(*** MAKROER BEGYNDER ***)

1 Makrodefinitioner

Dette afsnit indeholder dé makrodefinitioner, som vi vil gøre brug af i resten af rapporten. Definitionerne drejer sig for det meste om mængdeteoretiske begreber, f.eks. “ækvivalensklasse” og “partition”. Til sidst i afsnittet formulerer vi hovedresultatet — at der til enhver ækvivalensrelation svarer en partition — som et formelt teorem.

1.1 Konnektiver

Ud fra de to basale konnektiver $[\dot{\neg}\,\mathsf{x}]$ og $[\mathsf{x} \Rightarrow \mathsf{y}]$ definerer vi konjunktion, disjunktion og dobbeltimplikation:

$$[\mathsf{x} \mathrel{\dot{\wedge}} \mathsf{y} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \mathrel{\dot{\wedge}} \mathsf{y} \mathrel{\ddot{=}} \dot{\neg}\,(\mathsf{x} \Rightarrow \dot{\neg}\,\mathsf{y})]\rceil)]$$

$$[\mathsf{x} \mathrel{\dot{\vee}} \mathsf{y} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \mathrel{\dot{\vee}} \mathsf{y} \mathrel{\ddot{=}} \dot{\neg}\,\mathsf{x} \Rightarrow \mathsf{y}]\rceil)]$$

$$[\mathsf{x} \mathrel{\dot{\Leftrightarrow}} \mathsf{y} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \mathrel{\dot{\Leftrightarrow}} \mathsf{y} \mathrel{\ddot{=}} (\mathsf{x} \Rightarrow \mathsf{y}) \mathrel{\dot{\wedge}} (\mathsf{y} \Rightarrow \mathsf{x})]\rceil)]$$

1.2 Negerede formler

Det er ganske enkelt at definere negeret lighed $(\dot{\neg}\,\mathsf{x}{=}{=}\mathsf{y})$ og negeret medlemskab $(\dot{\neg}\,\mathsf{x} \in \mathsf{y})$:

$$[\mathsf{x} \neq \mathsf{y} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \neq \mathsf{y} \mathrel{\ddot{=}} \dot{\neg}\,\mathsf{x}{=}{=}\mathsf{y}]\rceil)]$$

$$[\mathsf{x} \notin \mathsf{y} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \notin \mathsf{y} \mathrel{\ddot{=}} \dot{\neg}\,\mathsf{x} \in \mathsf{y}]\rceil)]$$ [1]

1.3 Delmængde

Mængden x er en delmængde af y hviss ethvert medlem af x også tilhører y:

$$[\mathsf{x} \subseteq \mathsf{y} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \subseteq \mathsf{y} \mathrel{\ddot{=}} (s \in \mathsf{x} \Rightarrow s \in \mathsf{y})]\rceil)]$$

1.4 Singleton-mængde

$[\{\mathsf{x},\mathsf{x}\}]$ er mængden, der indeholder x som sit eneste element. Vi definerer $[\{\mathsf{x},\mathsf{x}\}]$ ved at parre x med sig selv:

$$[\{\mathsf{x}\} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\{\mathsf{x}\} \mathrel{\ddot{=}} \{\mathsf{x},\mathsf{x}\}]\rceil)]$$

1.5 Binær foreningsmængde og fællesmængde

Vi definerer foreningsmængden mellem to mængder x og y som følger:

$$[\mathsf{x} \cup \mathsf{y} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \cup \mathsf{y} \mathrel{\ddot{=}} \cup\{\{\mathsf{x}\},\{\mathsf{y}\}\}]\rceil)]$$

[1] Højresiderne i disse definitioner skal læses som hhv. $[\dot{\neg}\,\mathsf{x}{=}{=}\mathsf{y}]$ og $[\dot{\neg}\,\mathsf{x} \in \mathsf{y}]$.

Fællesmængden mellem to mængder x og y er en delmængde af deres foreningsmængde:

$$[\mathsf{x}{\cap}\mathsf{y} \stackrel{\text{macro}}{\to} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x}{\cap}\mathsf{y} \ddot{=} \{\mathrm{ph} \in \mathsf{x} \cup \mathsf{y} \mid \mathrm{ph}_3 \in \mathsf{x} \dot{\wedge} \mathrm{ph}_3 \in \mathsf{y}\}]\rceil)]$$

1.6 Relation

Det ordnede par $\{\{\mathsf{x},\mathsf{x}\},\{\mathsf{x},\mathsf{y}\}\}$ indeholder x som "førstekomponent" og y som "andenkomponent". Den følgende definition af $\{\{\mathsf{x},\mathsf{x}\},\{\mathsf{x},\mathsf{y}\}\}$ er den mest udbredte i litteraturen (se f.eks. afsnit 4.3 i [**?**] og afsnit 2.1 i [**?**]):

$$[\langle\mathsf{x},\mathsf{y}\rangle \stackrel{\text{macro}}{\to} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\langle\mathsf{x},\mathsf{y}\rangle \ddot{=} \{\{\mathsf{x}\},\{\mathsf{x},\mathsf{y}\}\}]\rceil)]$$

Vi kan nu definere en "relation" som en mængde af ordnede par. Vi udtrykker denne definition ved at formalisere, hvad det vil sige, at x er relateret til y i kraft af relationen r:

$$[\mathsf{r}(\mathsf{x},\mathsf{y}) \stackrel{\text{macro}}{\to} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{r}(\mathsf{x},\mathsf{y}) \ddot{=} \langle\mathsf{x},\mathsf{y}\rangle \in \mathsf{r}]\rceil)]$$

Vi kommer faktisk ikke til at bruge disse to definitioner i rapporten; beviserne vil behandle $[\{\{\mathsf{x},\mathsf{x}\},\{\mathsf{x},\mathsf{y}\}\} \in \mathsf{r}]$ som en primitiv konstruktion. Men det er alligevel betryggende at have det formelle grundlag for relationsbegrebet på plads.

1.7 Ækvivalensrelation

At en relation er refleksiv på en mængde x vil sige, at alle elementer i x er relateret til sig selv:

$$[\mathrm{ReflRel}(\mathsf{r},\mathsf{x}) \overset{\mathrm{macro}}{\to} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathrm{ReflRel}(\mathsf{r},\mathsf{x}) \ddot{=} \forall s\colon (s \in \mathsf{x} \Rightarrow \mathsf{r}(s,s))]\rceil)]$$

At en relation er symmetrisk på en mængde x vil sige, at alle elementer i x opfylder den følgende implikation:

$$[\mathrm{SymRel}(\mathsf{r},\mathsf{x}) \overset{\mathrm{macro}}{\to} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathrm{SymRel}(\mathsf{r},\mathsf{x}) \ddot{=} \forall s, t\colon (s \in \mathsf{x} \Rightarrow t \in \mathsf{x} \Rightarrow \mathsf{r}(s,t) \Rightarrow \mathsf{r}(t,s))]\rceil)]$$

At en relation er transitiv på en mængde x vil sige, at alle elementer i x opfylder den følgende implikation:

$$[\mathrm{TransRel}(\mathsf{r},\mathsf{x}) \overset{\mathrm{macro}}{\to} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathrm{TransRel}(\mathsf{r},\mathsf{x}) \ddot{=} \forall s, t, u\colon (s \in \mathsf{x} \Rightarrow t \in \mathsf{x} \Rightarrow u \in \mathsf{x} \Rightarrow \mathsf{r}(s,t) \Rightarrow \mathsf{r}(t,u) \Rightarrow \mathsf{r}(s,u))]\rceil)]$$

Endelig er en ækvivalensrelation det samme som en relation, der er refleksiv, symmetrisk og transitiv:

$$[\mathrm{EqRel}(\mathsf{r},\mathsf{x}) \overset{\mathrm{macro}}{\to} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathrm{EqRel}(\mathsf{r},\mathsf{x}) \ddot{=} \mathrm{ReflRel}(\mathsf{r},\mathsf{x}) \dot{\wedge} \mathrm{SymRel}(\mathsf{r},\mathsf{x}) \dot{\wedge} \mathrm{TransRel}(\mathsf{r},\mathsf{x})]\rceil)]$$

1.8 Mængde-variable

Mange af rapportens beviser sker i forhold til en uspecificeret mængde. Vi vil referere til denne mængde med metavariablen $\underline{\mathsf{bs}}$ og objektvariablen $\overline{\mathsf{bs}}$:

$$[\mathcal{BS} \overset{\mathrm{macro}}{\to} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathcal{BS} \ddot{=} \underline{\mathsf{bs}}]\rceil)]$$

$$[\mathsf{OBS} \overset{\mathrm{macro}}{\to} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{OBS} \ddot{=} \overline{\mathsf{bs}}]\rceil)]^2$$

Vi vil så vidt muligt bruge metavariablen, men i afsnit **??** og senere bliver det nødvendigt at gå over til objektvariablen.

1.9 Ækvivalensklasse

Lad r være en ækvivalensrelation defineret på bs, og lad x være et medlem af bs. Vi definerer ækvivalensklassen $\{\mathrm{ph} \in \mathsf{bs} \mid \{\{\mathrm{ph}_1, \mathrm{ph}_1\}, \{\mathrm{ph}_1, \mathsf{x}\}\} \in \mathsf{r}\}$ som den delmængde af bs, hvis medlemmer står i forhold til x:

$$[[\mathsf{x} \in \mathsf{bs}]_{\mathsf{r}} \overset{\mathrm{macro}}{\to} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[[\mathsf{x} \in \mathsf{bs}]_{\mathsf{r}} \ddot{=} \{\mathrm{ph} \in \mathsf{bs} \mid \mathsf{r}(\mathrm{ph}_1,\mathsf{x})\}]\rceil)]$$

[2] Navnene "$\underline{\mathsf{bs}}$" og "$\overline{\mathsf{bs}}$" står for hhv. for "big set" og "object big set". Konstruktionerne $[\underline{\mathsf{x}}]$ og $[\overline{\mathsf{x}}]$ omdanner x til hhv. en meta- og en objektvariabel. Variablen $[\mathsf{bs}]$ vil også blive brugt i nogle af de kommende definitioner, men ikke i selve beviserne.

Ækvivalenssystemet $\{\mathrm{ph} \in \mathrm{P}(\mathsf{bs}) \mid \dot{\neg}\, \mathsf{t}_{\mathrm{Ex}} \in \mathsf{bs} \Rightarrow \dot{\neg}\, \{\mathrm{ph} \in \mathsf{bs} \mid \{\{\mathrm{ph}_1, \mathrm{ph}_1\}, \{\mathrm{ph}_1, \mathsf{t}_{\mathrm{Ex}}\}\} \in \mathsf{r}\} == \mathrm{ph}_2\}$ er mængden af alle de ækvivalensklasser, som bs definerer på r. Vi definerer $\{\mathrm{ph} \in \mathrm{P}(\mathsf{bs}) \mid \dot{\neg}\, \mathsf{t}_{\mathrm{Ex}} \in \mathsf{bs} \Rightarrow \dot{\neg}\, \{\mathrm{ph} \in \mathsf{bs} \mid \{\{\mathrm{ph}_1, \mathrm{ph}_1\}, \{\mathrm{ph}_1, \mathsf{t}_{\mathrm{Ex}}\}\} \in \mathsf{r}\} == \mathrm{ph}_2\}$ som en delmængde af potensmængden $\mathrm{P}(\mathsf{bs})$:

$$[\mathsf{bs}/\mathsf{r} \stackrel{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil [\mathsf{bs}/\mathsf{r} \stackrel{..}{=} \{\mathrm{ph} \in \mathrm{P}(\mathsf{bs}) \mid \mathrm{Ex}_{20} \in \mathsf{bs} \dot{\wedge} [\mathrm{Ex}_{20} \in \mathsf{bs}]_{\mathsf{r}} == \mathrm{ph}_2\}] \rceil)]$$

1.10 Partition

En partition af en mængde bs er en mængde p, som opfylder tre krav:

1. Ingen af mængderne i p er tomme.
2. Alle mængderne i p er indbyrdes disjunkte.
3. Foreningsmængden af alle mængderne i p er lig med bs.

Den formelle version af denne definition ser således ud:

$$[\text{Partition}(\mathsf{p},\mathsf{bs}) \stackrel{\text{macro}}{\to} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\text{Partition}(\mathsf{p},\mathsf{bs}) \ddot{=} (\forall s\colon (s \in \mathsf{p} \Rightarrow s \neq \emptyset))\, \dot{\wedge}$$
$$(\forall s,t\colon (s \in \mathsf{p} \Rightarrow t \in \mathsf{p} \Rightarrow s \neq t \Rightarrow s \cap t {=}{=} \emptyset))\, \dot{\wedge}$$
$$\cup\, \mathsf{p} {=}{=} \mathsf{bs}]\rceil)]$$

(*** MAKROER SLUTTER ***)

2 Deduktionsreglen

Dette bilag præsenterer dén version af deduktionsreglen fra [?], som jeg har gjort brug af. Underafsnit **??** forklarer, hvorfor jeg har ændret på den oprindelige regel, og underafsnit 2.1 indeholder selve den ændrede kode (som er skrevet i L).

