{\mathsf{t}}\rceil\#^0\lceil\underline{\mathsf{bs}}\rceil \Vdash \lceil\bar{\mathsf{u}}\rceil\#^0\lceil\underline{\mathsf{r}}\rceil \Vdash \lceil\bar{\mathsf{u}}\rceil\#^0\lceil\underline{\mathsf{bs}}\rceil \Vdash \underline{\mathsf{x}} \in \underline{\mathsf{bs}} \vdash \underline{\mathsf{y}} \in \underline{\mathsf{bs}} \vdash$

$\dot{\neg}\dot{\neg}\forall_{obj}\bar{s}\colon\bar{s}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\}\in\underline{r}\Rightarrow\dot{\neg}\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\underline{r}\Rightarrow\dot{\neg}\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon\forall_{obj}\bar{u}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\bar{u}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\underline{r}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\underline{r}\vdash$ HelperEqNecessary $\unrhd\lceil\bar{s}\rceil\#^0\lceil\underline{r}\rceil\unrhd\lceil\bar{s}\rceil\#^0\lceil\underline{bs}\rceil\unrhd\lceil\bar{t}\rceil\#^0\lceil\underline{r}\rceil\unrhd\lceil\bar{t}\rceil\#^0\lceil\underline{bs}\rceil\unrhd\lceil\bar{u}\rceil\#^0\lceil\underline{r}\rceil\unrhd\lceil\bar{u}\rceil\#^0\lceil\underline{bs}\rceil\gg\underline{x}\in\underline{bs}\Rightarrow\underline{y}\in\underline{bs}\Rightarrow\dot{\neg}\dot{\neg}\forall_{obj}\bar{s}\colon\bar{s}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\}\in\underline{r}\Rightarrow\dot{\neg}\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\underline{r}\Rightarrow\dot{\neg}\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon\forall_{obj}\bar{u}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\bar{u}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\underline{r}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\underline{r}\Rightarrow\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\}\in\underline{r}\Rightarrow\{ph\in\underline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{x}\}\}\in\underline{r}\}=\{ph\in\underline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{y}\}\}\in\underline{r}\};MP3\rhd\underline{x}\in\underline{bs}\Rightarrow\underline{y}\in\underline{bs}\Rightarrow\dot{\neg}\dot{\neg}\forall_{obj}\bar{s}\colon\bar{s}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\}\in\underline{r}\Rightarrow\dot{\neg}\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\underline{r}\Rightarrow\dot{\neg}\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon\forall_{obj}\bar{u}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\bar{u}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\underline{r}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\underline{r}\Rightarrow\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\}\in\underline{r}\Rightarrow\{ph\in\underline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{x}\}\}\in\underline{r}\}=\{ph\in\underline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{y}\}\}\in\underline{r}\}\rhd\underline{x}\in\underline{bs}\rhd\underline{y}\in\underline{bs}\rhd\dot{\neg}\dot{\neg}\forall_{obj}\bar{s}\colon\bar{s}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\}\in\underline{r}\Rightarrow\dot{\neg}\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\underline{r}\Rightarrow\dot{\neg}\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon\forall_{obj}\bar{u}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\bar{u}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\underline{r}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\underline{r}\gg\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\}\in\underline{r}\Rightarrow\{ph\in\underline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{x}\}\}\in\underline{r}\}=\{ph\in\underline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{y}\}\}\in\underline{r}\}\rceil,p_0,c)]$

11.2.3 Lemmaet "NoneEqNecessary"

Resultatet fra dette under-underafsnit kan formuleres som følger: "Lad $\underline{x}$ og $\underline{y}$ være to medlemmer af $\underline{bs}$, som ikke opfylder $\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\}\in\underline{r}$. Da er de to ækvivalensklasser $\{ph\in\underline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{x}\}\}\in\underline{r}\}$ og $\{ph\in\underline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{y}\}\}\in\underline{r}\}$ indbyrdes disjunkte."[21]
Vi viser dette ved at vise den kontrapositive påstand: Hvis ækvivalensklasserne har et medlem tilfælles, så gælder $\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\}\in\underline{r}$.
Dette varetages af lemmaet HelperNoneEqNecessary:

[HelperNoneEqNecessary $\overset{stmt}{\rightarrow}$ ZFsub $\vdash\forall\underline{r}\colon\forall\underline{s}\colon\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{bs}\colon\lceil\bar{s}\rceil\#^0\lceil\underline{r}\rceil\Vdash\lceil\bar{s}\rceil\#^0\lceil\underline{bs}\rceil\Vdash\lceil\bar{t}\rceil\#^0\lceil\underline{r}\rceil\Vdash\lceil\bar{t}\rceil\#^0\lceil\underline{bs}\rceil\Vdash\lceil\bar{u}\rceil\#^0\lceil\underline{r}\rceil\Vdash\lceil\bar{u}\rceil\#^0\lceil\underline{bs}\rceil\Vdash\underline{x}\in\underline{bs}\Rightarrow\underline{y}\in\underline{bs}\Rightarrow\dot{\neg}\dot{\neg}\forall_{obj}\bar{s}\colon\bar{s}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\}\in\underline{r}\Rightarrow\dot{\neg}\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\underline{r}\Rightarrow\dot{\neg}\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon\forall_{obj}\bar{u}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\bar{u}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\underline{r}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\underline{r}\Rightarrow\underline{s}\in\{ph\in\cup\{\{\{ph\in\underline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{x}\}\}\in\underline{r}\},\{ph\in\underline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{x}\}\}\in\underline{r}\}\},\{\{ph\in\underline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{y}\}\}\in\underline{r}\},\{ph\in\underline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{y}\}\}\in\underline{r}\}\}\}\mid\dot{\neg}c_{Ph}\in\{ph\in\underline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{x}\}\}\in\underline{r}\}\Rightarrow\dot{\neg}c_{Ph}\in\{ph\in\underline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{y}\}\}\in\underline{r}\}\}\Rightarrow\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\}\in\underline{r}$]

Beviset for HelperEqNecessary kan beskrives som følger:

1. Antag at $\underline{s}$ tilhører både $\{ph\in\underline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{x}\}\}\in\underline{r}\}$ og $\{ph\in\underline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{y}\}\}\in\underline{r}\}$. Da gælder $\{\{\underline{s},\underline{s}\},\{\underline{s},\underline{x}\}\}\in\underline{r}$ (linie 17–19) og $\{\{\underline{s},\underline{s}\},\{\underline{s},\underline{y}\}\}\in\underline{r}$ (linie 20–22).

[21] Dette er lemma 4.4.b.1 i [5].

2. Da $\underline{r}$ er symmetrisk, følger $\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{s}\}\}\in\underline{r}$ af $\{\{\underline{s},\underline{s}\},\{\underline{s},\underline{x}\}\}\in\underline{r}$. (Linie 24).
3. Da $\underline{r}$ også er transitiv, medfører $\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{s}\}\}\in\underline{r}$ og $\{\{\underline{s},\underline{s}\},\{\underline{s},\underline{y}\}\}\in\underline{r}$ tilsammen $\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\}\in\underline{r}$, QED. (Linie 25).

Her er selve beviset:

[HelperNoneEqNecessary $\stackrel{\text{proof}}{\to}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{r}\colon \forall \underline{s}\colon \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{bs}\colon \lceil \bar{s}\rceil \#^0 \lceil \underline{r}\rceil \Vdash$ $\lceil \bar{s}\rceil\#^0\lceil \underline{bs}\rceil \Vdash \lceil \bar{t}\rceil\#^0\lceil \underline{r}\rceil \Vdash \lceil \bar{t}\rceil\#^0\lceil\underline{bs}\rceil \Vdash \lceil\bar{u}\rceil\#^0\lceil\underline{r}\rceil \Vdash \lceil\bar{u}\rceil\#^0\lceil\underline{bs}\rceil \Vdash \underline{x}\in\underline{bs}\vdash \underline{y}\in$ $\underline{bs} \vdash \dot\neg\dot\neg\forall_{\mathrm{obj}}\bar{s}\colon \bar{s}\in\underline{bs}\Rightarrow \{\{\bar s,\bar s\},\{\bar s,\bar s\}\}\in\underline r\Rightarrow\dot\neg\forall_{\mathrm{obj}}\bar s\colon\forall_{\mathrm{obj}}\bar t\colon \bar s\in\underline{bs}\Rightarrow\bar t\in\underline{bs}\Rightarrow$ $\{\{\bar s,\bar s\},\{\bar s,\bar t\}\}\in\underline r\Rightarrow\{\{\bar t,\bar t\},\{\bar t,\bar s\}\}\in\underline r\Rightarrow\dot\neg\forall_{\mathrm{obj}}\bar s\colon\forall_{\mathrm{obj}}\bar t\colon\forall_{\mathrm{obj}}\bar u\colon\bar s\in\underline{bs}\Rightarrow\bar t\in\underline{bs}\Rightarrow$ $\bar u\in\underline{bs}\Rightarrow\{\{\bar s,\bar s\},\{\bar s,\bar t\}\}\in\underline r\Rightarrow\{\{\bar t,\bar t\},\{\bar t,\bar u\}\}\in\underline r\Rightarrow\{\{\bar s,\bar s\},\{\bar s,\bar u\}\}\in\underline r\vdash\underline s\in$ $\{\mathrm{ph}\in\cup\{\{\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline x\}\}\in\underline r\},\{\mathrm{ph}\in\underline{bs}\mid$ $\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline x\}\}\in\underline r\}\},\{\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline y\}\}\in\underline r\},\{\mathrm{ph}\in\underline{bs}\mid$ $\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline y\}\}\in\underline r\}\}\}\mid\dot\neg c_{\mathrm{Ph}}\in\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline x\}\}\in\underline r\}\Rightarrow$ $\dot\neg c_{\mathrm{Ph}}\in\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline y\}\}\in\underline r\}\}\vdash$ ERisSymmetric $\rhd\dot\neg\dot\neg\forall_{\mathrm{obj}}\bar s\colon\bar s\in$ $\underline{bs}\Rightarrow\{\{\bar s,\bar s\},\{\bar s,\bar s\}\}\in\underline r\Rightarrow\dot\neg\forall_{\mathrm{obj}}\bar s\colon\forall_{\mathrm{obj}}\bar t\colon\bar s\in\underline{bs}\Rightarrow\bar t\in\underline{bs}\Rightarrow\{\{\bar s,\bar s\},\{\bar s,\bar t\}\}\in\underline r\Rightarrow$ $\{\{\bar t,\bar t\},\{\bar t,\bar s\}\}\in\underline r\Rightarrow\dot\neg\forall_{\mathrm{obj}}\bar s\colon\forall_{\mathrm{obj}}\bar t\colon\forall_{\mathrm{obj}}\bar u\colon\bar s\in\underline{bs}\Rightarrow\bar t\in\underline{bs}\Rightarrow\bar u\in\underline{bs}\Rightarrow$ $\{\{\bar s,\bar s\},\{\bar s,\bar t\}\}\in\underline r\Rightarrow\{\{\bar t,\bar t\},\{\bar t,\bar u\}\}\in\underline r\Rightarrow\{\{\bar s,\bar s\},\{\bar s,\bar u\}\}\in\underline r\gg\forall_{\mathrm{obj}}\bar s\colon\forall_{\mathrm{obj}}\bar t\colon\bar s\in$ $\underline{bs}\Rightarrow\bar t\in\underline{bs}\Rightarrow\{\{\bar s,\bar s\},\{\bar s,\bar t\}\}\in\underline r\Rightarrow\{\{\bar t,\bar t\},\{\bar t,\bar s\}\}\in\underline r;$ ERisTransitive $\rhd$ $\dot\neg\dot\neg\forall_{\mathrm{obj}}\bar s\colon\bar s\in\underline{bs}\Rightarrow\{\{\bar s,\bar s\},\{\bar s,\bar s\}\}\in\underline r\Rightarrow\dot\neg\forall_{\mathrm{obj}}\bar s\colon\forall_{\mathrm{obj}}\bar t\colon\bar s\in\underline{bs}\Rightarrow\bar t\in\underline{bs}\Rightarrow$ $\{\{\bar s,\bar s\},\{\bar s,\bar t\}\}\in\underline r\Rightarrow\{\{\bar t,\bar t\},\{\bar t,\bar s\}\}\in\underline r\Rightarrow\dot\neg\forall_{\mathrm{obj}}\bar s\colon\forall_{\mathrm{obj}}\bar t\colon\forall_{\mathrm{obj}}\bar u\colon\bar s\in\underline{bs}\Rightarrow\bar t\in\underline{bs}\Rightarrow$ $\bar u\in\underline{bs}\Rightarrow\{\{\bar s,\bar s\},\{\bar s,\bar t\}\}\in\underline r\Rightarrow\{\{\bar t,\bar t\},\{\bar t,\bar u\}\}\in\underline r\Rightarrow\{\{\bar s,\bar s\},\{\bar s,\bar u\}\}\in\underline r\gg$ $\forall_{\mathrm{obj}}\bar s\colon\forall_{\mathrm{obj}}\bar t\colon\forall_{\mathrm{obj}}\bar u\colon\bar s\in\underline{bs}\Rightarrow\bar t\in\underline{bs}\Rightarrow\bar u\in\underline{bs}\Rightarrow\{\{\bar s,\bar s\},\{\bar s,\bar t\}\}\in\underline r\Rightarrow$ $\{\{\bar t,\bar t\},\{\bar t,\bar u\}\}\in\underline r\Rightarrow\{\{\bar s,\bar s\},\{\bar s,\bar u\}\}\in\underline r;$ Sep2Formula $\rhd\underline s\in\{\mathrm{ph}\in\cup\{\{\{\mathrm{ph}\in\underline{bs}\mid$ $\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline x\}\}\in\underline r\},\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline x\}\}\in\underline r\}\},\{\{\mathrm{ph}\in\underline{bs}\mid$ $\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline y\}\}\in\underline r\},\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline y\}\}\in\underline r\}\}\}\mid\dot\neg c_{\mathrm{Ph}}\in$ $\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline x\}\}\in\underline r\}\Rightarrow\dot\neg c_{\mathrm{Ph}}\in\{\mathrm{ph}\in\underline{bs}\mid$ $\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline y\}\}\in\underline r\}\}\gg\dot\neg\underline s\in\cup\{\{\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline x\}\}\in$ $\underline r\},\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline x\}\}\in\underline r\}\},\{\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline y\}\}\in$ $\underline r\},\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline y\}\}\in\underline r\}\}\}\Rightarrow\dot\neg\dot\neg\underline s\in\{\mathrm{ph}\in\underline{bs}\mid$ $\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline x\}\}\in\underline r\}\Rightarrow\dot\neg\underline s\in\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline y\}\}\in$ $\underline r\};$ SecondConjunct $\rhd\dot\neg\underline s\in\cup\{\{\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline x\}\}\in\underline r\},\{\mathrm{ph}\in\underline{bs}\mid$ $\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline x\}\}\in\underline r\}\},\{\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline y\}\}\in\underline r\},\{\mathrm{ph}\in\underline{bs}\mid$ $\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline y\}\}\in\underline r\}\}\}\Rightarrow\dot\neg\dot\neg\underline s\in\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline x\}\}\in\underline r\}\Rightarrow$ $\dot\neg\underline s\in\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline y\}\}\in\underline r\}\gg\dot\neg\underline s\in\{\mathrm{ph}\in\underline{bs}\mid$ $\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline x\}\}\in\underline r\}\Rightarrow\dot\neg\underline s\in\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline y\}\}\in$ $\underline r\};$ FirstConjunct $\rhd\dot\neg\underline s\in\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline x\}\}\in\underline r\}\Rightarrow\dot\neg\underline s\in\{\mathrm{ph}\in$ $\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline y\}\}\in\underline r\}\gg\underline s\in\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline x\}\}\in$ $\underline r\};$ Sep2Formula $\rhd\underline s\in\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline x\}\}\in\underline r\}\gg\dot\neg\underline s\in\underline{bs}\Rightarrow$ $\dot\neg\{\{\underline s,\underline s\},\{\underline s,\underline x\}\}\in\underline r;$ SecondConjunct $\rhd\dot\neg\underline s\in\underline{bs}\Rightarrow\dot\neg\{\{\underline s,\underline s\},\{\underline s,\underline x\}\}\in\underline r\gg$ $\{\{\underline s,\underline s\},\{\underline s,\underline x\}\}\in\underline r;$ SecondConjunct $\rhd\dot\neg\underline s\in\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline x\}\}\in$ $\underline r\}\Rightarrow\dot\neg\underline s\in\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline y\}\}\in\underline r\}\gg\underline s\in\{\mathrm{ph}\in\underline{bs}\mid$ $\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline y\}\}\in\underline r\};$ Sep2Formula $\rhd\underline s\in\{\mathrm{ph}\in\underline{bs}\mid$ $\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline y\}\}\in\underline r\}\gg\dot\neg\underline s\in\underline{bs}\Rightarrow\dot\neg\{\{\underline s,\underline s\},\{\underline s,\underline y\}\}\in$ $\underline r;$ SecondConjunct $\rhd\dot\neg\underline s\in\underline{bs}\Rightarrow\dot\neg\{\{\underline s,\underline s\},\{\underline s,\underline y\}\}\in\underline r\gg\{\{\underline s,\underline s\},\{\underline s,\underline y\}\}\in$

$\underline{r}$; FirstConjunct $\rhd \dot{\neg}\,\underline{s}\in\underline{bs}\Rightarrow\dot{\neg}\,\{\{\underline{s},\underline{s}\},\{\underline{s},\underline{x}\}\}\in\underline{r}\gg\underline{s}\in\underline{bs}$; Symmetry $\unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{r}\rceil \unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{bs}\rceil \unrhd \lceil\bar{t}\rceil\#^0\lceil\underline{r}\rceil \unrhd \lceil\bar{t}\rceil\#^0\lceil\underline{bs}\rceil \rhd \forall_{\mathrm{obj}}\bar{s}\colon\forall_{\mathrm{obj}}\bar{t}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\underline{r}\rhd\underline{s}\in\underline{bs}\rhd\underline{x}\in\underline{bs}\rhd\{\{\underline{s},\underline{s}\},\{\underline{s},\underline{x}\}\}\in\underline{r}\gg\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{s}\}\}\in\underline{r}$; Transitivity $\unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{r}\rceil \unrhd \lceil\bar{s}\rceil\#^0\lceil\underline{bs}\rceil \unrhd \lceil\bar{t}\rceil\#^0\lceil\underline{r}\rceil \unrhd \lceil\bar{t}\rceil\#^0\lceil\underline{bs}\rceil \unrhd \lceil\bar{u}\rceil\#^0\lceil\underline{r}\rceil \unrhd \lceil\bar{u}\rceil\#^0\lceil\underline{bs}\rceil \rhd \forall_{\mathrm{obj}}\bar{s}\colon\forall_{\mathrm{obj}}\bar{t}\colon\forall_{\mathrm{obj}}\bar{u}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\bar{u}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\underline{r}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\underline{r}\rhd\underline{x}\in\underline{bs}\rhd\underline{s}\in\underline{bs}\rhd\underline{y}\in\underline{bs}\rhd\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{s}\}\}\in\underline{r}\rhd\{\{\underline{s},\underline{s}\},\{\underline{s},\underline{y}\}\}\in\underline{r}\gg\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\}\in\underline{r}$; $\forall\underline{r}\colon\forall\underline{s}\colon\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{bs}\colon$ Ded $\rhd\forall\underline{r}\colon\forall\underline{s}\colon\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{bs}\colon\lceil\bar{s}\rceil\#^0\lceil\underline{r}\rceil\Vdash\lceil\bar{s}\rceil\#^0\lceil\underline{bs}\rceil\Vdash\lceil\bar{t}\rceil\#^0\lceil\underline{r}\rceil\Vdash\lceil\bar{t}\rceil\#^0\lceil\underline{bs}\rceil\Vdash\lceil\bar{u}\rceil\#^0\lceil\underline{r}\rceil\Vdash\lceil\bar{u}\rceil\#^0\lceil\underline{bs}\rceil\Vdash\underline{x}\in\underline{bs}\vdash\underline{y}\in\underline{bs}\vdash\dot{\neg}\,\dot{\neg}\,\forall_{\mathrm{obj}}\bar{s}\colon\bar{s}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\}\in\underline{r}\Rightarrow\dot{\neg}\,\forall_{\mathrm{obj}}\bar{s}\colon\forall_{\mathrm{obj}}\bar{t}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\underline{r}\Rightarrow\dot{\neg}\,\forall_{\mathrm{obj}}\bar{s}\colon\forall_{\mathrm{obj}}\bar{t}\colon\forall_{\mathrm{obj}}\bar{u}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\bar{u}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\underline{r}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\underline{r}\vdash\underline{s}\in\{\mathrm{ph}\in\cup\{\{\{\mathrm{ph}\in\underline{bs}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\},\{\mathrm{ph}\in\underline{bs}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\}\},\{\{\mathrm{ph}\in\underline{bs}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\},\{\mathrm{ph}\in\underline{bs}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\}\}\}\mid\dot{\neg}\,\mathsf{c}_{\mathrm{Ph}}\in\{\mathrm{ph}\in\underline{bs}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\}\Rightarrow\dot{\neg}\,\mathsf{c}_{\mathrm{Ph}}\in\{\mathrm{ph}\in\underline{bs}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\}\}\vdash\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\}\in\underline{r}\gg\lceil\bar{s}\rceil\#^0\lceil\underline{r}\rceil\Vdash\lceil\bar{s}\rceil\#^0\lceil\underline{bs}\rceil\Vdash\lceil\bar{t}\rceil\#^0\lceil\underline{r}\rceil\Vdash\lceil\bar{t}\rceil\#^0\lceil\underline{bs}\rceil\Vdash\lceil\bar{u}\rceil\#^0\lceil\underline{r}\rceil\Vdash\lceil\bar{u}\rceil\#^0\lceil\underline{bs}\rceil\Vdash\underline{x}\in\underline{bs}\Rightarrow\underline{y}\in\underline{bs}\Rightarrow\dot{\neg}\,\dot{\neg}\,\forall_{\mathrm{obj}}\bar{s}\colon\bar{s}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\}\in\underline{r}\Rightarrow\dot{\neg}\,\forall_{\mathrm{obj}}\bar{s}\colon\forall_{\mathrm{obj}}\bar{t}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\underline{r}\Rightarrow\dot{\neg}\,\forall_{\mathrm{obj}}\bar{s}\colon\forall_{\mathrm{obj}}\bar{t}\colon\forall_{\mathrm{obj}}\bar{u}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\bar{u}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\underline{r}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\underline{r}\Rightarrow\underline{s}\in\{\mathrm{ph}\in\cup\{\{\{\mathrm{ph}\in\underline{bs}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\},\{\mathrm{ph}\in\underline{bs}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\}\},\{\{\mathrm{ph}\in\underline{bs}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\},\{\mathrm{ph}\in\underline{bs}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\}\}\}\mid\dot{\neg}\,\mathsf{c}_{\mathrm{Ph}}\in\{\mathrm{ph}\in\underline{bs}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\}\Rightarrow\dot{\neg}\,\mathsf{c}_{\mathrm{Ph}}\in\{\mathrm{ph}\in\underline{bs}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\}\}\Rightarrow\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\}\in\underline{r}\rceil,\mathrm{p}_0,\mathsf{c})]$

Vi bruger nu modus tollens til at vise en kontrapositiv version af HelperNoneEqNecessary. Dette sker i beviset for en lemmastump ved navn Helper(2)NoneEqNecessary:

[Helper(2)NoneEqNecessary $\stackrel{\mathrm{stmt}}{\rightarrow}$ ZFsub $\vdash\forall\underline{r}\colon\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{bs}\colon\lceil\bar{s}\rceil\#^0\lceil\underline{r}\rceil\Vdash\lceil\bar{s}\rceil\#^0\lceil\underline{bs}\rceil\Vdash\lceil\bar{t}\rceil\#^0\lceil\underline{r}\rceil\Vdash\lceil\bar{t}\rceil\#^0\lceil\underline{bs}\rceil\Vdash\lceil\bar{u}\rceil\#^0\lceil\underline{r}\rceil\Vdash\lceil\bar{u}\rceil\#^0\lceil\underline{bs}\rceil\Vdash\underline{x}\in\underline{bs}\Rightarrow\underline{y}\in\underline{bs}\Rightarrow\dot{\neg}\,\dot{\neg}\,\forall_{\mathrm{obj}}\bar{s}\colon\bar{s}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\}\in\underline{r}\Rightarrow\dot{\neg}\,\forall_{\mathrm{obj}}\bar{s}\colon\forall_{\mathrm{obj}}\bar{t}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\underline{r}\Rightarrow\dot{\neg}\,\forall_{\mathrm{obj}}\bar{s}\colon\forall_{\mathrm{obj}}\bar{t}\colon\forall_{\mathrm{obj}}\bar{u}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\bar{u}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\underline{r}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\underline{r}\Rightarrow\dot{\neg}\,\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\}\in\underline{r}\Rightarrow\{\mathrm{ph}\in\cup\{\{\{\mathrm{ph}\in\underline{bs}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\},\{\mathrm{ph}\in\underline{bs}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\}\},\{\{\mathrm{ph}\in\underline{bs}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\},\{\mathrm{ph}\in\underline{bs}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\}\}\}\mid\dot{\neg}\,\mathsf{c}_{\mathrm{Ph}}\in\{\mathrm{ph}\in\underline{bs}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\}\Rightarrow\dot{\neg}\,\mathsf{c}_{\mathrm{Ph}}\in\{\mathrm{ph}\in\underline{bs}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\}\}=\emptyset$]

[Helper(2)NoneEqNecessary $\stackrel{\mathrm{proof}}{\rightarrow}\lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil$ZFsub $\vdash\forall\underline{r}\colon\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{bs}\colon\lceil\bar{s}\rceil\#^0\lceil\underline{r}\rceil\Vdash\lceil\bar{s}\rceil\#^0\lceil\underline{bs}\rceil\Vdash\lceil\bar{t}\rceil\#^0\lceil\underline{r}\rceil\Vdash\lceil\bar{t}\rceil\#^0\lceil\underline{bs}\rceil\Vdash\lceil\bar{u}\rceil\#^0\lceil\underline{r}\rceil\Vdash\lceil\bar{u}\rceil\#^0\lceil\underline{bs}\rceil\Vdash\underline{x}\in\underline{bs}\vdash\underline{y}\in\underline{bs}\vdash\dot{\neg}\,\dot{\neg}\,\forall_{\mathrm{obj}}\bar{s}\colon\bar{s}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\}\in\underline{r}\Rightarrow\dot{\neg}\,\forall_{\mathrm{obj}}\bar{s}\colon\forall_{\mathrm{obj}}\bar{t}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\underline{r}\Rightarrow\dot{\neg}\,\forall_{\mathrm{obj}}\bar{s}\colon\forall_{\mathrm{obj}}\bar{t}\colon\forall_{\mathrm{obj}}\bar{u}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\bar{u}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\underline{r}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\underline{r}\vdash\dot{\neg}\,\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\}\in\underline{r}\vdash$ HelperNoneEqNecessary $\unrhd\lceil\bar{s}\rceil\#^0\lceil\underline{r}\rceil\unrhd\lceil\bar{s}\rceil\#^0\lceil\underline{bs}\rceil\unrhd\lceil\bar{t}\rceil\#^0\lceil\underline{r}\rceil\unrhd\lceil\bar{t}\rceil\#^0\lceil\underline{bs}\rceil\unrhd\lceil\bar{u}\rceil\#^0\lceil\underline{r}\rceil\unrhd\lceil\bar{u}\rceil\#^0\lceil\underline{bs}\rceil\gg\underline{x}\in\underline{bs}\Rightarrow\underline{y}\in\underline{bs}\Rightarrow$

$\dot\neg\dot\neg\forall_{\mathrm{obj}}\bar{s}{:}\,\bar{s}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\}\in\underline{r}\Rightarrow\dot\neg\forall_{\mathrm{obj}}\bar{s}{:}\,\forall_{\mathrm{obj}}\bar{t}{:}\,\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow$
$\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\underline{r}\Rightarrow\dot\neg\forall_{\mathrm{obj}}\bar{s}{:}\,\forall_{\mathrm{obj}}\bar{t}{:}\,\forall_{\mathrm{obj}}\bar{u}{:}\,\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow$
$\bar{u}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\underline{r}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\underline{r}\Rightarrow\bar{s}\in$
$\{\mathrm{ph}\in\cup\{\{\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\},\{\mathrm{ph}\in\underline{bs}\mid$
$\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\}\},\{\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\},\{\mathrm{ph}\in\underline{bs}\mid$
$\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\}\}\}\mid\dot\neg c_{\mathrm{Ph}}\in\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\}\Rightarrow$
$\dot\neg c_{\mathrm{Ph}}\in\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\}\}\Rightarrow\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\}\in\underline{r};\mathrm{MP3}\rhd\underline{x}\in$
$\underline{bs}\Rightarrow\underline{y}\in\underline{bs}\Rightarrow\dot\neg\dot\neg\forall_{\mathrm{obj}}\bar{s}{:}\,\bar{s}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\}\in\underline{r}\Rightarrow\dot\neg\forall_{\mathrm{obj}}\bar{s}{:}\,\forall_{\mathrm{obj}}\bar{t}{:}\,\bar{s}\in\underline{bs}\Rightarrow$
$\bar{t}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\underline{r}\Rightarrow\dot\neg\forall_{\mathrm{obj}}\bar{s}{:}\,\forall_{\mathrm{obj}}\bar{t}{:}\,\forall_{\mathrm{obj}}\bar{u}{:}\,\bar{s}\in\underline{bs}\Rightarrow$
$\bar{t}\in\underline{bs}\Rightarrow\bar{u}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\underline{r}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in$
$\underline{r}\Rightarrow\bar{s}\in\{\mathrm{ph}\in\cup\{\{\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\},\{\mathrm{ph}\in\underline{bs}\mid$
$\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\}\},\{\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\},\{\mathrm{ph}\in\underline{bs}\mid$
$\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\}\}\}\mid\dot\neg c_{\mathrm{Ph}}\in\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\}\Rightarrow$
$\dot\neg c_{\mathrm{Ph}}\in\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\}\}\Rightarrow\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\}\in\underline{r}\rhd\underline{x}\in$
$\underline{bs}\rhd\underline{y}\in\underline{bs}\rhd\dot\neg\dot\neg\forall_{\mathrm{obj}}\bar{s}{:}\,\bar{s}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\}\in\underline{r}\Rightarrow\dot\neg\forall_{\mathrm{obj}}\bar{s}{:}\,\forall_{\mathrm{obj}}\bar{t}{:}\,\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in$
$\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\underline{r}\Rightarrow\dot\neg\forall_{\mathrm{obj}}\bar{s}{:}\,\forall_{\mathrm{obj}}\bar{t}{:}\,\forall_{\mathrm{obj}}\bar{u}{:}\,\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in$
$\underline{bs}\Rightarrow\bar{u}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\underline{r}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\underline{r}\gg$
$\bar{s}\in\{\mathrm{ph}\in\cup\{\{\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\},\{\mathrm{ph}\in\underline{bs}\mid$
$\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\}\},\{\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\},\{\mathrm{ph}\in\underline{bs}\mid$
$\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\}\}\}\mid\dot\neg c_{\mathrm{Ph}}\in\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\}\Rightarrow$
$\dot\neg c_{\mathrm{Ph}}\in\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\}\}\Rightarrow\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\}\in\underline{r};\mathrm{MT}\rhd\bar{s}\in$
$\{\mathrm{ph}\in\cup\{\{\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\},\{\mathrm{ph}\in\underline{bs}\mid$
$\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\}\},\{\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\},\{\mathrm{ph}\in\underline{bs}\mid$
$\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\}\}\}\mid\dot\neg c_{\mathrm{Ph}}\in\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\}\Rightarrow$
$\dot\neg c_{\mathrm{Ph}}\in\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\}\}\Rightarrow\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\}\in$
$\underline{r}\rhd\dot\neg\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\}\in\underline{r}\gg\dot\neg\bar{s}\in\{\mathrm{ph}\in\cup\{\{\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\}\in$
$\underline{r}\},\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\}\},\{\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\}\in$
$\underline{r}\},\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\}\}\}\mid\dot\neg c_{\mathrm{Ph}}\in\{\mathrm{ph}\in\underline{bs}\mid$
$\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\}\Rightarrow\dot\neg c_{\mathrm{Ph}}\in\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\}\in$
$\underline{r}\}\};\mathrm{Unique}\emptyset\rhd\dot\neg\bar{s}\in\{\mathrm{ph}\in\cup\{\{\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\},\{\mathrm{ph}\in\underline{bs}\mid$
$\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\}\},\{\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\},\{\mathrm{ph}\in\underline{bs}\mid$
$\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\}\}\}\mid\dot\neg c_{\mathrm{Ph}}\in\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\}\Rightarrow$
$\dot\neg c_{\mathrm{Ph}}\in\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\}\}\gg\{\mathrm{ph}\in\cup\{\{\{\mathrm{ph}\in\underline{bs}\mid$
$\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\},\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\}\},\{\{\mathrm{ph}\in\underline{bs}\mid$
$\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\},\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\}\}\}\mid\dot\neg c_{\mathrm{Ph}}\in$
$\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\}\Rightarrow\dot\neg c_{\mathrm{Ph}}\in\{\mathrm{ph}\in\underline{bs}\mid$
$\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\}\}=\emptyset;\forall\underline{r}{:}\,\forall\underline{x}{:}\,\forall\underline{y}{:}\,\forall\underline{bs}{:}\,\mathrm{Ded}\rhd\forall\underline{r}{:}\,\forall\underline{x}{:}\,\forall\underline{y}{:}\,\forall\underline{bs}{:}\,\lceil\bar{s}\rceil\#^0\lceil\underline{r}\rceil\Vdash$
$\lceil\bar{s}\rceil\#^0\lceil\underline{bs}\rceil\Vdash\lceil\bar{t}\rceil\#^0\lceil\underline{r}\rceil\Vdash\lceil\bar{t}\rceil\#^0\lceil\underline{bs}\rceil\Vdash\lceil\bar{u}\rceil\#^0\lceil\underline{r}\rceil\Vdash\lceil\bar{u}\rceil\#^0\lceil\underline{bs}\rceil\Vdash\underline{x}\in\underline{bs}\vdash\underline{y}\in$
$\underline{bs}\vdash\dot\neg\dot\neg\forall_{\mathrm{obj}}\bar{s}{:}\,\bar{s}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\}\in\underline{r}\Rightarrow\dot\neg\forall_{\mathrm{obj}}\bar{s}{:}\,\forall_{\mathrm{obj}}\bar{t}{:}\,\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow$
$\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\underline{r}\Rightarrow\dot\neg\forall_{\mathrm{obj}}\bar{s}{:}\,\forall_{\mathrm{obj}}\bar{t}{:}\,\forall_{\mathrm{obj}}\bar{u}{:}\,\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow$
$\bar{u}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\underline{r}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\underline{r}\vdash$
$\dot\neg\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\}\in\underline{r}\vdash\{\mathrm{ph}\in\cup\{\{\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\},\{\mathrm{ph}\in$
$\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\}\in\underline{r}\}\},\{\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\},\{\mathrm{ph}\in$
$\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\}\}\}\mid\dot\neg c_{\mathrm{Ph}}\in\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\}\in$
$\underline{r}\}\Rightarrow\dot\neg c_{\mathrm{Ph}}\in\{\mathrm{ph}\in\underline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\}\in\underline{r}\}\}=\emptyset\gg\lceil\bar{s}\rceil\#^0\lceil\underline{r}\rceil\Vdash$

$\lceil \bar{s} \rceil \#^0 \lceil \underline{bs} \rceil \Vdash \lceil \bar{t} \rceil \#^0 \lceil \underline{r} \rceil \Vdash \lceil \bar{t} \rceil \#^0 \lceil \underline{bs} \rceil \Vdash \lceil \bar{u} \rceil \#^0 \lceil \underline{r} \rceil \Vdash \lceil \bar{u} \rceil \#^0 \lceil \underline{bs} \rceil \Vdash \underline{x} \in \underline{bs} \Rightarrow \underline{y} \in \underline{bs} \Rightarrow \dot{\neg}\dot{\neg}\forall_{obj}\bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon\forall_{obj}\bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r} \Rightarrow \dot{\neg}\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\} \in \underline{r} \Rightarrow \{ph \in \cup\{\{\{ph \in \underline{bs} \mid \{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{x}\}\} \in \underline{r}\},\{ph \in \underline{bs} \mid \{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{x}\}\} \in \underline{r}\}\},\{\{ph \in \underline{bs} \mid \{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{y}\}\} \in \underline{r}\},\{ph \in \underline{bs} \mid \{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{y}\}\} \in \underline{r}\}\}\} \mid \dot{\neg} c_{Ph} \in \{ph \in \underline{bs} \mid \{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{x}\}\} \in \underline{r}\} \Rightarrow \dot{\neg} c_{Ph} \in \{ph \in \underline{bs} \mid \{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{y}\}\} \in \underline{r}\}\} = \emptyset\rceil, p_0, c)]$

I beviset for hovedlemmaet NoneEqNecessary foretager vi den sædvanlige konvertering fra $[x \Rightarrow y]$ til $[x \vdash y]$:

$[\text{NoneEqNecessary} \overset{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall\underline{r}\colon\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{bs}\colon \lceil \bar{s} \rceil \#^0 \lceil \underline{r} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{bs} \rceil \Vdash \lceil \bar{t} \rceil \#^0 \lceil \underline{r} \rceil \Vdash \lceil \bar{t} \rceil \#^0 \lceil \underline{bs} \rceil \Vdash \lceil \bar{u} \rceil \#^0 \lceil \underline{r} \rceil \Vdash \lceil \bar{u} \rceil \#^0 \lceil \underline{bs} \rceil \Vdash \underline{x} \in \underline{bs} \vdash \underline{y} \in \underline{bs} \vdash \dot{\neg}\dot{\neg}\forall_{obj}\bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon\forall_{obj}\bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r} \vdash \dot{\neg}\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\} \in \underline{r} \Rightarrow \{ph \in \cup\{\{\{ph \in \underline{bs} \mid \{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{x}\}\} \in \underline{r}\},\{ph \in \underline{bs} \mid \{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{x}\}\} \in \underline{r}\}\},\{\{ph \in \underline{bs} \mid \{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{y}\}\} \in \underline{r}\},\{ph \in \underline{bs} \mid \{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{y}\}\} \in \underline{r}\}\}\} \mid \dot{\neg} c_{Ph} \in \{ph \in \underline{bs} \mid \{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{x}\}\} \in \underline{r}\} \Rightarrow \dot{\neg} c_{Ph} \in \{ph \in \underline{bs} \mid \{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{y}\}\} \in \underline{r}\}\} = \emptyset]$

$[\text{NoneEqNecessary} \overset{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall\underline{r}\colon\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{bs}\colon \lceil \bar{s} \rceil \#^0 \lceil \underline{r} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{bs} \rceil \Vdash \lceil \bar{t} \rceil \#^0 \lceil \underline{r} \rceil \Vdash \lceil \bar{t} \rceil \#^0 \lceil \underline{bs} \rceil \Vdash \lceil \bar{u} \rceil \#^0 \lceil \underline{r} \rceil \Vdash \lceil \bar{u} \rceil \#^0 \lceil \underline{bs} \rceil \Vdash \underline{x} \in \underline{bs} \vdash \underline{y} \in \underline{bs} \vdash \dot{\neg}\dot{\neg}\forall_{obj}\bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon\forall_{obj}\bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r} \vdash \text{Helper(2)NoneEqNecessary} \unrhd \lceil \bar{s} \rceil \#^0 \lceil \underline{r} \rceil \unrhd \lceil \bar{s} \rceil \#^0 \lceil \underline{bs} \rceil \unrhd \lceil \bar{t} \rceil \#^0 \lceil \underline{r} \rceil \unrhd \lceil \bar{t} \rceil \#^0 \lceil \underline{bs} \rceil \unrhd \lceil \bar{u} \rceil \#^0 \lceil \underline{r} \rceil \unrhd \lceil \bar{u} \rceil \#^0 \lceil \underline{bs} \rceil \gg \underline{x} \in \underline{bs} \Rightarrow \underline{y} \in \underline{bs} \Rightarrow \dot{\neg}\dot{\neg}\forall_{obj}\bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon\forall_{obj}\bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r} \Rightarrow \dot{\neg}\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\} \in \underline{r} \Rightarrow \{ph \in \cup\{\{\{ph \in \underline{bs} \mid \{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{x}\}\} \in \underline{r}\},\{ph \in \underline{bs} \mid \{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{x}\}\} \in \underline{r}\}\},\{\{ph \in \underline{bs} \mid \{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{y}\}\} \in \underline{r}\},\{ph \in \underline{bs} \mid \{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{y}\}\} \in \underline{r}\}\}\} \mid \dot{\neg} c_{Ph} \in \{ph \in \underline{bs} \mid \{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{x}\}\} \in \underline{r}\} \Rightarrow \dot{\neg} c_{Ph} \in \{ph \in \underline{bs} \mid \{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{y}\}\} \in \underline{r}\}\} = \emptyset; \text{MP3} \rhd \underline{x} \in \underline{bs} \Rightarrow \underline{y} \in \underline{bs} \Rightarrow \dot{\neg}\dot{\neg}\forall_{obj}\bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon\forall_{obj}\bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r} \Rightarrow \dot{\neg}\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\} \in \underline{r} \Rightarrow \{ph \in \cup\{\{\{ph \in \underline{bs} \mid \{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{x}\}\} \in \underline{r}\},\{ph \in \underline{bs} \mid \{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{x}\}\} \in \underline{r}\}\},\{\{ph \in \underline{bs} \mid \{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{y}\}\} \in \underline{r}\},\{ph \in \underline{bs} \mid \{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{y}\}\} \in \underline{r}\}\}\} \mid \dot{\neg} c_{Ph} \in \{ph \in \underline{bs} \mid \{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{x}\}\} \in \underline{r}\} \Rightarrow \dot{\neg} c_{Ph} \in \{ph \in \underline{bs} \mid \{\{a_{Ph},a_{Ph}\},\{a_{Ph},\underline{y}\}\} \in \underline{r}\}\} = \emptyset \rhd \underline{x} \in \underline{bs} \rhd \underline{y} \in \underline{bs} \rhd \dot{\neg}\dot{\neg}\forall_{obj}\bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon\forall_{obj}\bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r} \gg$

$\dot\neg\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\} \in \underline{r} \Rightarrow \{\mathrm{ph} \in \cup\{\{\{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\} \in \underline{r}\}, \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\} \in \underline{r}\}\}, \{\{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\} \in \underline{r}\}, \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\} \in \underline{r}\}\}\} \mid \dot\neg c_{\mathrm{Ph}} \in \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\} \in \underline{r}\} \Rightarrow \dot\neg c_{\mathrm{Ph}} \in \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\} \in \underline{r}\}\} = \emptyset\rceil, p_0, c)]$

11.2.4 To ækvivalensklasser er disjunkte

Vi kan nu bevise, at to forskellige ækvivalensklasser er disjunkte:

[EqClassesAreDisjoint $\overset{\mathrm{stmt}}{\rightarrow}$ ZFsub $\vdash \forall\underline{r}\colon \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{bs}\colon \lceil\bar{s}\rceil\#^0\lceil\underline{r}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{bs}\rceil \Vdash \lceil\bar{t}\rceil\#^0\lceil\underline{r}\rceil \Vdash \lceil\bar{t}\rceil\#^0\lceil\underline{bs}\rceil \Vdash \lceil\bar{u}\rceil\#^0\lceil\underline{r}\rceil \Vdash \lceil\bar{u}\rceil\#^0\lceil\underline{bs}\rceil \Vdash \underline{x} \in \underline{bs} \vdash \underline{y} \in \underline{bs} \vdash \dot\neg\dot\neg\forall_{\mathrm{obj}}\bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow \dot\neg\forall_{\mathrm{obj}}\bar{s}\colon \forall_{\mathrm{obj}}\bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \Rightarrow \dot\neg\forall_{\mathrm{obj}}\bar{s}\colon \forall_{\mathrm{obj}}\bar{t}\colon \forall_{\mathrm{obj}}\bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r} \vdash \dot\neg\{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\} \in \underline{r}\} = \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\} \in \underline{r}\} \vdash \{\mathrm{ph} \in \cup\{\{\{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\} \in \underline{r}\}, \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\} \in \underline{r}\}\}, \{\{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\} \in \underline{r}\}, \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\} \in \underline{r}\}\}\} \mid \dot\neg c_{\mathrm{Ph}} \in \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\} \in \underline{r}\} \Rightarrow \dot\neg c_{\mathrm{Ph}} \in \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\} \in \underline{r}\}\} = \emptyset$]

Beviset for EqClassesAreDisjoint forløber som følger:

1. Vi antager, at $\{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\} \in \underline{r}\}$ og $\{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\} \in \underline{r}\}$ ikke er lig med hinanden. Fra EqNecessary ved vi, at hvis der gjaldt $\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\} \in \underline{r}$, så ville $\{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\} \in \underline{r}\}$ og $\{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\} \in \underline{r}\}$ være lig med hinanden. Derfor må der gælde $\dot\neg\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\} \in \underline{r}$. (Linie 12–13).
2. Fra NoneEqNecessary ved vi, at $\dot\neg\{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\} \in \underline{r}$ medfører, at de to ækvivalensklasser er disjunkte; QED. (Linie 14–15).

Her er beviset:

[EqClassesAreDisjoint $\overset{\mathrm{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash \forall\underline{r}\colon \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{bs}\colon \lceil\bar{s}\rceil\#^0\lceil\underline{r}\rceil \Vdash \lceil\bar{s}\rceil\#^0\lceil\underline{bs}\rceil \Vdash \lceil\bar{t}\rceil\#^0\lceil\underline{r}\rceil \Vdash \lceil\bar{t}\rceil\#^0\lceil\underline{bs}\rceil \Vdash \lceil\bar{u}\rceil\#^0\lceil\underline{r}\rceil \Vdash \lceil\bar{u}\rceil\#^0\lceil\underline{bs}\rceil \Vdash \underline{x} \in \underline{bs} \vdash \underline{y} \in \underline{bs} \vdash \dot\neg\dot\neg\forall_{\mathrm{obj}}\bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow \dot\neg\forall_{\mathrm{obj}}\bar{s}\colon \forall_{\mathrm{obj}}\bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \Rightarrow \dot\neg\forall_{\mathrm{obj}}\bar{s}\colon \forall_{\mathrm{obj}}\bar{t}\colon \forall_{\mathrm{obj}}\bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r} \vdash \dot\neg\{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\} \in \underline{r}\} = \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\} \in \underline{r}\} \vdash$ EqNecessary $\mathrel{|\!\rhd} \lceil\bar{s}\rceil\#^0\lceil\underline{r}\rceil \mathrel{|\!\rhd} \lceil\bar{s}\rceil\#^0\lceil\underline{bs}\rceil \mathrel{|\!\rhd} \lceil\bar{t}\rceil\#^0\lceil\underline{r}\rceil \mathrel{|\!\rhd} \lceil\bar{t}\rceil\#^0\lceil\underline{bs}\rceil \mathrel{|\!\rhd} \lceil\bar{u}\rceil\#^0\lceil\underline{r}\rceil \mathrel{|\!\rhd} \lceil\bar{u}\rceil\#^0\lceil\underline{bs}\rceil \rhd \underline{x} \in \underline{bs} \rhd \underline{y} \in \underline{bs} \rhd \dot\neg\dot\neg\forall_{\mathrm{obj}}\bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow \dot\neg\forall_{\mathrm{obj}}\bar{s}\colon \forall_{\mathrm{obj}}\bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \Rightarrow \dot\neg\forall_{\mathrm{obj}}\bar{s}\colon \forall_{\mathrm{obj}}\bar{t}\colon \forall_{\mathrm{obj}}\bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r} \gg \{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\} \in \underline{r} \Rightarrow \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\} \in \underline{r}\} = \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\} \in \underline{r}\}; \mathrm{MT} \rhd \{\{\underline{x},\underline{x}\},\{\underline{x},\underline{y}\}\} \in \underline{r} \Rightarrow \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\} \in \underline{r}\} = \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{y}\}\} \in \underline{r}\} \rhd \dot\neg\{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\} \in \underline{r}\} = \{\mathrm{ph} \in$

$\underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{y}}\}\} \in \underline{\mathsf{r}}\} \gg \dot{\neg}\{\{\underline{\mathsf{x}},\underline{\mathsf{x}}\},\{\underline{\mathsf{x}},\underline{\mathsf{y}}\}\} \in \underline{\mathsf{r}}$; NoneEqNecessary $\unrhd \lceil\bar{\mathsf{s}}\rceil \#^0 \lceil\underline{\mathsf{r}}\rceil \unrhd \lceil\bar{\mathsf{s}}\rceil \#^0 \lceil\underline{\mathsf{bs}}\rceil \unrhd \lceil\bar{\mathsf{t}}\rceil \#^0 \lceil\underline{\mathsf{r}}\rceil \unrhd \lceil\bar{\mathsf{t}}\rceil \#^0 \lceil\underline{\mathsf{bs}}\rceil \unrhd \lceil\bar{\mathsf{u}}\rceil \#^0 \lceil\underline{\mathsf{r}}\rceil \unrhd \lceil\bar{\mathsf{u}}\rceil \#^0 \lceil\underline{\mathsf{bs}}\rceil \rhd \underline{\mathsf{x}} \in \underline{\mathsf{bs}} \rhd \underline{\mathsf{y}} \in \underline{\mathsf{bs}} \rhd \dot{\neg}\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{u}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{u}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \gg \dot{\neg}\{\{\underline{\mathsf{x}},\underline{\mathsf{x}}\},\{\underline{\mathsf{x}},\underline{\mathsf{y}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\mathrm{ph} \in \cup\{\{\{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{x}}\}\} \in \underline{\mathsf{r}}\},\{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{x}}\}\} \in \underline{\mathsf{r}}\}\},\{\{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{y}}\}\} \in \underline{\mathsf{r}}\},\{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{y}}\}\} \in \underline{\mathsf{r}}\}\}\} \mid \dot{\neg}\mathsf{c}_{\mathrm{Ph}} \in \{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{x}}\}\} \in \underline{\mathsf{r}}\} \Rightarrow \dot{\neg}\mathsf{c}_{\mathrm{Ph}} \in \{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{y}}\}\} \in \underline{\mathsf{r}}\}\} = \emptyset$; MP $\rhd \dot{\neg}\{\{\underline{\mathsf{x}},\underline{\mathsf{x}}\},\{\underline{\mathsf{x}},\underline{\mathsf{y}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\mathrm{ph} \in \cup\{\{\{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{x}}\}\} \in \underline{\mathsf{r}}\},\{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{x}}\}\} \in \underline{\mathsf{r}}\}\},\{\{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{y}}\}\} \in \underline{\mathsf{r}}\},\{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{y}}\}\} \in \underline{\mathsf{r}}\}\}\} \mid \dot{\neg}\mathsf{c}_{\mathrm{Ph}} \in \{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{x}}\}\} \in \underline{\mathsf{r}}\} \Rightarrow \dot{\neg}\mathsf{c}_{\mathrm{Ph}} \in \{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{y}}\}\} \in \underline{\mathsf{r}}\}\} = \emptyset \rhd \dot{\neg}\{\{\underline{\mathsf{x}},\underline{\mathsf{x}}\},\{\underline{\mathsf{x}},\underline{\mathsf{y}}\}\} \in \underline{\mathsf{r}} \gg \{\mathrm{ph} \in \cup\{\{\{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{x}}\}\} \in \underline{\mathsf{r}}\},\{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{x}}\}\} \in \underline{\mathsf{r}}\}\},\{\{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{y}}\}\} \in \underline{\mathsf{r}}\},\{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{y}}\}\} \in \underline{\mathsf{r}}\}\}\} \mid \dot{\neg}\mathsf{c}_{\mathrm{Ph}} \in \{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{x}}\}\} \in \underline{\mathsf{r}}\} \Rightarrow \dot{\neg}\mathsf{c}_{\mathrm{Ph}} \in \{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\underline{\mathsf{y}}\}\} \in \underline{\mathsf{r}}\}\} = \emptyset\rceil, \mathrm{p}_0, \mathsf{c})]$

11.2.5 Alle ækvivalensklasser er disjunkte

Ud fra EqClassesAreDisjoint kan vi nu vise, at alle medlemmer af et ækvivalenssystem er parvis disjunkte. Da beviset afhænger af SameIntersection, som igen afhænger af SameBinaryUnion, bruger vi objektvariable (jvf. afsnit 10.4):

[AllDisjoint $\stackrel{\mathrm{stmt}}{\rightarrow}$ ZFsub $\vdash \dot{\neg}\dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \overline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{s}}\}\} \in \bar{\mathsf{r}} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon \bar{\mathsf{s}} \in \overline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \overline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\} \in \bar{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{s}}\}\} \in \bar{\mathsf{r}} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{u}}\colon \bar{\mathsf{s}} \in \overline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \overline{\mathsf{bs}} \Rightarrow \bar{\mathsf{u}} \in \overline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\} \in \bar{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{u}}\}\} \in \bar{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{u}}\}\} \in \bar{\mathsf{r}} \vdash \bar{\mathsf{x}} \in \{\mathrm{ph} \in \mathrm{P}(\overline{\mathsf{bs}}) \mid \dot{\neg}\mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot{\neg}\{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \vdash \bar{\mathsf{y}} \in \{\mathrm{ph} \in \mathrm{P}(\overline{\mathsf{bs}}) \mid \dot{\neg}\mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot{\neg}\{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \vdash \dot{\neg}\bar{\mathsf{x}} = \bar{\mathsf{y}} \vdash \{\mathrm{ph} \in \cup\{\{\bar{\mathsf{x}},\bar{\mathsf{x}}\},\{\bar{\mathsf{y}},\bar{\mathsf{y}}\}\} \mid \dot{\neg}\mathsf{c}_{\mathrm{Ph}} \in \bar{\mathsf{x}} \Rightarrow \dot{\neg}\mathsf{c}_{\mathrm{Ph}} \in \bar{\mathsf{y}}\} = \emptyset$]

Beviset forløber som følger:

1. Lad $\bar{\mathsf{x}}$ og $\bar{\mathsf{y}}$ være to forskellige medlemmer af et ækvivalenssystem $\{\mathrm{ph} \in \mathrm{P}(\overline{\mathsf{bs}}) \mid \dot{\neg}\mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot{\neg}\{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\}$ (linie 2–4).
2. Mængden $\bar{\mathsf{x}}$ må være en ækvivalensklasse; vi kan altså skrive $\bar{\mathsf{x}} = \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\}$ for et a_{Ex}, som tilhører $\overline{\mathsf{bs}}$. (Linie 5–9).
3. Det samme gælder for $\bar{\mathsf{y}}$; her skriver vi $\bar{\mathsf{y}} = \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{b}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\}$. (Linie 10–14).
4. Da $\bar{\mathsf{x}}$ og $\bar{\mathsf{y}}$ er forskellige, må $\{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\}$ og $\{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{b}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\}$ også være forskellige. (Linie 15).

5. Heraf følger, at fællesmængden mellem
   $\{ph \in \overline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, a_{Ex}\}\} \in \bar{r}\}$ og
   $\{ph \in \overline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, b_{Ex}\}\} \in \bar{r}\}$ er tom. (Linie 16).

6. Af lighedslemmaet SameIntersection får vi
   $\{ph \in \cup\{\{\bar{x}, \bar{x}\}, \{\bar{y}, \bar{y}\}\} \mid \dot\neg c_{Ph} \in \bar{x} \Rightarrow \dot\neg c_{Ph} \in \bar{y}\} = \{ph \in \cup\{\{\{ph \in \overline{bs} \mid \{\{a_{Ph},$
   (Linie 17).

7. Da $\{ph \in \cup\{\{\{ph \in \overline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, a_{Ex}\}\} \in \bar{r}\}, \{ph \in \overline{bs} \mid$
   $\{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, a_{Ex}\}\} \in \bar{r}\}\}, \{\{ph \in \overline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, a_{Ex}\}\} \in$
   $\bar{r}\}, \{ph \in \overline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, a_{Ex}\}\} \in \bar{r}\}\}\} \mid \dot\neg c_{Ph} \in \{ph \in \overline{bs} \mid$
   $\{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, a_{Ex}\}\} \in \bar{r}\} \Rightarrow \dot\neg c_{Ph} \in \{ph \in \overline{bs} \mid$
   $\{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, a_{Ex}\}\} \in \bar{r}\}\}$ er tom (punkt 5), må
   $\{ph \in \cup\{\{\bar{x}, \bar{x}\}, \{\bar{y}, \bar{y}\}\} \mid \dot\neg c_{Ph} \in \bar{x} \Rightarrow \dot\neg c_{Ph} \in \bar{y}\}$ derfor også være tom,
   QED. (Linie 18).

Her er beviset:

[AllDisjoint $\overset{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash \dot\neg\dot\neg\forall_{obj}\bar{s}\colon \bar{s} \in \overline{bs} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{s}\}\} \in \bar{r} \Rightarrow$
$\dot\neg\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \bar{s} \in \overline{bs} \Rightarrow \bar{t} \in \overline{bs} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{t}\}\} \in \bar{r} \Rightarrow \{\{\bar{t}, \bar{t}\}, \{\bar{t}, \bar{s}\}\} \in \bar{r} \Rightarrow$
$\dot\neg\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \forall_{obj}\bar{u}\colon \bar{s} \in \overline{bs} \Rightarrow \bar{t} \in \overline{bs} \Rightarrow \bar{u} \in \overline{bs} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{t}\}\} \in \bar{r} \Rightarrow$
$\{\{\bar{t}, \bar{t}\}, \{\bar{t}, \bar{u}\}\} \in \bar{r} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{u}\}\} \in \bar{r} \vdash \bar{x} \in \{ph \in P(\overline{bs}) \mid \dot\neg t_{Ex} \in \overline{bs} \Rightarrow$
$\dot\neg\{ph \in \overline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, t_{Ex}\}\} \in \bar{r}\} = b_{Ph}\} \vdash \bar{y} \in \{ph \in P(\overline{bs}) \mid \dot\neg t_{Ex} \in$
$\overline{bs} \Rightarrow \dot\neg\{ph \in \overline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, t_{Ex}\}\} \in \bar{r}\} = b_{Ph}\} \vdash \dot\neg\bar{x} = \bar{y} \vdash$
Sep2Formula $\rhd \bar{x} \in \{ph \in P(\overline{bs}) \mid \dot\neg t_{Ex} \in \overline{bs} \Rightarrow \dot\neg\{ph \in \overline{bs} \mid$
$\{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, t_{Ex}\}\} \in \bar{r}\} = b_{Ph}\} \gg \dot\neg\bar{x} \in P(\overline{bs}) \Rightarrow \dot\neg\dot\neg a_{Ex} \in \overline{bs} \Rightarrow \dot\neg\{ph \in$
$\overline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, a_{Ex}\}\} \in \bar{r}\} = \bar{x}$; SecondConjunct $\rhd \dot\neg\bar{x} \in P(\overline{bs}) \Rightarrow$
$\dot\neg\dot\neg a_{Ex} \in \overline{bs} \Rightarrow \dot\neg\{ph \in \overline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, a_{Ex}\}\} \in \bar{r}\} = \bar{x} \gg \dot\neg a_{Ex} \in \overline{bs} \Rightarrow$
$\dot\neg\{ph \in \overline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, a_{Ex}\}\} \in \bar{r}\} = \bar{x}$; FirstConjunct $\rhd \dot\neg a_{Ex} \in \overline{bs} \Rightarrow$
$\dot\neg\{ph \in \overline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, a_{Ex}\}\} \in \bar{r}\} = \bar{x} \gg a_{Ex} \in$
$\overline{bs}$; SecondConjunct $\rhd \dot\neg a_{Ex} \in \overline{bs} \Rightarrow \dot\neg\{ph \in \overline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, a_{Ex}\}\} \in \bar{r}\} =$
$\bar{x} \gg \{ph \in \overline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, a_{Ex}\}\} \in \bar{r}\} = \bar{x}$; =Symmetry $\rhd \{ph \in \overline{bs} \mid$
$\{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, a_{Ex}\}\} \in \bar{r}\} = \bar{x} \gg \bar{x} = \{ph \in \overline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, a_{Ex}\}\} \in$
$\bar{r}\}$; Sep2Formula $\rhd \bar{y} \in \{ph \in P(\overline{bs}) \mid \dot\neg t_{Ex} \in \overline{bs} \Rightarrow \dot\neg\{ph \in \overline{bs} \mid$
$\{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, t_{Ex}\}\} \in \bar{r}\} = b_{Ph}\} \gg \dot\neg\bar{y} \in P(\overline{bs}) \Rightarrow \dot\neg\dot\neg b_{Ex} \in \overline{bs} \Rightarrow \dot\neg\{ph \in$
$\overline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, b_{Ex}\}\} \in \bar{r}\} = \bar{y}$; SecondConjunct $\rhd \dot\neg\bar{y} \in P(\overline{bs}) \Rightarrow$
$\dot\neg\dot\neg b_{Ex} \in \overline{bs} \Rightarrow \dot\neg\{ph \in \overline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, b_{Ex}\}\} \in \bar{r}\} = \bar{y} \gg \dot\neg b_{Ex} \in \overline{bs} \Rightarrow$
$\dot\neg\{ph \in \overline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, b_{Ex}\}\} \in \bar{r}\} = \bar{y}$; FirstConjunct $\rhd \dot\neg b_{Ex} \in \overline{bs} \Rightarrow$
$\dot\neg\{ph \in \overline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, b_{Ex}\}\} \in \bar{r}\} = \bar{y} \gg b_{Ex} \in$
$\overline{bs}$; SecondConjunct $\rhd \dot\neg b_{Ex} \in \overline{bs} \Rightarrow \dot\neg\{ph \in \overline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, b_{Ex}\}\} \in \bar{r}\} =$
$\bar{y} \gg \{ph \in \overline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, b_{Ex}\}\} \in \bar{r}\} = \bar{y}$; =Symmetry $\rhd \{ph \in \overline{bs} \mid$
$\{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, b_{Ex}\}\} \in \bar{r}\} = \bar{y} \gg \bar{y} = \{ph \in \overline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, b_{Ex}\}\} \in$
$\bar{r}\}$; TransferNotEq $\rhd \dot\neg\bar{x} = \bar{y} \rhd \bar{x} = \{ph \in \overline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, a_{Ex}\}\} \in \bar{r}\} \rhd \bar{y} =$
$\{ph \in \overline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, b_{Ex}\}\} \in \bar{r}\} \gg \dot\neg\{ph \in \overline{bs} \mid$
$\{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, a_{Ex}\}\} \in \bar{r}\} = \{ph \in \overline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, b_{Ex}\}\} \in$
$\bar{r}\}$; EqClassesAreDisjoint $\rhd a_{Ex} \in \overline{bs} \rhd b_{Ex} \in \overline{bs} \rhd \dot\neg\dot\neg\forall_{obj}\bar{s}\colon \bar{s} \in \overline{bs} \Rightarrow$
$\{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{s}\}\} \in \bar{r} \Rightarrow \dot\neg\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \bar{s} \in \overline{bs} \Rightarrow \bar{t} \in \overline{bs} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{t}\}\} \in \bar{r} \Rightarrow$
$\{\{\bar{t}, \bar{t}\}, \{\bar{t}, \bar{s}\}\} \in \bar{r} \Rightarrow \dot\neg\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \forall_{obj}\bar{u}\colon \bar{s} \in \overline{bs} \Rightarrow \bar{t} \in \overline{bs} \Rightarrow \bar{u} \in \overline{bs} \Rightarrow$