2.1 Kode

Funktionen $[\lambda\mathsf{x}.\text{Dedu}_0(\lceil\mathsf{p}\rceil,\lceil\mathsf{c}\rceil)]$ er en kopi af $[\lambda\mathsf{x}.\text{Ded}_0(\lceil\mathsf{p}\rceil,\lceil\mathsf{c}\rceil)]$ fra [?]:

$$[\text{Dedu}(\mathsf{p},\mathsf{c}) \stackrel{\text{macro}}{\to} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\text{Dedu}(\mathsf{p},\mathsf{c}) \ddot{=} \lambda\mathsf{x}.\text{Dedu}_0(\lceil\mathsf{p}\rceil,\lceil\mathsf{c}\rceil)]\rceil)]$$

Jeg har ændret funktionen $[\text{Ded}_0(\mathsf{p},\mathsf{c})]$, så den kalder $[\text{Dedu}_\text{s}(\text{Dedu}_7(\mathsf{p}),\mathsf{c},\mathsf{T})]$ i stedet for $[\text{Ded}_1(\text{Ded}_7(\mathsf{p}),\mathsf{c},\mathsf{T})]$:

$$[\text{Dedu}_0(\mathsf{p},\mathsf{c}) \stackrel{\text{val}}{\to} \mathsf{c}!\text{If}(\text{Dedu}_8(\mathsf{p},\mathsf{T}),\text{Dedu}_\text{s}(\text{Dedu}_7(\mathsf{p}),\mathsf{c},\mathsf{T}),\mathsf{F})]$$

Funktionen $[\text{Dedu}_\text{s}(\mathsf{p},\mathsf{c},\mathsf{s})]$ giver straks kontrollen videre til $[\text{Ded}_1(\mathsf{p},\mathsf{c},\mathsf{s})]$ — medmindre p og c begynder med et antal identiske sidebetingelser. I så fald flyttes disse sidebetingelser fra p og c over til listen s, før kontrollen går videre til $[\text{Ded}_1(\mathsf{p},\mathsf{c},\mathsf{s})]$:

$$[\text{Dedu}_\text{s}(\mathsf{p},\mathsf{c},\mathsf{s}) \stackrel{\text{val}}{\to} \text{If}(\mathsf{p} \stackrel{\text{r}}{=} \lceil\mathsf{x} \Vdash \mathsf{y}\rceil, \mathsf{c} \stackrel{\text{r}}{=} \lceil\mathsf{x} \Vdash \mathsf{y}\rceil \wedge \mathsf{p}^1 \stackrel{\text{t}}{=} \mathsf{c}^1 \wedge \text{Dedu}_\text{s}(\mathsf{p}^2,\mathsf{c}^2,\mathsf{c}^1 :: \mathsf{s}),$$
$$\text{Dedu}_1(\mathsf{p},\mathsf{c},\mathsf{s}))]$$

Fra og med $[\text{Ded}_1(\mathsf{p},\mathsf{c},\mathsf{s})]$ er koden kopieret fra appendikset til [?]:

$$[\text{Dedu}_1(\mathsf{p},\mathsf{c},\mathsf{s}) \stackrel{\text{val}}{\to} \text{If}(\mathsf{c} \stackrel{\text{r}}{=} \lceil\mathsf{x} \Vdash \mathsf{y}\rceil, \text{Dedu}_1(\mathsf{p},\mathsf{c}^2,\mathsf{c}^1 :: \mathsf{s}), \text{Dedu}_2(\mathsf{p},\mathsf{c},\mathsf{s}))]$$

$[\mathrm{Dedu}_2(\mathsf{p},\mathsf{c},\mathsf{s}) \overset{\mathrm{val}}{\to} \mathsf{s}!\mathsf{p} \overset{\mathrm{r}}{\doteq} \lceil \mathsf{x} \vdash \mathsf{y} \rceil \wedge \mathsf{c} \overset{\mathrm{r}}{\doteq} \lceil \mathsf{x} \Rightarrow \mathsf{y} \rceil \left\{ \begin{array}{l} \mathrm{Dedu}_3(\mathsf{p}^1,\mathsf{c}^1,\mathsf{s},\mathsf{T}) \wedge \mathrm{Dedu}_2(\mathsf{p}^2,\mathsf{c}^2,\mathsf{s}) \\ \mathrm{Dedu}_4(\mathsf{p},\mathsf{c},\mathsf{s},\mathrm{Dedu}_6(\mathsf{p},\mathsf{c},\mathsf{T},\mathsf{T})) \end{array} \right.]$

$[\mathrm{Dedu}_3(\mathsf{p},\mathsf{c},\mathsf{s},\mathsf{b}) \overset{\mathrm{val}}{\to} \mathrm{If}(\neg \mathsf{c} \overset{\mathrm{r}}{\doteq} \lceil \forall_{\mathrm{obj}} \mathsf{x}\colon \mathsf{y} \rceil, \mathrm{Dedu}_4(\mathsf{p},\mathsf{c},\mathsf{s},\mathsf{b}),$
$\mathrm{If}(\mathsf{p} \overset{\mathrm{r}}{\doteq} \lceil \forall_{\mathrm{obj}} \mathsf{x}\colon \mathsf{y} \rceil \wedge \mathsf{p}^1 \overset{\mathrm{t}}{\doteq} \mathsf{c}^1, \mathrm{Dedu}_4(\mathsf{p},\mathsf{c},\mathsf{s},\mathsf{b}), \mathrm{Dedu}_3(\mathsf{p},\mathsf{c}^2,\mathsf{s},\mathsf{c}^1 :: \mathsf{c}^1 :: \mathsf{b})))]$

$[\mathrm{Dedu}_4(\mathsf{p},\mathsf{c},\mathsf{s},\mathsf{b}) \overset{\mathrm{val}}{\to} \mathsf{s}!\mathsf{b}!\mathrm{If}(\mathsf{p} \overset{\mathrm{r}}{\doteq} \lceil \overline{\mathsf{x}} \rceil, \mathbf{lookup}(\mathsf{p},\mathsf{b},\mathsf{T}) \overset{\mathrm{t}}{\doteq} \mathsf{c}, \mathrm{If}(\neg \mathsf{p} \overset{\mathrm{r}}{\doteq} \mathsf{c}, \mathsf{F},$
$\mathrm{If}(\mathsf{p} \overset{\mathrm{r}}{\doteq} \lceil \forall_{\mathrm{obj}} \mathsf{x}\colon \mathsf{y} \rceil, \mathsf{p}^1 \overset{\mathrm{t}}{\doteq} \mathsf{c}^1 \wedge \mathrm{Dedu}_4(\mathsf{p}^2,\mathsf{c}^2,\mathsf{s},\mathsf{p}^1 :: \mathsf{p}^1 :: \mathsf{b}), \mathrm{If}(\neg \mathsf{p} \overset{\mathrm{r}}{\doteq} \lceil \underline{\mathsf{x}} \rceil,$
$\mathrm{Dedu}_4^*(\mathsf{p}^{\mathrm{t}},\mathsf{c}^{\mathrm{t}},\mathsf{s},\mathsf{b}), \mathsf{p}^1 \overset{\mathrm{t}}{\doteq} \mathsf{c}^1 \wedge \mathrm{Dedu}_5(\mathsf{p},\mathsf{s},\mathsf{b})))))]$

$[\mathrm{Dedu}_4^*(\mathsf{p},\mathsf{c},\mathsf{s},\mathsf{b}) \overset{\mathrm{val}}{\to} \mathsf{c}!\mathsf{s}!\mathsf{b}!\mathrm{If}(\mathsf{p},\mathsf{T},\mathrm{Dedu}_4(\mathsf{p}^{\mathrm{h}},\mathsf{c}^{\mathrm{h}},\mathsf{s},\mathsf{b}) \wedge \mathrm{Dedu}_4^*(\mathsf{p}^{\mathrm{t}},\mathsf{c}^{\mathrm{t}},\mathsf{s},\mathsf{b}))]$

$[\mathrm{Dedu}_5(\mathsf{p},\mathsf{s},\mathsf{b}) \overset{\mathrm{val}}{\to} \mathsf{p}!\mathsf{s}!\mathrm{If}(\mathsf{b},\mathsf{T},$
$\lceil\lceil \mathsf{x} \rceil \#^0 \lceil \mathsf{y} \rceil\rceil^{\mathrm{h}} :: \lceil\lceil * \rceil\rceil^{\mathrm{h}} :: \mathsf{b}^{\mathrm{hh}} :: \mathsf{T} :: \lceil\lceil \mathsf{x} \rceil\rceil^{\mathrm{h}} :: \mathsf{p} :: \mathsf{T} :: \mathsf{T} \in_{\mathrm{t}} \mathsf{s} \wedge \mathrm{Dedu}_5(\mathsf{p},\mathsf{s},\mathsf{b}^{\mathrm{t}}))]$

$[\mathrm{Dedu}_6(\mathsf{p},\mathsf{c},\mathsf{e},\mathsf{b}) \overset{\mathrm{val}}{\to} \mathsf{p}!\mathsf{c}!\mathsf{b}!\mathsf{e}!\mathrm{If}(\mathsf{p} \overset{\mathrm{r}}{\doteq} \lceil \overline{\mathsf{x}} \rceil, \mathsf{p} \in_{\mathrm{t}} \mathsf{e} \left\{ \begin{array}{l} \mathsf{b} \\ \mathsf{p} :: \mathsf{c} :: \mathsf{b} \end{array} \right., \mathrm{If}(\neg \mathsf{p} \overset{\mathrm{r}}{\doteq} \mathsf{c}, \mathsf{T},$
$\mathrm{If}(\mathsf{p} \overset{\mathrm{r}}{\doteq} \lceil \underline{\mathsf{a}} \rceil, \mathsf{b}, \mathrm{If}(\mathsf{p} \overset{\mathrm{r}}{\doteq} \lceil \forall_{\mathrm{obj}} \mathsf{x}\colon \mathsf{y} \rceil, \mathrm{Dedu}_6(\mathsf{p}^2,\mathsf{c}^2,\mathsf{c}^1 :: \mathsf{e},\mathsf{b}), \mathrm{Dedu}_6^*(\mathsf{p}^{\mathrm{t}},\mathsf{c}^{\mathrm{t}},\mathsf{e},\mathsf{b})))))]$

$[\mathrm{Dedu}_6^*(\mathsf{p},\mathsf{c},\mathsf{e},\mathsf{b}) \overset{\mathrm{val}}{\to} \mathsf{p}!\mathsf{c}!\mathsf{b}!\mathsf{e}!\mathrm{If}(\mathsf{p},\mathsf{b},\mathrm{Dedu}_6^*(\mathsf{p}^{\mathrm{t}},\mathsf{c}^{\mathrm{t}},\mathsf{e},\mathrm{Dedu}_6(\mathsf{p}^{\mathrm{h}},\mathsf{c}^{\mathrm{h}},\mathsf{e},\mathsf{b})))]$

$[\mathrm{Dedu}_7(\mathsf{p}) \overset{\mathrm{val}}{\to} \mathsf{p} \overset{\mathrm{r}}{\doteq} \lceil \forall \mathsf{x}\colon \mathsf{y} \rceil \left\{ \begin{array}{l} \mathrm{Dedu}_7(\mathsf{p}^2) \\ \mathsf{p} \end{array} \right.]$

$[\mathrm{Dedu}_8(\mathsf{p},\mathsf{b}) \overset{\mathrm{val}}{\to} \mathrm{If}(\mathsf{p} \overset{\mathrm{r}}{\doteq} \lceil \forall \mathsf{x}\colon \mathsf{y} \rceil, \mathrm{Dedu}_8(\mathsf{p}^2,\mathsf{p}^1 :: \mathsf{b}), \mathrm{If}(\mathsf{p} \overset{\mathrm{r}}{\doteq} \lceil \underline{\mathsf{a}} \rceil, \mathsf{p} \in_{\mathrm{t}} \mathsf{b},$
$\mathrm{Dedu}_8^*(\mathsf{p}^{\mathrm{t}},\mathsf{b})))]$

$[\mathrm{Dedu}_8^*(\mathsf{p},\mathsf{b}) \overset{\mathrm{val}}{\to} \mathsf{b}!\mathrm{If}(\mathsf{p},\mathsf{T},\mathrm{If}(\mathrm{Dedu}_8(\mathsf{p}^{\mathrm{h}},\mathsf{b}),\mathrm{Dedu}_8^*(\mathsf{p}^{\mathrm{t}},\mathsf{b}),\mathsf{F}))]$

(*** EKSISTENS-VARIABLE ***)

$$[\mathsf{x}^{\mathrm{Ex}} \overset{\mathrm{val}}{\to} \mathsf{x} \overset{\mathrm{r}}{\doteq} \lceil \mathsf{x}_{\mathrm{Ex}} \rceil]$$

Vi kan da definere de fire eksistens-variable, som denne rapport vil gøre brug af (jvf. bilag **??**):

$$[\mathrm{Ex}_1 \overset{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil [\mathrm{Ex}_1 \ddot{=} \mathsf{a}_{\mathrm{Ex}}] \rceil)]$$

$[\mathrm{Ex}_2 \overset{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil [\mathrm{Ex}_2 \ddot{=} \mathsf{b}_{\mathrm{Ex}}] \rceil)]$
$[\mathrm{Ex}_{10} \overset{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil [\mathrm{Ex}_{10} \ddot{=} \mathsf{j}_{\mathrm{Ex}}] \rceil)]$
$[\mathrm{Ex}_{20} \overset{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil [\mathrm{Ex}_{20} \ddot{=} \mathsf{t}_{\mathrm{Ex}}] \rceil)]$
$[\langle \mathsf{a} {\equiv} \mathsf{b} | \mathsf{x} {:==} \mathsf{t} \rangle_{\mathrm{Ex}} \overset{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil [\langle \mathsf{a} {\equiv} \mathsf{b} | \mathsf{x} {:==} \mathsf{t} \rangle_{\mathrm{Ex}} \ddot{=}$
$\langle \lceil \mathsf{a} \rceil {\equiv}^0 \lceil \mathsf{b} \rceil | \lceil \mathsf{x} \rceil {:==} \lceil \mathsf{t} \rceil \rangle_{\mathrm{Ex}}] \rceil)]$

$[\langle \mathsf{a} {\equiv}^0 \mathsf{b} | \mathsf{x} {:==} \mathsf{t} \rangle_{\mathrm{Ex}} \overset{\mathrm{val}}{\to} \lambda \mathsf{c}.\mathsf{x}^{\mathrm{Ex}} \wedge \langle \mathsf{a} {\equiv}^1 \mathsf{b} | \mathsf{x} {:==} \mathsf{t} \rangle_{\mathrm{Ex}}]$

$[\langle a\equiv^1 b|x{:}{=}{=}t\rangle_{\mathrm{Ex}} \stackrel{\mathrm{val}}{\to} a!x!t!$
$\mathrm{If}(b \stackrel{\mathrm{r}}{=} \lceil \forall_{\mathrm{obj}} u{:}\, v\rceil, F,$
$\mathrm{If}(b^{\mathrm{Ex}} \wedge b \stackrel{\mathrm{t}}{=} x, a \stackrel{\mathrm{t}}{=} t, \mathrm{If}($
$a \stackrel{\mathrm{r}}{=} b, \langle a^{\mathrm{t}}\equiv^* b^{\mathrm{t}}|x{:}{=}{=}t\rangle_{\mathrm{Ex}}, F)))]$