$\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\bar{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{u}}\}\}\in\bar{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{u}}\}\}\in\bar{\mathsf{r}}\rhd\dot{\neg}\{\mathrm{ph}\in\overline{\mathsf{bs}}\mid$
$\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ex}}\}\}\in\bar{\mathsf{r}}\}=\{\mathrm{ph}\in\overline{\mathsf{bs}}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{b}_{\mathrm{Ex}}\}\}\in\bar{\mathsf{r}}\}\gg$
$\{\mathrm{ph}\in\cup\{\{\{\mathrm{ph}\in\overline{\mathsf{bs}}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ex}}\}\}\in\bar{\mathsf{r}}\},\{\mathrm{ph}\in\overline{\mathsf{bs}}\mid$
$\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ex}}\}\}\in\bar{\mathsf{r}}\}\},\{\{\mathrm{ph}\in\overline{\mathsf{bs}}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{b}_{\mathrm{Ex}}\}\}\in\bar{\mathsf{r}}\},\{\mathrm{ph}\in$
$\overline{\mathsf{bs}}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{b}_{\mathrm{Ex}}\}\}\in\bar{\mathsf{r}}\}\}\}\mid\dot{\neg}\,\mathsf{c}_{\mathrm{Ph}}\in\{\mathrm{ph}\in\overline{\mathsf{bs}}\mid$
$\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ex}}\}\}\in\bar{\mathsf{r}}\}\Rightarrow\dot{\neg}\,\mathsf{c}_{\mathrm{Ph}}\in\{\mathrm{ph}\in\overline{\mathsf{bs}}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{b}_{\mathrm{Ex}}\}\}\in$
$\bar{\mathsf{r}}\}\}=\emptyset;\mathrm{SameIntersection}\rhd\bar{\mathsf{x}}=\{\mathrm{ph}\in\overline{\mathsf{bs}}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ex}}\}\}\in\bar{\mathsf{r}}\}\rhd\bar{\mathsf{y}}=$
$\{\mathrm{ph}\in\overline{\mathsf{bs}}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{b}_{\mathrm{Ex}}\}\}\in\bar{\mathsf{r}}\}\gg\{\mathrm{ph}\in\cup\{\{\bar{\mathsf{x}},\bar{\mathsf{x}}\},\{\bar{\mathsf{y}},\bar{\mathsf{y}}\}\}\mid\dot{\neg}\,\mathsf{c}_{\mathrm{Ph}}\in\bar{\mathsf{x}}\Rightarrow$
$\dot{\neg}\,\mathsf{c}_{\mathrm{Ph}}\in\bar{\mathsf{y}}\}=\{\mathrm{ph}\in\cup\{\{\{\mathrm{ph}\in\overline{\mathsf{bs}}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ex}}\}\}\in\bar{\mathsf{r}}\},\{\mathrm{ph}\in\overline{\mathsf{bs}}\mid$
$\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ex}}\}\}\in\bar{\mathsf{r}}\}\},\{\{\mathrm{ph}\in\overline{\mathsf{bs}}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{b}_{\mathrm{Ex}}\}\}\in\bar{\mathsf{r}}\},\{\mathrm{ph}\in$
$\overline{\mathsf{bs}}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{b}_{\mathrm{Ex}}\}\}\in\bar{\mathsf{r}}\}\}\}\mid\dot{\neg}\,\mathsf{c}_{\mathrm{Ph}}\in\{\mathrm{ph}\in\overline{\mathsf{bs}}\mid$
$\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ex}}\}\}\in\bar{\mathsf{r}}\}\Rightarrow\dot{\neg}\,\mathsf{c}_{\mathrm{Ph}}\in\{\mathrm{ph}\in\overline{\mathsf{bs}}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{b}_{\mathrm{Ex}}\}\}\in$
$\bar{\mathsf{r}}\}\};=\mathrm{Transitivity}\rhd\{\mathrm{ph}\in\cup\{\{\bar{\mathsf{x}},\bar{\mathsf{x}}\},\{\bar{\mathsf{y}},\bar{\mathsf{y}}\}\}\mid\dot{\neg}\,\mathsf{c}_{\mathrm{Ph}}\in\bar{\mathsf{x}}\Rightarrow\dot{\neg}\,\mathsf{c}_{\mathrm{Ph}}\in\bar{\mathsf{y}}\}=\{\mathrm{ph}\in$
$\cup\{\{\{\mathrm{ph}\in\overline{\mathsf{bs}}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ex}}\}\}\in\bar{\mathsf{r}}\},\{\mathrm{ph}\in\overline{\mathsf{bs}}\mid$
$\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ex}}\}\}\in\bar{\mathsf{r}}\}\},\{\{\mathrm{ph}\in\overline{\mathsf{bs}}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{b}_{\mathrm{Ex}}\}\}\in\bar{\mathsf{r}}\},\{\mathrm{ph}\in$
$\overline{\mathsf{bs}}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{b}_{\mathrm{Ex}}\}\}\in\bar{\mathsf{r}}\}\}\}\mid\dot{\neg}\,\mathsf{c}_{\mathrm{Ph}}\in\{\mathrm{ph}\in\overline{\mathsf{bs}}\mid$
$\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ex}}\}\}\in\bar{\mathsf{r}}\}\Rightarrow\dot{\neg}\,\mathsf{c}_{\mathrm{Ph}}\in\{\mathrm{ph}\in\overline{\mathsf{bs}}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{b}_{\mathrm{Ex}}\}\}\in$
$\bar{\mathsf{r}}\}\}\rhd\{\mathrm{ph}\in\cup\{\{\{\mathrm{ph}\in\overline{\mathsf{bs}}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ex}}\}\}\in\bar{\mathsf{r}}\},\{\mathrm{ph}\in\overline{\mathsf{bs}}\mid$
$\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ex}}\}\}\in\bar{\mathsf{r}}\}\},\{\{\mathrm{ph}\in\overline{\mathsf{bs}}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{b}_{\mathrm{Ex}}\}\}\in\bar{\mathsf{r}}\},\{\mathrm{ph}\in$
$\overline{\mathsf{bs}}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{b}_{\mathrm{Ex}}\}\}\in\bar{\mathsf{r}}\}\}\}\mid\dot{\neg}\,\mathsf{c}_{\mathrm{Ph}}\in\{\mathrm{ph}\in\overline{\mathsf{bs}}\mid$
$\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ex}}\}\}\in\bar{\mathsf{r}}\}\Rightarrow\dot{\neg}\,\mathsf{c}_{\mathrm{Ph}}\in\{\mathrm{ph}\in\overline{\mathsf{bs}}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{b}_{\mathrm{Ex}}\}\}\in$
$\bar{\mathsf{r}}\}\}=\emptyset\gg\{\mathrm{ph}\in\cup\{\{\bar{\mathsf{x}},\bar{\mathsf{x}}\},\{\bar{\mathsf{y}},\bar{\mathsf{y}}\}\}\mid\dot{\neg}\,\mathsf{c}_{\mathrm{Ph}}\in\bar{\mathsf{x}}\Rightarrow\dot{\neg}\,\mathsf{c}_{\mathrm{Ph}}\in\bar{\mathsf{y}}\}=\emptyset],\mathrm{p}_0,\mathsf{c})]$

Bemærk at vi i dette bevis introducerer både a_{Ex} og b_{Ex} ud fra definitionen af $[\{\mathrm{ph}\in\mathrm{P}(\overline{\mathsf{bs}})\mid\dot{\neg}\,\mathsf{t}_{\mathrm{Ex}}\in\overline{\mathsf{bs}}\Rightarrow\dot{\neg}\{\mathrm{ph}\in\overline{\mathsf{bs}}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{t}_{\mathrm{Ex}}\}\}\in\bar{\mathsf{r}}\}=\mathsf{b}_{\mathrm{Ph}}\}]$, selvom denne definition bruger eksistensvariablen t_{Ex}. Vi nyder her gavn af, at separationsaksiomet tillader introduktion af ubrugte eksistens-variable (som nævnt i afsnit 4.2.1).

11.2.6 Implikation i stedet for inferens

Det sidste skridt i dette afsnit er at skifte inferenserne fra AllDisjoint ud med implikationer. (Dette er påkrævet for, at resultatet fra AllDisjoint kan bruges i beviset for hovedresultatet). Vi plejer jo at bruge deduktionsreglen til dén slags konverteringer; men der et problem. Vi kan ikke konkludere

$$[\dot{\neg}\,\dot{\neg}\,\forall_{\mathrm{obj}}\bar{\mathsf{s}}{:}\,\bar{\mathsf{s}}\in\overline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{s}}\}\}\in\bar{\mathsf{r}}\Rightarrow\dot{\neg}\,\forall_{\mathrm{obj}}\bar{\mathsf{s}}{:}\,\forall_{\mathrm{obj}}\bar{\mathsf{t}}{:}\,\bar{\mathsf{s}}\in\overline{\mathsf{bs}}\Rightarrow\bar{\mathsf{t}}\in\overline{\mathsf{bs}}\Rightarrow \{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\bar{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{s}}\}\}\in\bar{\mathsf{r}}\Rightarrow\dot{\neg}\,\forall_{\mathrm{obj}}\bar{\mathsf{s}}{:}\,\forall_{\mathrm{obj}}\bar{\mathsf{t}}{:}\,\forall_{\mathrm{obj}}\bar{\mathsf{u}}{:}\,\bar{\mathsf{s}}\in\overline{\mathsf{bs}}\Rightarrow\bar{\mathsf{t}}\in \overline{\mathsf{bs}}\Rightarrow\bar{\mathsf{u}}\in\overline{\mathsf{bs}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{t}}\}\}\in\bar{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{t}},\bar{\mathsf{t}}\},\{\bar{\mathsf{t}},\bar{\mathsf{u}}\}\}\in\bar{\mathsf{r}}\Rightarrow\{\{\bar{\mathsf{s}},\bar{\mathsf{s}}\},\{\bar{\mathsf{s}},\bar{\mathsf{u}}\}\}\in \bar{\mathsf{r}}\Rightarrow\bar{\mathsf{x}}\in\{\mathrm{ph}\in\mathrm{P}(\overline{\mathsf{bs}})\mid\dot{\neg}\,\mathsf{t}_{\mathrm{Ex}}\in\overline{\mathsf{bs}}\Rightarrow\dot{\neg}\{\mathrm{ph}\in\overline{\mathsf{bs}}\mid \{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{t}_{\mathrm{Ex}}\}\}\in\bar{\mathsf{r}}\}=\mathsf{b}_{\mathrm{Ph}}\}\Rightarrow\bar{\mathsf{y}}\in\{\mathrm{ph}\in\mathrm{P}(\overline{\mathsf{bs}})\mid\dot{\neg}\,\mathsf{t}_{\mathrm{Ex}}\in\overline{\mathsf{bs}}\Rightarrow \dot{\neg}\{\mathrm{ph}\in\overline{\mathsf{bs}}\mid\{\{\mathsf{a}_{\mathrm{Ph}},\mathsf{a}_{\mathrm{Ph}}\},\{\mathsf{a}_{\mathrm{Ph}},\mathsf{t}_{\mathrm{Ex}}\}\}\in\bar{\mathsf{r}}\}=\mathsf{b}_{\mathrm{Ph}}\}\Rightarrow\dot{\neg}\,\bar{\mathsf{x}}=\bar{\mathsf{y}}\Rightarrow\{\mathrm{ph}\in \cup\{\{\bar{\mathsf{x}},\bar{\mathsf{x}}\},\{\bar{\mathsf{y}},\bar{\mathsf{y}}\}\}\mid\dot{\neg}\,\mathsf{c}_{\mathrm{Ph}}\in\bar{\mathsf{x}}\Rightarrow\dot{\neg}\,\mathsf{c}_{\mathrm{Ph}}\in\bar{\mathsf{y}}\}=\emptyset]$$

ud fra Ded; vi kan kun konkludere

$$[\forall_{obj}\bar{r}\colon\forall_{obj}\overline{bs}\colon\dot\neg\,\dot\neg\,\forall_{obj}\bar{s}\colon\bar{s}\in\overline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\}\in\bar{r}\Rightarrow\dot\neg\,\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon\bar{s}\in\overline{bs}\Rightarrow\bar{t}\in\overline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\bar{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\bar{r}\Rightarrow\dot\neg\,\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon\forall_{obj}\bar{u}\colon\bar{s}\in\overline{bs}\Rightarrow\bar{t}\in\overline{bs}\Rightarrow\bar{u}\in\overline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\bar{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\bar{r}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\bar{r}\Rightarrow\forall_{obj}\bar{r}\colon\forall_{obj}\bar{x}\colon\forall_{obj}\overline{bs}\colon\bar{x}\in\{ph\in P(\overline{bs})\mid\dot\neg\,t_{Ex}\in\overline{bs}\Rightarrow\dot\neg\,\{ph\in\overline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},t_{Ex}\}\}\in\bar{r}\}=b_{Ph}\}\Rightarrow\forall_{obj}\bar{r}\colon\forall_{obj}\bar{y}\colon\forall_{obj}\overline{bs}\colon\bar{y}\in\{ph\in P(\overline{bs})\mid\dot\neg\,t_{Ex}\in\overline{bs}\Rightarrow\dot\neg\,\{ph\in\overline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},t_{Ex}\}\}\in\bar{r}\}=b_{Ph}\}\Rightarrow\forall_{obj}\bar{x}\colon\forall_{obj}\bar{y}\colon\dot\neg\,\bar{x}=\bar{y}\Rightarrow\{ph\in\cup\{\{\bar{x},\bar{x}\},\{\bar{y},\bar{y}\}\}\mid\dot\neg\,c_{Ph}\in\bar{x}\Rightarrow\dot\neg\,c_{Ph}\in\bar{y}\}=\emptyset].$$

Deduktionsreglen tillader altså ikke frie objektvariable i antecedenterne. Dette er også et rimeligt krav; ellers ville vi f.eks. kunne slutte den falske implikation $[\bar{s}=\emptyset\Rightarrow\forall_{obj}\bar{s}\colon\bar{s}=\emptyset]$ ud fra lemmaet $[\bar{s}=\emptyset\vdash\forall_{obj}\bar{s}\colon\bar{s}=\emptyset]$. Problemet er, at beviset for hovedresultatet kræver, at implikationen er renset for objektkvantorer. Så vi er altså havnet i en blindgyde.
Vi løser problemet ved at indføre følgende "snyde-aksiom":

$$[\text{CheatAllDisjoint}\overset{\text{stmt}}{\rightarrow}\text{ZFsub}\vdash\forall\underline{r}\colon\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{bs}\colon\dot\neg\,\dot\neg\,\forall_{obj}\bar{s}\colon\bar{s}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\}\in\underline{r}\Rightarrow\dot\neg\,\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\underline{r}\Rightarrow\dot\neg\,\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon\forall_{obj}\bar{u}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\bar{u}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\underline{r}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\underline{r}\vdash\underline{x}\in\{ph\in P(\underline{bs})\mid\dot\neg\,t_{Ex}\in\underline{bs}\Rightarrow\dot\neg\,\{ph\in\underline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},t_{Ex}\}\}\in\underline{r}\}=b_{Ph}\}\vdash\underline{y}\in\{ph\in P(\underline{bs})\mid\dot\neg\,t_{Ex}\in\underline{bs}\Rightarrow\dot\neg\,\{ph\in\underline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},t_{Ex}\}\}\in\underline{r}\}=b_{Ph}\}\vdash\dot\neg\,\underline{x}=\underline{y}\vdash\{ph\in\cup\{\{\underline{x},\underline{x}\},\{\underline{y},\underline{y}\}\}\mid\dot\neg\,c_{Ph}\in\underline{x}\Rightarrow\dot\neg\,c_{Ph}\in\underline{y}\}=\emptyset][\text{CheatAllDisjoint}\overset{\text{proof}}{\rightarrow}\text{Rule tactic}]$$

I CheatAllDisjoint formulerer vi AllDisjoint ved hjælp af metavariable. Vi kan opfatte CheatAllDisjoint som dét resultat, vi var nået frem til, hvis vi havde kæmpet videre med de mange sidebetingelser (bortset fra at der ikke er nogen sidebetingelser i formuleringen af CheatAllDisjoint). Ud fra denne formulering af AllDisjoint er det nemt at aflede en implikation:

$$[\text{AllDisjointImply}\overset{\text{stmt}}{\rightarrow}\text{ZFsub}\vdash\forall\underline{r}\colon\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{bs}\colon\lceil\bar{s}\rceil\#^0\lceil\underline{r}\rceil\Vdash\lceil\bar{s}\rceil\#^0\lceil\underline{bs}\rceil\Vdash\lceil\bar{t}\rceil\#^0\lceil\underline{r}\rceil\Vdash\lceil\bar{t}\rceil\#^0\lceil\underline{bs}\rceil\Vdash\lceil\bar{u}\rceil\#^0\lceil\underline{r}\rceil\Vdash\lceil\bar{u}\rceil\#^0\lceil\underline{bs}\rceil\Vdash\dot\neg\,\dot\neg\,\forall_{obj}\bar{s}\colon\bar{s}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\}\in\underline{r}\Rightarrow\dot\neg\,\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\underline{r}\Rightarrow\dot\neg\,\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon\forall_{obj}\bar{u}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\bar{u}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\underline{r}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\underline{r}\Rightarrow\underline{x}\in\{ph\in P(\underline{bs})\mid\dot\neg\,t_{Ex}\in\underline{bs}\Rightarrow\dot\neg\,\{ph\in\underline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},t_{Ex}\}\}\in\underline{r}\}=b_{Ph}\}\Rightarrow\underline{y}\in\{ph\in P(\underline{bs})\mid\dot\neg\,t_{Ex}\in\underline{bs}\Rightarrow\dot\neg\,\{ph\in\underline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},t_{Ex}\}\}\in\underline{r}\}=b_{Ph}\}\Rightarrow\dot\neg\,\underline{x}=\underline{y}\Rightarrow\{ph\in\cup\{\{\underline{x},\underline{x}\},\{\underline{y},\underline{y}\}\}\mid\dot\neg\,c_{Ph}\in\underline{x}\Rightarrow\dot\neg\,c_{Ph}\in\underline{y}\}=\emptyset]$$

$$[\text{AllDisjointImply}\overset{\text{proof}}{\rightarrow}\lambda c.\lambda x.\mathcal{P}(\lceil\text{ZFsub}\vdash\forall\underline{r}\colon\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{bs}\colon\dot\neg\,\dot\neg\,\forall_{obj}\bar{s}\colon\bar{s}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\}\in\underline{r}\Rightarrow\dot\neg\,\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\underline{r}\Rightarrow\dot\neg\,\forall_{obj}\bar{s}\colon\forall_{obj}\bar{t}\colon\forall_{obj}\bar{u}\colon\bar{s}\in\underline{bs}\Rightarrow\bar{t}\in\underline{bs}\Rightarrow\bar{u}\in\underline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\underline{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\underline{r}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\underline{r}\vdash\underline{x}\in\{ph\in P(\underline{bs})\mid\dot\neg\,t_{Ex}\in\underline{bs}\Rightarrow\dot\neg\,\{ph\in\underline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},t_{Ex}\}\}\in\underline{r}\}=b_{Ph}\}\vdash\underline{y}\in\{ph\in P(\underline{bs})\mid\dot\neg\,t_{Ex}\in\underline{bs}\Rightarrow\dot\neg\,\{ph\in\underline{bs}\mid\{\{a_{Ph},a_{Ph}\},\{a_{Ph},t_{Ex}\}\}\in\underline{r}\}=b_{Ph}\}\vdash\dot\neg\,\underline{x}=$$

$$\underline{y} \vdash \mathrm{CheatAllDisjoint} \rhd \dot{\neg}\,\dot{\neg}\,\forall_{\mathrm{obj}}\bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}}\bar{s}\colon \forall_{\mathrm{obj}}\bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}}\bar{s}\colon \forall_{\mathrm{obj}}\bar{t}\colon \forall_{\mathrm{obj}}\bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r} \rhd \underline{x} \in \{\mathrm{ph} \in \mathrm{P}(\underline{bs}) \mid \dot{\neg}\, t_{\mathrm{Ex}} \in \underline{bs} \Rightarrow \dot{\neg}\{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},t_{\mathrm{Ex}}\}\} \in \underline{r}\} = b_{\mathrm{Ph}}\} \rhd \underline{y} \in \{\mathrm{ph} \in \mathrm{P}(\underline{bs}) \mid \dot{\neg}\, t_{\mathrm{Ex}} \in \underline{bs} \Rightarrow \dot{\neg}\{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},t_{\mathrm{Ex}}\}\} \in \underline{r}\} = b_{\mathrm{Ph}}\} \rhd \dot{\neg}\,\underline{x} = \underline{y} \gg \{\mathrm{ph} \in \cup\{\{\underline{x},\underline{x}\},\{\underline{y},\underline{y}\}\} \mid \dot{\neg}\, c_{\mathrm{Ph}} \in \underline{x} \Rightarrow \dot{\neg}\, c_{\mathrm{Ph}} \in \underline{y}\} = \emptyset; \forall\underline{r}\colon\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{bs}\colon \mathrm{Ded} \rhd \forall\underline{r}\colon\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{bs}\colon \dot{\neg}\,\dot{\neg}\,\forall_{\mathrm{obj}}\bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}}\bar{s}\colon \forall_{\mathrm{obj}}\bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}}\bar{s}\colon \forall_{\mathrm{obj}}\bar{t}\colon \forall_{\mathrm{obj}}\bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r} \vdash \underline{x} \in \{\mathrm{ph} \in \mathrm{P}(\underline{bs}) \mid \dot{\neg}\, t_{\mathrm{Ex}} \in \underline{bs} \Rightarrow \dot{\neg}\{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},t_{\mathrm{Ex}}\}\} \in \underline{r}\} = b_{\mathrm{Ph}}\} \vdash \underline{y} \in \{\mathrm{ph} \in \mathrm{P}(\underline{bs}) \mid \dot{\neg}\, t_{\mathrm{Ex}} \in \underline{bs} \Rightarrow \dot{\neg}\{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},t_{\mathrm{Ex}}\}\} \in \underline{r}\} = b_{\mathrm{Ph}}\} \vdash \dot{\neg}\,\underline{x} = \underline{y} \vdash \{\mathrm{ph} \in \cup\{\{\underline{x},\underline{x}\},\{\underline{y},\underline{y}\}\} \mid \dot{\neg}\, c_{\mathrm{Ph}} \in \underline{x} \Rightarrow \dot{\neg}\, c_{\mathrm{Ph}} \in \underline{y}\} = \emptyset \gg \lceil\bar{s}\rceil \#^0 \lceil\underline{r}\rceil \Vdash \lceil\bar{s}\rceil \#^0 \lceil\underline{bs}\rceil \Vdash \lceil\bar{t}\rceil \#^0 \lceil\underline{r}\rceil \Vdash \lceil\bar{t}\rceil \#^0 \lceil\underline{bs}\rceil \Vdash \lceil\bar{u}\rceil \#^0 \lceil\underline{r}\rceil \Vdash \lceil\bar{u}\rceil \#^0 \lceil\underline{bs}\rceil \Vdash \dot{\neg}\,\dot{\neg}\,\forall_{\mathrm{obj}}\bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}}\bar{s}\colon \forall_{\mathrm{obj}}\bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}}\bar{s}\colon \forall_{\mathrm{obj}}\bar{t}\colon \forall_{\mathrm{obj}}\bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r} \Rightarrow \underline{x} \in \{\mathrm{ph} \in \mathrm{P}(\underline{bs}) \mid \dot{\neg}\, t_{\mathrm{Ex}} \in \underline{bs} \Rightarrow \dot{\neg}\{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},t_{\mathrm{Ex}}\}\} \in \underline{r}\} = b_{\mathrm{Ph}}\} \Rightarrow \underline{y} \in \{\mathrm{ph} \in \mathrm{P}(\underline{bs}) \mid \dot{\neg}\, t_{\mathrm{Ex}} \in \underline{bs} \Rightarrow \dot{\neg}\{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},t_{\mathrm{Ex}}\}\} \in \underline{r}\} = b_{\mathrm{Ph}}\} \Rightarrow \dot{\neg}\,\underline{x} = \underline{y} \Rightarrow \{\mathrm{ph} \in \cup\{\{\underline{x},\underline{x}\},\{\underline{y},\underline{y}\}\} \mid \dot{\neg}\, c_{\mathrm{Ph}} \in \underline{x} \Rightarrow \dot{\neg}\, c_{\mathrm{Ph}} \in \underline{y}\} = \emptyset\rceil, \mathrm{p}_0, c)]$$

Vores brug af metavariable betyder, at sidebetingelserne straks vender tilbage — dog i et tåleligt omfang. Det er lemmaet AllDisjointImply, som vi kommer til at bruge i beviset for EqSysIsPartition.

11.3 Ækvivalensklassernes foreningsmængde

I dette underafsnit viser vi først, at $\underline{bs}$ er en delmængde af ækvivalensklassernes foreningsmængde $\cup\{\mathrm{ph} \in \mathrm{P}(\underline{bs}) \mid \dot{\neg}\, t_{\mathrm{Ex}} \in \underline{bs} \Rightarrow \dot{\neg}\{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},t_{\mathrm{Ex}}\}\} \in \underline{r}\} = b_{\mathrm{Ph}}\}$ (under-underafsnit 11.3.1), og vice versa (under-underafsnit 11.3.2). Herudfra kan vi let vise, at de to mængder er lig hinanden (under-underafsnit 11.3.3).

11.3.1 Den ene halvdel

Lemmaet EqClassIsSubset udsiger, at en ækvivalensklasse $\{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\} \in \underline{r}\}$ defineret på en mængde $\underline{bs}$ er en delmængde af $\underline{bs}$:

$$[\mathrm{EqClassIsSubset} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall\underline{r}\colon\forall\underline{s}\colon\forall\underline{x}\colon\forall\underline{bs}\colon \underline{s} \in \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\} \in \underline{r}\} \Rightarrow \underline{s} \in \underline{bs}]$$

Lemmaet følger direkte af definitionen af $[\{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\} \in \underline{r}\}]$ fra afsnit 5.9:

$$[\mathrm{EqClassIsSubset} \stackrel{\mathrm{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall\underline{r}\colon\forall\underline{s}\colon\forall\underline{x}\colon\forall\underline{bs}\colon \underline{s} \in \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},\underline{x}\}\} \in \underline{r}\} \vdash \mathrm{Sep2Formula} \rhd \underline{s} \in \{\mathrm{ph} \in \underline{bs} \mid$$

$\{\{a_{\mathrm{Ph}}, a_{\mathrm{Ph}}\}, \{a_{\mathrm{Ph}}, \underline{x}\}\} \in \underline{r}\} \gg \dot{\neg}\, \underline{s} \in \underline{bs} \Rightarrow \dot{\neg}\, \{\{\underline{s}, \underline{s}\}, \{\underline{s}, \underline{x}\}\} \in \underline{r}; \mathrm{FirstConjunct} \rhd \dot{\neg}\, \underline{s} \in \underline{bs} \Rightarrow \dot{\neg}\, \{\{\underline{s}, \underline{s}\}, \{\underline{s}, \underline{x}\}\} \in \underline{r} \gg \underline{s} \in \underline{bs}; \forall \underline{r}\colon \forall \underline{s}\colon \forall \underline{x}\colon \forall \underline{bs}\colon \mathrm{Ded} \rhd \forall \underline{r}\colon \forall \underline{s}\colon \forall \underline{x}\colon \forall \underline{bs}\colon \underline{s} \in \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}}, a_{\mathrm{Ph}}\}, \{a_{\mathrm{Ph}}, \underline{x}\}\} \in \underline{r}\} \vdash \underline{s} \in \underline{bs} \gg \underline{s} \in \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}}, a_{\mathrm{Ph}}\}, \{a_{\mathrm{Ph}}, \underline{x}\}\} \in \underline{r}\} \Rightarrow \underline{s} \in \underline{bs}\rceil, \mathrm{p}_0, c)]$

Kommet så vidt kan vi bevise, at $\underline{bs}$ er en delmængde af $\cup\{\mathrm{ph} \in \mathrm{P}(\underline{bs}) \mid \dot{\neg}\, t_{\mathrm{Ex}} \in \underline{bs} \Rightarrow \dot{\neg}\, \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}}, a_{\mathrm{Ph}}\}, \{a_{\mathrm{Ph}}, t_{\mathrm{Ex}}\}\} \in \underline{r}\} = b_{\mathrm{Ph}}\}$:

$[\mathrm{BSsubset} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{r}\colon \forall \underline{s}\colon \forall \underline{bs}\colon \lceil \bar{s} \rceil \#^0 \lceil \underline{r} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{bs} \rceil \Vdash \lceil \bar{t} \rceil \#^0 \lceil \underline{r} \rceil \Vdash \lceil \bar{t} \rceil \#^0 \lceil \underline{bs} \rceil \Vdash \lceil \bar{u} \rceil \#^0 \lceil \underline{r} \rceil \Vdash \lceil \bar{u} \rceil \#^0 \lceil \underline{bs} \rceil \Vdash \dot{\neg}\, \dot{\neg}\, \forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\, \forall_{\mathrm{obj}} \bar{s}\colon \forall_{\mathrm{obj}} \bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t}, \bar{t}\}, \{\bar{t}, \bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\, \forall_{\mathrm{obj}} \bar{s}\colon \forall_{\mathrm{obj}} \bar{t}\colon \forall_{\mathrm{obj}} \bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t}, \bar{t}\}, \{\bar{t}, \bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{u}\}\} \in \underline{r} \Rightarrow \underline{s} \in \underline{bs} \Rightarrow \underline{s} \in \cup\{\mathrm{ph} \in \mathrm{P}(\underline{bs}) \mid \dot{\neg}\, t_{\mathrm{Ex}} \in \underline{bs} \Rightarrow \dot{\neg}\, \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}}, a_{\mathrm{Ph}}\}, \{a_{\mathrm{Ph}}, t_{\mathrm{Ex}}\}\} \in \underline{r}\} = b_{\mathrm{Ph}}\}]$

Beviset for BSsubset falder i tre dele:

1. Lad $\underline{s}$ være et vilkårligt medlem af $\underline{bs}$. Ud fra AutoMember har vi, at $\underline{s}$ tilhører sin egen ækvivalensklasse $\{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}}, a_{\mathrm{Ph}}\}, \{a_{\mathrm{Ph}}, \underline{s}\}\} \in \underline{r}\}$ (linie 6–7).
2. Videre har vi, at denne ækvivalensklasse tilhører ækvivalenssystemet $\{\mathrm{ph} \in \mathrm{P}(\underline{bs}) \mid \dot{\neg}\, t_{\mathrm{Ex}} \in \underline{bs} \Rightarrow \dot{\neg}\, \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}}, a_{\mathrm{Ph}}\}, \{a_{\mathrm{Ph}}, t_{\mathrm{Ex}}\}\} \in \underline{r}\} = b_{\mathrm{Ph}}\}$ (linie 8–13).
3. Ud fra punkt 1 og 2 og definitionen af "foreningsmængde" får vi, at $\underline{s}$ tilhører ækvivalenssystemets foreningsmængde $\cup\{\mathrm{ph} \in \mathrm{P}(\underline{bs}) \mid \dot{\neg}\, t_{\mathrm{Ex}} \in \underline{bs} \Rightarrow \dot{\neg}\, \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}}, a_{\mathrm{Ph}}\}, \{a_{\mathrm{Ph}}, t_{\mathrm{Ex}}\}\} \in \underline{r}\} = b_{\mathrm{Ph}}\}$, QED (linie 14–16).