$[\langle a\equiv^* b|x{:}{=}{=}t\rangle_{\mathrm{Ex}} \stackrel{\mathrm{val}}{\to} b!x!t!\mathrm{If}(a, T, \mathrm{If}(\langle a^{\mathrm{h}}\equiv^1 b^{\mathrm{h}}|x{:}{=}{=}t\rangle_{\mathrm{Ex}}, \langle a^{\mathrm{t}}\equiv^* b^{\mathrm{t}}|x{:}{=}{=}t\rangle_{\mathrm{Ex}}, F))]$

(*** AKSIOMATISK SYSTEM ***)

$[\mathrm{ZFsub} \stackrel{\mathrm{stmt}}{\to} \forall\underline{x}{:}\forall\underline{y}{:}\forall\underline{z}{:}\, \underline{x}*\underline{y}*\underline{z} = \underline{x}*\underline{y}*\underline{z} \oplus \forall\overline{(fx)}{:}\forall\overline{(fy)}{:}\, R(\overline{(fx)}) << R(\overline{(fy)}) \vdash$
$\overline{(fx)} <_{\mathrm{f}} \overline{(fy)} \oplus \forall\underline{m}{:}\forall\overline{(\epsilon)}{:}\forall\overline{(fx)}{:}\forall\overline{(fy)}{:}\, \overline{(fx)} <_{\mathrm{f}} \overline{(fy)} \vdash \dot\neg 0 <= \overline{(\epsilon)} \Rightarrow \dot\neg\dot\neg 0 = \overline{(\epsilon)} \vdash$
$c_{\mathrm{Ex}} <= \underline{m} \Rightarrow \overline{(fx)}[\underline{m}] <= \overline{(fy)}[\underline{m}] + -\overline{(\epsilon)} \oplus \forall\overline{(rx)}{:}\forall\overline{(ry)}{:}\forall\overline{(rz)}{:}\, \overline{(rx)} == \overline{(ry)} \vdash$
$\overline{(ry)} == \overline{(rz)} \vdash \overline{(rx)} == \overline{(rz)} \oplus \forall\underline{x}{:}\forall\underline{y}{:}\forall\underline{z}{:}\, \underline{x} <= \underline{y} \Rightarrow \underline{y} <= \underline{z} \Rightarrow \underline{x} <= \underline{z} \oplus$
$\forall\underline{x}{:}\forall\underline{t}{:}\forall\underline{a}{:}\forall\underline{b}{:}\, \langle\lceil\underline{a}\rceil\equiv^0\lceil\underline{b}\rceil|\lceil\underline{x}\rceil{:}{=}{=}\lceil\underline{t}\rceil\rangle_{\mathrm{Ex}} \Vdash \underline{a} \vdash \underline{b} \oplus \forall\underline{m}{:}\forall\overline{(\epsilon)}{:}\forall\overline{(fx)}{:}\forall\overline{(fy)}{:}\, \dot\neg 0 <=$
$\overline{(\epsilon)} \Rightarrow \dot\neg\dot\neg 0 = \overline{(\epsilon)} \Rightarrow c_{\mathrm{Ex}} <= \underline{m} \Rightarrow \dot\neg\,\mathrm{if}(0 <=$
$\overline{(fx)}[\underline{m}] + -\overline{(fy)}[\underline{m}], \overline{(fx)}[\underline{m}] + -\overline{(fy)}[\underline{m}], -\overline{(fx)}[\underline{m}] + -\overline{(fy)}[\underline{m}]) <= \overline{(\epsilon)} \Rightarrow$
$\dot\neg\dot\neg\,\mathrm{if}(0 <= \overline{(fx)}[\underline{m}] + -\overline{(fy)}[\underline{m}], \overline{(fx)}[\underline{m}] + -\overline{(fy)}[\underline{m}], -\overline{(fx)}[\underline{m}] + -\overline{(fy)}[\underline{m}]) = \overline{(\epsilon)} \vdash$
$\mathrm{SF}(\overline{(fx)}, \overline{(fy)}) \oplus \forall\underline{x}{:}\forall\underline{y}{:}\forall\underline{z}{:}\, \underline{x} = \underline{y} \Rightarrow \underline{x}+\underline{z} = \underline{y}+\underline{z} \oplus \forall\underline{x}{:}\forall\underline{y}{:}\forall\underline{z}{:}\, \underline{x}+\underline{y}+\underline{z} =$
$\underline{x}+\underline{y}+\underline{z} \oplus \forall\underline{s}{:}\forall\underline{x}{:}\, \dot\neg\underline{s} \in \cup\underline{x} \Rightarrow \dot\neg\underline{s} \in j_{\mathrm{Ex}} \Rightarrow \dot\neg j_{\mathrm{Ex}} \in \underline{x} \Rightarrow \dot\neg\dot\neg\underline{s} \in j_{\mathrm{Ex}} \Rightarrow \dot\neg j_{\mathrm{Ex}} \in \underline{x} \Rightarrow$
$\underline{s} \in \cup\underline{x} \oplus \forall\underline{m}{:}\forall\overline{(\epsilon)}{:}\forall\overline{(fx)}{:}\forall\overline{(fy)}{:}\forall\overline{(rx)}{:}\forall\overline{(ry)}{:}\, \overline{(rx)} << \overline{(ry)} \vdash \overline{(fx)} \in \overline{(rx)} \vdash \overline{(fy)} \in$
$\overline{(ry)} \vdash \dot\neg 0 <= \overline{(\epsilon)} \Rightarrow \dot\neg\dot\neg 0 = \overline{(\epsilon)} \vdash a_{\mathrm{Ex}} <= \underline{m} \Rightarrow \overline{(fx)}[\underline{m}] <= \overline{(fy)}[\underline{m}] + -\overline{(\epsilon)} \oplus$
$\forall\underline{a}{:}\forall\underline{x}{:}\forall\underline{y}{:}\, \underline{a} \vdash \mathrm{if}(\underline{a}, \underline{x}, \underline{y}) = \underline{x} \oplus \forall\underline{x}{:}\forall\underline{a}{:}\, \underline{a} \vdash \forall_{\mathrm{obj}}\underline{x}{:}\, \underline{a} \oplus \forall\underline{m}{:}\forall\overline{(fx)}{:}\, -_{\mathrm{f}}\overline{(fx)}[\underline{m}] =$
$-\overline{(fx)}[\underline{m}] \oplus \forall\overline{(fx)}{:}\forall\overline{(fy)}{:}\, R(\overline{(fx)}) ** R(\overline{(fy)}) == R(\overline{(fx)} *_{\mathrm{f}} \overline{(fy)}) \oplus \forall\underline{x}{:}\forall\underline{y}{:}\forall\underline{z}{:}\, \underline{x}*\underline{y} +$
$\underline{z} = \underline{x}*\underline{y} + \underline{x}*\underline{z} \oplus \forall\underline{x}{:}\, \underline{x} + -\underline{x} = 0 \oplus \forall\underline{a}{:}\forall\underline{b}{:}\forall\underline{p}{:}\forall\underline{x}{:}\forall\underline{z}{:}\, \underline{p}^{\mathrm{Ph}} \wedge \langle\underline{b}\equiv\underline{a}|\underline{p}{:}{=}{=}\underline{z}\rangle_{\mathrm{Ph}} \Vdash$
$\dot\neg\underline{z} \in \{ph \in \underline{x} \mid \underline{a}\} \Rightarrow \dot\neg\underline{z} \in \underline{x} \Rightarrow \dot\neg\underline{b} \Rightarrow \dot\neg\dot\neg\underline{z} \in \underline{x} \Rightarrow \dot\neg\underline{b} \Rightarrow \underline{z} \in \{ph \in \underline{x} \mid \underline{a}\} \oplus$
$\forall\overline{(rx)}{:}\forall\overline{(ry)}{:}\, \overline{(rx)} << \overline{(ry)} \vdash t_{\mathrm{Ex}} \in \overline{(ry)} \oplus \forall\underline{m}{:}\forall\overline{(fx)}{:}\forall\overline{(fy)}{:}\, \overline{(fx)} =_{\mathrm{f}} \overline{(fy)} \vdash$
$\overline{(fx)}[\underline{m}] = \overline{(fy)}[\underline{m}] \oplus \forall\overline{(rx)}{:}\forall\overline{(ry)}{:}\, \overline{(rx)} == \overline{(ry)} \vdash \overline{(ry)} == \overline{(rx)} \oplus \forall\underline{x}{:}\, \underline{x} <= \underline{x} \oplus$
$\forall\underline{a}{:}\forall\underline{b}{:}\, \dot\neg\underline{b} \Rightarrow \underline{a} \vdash \dot\neg\underline{b} \Rightarrow \dot\neg\underline{a} \vdash \underline{b} \oplus \forall\underline{m}{:}\, 1\mathrm{f}[\underline{m}] = 1 \oplus \forall\underline{x}{:}\forall\underline{y}{:}\forall\underline{z}{:}\, \underline{x} = \underline{y} \Rightarrow \underline{x} = \underline{z} \Rightarrow$
$\underline{y} = \underline{z} \oplus \forall\underline{x}{:}\forall\underline{y}{:}\forall\underline{z}{:}\, \underline{x} <= \underline{y} \Rightarrow \underline{x}+\underline{z} <= \underline{y}+\underline{z} \oplus \forall\underline{s}{:}\, \dot\neg\underline{s} \in \emptyset \oplus$
$\forall\overline{(fx)}{:}\forall\overline{(fy)}{:}\, R(\overline{(fx)}) == R(\overline{(fy)}) \vdash \mathrm{SF}(\overline{(fx)}, \overline{(fy)}) \oplus \forall\overline{(fx)}{:}\forall\overline{(rx)}{:}\forall\overline{(ry)}{:}\, \overline{(rx)} ==$
$\overline{(ry)} \vdash \overline{(fx)} \in \overline{(rx)} \vdash \overline{(fx)} \in \overline{(ry)} \oplus \forall\underline{m}{:}\forall\overline{(fx)}{:}\forall\overline{(fy)}{:}\, \overline{(fx)} +_{\mathrm{f}} \overline{(fy)}[\underline{m}] =$
$\overline{(fx)}[\underline{m}] + \overline{(fy)}[\underline{m}] \oplus \forall\overline{(fx)}{:}\forall\overline{(fy)}{:}\, \overline{(fx)} \in R(\overline{(fy)}) \vdash \mathrm{SF}(\overline{(fx)}, \overline{(fy)}) \oplus \forall\underline{x}{:}\, \dot\neg\underline{x} = 0 \Rightarrow$
$\underline{x} * \mathrm{rec}\underline{x} = 1 \oplus \forall\underline{x}{:}\, \underline{x} + 0 = \underline{x} \oplus \forall\underline{m}{:}\forall\underline{n}{:}\forall\overline{(\epsilon)}{:}\forall\overline{(fx)}{:}\forall\overline{(fy)}{:}\forall\overline{(rx)}{:}\forall\overline{(ry)}{:}\, \overline{(fx)} \in$
$\overline{(rx)} \Rightarrow \overline{(fy)} \in \overline{(ry)} \Rightarrow \dot\neg 0 <= \overline{(\epsilon)} \Rightarrow \dot\neg\dot\neg 0 = \overline{(\epsilon)} \Rightarrow a_{\mathrm{Ex}} <= \underline{m} \Rightarrow \overline{(fx)}[\underline{m}] <=$
$\overline{(fy)}[\underline{m}] + -\overline{(\epsilon)} \vdash \overline{(rx)} << \overline{(ry)} \oplus \forall\underline{m}{:}\forall\overline{(fx)}{:}\forall\overline{(fy)}{:}\, \overline{(fx)}[\underline{m}] = \overline{(fy)}[\underline{m}] \vdash \overline{(fx)} =_{\mathrm{f}}$
$\overline{(fy)} \oplus \forall\underline{x}{:}\forall\underline{y}{:}\, \underline{x} <= \underline{y} \Rightarrow \underline{y} <= \underline{x} \Rightarrow \underline{x} = \underline{y} \oplus \forall\underline{a}{:}\forall\underline{b}{:}\, \lambda x.\mathrm{Dedu}_0(\lceil\underline{a}\rceil, \lceil\underline{b}\rceil) \Vdash \underline{a} \vdash \underline{b} \oplus$
$\forall\underline{m}{:}\forall\overline{(\epsilon)}{:}\forall\overline{(fx)}{:}\forall\overline{(fy)}{:}\, \mathrm{SF}(\overline{(fx)}, \overline{(fy)}) \vdash \dot\neg 0 <= \overline{(\epsilon)} \Rightarrow \dot\neg\dot\neg 0 = \overline{(\epsilon)} \vdash c_{\mathrm{Ex}} <= \underline{m} \Rightarrow$
$\dot\neg\,\mathrm{if}(0 <= \overline{(fx)}[\underline{m}] + -\overline{(fy)}[\underline{m}], \overline{(fx)}[\underline{m}] + -\overline{(fy)}[\underline{m}], -\overline{(fx)}[\underline{m}] + -\overline{(fy)}[\underline{m}]) <= \overline{(\epsilon)} \Rightarrow$
$\dot\neg\dot\neg\,\mathrm{if}(0 <= \overline{(fx)}[\underline{m}] + -\overline{(fy)}[\underline{m}], \overline{(fx)}[\underline{m}] + -\overline{(fy)}[\underline{m}], -\overline{(fx)}[\underline{m}] + -\overline{(fy)}[\underline{m}]) = \overline{(\epsilon)} \oplus$
$\forall\underline{x}{:}\forall\underline{y}{:}\, \underline{x} = \underline{y} \Rightarrow \underline{x} <= \underline{y} \oplus \forall\underline{x}{:}\forall\underline{y}{:}\forall\underline{z}{:}\, 0 <= \underline{z} \Rightarrow \underline{x} <= \underline{y} \Rightarrow \underline{x}*\underline{z} <= \underline{y}*\underline{z} \oplus$
$\forall\underline{s}{:}\forall\underline{x}{:}\forall\underline{y}{:}\, \dot\neg\underline{s} \in \{\underline{x}, \underline{y}\} \Rightarrow \dot\neg\underline{s}{=}{=}\underline{x} \Rightarrow \underline{s}{=}{=}\underline{y} \Rightarrow \dot\neg\dot\neg\underline{s}{=}{=}\underline{x} \Rightarrow \underline{s}{=}{=}\underline{y} \Rightarrow \underline{s} \in$
$\{\underline{x}, \underline{y}\} \oplus \forall\overline{(fx)}{:}\forall\overline{(fy)}{:}\, \mathrm{SF}(\overline{(fx)}, \overline{(fy)}) \vdash R(\overline{(fx)}) == R(\overline{(fy)}) \oplus$
$\forall\overline{(fx)}{:}\forall\overline{(rx)}{:}\forall\overline{(ry)}{:}\, \overline{(rx)} == \overline{(ry)} \vdash \overline{(fx)} \in \overline{(ry)} \vdash \overline{(fx)} \in \overline{(rx)} \oplus \forall\underline{a}{:}\forall\underline{b}{:}\, \underline{a} \Rightarrow \underline{b} \vdash \underline{a} \vdash$
$\underline{b} \oplus \forall\underline{m}{:}\forall\overline{(fx)}{:}\forall\overline{(fy)}{:}\, \overline{(fx)} *_{\mathrm{f}} \overline{(fy)}[\underline{m}] = \overline{(fx)}[\underline{m}] * \overline{(fy)}[\underline{m}] \oplus$
$\forall\overline{(fx)}{:}\forall\overline{(fy)}{:}\, R(\overline{(fx)} +_{\mathrm{f}} \overline{(fy)}) == R(\overline{(fx)} +_{\mathrm{f}} \overline{(fy)}) \oplus \forall\underline{x}{:}\, \underline{x} * 1 = \underline{x} \oplus \forall\underline{x}{:}\forall\underline{y}{:}\, \underline{x} + \underline{y} =$
$\underline{y} + \underline{x} \oplus \forall\underline{s}{:}\forall\underline{x}{:}\, \dot\neg\underline{s} \in P(\underline{x}) \Rightarrow \forall_{\mathrm{obj}}\bar{s}{:}\, \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \dot\neg\forall_{\mathrm{obj}}\bar{s}{:}\, \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \underline{s} \in$

$P(\underline{x}) \oplus \forall \overline{(rx)}: \forall \overline{(ry)}: \overline{(rx)} << \overline{(ry)} \vdash j_{Ex} \in \overline{(rx)} \oplus \forall \underline{a}: \forall \underline{x}: \forall \underline{y}: \dot{\neg}\underline{a} \vdash \mathrm{if}(\underline{a}, \underline{x}, \underline{y}) = \underline{y} \oplus \forall \overline{(rx)}: \overline{(rx)} == \overline{(rx)} \oplus \forall \underline{a}: \underline{a} \vdash \underline{a} \oplus \forall \underline{m}: 0f[\underline{m}] = 0 \oplus \dot{\neg} 0 = 1 \oplus \forall \underline{x}: \forall \underline{y}: \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \underline{y} <= \underline{x} \oplus \forall \underline{x}: \forall \underline{y}: \dot{\neg}\underline{x} == \underline{y} \Rightarrow \forall_{obj}\bar{s}: \dot{\neg}\bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y} \Rightarrow \dot{\neg}\bar{s} \in \underline{y} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}: \dot{\neg}\bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y} \Rightarrow \dot{\neg}\bar{s} \in \underline{y} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \underline{x} == \underline{y} \oplus \forall \overline{(fx)}: \forall \overline{(fy)}: \forall \overline{(rx)}: \forall \overline{(ry)}: \overline{(fx)} \in \overline{(rx)} \Rightarrow \overline{(fy)} \in \overline{(ry)} \Rightarrow SF(\overline{(fx)}, \overline{(fy)}) \vdash \overline{(rx)} == \overline{(ry)} \oplus \forall \underline{x}: \forall \underline{y}: \forall \underline{z}: \underline{x} = \underline{y} \Rightarrow \underline{x} * \underline{z} = \underline{y} * \underline{z} \oplus \forall \underline{m}: \forall \overline{(\epsilon)}: \forall \overline{(fx)}: \forall \overline{(fy)}: \dot{\neg} 0 <= \overline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \overline{(\epsilon)} \Rightarrow c_{Ex} <= \underline{m} \Rightarrow \overline{(fx)}[\underline{m}] <= \overline{(fy)}[\underline{m}] + -\overline{(\epsilon)} \vdash \overline{(fx)} <_f \overline{(fy)} \oplus \forall \overline{(fx)}: \forall \overline{(fy)}: \overline{(fx)} <_f \overline{(fy)} \vdash R(\overline{(fx)}) << R(\overline{(fy)}) \oplus \forall \underline{x}: \forall \underline{y}: \underline{x} * \underline{y} = \underline{y} * \underline{x}]$

[MP $\overset{stmt}{\to}$ ZFsub $\vdash \forall \underline{a}: \forall \underline{b}: \underline{a} \Rightarrow \underline{b} \vdash \underline{a} \vdash \underline{b}$][MP $\overset{proof}{\to}$ Rule tactic]

[Gen $\overset{stmt}{\to}$ ZFsub $\vdash \forall \underline{x}: \forall \underline{a}: \underline{a} \vdash \forall_{obj}\underline{x}: \underline{a}$][Gen $\overset{proof}{\to}$ Rule tactic]

[Repetition $\overset{stmt}{\to}$ ZFsub $\vdash \forall \underline{a}: \underline{a} \vdash \underline{a}$][Repetition $\overset{proof}{\to}$ Rule tactic]

[Neg $\overset{stmt}{\to}$ ZFsub $\vdash \forall \underline{a}: \forall \underline{b}: \dot{\neg}\underline{b} \Rightarrow \underline{a} \vdash \dot{\neg}\underline{b} \Rightarrow \dot{\neg}\underline{a} \vdash \underline{b}$][Neg $\overset{proof}{\to}$ Rule tactic]

[Ded $\overset{stmt}{\to}$ ZFsub $\vdash \forall \underline{a}: \forall \underline{b}: \lambda x.\mathrm{Dedu}_0(\lceil \underline{a} \rceil, \lceil \underline{b} \rceil) \Vdash \underline{a} \vdash \underline{b}$][Ded $\overset{proof}{\to}$ Rule tactic]

[ExistIntro $\overset{stmt}{\to}$ ZFsub $\vdash \forall \underline{x}: \forall \underline{t}: \forall \underline{a}: \forall \underline{b}: \langle \lceil \underline{a} \rceil \equiv^0 \lceil \underline{b} \rceil | \lceil \underline{x} \rceil :== \lceil \underline{t} \rceil \rangle_{Ex} \Vdash \underline{a} \vdash \underline{b}$][ExistIntro $\overset{proof}{\to}$ Rule tactic]

[Extensionality $\overset{stmt}{\to}$ ZFsub $\vdash \forall \underline{x}: \forall \underline{y}: \dot{\neg}\underline{x} == \underline{y} \Rightarrow \forall_{obj}\bar{s}: \dot{\neg}\bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y} \Rightarrow \dot{\neg}\bar{s} \in \underline{y} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}: \dot{\neg}\bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y} \Rightarrow \dot{\neg}\bar{s} \in \underline{y} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \underline{x} == \underline{y}$][Extensionality $\overset{proof}{\to}$ Rule tactic]

[Ødef $\overset{stmt}{\to}$ ZFsub $\vdash \forall \underline{s}: \dot{\neg}\underline{s} \in \emptyset$][Ødef $\overset{proof}{\to}$ Rule tactic]

[PairDef $\overset{stmt}{\to}$ ZFsub $\vdash \forall \underline{s}: \forall \underline{x}: \forall \underline{y}: \dot{\neg}\underline{s} \in \{\underline{x}, \underline{y}\} \Rightarrow \dot{\neg}\underline{s} == \underline{x} \Rightarrow \underline{s} == \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{s} == \underline{x} \Rightarrow \underline{s} == \underline{y} \Rightarrow \underline{s} \in \{\underline{x}, \underline{y}\}$][PairDef $\overset{proof}{\to}$ Rule tactic]

[UnionDef $\overset{stmt}{\to}$ ZFsub $\vdash \forall \underline{s}: \forall \underline{x}: \dot{\neg}\underline{s} \in \cup\underline{x} \Rightarrow \dot{\neg}\underline{s} \in j_{Ex} \Rightarrow \dot{\neg} j_{Ex} \in \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\underline{s} \in j_{Ex} \Rightarrow \dot{\neg} j_{Ex} \in \underline{x} \Rightarrow \underline{s} \in \cup\underline{x}$][UnionDef $\overset{proof}{\to}$ Rule tactic]

[PowerDef $\overset{stmt}{\to}$ ZFsub $\vdash \forall \underline{s}: \forall \underline{x}: \dot{\neg}\underline{s} \in P(\underline{x}) \Rightarrow \forall_{obj}\bar{s}: \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \dot{\neg}\forall_{obj}\bar{s}: \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \underline{s} \in P(\underline{x})$][PowerDef $\overset{proof}{\to}$ Rule tactic]

[SeparationDef $\overset{stmt}{\to}$ ZFsub $\vdash \forall \underline{a}: \forall \underline{b}: \forall \underline{p}: \forall \underline{x}: \forall \underline{z}: \underline{p}^{Ph} \wedge \langle \underline{b} \equiv \underline{a} | \underline{p} :== \underline{z} \rangle_{Ph} \Vdash \dot{\neg}\underline{z} \in \{ph \in \underline{x} \mid \underline{a}\} \Rightarrow \dot{\neg}\underline{z} \in \underline{x} \Rightarrow \dot{\neg}\underline{b} \Rightarrow \dot{\neg}\dot{\neg}\underline{z} \in \underline{x} \Rightarrow \dot{\neg}\underline{b} \Rightarrow \underline{z} \in \{ph \in \underline{x} \mid \underline{a}\}$][SeparationDef $\overset{proof}{\to}$ Rule tactic]

3 Udsagnslogisk bibliotek

I dette afsnit vil jeg bevise en samling af udsagnslogiske sandheder (eller "tautologier"), som vil blive brugt i de følgende afsnit. De fleste af disse tautologier har mange andre anvendelser end lige netop mængdelære. Beviserne er fordelt på syv underafsnit; figur 1 giver et overblik over, hvordan beviserne forholder sig til hinanden. Jeg vil kommentere de fleste af beviserne; dog er nogle af dem så tekniske, at jeg har ladet dem stå alene.

3.1 MP-lemmaer

Man får ofte brug for at anvende slutningsreglen MP flere gange i træk. Derfor vil jeg begynde med at vise fire lemmaer, der kan klare mellem 2 og 5 anvendelser af MP[3]:

$[\mathrm{MP2} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \forall \underline{\mathsf{c}} \colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \vdash \underline{\mathsf{a}} \vdash \underline{\mathsf{b}} \vdash \underline{\mathsf{c}}]$

$[\mathrm{MP3} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \forall \underline{\mathsf{c}} \colon \forall \underline{\mathsf{d}} \colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \Rightarrow \underline{\mathsf{d}} \vdash \underline{\mathsf{a}} \vdash \underline{\mathsf{b}} \vdash \underline{\mathsf{c}} \vdash \underline{\mathsf{d}}]$

$[\mathrm{MP4} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \forall \underline{\mathsf{c}} \colon \forall \underline{\mathsf{d}} \colon \forall \underline{\mathsf{e}} \colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \Rightarrow \underline{\mathsf{d}} \Rightarrow \underline{\mathsf{e}} \vdash \underline{\mathsf{a}} \vdash \underline{\mathsf{b}} \vdash \underline{\mathsf{c}} \vdash \underline{\mathsf{d}} \vdash \underline{\mathsf{e}}]$

$[\mathrm{MP5} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \forall \underline{\mathsf{c}} \colon \forall \underline{\mathsf{d}} \colon \forall \underline{\mathsf{e}} \colon \forall \underline{\mathsf{f}} \colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \Rightarrow \underline{\mathsf{d}} \Rightarrow \underline{\mathsf{e}} \Rightarrow \underline{\mathsf{f}} \vdash \underline{\mathsf{a}} \vdash \underline{\mathsf{b}} \vdash \underline{\mathsf{c}} \vdash \underline{\mathsf{d}} \vdash \underline{\mathsf{e}} \vdash \underline{\mathsf{f}}]$

3.1.1 Det første bevis

Vi begynder med at bevise MP2:

$[\mathrm{MP2} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \forall \underline{\mathsf{c}} \colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \vdash \underline{\mathsf{a}} \vdash \underline{\mathsf{b}} \vdash \underline{\mathsf{c}}]$

$[\mathrm{MP2} \overset{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \forall \underline{\mathsf{c}} \colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \vdash \underline{\mathsf{a}} \vdash \underline{\mathsf{b}} \vdash \mathrm{MP} \rhd \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \rhd \underline{\mathsf{a}} \gg \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}}; \mathrm{MP} \rhd \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \rhd \underline{\mathsf{b}} \gg \underline{\mathsf{c}} \rceil, p_0, \mathsf{c})]$

Da dette er rapportens første bevis, vil jeg bringe nogle ekstra kommentarer[4]. Oven over beviset har jeg gentaget definitionen af det, der skal bevises; dette er kun for overblikkets skyld — det er ikke en formel nødvendighed. Selve beviset for MP2 består af seks linier, nummereret fra 1 til 6. En bevislinie kan have to former. Den første form er:

`Argumentation` $\gg$ `Konklusion`

hvor `Konklusion` er det som linien beviser, mens teksten i `Argumentation` udgør en begrundelse for, at `Konklusion` gælder. F.eks. siger linie 5, at meta-formlen $[\underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}}]$ gælder, fordi den kan udledes fra slutningsreglen MP ved substitution. Argumentationen skal læses på den måde, at konklusionerne fra

[3] I afsnit ?? får vi faktisk brug for at anvende MP 6 gange i træk; men et eller andet sted skal man jo stoppe.

[4] Denne beskrivelse er en revideret udgave af afsnit 5.1 i [?].

linie 2 og 3 bliver brugt som præmisser til MP. Den generelle betydning af konstruktionen $[\mathsf{x} \rhd \mathsf{y}]$ er, at konklusionen fra linie y bliver brugt som præmis i forhold til x.
Den anden form, en bevislinie kan have, er:

$$\texttt{Nøgleord} \gg \texttt{Konklusion}$$

hvor `Nøgleord` er et af de tre ord "Arbitrary", "Premise" eller "Side-condition". Betydningen af ordene "Premise" og "Side-condition" er åbenlys: De angiver, at liniens konklusion indgår som en præmis (hhv. sidebetingelse) i den sætning, der skal bevises. F.eks. siger bevisets linie 2, at MP2 bruger meta-formlen
$[\underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}}]$ som præmis. Når ordet "Arbitrary" bruges, består konklusionen af en liste af meta-variable (f.eks. $[\underline{\mathsf{a}}, \underline{\mathsf{b}}, \underline{\mathsf{c}}]$ i linie 1). Ideen hermed er at udtrykke, at vi ikke antager noget om de pågældende meta-variable, og at vi derfor har ret til at binde dem med en meta-alkvantor i den sætning, der skal bevises. I det forhåndenværende bevis berettiger linien med "Arbitrary" altså, at MP2 er kvantificeret med $[\forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \forall \underline{\mathsf{c}}\colon (\cdots)]$.
Alt dette har drejet sig om den formelle syntaks for et Logiweb bevis. Der er ikke så meget at sige om selve beviset for MP2; vi indkapsler simpelthen to på hinanden følgende anvendelser af MP.