Her er selve beviset:

$[\mathrm{BSsubset} \stackrel{\mathrm{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{r}\colon \forall \underline{s}\colon \forall \underline{bs}\colon \lceil \bar{s} \rceil \#^0 \lceil \underline{r} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{bs} \rceil \Vdash \dot{\neg}\, \dot{\neg}\, \forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\, \forall_{\mathrm{obj}} \bar{s}\colon \forall_{\mathrm{obj}} \bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t}, \bar{t}\}, \{\bar{t}, \bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\, \forall_{\mathrm{obj}} \bar{s}\colon \forall_{\mathrm{obj}} \bar{t}\colon \forall_{\mathrm{obj}} \bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t}, \bar{t}\}, \{\bar{t}, \bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{u}\}\} \in \underline{r} \vdash \underline{s} \in \underline{bs} \vdash \mathrm{AutoMember} \unrhd \lceil \bar{s} \rceil \#^0 \lceil \underline{r} \rceil \unrhd \lceil \bar{s} \rceil \#^0 \lceil \underline{bs} \rceil \rhd \dot{\neg}\, \dot{\neg}\, \forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\, \forall_{\mathrm{obj}} \bar{s}\colon \forall_{\mathrm{obj}} \bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t}, \bar{t}\}, \{\bar{t}, \bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\, \forall_{\mathrm{obj}} \bar{s}\colon \forall_{\mathrm{obj}} \bar{t}\colon \forall_{\mathrm{obj}} \bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t}, \bar{t}\}, \{\bar{t}, \bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{u}\}\} \in \underline{r} \rhd \underline{s} \in \underline{bs} \gg \underline{s} \in \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}}, a_{\mathrm{Ph}}\}, \{a_{\mathrm{Ph}}, \underline{s}\}\} \in \underline{r}\}; \mathrm{EqClassIsSubset} \gg \bar{s} \in \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}}, a_{\mathrm{Ph}}\}, \{a_{\mathrm{Ph}}, \underline{s}\}\} \in \underline{r}\} \Rightarrow \bar{s} \in \underline{bs}; \mathrm{SubsetInPower} \rhd \bar{s} \in \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}}, a_{\mathrm{Ph}}\}, \{a_{\mathrm{Ph}}, \underline{s}\}\} \in \underline{r}\} \Rightarrow \bar{s} \in \underline{bs} \gg \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}}, a_{\mathrm{Ph}}\}, \{a_{\mathrm{Ph}}, \underline{s}\}\} \in \underline{r}\} \in \mathrm{P}(\underline{bs}); = \mathrm{Reflexivity} \gg \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}}, a_{\mathrm{Ph}}\}, \{a_{\mathrm{Ph}}, \underline{s}\}\} \in \underline{r}\} = \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}}, a_{\mathrm{Ph}}\}, \{a_{\mathrm{Ph}}, \underline{s}\}\} \in \underline{r}\}; \mathrm{JoinConjuncts} \rhd \underline{s} \in \underline{bs} \rhd \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}}, a_{\mathrm{Ph}}\}, \{a_{\mathrm{Ph}}, \underline{s}\}\} \in \underline{r}\} = \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}}, a_{\mathrm{Ph}}\}, \{a_{\mathrm{Ph}}, \underline{s}\}\} \in \underline{r}\} \gg \dot{\neg}\, \underline{s} \in \underline{bs} \Rightarrow \dot{\neg}\, \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}}, a_{\mathrm{Ph}}\}, \{a_{\mathrm{Ph}}, \underline{s}\}\} \in \underline{r}\} = \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}}, a_{\mathrm{Ph}}\}, \{a_{\mathrm{Ph}}, \underline{s}\}\} \in \underline{r}\}; \mathrm{ExistIntro} @ a_{\mathrm{Ex}} @ \underline{s} \rhd \dot{\neg}\, \underline{s} \in \underline{bs} \Rightarrow \dot{\neg}\, \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}}, a_{\mathrm{Ph}}\}, \{a_{\mathrm{Ph}}, \underline{s}\}\} \in \underline{r}\} = \{\mathrm{ph} \in \underline{bs} \mid \{\{a_{\mathrm{Ph}}, a_{\mathrm{Ph}}\}, \{a_{\mathrm{Ph}}, \underline{s}\}\} \in \underline{r}\} \gg \dot{\neg}\, a_{\mathrm{Ex}} \in \underline{bs} \Rightarrow \dot{\neg}\, \{\mathrm{ph} \in \underline{bs} \mid$

$\{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, a_{Ex}\}\} \in \underline{r}\} = \{ph \in \underline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, \underline{s}\}\} \in \underline{r}\}$; Formula2Sep $\rhd \{ph \in \underline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, \underline{s}\}\} \in \underline{r}\} \in P(\underline{bs}) \rhd \dot{\neg}\, a_{Ex} \in \underline{bs} \Rightarrow \dot{\neg}\, \{ph \in \underline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, a_{Ex}\}\} \in \underline{r}\} = \{ph \in \underline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, \underline{s}\}\} \in \underline{r}\} \gg \{ph \in \underline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, \underline{s}\}\} \in \underline{r}\} \in \{ph \in P(\underline{bs}) \mid \dot{\neg}\, t_{Ex} \in \underline{bs} \Rightarrow \dot{\neg}\, \{ph \in \underline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, t_{Ex}\}\} \in \underline{r}\} = b_{Ph}\}$; ExistIntro $@\, j_{Ex}\, @\{ph \in \underline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, \underline{s}\}\} \in \underline{r}\} \rhd \underline{s} \in \{ph \in \underline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, \underline{s}\}\} \in \underline{r}\} \gg \underline{s} \in j_{Ex}$; ExistIntro $@\, j_{Ex}\, @\{ph \in \underline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, \underline{s}\}\} \in \underline{r}\} \rhd \{ph \in \underline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, \underline{s}\}\} \in \underline{r}\} \in \{ph \in P(\underline{bs}) \mid \dot{\neg}\, t_{Ex} \in \underline{bs} \Rightarrow \dot{\neg}\, \{ph \in \underline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, t_{Ex}\}\} \in \underline{r}\} = b_{Ph}\} \gg j_{Ex} \in \{ph \in P(\underline{bs}) \mid \dot{\neg}\, t_{Ex} \in \underline{bs} \Rightarrow \dot{\neg}\, \{ph \in \underline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, t_{Ex}\}\} \in \underline{r}\} = b_{Ph}\}$; Formula2Union $\rhd \underline{s} \in j_{Ex} \rhd j_{Ex} \in \{ph \in P(\underline{bs}) \mid \dot{\neg}\, t_{Ex} \in \underline{bs} \Rightarrow \dot{\neg}\, \{ph \in \underline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, t_{Ex}\}\} \in \underline{r}\} = b_{Ph}\} \gg \underline{s} \in \cup\{ph \in P(\underline{bs}) \mid \dot{\neg}\, t_{Ex} \in \underline{bs} \Rightarrow \dot{\neg}\, \{ph \in \underline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, t_{Ex}\}\} \in \underline{r}\} = b_{Ph}\}$; $\forall \underline{r}\colon \forall \underline{s}\colon \forall \underline{bs}\colon$ Ded $\rhd \forall \underline{r}\colon \forall \underline{s}\colon \forall \underline{bs}\colon \lceil \bar{s} \rceil \#^0 \lceil \underline{r} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{bs} \rceil \Vdash \dot{\neg}\dot{\neg} \forall_{obj} \bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg} \forall_{obj} \bar{s}\colon \forall_{obj} \bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t}, \bar{t}\}, \{\bar{t}, \bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg} \forall_{obj} \bar{s}\colon \forall_{obj} \bar{t}\colon \forall_{obj} \bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t}, \bar{t}\}, \{\bar{t}, \bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{u}\}\} \in \underline{r} \vdash \underline{s} \in \underline{bs} \vdash \underline{s} \in \cup\{ph \in P(\underline{bs}) \mid \dot{\neg}\, t_{Ex} \in \underline{bs} \Rightarrow \dot{\neg}\, \{ph \in \underline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, t_{Ex}\}\} \in \underline{r}\} = b_{Ph}\} \gg \lceil \bar{s} \rceil \#^0 \lceil \underline{r} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{bs} \rceil \Vdash \lceil \bar{t} \rceil \#^0 \lceil \underline{r} \rceil \Vdash \lceil \bar{t} \rceil \#^0 \lceil \underline{bs} \rceil \Vdash \lceil \bar{u} \rceil \#^0 \lceil \underline{r} \rceil \Vdash \lceil \bar{u} \rceil \#^0 \lceil \underline{bs} \rceil \Vdash \dot{\neg}\dot{\neg} \forall_{obj} \bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg} \forall_{obj} \bar{s}\colon \forall_{obj} \bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t}, \bar{t}\}, \{\bar{t}, \bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg} \forall_{obj} \bar{s}\colon \forall_{obj} \bar{t}\colon \forall_{obj} \bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t}, \bar{t}\}, \{\bar{t}, \bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{u}\}\} \in \underline{r} \Rightarrow \underline{s} \in \underline{bs} \Rightarrow \underline{s} \in \cup\{ph \in P(\underline{bs}) \mid \dot{\neg}\, t_{Ex} \in \underline{bs} \Rightarrow \dot{\neg}\, \{ph \in \underline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, t_{Ex}\}\} \in \underline{r}\} = b_{Ph}\}\rceil, p_0, c)]$

I beviset bruger vi for første gang slutningsreglen ExistIntro. Vi anvender her konstruktionen $[x\,@\,y]$ til at fortælle bevischeckeren, hvordan metavariablene $\underline{x}$ og $\underline{t}$ i ExistIntro skal instantieres. Dette kan bevischeckeren ikke selv finde frem til, da både "$\underline{x}$" og "$\underline{t}$" kun optræder i reglens sidebetingelse. Det er med andre ord det samme problem som ved separationsaksiomet i afsnit 4.2.1, som vi her har løst på en anden måde.

11.3.2 Den anden halvdel

Vi viser nu det modsatte resultat af BSsubset: $\underline{bs}$ er en delmængde af $\cup\{ph \in P(\underline{bs}) \mid \dot{\neg}\, t_{Ex} \in \underline{bs} \Rightarrow \dot{\neg}\, \{ph \in \underline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, t_{Ex}\}\} \in \underline{r}\} = b_{Ph}\}$:

[Union(BS/R)subset $\stackrel{stmt}{\to}$ ZFsub $\vdash \forall \underline{r}\colon \forall \underline{s}\colon \forall \underline{bs}\colon \lceil \bar{s} \rceil \#^0 \lceil \underline{bs} \rceil \Vdash \lceil \bar{s} \rceil \#^0 \lceil \underline{r} \rceil \Vdash \lceil \bar{t} \rceil \#^0 \lceil \underline{r} \rceil \Vdash \lceil \bar{t} \rceil \#^0 \lceil \underline{bs} \rceil \Vdash \lceil \bar{u} \rceil \#^0 \lceil \underline{r} \rceil \Vdash \lceil \bar{u} \rceil \#^0 \lceil \underline{bs} \rceil \Vdash \dot{\neg}\dot{\neg} \forall_{obj} \bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg} \forall_{obj} \bar{s}\colon \forall_{obj} \bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t}, \bar{t}\}, \{\bar{t}, \bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg} \forall_{obj} \bar{s}\colon \forall_{obj} \bar{t}\colon \forall_{obj} \bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t}, \bar{t}\}, \{\bar{t}, \bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s}, \bar{s}\}, \{\bar{s}, \bar{u}\}\} \in \underline{r} \Rightarrow \underline{s} \in \cup\{ph \in P(\underline{bs}) \mid \dot{\neg}\, t_{Ex} \in \underline{bs} \Rightarrow \dot{\neg}\, \{ph \in \underline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, t_{Ex}\}\} \in \underline{r}\} = b_{Ph}\} \Rightarrow \underline{s} \in \underline{bs}$]

Beviset for Union(BS/R)subset kan opdeles i fire dele:

1. Lad $\underline{s}$ være et medlem af $\cup\{ph \in P(\underline{bs}) \mid \dot{\neg}\, t_{Ex} \in \underline{bs} \Rightarrow \dot{\neg}\, \{ph \in \underline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, t_{Ex}\}\} \in \underline{r}\} = b_{Ph}\}$. $[\underline{s}]$ må tilhøre en ækvivalensklasse, som vi kalder for j_{Ex} (linie 5–7).

2. [j_{Ex}] tilhører ækvivalenssystemet $\{ph \in P(\underline{bs}) \mid \dot{\neg}\, t_{Ex} \in \underline{bs} \Rightarrow \dot{\neg}\{ph \in \underline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, t_{Ex}\}\} \in \underline{r}\} = b_{Ph}\}$, som igen er en delmængde af potensmængden $P(\underline{bs})$. Derfor må j_{Ex} tilhøre $P(\underline{bs})$ (linie 8–10).
3. Dette medfører igen, at j_{Ex} er en delmængde af $\underline{bs}$ (linie 11).
4. Ud fra punkt 1 og 3 får vi, at $\underline{s}$ tilhører $\underline{bs}$, QED (linie 12–14).

Her er selve beviset:

$$
\begin{array}{l}
[\text{Union(BS/R)subset} \stackrel{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{r}\colon \forall \underline{s}\colon \forall \underline{bs}\colon \lceil \bar{s}\rceil \#^0 \lceil \underline{bs}\rceil \Vdash \\
\dot{\neg}\dot{\neg}\forall_{obj}\bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \\
\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \forall_{obj}\bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \\
\bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r} \vdash \underline{s} \in \\
\cup\{ph \in P(\underline{bs}) \mid \dot{\neg}\, t_{Ex} \in \underline{bs} \Rightarrow \dot{\neg}\{ph \in \underline{bs} \mid \{\{a_{Ph},a_{Ph}\},\{a_{Ph},t_{Ex}\}\} \in \underline{r}\} = b_{Ph}\} \vdash \\
\text{Union2Formula} \rhd \underline{s} \in \cup\{ph \in P(\underline{bs}) \mid \dot{\neg}\, t_{Ex} \in \underline{bs} \Rightarrow \dot{\neg}\{ph \in \underline{bs} \mid \\
\{\{a_{Ph},a_{Ph}\},\{a_{Ph},t_{Ex}\}\} \in \underline{r}\} = b_{Ph}\} \gg \dot{\neg}\underline{s} \in j_{Ex} \Rightarrow \dot{\neg} j_{Ex} \in \{ph \in P(\underline{bs}) \mid \\
\dot{\neg}\, t_{Ex} \in \underline{bs} \Rightarrow \dot{\neg}\{ph \in \underline{bs} \mid \{\{a_{Ph},a_{Ph}\},\{a_{Ph},t_{Ex}\}\} \in \underline{r}\} = \\
b_{Ph}\}; \text{FirstConjunct} \rhd \dot{\neg}\underline{s} \in j_{Ex} \Rightarrow \dot{\neg} j_{Ex} \in \{ph \in P(\underline{bs}) \mid \dot{\neg}\, t_{Ex} \in \underline{bs} \Rightarrow \dot{\neg}\{ph \in \\
\underline{bs} \mid \{\{a_{Ph},a_{Ph}\},\{a_{Ph},t_{Ex}\}\} \in \underline{r}\} = b_{Ph}\} \gg \underline{s} \in j_{Ex}; \text{SecondConjunct} \rhd \dot{\neg}\underline{s} \in \\
j_{Ex} \Rightarrow \dot{\neg} j_{Ex} \in \{ph \in P(\underline{bs}) \mid \dot{\neg}\, t_{Ex} \in \underline{bs} \Rightarrow \dot{\neg}\{ph \in \underline{bs} \mid \\
\{\{a_{Ph},a_{Ph}\},\{a_{Ph},t_{Ex}\}\} \in \underline{r}\} = b_{Ph}\} \gg j_{Ex} \in \{ph \in P(\underline{bs}) \mid \dot{\neg}\, t_{Ex} \in \underline{bs} \Rightarrow \\
\dot{\neg}\{ph \in \underline{bs} \mid \{\{a_{Ph},a_{Ph}\},\{a_{Ph},t_{Ex}\}\} \in \underline{r}\} = b_{Ph}\}; \text{Sep2Formula} \rhd j_{Ex} \in \{ph \in \\
P(\underline{bs}) \mid \dot{\neg}\, t_{Ex} \in \underline{bs} \Rightarrow \dot{\neg}\{ph \in \underline{bs} \mid \{\{a_{Ph},a_{Ph}\},\{a_{Ph},t_{Ex}\}\} \in \underline{r}\} = b_{Ph}\} \gg \dot{\neg} j_{Ex} \in \\
P(\underline{bs}) \Rightarrow \dot{\neg}\dot{\neg} a_{Ex} \in \underline{bs} \Rightarrow \dot{\neg}\{ph \in \underline{bs} \mid \{\{a_{Ph},a_{Ph}\},\{a_{Ph},a_{Ex}\}\} \in \underline{r}\} = \\
j_{Ex}; \text{FirstConjunct} \rhd \dot{\neg} j_{Ex} \in P(\underline{bs}) \Rightarrow \dot{\neg}\dot{\neg} a_{Ex} \in \underline{bs} \Rightarrow \dot{\neg}\{ph \in \underline{bs} \mid \\
\{\{a_{Ph},a_{Ph}\},\{a_{Ph},a_{Ex}\}\} \in \underline{r}\} = j_{Ex} \gg j_{Ex} \in P(\underline{bs}); \text{(Switch)PowerIsSub} \mathrel{|\!\!\rhd} \\
\lceil \bar{s}\rceil \#^0 \lceil \underline{bs}\rceil \rhd j_{Ex} \in P(\underline{bs}) \gg \bar{s} \in j_{Ex} \Rightarrow \bar{s} \in \underline{bs}; \text{Gen} \rhd \bar{s} \in j_{Ex} \Rightarrow \bar{s} \in \underline{bs} \gg \\
\forall_{obj}\bar{s}\colon \bar{s} \in j_{Ex} \Rightarrow \bar{s} \in \underline{bs}; \text{(Switch)HelperPowerIsSub} \mathrel{|\!\!\rhd} \lceil \bar{s}\rceil \#^0 \lceil \underline{bs}\rceil \gg \forall_{obj}\bar{s}\colon \bar{s} \in \\
j_{Ex} \Rightarrow \bar{s} \in \underline{bs} \Rightarrow \underline{s} \in j_{Ex} \Rightarrow \underline{s} \in \underline{bs}; \text{MP2} \rhd \forall_{obj}\bar{s}\colon \bar{s} \in j_{Ex} \Rightarrow \bar{s} \in \underline{bs} \Rightarrow \underline{s} \in j_{Ex} \Rightarrow \underline{s} \in \\
\underline{bs} \rhd \forall_{obj}\bar{s}\colon \bar{s} \in j_{Ex} \Rightarrow \bar{s} \in \underline{bs} \rhd \underline{s} \in j_{Ex} \gg \underline{s} \in \\
\underline{bs}; \forall \underline{r}\colon \forall \underline{s}\colon \forall \underline{bs}\colon \text{Ded} \rhd \forall \underline{r}\colon \forall \underline{s}\colon \forall \underline{bs}\colon \lceil \bar{s}\rceil \#^0 \lceil \underline{bs}\rceil \Vdash \dot{\neg}\dot{\neg}\forall_{obj}\bar{s}\colon \bar{s} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \\
\underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \Rightarrow \\
\dot{\neg}\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \forall_{obj}\bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \\
\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r} \vdash \underline{s} \in \cup\{ph \in P(\underline{bs}) \mid \dot{\neg}\, t_{Ex} \in \underline{bs} \Rightarrow \\
\dot{\neg}\{ph \in \underline{bs} \mid \{\{a_{Ph},a_{Ph}\},\{a_{Ph},t_{Ex}\}\} \in \underline{r}\} = b_{Ph}\} \vdash \underline{s} \in \underline{bs} \gg \lceil \bar{s}\rceil \#^0 \lceil \underline{bs}\rceil \Vdash \\
\lceil \bar{s}\rceil \#^0 \lceil \underline{r}\rceil \Vdash \lceil \bar{t}\rceil \#^0 \lceil \underline{r}\rceil \Vdash \lceil \bar{t}\rceil \#^0 \lceil \underline{bs}\rceil \Vdash \lceil \bar{u}\rceil \#^0 \lceil \underline{r}\rceil \Vdash \lceil \bar{u}\rceil \#^0 \lceil \underline{bs}\rceil \Vdash \dot{\neg}\dot{\neg}\forall_{obj}\bar{s}\colon \bar{s} \in \\
\underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \\
\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\} \in \underline{r} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}\colon \forall_{obj}\bar{t}\colon \forall_{obj}\bar{u}\colon \bar{s} \in \underline{bs} \Rightarrow \bar{t} \in \underline{bs} \Rightarrow \bar{u} \in \underline{bs} \Rightarrow \\
\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\} \in \underline{r} \Rightarrow \{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\} \in \underline{r} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\} \in \underline{r} \Rightarrow \underline{s} \in \cup\{ph \in \\
P(\underline{bs}) \mid \dot{\neg}\, t_{Ex} \in \underline{bs} \Rightarrow \dot{\neg}\{ph \in \underline{bs} \mid \{\{a_{Ph},a_{Ph}\},\{a_{Ph},t_{Ex}\}\} \in \underline{r}\} = b_{Ph}\} \Rightarrow \underline{s} \in \\
\underline{bs}\rceil, p_0, c)]
\end{array}
$$

11.3.3 De to halvdele sættes sammen

Nu hvor vi har vist, at $\underline{bs}$ og $\cup\{ph \in P(\underline{bs}) \mid \dot{\neg}\, t_{Ex} \in \underline{bs} \Rightarrow \dot{\neg}\{ph \in \underline{bs} \mid \{\{a_{Ph}, a_{Ph}\}, \{a_{Ph}, t_{Ex}\}\} \in \underline{r}\} = b_{Ph}\}$ er hinandens delmængder, er det let at vise, at de er lig med hinanden:

[UnionIdentity $\overset{\mathrm{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{\mathsf{r}}\colon \forall \underline{\mathsf{bs}}\colon \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{r}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{bs}} \rceil \Vdash \lceil \bar{\mathsf{t}} \rceil \#^0 \lceil \underline{\mathsf{r}} \rceil \Vdash \lceil \bar{\mathsf{t}} \rceil \#^0 \lceil \underline{\mathsf{bs}} \rceil \Vdash \lceil \bar{\mathsf{u}} \rceil \#^0 \lceil \underline{\mathsf{r}} \rceil \Vdash \lceil \bar{\mathsf{u}} \rceil \#^0 \lceil \underline{\mathsf{bs}} \rceil \Vdash \dot{\neg}\dot{\neg}\forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \forall_{\mathrm{obj}} \bar{\mathsf{t}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \forall_{\mathrm{obj}} \bar{\mathsf{t}}\colon \forall_{\mathrm{obj}} \bar{\mathsf{u}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{u}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \vdash \cup\{\mathrm{ph} \in \mathrm{P}(\underline{\mathsf{bs}}) \mid \dot{\neg}\, \mathsf{t}_{\mathrm{Ex}} \in \underline{\mathsf{bs}} \Rightarrow \dot{\neg}\{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \underline{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} = \underline{\mathsf{bs}}$]

[UnionIdentity $\overset{\mathrm{proof}}{\rightarrow}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{\mathsf{r}}\colon \forall \underline{\mathsf{bs}}\colon \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{r}} \rceil \Vdash \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{bs}} \rceil \Vdash \lceil \bar{\mathsf{t}} \rceil \#^0 \lceil \underline{\mathsf{r}} \rceil \Vdash \lceil \bar{\mathsf{t}} \rceil \#^0 \lceil \underline{\mathsf{bs}} \rceil \Vdash \lceil \bar{\mathsf{u}} \rceil \#^0 \lceil \underline{\mathsf{r}} \rceil \Vdash \lceil \bar{\mathsf{u}} \rceil \#^0 \lceil \underline{\mathsf{bs}} \rceil \Vdash \dot{\neg}\dot{\neg}\forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \forall_{\mathrm{obj}} \bar{\mathsf{t}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \forall_{\mathrm{obj}} \bar{\mathsf{t}}\colon \forall_{\mathrm{obj}} \bar{\mathsf{u}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{u}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \vdash$ BSsubset $\unrhd \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{r}} \rceil \unrhd \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{bs}} \rceil \unrhd \lceil \bar{\mathsf{t}} \rceil \#^0 \lceil \underline{\mathsf{r}} \rceil \unrhd \lceil \bar{\mathsf{t}} \rceil \#^0 \lceil \underline{\mathsf{bs}} \rceil \unrhd \lceil \bar{\mathsf{u}} \rceil \#^0 \lceil \underline{\mathsf{r}} \rceil \unrhd \lceil \bar{\mathsf{u}} \rceil \#^0 \lceil \underline{\mathsf{bs}} \rceil \gg \dot{\neg}\dot{\neg}\forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \forall_{\mathrm{obj}} \bar{\mathsf{t}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \forall_{\mathrm{obj}} \bar{\mathsf{t}}\colon \forall_{\mathrm{obj}} \bar{\mathsf{u}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{u}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{s}} \in \cup\{\mathrm{ph} \in \mathrm{P}(\underline{\mathsf{bs}}) \mid \dot{\neg}\, \mathsf{t}_{\mathrm{Ex}} \in \underline{\mathsf{bs}} \Rightarrow \dot{\neg}\{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \underline{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\}$; MP $\rhd \dot{\neg}\dot{\neg}\forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \forall_{\mathrm{obj}} \bar{\mathsf{t}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \forall_{\mathrm{obj}} \bar{\mathsf{t}}\colon \forall_{\mathrm{obj}} \bar{\mathsf{u}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{u}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{s}} \in \cup\{\mathrm{ph} \in \mathrm{P}(\underline{\mathsf{bs}}) \mid \dot{\neg}\, \mathsf{t}_{\mathrm{Ex}} \in \underline{\mathsf{bs}} \Rightarrow \dot{\neg}\{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \underline{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \rhd \dot{\neg}\dot{\neg}\forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \forall_{\mathrm{obj}} \bar{\mathsf{t}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \forall_{\mathrm{obj}} \bar{\mathsf{t}}\colon \forall_{\mathrm{obj}} \bar{\mathsf{u}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{u}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \gg \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{s}} \in \cup\{\mathrm{ph} \in \mathrm{P}(\underline{\mathsf{bs}}) \mid \dot{\neg}\, \mathsf{t}_{\mathrm{Ex}} \in \underline{\mathsf{bs}} \Rightarrow \dot{\neg}\{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \underline{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\}$; Union(BS/R)subset $\unrhd \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{bs}} \rceil \unrhd \lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{r}} \rceil \unrhd \lceil \bar{\mathsf{t}} \rceil \#^0 \lceil \underline{\mathsf{r}} \rceil \unrhd \lceil \bar{\mathsf{t}} \rceil \#^0 \lceil \underline{\mathsf{bs}} \rceil \unrhd \lceil \bar{\mathsf{u}} \rceil \#^0 \lceil \underline{\mathsf{r}} \rceil \unrhd \lceil \bar{\mathsf{u}} \rceil \#^0 \lceil \underline{\mathsf{bs}} \rceil \gg \dot{\neg}\dot{\neg}\forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \forall_{\mathrm{obj}} \bar{\mathsf{t}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \forall_{\mathrm{obj}} \bar{\mathsf{t}}\colon \forall_{\mathrm{obj}} \bar{\mathsf{u}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{u}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \Rightarrow \bar{\mathsf{s}} \in \cup\{\mathrm{ph} \in \mathrm{P}(\underline{\mathsf{bs}}) \mid \dot{\neg}\, \mathsf{t}_{\mathrm{Ex}} \in \underline{\mathsf{bs}} \Rightarrow \dot{\neg}\{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \underline{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{bs}}$; MP $\rhd \dot{\neg}\dot{\neg}\forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \forall_{\mathrm{obj}} \bar{\mathsf{t}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \forall_{\mathrm{obj}} \bar{\mathsf{t}}\colon \forall_{\mathrm{obj}} \bar{\mathsf{u}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{u}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \Rightarrow \bar{\mathsf{s}} \in \cup\{\mathrm{ph} \in \mathrm{P}(\underline{\mathsf{bs}}) \mid \dot{\neg}\, \mathsf{t}_{\mathrm{Ex}} \in \underline{\mathsf{bs}} \Rightarrow \dot{\neg}\{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \underline{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \rhd \dot{\neg}\dot{\neg}\forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \forall_{\mathrm{obj}} \bar{\mathsf{t}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}} \bar{\mathsf{s}}\colon \forall_{\mathrm{obj}} \bar{\mathsf{t}}\colon \forall_{\mathrm{obj}} \bar{\mathsf{u}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{u}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \gg \bar{\mathsf{s}} \in \cup\{\mathrm{ph} \in \mathrm{P}(\underline{\mathsf{bs}}) \mid \dot{\neg}\, \mathsf{t}_{\mathrm{Ex}} \in \underline{\mathsf{bs}} \Rightarrow \dot{\neg}\{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \underline{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{bs}}$; ToSetEquality $\rhd \bar{\mathsf{s}} \in \cup\{\mathrm{ph} \in \mathrm{P}(\underline{\mathsf{bs}}) \mid \dot{\neg}\, \mathsf{t}_{\mathrm{Ex}} \in \underline{\mathsf{bs}} \Rightarrow \dot{\neg}\{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \underline{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \rhd \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{s}} \in \cup\{\mathrm{ph} \in \mathrm{P}(\underline{\mathsf{bs}}) \mid \dot{\neg}\, \mathsf{t}_{\mathrm{Ex}} \in \underline{\mathsf{bs}} \Rightarrow \dot{\neg}\{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \underline{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \gg \cup\{\mathrm{ph} \in \mathrm{P}(\underline{\mathsf{bs}}) \mid \dot{\neg}\, \mathsf{t}_{\mathrm{Ex}} \in \underline{\mathsf{bs}} \Rightarrow \dot{\neg}\{\mathrm{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \underline{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} = \underline{\mathsf{bs}}\rceil, \mathrm{p}_0, \mathsf{c})$]

11.4 Det sidste bevis

Rapportens sidste bevis vedrører naturligvis hovedresultatet:

[EqSysIsPartition $\overset{\mathrm{stmt}}{\rightarrow}$ ZFsub $\vdash \dot\neg\dot\neg\forall_{\mathrm{obj}}\bar{s}\colon \bar{s}\in\overline{bs} \Rightarrow \{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\}\in\bar{r}\Rightarrow$
$\dot\neg\forall_{\mathrm{obj}}\bar{s}\colon\forall_{\mathrm{obj}}\bar{t}\colon \bar{s}\in\overline{bs}\Rightarrow\bar{t}\in\overline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\bar{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\bar{r}\Rightarrow$
$\dot\neg\forall_{\mathrm{obj}}\bar{s}\colon\forall_{\mathrm{obj}}\bar{t}\colon\forall_{\mathrm{obj}}\bar{u}\colon \bar{s}\in\overline{bs}\Rightarrow\bar{t}\in\overline{bs}\Rightarrow\bar{u}\in\overline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\bar{r}\Rightarrow$
$\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\bar{r}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\bar{r}\vdash\dot\neg\dot\neg\forall_{\mathrm{obj}}\bar{s}\colon\bar{s}\in\{\mathrm{ph}\in\mathrm{P}(\overline{bs})\mid\dot\neg t_{\mathrm{Ex}}\in$
$\overline{bs}\Rightarrow\dot\neg\{\mathrm{ph}\in\overline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},t_{\mathrm{Ex}}\}\}\in\bar{r}\}=b_{\mathrm{Ph}}\}\Rightarrow\dot\neg\bar{s}=\emptyset\Rightarrow$
$\dot\neg\forall_{\mathrm{obj}}\bar{s}\colon\forall_{\mathrm{obj}}\bar{t}\colon\bar{s}\in\{\mathrm{ph}\in\mathrm{P}(\overline{bs})\mid\dot\neg t_{\mathrm{Ex}}\in\overline{bs}\Rightarrow\dot\neg\{\mathrm{ph}\in\overline{bs}\mid$
$\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},t_{\mathrm{Ex}}\}\}\in\bar{r}\}=b_{\mathrm{Ph}}\}\Rightarrow\bar{t}\in\{\mathrm{ph}\in\mathrm{P}(\overline{bs})\mid\dot\neg t_{\mathrm{Ex}}\in\overline{bs}\Rightarrow\dot\neg\{\mathrm{ph}\in$
$\overline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},t_{\mathrm{Ex}}\}\}\in\bar{r}\}=b_{\mathrm{Ph}}\}\Rightarrow\dot\neg\bar{s}=\bar{t}\Rightarrow\{\mathrm{ph}\in\cup\{\{\bar{s},\bar{s}\},\{\bar{t},\bar{t}\}\}\mid$
$\dot\neg c_{\mathrm{Ph}}\in\bar{s}\Rightarrow\dot\neg c_{\mathrm{Ph}}\in\bar{t}\}=\emptyset\Rightarrow\dot\neg\cup\{\mathrm{ph}\in\mathrm{P}(\overline{bs})\mid\dot\neg t_{\mathrm{Ex}}\in\overline{bs}\Rightarrow\dot\neg\{\mathrm{ph}\in\overline{bs}\mid$
$\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},t_{\mathrm{Ex}}\}\}\in\bar{r}\}=b_{\mathrm{Ph}}\}=\overline{bs}$]

I beviset sammensætter vi resultaterne fra underafsnit 11.1 (linie 2–4), underafsnit 11.2 (linie 5–8) og underafsnit 11.3 (linie 9):

[EqSysIsPartition $\overset{\mathrm{proof}}{\rightarrow}\lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash\dot\neg\dot\neg\forall_{\mathrm{obj}}\bar{s}\colon\bar{s}\in\overline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\}\in$
$\bar{r}\Rightarrow\dot\neg\forall_{\mathrm{obj}}\bar{s}\colon\forall_{\mathrm{obj}}\bar{t}\colon\bar{s}\in\overline{bs}\Rightarrow\bar{t}\in\overline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\bar{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\bar{r}\Rightarrow$
$\dot\neg\forall_{\mathrm{obj}}\bar{s}\colon\forall_{\mathrm{obj}}\bar{t}\colon\forall_{\mathrm{obj}}\bar{u}\colon\bar{s}\in\overline{bs}\Rightarrow\bar{t}\in\overline{bs}\Rightarrow\bar{u}\in\overline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\bar{r}\Rightarrow$
$\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\bar{r}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\bar{r}\vdash$ EqSysNotØ $\rhd\dot\neg\dot\neg\forall_{\mathrm{obj}}\bar{s}\colon\bar{s}\in\overline{bs}\Rightarrow$
$\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\}\in\bar{r}\Rightarrow\dot\neg\forall_{\mathrm{obj}}\bar{s}\colon\forall_{\mathrm{obj}}\bar{t}\colon\bar{s}\in\overline{bs}\Rightarrow\bar{t}\in\overline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\bar{r}\Rightarrow$
$\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\bar{r}\Rightarrow\dot\neg\forall_{\mathrm{obj}}\bar{s}\colon\forall_{\mathrm{obj}}\bar{t}\colon\forall_{\mathrm{obj}}\bar{u}\colon\bar{s}\in\overline{bs}\Rightarrow\bar{t}\in\overline{bs}\Rightarrow\bar{u}\in\overline{bs}\Rightarrow$
$\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\bar{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\bar{r}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\bar{r}\gg\bar{x}\in\{\mathrm{ph}\in$
$\mathrm{P}(\overline{bs})\mid\dot\neg t_{\mathrm{Ex}}\in\overline{bs}\Rightarrow\dot\neg\{\mathrm{ph}\in\overline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},t_{\mathrm{Ex}}\}\}\in\bar{r}\}=b_{\mathrm{Ph}}\}\Rightarrow\dot\neg\bar{x}=$
$\emptyset;$ Ded $\rhd\bar{x}\in\{\mathrm{ph}\in\mathrm{P}(\overline{bs})\mid\dot\neg t_{\mathrm{Ex}}\in\overline{bs}\Rightarrow\dot\neg\{\mathrm{ph}\in\overline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},t_{\mathrm{Ex}}\}\}\in$
$\bar{r}\}=b_{\mathrm{Ph}}\}\Rightarrow\dot\neg\bar{x}=\emptyset\gg\bar{s}\in\{\mathrm{ph}\in\mathrm{P}(\overline{bs})\mid\dot\neg t_{\mathrm{Ex}}\in\overline{bs}\Rightarrow\dot\neg\{\mathrm{ph}\in\overline{bs}\mid$
$\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},t_{\mathrm{Ex}}\}\}\in\bar{r}\}=b_{\mathrm{Ph}}\}\Rightarrow\dot\neg\bar{s}=\emptyset;$ Gen $\rhd\bar{s}\in\{\mathrm{ph}\in\mathrm{P}(\overline{bs})\mid\dot\neg t_{\mathrm{Ex}}\in$
$\overline{bs}\Rightarrow\dot\neg\{\mathrm{ph}\in\overline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},t_{\mathrm{Ex}}\}\}\in\bar{r}\}=b_{\mathrm{Ph}}\}\Rightarrow\dot\neg\bar{s}=\emptyset\gg\forall_{\mathrm{obj}}\bar{s}\colon\bar{s}\in$
$\{\mathrm{ph}\in\mathrm{P}(\overline{bs})\mid\dot\neg t_{\mathrm{Ex}}\in\overline{bs}\Rightarrow\dot\neg\{\mathrm{ph}\in\overline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},t_{\mathrm{Ex}}\}\}\in\bar{r}\}=b_{\mathrm{Ph}}\}\Rightarrow$
$\dot\neg\bar{s}=\emptyset;$ AllDisjointImply $\gg\dot\neg\dot\neg\forall_{\mathrm{obj}}\bar{s}\colon\bar{s}\in\overline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\}\in\bar{r}\Rightarrow$
$\dot\neg\forall_{\mathrm{obj}}\bar{s}\colon\forall_{\mathrm{obj}}\bar{t}\colon\bar{s}\in\overline{bs}\Rightarrow\bar{t}\in\overline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\bar{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\bar{r}\Rightarrow$
$\dot\neg\forall_{\mathrm{obj}}\bar{s}\colon\forall_{\mathrm{obj}}\bar{t}\colon\forall_{\mathrm{obj}}\bar{u}\colon\bar{s}\in\overline{bs}\Rightarrow\bar{t}\in\overline{bs}\Rightarrow\bar{u}\in\overline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\bar{r}\Rightarrow$
$\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\bar{r}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\bar{r}\Rightarrow\bar{s}\in\{\mathrm{ph}\in\mathrm{P}(\overline{bs})\mid\dot\neg t_{\mathrm{Ex}}\in\overline{bs}\Rightarrow$
$\dot\neg\{\mathrm{ph}\in\overline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},t_{\mathrm{Ex}}\}\}\in\bar{r}\}=b_{\mathrm{Ph}}\}\Rightarrow\bar{t}\in\{\mathrm{ph}\in\mathrm{P}(\overline{bs})\mid\dot\neg t_{\mathrm{Ex}}\in$
$\overline{bs}\Rightarrow\dot\neg\{\mathrm{ph}\in\overline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},t_{\mathrm{Ex}}\}\}\in\bar{r}\}=b_{\mathrm{Ph}}\}\Rightarrow\dot\neg\bar{s}=\bar{t}\Rightarrow\{\mathrm{ph}\in$
$\cup\{\{\bar{s},\bar{s}\},\{\bar{t},\bar{t}\}\}\mid\dot\neg c_{\mathrm{Ph}}\in\bar{s}\Rightarrow\dot\neg c_{\mathrm{Ph}}\in\bar{t}\}=\emptyset;$ MP $\rhd\dot\neg\dot\neg\forall_{\mathrm{obj}}\bar{s}\colon\bar{s}\in\overline{bs}\Rightarrow$
$\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\}\in\bar{r}\Rightarrow\dot\neg\forall_{\mathrm{obj}}\bar{s}\colon\forall_{\mathrm{obj}}\bar{t}\colon\bar{s}\in\overline{bs}\Rightarrow\bar{t}\in\overline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\bar{r}\Rightarrow$
$\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\bar{r}\Rightarrow\dot\neg\forall_{\mathrm{obj}}\bar{s}\colon\forall_{\mathrm{obj}}\bar{t}\colon\forall_{\mathrm{obj}}\bar{u}\colon\bar{s}\in\overline{bs}\Rightarrow\bar{t}\in\overline{bs}\Rightarrow\bar{u}\in\overline{bs}\Rightarrow$
$\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\bar{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\bar{r}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\bar{r}\Rightarrow\bar{s}\in\{\mathrm{ph}\in$
$\mathrm{P}(\overline{bs})\mid\dot\neg t_{\mathrm{Ex}}\in\overline{bs}\Rightarrow\dot\neg\{\mathrm{ph}\in\overline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},t_{\mathrm{Ex}}\}\}\in\bar{r}\}=b_{\mathrm{Ph}}\}\Rightarrow\bar{t}\in$
$\{\mathrm{ph}\in\mathrm{P}(\overline{bs})\mid\dot\neg t_{\mathrm{Ex}}\in\overline{bs}\Rightarrow\dot\neg\{\mathrm{ph}\in\overline{bs}\mid\{\{a_{\mathrm{Ph}},a_{\mathrm{Ph}}\},\{a_{\mathrm{Ph}},t_{\mathrm{Ex}}\}\}\in\bar{r}\}=b_{\mathrm{Ph}}\}\Rightarrow$
$\dot\neg\bar{s}=\bar{t}\Rightarrow\{\mathrm{ph}\in\cup\{\{\bar{s},\bar{s}\},\{\bar{t},\bar{t}\}\}\mid\dot\neg c_{\mathrm{Ph}}\in\bar{s}\Rightarrow\dot\neg c_{\mathrm{Ph}}\in\bar{t}\}=\emptyset\rhd\dot\neg\dot\neg\forall_{\mathrm{obj}}\bar{s}\colon\bar{s}\in$
$\overline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{s}\}\}\in\bar{r}\Rightarrow\dot\neg\forall_{\mathrm{obj}}\bar{s}\colon\forall_{\mathrm{obj}}\bar{t}\colon\bar{s}\in\overline{bs}\Rightarrow\bar{t}\in\overline{bs}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\bar{r}\Rightarrow$
$\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{s}\}\}\in\bar{r}\Rightarrow\dot\neg\forall_{\mathrm{obj}}\bar{s}\colon\forall_{\mathrm{obj}}\bar{t}\colon\forall_{\mathrm{obj}}\bar{u}\colon\bar{s}\in\overline{bs}\Rightarrow\bar{t}\in\overline{bs}\Rightarrow\bar{u}\in\overline{bs}\Rightarrow$
$\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{t}\}\}\in\bar{r}\Rightarrow\{\{\bar{t},\bar{t}\},\{\bar{t},\bar{u}\}\}\in\bar{r}\Rightarrow\{\{\bar{s},\bar{s}\},\{\bar{s},\bar{u}\}\}\in\bar{r}\gg\bar{s}\in\{\mathrm{ph}\in$

$\mathrm{P}(\overline{\mathsf{bs}}) \mid \dot\neg\, \mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot\neg \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \Rightarrow \bar{\mathsf{t}} \in$
$\{\mathrm{ph} \in \mathrm{P}(\overline{\mathsf{bs}}) \mid \dot\neg\, \mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot\neg \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \Rightarrow$
$\dot\neg\, \bar{\mathsf{s}} = \bar{\mathsf{t}} \Rightarrow \{\mathrm{ph} \in \cup\{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}\} \mid \dot\neg\, \mathsf{c}_{\mathrm{Ph}} \in \bar{\mathsf{s}} \Rightarrow \dot\neg\, \mathsf{c}_{\mathrm{Ph}} \in \bar{\mathsf{t}}\} = \emptyset; \mathrm{Gen} \rhd \bar{\mathsf{s}} \in \{\mathrm{ph} \in$
$\mathrm{P}(\overline{\mathsf{bs}}) \mid \dot\neg\, \mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot\neg \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \Rightarrow \bar{\mathsf{t}} \in$
$\{\mathrm{ph} \in \mathrm{P}(\overline{\mathsf{bs}}) \mid \dot\neg\, \mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot\neg \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \Rightarrow$
$\dot\neg\, \bar{\mathsf{s}} = \bar{\mathsf{t}} \Rightarrow \{\mathrm{ph} \in \cup\{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}\} \mid \dot\neg\, \mathsf{c}_{\mathrm{Ph}} \in \bar{\mathsf{s}} \Rightarrow \dot\neg\, \mathsf{c}_{\mathrm{Ph}} \in \bar{\mathsf{t}}\} = \emptyset \gg \forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon \bar{\mathsf{s}} \in \{\mathrm{ph} \in$
$\mathrm{P}(\overline{\mathsf{bs}}) \mid \dot\neg\, \mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot\neg \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \Rightarrow \bar{\mathsf{t}} \in$
$\{\mathrm{ph} \in \mathrm{P}(\overline{\mathsf{bs}}) \mid \dot\neg\, \mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot\neg \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \Rightarrow$
$\dot\neg\, \bar{\mathsf{s}} = \bar{\mathsf{t}} \Rightarrow \{\mathrm{ph} \in \cup\{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}\} \mid \dot\neg\, \mathsf{c}_{\mathrm{Ph}} \in \bar{\mathsf{s}} \Rightarrow \dot\neg\, \mathsf{c}_{\mathrm{Ph}} \in \bar{\mathsf{t}}\} = \emptyset; \mathrm{Gen} \rhd \forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon \bar{\mathsf{s}} \in$
$\{\mathrm{ph} \in \mathrm{P}(\overline{\mathsf{bs}}) \mid \dot\neg\, \mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot\neg \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \Rightarrow$
$\bar{\mathsf{t}} \in \{\mathrm{ph} \in \mathrm{P}(\overline{\mathsf{bs}}) \mid \dot\neg\, \mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot\neg \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} =$
$\mathsf{b}_{\mathrm{Ph}}\} \Rightarrow \dot\neg\, \bar{\mathsf{s}} = \bar{\mathsf{t}} \Rightarrow \{\mathrm{ph} \in \cup\{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}\} \mid \dot\neg\, \mathsf{c}_{\mathrm{Ph}} \in \bar{\mathsf{s}} \Rightarrow \dot\neg\, \mathsf{c}_{\mathrm{Ph}} \in \bar{\mathsf{t}}\} = \emptyset \gg$
$\forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon \bar{\mathsf{s}} \in \{\mathrm{ph} \in \mathrm{P}(\overline{\mathsf{bs}}) \mid \dot\neg\, \mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot\neg \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid$
$\{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \Rightarrow \bar{\mathsf{t}} \in \{\mathrm{ph} \in \mathrm{P}(\overline{\mathsf{bs}}) \mid \dot\neg\, \mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot\neg \{\mathrm{ph} \in$
$\overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \Rightarrow \dot\neg\, \bar{\mathsf{s}} = \bar{\mathsf{t}} \Rightarrow \{\mathrm{ph} \in \cup\{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}\} \mid$
$\dot\neg\, \mathsf{c}_{\mathrm{Ph}} \in \bar{\mathsf{s}} \Rightarrow \dot\neg\, \mathsf{c}_{\mathrm{Ph}} \in \bar{\mathsf{t}}\} = \emptyset; \mathrm{UnionIdentity} \rhd \dot\neg\, \dot\neg\, \forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \overline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}\} \in$
$\bar{\mathsf{r}} \Rightarrow \dot\neg\, \forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon \bar{\mathsf{s}} \in \overline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \overline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{t}}\}\} \in \bar{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{s}}\}\} \in \bar{\mathsf{r}} \Rightarrow$
$\dot\neg\, \forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{u}}\colon \bar{\mathsf{s}} \in \overline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \overline{\mathsf{bs}} \Rightarrow \bar{\mathsf{u}} \in \overline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{t}}\}\} \in \bar{\mathsf{r}} \Rightarrow$
$\{\{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{u}}\}\} \in \bar{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{u}}\}\} \in \bar{\mathsf{r}} \gg \cup\{\mathrm{ph} \in \mathrm{P}(\overline{\mathsf{bs}}) \mid \dot\neg\, \mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot\neg \{\mathrm{ph} \in$
$\overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} = \overline{\mathsf{bs}}; \mathrm{JoinConjuncts} \rhd \forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \{\mathrm{ph} \in$
$\mathrm{P}(\overline{\mathsf{bs}}) \mid \dot\neg\, \mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot\neg \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \Rightarrow \dot\neg\, \bar{\mathsf{s}} =$
$\emptyset \rhd \forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon \bar{\mathsf{s}} \in \{\mathrm{ph} \in \mathrm{P}(\overline{\mathsf{bs}}) \mid \dot\neg\, \mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot\neg \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid$
$\{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \Rightarrow \bar{\mathsf{t}} \in \{\mathrm{ph} \in \mathrm{P}(\overline{\mathsf{bs}}) \mid \dot\neg\, \mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot\neg \{\mathrm{ph} \in$
$\overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \Rightarrow \dot\neg\, \bar{\mathsf{s}} = \bar{\mathsf{t}} \Rightarrow \{\mathrm{ph} \in \cup\{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}\} \mid$
$\dot\neg\, \mathsf{c}_{\mathrm{Ph}} \in \bar{\mathsf{s}} \Rightarrow \dot\neg\, \mathsf{c}_{\mathrm{Ph}} \in \bar{\mathsf{t}}\} = \emptyset \gg \dot\neg\, \forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \{\mathrm{ph} \in \mathrm{P}(\overline{\mathsf{bs}}) \mid \dot\neg\, \mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot\neg \{\mathrm{ph} \in$
$\overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \Rightarrow \dot\neg\, \bar{\mathsf{s}} = \emptyset \Rightarrow \dot\neg\, \forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon \bar{\mathsf{s}} \in \{\mathrm{ph} \in$
$\mathrm{P}(\overline{\mathsf{bs}}) \mid \dot\neg\, \mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot\neg \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \Rightarrow \bar{\mathsf{t}} \in$
$\{\mathrm{ph} \in \mathrm{P}(\overline{\mathsf{bs}}) \mid \dot\neg\, \mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot\neg \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \Rightarrow$
$\dot\neg\, \bar{\mathsf{s}} = \bar{\mathsf{t}} \Rightarrow \{\mathrm{ph} \in \cup\{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}\} \mid \dot\neg\, \mathsf{c}_{\mathrm{Ph}} \in \bar{\mathsf{s}} \Rightarrow \dot\neg\, \mathsf{c}_{\mathrm{Ph}} \in \bar{\mathsf{t}}\} =$
$\emptyset; \mathrm{JoinConjuncts} \rhd \dot\neg\, \forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \{\mathrm{ph} \in \mathrm{P}(\overline{\mathsf{bs}}) \mid \dot\neg\, \mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot\neg \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid$
$\{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \Rightarrow \dot\neg\, \bar{\mathsf{s}} = \emptyset \Rightarrow \dot\neg\, \forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon \bar{\mathsf{s}} \in \{\mathrm{ph} \in$
$\mathrm{P}(\overline{\mathsf{bs}}) \mid \dot\neg\, \mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot\neg \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \Rightarrow \bar{\mathsf{t}} \in$
$\{\mathrm{ph} \in \mathrm{P}(\overline{\mathsf{bs}}) \mid \dot\neg\, \mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot\neg \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \Rightarrow$
$\dot\neg\, \bar{\mathsf{s}} = \bar{\mathsf{t}} \Rightarrow \{\mathrm{ph} \in \cup\{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}\} \mid \dot\neg\, \mathsf{c}_{\mathrm{Ph}} \in \bar{\mathsf{s}} \Rightarrow \dot\neg\, \mathsf{c}_{\mathrm{Ph}} \in \bar{\mathsf{t}}\} = \emptyset \rhd \cup\{\mathrm{ph} \in \mathrm{P}(\overline{\mathsf{bs}}) \mid$
$\dot\neg\, \mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot\neg \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} = \overline{\mathsf{bs}} \gg$
$\dot\neg\, \dot\neg\, \forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \{\mathrm{ph} \in \mathrm{P}(\overline{\mathsf{bs}}) \mid \dot\neg\, \mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot\neg \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in$
$\bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \Rightarrow \dot\neg\, \bar{\mathsf{s}} = \emptyset \Rightarrow \dot\neg\, \forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon \bar{\mathsf{s}} \in \{\mathrm{ph} \in \mathrm{P}(\overline{\mathsf{bs}}) \mid \dot\neg\, \mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot\neg \{\mathrm{ph} \in$
$\overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \Rightarrow \bar{\mathsf{t}} \in \{\mathrm{ph} \in \mathrm{P}(\overline{\mathsf{bs}}) \mid \dot\neg\, \mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow$
$\dot\neg \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \Rightarrow \dot\neg\, \bar{\mathsf{s}} = \bar{\mathsf{t}} \Rightarrow \{\mathrm{ph} \in$
$\cup\{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}\} \mid \dot\neg\, \mathsf{c}_{\mathrm{Ph}} \in \bar{\mathsf{s}} \Rightarrow \dot\neg\, \mathsf{c}_{\mathrm{Ph}} \in \bar{\mathsf{t}}\} = \emptyset \Rightarrow \dot\neg \cup \{\mathrm{ph} \in \mathrm{P}(\overline{\mathsf{bs}}) \mid \dot\neg\, \mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow$
$\dot\neg \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} = \overline{\mathsf{bs}}; \mathrm{Repetition} \rhd \dot\neg\, \dot\neg\, \forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in$
$\{\mathrm{ph} \in \mathrm{P}(\overline{\mathsf{bs}}) \mid \dot\neg\, \mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot\neg \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \Rightarrow$
$\dot\neg\, \bar{\mathsf{s}} = \emptyset \Rightarrow \dot\neg\, \forall_{\mathrm{obj}}\bar{\mathsf{s}}\colon \forall_{\mathrm{obj}}\bar{\mathsf{t}}\colon \bar{\mathsf{s}} \in \{\mathrm{ph} \in \mathrm{P}(\overline{\mathsf{bs}}) \mid \dot\neg\, \mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot\neg \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid$
$\{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \Rightarrow \bar{\mathsf{t}} \in \{\mathrm{ph} \in \mathrm{P}(\overline{\mathsf{bs}}) \mid \dot\neg\, \mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot\neg \{\mathrm{ph} \in$
$\overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \Rightarrow \dot\neg\, \bar{\mathsf{s}} = \bar{\mathsf{t}} \Rightarrow \{\mathrm{ph} \in \cup\{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}\} \mid$
$\dot\neg\, \mathsf{c}_{\mathrm{Ph}} \in \bar{\mathsf{s}} \Rightarrow \dot\neg\, \mathsf{c}_{\mathrm{Ph}} \in \bar{\mathsf{t}}\} = \emptyset \Rightarrow \dot\neg \cup \{\mathrm{ph} \in \mathrm{P}(\overline{\mathsf{bs}}) \mid \dot\neg\, \mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot\neg \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid$

$\underline{\{\{}\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} = \overline{\mathsf{bs}} \gg \dot{\neg}\, \dot{\neg}\, \forall_{\mathrm{obj}} \bar{\mathsf{s}} \colon \bar{\mathsf{s}} \in \{\mathrm{ph} \in \mathrm{P}(\overline{\mathsf{bs}}) \mid \dot{\neg}\, \mathsf{t}_{\mathrm{Ex}} \in$
$\overline{\mathsf{bs}} \Rightarrow \dot{\neg}\, \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \Rightarrow \dot{\neg}\, \bar{\mathsf{s}} = \emptyset \Rightarrow$
$\dot{\neg}\, \forall_{\mathrm{obj}} \bar{\mathsf{s}} \colon \forall_{\mathrm{obj}} \bar{\mathsf{t}} \colon \bar{\mathsf{s}} \in \{\mathrm{ph} \in \mathrm{P}(\overline{\mathsf{bs}}) \mid \dot{\neg}\, \mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot{\neg}\, \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid$
$\underline{\{\{}\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \Rightarrow \bar{\mathsf{t}} \in \{\mathrm{ph} \in \mathrm{P}(\overline{\mathsf{bs}}) \mid \dot{\neg}\, \mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot{\neg}\, \{\mathrm{ph} \in$
$\overline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} \Rightarrow \dot{\neg}\, \bar{\mathsf{s}} = \bar{\mathsf{t}} \Rightarrow \{\mathrm{ph} \in \cup\{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}\} \mid$
$\dot{\neg}\, \mathsf{c}_{\mathrm{Ph}} \in \bar{\mathsf{s}} \Rightarrow \dot{\neg}\, \mathsf{c}_{\mathrm{Ph}} \in \bar{\mathsf{t}}\} = \emptyset \Rightarrow \dot{\neg} \cup \{\mathrm{ph} \in \mathrm{P}(\overline{\mathsf{bs}}) \mid \dot{\neg}\, \mathsf{t}_{\mathrm{Ex}} \in \overline{\mathsf{bs}} \Rightarrow \dot{\neg}\, \{\mathrm{ph} \in \overline{\mathsf{bs}} \mid$
$\{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \bar{\mathsf{r}}\} = \mathsf{b}_{\mathrm{Ph}}\} = \overline{\mathsf{bs}}\rceil, \mathrm{p}_0, \mathsf{c})]$

Læg mærke til, at vi i linie 2 instantierer meta-variablen $\underline{\mathsf{s}}$ i EqSysNotØ til objektvariablen $\bar{\mathsf{x}}$, selvom det egentlig er $\bar{\mathsf{s}}$, vi gerne ville have fat i. Dette skyldes, at EqSysNotØ indeholder sidebetingelsen $[\lceil\bar{\mathsf{s}}\rceil \#^0 \lceil\underline{\mathsf{s}}\rceil]$ (jvf. afsnit 11.1). Vi løser problemet i linie 3, hvor deduktionsreglen skifter $\bar{\mathsf{x}}$ ud med $\bar{\mathsf{s}}$. Deduktionsreglen tillader kun sådanne variabelskift, når der ikke er nogen metavariable i nærheden; det er derfor, at vi ikke tidligere har brugt reglen på denne måde.

12 Konklusion

Vi har ikke haft 100% succes med at bevise hovedresultatet; der måtte et "snyde-aksiom" til hjælp. Problemet er, at systemet kræver et stort antal sidebetingelser for at sikre, at objekt- og metavariable undgår hinanden. Et spørgsmål er, om vi kunne have undgået dette problem ved at håndtere aksiomerne og definitionerne fra afsnit 4 og 5 anderledes. Her er en kort diskussion af et par alternative strategier:
Det er næppe holdbart kun at arbejde med objektvariable. Deduktionsreglen kan gennemføre variabelskift som f.eks. "$\bar{\mathsf{y}}$ i stedet for $\bar{\mathsf{x}}$", men ikke instantieringer som f.eks. "$\{\bar{\mathsf{y}}, \bar{\mathsf{z}}\}$ i stedet for $\bar{\mathsf{x}}$". (Jvf. formuleringen af SameIntersection i afsnit 10.6, hvor vi måtte instantiere objektvariablene "på forhånd"). Vi så også i afsnit 11.2.6, at det er svært at skifte fra inferens til implikation med objektvariable; deduktionsreglen kan ikke anvendes.
Det andet ekstrem er kun at arbejde med metavariable og -kvantorer. Man kunne da f.eks. formulere den ene halvdel af Extensionality som

$$[\forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \forall \underline{\mathsf{s}}\colon \dot{\neg}\, \underline{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \underline{\mathsf{s}} \in \underline{\mathsf{y}} \Rightarrow \dot{\neg}\, \underline{\mathsf{s}} \in \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{s}} \in \underline{\mathsf{x}} \vdash \underline{\mathsf{x}} = \underline{\mathsf{y}}].$$

Logiwebs bevischecker er imidlertid ikke gearet til denne løsning, og det virker heller ikke som om, at vi på denne måde løser det egentlige problem: F.eks. giver det noget sludder, hvis vi instantierer "$\underline{\mathsf{s}}$" til "$\underline{\mathsf{x}}$" i det ovenstående. Den samme kritik kan rettes mod forslaget "lad objektkvantorerne binde metavariable i stedet for objektvariable".
Den bedste løsning må derfor være at lade bevischeckeren få ansvaret for at administrere sidebetingelserne, således at brugeren kun lægger mærke til dem, hvis han overtræder dem. Jeg har imidlertid ikke noget overblik over, hvor krævende det er at implementere denne løsning.
Min anden hovedkonklusion er, at deduktionsreglen skal suppleres med A4 fra [6], hvis eksistenskvantorer skal håndteres tilfredsstillende. Min løsning med eksistens-variable har fungeret i denne rapport, men det er ikke nogen sikker metode. Der er f.eks. intet, som forhindrer brugeren i at introducere allerede brugte eksistens-variable og dermed opnå systemets godkendelse af et ukorrekt bevis. Det er et åbent spørgsmål, hvor brugervenlig kombinationen "Ded og A4" er; den interesserede læser skal være velkommen til at gøre sine egne erfaringer.

Litteratur

[1] Frederik Eriksen. Et bevis i peano aritmetik, 2005. Logiweb-dokument tilgængelig på hjemmesiden: `http://www.diku.dk/~grue/logiweb/20050502/home/eriksen/peano-commutativity/GRD-2005-06-29-UTC-11-26-23-740136`.

[2] Derek Goldrei. *Classic Set Theory — For Guided Independent Study*. Chapman & Hall, 1996.

[3] Klaus Grue. A logiweb base page. Logiweb-dokument tilgængelig på hjemmesiden: `http://www.diku.dk/~grue/logiweb/20060417/home/grue/base/GRD-2006-06-06-UTC-07-18-19-012409`.

[4] Klaus Grue. The logiweb sequent calculus. Logiweb-dokument tilgængelig på hjemmesiden: `http://www.diku.dk/~grue/logiweb/20060417/home/grue/check/GRD-2006-06-06-UTC-07-49-23-045248`. Dokumentet er fordelt på tre .pdf-filer, hvis adresser (relativt til den ovenstående adresse) er: body/tex/page.pdf, body/tex/appendix.pdf og body/tex/chores.pdf. De to første filer er så langt de vigtigste.