3.1.2 Beviser for de andre MP-lemmaer

Beviserne for de øvrige MP-lemmaer er lige ud ad landevejen:

$$[\mathrm{MP3} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \forall \underline{\mathsf{c}}\colon \forall \underline{\mathsf{d}}\colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \Rightarrow \underline{\mathsf{d}} \vdash \underline{\mathsf{a}} \vdash \underline{\mathsf{b}} \vdash \underline{\mathsf{c}} \vdash \underline{\mathsf{d}}]$$

$$[\mathrm{MP3} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \forall \underline{\mathsf{c}}\colon \forall \underline{\mathsf{d}}\colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \Rightarrow \underline{\mathsf{d}} \vdash \underline{\mathsf{a}} \vdash \underline{\mathsf{b}} \vdash \underline{\mathsf{c}} \vdash \mathrm{MP2} \rhd \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \Rightarrow \underline{\mathsf{d}} \rhd \underline{\mathsf{a}} \rhd \underline{\mathsf{b}} \gg \underline{\mathsf{c}} \Rightarrow \underline{\mathsf{d}}; \mathrm{MP} \rhd \underline{\mathsf{c}} \Rightarrow \underline{\mathsf{d}} \rhd \underline{\mathsf{c}} \gg \underline{\mathsf{d}}\rceil, \mathrm{p}_0, \mathsf{c})]$$

$$[\mathrm{MP4} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \forall \underline{\mathsf{c}}\colon \forall \underline{\mathsf{d}}\colon \forall \underline{\mathsf{e}}\colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \Rightarrow \underline{\mathsf{d}} \Rightarrow \underline{\mathsf{e}} \vdash \underline{\mathsf{a}} \vdash \underline{\mathsf{b}} \vdash \underline{\mathsf{c}} \vdash \underline{\mathsf{d}} \vdash \underline{\mathsf{e}}]$$

$$[\mathrm{MP4} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \forall \underline{\mathsf{c}}\colon \forall \underline{\mathsf{d}}\colon \forall \underline{\mathsf{e}}\colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \Rightarrow \underline{\mathsf{d}} \Rightarrow \underline{\mathsf{e}} \vdash \underline{\mathsf{a}} \vdash \underline{\mathsf{b}} \vdash \underline{\mathsf{c}} \vdash \underline{\mathsf{d}} \vdash \mathrm{MP2} \rhd \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \Rightarrow \underline{\mathsf{d}} \Rightarrow \underline{\mathsf{e}} \rhd \underline{\mathsf{a}} \rhd \underline{\mathsf{b}} \gg \underline{\mathsf{c}} \Rightarrow \underline{\mathsf{d}} \Rightarrow \underline{\mathsf{e}}; \mathrm{MP2} \rhd \underline{\mathsf{c}} \Rightarrow \underline{\mathsf{d}} \Rightarrow \underline{\mathsf{e}} \rhd \underline{\mathsf{c}} \rhd \underline{\mathsf{d}} \gg \underline{\mathsf{e}}\rceil, \mathrm{p}_0, \mathsf{c})]$$

$$[\mathrm{MP5} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \forall \underline{\mathsf{c}}\colon \forall \underline{\mathsf{d}}\colon \forall \underline{\mathsf{e}}\colon \forall \underline{\mathsf{f}}\colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \Rightarrow \underline{\mathsf{d}} \Rightarrow \underline{\mathsf{e}} \Rightarrow \underline{\mathsf{f}} \vdash \underline{\mathsf{a}} \vdash \underline{\mathsf{b}} \vdash \underline{\mathsf{c}} \vdash \underline{\mathsf{d}} \vdash \underline{\mathsf{e}} \vdash \underline{\mathsf{f}}]$$

$$[\mathrm{MP5} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \forall \underline{\mathsf{c}}\colon \forall \underline{\mathsf{d}}\colon \forall \underline{\mathsf{e}}\colon \forall \underline{\mathsf{f}}\colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \Rightarrow \underline{\mathsf{d}} \Rightarrow \underline{\mathsf{e}} \Rightarrow \underline{\mathsf{f}} \vdash \underline{\mathsf{a}} \vdash \underline{\mathsf{b}} \vdash \underline{\mathsf{c}} \vdash \underline{\mathsf{d}} \vdash \underline{\mathsf{e}} \vdash \mathrm{MP3} \rhd \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \Rightarrow \underline{\mathsf{d}} \Rightarrow \underline{\mathsf{e}} \Rightarrow \underline{\mathsf{f}} \rhd \underline{\mathsf{a}} \rhd \underline{\mathsf{b}} \rhd \underline{\mathsf{c}} \gg \underline{\mathsf{d}} \Rightarrow \underline{\mathsf{e}} \Rightarrow \underline{\mathsf{f}}; \mathrm{MP2} \rhd \underline{\mathsf{d}} \Rightarrow \underline{\mathsf{e}} \Rightarrow \underline{\mathsf{f}} \rhd \underline{\mathsf{d}} \rhd \underline{\mathsf{e}} \gg \underline{\mathsf{f}}\rceil, \mathrm{p}_0, \mathsf{c})]$$

3.2 Implikation

Dette afsnit indeholder en række lemmaer vedr. implikation, grupperet i fire under-underafsnit.

3.2.1 Refleksivitet; blok-konstruktionen

Lemmaet AutoImply udsiger, at implikations-relationen er refleksiv:

$[\text{AutoImply} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{a}\colon \underline{a} \Rightarrow \underline{a}]$

$[\text{AutoImply} \stackrel{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{a}\colon \underline{a} \vdash \text{Repetition} \rhd \underline{a} \gg \underline{a}; \forall \underline{a}\colon \text{Ded} \rhd \forall \underline{a}\colon \underline{a} \vdash \underline{a} \gg \underline{a} \Rightarrow \underline{a}\rceil, p_0, c)]$

Beviset for AutoImply indeholder to nye ting i forhold til de hidtidige beviser: En bevisblok, og en anvendelse af deduktions-reglen. En bevisblok er selvstændig enhed i et bevis; den afhænger ikke af den øvrige del af beviset. Den ovenstående bevisblok indeholder et bevis for lemmaet $[\forall \underline{a}\colon \underline{a} \vdash \underline{a}]$. Pointen er nu, at blokkens sidste linie (linie 5) fungerer som en forkortelse for dette lemma. Vi kan da anvende deduktionsreglen på denne linie til at omdanne inferensen $[\forall \underline{a}\colon \underline{a} \vdash \underline{a}]$ til implikationen $[\underline{a} \Rightarrow \underline{a}]$. Det vigtigste formål med deduktionsreglen er netop, at vi let kan skifte fra inferens til implikation.

3.2.2 Transitivitet

Lemmaet ImplyTransitivity udsiger, at implikations-relationen er transitiv:

$[\text{ImplyTransitivity} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} \Rightarrow \underline{b} \vdash \underline{b} \Rightarrow \underline{c} \vdash \underline{a} \Rightarrow \underline{c}]$

Vi viser ImplyTransitivity ved hjælp af MP og deduktionsreglen:

$[\text{ImplyTransitivity} \stackrel{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} \Rightarrow \underline{b} \vdash \underline{b} \Rightarrow \underline{c} \vdash \underline{a} \vdash \text{MP} \rhd \underline{a} \Rightarrow \underline{b} \rhd \underline{a} \gg \underline{b}; \text{MP} \rhd \underline{b} \Rightarrow \underline{c} \rhd \underline{b} \gg \underline{c}; \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} \Rightarrow \underline{b} \vdash \underline{b} \Rightarrow \underline{c} \vdash \text{Ded} \rhd \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \underline{a} \Rightarrow \underline{b} \vdash \underline{b} \Rightarrow \underline{c} \vdash \underline{a} \vdash \underline{c} \gg \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{a} \Rightarrow \underline{c}; \text{MP2} \rhd \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{a} \Rightarrow \underline{c} \rhd \underline{a} \Rightarrow \underline{b} \rhd \underline{b} \Rightarrow \underline{c} \gg \underline{a} \Rightarrow \underline{c}\rceil, p_0, c)]$

3.2.3 Svækkelse

Vi får ofte brug for det følgende ræsonnement: Hvis formlen $\underline{a}$ gælder ubetinget, så gælder den også under antagelse af en vilkårlig anden formel $\underline{b}$. Lemmaet Weakening udtrykker dette ræsonnement som følger:

$[\text{Weakening} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{b} \vdash \underline{a} \Rightarrow \underline{b}]$

Vi beviser Weakening ved hjælp af deduktionsreglen:

$[\text{Weakening} \stackrel{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{b} \vdash \underline{a} \vdash \text{Repetition} \rhd \underline{b} \gg \underline{b}; \forall \underline{a}\colon \forall \underline{b}\colon \text{Ded} \rhd \forall \underline{a}\colon \forall \underline{b}\colon \underline{b} \vdash \underline{a} \vdash \underline{b} \gg \underline{b} \Rightarrow \underline{a} \Rightarrow \underline{b}; \underline{b} \vdash \text{MP} \rhd \underline{b} \Rightarrow \underline{a} \Rightarrow \underline{b} \rhd \underline{b} \gg \underline{a} \Rightarrow \underline{b}\rceil, p_0, c)]$

3.2.4 Modsigelse

Det sidste lemma i dette afsnit vedrører strengt taget ikke implikation, men derimod inferens ($\mathsf{x} \vdash \mathsf{y}$). Lemmaet FromContradiction udsiger, at vi kan bevise hvad som helst, hvis vi har bevist to formler, der modsiger hinanden:

$$[\text{FromContradiction} \overset{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \vdash \dot\neg\, \underline{\mathsf{a}} \vdash \underline{\mathsf{b}}]$$

Beviset bruger Weakening og slutningsreglen Neg:

$$[\text{FromContradiction} \overset{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \vdash \dot\neg\, \underline{\mathsf{a}} \vdash \text{Weakening} \rhd \underline{\mathsf{a}} \gg \dot\neg\, \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}}; \text{Weakening} \rhd \dot\neg\, \underline{\mathsf{a}} \gg \dot\neg\, \underline{\mathsf{b}} \Rightarrow \dot\neg\, \underline{\mathsf{a}}; \text{Neg} \rhd \dot\neg\, \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}} \rhd \dot\neg\, \underline{\mathsf{b}} \Rightarrow \dot\neg\, \underline{\mathsf{a}} \gg \underline{\mathsf{b}} \rceil, \mathrm{p}_0, \mathsf{c})]$$

3.3 Håndtering af dobbeltnegationer

De to lemmaer RemoveDoubleNeg og AddDoubleNeg tillader os hhv. at fjerne og tilføje dobbeltnegationer. Jeg vil ikke kommentere beviserne:

$$[\text{RemoveDoubleNeg} \overset{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \dot\neg\, \dot\neg\, \underline{\mathsf{a}} \vdash \underline{\mathsf{a}}]$$

$$[\text{RemoveDoubleNeg} \overset{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \dot\neg\, \dot\neg\, \underline{\mathsf{a}} \vdash \text{Weakening} \rhd \dot\neg\, \dot\neg\, \underline{\mathsf{a}} \gg \dot\neg\, \underline{\mathsf{a}} \Rightarrow \dot\neg\, \dot\neg\, \underline{\mathsf{a}}; \text{AutoImply} \gg \dot\neg\, \underline{\mathsf{a}} \Rightarrow \dot\neg\, \underline{\mathsf{a}}; \text{Neg} \rhd \dot\neg\, \underline{\mathsf{a}} \Rightarrow \dot\neg\, \underline{\mathsf{a}} \rhd \dot\neg\, \underline{\mathsf{a}} \Rightarrow \dot\neg\, \dot\neg\, \underline{\mathsf{a}} \gg \underline{\mathsf{a}} \rceil, \mathrm{p}_0, \mathsf{c})]$$

$$[\text{AddDoubleNeg} \overset{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \underline{\mathsf{a}} \vdash \dot\neg\, \dot\neg\, \underline{\mathsf{a}}]$$

$$[\text{AddDoubleNeg} \overset{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \dot\neg\, \dot\neg\, \dot\neg\, \underline{\mathsf{a}} \vdash \text{RemoveDoubleNeg} \rhd \dot\neg\, \dot\neg\, \dot\neg\, \underline{\mathsf{a}} \gg \dot\neg\, \underline{\mathsf{a}}; \forall \underline{\mathsf{a}} \colon \text{Ded} \rhd \forall \underline{\mathsf{a}} \colon \dot\neg\, \dot\neg\, \dot\neg\, \underline{\mathsf{a}} \vdash \dot\neg\, \underline{\mathsf{a}} \gg \dot\neg\, \dot\neg\, \dot\neg\, \underline{\mathsf{a}} \Rightarrow \dot\neg\, \underline{\mathsf{a}}; \underline{\mathsf{a}} \vdash \text{Weakening} \rhd \underline{\mathsf{a}} \gg \dot\neg\, \dot\neg\, \dot\neg\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{a}}; \text{Neg} \rhd \dot\neg\, \dot\neg\, \dot\neg\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{a}} \rhd \dot\neg\, \dot\neg\, \dot\neg\, \underline{\mathsf{a}} \Rightarrow \dot\neg\, \underline{\mathsf{a}} \gg \dot\neg\, \dot\neg\, \underline{\mathsf{a}} \rceil, \mathrm{p}_0, \mathsf{c})]$$

3.4 Modus tollens og beslægtede lemmaer

Hovedresultatet fra dette afsnit er slutningsreglen modus tollens, bevist som et lemma:

$$[\text{MT} \overset{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \dot\neg\, \underline{\mathsf{b}} \vdash \dot\neg\, \underline{\mathsf{a}}]$$