[5] Karel Hrbacek and Thomas Jech. *Introduction to Set Theory*. Marcel Dekker, third edition, 1999.

[6] Elliott Mendelson. *Introduction to Mathematical Logic*. Chapman & Hall, fourth edition, 1997.

A Oversigt over variabelnavne

- De fire vigtigste objektvariable er: $[\overline{\mathsf{s}}]$, $[\overline{\mathsf{t}}]$, $[\overline{\mathsf{u}}]$ og $[\overline{\mathsf{bs}}]$.
- Eksistens-variablen $[\mathsf{j}_{\mathrm{Ex}}]$ bruges i aksiomet [UnionDef] fra afsnit 4.2.
- Ekistens-variablen $[\mathsf{t}_{\mathrm{Ex}}]$ bruges i definitionen af $[\{\mathrm{ph} \in \mathrm{P}(\mathsf{bs}) \mid \dot{\neg}\, \mathsf{t}_{\mathrm{Ex}} \in \mathsf{bs} \Rightarrow \dot{\neg}\, \{\mathrm{ph} \in \mathsf{bs} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \mathsf{r}\} = \mathsf{b}_{\mathrm{Ph}}\}]$ fra afsnit 5.9.
- Eksistens-variablene $[\mathsf{a}_{\mathrm{Ex}}]$ og $[\mathsf{b}_{\mathrm{Ex}}]$ bruges i beviser, når $[\mathsf{j}_{\mathrm{Ex}}]$ eller $[\mathsf{t}_{\mathrm{Ex}}]$ ikke er påkrævede.
- Pladsholder-variablen $[\mathsf{a}_{\mathrm{Ph}}]$ bruges i definitionen af $[\{\mathrm{ph} \in \mathsf{bs} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{x}\}\} \in \mathsf{r}\}]$ fra afsnit 5.9.
- Pladsholder-variablen $[\mathsf{b}_{\mathrm{Ph}}]$ bruges i definitionen af $[\{\mathrm{ph} \in \mathrm{P}(\mathsf{bs}) \mid \dot{\neg}\, \mathsf{t}_{\mathrm{Ex}} \in \mathsf{bs} \Rightarrow \dot{\neg}\, \{\mathrm{ph} \in \mathsf{bs} \mid \{\{\mathsf{a}_{\mathrm{Ph}}, \mathsf{a}_{\mathrm{Ph}}\}, \{\mathsf{a}_{\mathrm{Ph}}, \mathsf{t}_{\mathrm{Ex}}\}\} \in \mathsf{r}\} = \mathsf{b}_{\mathrm{Ph}}\}]$ fra afsnit 5.9.
- Pladsholder-variablen $[\mathsf{c}_{\mathrm{Ph}}]$ bruges i definitionen af $[\{\mathrm{ph} \in \cup\{\{\mathsf{x}, \mathsf{x}\}, \{\mathsf{y}, \mathsf{y}\}\} \mid \dot{\neg}\, \mathsf{c}_{\mathrm{Ph}} \in \mathsf{x} \Rightarrow \dot{\neg}\, \mathsf{c}_{\mathrm{Ph}} \in \mathsf{y}\}]$ fra afsnit 5.5.

- Metavariablene [$\underline{\mathsf{a}}$], [$\underline{\mathsf{b}}$] $\cdots$ [$\underline{\mathsf{f}}$] varierer over formler.
- Metavariabelene [$\underline{\mathsf{r}}$], [$\underline{\mathsf{s}}$] $\cdots$ [$\underline{\mathsf{z}}$] samt [$\underline{\mathsf{bs}}$] varierer over termer. ([$\underline{\mathsf{r}}$] står altid for en relation).
- Metavariablen [$\underline{\mathsf{p}}$] varierer over pladsholder-variable.
- Endelig kan de "rå" variable [a], [b] $\cdots$ [z] samt [bs] variere over hvad som helst; disse variable er ikke tilknyttet nogen semantik.

B Det samlede aksiomsystem

[MP $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall\underline{\mathsf{a}}\colon \forall\underline{\mathsf{b}}\colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \underline{\mathsf{a}} \vdash \underline{\mathsf{b}}$][MP $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[Gen $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall\underline{\mathsf{x}}\colon \forall\underline{\mathsf{a}}\colon \underline{\mathsf{a}} \vdash \forall_{\text{obj}}\underline{\mathsf{x}}\colon \underline{\mathsf{a}}$][Gen $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[Repetition $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall\underline{\mathsf{a}}\colon \underline{\mathsf{a}} \vdash \underline{\mathsf{a}}$][Repetition $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[Neg $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall\underline{\mathsf{a}}\colon \forall\underline{\mathsf{b}}\colon \dot{\neg}\,\underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}} \vdash \dot{\neg}\,\underline{\mathsf{b}} \Rightarrow \dot{\neg}\,\underline{\mathsf{a}} \vdash \underline{\mathsf{b}}$][Neg $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[Ded $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall\underline{\mathsf{a}}\colon \forall\underline{\mathsf{b}}\colon \lambda \mathrm{x}.\mathrm{Dedu}_0(\lceil\underline{\mathsf{a}}\rceil, \lceil\underline{\mathsf{b}}\rceil) \Vdash \underline{\mathsf{a}} \vdash \underline{\mathsf{b}}$][Ded $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[ExistIntro $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall\underline{\mathsf{x}}\colon \forall\underline{\mathsf{t}}\colon \forall\underline{\mathsf{a}}\colon \forall\underline{\mathsf{b}}\colon \langle\lceil\underline{\mathsf{a}}\rceil \equiv^0 \lceil\underline{\mathsf{b}}\rceil | \lceil\underline{\mathsf{x}}\rceil := \lceil\underline{\mathsf{t}}\rceil\rangle_{\mathrm{Ex}} \Vdash \underline{\mathsf{a}} \vdash \underline{\mathsf{b}}$][ExistIntro $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[Extensionality $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall\underline{\mathsf{x}}\colon \forall\underline{\mathsf{y}}\colon \dot{\neg}\,\underline{\mathsf{x}} = \underline{\mathsf{y}} \Rightarrow \forall_{\text{obj}}\bar{\mathsf{s}}\colon \dot{\neg}\,\bar{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{y}} \Rightarrow \dot{\neg}\,\bar{\mathsf{s}} \in \underline{\mathsf{y}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \dot{\neg}\forall_{\text{obj}}\bar{\mathsf{s}}\colon \dot{\neg}\,\bar{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{y}} \Rightarrow \dot{\neg}\,\bar{\mathsf{s}} \in \underline{\mathsf{y}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \underline{\mathsf{x}} = \underline{\mathsf{y}}$][Extensionality $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[Ødef $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall\underline{\mathsf{s}}\colon \dot{\neg}\,\underline{\mathsf{s}} \in \emptyset$][Ødef $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[PairDef $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall\underline{\mathsf{s}}\colon \forall\underline{\mathsf{x}}\colon \forall\underline{\mathsf{y}}\colon \dot{\neg}\,\underline{\mathsf{s}} \in \{\underline{\mathsf{x}}, \underline{\mathsf{y}}\} \Rightarrow \dot{\neg}\,\underline{\mathsf{s}} = \underline{\mathsf{x}} \Rightarrow \underline{\mathsf{s}} = \underline{\mathsf{y}} \Rightarrow \dot{\neg}\,\dot{\neg}\,\underline{\mathsf{s}} = \underline{\mathsf{x}} \Rightarrow \underline{\mathsf{s}} = \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{s}} \in \{\underline{\mathsf{x}}, \underline{\mathsf{y}}\}$][PairDef $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[UnionDef $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall\underline{\mathsf{s}}\colon \forall\underline{\mathsf{x}}\colon \dot{\neg}\,\underline{\mathsf{s}} \in \cup\underline{\mathsf{x}} \Rightarrow \dot{\neg}\,\underline{\mathsf{s}} \in \mathsf{j}_{\mathrm{Ex}} \Rightarrow \dot{\neg}\,\mathsf{j}_{\mathrm{Ex}} \in \underline{\mathsf{x}} \Rightarrow \dot{\neg}\,\dot{\neg}\,\underline{\mathsf{s}} \in \mathsf{j}_{\mathrm{Ex}} \Rightarrow \dot{\neg}\,\mathsf{j}_{\mathrm{Ex}} \in \underline{\mathsf{x}} \Rightarrow \underline{\mathsf{s}} \in \cup\underline{\mathsf{x}}$][UnionDef $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[PowerDef $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall\underline{\mathsf{s}}\colon \forall\underline{\mathsf{x}}\colon \dot{\neg}\,\underline{\mathsf{s}} \in \mathrm{P}(\underline{\mathsf{x}}) \Rightarrow \forall_{\text{obj}}\bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{s}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \dot{\neg}\forall_{\text{obj}}\bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{s}} \Rightarrow \bar{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \underline{\mathsf{s}} \in \mathrm{P}(\underline{\mathsf{x}})$][PowerDef $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[SeparationDef $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall\underline{\mathsf{a}}\colon \forall\underline{\mathsf{b}}\colon \forall\underline{\mathsf{p}}\colon \forall\underline{\mathsf{x}}\colon \forall\underline{\mathsf{z}}\colon \underline{\mathsf{p}}^{\mathrm{Ph}} \wedge \langle\lceil\underline{\mathsf{b}}\rceil \equiv^0 \lceil\underline{\mathsf{a}}\rceil | \lceil\underline{\mathsf{p}}\rceil := \lceil\underline{\mathsf{z}}\rceil\rangle_{\mathrm{Ph}} \Vdash \dot{\neg}\,\underline{\mathsf{z}} \in \{\mathrm{ph} \in \underline{\mathsf{x}} \mid \underline{\mathsf{a}}\} \Rightarrow \dot{\neg}\,\underline{\mathsf{z}} \in \underline{\mathsf{x}} \Rightarrow \dot{\neg}\,\underline{\mathsf{b}} \Rightarrow \dot{\neg}\,\dot{\neg}\,\underline{\mathsf{z}} \in \underline{\mathsf{x}} \Rightarrow \dot{\neg}\,\underline{\mathsf{b}} \Rightarrow \underline{\mathsf{z}} \in \{\mathrm{ph} \in \underline{\mathsf{x}} \mid \underline{\mathsf{a}}\}$][SeparationDef $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[CheatAllDisjoint $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{\mathsf{r}}\colon \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \forall \underline{\mathsf{bs}}\colon \dot{\neg}\, \dot{\neg}\, \forall_{\text{obj}} \bar{\mathsf{s}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\, \forall_{\text{obj}} \bar{\mathsf{s}}\colon \forall_{\text{obj}} \bar{\mathsf{t}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{s}}\}\} \in \underline{\mathsf{r}} \Rightarrow \dot{\neg}\, \forall_{\text{obj}} \bar{\mathsf{s}}\colon \forall_{\text{obj}} \bar{\mathsf{t}}\colon \forall_{\text{obj}} \bar{\mathsf{u}}\colon \bar{\mathsf{s}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{t}} \in \underline{\mathsf{bs}} \Rightarrow \bar{\mathsf{u}} \in \underline{\mathsf{bs}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{t}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{t}}, \bar{\mathsf{t}}\}, \{\bar{\mathsf{t}}, \bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \Rightarrow \{\{\bar{\mathsf{s}}, \bar{\mathsf{s}}\}, \{\bar{\mathsf{s}}, \bar{\mathsf{u}}\}\} \in \underline{\mathsf{r}} \vdash \underline{\mathsf{x}} \in \{\text{ph} \in \text{P}(\underline{\mathsf{bs}}) \mid \dot{\neg}\, \mathsf{t}_{\text{Ex}} \in \underline{\mathsf{bs}} \Rightarrow \dot{\neg}\, \{\text{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\text{Ph}}, \mathsf{a}_{\text{Ph}}\}, \{\mathsf{a}_{\text{Ph}}, \mathsf{t}_{\text{Ex}}\}\} \in \underline{\mathsf{r}}\} = \mathsf{b}_{\text{Ph}}\} \vdash \underline{\mathsf{y}} \in \{\text{ph} \in \text{P}(\underline{\mathsf{bs}}) \mid \dot{\neg}\, \mathsf{t}_{\text{Ex}} \in \underline{\mathsf{bs}} \Rightarrow \dot{\neg}\, \{\text{ph} \in \underline{\mathsf{bs}} \mid \{\{\mathsf{a}_{\text{Ph}}, \mathsf{a}_{\text{Ph}}\}, \{\mathsf{a}_{\text{Ph}}, \mathsf{t}_{\text{Ex}}\}\} \in \underline{\mathsf{r}}\} = \mathsf{b}_{\text{Ph}}\} \vdash \dot{\neg}\, \underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \{\text{ph} \in \cup\{\{\underline{\mathsf{x}}, \underline{\mathsf{x}}\}, \{\underline{\mathsf{y}}, \underline{\mathsf{y}}\}\} \mid \dot{\neg}\, \mathsf{c}_{\text{Ph}} \in \underline{\mathsf{x}} \Rightarrow \dot{\neg}\, \mathsf{c}_{\text{Ph}} \in \underline{\mathsf{y}}\} = \emptyset$][CheatAllDisjoint $\overset{\text{proof}}{\rightarrow}$ Rule tactic]

C Deduktionsreglen

Dette bilag præsenterer dén version af deduktionsreglen fra [4], som jeg har brugt af. Underafsnit C.1 forklarer, hvorfor jeg har ændret på den oprindelige regel, og underafsnit C.2 indeholder selve den ændrede kode (som er skrevet i L).

C.1 Motivering

I beviset for HelperMemberNotØ i afsnit 9.1.3 konkluderer vi $[\lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \Vdash \underline{\mathsf{s}} \in \underline{\mathsf{x}} \Rightarrow \underline{\mathsf{x}} = \emptyset \Rightarrow \underline{\mathsf{s}} \in \emptyset]$ ud fra præmissen $[\lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil \Vdash \underline{\mathsf{s}} \in \underline{\mathsf{x}} \vdash \underline{\mathsf{x}} = \emptyset \vdash \underline{\mathsf{s}} \in \emptyset]$ ved hjælp af deduktionsreglen; det er et klassisk skift fra inferens til implikation. Denne slutning kan imidlertid ikke gennemføres med deduktionsreglen fra [4], fordi denne regel fjerner alle sidebetingelser fra konklusionen, før præmis og konklusion sammenlignes. Da præmissen stadigvæk begynder med sidebetingelsen $[\lceil \bar{\mathsf{s}} \rceil \#^0 \lceil \underline{\mathsf{x}} \rceil]$, matcher præmis og konklusion ikke hinanden, og slutningen fra præmis til konklusion forkastes. Deduktionsreglen fra dette bilag tillader derimod, at præmis og konklusion begynder med et antal identiske sidebetingelser. Dermed kan vi uden problemer gennemføre slutninger som den ovenstående, hvor præmissen indeholder sidebetingelser.

C.2 Kode

Funktionen $[\lambda \mathsf{x}.\mathrm{Dedu}_0(\lceil \mathsf{p} \rceil, \lceil \mathsf{c} \rceil)]$ er en kopi af $[\lambda \mathsf{x}.\mathrm{Ded}_0(\lceil \mathsf{p} \rceil, \lceil \mathsf{c} \rceil)]$ fra [4]:

$$[\mathrm{Dedu}(\mathsf{p}, \mathsf{c}) \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil [\mathrm{Dedu}(\mathsf{p}, \mathsf{c}) \ddot{=} \lambda \mathsf{x}.\mathrm{Dedu}_0(\lceil \mathsf{p} \rceil, \lceil \mathsf{c} \rceil)] \rceil)]$$

Jeg har ændret funktionen $[\mathrm{Ded}_0(\mathsf{p}, \mathsf{c})]$, så den kalder $[\mathrm{Dedu}_\mathrm{s}(\mathrm{Dedu}_7(\mathsf{p}), \mathsf{c}, \mathsf{T})]$ i stedet for $[\mathrm{Ded}_1(\mathrm{Ded}_7(\mathsf{p}), \mathsf{c}, \mathsf{T})]$:

$$[\mathrm{Dedu}_0(\mathsf{p}, \mathsf{c}) \stackrel{\text{val}}{\to} \mathsf{c}!\mathrm{If}(\mathrm{Dedu}_8(\mathsf{p}, \mathsf{T}), \mathrm{Dedu}_\mathrm{s}(\mathrm{Dedu}_7(\mathsf{p}), \mathsf{c}, \mathsf{T}), \mathsf{F})]$$

Funktionen $[\mathrm{Dedu}_\mathrm{s}(\mathsf{p}, \mathsf{c}, \mathsf{s})]$ giver straks kontrollen videre til $[\mathrm{Ded}_1(\mathsf{p}, \mathsf{c}, \mathsf{s})]$ — medmindre p og c begynder med et antal identiske sidebetingelser. I så fald flyttes disse sidebetingelser fra p og c over til listen s, før kontrollen går videre til $[\mathrm{Ded}_1(\mathsf{p}, \mathsf{c}, \mathsf{s})]$:

$$[\mathrm{Dedu}_\mathrm{s}(\mathsf{p}, \mathsf{c}, \mathsf{s}) \stackrel{\text{val}}{\to} \mathrm{If}(\mathsf{p} \stackrel{\text{r}}{=} \lceil \mathsf{x} \Vdash \mathsf{y} \rceil, \mathsf{c} \stackrel{\text{r}}{=} \lceil \mathsf{x} \Vdash \mathsf{y} \rceil \wedge \mathsf{p}^1 \stackrel{\text{t}}{=} \mathsf{c}^1 \wedge \mathrm{Dedu}_\mathrm{s}(\mathsf{p}^2, \mathsf{c}^2, \mathsf{c}^1 :: \mathsf{s}), \mathrm{Dedu}_1(\mathsf{p}, \mathsf{c}, \mathsf{s}))]$$

Fra og med $[\mathrm{Ded}_1(\mathsf{p}, \mathsf{c}, \mathsf{s})]$ er koden kopieret fra appendikset til [4]:

$$[\mathrm{Dedu}_1(\mathsf{p}, \mathsf{c}, \mathsf{s}) \stackrel{\text{val}}{\to} \mathrm{If}(\mathsf{c} \stackrel{\text{r}}{=} \lceil \mathsf{x} \Vdash \mathsf{y} \rceil, \mathrm{Dedu}_1(\mathsf{p}, \mathsf{c}^2, \mathsf{c}^1 :: \mathsf{s}), \mathrm{Dedu}_2(\mathsf{p}, \mathsf{c}, \mathsf{s}))]$$

$$[\mathrm{Dedu}_2(\mathsf{p}, \mathsf{c}, \mathsf{s}) \stackrel{\text{val}}{\to} \mathsf{s}!\mathsf{p} \stackrel{\text{r}}{=} \lceil \mathsf{x} \vdash \mathsf{y} \rceil \wedge \mathsf{c} \stackrel{\text{r}}{=} \lceil \mathsf{x} \Rightarrow \mathsf{y} \rceil \left\{ \begin{array}{l} \mathrm{Dedu}_3(\mathsf{p}^1, \mathsf{c}^1, \mathsf{s}, \mathsf{T}) \wedge \mathrm{Dedu}_2(\mathsf{p}^2, \mathsf{c}^2, \mathsf{s}) \\ \mathrm{Dedu}_4(\mathsf{p}, \mathsf{c}, \mathsf{s}, \mathrm{Dedu}_6(\mathsf{p}, \mathsf{c}, \mathsf{T}, \mathsf{T})) \end{array} \right.]$$

$[\mathrm{Dedu}_3(\mathsf{p},\mathsf{c},\mathsf{s},\mathsf{b}) \overset{\mathrm{val}}{\to} \mathrm{If}(\neg \mathsf{c} \overset{\mathrm{r}}{=} \lceil \forall_{\mathrm{obj}} \mathsf{x} \colon \mathsf{y} \rceil, \mathrm{Dedu}_4(\mathsf{p},\mathsf{c},\mathsf{s},\mathsf{b}),$
$\mathrm{If}(\mathsf{p} \overset{\mathrm{r}}{=} \lceil \forall_{\mathrm{obj}} \mathsf{x} \colon \mathsf{y} \rceil \wedge \mathsf{p}^1 \overset{\mathrm{t}}{=} \mathsf{c}^1, \mathrm{Dedu}_4(\mathsf{p},\mathsf{c},\mathsf{s},\mathsf{b}), \mathrm{Dedu}_3(\mathsf{p},\mathsf{c}^2,\mathsf{s},\mathsf{c}^1 :: \mathsf{c}^1 :: \mathsf{b})))]$

$[\mathrm{Dedu}_4(\mathsf{p},\mathsf{c},\mathsf{s},\mathsf{b}) \overset{\mathrm{val}}{\to} \mathsf{s}!\mathsf{b}!\mathrm{If}(\mathsf{p} \overset{\mathrm{r}}{=} \lceil \overline{\mathsf{x}} \rceil, \mathbf{lookup}(\mathsf{p},\mathsf{b},\mathsf{T}) \overset{\mathrm{t}}{=} \mathsf{c}, \mathrm{If}(\neg \mathsf{p} \overset{\mathrm{r}}{=} \mathsf{c}, \mathsf{F},$
$\mathrm{If}(\mathsf{p} \overset{\mathrm{r}}{=} \lceil \forall_{\mathrm{obj}} \mathsf{x} \colon \mathsf{y} \rceil, \mathsf{p}^1 \overset{\mathrm{t}}{=} \mathsf{c}^1 \wedge \mathrm{Dedu}_4(\mathsf{p}^2,\mathsf{c}^2,\mathsf{s},\mathsf{p}^1 :: \mathsf{p}^1 :: \mathsf{b}), \mathrm{If}(\neg \mathsf{p} \overset{\mathrm{r}}{=} \lceil \underline{\mathsf{x}} \rceil,$
$\mathrm{Dedu}_4^*(\mathsf{p}^{\mathrm{t}},\mathsf{c}^{\mathrm{t}},\mathsf{s},\mathsf{b}), \mathsf{p}^1 \overset{\mathrm{t}}{=} \mathsf{c}^1 \wedge \mathrm{Dedu}_5(\mathsf{p},\mathsf{s},\mathsf{b}))))))]$

$[\mathrm{Dedu}_4^*(\mathsf{p},\mathsf{c},\mathsf{s},\mathsf{b}) \overset{\mathrm{val}}{\to} \mathsf{c}!\mathsf{s}!\mathsf{b}!\mathrm{If}(\mathsf{p},\mathsf{T},\mathrm{Dedu}_4(\mathsf{p}^{\mathrm{h}},\mathsf{c}^{\mathrm{h}},\mathsf{s},\mathsf{b}) \wedge \mathrm{Dedu}_4^*(\mathsf{p}^{\mathrm{t}},\mathsf{c}^{\mathrm{t}},\mathsf{s},\mathsf{b}))]$

$[\mathrm{Dedu}_5(\mathsf{p},\mathsf{s},\mathsf{b}) \overset{\mathrm{val}}{\to} \mathsf{p}!\mathsf{s}!\mathrm{If}(\mathsf{b},\mathsf{T},$
$\lceil \lceil \mathsf{x} \rceil \#^0 \lceil \mathsf{y} \rceil \rceil^{\mathrm{h}} :: \lceil \lceil * \rceil \rceil^{\mathrm{h}} :: \mathsf{b}^{\mathrm{hh}} :: \mathsf{T} :: \lceil \lceil \mathsf{x} \rceil \rceil^{\mathrm{h}} :: \mathsf{p} :: \mathsf{T} :: \mathsf{T} \in_{\mathrm{t}} \mathsf{s} \wedge \mathrm{Dedu}_5(\mathsf{p},\mathsf{s},\mathsf{b}^{\mathrm{t}}))]$

$[\mathrm{Dedu}_6(\mathsf{p},\mathsf{c},\mathsf{e},\mathsf{b}) \overset{\mathrm{val}}{\to} \mathsf{p}!\mathsf{c}!\mathsf{b}!\mathsf{e}!\mathrm{If}(\mathsf{p} \overset{\mathrm{r}}{=} \lceil \overline{\mathsf{x}} \rceil, \mathsf{p} \in_{\mathrm{t}} \mathsf{e} \begin{cases} \mathsf{b} \\ \mathsf{p} :: \mathsf{c} :: \mathsf{b} \end{cases}, \mathrm{If}(\neg \mathsf{p} \overset{\mathrm{r}}{=} \mathsf{c}, \mathsf{T},$
$\mathrm{If}(\mathsf{p} \overset{\mathrm{r}}{=} \lceil \underline{\mathsf{a}} \rceil, \mathsf{b}, \mathrm{If}(\mathsf{p} \overset{\mathrm{r}}{=} \lceil \forall_{\mathrm{obj}} \mathsf{x} \colon \mathsf{y} \rceil, \mathrm{Dedu}_6(\mathsf{p}^2,\mathsf{c}^2,\mathsf{c}^1 :: \mathsf{e},\mathsf{b}), \mathrm{Dedu}_6^*(\mathsf{p}^{\mathrm{t}},\mathsf{c}^{\mathrm{t}},\mathsf{e},\mathsf{b})))))]$

$[\mathrm{Dedu}_6^*(\mathsf{p},\mathsf{c},\mathsf{e},\mathsf{b}) \overset{\mathrm{val}}{\to} \mathsf{p}!\mathsf{c}!\mathsf{b}!\mathsf{e}!\mathrm{If}(\mathsf{p},\mathsf{b},\mathrm{Dedu}_6^*(\mathsf{p}^{\mathrm{t}},\mathsf{c}^{\mathrm{t}},\mathsf{e},\mathrm{Dedu}_6(\mathsf{p}^{\mathrm{h}},\mathsf{c}^{\mathrm{h}},\mathsf{e},\mathsf{b})))]$

$[\mathrm{Dedu}_7(\mathsf{p}) \overset{\mathrm{val}}{\to} \mathsf{p} \overset{\mathrm{r}}{=} \lceil \forall \mathsf{x} \colon \mathsf{y} \rceil \begin{cases} \mathrm{Dedu}_7(\mathsf{p}^2) \\ \mathsf{p} \end{cases}]$

$[\mathrm{Dedu}_8(\mathsf{p},\mathsf{b}) \overset{\mathrm{val}}{\to} \mathrm{If}(\mathsf{p} \overset{\mathrm{r}}{=} \lceil \forall \mathsf{x} \colon \mathsf{y} \rceil, \mathrm{Dedu}_8(\mathsf{p}^2,\mathsf{p}^1 :: \mathsf{b}), \mathrm{If}(\mathsf{p} \overset{\mathrm{r}}{=} \lceil \underline{\mathsf{a}} \rceil, \mathsf{p} \in_{\mathrm{t}} \mathsf{b},$
$\mathrm{Dedu}_8^*(\mathsf{p}^{\mathrm{t}},\mathsf{b})))]$

$[\mathrm{Dedu}_8^*(\mathsf{p},\mathsf{b}) \overset{\mathrm{val}}{\to} \mathsf{b}!\mathrm{If}(\mathsf{p},\mathsf{T},\mathrm{If}(\mathrm{Dedu}_8(\mathsf{p}^{\mathrm{h}},\mathsf{b}),\mathrm{Dedu}_8^*(\mathsf{p}^{\mathrm{t}},\mathsf{b}),\mathsf{F}))]$

D Pyk definitioner

[$(\cdots)$ $\overset{\mathrm{pyk}}{\to}$ “cdots”]
[`Objekt-var` $\overset{\mathrm{pyk}}{\to}$ “object-var”]
[`Ex-var` $\overset{\mathrm{pyk}}{\to}$ “ex-var”]
[`Ph-var` $\overset{\mathrm{pyk}}{\to}$ “ph-var”]
[`Værdi` $\overset{\mathrm{pyk}}{\to}$ “vaerdi”]
[`Variabel` $\overset{\mathrm{pyk}}{\to}$ “variabel”]
[Op($*$) $\overset{\mathrm{pyk}}{\to}$ “op " end op”]
[Op($*, *$) $\overset{\mathrm{pyk}}{\to}$ “op2 " comma " end op2”]
[$* \ddot{=} *$ $\overset{\mathrm{pyk}}{\to}$ “define-equal " comma " end equal”]
[ContainsEmpty($*$) $\overset{\mathrm{pyk}}{\to}$ “contains-empty " end empty”]
[Dedu($*, *$) $\overset{\mathrm{pyk}}{\to}$ “1deduction " conclude " end 1deduction”]
[$\mathrm{Dedu}_0(*, *)$ $\overset{\mathrm{pyk}}{\to}$ “1deduction zero " conclude " end 1deduction”]
[$\mathrm{Dedu}_{\mathrm{s}}(*, *, *)$ $\overset{\mathrm{pyk}}{\to}$ “1deduction side " conclude " condition " end 1deduction”]

[$\mathrm{Dedu}_1(*,*,*) \stackrel{\mathrm{pyk}}{\to}$ "1deduction one " conclude " condition " end 1deduction"]

[$\mathrm{Dedu}_2(*,*,*) \stackrel{\mathrm{pyk}}{\to}$ "1deduction two " conclude " condition " end 1deduction"]

[$\mathrm{Dedu}_3(*,*,*,*) \stackrel{\mathrm{pyk}}{\to}$ "1deduction three " conclude " condition " bound " end 1deduction"]

[$\mathrm{Dedu}_4(*,*,*,*) \stackrel{\mathrm{pyk}}{\to}$ "1deduction four " conclude " condition " bound " end 1deduction"]

[$\mathrm{Dedu}_4^*(*,*,*,*) \stackrel{\mathrm{pyk}}{\to}$ "1deduction four star " conclude " condition " bound " end 1deduction"]

[$\mathrm{Dedu}_5(*,*,*) \stackrel{\mathrm{pyk}}{\to}$ "1deduction five " condition " bound " end 1deduction"]

[$\mathrm{Dedu}_6(*,*,*,*) \stackrel{\mathrm{pyk}}{\to}$ "1deduction six " conclude " exception " bound " end 1deduction"]

[$\mathrm{Dedu}_6^*(*,*,*,*) \stackrel{\mathrm{pyk}}{\to}$ "1deduction six star " conclude " exception " bound " end 1deduction"]

[$\mathrm{Dedu}_7(*) \stackrel{\mathrm{pyk}}{\to}$ "1deduction seven " end 1deduction"]

[$\mathrm{Dedu}_8(*,*) \stackrel{\mathrm{pyk}}{\to}$ "1deduction eight " bound " end 1deduction"]

[$\mathrm{Dedu}_8^*(*,*) \stackrel{\mathrm{pyk}}{\to}$ "1deduction eight star " bound " end 1deduction"]

[$\mathrm{Ex}_1 \stackrel{\mathrm{pyk}}{\to}$ "ex1"]

[$\mathrm{Ex}_2 \stackrel{\mathrm{pyk}}{\to}$ "ex2"]

[$\mathrm{Ex}_{10} \stackrel{\mathrm{pyk}}{\to}$ "ex10"]

[$\mathrm{Ex}_{20} \stackrel{\mathrm{pyk}}{\to}$ "ex20"]

[$*_{\mathrm{Ex}} \stackrel{\mathrm{pyk}}{\to}$ "existential var " end var"]

[$*^{\mathrm{Ex}} \stackrel{\mathrm{pyk}}{\to}$ "" is existential var"]

[$\langle * \equiv * \,|\, * := * \rangle_{\mathrm{Ex}} \stackrel{\mathrm{pyk}}{\to}$ "exist-sub " is " where " is " end sub"]

[$\langle * \equiv^0 * \,|\, * := * \rangle_{\mathrm{Ex}} \stackrel{\mathrm{pyk}}{\to}$ "exist-sub0 " is " where " is " end sub"]

[$\langle * \equiv^1 * \,|\, * := * \rangle_{\mathrm{Ex}} \stackrel{\mathrm{pyk}}{\to}$ "exist-sub1 " is " where " is " end sub"]

[$\langle * \equiv^* * \,|\, * := * \rangle_{\mathrm{Ex}} \stackrel{\mathrm{pyk}}{\to}$ "exist-sub* " is " where " is " end sub"]

[$\mathrm{ph}_1 \stackrel{\mathrm{pyk}}{\to}$ "placeholder-var1"]

[$\mathrm{ph}_2 \stackrel{\mathrm{pyk}}{\to}$ "placeholder-var2"]

[$\mathrm{ph}_3 \stackrel{\mathrm{pyk}}{\to}$ "placeholder-var3"]

[$*_{\mathrm{Ph}} \stackrel{\mathrm{pyk}}{\to}$ "placeholder-var " end var"]

[$*^{\mathrm{Ph}} \stackrel{\mathrm{pyk}}{\to}$ "" is placeholder-var"]

[$\langle * \equiv * \,|\, * := * \rangle_{\mathrm{Ph}} \stackrel{\mathrm{pyk}}{\to}$ "ph-sub " is " where " is " end sub"]

[$\langle * \equiv^0 * \,|\, * := * \rangle_{\mathrm{Ph}} \stackrel{\mathrm{pyk}}{\to}$ "ph-sub0 " is " where " is " end sub"]

[$\langle * \equiv^1 * \,|\, * := * \rangle_{\mathrm{Ph}} \stackrel{\mathrm{pyk}}{\to}$ "ph-sub1 " is " where " is " end sub"]

[$\langle * \equiv^* * \,|\, * := * \rangle_{\mathrm{Ph}} \stackrel{\mathrm{pyk}}{\to}$ "ph-sub* " is " where " is " end sub"]

[$\mathsf{bs} \stackrel{\mathrm{pyk}}{\to}$ "var big set"]

[$\mathsf{OBS} \stackrel{\text{pyk}}{\to}$ "object big set"]
[$\mathcal{BS} \stackrel{\text{pyk}}{\to}$ "meta big set"]
[$\emptyset \stackrel{\text{pyk}}{\to}$ "zermelo empty set"]
[ZFsub $\stackrel{\text{pyk}}{\to}$ "system zf"]
[MP $\stackrel{\text{pyk}}{\to}$ "1rule mp"]
[Gen $\stackrel{\text{pyk}}{\to}$ "1rule gen"]
[Repetition $\stackrel{\text{pyk}}{\to}$ "1rule repetition"]
[Neg $\stackrel{\text{pyk}}{\to}$ "1rule ad absurdum"]
[Ded $\stackrel{\text{pyk}}{\to}$ "1rule deduction"]
[ExistIntro $\stackrel{\text{pyk}}{\to}$ "1rule exist intro"]
[Extensionality $\stackrel{\text{pyk}}{\to}$ "axiom extensionality"]
[Ødef $\stackrel{\text{pyk}}{\to}$ "axiom empty set"]
[PairDef $\stackrel{\text{pyk}}{\to}$ "axiom pair definition"]
[UnionDef $\stackrel{\text{pyk}}{\to}$ "axiom union definition"]
[PowerDef $\stackrel{\text{pyk}}{\to}$ "axiom power definition"]
[SeparationDef $\stackrel{\text{pyk}}{\to}$ "axiom separation definition"]
[CheatAllDisjoint $\stackrel{\text{pyk}}{\to}$ "cheating axiom all disjoint"]
[AddDoubleNeg $\stackrel{\text{pyk}}{\to}$ "prop lemma add double neg"]
[RemoveDoubleNeg $\stackrel{\text{pyk}}{\to}$ "prop lemma remove double neg"]
[AndCommutativity $\stackrel{\text{pyk}}{\to}$ "prop lemma and commutativity"]
[AutoImply $\stackrel{\text{pyk}}{\to}$ "prop lemma auto imply"]
[Contrapositive $\stackrel{\text{pyk}}{\to}$ "prop lemma contrapositive"]
[FirstConjunct $\stackrel{\text{pyk}}{\to}$ "prop lemma first conjunct"]
[SecondConjunct $\stackrel{\text{pyk}}{\to}$ "prop lemma second conjunct"]
[FromContradiction $\stackrel{\text{pyk}}{\to}$ "prop lemma from contradiction"]
[FromDisjuncts $\stackrel{\text{pyk}}{\to}$ "prop lemma from disjuncts"]
[IffCommutativity $\stackrel{\text{pyk}}{\to}$ "prop lemma iff commutativity"]
[IffFirst $\stackrel{\text{pyk}}{\to}$ "prop lemma iff first"]
[IffSecond $\stackrel{\text{pyk}}{\to}$ "prop lemma iff second"]
[ImplyTransitivity $\stackrel{\text{pyk}}{\to}$ "prop lemma imply transitivity"]
[JoinConjuncts $\stackrel{\text{pyk}}{\to}$ "prop lemma join conjuncts"]
[MP2 $\stackrel{\text{pyk}}{\to}$ "prop lemma mp2"]
[MP3 $\stackrel{\text{pyk}}{\to}$ "prop lemma mp3"]
[MP4 $\stackrel{\text{pyk}}{\to}$ "prop lemma mp4"]
[MP5 $\stackrel{\text{pyk}}{\to}$ "prop lemma mp5"]

[MT $\stackrel{\text{pyk}}{\to}$ "prop lemma mt"]
[NegativeMT $\stackrel{\text{pyk}}{\to}$ "prop lemma negative mt"]
[Technicality $\stackrel{\text{pyk}}{\to}$ "prop lemma technicality"]
[Weakening $\stackrel{\text{pyk}}{\to}$ "prop lemma weakening"]
[WeakenOr1 $\stackrel{\text{pyk}}{\to}$ "prop lemma weaken or first"]
[WeakenOr2 $\stackrel{\text{pyk}}{\to}$ "prop lemma weaken or second"]
[Formula2Pair $\stackrel{\text{pyk}}{\to}$ "lemma formula2pair"]
[Pair2Formula $\stackrel{\text{pyk}}{\to}$ "lemma pair2formula"]
[Formula2Union $\stackrel{\text{pyk}}{\to}$ "lemma formula2union"]
[Union2Formula $\stackrel{\text{pyk}}{\to}$ "lemma union2formula"]
[Formula2Sep $\stackrel{\text{pyk}}{\to}$ "lemma formula2separation"]
[Sep2Formula $\stackrel{\text{pyk}}{\to}$ "lemma separation2formula"]
[SubsetInPower $\stackrel{\text{pyk}}{\to}$ "lemma subset in power set"]
[HelperPowerIsSub $\stackrel{\text{pyk}}{\to}$ "lemma power set is subset0"]
[PowerIsSub $\stackrel{\text{pyk}}{\to}$ "lemma power set is subset"]
[(Switch)HelperPowerIsSub $\stackrel{\text{pyk}}{\to}$ "lemma power set is subset0-switch"]
[(Switch)PowerIsSub $\stackrel{\text{pyk}}{\to}$ "lemma power set is subset-switch"]
[ToSetEquality $\stackrel{\text{pyk}}{\to}$ "lemma set equality suff condition"]
[HelperToSetEquality(t) $\stackrel{\text{pyk}}{\to}$ "lemma set equality suff condition(t)0"]
[ToSetEquality(t) $\stackrel{\text{pyk}}{\to}$ "lemma set equality suff condition(t)"]
[HelperFromSetEquality $\stackrel{\text{pyk}}{\to}$ "lemma set equality skip quantifier"]
[FromSetEquality $\stackrel{\text{pyk}}{\to}$ "lemma set equality nec condition"]
[HelperReflexivity $\stackrel{\text{pyk}}{\to}$ "lemma reflexivity0"]
[Reflexivity $\stackrel{\text{pyk}}{\to}$ "lemma reflexivity"]
[HelperSymmetry $\stackrel{\text{pyk}}{\to}$ "lemma symmetry0"]
[Symmetry $\stackrel{\text{pyk}}{\to}$ "lemma symmetry"]
[HelperTransitivity $\stackrel{\text{pyk}}{\to}$ "lemma transitivity0"]
[Transitivity $\stackrel{\text{pyk}}{\to}$ "lemma transitivity"]
[ERisReflexive $\stackrel{\text{pyk}}{\to}$ "lemma er is reflexive"]
[ERisSymmetric $\stackrel{\text{pyk}}{\to}$ "lemma er is symmetric"]
[ERisTransitive $\stackrel{\text{pyk}}{\to}$ "lemma er is transitive"]
[ØisSubset $\stackrel{\text{pyk}}{\to}$ "lemma empty set is subset"]
[HelperMemberNotØ $\stackrel{\text{pyk}}{\to}$ "lemma member not empty0"]
[MemberNotØ $\stackrel{\text{pyk}}{\to}$ "lemma member not empty"]
[HelperUniqueØ $\stackrel{\text{pyk}}{\to}$ "lemma unique empty set0"]

[Unique∅ $\stackrel{\text{pyk}}{\rightarrow}$ “lemma unique empty set”]

[=Reflexivity $\stackrel{\text{pyk}}{\rightarrow}$ “lemma =reflexivity”]

[=Symmetry $\stackrel{\text{pyk}}{\rightarrow}$ “lemma =symmetry”]

[Helper=Transitivity $\stackrel{\text{pyk}}{\rightarrow}$ “lemma =transitivity0”]

[=Transitivity $\stackrel{\text{pyk}}{\rightarrow}$ “lemma =transitivity”]

[HelperTransferNotEq $\stackrel{\text{pyk}}{\rightarrow}$ “lemma transfer ˜is0”]

[TransferNotEq $\stackrel{\text{pyk}}{\rightarrow}$ “lemma transfer ˜is”]

[HelperPairSubset $\stackrel{\text{pyk}}{\rightarrow}$ “lemma pair subset0”]

[Helper(2)PairSubset $\stackrel{\text{pyk}}{\rightarrow}$ “lemma pair subset1”]

[PairSubset $\stackrel{\text{pyk}}{\rightarrow}$ “lemma pair subset”]

[SamePair $\stackrel{\text{pyk}}{\rightarrow}$ “lemma same pair”]

[SameSingleton $\stackrel{\text{pyk}}{\rightarrow}$ “lemma same singleton”]

[UnionSubset $\stackrel{\text{pyk}}{\rightarrow}$ “lemma union subset”]

[SameUnion $\stackrel{\text{pyk}}{\rightarrow}$ “lemma same union”]

[SeparationSubset $\stackrel{\text{pyk}}{\rightarrow}$ “lemma separation subset”]

[SameSeparation $\stackrel{\text{pyk}}{\rightarrow}$ “lemma same separation”]

[SameBinaryUnion $\stackrel{\text{pyk}}{\rightarrow}$ “lemma same binary union”]

[IntersectionSubset $\stackrel{\text{pyk}}{\rightarrow}$ “lemma intersection subset”]

[SameIntersection $\stackrel{\text{pyk}}{\rightarrow}$ “lemma same intersection”]

[AutoMember $\stackrel{\text{pyk}}{\rightarrow}$ “lemma auto member”]

[HelperEqSysNot∅ $\stackrel{\text{pyk}}{\rightarrow}$ “lemma eq-system not empty0”]

[EqSysNot∅ $\stackrel{\text{pyk}}{\rightarrow}$ “lemma eq-system not empty”]

[HelperEqSubset $\stackrel{\text{pyk}}{\rightarrow}$ “lemma eq subset0”]

[EqSubset $\stackrel{\text{pyk}}{\rightarrow}$ “lemma eq subset”]

[HelperEqNecessary $\stackrel{\text{pyk}}{\rightarrow}$ “lemma equivalence nec condition0”]

[EqNecessary $\stackrel{\text{pyk}}{\rightarrow}$ “lemma equivalence nec condition”]

[HelperNoneEqNecessary $\stackrel{\text{pyk}}{\rightarrow}$ “lemma none-equivalence nec condition0”]

[Helper(2)NoneEqNecessary $\stackrel{\text{pyk}}{\rightarrow}$ “lemma none-equivalence nec condition1”]

[NoneEqNecessary $\stackrel{\text{pyk}}{\rightarrow}$ “lemma none-equivalence nec condition”]

[EqClassIsSubset $\stackrel{\text{pyk}}{\rightarrow}$ “lemma equivalence class is subset”]

[EqClassesAreDisjoint $\stackrel{\text{pyk}}{\rightarrow}$ “lemma equivalence classes are disjoint”]

[AllDisjoint $\stackrel{\text{pyk}}{\rightarrow}$ “lemma all disjoint”]

[AllDisjointImply $\stackrel{\text{pyk}}{\rightarrow}$ “lemma all disjoint-imply”]

[BSsubset $\stackrel{\text{pyk}}{\rightarrow}$ “lemma bs subset union(bs/r)”]

[Union(BS/R)subset $\stackrel{\text{pyk}}{\rightarrow}$ “lemma union(bs/r) subset bs”]

[UnionIdentity $\stackrel{\text{pyk}}{\to}$ "lemma union(bs/r) is bs"]

[EqSysIsPartition $\stackrel{\text{pyk}}{\to}$ "theorem eq-system is partition"]

[$*/*$ $\stackrel{\text{pyk}}{\to}$ "eq-system of " modulo ""]

[$*\cap*$ $\stackrel{\text{pyk}}{\to}$ "intersection " comma " end intersection"]

[$\cup*$ $\stackrel{\text{pyk}}{\to}$ "union " end union"]

[$*\cup*$ $\stackrel{\text{pyk}}{\to}$ "binary-union " comma " end union"]

[$\mathrm{P}(*)$ $\stackrel{\text{pyk}}{\to}$ "power " end power"]

[$\{*\}$ $\stackrel{\text{pyk}}{\to}$ "zermelo singleton " end singleton"]

[$\{*,*\}$ $\stackrel{\text{pyk}}{\to}$ "zermelo pair " comma " end pair"]

[$\langle *,*\rangle$ $\stackrel{\text{pyk}}{\to}$ "zermelo ordered pair " comma " end pair"]

[$*\in*$ $\stackrel{\text{pyk}}{\to}$ "" zermelo in ""]

[$*(*,*)$ $\stackrel{\text{pyk}}{\to}$ "" is related to " under ""]

[ReflRel$(*,*)$ $\stackrel{\text{pyk}}{\to}$ "" is reflexive relation in ""]

[SymRel$(*,*)$ $\stackrel{\text{pyk}}{\to}$ "" is symmetric relation in ""]

[TransRel$(*,*)$ $\stackrel{\text{pyk}}{\to}$ "" is transitive relation in ""]

[EqRel$(*,*)$ $\stackrel{\text{pyk}}{\to}$ "" is equivalence relation in ""]

[$[*\in*]_*$ $\stackrel{\text{pyk}}{\to}$ "equivalence class of " in " modulo ""]

[Partition$(*,*)$ $\stackrel{\text{pyk}}{\to}$ "" is partition of ""]

[$*{=}*$ $\stackrel{\text{pyk}}{\to}$ "" zermelo is ""]

[$*\subseteq*$ $\stackrel{\text{pyk}}{\to}$ "" is subset of ""]

[$\dot{\neg}*$ $\stackrel{\text{pyk}}{\to}$ "not0 ""]

[$*\notin*$ $\stackrel{\text{pyk}}{\to}$ "" zermelo ~in ""]

[$*\neq*$ $\stackrel{\text{pyk}}{\to}$ "" zermelo ~is ""]

[$*\mathrel{\dot{\wedge}}*$ $\stackrel{\text{pyk}}{\to}$ "" and0 ""]

[$*\mathrel{\dot{\vee}}*$ $\stackrel{\text{pyk}}{\to}$ "" or0 ""]

[$*\mathrel{\dot{\Leftrightarrow}}*$ $\stackrel{\text{pyk}}{\to}$ "" iff ""]

[$\{\mathrm{ph}\in*\mid*\}$ $\stackrel{\text{pyk}}{\to}$ "the set of ph in " such that " end set"]

[EquivalenceRelations $\stackrel{\text{pyk}}{\to}$ "equivalence-relations"]

E Prioritetstabel

Den nedenstående tabel indeholder alle de formelle konstruktioner, som er til rådighed i dette dokument. Konstruktionerne er grupperet efter prioritet; de førstnævnte grupper har størst prioritet. Hver gruppe er markeret med et af de to ord "**Preassociative**" eller "**Postassociative**", der angiver, om konstruktionerne er venstre- eller højreassociative.

[EquivalenceRelations $\stackrel{\text{prio}}{\rightarrow}$

Preassociative

[EquivalenceRelations], [base], [bracket $*$ end bracket], [big bracket $*$ end bracket], [\$ $*$ \$], [**flush left** [$*$]], [x], [y], [z], [[$* \bowtie *$]], [[$* \stackrel{*}{\rightarrow} *$]], [pyk], [tex], [name], [prio], [$*$], [T], [if$(*,*,*)$], [[$* \stackrel{*}{\Rightarrow} *$]], [val], [claim], [$\bot$], [f$(*)$], [$(*)^{\mathrm{I}}$], [$\mathsf{F}$], [$\underline{0}$], [$\underline{1}$], [$\underline{2}$], [$\underline{3}$], [$\underline{4}$], [$\underline{5}$], [$\underline{6}$], [$\underline{7}$], [$\underline{8}$], [$\underline{9}$], [0], [1], [2], [3], [4], [5], [6], [7], [8], [9], [a], [b], [c], [d], [e], [f], [g], [h], [i], [j], [k], [l], [m], [n], [o], [p], [q], [r], [s], [t], [u], [v], [w], [$(*)^{\mathrm{M}}$], [If$(*,*,*)$], [array$\{*\}$ $*$ end array], [l], [c], [r], [empty], [$\langle * \mid * := * \rangle$], [$\mathcal{M}(*)$], [$\tilde{\mathcal{U}}(*)$], [$\mathcal{U}(*)$], [$\mathcal{U}^{\mathrm{M}}(*)$], [**apply**$(*,*)$], [$\mathbf{apply}_1(*,*)$], [identifier$(*)$], [identifier$_1(*,*)$], [array-plus$(*,*)$], [array-remove$(*,*,*)$], [array-put$(*,*,*,*)$], [array-add$(*,*,*,*,*)$], [bit$(*,*)$], [bit$_1(*,*)$], [rack], ["vector"], ["bibliography"], ["dictionary"], ["body"], ["codex"], ["expansion"], ["code"], ["cache"], ["diagnose"], ["pyk"], ["tex"], ["texname"], ["value"], ["message"], ["macro"], ["definition"], ["unpack"], ["claim"], ["priority"], ["lambda"], ["apply"], ["true"], ["if"], ["quote"], ["proclaim"], ["define"], ["introduce"], ["hide"], ["pre"], ["post"], [$\mathcal{E}(*,*,*)$], [$\mathcal{E}_2(*,*,*,*,*)$], [$\mathcal{E}_3(*,*,*,*)$], [$\mathcal{E}_4(*,*,*,*)$], [**lookup**$(*,*,*)$], [**abstract**$(*,*,*,*)$], [[$\lceil * \rceil$]], [$\mathcal{M}(*,*,*)$], [$\mathcal{M}_2(*,*,*,*)$], [$\mathcal{M}^*(*,*,*)$], [macro], [s$_0$], [**zip**$(*,*)$], [$\mathbf{assoc}_1(*,*,*)$], [$(*)^{\mathbf{P}}$], [self], [[$* \stackrel{..}{=} *$]], [[$* \doteq *$]], [[$* \acute{=} *$]], [[$* \stackrel{\text{pyk}}{=} *$]], [[$* \stackrel{\text{tex}}{=} *$]], [[$* \stackrel{\text{name}}{=} *$]], [**Priority table**[$*$]], [$\tilde{\mathcal{M}}_1$], [$\tilde{\mathcal{M}}_2(*)$], [$\tilde{\mathcal{M}}_3(*)$], [$\tilde{\mathcal{M}}_4(*,*,*,*)$], [$\tilde{\mathcal{M}}(*,*,*)$], [$\tilde{\mathcal{Q}}(*,*,*)$], [$\tilde{\mathcal{Q}}_2(*,*,*)$], [$\tilde{\mathcal{Q}}_3(*,*,*,*)$], [$\tilde{\mathcal{Q}}^*(*,*,*)$], [$(*)$], [$(*)$], [display$(*)$], [statement$(*)$], [[$*]^{\cdot}$], [[$*]^{-}$], [**aspect**$(*,*)$], [**aspect**$(*,*,*)$], [$\langle * \rangle$], [$\mathbf{tuple}_1(*)$], [$\mathbf{tuple}_2(*)$], [let$_2(*,*)$], [let$_1(*,*)$], [[$* \stackrel{\text{claim}}{=} *$]], [checker], [**check**$(*,*)$], [$\mathbf{check}_2(*,*,*)$], [$\mathbf{check}_3(*,*,*)$], [$\mathbf{check}^*(*,*)$], [$\mathbf{check}^*_2(*,*,*)$], [[$*]^{\cdot}$], [[$*]^{-}$], [[$*]^{\circ}$], [msg], [[$* \stackrel{\text{msg}}{=} *$]], [<stmt>], [stmt], [[$* \stackrel{\text{stmt}}{=} *$]], [HeadNil$'$], [HeadPair$'$], [Transitivity$'$], [$\perp\!\!\!\perp$], [Contra$'$], [$\mathrm{T}'_{\mathrm{E}}$], [L$_1$], [$\underline{*}$], [$\mathcal{A}$], [$\mathcal{B}$], [$\mathcal{C}$], [$\mathcal{D}$], [$\mathcal{E}$], [$\mathcal{F}$], [$\mathcal{G}$], [$\mathcal{H}$], [$\mathcal{I}$], [$\mathcal{J}$], [$\mathcal{K}$], [$\mathcal{L}$], [$\mathcal{M}$], [$\mathcal{N}$], [$\mathcal{O}$], [$\mathcal{P}$], [$\mathcal{Q}$], [$\mathcal{R}$], [$\mathcal{S}$], [$\mathcal{T}$], [$\mathcal{U}$], [$\mathcal{V}$], [$\mathcal{W}$], [$\mathcal{X}$], [$\mathcal{Y}$], [$\mathcal{Z}$], [$\langle * \mid * := * \rangle$], [$\langle^* * \mid * := * \rangle$], [$\emptyset$], [Remainder], [$(*)^{\mathbf{v}}$], [intro$(*,*,*,*)$], [intro$(*,*,*)$], [error$(*,*)$], [error$_2(*,*)$], [proof$(*,*,*)$], [proof$_2(*,*)$], [$\mathcal{S}(*,*)$], [$\mathcal{S}^{\mathrm{I}}(*,*)$], [$\mathcal{S}^{\rhd}(*,*)$], [$\mathcal{S}^{\rhd}_1(*,*,*)$], [$\mathcal{S}^{\mathrm{E}}(*,*)$], [$\mathcal{S}^{\mathrm{E}}_1(*,*,*)$], [$\mathcal{S}^{+}(*,*)$], [$\mathcal{S}^{+}_1(*,*,*)$], [$\mathcal{S}^{-}(*,*)$], [$\mathcal{S}^{-}_1(*,*,*)$], [$\mathcal{S}^{*}(*,*)$], [$\mathcal{S}^{*}_1(*,*,*)$], [$\mathcal{S}^{*}_2(*,*,*,*)$], [$\mathcal{S}^{@}(*,*)$], [$\mathcal{S}^{@}_1(*,*,*)$], [$\mathcal{S}^{\vdash}(*,*)$], [$\mathcal{S}^{\vdash}_1(*,*,*,*)$], [$\mathcal{S}^{\Vdash}(*,*)$], [$\mathcal{S}^{\Vdash}_1(*,*,*,*)$], [$\mathcal{S}^{\text{i.e.}}(*,*)$], [$\mathcal{S}^{\text{i.e.}}_1(*,*,*,*)$], [$\mathcal{S}^{\text{i.e.}}_2(*,*,*,*,*)$], [$\mathcal{S}^{\forall}(*,*)$], [$\mathcal{S}^{\forall}_1(*,*,*,*)$], [$\mathcal{S}^{;}(*,*)$], [$\mathcal{S}^{;}_1(*,*,*)$], [$\mathcal{S}^{;}_2(*,*,*,*)$], [$\mathcal{T}(*)$], [claims$(*,*,*)$], [claims$_2(*,*,*)$], [<proof>], [proof], [[**Lemma** $*: *$]], [[**Proof of** $*: *$]], [[$*$ **lemma** $*: *$]], [[$*$ **antilemma** $*: *$]], [[$*$ **rule** $*: *$]], [[$*$ **antirule** $*: *$]], [verifier], [$\mathcal{V}_1(*)$], [$\mathcal{V}_2(*,*)$], [$\mathcal{V}_3(*,*,*,*)$], [$\mathcal{V}_4(*,*)$], [$\mathcal{V}_5(*,*,*,*)$], [$\mathcal{V}_6(*,*,*,*)$], [$\mathcal{V}_7(*,*,*,*)$], [Cut$(*,*)$], [Head$_\oplus(*)$], [Tail$_\oplus(*)$], [rule$_1(*,*)$], [rule$(*,*)$], [Rule tactic], [Plus$(*,*)$], [[**Theory** $*$]], [theory$_2(*,*)$], [theory$_3(*,*)$], [theory$_4(*,*,*)$], [HeadNil$''$], [HeadPair$''$], [Transitivity$''$], [Contra$''$], [HeadNil], [HeadPair], [Transitivity], [Contra], [$\mathrm{T_E}$], [ragged right], [ragged right expansion], [parm$(*,*,*)$], [parm$^*(*,*,*)$], [inst$(*,*)$], [inst$^*(*,*)$], [occur$(*,*,*)$], [occur$^*(*,*,*)$], [unify$(*=*,*)$], [unify$^*(*=*,*)$], [unify$_2(*=*,*)$], [$\mathrm{L_a}$], [$\mathrm{L_b}$], [$\mathrm{L_c}$], [$\mathrm{L_d}$], [$\mathrm{L_e}$], [$\mathrm{L_f}$], [$\mathrm{L_g}$], [$\mathrm{L_h}$], [$\mathrm{L_i}$], [$\mathrm{L_j}$], [$\mathrm{L_k}$], [$\mathrm{L_l}$], [$\mathrm{L_m}$], [$\mathrm{L_n}$], [$\mathrm{L_o}$], [$\mathrm{L_p}$], [$\mathrm{L_q}$], [$\mathrm{L_r}$], [$\mathrm{L_s}$], [$\mathrm{L_t}$], [$\mathrm{L_u}$], [$\mathrm{L_v}$], [$\mathrm{L_w}$], [$\mathrm{L_x}$], [$\mathrm{L_y}$], [$\mathrm{L_z}$], [$\mathrm{L_A}$], [$\mathrm{L_B}$], [$\mathrm{L_C}$], [$\mathrm{L_D}$], [$\mathrm{L_E}$], [$\mathrm{L_F}$], [$\mathrm{L_G}$], [$\mathrm{L_H}$], [$\mathrm{L_I}$], [$\mathrm{L_J}$], [$\mathrm{L_K}$], [$\mathrm{L_L}$], [$\mathrm{L_M}$], [$\mathrm{L_N}$], [$\mathrm{L_O}$], [$\mathrm{L_P}$], [$\mathrm{L_Q}$], [$\mathrm{L_R}$], [$\mathrm{L_S}$], [$\mathrm{L_T}$], [$\mathrm{L_U}$], [$\mathrm{L_V}$], [$\mathrm{L_W}$], [$\mathrm{L_X}$], [$\mathrm{L_Y}$], [$\mathrm{L_Z}$], [$\mathrm{L_?}$], [Reflexivity], [Reflexivity$_1$], [Commutativity], [Commutativity$_1$], [<tactic>], [tactic], [[$* \overset{\text{tactic}}{=} *$]], [$\mathcal{P}(*,*,*)$], [$\mathcal{P}^*(*,*,*)$], [$\mathrm{p_0}$], [conclude$_1(*,*)$], [conclude$_2(*,*,*)$], [conclude$_3(*,*,*,*)$], [conclude$_4(*,*)$], [check], [[$* \overset{\circ}{=} *$]], [RootVisible$(*)$], [A], [R], [C], [T], [L], [$\{*\}$], [$\overline{*}$], [a], [b], [c], [d], [e], [f], [g], [h], [i], [j], [k], [l], [m], [n], [o], [p], [q], [r], [s], [t], [u], [v], [w], [x], [y], [z], [$\langle * \equiv * \mid * := * \rangle$], [$\langle * \equiv^0 * \mid * := * \rangle$], [$\langle * \equiv^1 * \mid * := * \rangle$], [$\langle * \equiv^* * \mid * := * \rangle$], [Ded$(*,*)$], [Ded$_0(*,*)$], [Ded$_1(*,*,*)$], [Ded$_2(*,*,*)$], [Ded$_3(*,*,*,*)$], [Ded$_4(*,*,*,*)$], [Ded$^*_4(*,*,*,*)$], [Ded$_5(*,*,*)$], [Ded$_6(*,*,*,*)$], [Ded$^*_6(*,*,*,*)$], [Ded$_7(*)$], [Ded$_8(*,*)$], [Ded$^*_8(*,*)$], [S], [Neg], [MP], [Gen], [Ded], [S1], [S2], [S3], [S4], [S5], [S6], [S7], [S8], [S9], [Repetition], [A1$'$], [A2$'$], [A4$'$], [A5$'$], [Prop 3.2a], [Prop 3.2b], [Prop 3.2c], [Prop 3.2d], [Prop 3.2e$_1$], [Prop 3.2e$_2$], [Prop 3.2e], [Prop 3.2f$_1$], [Prop 3.2f$_2$], [Prop 3.2f], [Prop 3.2g$_1$], [Prop 3.2g$_2$], [Prop 3.2g], [Prop 3.2h$_1$], [Prop 3.2h$_2$], [Prop 3.2h], [Block$_1(*,*,*)$], [Block$_2(*)$], [$(\cdots)$], [`Objekt-var`], [`Ex-var`], [`Ph-var`], [`Værdi`], [`Variabel`], [Op$(*)$], [Op$(*,*)$], [$* \overset{..}{=} *$], [ContainsEmpty$(*)$], [Dedu$(*,*)$], [Dedu$_0(*,*)$], [Dedu$_\mathrm{s}(*,*,*)$], [Dedu$_1(*,*,*)$], [Dedu$_2(*,*,*)$], [Dedu$_3(*,*,*,*)$], [Dedu$_4(*,*,*,*)$], [Dedu$^*_4(*,*,*,*)$], [Dedu$_5(*,*,*)$], [Dedu$_6(*,*,*,*)$], [Dedu$^*_6(*,*,*,*)$], [Dedu$_7(*)$], [Dedu$_8(*,*)$], [Dedu$^*_8(*,*)$], [Ex$_1$], [Ex$_2$], [Ex$_{10}$], [Ex$_{20}$], [$*_{\mathrm{Ex}}$], [$*^{\mathrm{Ex}}$], [$\langle * \equiv * \mid * := * \rangle_{\mathrm{Ex}}$], [$\langle * \equiv^0 * \mid * := * \rangle_{\mathrm{Ex}}$], [$\langle * \equiv^1 * \mid * := * \rangle_{\mathrm{Ex}}$], [$\langle * \equiv^* * \mid * := * \rangle_{\mathrm{Ex}}$], [ph$_1$], [ph$_2$], [ph$_3$], [$*_{\mathrm{Ph}}$], [$*^{\mathrm{Ph}}$], [$\langle * \equiv * \mid * := * \rangle_{\mathrm{Ph}}$], [$\langle * \equiv^0 * \mid * := * \rangle_{\mathrm{Ph}}$], [$\langle * \equiv^1 * \mid * := * \rangle_{\mathrm{Ph}}$], [$\langle * \equiv^* * \mid * := * \rangle_{\mathrm{Ph}}$], [bs], [OBS], [$\mathcal{BS}$], [Ø], [ZFsub], [MP], [Gen], [Repetition], [Neg], [Ded], [ExistIntro], [Extensionality], [Ødef], [PairDef], [UnionDef], [PowerDef], [SeparationDef], [CheatAllDisjoint], [AddDoubleNeg], [RemoveDoubleNeg], [AndCommutativity], [AutoImply], [Contrapositive], [FirstConjunct], [SecondConjunct], [FromContradiction], [FromDisjuncts], [IffCommutativity], [IffFirst], [IffSecond], [ImplyTransitivity], [JoinConjuncts], [MP2], [MP3], [MP4], [MP5], [MT], [NegativeMT], [Technicality], [Weakening], [WeakenOr1], [WeakenOr2], [Formula2Pair], [Pair2Formula], [Formula2Union], [Union2Formula], [Formula2Sep], [Sep2Formula], [SubsetInPower], [HelperPowerIsSub], [PowerIsSub], [(Switch)HelperPowerIsSub], [(Switch)PowerIsSub], [ToSetEquality], [HelperToSetEquality(t)], [ToSetEquality(t)], [HelperFromSetEquality], [FromSetEquality], [HelperReflexivity], [Reflexivity], [HelperSymmetry], [Symmetry], [HelperTransitivity], [Transitivity], [ERisReflexive], [ERisSymmetric], [ERisTransitive], [ØisSubset], [HelperMemberNotØ], [MemberNotØ], [HelperUniqueØ], [UniqueØ], [=Reflexivity], [=Symmetry], [Helper = Transitivity], [= Transitivity], [HelperTransferNotEq], [TransferNotEq], [HelperPairSubset], [Helper(2)PairSubset], [PairSubset], [SamePair], [SameSingleton], [UnionSubset], [SameUnion], [SeparationSubset], [SameSeparation], [SameBinaryUnion], [IntersectionSubset], [SameIntersection], [AutoMember], [HelperEqSysNot∅], [EqSysNot∅], [HelperEqSubset], [EqSubset], [HelperEqNecessary], [EqNecessary], [HelperNoneEqNecessary], [Helper(2)NoneEqNecessary], [NoneEqNecessary], [EqClassIsSubset], [EqClassesAreDisjoint], [AllDisjoint], [AllDisjointImply], [BSsubset], [Union(BS/R)subset], [UnionIdentity], [EqSysIsPartition];