For at vise MT begynder vi med et teknisk lemma, der ikke har den store værdi i sig selv:

$$[\text{Technicality} \overset{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \dot\neg\, \dot\neg\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}]$$

$$[\text{Technicality} \overset{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \dot\neg\, \dot\neg\, \underline{\mathsf{a}} \vdash \text{RemoveDoubleNeg} \rhd \dot\neg\, \dot\neg\, \underline{\mathsf{a}} \gg \underline{\mathsf{a}}; \text{MP} \rhd \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \rhd \underline{\mathsf{a}} \gg \underline{\mathsf{b}}; \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \text{Ded} \rhd \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \dot\neg\, \dot\neg\, \underline{\mathsf{a}} \vdash \underline{\mathsf{b}} \gg \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \dot\neg\, \dot\neg\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}; \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \text{MP} \rhd \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \dot\neg\, \dot\neg\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \rhd \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \gg \dot\neg\, \dot\neg\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \rceil, \mathrm{p}_0, \mathsf{c})]$$

Uafhængigt af Technicality kan vi vise en version af MT, hvor $\underline{\mathsf{a}}$ optræder i negeret form:

$[\mathrm{NegativeMT} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \dot{\neg}\, \underline{\mathsf{b}} \vdash \underline{\mathsf{a}}]$

$[\mathrm{NegativeMT} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \dot{\neg}\, \underline{\mathsf{b}} \vdash \mathrm{Weakening} \rhd \dot{\neg}\, \underline{\mathsf{b}} \gg \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}}; \mathrm{Neg} \rhd \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \rhd \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \gg \underline{\mathsf{a}} \rceil, \mathrm{p}_0, \mathsf{c})]$

Ud fra Technicality og NegativeMT kan vi nu vise MT:

$[\mathrm{MT} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \dot{\neg}\, \underline{\mathsf{b}} \vdash \dot{\neg}\, \underline{\mathsf{a}}]$

$[\mathrm{MT} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \dot{\neg}\, \underline{\mathsf{b}} \vdash \mathrm{Technicality} \gg \dot{\neg}\, \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}; \mathrm{NegativeMT} \rhd \dot{\neg}\, \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \rhd \dot{\neg}\, \underline{\mathsf{b}} \gg \dot{\neg}\, \underline{\mathsf{a}} \rceil, \mathrm{p}_0, \mathsf{c})]$

Vi slutter dette underafsnit med en variant af MT, som erstatter en inferens med en implikation:

$[\mathrm{Contrapositive} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{a}}]$

Når en inferens skal erstattes med en implikation, er det altid deduktionsreglen, der skal i spil:

$[\mathrm{Contrapositive} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \dot{\neg}\, \underline{\mathsf{b}} \vdash \mathrm{MT} \rhd \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \rhd \dot{\neg}\, \underline{\mathsf{b}} \gg \dot{\neg}\, \underline{\mathsf{a}}; \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \mathrm{Ded} \rhd \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \dot{\neg}\, \underline{\mathsf{b}} \vdash \dot{\neg}\, \underline{\mathsf{a}} \gg \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{a}}; \mathrm{MP} \rhd \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{a}} \rhd \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \gg \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{a}} \rceil, \mathrm{p}_0, \mathsf{c})]$

3.5 Konjunktion

Hovedmålet med dette underafsnit er at konvertere mellem formlerne $\underline{\mathsf{a}}$ og $\underline{\mathsf{b}}$ og deres konjunktion $[\dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}}]$.

3.5.1 Forening af konjunkter

Vi begynder med at slå $\underline{\mathsf{a}}$ og $\underline{\mathsf{b}}$ sammen til $[\dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}}]$:

$[\mathrm{JoinConjuncts} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \vdash \underline{\mathsf{b}} \vdash \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}}]$

Beviset for JoinConjuncts er af teknisk karakter. Vi viser den makroekspanderede form $[\dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}}]$, som vi i bevisets sidste linie konverterer til $[\dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}}]$. Denne sidste linie er ikke nødvendig for bevischeckeren, men den gør beviset lidt nemmere at læse:

$[\mathrm{JoinConjuncts} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \vdash \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \vdash \mathrm{MP} \rhd \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \rhd \underline{\mathsf{a}} \gg \dot{\neg}\, \underline{\mathsf{b}}; \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \mathrm{Ded} \rhd \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \vdash \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \vdash \dot{\neg}\, \underline{\mathsf{b}} \gg \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}}; \underline{\mathsf{a}} \vdash \underline{\mathsf{b}} \vdash \mathrm{MP} \rhd \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \rhd \underline{\mathsf{a}} \gg \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}}; \mathrm{AddDoubleNeg} \rhd \underline{\mathsf{b}} \gg \dot{\neg}\, \dot{\neg}\, \underline{\mathsf{b}}; \mathrm{MT} \rhd \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \rhd \dot{\neg}\, \dot{\neg}\, \underline{\mathsf{b}} \gg \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}}; \mathrm{Repetition} \rhd \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \gg \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \rceil, \mathrm{p}_0, \mathsf{c})]$

3.5.2 Udskilning af anden konjunkt

Tautologien SecondConjunct lader os udskille den anden konjunkt fra $[\dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}}]$. Jeg vil ikke kommentere beviset:

$[\mathrm{SecondConjunct} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \vdash \underline{\mathsf{b}}]$

$[\mathrm{SecondConjunct} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \dot{\neg}\, \underline{\mathsf{b}} \vdash \mathrm{Weakening} \rhd \dot{\neg}\, \underline{\mathsf{b}} \gg \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}}; \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \mathrm{Ded} \rhd \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \dot{\neg}\, \underline{\mathsf{b}} \vdash \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \gg \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}}; \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \vdash \mathrm{Repetition} \rhd \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \gg \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}}; \mathrm{NegativeMT} \rhd \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \rhd \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \gg \underline{\mathsf{b}}\rceil, \mathrm{p}_0, \mathsf{c})]$

3.5.3 Udskilning af første konjunkt

For at udskille $\underline{\mathsf{a}}$ fra $[\dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}}]$ viser vi først, at $[\dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}}]$ er kommutativ. Jeg vil ikke kommentere beviset:

$[\mathrm{AndCommutativity} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \vdash \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{a}}]$

$[\mathrm{AndCommutativity} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{a}} \vdash \underline{\mathsf{a}} \vdash \mathrm{AddDoubleNeg} \rhd \underline{\mathsf{a}} \gg \dot{\neg}\, \dot{\neg}\, \underline{\mathsf{a}}; \mathrm{MT} \rhd \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{a}} \rhd \dot{\neg}\, \dot{\neg}\, \underline{\mathsf{a}} \gg \dot{\neg}\, \underline{\mathsf{b}}; \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \mathrm{Ded} \rhd \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{a}} \vdash \underline{\mathsf{a}} \vdash \dot{\neg}\, \underline{\mathsf{b}} \gg \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}}; \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \vdash \mathrm{Repetition} \gg \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}}; \mathrm{MT} \rhd \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \rhd \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \gg \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{a}}; \mathrm{Repetition} \rhd \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{a}} \gg \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{a}}\rceil, \mathrm{p}_0, \mathsf{c})]$

Nu er det let at udskille den første konjunkt fra $[\dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}}]$: Først vender vi konjunktionen om til $[\dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{a}}]$ ved hjælp af AndCommutativity, og så udskiller vi $\underline{\mathsf{a}}$ ved hjælp af SecondConjunct:

$[\mathrm{FirstConjunct} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \vdash \underline{\mathsf{a}}]$

$[\mathrm{FirstConjunct} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \vdash \mathrm{AndCommutativity} \rhd \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \gg \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{a}}; \mathrm{SecondConjunct} \rhd \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{a}} \gg \underline{\mathsf{a}}\rceil, \mathrm{p}_0, \mathsf{c})]$

3.6 Dobbeltimplikation

I dette underafsnit viser vi tre enkle resultater vedr. dobbeltimplikation.

3.6.1 Brug sammen med modus ponens

De følgende to tautologier gør det let at bruge anvende slutningsreglen MP på dobbeltimplikationer. Beviserne er enkle og kræver ingen kommentarer:

$[\mathrm{IffFirst} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}} \vdash \underline{\mathsf{b}} \vdash \underline{\mathsf{a}}]$

$[\mathrm{IffFirst} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}} \vdash \underline{\mathsf{b}} \vdash \mathrm{SecondConjunct} \rhd \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}} \gg \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}}; \mathrm{MP} \rhd \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}} \rhd \underline{\mathsf{b}} \gg \underline{\mathsf{a}}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{IffSecond} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{a} \vdash \underline{a} \vdash \underline{b}]$

$[\mathrm{IffSecond} \stackrel{\mathrm{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{a} \vdash \underline{a} \vdash \mathrm{FirstConjunct} \rhd \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{a} \gg \underline{a} \Rightarrow \underline{b}; \mathrm{MP} \rhd \underline{a} \Rightarrow \underline{b} \rhd \underline{a} \gg \underline{b} \rceil, \mathrm{p}_0, c)]$

3.6.2 Kommutativitet

Lemmaet IffCommutativity følger direkte af, at operatoren $[\dot{\neg}\, x \Rightarrow \dot{\neg}\, y]$ er kommutativ:

$[\mathrm{IffCommutativity} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{a} \vdash \dot{\neg}\, \underline{b} \Rightarrow \underline{a} \Rightarrow \dot{\neg}\, \underline{a} \Rightarrow \underline{b}]$

$[\mathrm{IffCommutativity} \stackrel{\mathrm{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{a} \vdash \mathrm{Repetition} \rhd \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{a} \gg \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{a}; \mathrm{AndCommutativity} \rhd \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{a} \gg \dot{\neg}\, \underline{b} \Rightarrow \underline{a} \Rightarrow \dot{\neg}\, \underline{a} \Rightarrow \underline{b}; \mathrm{Repetition} \rhd \dot{\neg}\, \underline{b} \Rightarrow \underline{a} \Rightarrow \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \gg \dot{\neg}\, \underline{b} \Rightarrow \underline{a} \Rightarrow \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \rceil, \mathrm{p}_0, c)]$

3.7 Disjunktion

Dette underafsnit indeholder tre lemmaer vedr. disjunktion, som vi fordeler på to under-underafsnit.

3.7.1 Svækkelse

Givet en påstand $\underline{b}$ vil vi gerne udlede de svagere påstande $[\dot{\neg}\, \underline{a} \Rightarrow \underline{b}]$ og $[\dot{\neg}\, \underline{b} \Rightarrow \underline{a}]$. Den første slutning varetages af lemmaet WeakenOr1:

$[\mathrm{WeakenOr1} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{b} \vdash \dot{\neg}\, \underline{a} \Rightarrow \underline{b}]$

Beviset består af en simpel anvendelse af Weakening:

$[\mathrm{WeakenOr1} \stackrel{\mathrm{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{b} \vdash \mathrm{Weakening} \rhd \underline{b} \gg \dot{\neg}\, \underline{a} \Rightarrow \underline{b}; \mathrm{Repetition} \rhd \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \gg \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \rceil, \mathrm{p}_0, c)]$

Slutningen fra $\underline{a}$ til $[\dot{\neg}\, \underline{a} \Rightarrow \underline{b}]$ varetages af lemmaet WeakenOr2:

$[\mathrm{WeakenOr2} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \vdash \dot{\neg}\, \underline{a} \Rightarrow \underline{b}]$

Kernen i beviset for WeakenOr2 er en anvendelse af FromContradiction:

$[\mathrm{WeakenOr2} \stackrel{\mathrm{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \vdash \dot{\neg}\, \underline{a} \vdash \mathrm{FromContradiction} \rhd \underline{a} \rhd \dot{\neg}\, \underline{a} \gg \underline{b}; \forall \underline{a}\colon \forall \underline{b}\colon \mathrm{Ded} \rhd \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \vdash \dot{\neg}\, \underline{a} \vdash \underline{b} \gg \underline{a} \Rightarrow \dot{\neg}\, \underline{a} \Rightarrow \underline{b}; \underline{a} \vdash \mathrm{MP} \rhd \underline{a} \Rightarrow \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \rhd \underline{a} \gg \dot{\neg}\, \underline{a} \Rightarrow \underline{b}; \mathrm{Repetition} \rhd \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \gg \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \rceil, \mathrm{p}_0, c)]$

3.7.2 Slutning ud fra disjunktion

Lemmaet FromDisjuncts lader os drage slutninger ud fra en disjunktion:

$[\text{FromDisjuncts} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \forall \underline{\mathsf{c}}\colon \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{c}} \vdash \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \vdash \underline{\mathsf{c}}]$

Om beviset vil jeg kun sige, at det er en ret elegant øvelse i bevisteknik:

$[\text{FromDisjuncts} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \forall \underline{\mathsf{c}}\colon \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{c}} \vdash \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \vdash$
$\text{Repetition} \rhd \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \gg \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}; \text{Contrapositive} \rhd \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \gg \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow$
$\dot{\neg}\, \dot{\neg}\, \underline{\mathsf{a}}; \text{Technicality} \rhd \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{c}} \gg \dot{\neg}\, \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{c}}; \text{ImplyTransitivity} \rhd \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow$
$\dot{\neg}\, \dot{\neg}\, \underline{\mathsf{a}} \rhd \dot{\neg}\, \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{c}} \gg \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}}; \text{Contrapositive} \rhd \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \gg \dot{\neg}\, \underline{\mathsf{c}} \Rightarrow$
$\dot{\neg}\, \dot{\neg}\, \underline{\mathsf{b}}; \text{Contrapositive} \rhd \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \gg \dot{\neg}\, \underline{\mathsf{c}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}}; \text{Neg} \rhd \dot{\neg}\, \underline{\mathsf{c}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \rhd \dot{\neg}\, \underline{\mathsf{c}} \Rightarrow \dot{\neg}\, \dot{\neg}\, \underline{\mathsf{b}} \gg$
$\underline{\mathsf{c}}\rceil, \mathrm{p}_0, \mathsf{c})]$

(*******************)

$[\text{am} \stackrel{\text{prio}}{\rightarrow}$

Preassociative

[am], [base], [bracket $*$ end bracket], [big bracket $*$ end bracket], $[\ \$ * \$\ ]$, [**flush left** $[*]$], $[\mathsf{x}]$, $[\mathsf{y}]$, $[\mathsf{z}]$, $[[* \bowtie *]]$, $[[* \stackrel{*}{\rightarrow} *]]$, [pyk], [tex], [name], [prio], $[*]$, $[\mathsf{T}]$, $[\text{if}(*,*,*)]$, $[[* \stackrel{*}{\Rightarrow} *]]$, [val], [claim], $[\bot]$, $[\text{f}(*)]$, $[(*)^{\text{I}}]$, $[\mathsf{F}]$, $[\underline{0}]$, $[\underline{1}]$, $[\underline{2}]$, $[\underline{3}]$, $[\underline{4}]$, $[\underline{5}]$, $[\underline{6}]$, $[\underline{7}]$, $[\underline{8}]$, $[\underline{9}]$, [0], [1], [2], [3], [4], [5], [6], [7], [8], [9], $[\mathsf{a}]$, $[\mathsf{b}]$, $[\mathsf{c}]$, $[\mathsf{d}]$, $[\mathsf{e}]$, $[\mathsf{f}]$, $[\mathsf{g}]$, $[\mathsf{h}]$, $[\mathsf{i}]$, $[\mathsf{j}]$, $[\mathsf{k}]$, $[\mathsf{l}]$, $[\mathsf{m}]$, $[\mathsf{n}]$, $[\mathsf{o}]$, $[\mathsf{p}]$, $[\mathsf{q}]$, $[\mathsf{r}]$, $[\mathsf{s}]$, $[\mathsf{t}]$, $[\mathsf{u}]$, $[\mathsf{v}]$, $[\mathsf{w}]$, $[(*)^{\text{M}}]$, $[\text{If}(*,*,*)]$, [array$\{*\}$ $*$ end array], [l], [c], [r], [empty], $[\langle * \mid * := * \rangle]$, $[\mathcal{M}(*)]$, $[\tilde{\mathcal{U}}(*)]$, $[\mathcal{U}(*)]$, $[\mathcal{U}^{\text{M}}(*)]$, [$\textbf{apply}(*,*)$], [$\textbf{apply}_1(*,*)$], [identifier$(*)$], [identifier$_1(*,*)$], [array-plus$(*,*)$], [array-remove$(*,*,*)$], [array-put$(*,*,*,*)$], [array-add$(*,*,*,*,*)$], [bit$(*,*)$], [bit$_1(*,*)$], [rack], ["vector"], ["bibliography"], ["dictionary"], ["body"], ["codex"], ["expansion"], ["code"], ["cache"], ["diagnose"], ["pyk"], ["tex"], ["texname"], ["value"], ["message"], ["macro"], ["definition"], ["unpack"], ["claim"], ["priority"], ["lambda"], ["apply"], ["true"], ["if"], ["quote"], ["proclaim"], ["define"], ["introduce"], ["hide"], ["pre"], ["post"], $[\mathcal{E}(*,*,*)]$, $[\mathcal{E}_2(*,*,*,*,*)]$, $[\mathcal{E}_3(*,*,*,*)]$, $[\mathcal{E}_4(*,*,*,*)]$, [$\textbf{lookup}(*,*,*)$], [$\textbf{abstract}(*,*,*,*)$], $[\lceil * \rceil]$, $[\mathcal{M}(*,*,*)]$, $[\mathcal{M}_2(*,*,*,*)]$, $[\mathcal{M}^*(*,*,*)]$, [macro], $[\text{s}_0]$, [$\textbf{zip}(*,*)$], [$\textbf{assoc}_1(*,*,*)$], $[(*)^{\mathbf{P}}]$, [self], $[[* \ddot{=} *]]$, $[[* \doteq *]]$, $[[* \acute{=} *]]$, $[[* \stackrel{\text{pyk}}{=} *]]$, $[[* \stackrel{\text{tex}}{=} *]]$, $[[* \stackrel{\text{name}}{=} *]]$, [**Priority table**$[*]$], $[\tilde{\mathcal{M}}_1]$, $[\tilde{\mathcal{M}}_2(*)]$, $[\tilde{\mathcal{M}}_3(*)]$, $[\tilde{\mathcal{M}}_4(*,*,*,*)]$, $[\tilde{\mathcal{M}}(*,*,*)]$, $[\tilde{\mathcal{Q}}(*,*,*)]$, $[\tilde{\mathcal{Q}}_2(*,*,*)]$, $[\tilde{\mathcal{Q}}_3(*,*,*,*)]$, $[\tilde{\mathcal{Q}}^*(*,*,*)]$, $[(*)]$, $[(*)]$, [display$(*)$], [statement$(*)$], $[[*]^{\cdot}]$, $[[*]^{-}]$, [$\textbf{aspect}(*,*)$], [$\textbf{aspect}(*,*,*)$], $[\langle * \rangle]$, [$\textbf{tuple}_1(*)$], [$\textbf{tuple}_2(*)$], [let$_2(*,*)$], [let$_1(*,*)$], $[[* \stackrel{\text{claim}}{=} *]]$, [checker], [$\textbf{check}(*,*)$], [$\textbf{check}_2(*,*,*)$], [$\textbf{check}_3(*,*,*)$], [$\textbf{check}^*(*,*)$], [$\textbf{check}^*_2(*,*,*)$], $[[*]^{\cdot}]$, $[[*]^{-}]$, $[[*]^{\circ}]$, [msg], $[[* \stackrel{\text{msg}}{=} *]]$, [<stmt>], [stmt], $[[* \stackrel{\text{stmt}}{=} *]]$, [HeadNil$'$], [HeadPair$'$], [Transitivity$'$], $[\bot\!\!\bot]$, [Contra$'$], $[\text{T}'_{\text{E}}]$, $[\text{L}_1]$, $[\underline{*}]$, $[\mathcal{A}]$, $[\mathcal{B}]$, $[\mathcal{C}]$, $[\mathcal{D}]$, $[\mathcal{E}]$, $[\mathcal{F}]$, $[\mathcal{G}]$, $[\mathcal{H}]$, $[\mathcal{I}]$, $[\mathcal{J}]$, $[\mathcal{K}]$, $[\mathcal{L}]$, $[\mathcal{M}]$, $[\mathcal{N}]$, $[\mathcal{O}]$, $[\mathcal{P}]$, $[\mathcal{Q}]$, $[\mathcal{R}]$, $[\mathcal{S}]$, $[\mathcal{T}]$, $[\mathcal{U}]$, $[\mathcal{V}]$, $[\mathcal{W}]$, $[\mathcal{X}]$, $[\mathcal{Y}]$, $[\mathcal{Z}]$, $[\langle * \mid * := * \rangle]$, $[\langle^* * \mid * := * \rangle]$, $[\emptyset]$, [Remainder], $[(*)^{\mathbf{v}}]$, [intro$(*,*,*,*)$], [intro$(*,*,*)$], [error$(*,*)$], [error$_2(*,*)$], [proof$(*,*,*)$], [proof$_2(*,*)$], $[\mathcal{S}(*,*)]$, $[\mathcal{S}^{\text{I}}(*,*)]$, $[\mathcal{S}^{\rhd}(*,*)]$, $[\mathcal{S}^{\rhd}_1(*,*,*)]$, $[\mathcal{S}^{\text{E}}(*,*)]$, $[\mathcal{S}^{\text{E}}_1(*,*,*)]$,

$[\mathcal{S}^+(*,*)]$, $[\mathcal{S}^+_1(*,*,*)]$, $[\mathcal{S}^-(*,*)]$, $[\mathcal{S}^-_1(*,*,*)]$, $[\mathcal{S}^*(*,*)]$, $[\mathcal{S}^*_1(*,*,*)]$, $[\mathcal{S}^*_2(*,*,*,*)]$, $[\mathcal{S}^@(*,*)]$, $[\mathcal{S}^@_1(*,*,*)]$, $[\mathcal{S}^\vdash(*,*)]$, $[\mathcal{S}^\vdash_1(*,*,*,*)]$, $[\mathcal{S}^\Vdash(*,*)]$, $[\mathcal{S}^\Vdash_1(*,*,*,*)]$, $[\mathcal{S}^{\text{i.e.}}(*,*)]$, $[\mathcal{S}^{\text{i.e.}}_1(*,*,*,*)]$, $[\mathcal{S}^{\text{i.e.}}_2(*,*,*,*,*)]$, $[\mathcal{S}^\forall(*,*)]$, $[\mathcal{S}^\forall_1(*,*,*,*)]$, $[\mathcal{S}^;(*,*)]$, $[\mathcal{S}^;_1(*,*,*)]$, $[\mathcal{S}^;_2(*,*,*,*)]$, $[\mathcal{T}(*)]$, $[\text{claims}(*,*,*)]$, $[\text{claims}_2(*,*,*)]$, [<proof>], [proof], [[**Lemma** ∗: ∗]], [[**Proof of** ∗: ∗]], [[∗ **lemma** ∗: ∗]], [[∗ **antilemma** ∗: ∗]], [[∗ **rule** ∗: ∗]], [[∗ **antirule** ∗: ∗]], [verifier], $[\mathcal{V}_1(*)]$, $[\mathcal{V}_2(*,*)]$, $[\mathcal{V}_3(*,*,*,*)]$, $[\mathcal{V}_4(*,*)]$, $[\mathcal{V}_5(*,*,*,*)]$, $[\mathcal{V}_6(*,*,*,*)]$, $[\mathcal{V}_7(*,*,*,*)]$, $[\text{Cut}(*,*)]$, $[\text{Head}_\oplus(*)]$, $[\text{Tail}_\oplus(*)]$, $[\text{rule}_1(*,*)]$, $[\text{rule}(*,*)]$, [Rule tactic], $[\text{Plus}(*,*)]$, [[**Theory** ∗]], $[\text{theory}_2(*,*)]$, $[\text{theory}_3(*,*)]$, $[\text{theory}_4(*,*,*)]$, [HeadNil″], [HeadPair″], [Transitivity″], [Contra″], [HeadNil], [HeadPair], [Transitivity], [Contra], $[\text{T}_\text{E}]$, [ragged right], [ragged right expansion], $[\text{parm}(*,*,*)]$, $[\text{parm}^*(*,*,*)]$, $[\text{inst}(*,*)]$, $[\text{inst}^*(*,*)]$, $[\text{occur}(*,*,*)]$, $[\text{occur}^*(*,*,*)]$, $[\text{unify}(*=*,*)]$, $[\text{unify}^*(*=*,*)]$, $[\text{unify}_2(*=*,*)]$, $[\text{L}_\text{a}]$, $[\text{L}_\text{b}]$, $[\text{L}_\text{c}]$, $[\text{L}_\text{d}]$, $[\text{L}_\text{e}]$, $[\text{L}_\text{f}]$, $[\text{L}_\text{g}]$, $[\text{L}_\text{h}]$, $[\text{L}_\text{i}]$, $[\text{L}_\text{j}]$, $[\text{L}_\text{k}]$, $[\text{L}_\text{l}]$, $[\text{L}_\text{m}]$, $[\text{L}_\text{n}]$, $[\text{L}_\text{o}]$, $[\text{L}_\text{p}]$, $[\text{L}_\text{q}]$, $[\text{L}_\text{r}]$, $[\text{L}_\text{s}]$, $[\text{L}_\text{t}]$, $[\text{L}_\text{u}]$, $[\text{L}_\text{v}]$, $[\text{L}_\text{w}]$, $[\text{L}_\text{x}]$, $[\text{L}_\text{y}]$, $[\text{L}_\text{z}]$, $[\text{L}_\text{A}]$, $[\text{L}_\text{B}]$, $[\text{L}_\text{C}]$, $[\text{L}_\text{D}]$, $[\text{L}_\text{E}]$, $[\text{L}_\text{F}]$, $[\text{L}_\text{G}]$, $[\text{L}_\text{H}]$, $[\text{L}_\text{I}]$, $[\text{L}_\text{J}]$, $[\text{L}_\text{K}]$, $[\text{L}_\text{L}]$, $[\text{L}_\text{M}]$, $[\text{L}_\text{N}]$, $[\text{L}_\text{O}]$, $[\text{L}_\text{P}]$, $[\text{L}_\text{Q}]$, $[\text{L}_\text{R}]$, $[\text{L}_\text{S}]$, $[\text{L}_\text{T}]$, $[\text{L}_\text{U}]$, $[\text{L}_\text{V}]$, $[\text{L}_\text{W}]$, $[\text{L}_\text{X}]$, $[\text{L}_\text{Y}]$, $[\text{L}_\text{Z}]$, $[\text{L}_?]$, [Reflexivity], $[\text{Reflexivity}_1]$, [Commutativity], $[\text{Commutativity}_1]$, [<tactic>], [tactic], $[[* \overset{\text{tactic}}{=} *]]$, $[\mathcal{P}(*,*,*)]$, $[\mathcal{P}^*(*,*,*)]$, $[\text{p}_0]$, $[\text{conclude}_1(*,*)]$, $[\text{conclude}_2(*,*,*)]$, $[\text{conclude}_3(*,*,*,*)]$, $[\text{conclude}_4(*,*)]$, [check], $[[* \overset{\circ}{=} *]]$, $[\text{RootVisible}(*)]$, [A], [R], [C], [T], [L], $[\{*\}]$, $[\overline{*}]$, $[a]$, $[b]$, $[c]$, $[d]$, $[e]$, $[f]$, $[g]$, $[h]$, $[i]$, $[j]$, $[k]$, $[l]$, $[m]$, $[n]$, $[o]$, $[p]$, $[q]$, $[r]$, $[s]$, $[t]$, $[u]$, $[v]$, $[w]$, $[x]$, $[y]$, $[z]$, $[\langle * \equiv * \mid * := * \rangle]$, $[\langle * \equiv^0 * \mid * := * \rangle]$, $[\langle * \equiv^1 * \mid * := * \rangle]$, $[\langle * \equiv^* * \mid * := * \rangle]$, $[\text{Ded}(*,*)]$, $[\text{Ded}_0(*,*)]$, $[\text{Ded}_1(*,*,*)]$, $[\text{Ded}_2(*,*,*)]$, $[\text{Ded}_3(*,*,*,*)]$, $[\text{Ded}_4(*,*,*,*)]$, $[\text{Ded}^*_4(*,*,*,*)]$, $[\text{Ded}_5(*,*,*)]$, $[\text{Ded}_6(*,*,*,*)]$, $[\text{Ded}^*_6(*,*,*,*)]$, $[\text{Ded}_7(*)]$, $[\text{Ded}_8(*,*)]$, $[\text{Ded}^*_8(*,*)]$, [S], [Neg], [MP], [Gen], [Ded], [S1], [S2], [S3], [S4], [S5], [S6], [S7], [S8], [S9], [Repetition], [A1′], [A2′], [A4′], [A5′], [Prop 3.2a], [Prop 3.2b], [Prop 3.2c], [Prop 3.2d], $[\text{Prop 3.2e}_1]$, $[\text{Prop 3.2e}_2]$, [Prop 3.2e], $[\text{Prop 3.2f}_1]$, $[\text{Prop 3.2f}_2]$, [Prop 3.2f], $[\text{Prop 3.2g}_1]$, $[\text{Prop 3.2g}_2]$, [Prop 3.2g], $[\text{Prop 3.2h}_1]$, $[\text{Prop 3.2h}_2]$, [Prop 3.2h], $[\text{Block}_1(*,*,*)]$, $[\text{Block}_2(*)]$, $[(\cdots)]$, [`Objekt-var`], [`Ex-var`], [`Ph-var`], [`Værdi`], [`Variabel`], $[\text{Op}(*)]$, $[\text{Op}(*,*)]$, $[* \mathrel{\ddot{=}\!=} *]$, $[\text{ContainsEmpty}(*)]$, $[\text{Dedu}(*,*)]$, $[\text{Dedu}_0(*,*)]$, $[\text{Dedu}_\text{s}(*,*,*)]$, $[\text{Dedu}_1(*,*,*)]$, $[\text{Dedu}_2(*,*,*)]$, $[\text{Dedu}_3(*,*,*,*)]$, $[\text{Dedu}_4(*,*,*,*)]$, $[\text{Dedu}^*_4(*,*,*,*)]$, $[\text{Dedu}_5(*,*,*)]$, $[\text{Dedu}_6(*,*,*,*)]$, $[\text{Dedu}^*_6(*,*,*,*)]$, $[\text{Dedu}_7(*)]$, $[\text{Dedu}_8(*,*)]$, $[\text{Dedu}^*_8(*,*)]$, $[\text{Ex}_1]$, $[\text{Ex}_2]$, [Ex3], $[\text{Ex}_{10}]$, $[\text{Ex}_{20}]$, $[*_{\text{Ex}}]$, $[*^{\text{Ex}}]$, $[\langle * \equiv * \mid * :== * \rangle_{\text{Ex}}]$, $[\langle * \equiv^0 * \mid * :== * \rangle_{\text{Ex}}]$, $[\langle * \equiv^1 * \mid * :== * \rangle_{\text{Ex}}]$, $[\langle * \equiv^* * \mid * :== * \rangle_{\text{Ex}}]$, $[\text{ph}_1]$, $[\text{ph}_2]$, $[\text{ph}_3]$, $[*_{\text{Ph}}]$, $[*^{\text{Ph}}]$, $[\langle * \equiv * \mid * :== * \rangle_{\text{Ph}}]$, $[\langle * \equiv^0 * \mid * :== * \rangle_{\text{Ph}}]$, $[\langle * \equiv^1 * \mid * :== * \rangle_{\text{Ph}}]$, $[\langle * \equiv^* * \mid * :== * \rangle_{\text{Ph}}]$, $[\mathsf{bs}]$, $[\mathsf{OBS}]$, $[\mathcal{BS}]$, [Ø], [ZFsub], [MP], [Gen], [Repetition], [Neg], [Ded], [ExistIntro], [Extensionality], [Ødef], [PairDef], [UnionDef], [PowerDef], [SeparationDef], [AddDoubleNeg], [RemoveDoubleNeg], [AndCommutativity], [AutoImply], [Contrapositive], [FirstConjunct], [SecondConjunct], [FromContradiction], [FromDisjuncts], [IffCommutativity], [IffFirst], [IffSecond], [ImplyTransitivity], [JoinConjuncts], [MP2], [MP3], [MP4], [MP5], [MT], [NegativeMT], [Technicality], [Weakening], [WeakenOr1], [WeakenOr2], [Formula2Pair], [Pair2Formula], [Formula2Union], [Union2Formula], [Formula2Sep],