Preassociative

[*_{*}], [*/indexintro(*, *, *, *)], [*/intro(*, *, *)], [*/bothintro(*, *, *, *, *)], [*/nameintro(*, *, *, *)], [$*'$], [*[*]], [*[*→*]], [*[*⇒*]], [*0], [*1], [0b], [*-color(*)], [$*$-color*(*)], [$*^{\mathrm{H}}$], [$*^{\mathrm{T}}$], [$*^{\mathrm{U}}$], [$*^{\mathrm{h}}$], [$*^{\mathrm{t}}$], [$*^{\mathrm{s}}$], [$*^{\mathrm{c}}$], [$*^{\mathrm{d}}$], [$*^{\mathrm{a}}$], [$*^{\mathrm{C}}$], [$*^{\mathrm{M}}$], [$*^{\mathrm{B}}$], [$*^{\mathrm{r}}$], [$*^{\mathrm{i}}$], [$*^{\mathrm{d}}$], [$*^{\mathrm{R}}$], [$*^{0}$], [$*^{1}$], [$*^{2}$], [$*^{3}$], [$*^{4}$], [$*^{5}$], [$*^{6}$], [$*^{7}$], [$*^{8}$], [$*^{9}$], [$*^{\mathrm{E}}$], [$*^{\mathcal{V}}$], [$*^{\mathcal{C}}$], [$*^{\mathcal{C}^*}$], [$*^{\mathrm{hide}}$];

Preassociative

[“ * ”], [], [$(*)^{\mathbf{t}}$], [string(*) + *], [string(*) ++ *], [
*], [*], [!*], ["*], [#*], [$*], [%*], [&*], [’*], [(*], [)*], [**], [+*], [, *], [-*], [.*], [/*], [0*], [1*], [2*], [3*], [4*], [5*], [6*], [7*], [8*], [9*], [:*], [; *], [<*], [=*], [>*], [?*], [@*], [A*], [B*], [C*], [D*], [E*], [F*], [G*], [H*], [I*], [J*], [K*], [L*], [M*], [N*], [O*], [P*], [Q*], [R*], [S*], [T*], [U*], [V*], [W*], [X*], [Y*], [Z*], [[*], [*], []*], [^*], [_*], [‘*], [a*], [b*], [c*], [d*], [e*], [f*], [g*], [h*], [i*], [j*], [k*], [l*], [m*], [n*], [o*], [p*], [q*], [r*], [s*], [t*], [u*], [v*], [w*], [x*], [y*], [z*], [{*], [|*], [}*], [~*], [**Preassociative** *; *], [**Postassociative** *; *], [[*], *], [priority * end], [newline *], [macro newline *], [MacroIndent(*)];

Preassociative

[* ’ *], [* ‘ *];

Preassociative

[$*'$];

Preassociative

[*/*], [* ∩ *];

Preassociative

[∪*], [* ∪ *], [P(*)];

Preassociative

[{*}];

Preassociative

[{*, *}], [⟨*, *⟩];

Preassociative

[* ∈ *], [*(*, *)], [ReflRel(*, *)], [SymRel(*, *)], [TransRel(*, *)], [EqRel(*, *)], [$[* \in *]_*$], [Partition(*, *)];

Preassociative

[* · *], [$* \cdot_0 *$];

Preassociative

[* + *], [$* +_0 *$], [$* +_1 *$], [* − *], [$* -_0 *$], [$* -_1 *$];

Preassociative

[* ∪ {*}], [* ∪ *], [*\{*}];

Postassociative

$[* \therefore *], [* \underline{\therefore} *], [* \underline{::} *], [* \underline{+2*} *], [* :: *], [* +2* *];$

Postassociative

$[*, *];$

Preassociative

$[* \overset{B}{\approx} *], [* \overset{D}{\approx} *], [* \overset{C}{\approx} *], [* \overset{P}{\approx} *], [* \approx *], [* = *], [* \overset{+}{\rightarrow} *], [* \overset{t}{=} *], [* \overset{t^*}{=} *], [* \overset{r}{=} *],$
$[* \in_t *], [* \subseteq_T *], [* \overset{T}{=} *], [* \overset{s}{=} *], [* \text{ free in } *], [* \text{ free in}^* *], [* \text{ free for } * \text{ in } *],$
$[* \text{ free for}^* * \text{ in } *], [* \in_c *], [* < *], [* <' *], [* \leq' *], [* = *], [* \neq *], [*^{\text{var}}],$
$[* \#^0 *], [* \#^1 *], [* \#^* *], [* {=} *], [* \subseteq *];$

Preassociative

$[\neg *], [\dot{\neg} *], [* \notin *], [* \neq *];$

Preassociative

$[* \wedge *], [* \ddot{\wedge} *], [* \tilde{\wedge} *], [* \wedge_c *], [* \dot{\wedge} *];$

Preassociative

$[* \vee *], [* \parallel *], [* \ddot{\vee} *], [* \dot{\vee} *];$

Preassociative

$[\exists * \colon *], [\forall * \colon *], [\forall_{\text{obj}} * \colon *];$

Postassociative

$[* \ddot{\Rightarrow} *], [* \Rightarrow *], [* \Leftrightarrow *], [* \dot{\Leftrightarrow} *];$

Preassociative

$[\{\text{ph} \in * \mid *\}];$

Postassociative

$[* : *], [* \text{ spy } *], [* ! *];$

Preassociative

$[* \begin{cases} * \\ * \end{cases}];$

Preassociative

$[\lambda * . *], [\Lambda * . *], [\Lambda *], [\textbf{if } * \textbf{ then } * \textbf{ else } *], [\textbf{let } * = * \textbf{ in } *], [\textbf{let } * \ddot{=} * \text{ in } *];$

Preassociative

$[* \# *];$

Preassociative

$[*^{\text{I}}], [*^{\triangleright}], [*^{\text{V}}], [*^{+}], [*^{-}], [*^{*}];$

Preassociative

$[* @ *], [* \rhd *], [* \Rrightarrow *], [* \gg *], [* \trianglerighteq *];$

Postassociative

$[* \vdash *], [* \Vdash *], [* \text{ i.e. } *];$

Preassociative

$[\forall * \colon *], [\Pi * \colon *];$

Postassociative

$[* \oplus *];$

Postassociative

$[*; *];$

Preassociative

$[* \text{ proves } *];$

Preassociative

[$*$ **proof of** $* : *$], [Line $* : * \gg *; *$], [Last line $* \gg * \Box$],
[Line $*$: Premise $\gg *; *$], [Line $*$: Side-condition $\gg *; *$], [Arbitrary $\gg *; *$],
[Local $\gg * = *; *$], [Begin $*; *$: End; $*$], [Last block line $* \gg *$;],
[Arbitrary $\gg *; *$];
Postassociative
[$* \mid *$];
Postassociative
[$*$, $*$], [$*$[$*$]$*$];
Preassociative
[$*$&$*$];
Preassociative
[$*$\\$*$], [$*$ linebreak[4] $*$], [$*$\\$*$];]

F TEX definitioner

[EquivalenceRelations $\stackrel{\text{tex}}{\to}$ "EquivalenceRelations"]

[$(\cdots)$ $\stackrel{\text{tex}}{\to}$ "(\cdots{})"]

[`Objekt-var` $\stackrel{\text{tex}}{\to}$ "\texttt{Objekt-var}"]

[`Ex-var` $\stackrel{\text{tex}}{\to}$ "\texttt{Ex-var}"]

[`Ph-var` $\stackrel{\text{tex}}{\to}$ "\texttt{Ph-var}"]

[`Værdi` $\stackrel{\text{tex}}{\to}$ "\texttt{V\ae{}rdi}"]

[`Variabel` $\stackrel{\text{tex}}{\to}$ "\texttt{Variabel}"]

[Op(x) $\stackrel{\text{tex}}{\to}$ "Op(#1.
)"]

[Op(x, y) $\stackrel{\text{tex}}{\to}$ "Op(#1.
, #2.
)"]

[$\mathsf{x} \mathrel{\ddot{=}} \mathsf{y}$ $\stackrel{\text{tex}}{\to}$ "#1.
\mathrel {\ddot {=}} #2."]

[ContainsEmpty(x) $\stackrel{\text{tex}}{\to}$ "ContainsEmpty(#1.
)"]

[Dedu(x, y) $\stackrel{\text{tex}}{\to}$ "
Dedu(#1.
, #2.
)"]

[$\mathrm{Dedu}_0(\mathsf{x}, \mathsf{y}) \stackrel{\text{tex}}{\to}$ "
Dedu_0(#1.
, #2.
)"]

[$\mathrm{Dedu}_\mathrm{s}(\mathsf{x}, \mathsf{y}, \mathsf{z}) \stackrel{\text{tex}}{\to}$ "Dedu_{s}(#1.
, #2.
, #3.
)"]

[$\mathrm{Dedu}_1(\mathsf{x}, \mathsf{y}, \mathsf{z}) \stackrel{\text{tex}}{\to}$ "
Dedu_1(#1.
, #2.
, #3.
)"]

[$\mathrm{Dedu}_2(\mathsf{x}, \mathsf{y}, \mathsf{z}) \stackrel{\text{tex}}{\to}$ "
Dedu_2(#1.
, #2.
, #3.
)"]

[$\mathrm{Dedu}_3(\mathsf{x}, \mathsf{y}, \mathsf{z}, \mathsf{u}) \stackrel{\text{tex}}{\to}$ "
Dedu_3(#1.
, #2.
, #3.
, #4.
)"]

[$\mathrm{Dedu}_4(\mathsf{x}, \mathsf{y}, \mathsf{z}, \mathsf{u}) \stackrel{\text{tex}}{\to}$ "
Dedu_4(#1.
, #2.
, #3.
, #4.
)"]

[$\mathrm{Dedu}_4^*(\mathsf{x}, \mathsf{y}, \mathsf{z}, \mathsf{u}) \stackrel{\text{tex}}{\to}$ "
Dedu_4ˆ∗(#1.
, #2.
, #3.
, #4.
)"]

[$\mathrm{Dedu}_5(\mathsf{x}, \mathsf{y}, \mathsf{z}) \stackrel{\text{tex}}{\to}$ "
Dedu_5(#1.
, #2.
, #3.
)"]

[$\mathrm{Dedu}_6(\mathsf{p},\mathsf{c},\mathsf{e},\mathsf{b}) \stackrel{\text{tex}}{\rightarrow}$ “
Dedu_6(#1.
, #2.
, #3.
, #4.
)”]

[$\mathrm{Dedu}^*_6(\mathsf{p},\mathsf{c},\mathsf{e},\mathsf{b}) \stackrel{\text{tex}}{\rightarrow}$ “
Dedu_6^*(#1.
, #2.
, #3.
, #4.
)”]

[$\mathrm{Dedu}_7(\mathsf{p}) \stackrel{\text{tex}}{\rightarrow}$ “
Dedu_7(#1.
)”]

[$\mathrm{Dedu}_8(\mathsf{p},\mathsf{b}) \stackrel{\text{tex}}{\rightarrow}$ “
Dedu_8(#1.
, #2.
)”]

[$\mathrm{Dedu}^*_8(\mathsf{p},\mathsf{b}) \stackrel{\text{tex}}{\rightarrow}$ “
Dedu_8^*(#1.
, #2.
)”]

[$\mathrm{Ex}_1 \stackrel{\text{tex}}{\rightarrow}$ “Ex_{1}”]

[$\mathrm{Ex}_2 \stackrel{\text{tex}}{\rightarrow}$ “Ex_{2}”]

[$\mathrm{Ex}_{10} \stackrel{\text{tex}}{\rightarrow}$ “Ex_{10}”]

[$\mathrm{Ex}_{20} \stackrel{\text{tex}}{\rightarrow}$ “Ex_{20}”]

[$\mathsf{x}_{\mathrm{Ex}} \stackrel{\text{tex}}{\rightarrow}$ “#1.
_{Ex}”]

[$\mathsf{x}^{\mathrm{Ex}} \stackrel{\text{tex}}{\rightarrow}$ “#1.
^{Ex}”]

[$\langle \mathsf{x} {\equiv} \mathsf{y} | \mathsf{z} {:=} \mathsf{u} \rangle_{\mathrm{Ex}} \stackrel{\text{tex}}{\rightarrow}$ “\langle #1.
{\equiv} #2.
| #3.
{:=} #4.
\rangle_{Ex} ”]

[$\langle \mathsf{x} {\equiv}^0 \mathsf{y} | \mathsf{z} {:=} \mathsf{u} \rangle_{\mathrm{Ex}} \stackrel{\text{tex}}{\rightarrow}$ “\langle #1.
{\equiv}^0 #2.

| #3.
{:=} #4.
\rangle_{Ex} ”]

$[\langle x{\equiv}^1 y|z{:=}u\rangle_{\mathrm{Ex}} \stackrel{\mathrm{tex}}{\to}$ “\langle #1.
{\equiv}^1 #2.
| #3.
{:=} #4.
\rangle_{Ex} ”]

$[\langle x{\equiv}^* y|z{:=}u\rangle_{\mathrm{Ex}} \stackrel{\mathrm{tex}}{\to}$ “\langle #1.
{\equiv}^∗ #2.
| #3.
{:=} #4.
\rangle_{Ex} ”]

$[\mathrm{ph}_1 \stackrel{\mathrm{tex}}{\to}$ “ph_{1}”]

$[\mathrm{ph}_2 \stackrel{\mathrm{tex}}{\to}$ “ph_{2}”]

$[\mathrm{ph}_3 \stackrel{\mathrm{tex}}{\to}$ “ph_{3}”]

$[x_{\mathrm{Ph}} \stackrel{\mathrm{tex}}{\to}$ “#1.
_{Ph} ”]

$[x^{\mathrm{Ph}} \stackrel{\mathrm{tex}}{\to}$ “#1.
^{Ph}”]

$[\langle x{\equiv}y|z{:=}u\rangle_{\mathrm{Ph}} \stackrel{\mathrm{tex}}{\to}$ “\langle #1.
{\equiv} #2.
| #3.
{:=} #4.
\rangle_{Ph} ”]

$[\langle x{\equiv}^0 y|z{:=}u\rangle_{\mathrm{Ph}} \stackrel{\mathrm{tex}}{\to}$ “\langle #1.
{\equiv}^0 #2.
| #3.
{:=} #4.
\rangle_{Ph} ”]

$[\langle x{\equiv}^1 y|z{:=}u\rangle_{\mathrm{Ph}} \stackrel{\mathrm{tex}}{\to}$ “\langle #1.
{\equiv}^1 #2.
| #3.
{:=} #4.
\rangle_{Ph} ”]

$[\langle x{\equiv}^* y|z{:=}u\rangle_{\mathrm{Ph}} \stackrel{\mathrm{tex}}{\to}$ “\langle #1.
{\equiv}^∗ #2.
| #3.

{:=} #4.
\rangle_{Ph} ”]

[$\mathsf{bs} \stackrel{\text{tex}}{\to}$ “\mathsf {bs}”]

[$\mathsf{OBS} \stackrel{\text{tex}}{\to}$ “ \mathsf {OBS}”]

[$\mathcal{BS} \stackrel{\text{tex}}{\to}$ “{\cal BS}”]

[Ø $\stackrel{\text{tex}}{\to}$ “\mathrm{\O}”]

[ZFsub $\stackrel{\text{tex}}{\to}$ “ZFsub”]

[MP $\stackrel{\text{tex}}{\to}$ “MP”]

[Gen $\stackrel{\text{tex}}{\to}$ “Gen”]

[Repetition $\stackrel{\text{tex}}{\to}$ “Repetition”]

[Neg $\stackrel{\text{tex}}{\to}$ “Neg”]

[Ded $\stackrel{\text{tex}}{\to}$ “Ded”]

[ExistIntro $\stackrel{\text{tex}}{\to}$ “ExistIntro”]

[Extensionality $\stackrel{\text{tex}}{\to}$ “Extensionality”]

[Ødef $\stackrel{\text{tex}}{\to}$ “\O{}def”]

[PairDef $\stackrel{\text{tex}}{\to}$ “PairDef”]

[UnionDef $\stackrel{\text{tex}}{\to}$ “UnionDef”]

[PowerDef $\stackrel{\text{tex}}{\to}$ “PowerDef”]

[SeparationDef $\stackrel{\text{tex}}{\to}$ “SeparationDef”]

[CheatAllDisjoint $\stackrel{\text{tex}}{\to}$ “CheatAllDisjoint”]

[AddDoubleNeg $\stackrel{\text{tex}}{\to}$ “AddDoubleNeg”]

[RemoveDoubleNeg $\stackrel{\text{tex}}{\to}$ “RemoveDoubleNeg”]

[AndCommutativity $\stackrel{\text{tex}}{\to}$ “AndCommutativity”]

[AutoImply $\stackrel{\text{tex}}{\to}$ “AutoImply”]

[Contrapositive $\stackrel{\text{tex}}{\to}$ “Contrapositive”]

[FirstConjunct $\stackrel{\text{tex}}{\rightarrow}$ "FirstConjunct"]

[SecondConjunct $\stackrel{\text{tex}}{\rightarrow}$ "SecondConjunct"]

[FromContradiction $\stackrel{\text{tex}}{\rightarrow}$ "FromContradiction"]

[FromDisjuncts $\stackrel{\text{tex}}{\rightarrow}$ "FromDisjuncts"]

[IffCommutativity $\stackrel{\text{tex}}{\rightarrow}$ "IffCommutativity"]

[IffFirst $\stackrel{\text{tex}}{\rightarrow}$ "IffFirst"]

[IffSecond $\stackrel{\text{tex}}{\rightarrow}$ "IffSecond"]

[ImplyTransitivity $\stackrel{\text{tex}}{\rightarrow}$ "ImplyTransitivity"]

[JoinConjuncts $\stackrel{\text{tex}}{\rightarrow}$ "JoinConjuncts"]

[MP2 $\stackrel{\text{tex}}{\rightarrow}$ "MP2"]

[MP3 $\stackrel{\text{tex}}{\rightarrow}$ "MP3"]

[MP4 $\stackrel{\text{tex}}{\rightarrow}$ "MP4"]

[MP5 $\stackrel{\text{tex}}{\rightarrow}$ "MP5"]

[MT $\stackrel{\text{tex}}{\rightarrow}$ "MT"]

[NegativeMT $\stackrel{\text{tex}}{\rightarrow}$ "NegativeMT"]

[Technicality $\stackrel{\text{tex}}{\rightarrow}$ "Technicality"]

[Weakening $\stackrel{\text{tex}}{\rightarrow}$ "Weakening"]

[WeakenOr1 $\stackrel{\text{tex}}{\rightarrow}$ "WeakenOr1"]

[WeakenOr2 $\stackrel{\text{tex}}{\rightarrow}$ "WeakenOr2"]

[Pair2Formula $\stackrel{\text{tex}}{\rightarrow}$ "Pair2Formula"]

[Formula2Pair $\stackrel{\text{tex}}{\rightarrow}$ "Formula2Pair"]

[Union2Formula $\stackrel{\text{tex}}{\rightarrow}$ "Union2Formula"]

[Formula2Union $\stackrel{\text{tex}}{\rightarrow}$ "Formula2Union"]

[Sep2Formula $\stackrel{\text{tex}}{\rightarrow}$ "Sep2Formula"]

[Formula2Sep $\stackrel{\text{tex}}{\rightarrow}$ "Formula2Sep"]

[SubsetInPower $\overset{\text{tex}}{\to}$ "SubsetInPower"]

[HelperPowerIsSub $\overset{\text{tex}}{\to}$ "HelperPowerIsSub"]

[PowerIsSub $\overset{\text{tex}}{\to}$ "PowerIsSub"]

[(Switch)HelperPowerIsSub $\overset{\text{tex}}{\to}$ "(Switch)HelperPowerIsSub"]

[(Switch)PowerIsSub $\overset{\text{tex}}{\to}$ "(Switch)PowerIsSub"]

[ToSetEquality $\overset{\text{tex}}{\to}$ "ToSetEquality"]

[HelperToSetEquality(t) $\overset{\text{tex}}{\to}$ "HelperToSetEquality(t)"]

[ToSetEquality(t) $\overset{\text{tex}}{\to}$ "ToSetEquality(t)"]

[HelperFromSetEquality $\overset{\text{tex}}{\to}$ "HelperFromSetEquality"]

[FromSetEquality $\overset{\text{tex}}{\to}$ "FromSetEquality"]

[HelperReflexivity $\overset{\text{tex}}{\to}$ "HelperReflexivity"]

[Reflexivity $\overset{\text{tex}}{\to}$ "Reflexivity"]

[HelperSymmetry $\overset{\text{tex}}{\to}$ "HelperSymmetry"]

[Symmetry $\overset{\text{tex}}{\to}$ "Symmetry"]

[HelperTransitivity $\overset{\text{tex}}{\to}$ "HelperTransitivity"]

[Transitivity $\overset{\text{tex}}{\to}$ "Transitivity"],

[ERisReflexive $\overset{\text{tex}}{\to}$ "ERisReflexive"]

[ERisSymmetric $\overset{\text{tex}}{\to}$ "ERisSymmetric"]

[ERisTransitive $\overset{\text{tex}}{\to}$ "ERisTransitive"]

[ØisSubset $\overset{\text{tex}}{\to}$ "\O{}isSubset"]

[HelperMemberNotØ $\overset{\text{tex}}{\to}$ "HelperMemberNot\O{}"]

[MemberNotØ $\overset{\text{tex}}{\to}$ "MemberNot\O{}"]

[HelperUniqueØ $\overset{\text{tex}}{\to}$ "HelperUnique\O{}"]

[UniqueØ $\overset{\text{tex}}{\to}$ "Unique\O{}"]

[=Reflexivity $\overset{\text{tex}}{\to}$ "=\!{}Reflexivity"]

[=Symmetry $\overset{\text{tex}}{\to}$ "=\!{}Symmetry"]

[Helper=Transitivity $\overset{\text{tex}}{\to}$ "Helper\!{}=\!{}Transitivity"]

[=Transitivity $\overset{\text{tex}}{\to}$ "\!{}=\!{}Transitivity"]

[HelperTransferNotEq $\overset{\text{tex}}{\to}$ "HelperTransferNotEq"]

[TransferNotEq $\overset{\text{tex}}{\to}$ "TransferNotEq"]

[HelperPairSubset $\overset{\text{tex}}{\to}$ "HelperPairSubset"]

[Helper(2)PairSubset $\overset{\text{tex}}{\to}$ "Helper(2)PairSubset"]

[PairSubset $\overset{\text{tex}}{\to}$ "PairSubset"]

[SamePair $\overset{\text{tex}}{\to}$ "SamePair"]

[SameSingleton $\overset{\text{tex}}{\to}$ "SameSingleton"]

[UnionSubset $\overset{\text{tex}}{\to}$ "UnionSubset"]

[SameUnion $\overset{\text{tex}}{\to}$ "SameUnion"]

[SeparationSubset $\overset{\text{tex}}{\to}$ "SeparationSubset"]

[SameSeparation $\overset{\text{tex}}{\to}$ "SameSeparation"]

[SameBinaryUnion $\overset{\text{tex}}{\to}$ "SameBinaryUnion"]

[IntersectionSubset $\overset{\text{tex}}{\to}$ "IntersectionSubset"]

[SameIntersection $\overset{\text{tex}}{\to}$ "SameIntersection"]

[AutoMember $\overset{\text{tex}}{\to}$ "AutoMember"]

[HelperEqSysNotØ $\overset{\text{tex}}{\to}$ "HelperEqSysNot\O{}"]

[EqSysNotØ $\overset{\text{tex}}{\to}$ "EqSysNot\O{}"]

[HelperEqSubset $\overset{\text{tex}}{\to}$ "HelperEqSubset"]

[EqSubset $\overset{\text{tex}}{\to}$ "EqSubset"]

[EqNecessary $\overset{\text{tex}}{\to}$ "EqNecessary"]

[HelperEqNecessary $\overset{\text{tex}}{\to}$ "HelperEqNecessary"]

[HelperNoneEqNecessary $\overset{\text{tex}}{\to}$ "HelperNoneEqNecessary"]

[Helper(2)NoneEqNecessary $\stackrel{\text{tex}}{\to}$ “Helper(2)NoneEqNecessary”]

[NoneEqNecessary $\stackrel{\text{tex}}{\to}$ “NoneEqNecessary”]

[EqClassIsSubset $\stackrel{\text{tex}}{\to}$ “EqClassIsSubset”]

[EqClassesAreDisjoint $\stackrel{\text{tex}}{\to}$ “EqClassesAreDisjoint”]

[AllDisjoint $\stackrel{\text{tex}}{\to}$ “AllDisjoint”]

[AllDisjointImply $\stackrel{\text{tex}}{\to}$ “AllDisjointImply”]

[BSsubset $\stackrel{\text{tex}}{\to}$ “BSsubset”]

[Union(BS/R)subset $\stackrel{\text{tex}}{\to}$ “Union(BS/R)subset”]

[UnionIdentity $\stackrel{\text{tex}}{\to}$ “UnionIdentity”]

[EqSysIsPartition $\stackrel{\text{tex}}{\to}$ “EqSysIsPartition”]

[x/y $\stackrel{\text{tex}}{\to}$ “#1.
/ #2.”]

[$\mathsf{x} \cap \mathsf{y}$ $\stackrel{\text{tex}}{\to}$ “#1.
\cap #2.”]

[$\cup \mathsf{x}$ $\stackrel{\text{tex}}{\to}$ “\cup #1.”]

[$\mathsf{x} \cup \mathsf{y}$ $\stackrel{\text{tex}}{\to}$ “#1.
\mathrel{\cup} #2.”]

[P(x) $\stackrel{\text{tex}}{\to}$ “P(#1.
)”]

[$\{\mathsf{x}\}$ $\stackrel{\text{tex}}{\to}$ “\{#1.
\}”]

[$\{\mathsf{x}, \mathsf{y}\}$ $\stackrel{\text{tex}}{\to}$ “\{#1.
, #2.
\}”]

[$\langle \mathsf{x}, \mathsf{y} \rangle$ $\stackrel{\text{tex}}{\to}$ “\langle #1.
, #2.
\rangle”],

[$\mathsf{x} \in \mathsf{y}$ $\stackrel{\text{tex}}{\to}$ “#1.
\mathrel{\in} #2.”]

$[\mathsf{z}(\mathsf{x},\mathsf{y}) \stackrel{\text{tex}}{\to}$ “#3.
(#1.
, #2.
)”]

$[\mathrm{ReflRel}(\mathsf{r},\mathsf{x}) \stackrel{\text{tex}}{\to}$ “ReflRel(#1.
, #2.
)”]

$[\mathrm{SymRel}(\mathsf{r},\mathsf{x}) \stackrel{\text{tex}}{\to}$ “SymRel(#1.
, #2.
)”]

$[\mathrm{TransRel}(\mathsf{r},\mathsf{x}) \stackrel{\text{tex}}{\to}$ “TransRel(#1.
, #2.
)”]

$[\mathrm{EqRel}(\mathsf{r},\mathsf{x}) \stackrel{\text{tex}}{\to}$ “EqRel(#1.
, #2.
)”]

$[[\mathsf{x} \in \mathsf{bs}]_{\mathsf{r}} \stackrel{\text{tex}}{\to}$ “[#1.
\mathrel{\in} #2.
]_{#3.
}”]

$[\mathrm{Partition}(\mathsf{x},\mathsf{y}) \stackrel{\text{tex}}{\to}$ “Partition(#1.
, #2.
)”]

$[\mathsf{x}{=}\mathsf{y} \stackrel{\text{tex}}{\to}$ “#1.
\!\mathrel{=}\! #2.”]

$[\mathsf{x} \subseteq \mathsf{y} \stackrel{\text{tex}}{\to}$ “#1.
\mathrel{\subseteq} #2.”]

$[\dot{\neg}\,\mathsf{x} \stackrel{\text{tex}}{\to}$ “\dot{\neg}\, #1.”]

$[\mathsf{x} \notin \mathsf{y} \stackrel{\text{tex}}{\to}$ “#1.
\mathrel{\notin} #2.”]

$[\mathsf{x} \neq \mathsf{y} \stackrel{\text{tex}}{\to}$ “#1.
\mathrel{\neq} #2.”]

$[\mathsf{x} \mathrel{\dot{\wedge}} \mathsf{y} \stackrel{\text{tex}}{\to}$ “#1.
\mathrel{\dot{\wedge}} #2.”]

[$\mathsf{x} \mathrel{\dot{\vee}} \mathsf{y}$ $\overset{\text{tex}}{\rightarrow}$ "#1.
\mathrel{\dot{\vee}} #2."]

[$\mathsf{x} \mathrel{\dot{\Leftrightarrow}} \mathsf{y}$ $\overset{\text{tex}}{\rightarrow}$ "#1.
\mathrel{\dot{\Leftrightarrow}} #2."]

[$\{\mathrm{ph} \in \mathsf{x} \mid \mathsf{a}\}$ $\overset{\text{tex}}{\rightarrow}$ " \{ ph \mathrel{\in} #1.
\mid #2.
\}"]