[Sep2Formula], [SubsetInPower], [HelperPowerIsSub], [PowerIsSub], [(Switch)HelperPowerIsSub], [(Switch)PowerIsSub], [ToSetEquality], [HelperToSetEquality(t)], [ToSetEquality(t)], [HelperFromSetEquality], [FromSetEquality], [HelperReflexivity], [Reflexivity], [HelperSymmetry], [Symmetry], [HelperTransitivity], [Transitivity], [ERisReflexive], [ERisSymmetric], [ERisTransitive], [∅isSubset], [HelperMemberNot∅], [MemberNot∅], [HelperUnique∅], [Unique∅], [==Reflexivity], [==Symmetry], [Helper==Transitivity], [==Transitivity], [HelperTransferNotEq], [TransferNotEq], [HelperPairSubset], [Helper(2)PairSubset], [PairSubset], [SamePair], [SameSingleton], [UnionSubset], [SameUnion], [SeparationSubset], [SameSeparation], [SameBinaryUnion], [IntersectionSubset], [SameIntersection], [AutoMember], [HelperEqSysNot∅], [EqSysNot∅], [HelperEqSubset], [EqSubset], [HelperEqNecessary], [EqNecessary], [HelperNoneEqNecessary], [Helper(2)NoneEqNecessary], [NoneEqNecessary], [EqClassIsSubset], [EqClassesAreDisjoint], [AllDisjoint], [AllDisjointImply], [BSsubset], [Union(BS/R)subset], [UnionIdentity], [EqSysIsPartition], [(ϵ)], [(fx)], [(fy)], [(fz)], [(fv)], [var fv], [(rx)], [(ry)], [(rz)], [(ru)], [ϵ], [FX], [FY], [FZ], [FU], [FV], [RX], [RY], [RZ], [RU], [0], [1], [(-1)], [2], [1/2], [0f], [1f], [00], [01], [leqReflexivity], [leqAntisymmetryAxiom], [leqTransitivityAxiom], [leqTotality], [leqAdditionAxiom], [leqMultiplicationAxiom], [plusAssociativity], [plusCommutativity], [Negative], [plus0], [timesAssociativity], [timesCommutativity], [ReciprocalAxiom], [times1], [Distribution], [0not1], [equalityAxiom], [eqLeqAxiom], [eqAdditionAxiom], [eqMultiplicationAxiom], [SENC1], [SENC2], [IfThenElse(T)], [IfThenElse(F)], [From = f], [To = f], [From < f], [To < f], [PlusF], [TimesF], [MinusF], [0f], [1f], [FromSF], [ToSF], [To == XX], [From ==], [To ==], [From << XX], [From << (1)], [From << (2)], [to << XX], [From <<], [To <<], [FromInR], [PlusR], [TimesR], [leqAntisymmetry], [leqTransitivity], [leqAddition], [leqMultiplication], [Reciprocal], [Equality], [eqLeq], [eqAddition], [eqMultiplication], [ToNegatedImply], [TND], [ImplyNegation], [FromNegations], [From3Disjuncts], [From2 * 2Disjuncts], [NegateDisjunct1], [NegateDisjunct2], [ExpandDisjuncts], [eqReflexivity], [eqSymmetry], [eqTransitivity], [eqTransitivity4], [eqTransitivity5], [eqTransitivity6], [plus0Left], [times1Left], [lemma eqAdditionLeft], [EqMultiplicationLeft], [DistributionOut], [Three2twoTerms], [Three2threeTerms], [Three2threeFactors], [AddEquations], [SubtractEquations], [SubtractEquationsLeft], [EqNegated], [PositiveToRight(Eq)], [PositiveToLeft(Eq)(1term)], [NegativeToLeft(Eq)], [LessNeq], [NeqSymmetry], [NeqNegated], [SubNeqRight], [SubNeqLeft], [NeqAddition], [NeqMultiplication], [UniqueNegative], [DoubleMinus], [LeqLessEq], [LessLeq], [FromLeqGeq], [subLeqRight], [subLeqLeft], [Leq + 1], [PositiveToRight(Leq)], [PositiveToRight(Leq)(1term)], [negativeToLeft(Leq)], [LeqAdditionLeft], [leqSubtraction], [leqSubtractionLeft], [thirdGeq], [LeqNegated], [AddEquations(Leq)], [ThirdGeqSeries], [LeqNeqLess], [FromLess], [ToLess], [fromNotLess], [toNotLess], [NegativeLessPositive], [leqLessTransitivity], [LessLeqTransitivity], [LessTransitivity], [LessTotality], [SubLessRight], [SubLessLeft], [LessAddition], [LessAdditionLeft],

[LessMultiplication], [LessMultiplicationLeft], [LessDivision],
[AddEquations(Less)], [LessNegated], [PositiveNegated], [NonpositiveNegated],
[NegativeNegated], [NonnegativeNegated], [PositiveHalved],
[NonnegativeNumerical], [NegativeNumerical], [PositiveNumerical],
[lemma nonpositiveNumerical], [|0| = 0], [0 <= |x|], [SameNumerical],
[SignNumerical(+)], [SignNumerical], [NumericalDifference],
[SplitNumericalSumHelper], [splitNumericalSum(++)],
[splitNumericalSum(−−)], [splitNumericalSum(+ − small)],
[splitNumericalSum(+ − big)], [splitNumericalSum(+−)],
[splitNumericalSum(−+)], [splitNumericalSum],
[insertMiddleTerm(Numerical)], [x + y = zBackwards], [x ∗ y = zBackwards],
[x = x + (y − y)], [x = x + y − y], [], [insertMiddleTerm(Sum)],
[insertMiddleTerm(Difference)], [x ∗ 0 + x = x], [x ∗ 0 = 0],
[(−1) ∗ (−1) + (−1) ∗ 1 = 0], [(−1) ∗ (−1) = 1], [0 < 1Helper], [0 < 1], [0 < 2],
[0 < 1/2], [TwoWholes], [TwoHalves], [−x − y = −(x + y)], [MinusNegated],
[Times(−1)], [Times(−1)Left], [−0 = 0], [SFsymmetry], [SFtransitivity],
[= fToSameF], [PlusF(Sym)], [TimesF(Sym)], [f2R(Plus)], [f2R(Times)],
[PlusR(Sym)], [TimesR(Sym)], [LessLeq(R)], [eqLeq(R)], [SubLessRight(R)],
[SubLessLeft(R)], [<< TransitivityHelper(Q)], [<< Transitivity],
[<<== Reflexivity], [<<== AntisymmetryHelper(Q)],
[<<== Antisymmetry], [<<== Transitivity], [Plus0f], [Plus00], [== Addition],
[== AdditionLeft], [<< Addition], [<<== Addition], [PlusAssociativity(F)],
[PlusAssociativity(R)], [Negative(R)], [PlusCommutativity(F)],
[PlusCommutativity(R)], [TimesAssociativity(F)], [TimesAssociativity(R)],
[Times1f], [Times01], [TimesCommutativity(F)], [TimesCommutativity(R)],
[Distribution(F)], [Distribution(R)];

Preassociative

[∗_{∗}], [∗/indexintro(∗, ∗, ∗, ∗)], [∗/intro(∗, ∗, ∗)], [∗/bothintro(∗, ∗, ∗, ∗, ∗)],
[∗/nameintro(∗, ∗, ∗, ∗)], [∗′], [∗[∗]], [∗[∗→∗]], [∗[∗⇒∗]], [∗0], [∗1], [0b], [∗-color(∗)],
[∗-color*(∗)], [$*^{\mathrm{H}}$], [$*^{\mathrm{T}}$], [$*^{\mathrm{U}}$], [$*^{\mathrm{h}}$], [$*^{\mathrm{t}}$], [$*^{\mathrm{s}}$], [$*^{\mathrm{c}}$], [$*^{\mathrm{d}}$], [$*^{\mathrm{a}}$], [$*^{\mathrm{C}}$], [$*^{\mathrm{M}}$], [$*^{\mathrm{B}}$], [$*^{\mathrm{r}}$], [$*^{\mathrm{i}}$],
[$*^{\mathrm{d}}$], [$*^{\mathrm{R}}$], [$*^{0}$], [$*^{1}$], [$*^{2}$], [$*^{3}$], [$*^{4}$], [$*^{5}$], [$*^{6}$], [$*^{7}$], [$*^{8}$], [$*^{9}$], [$*^{\mathrm{E}}$], [$*^{\mathcal{V}}$], [$*^{\mathcal{C}}$], [$*^{\mathcal{C}^*}$],
[$*^{\mathrm{hide}}$];

Preassociative

[“ ∗ ”], [], [(∗)$^{\mathbf{t}}$], [string(∗) + ∗], [string(∗) ++ ∗], [
∗], [∗], [!∗], ["∗], [#∗], [$∗], [%∗], [&∗], [’∗], [(∗], [)∗], [∗∗], [+∗], [, ∗], [-∗], [.∗], [/∗],
[0∗], [1∗], [2∗], [3∗], [4∗], [5∗], [6∗], [7∗], [8∗], [9∗], [:∗], [; ∗], [<∗], [=∗], [>∗], [?∗],
[@∗], [A∗], [B∗], [C∗], [D∗], [E∗], [F∗], [G∗], [H∗], [I∗], [J∗], [K∗], [L∗], [M∗], [N∗],
[O∗], [P∗], [Q∗], [R∗], [S∗], [T∗], [U∗], [V∗], [W∗], [X∗], [Y∗], [Z∗], [[∗], [\∗], []∗], [ˆ∗],
[_∗], [‘∗], [a∗], [b∗], [c∗], [d∗], [e∗], [f∗], [g∗], [h∗], [i∗], [j∗], [k∗], [l∗], [m∗], [n∗], [o∗],
[p∗], [q∗], [r∗], [s∗], [t∗], [u∗], [v∗], [w∗], [x∗], [y∗], [z∗], [{∗], [|∗], [}∗], [˜∗],
[**Preassociative** ∗; ∗], [**Postassociative** ∗; ∗], [[∗], ∗], [priority ∗ end],
[newline ∗], [macro newline ∗], [MacroIndent(∗)];

Preassociative

[∗ ’ ∗], [∗ ‘ ∗];

Preassociative

[∗′], [R(∗)], [− − R(∗)], [rec∗];

Preassociative
$[*/*], [* \cap *], [*[*]];$
Preassociative
$[\cup *], [* \cup *], [\mathrm{P}(*)];$
Preassociative
$[\{*\}];$
Preassociative
$[\{*, *\}], [\langle *, * \rangle], [-*], [-_{\mathrm{f}} *];$
Preassociative
$[* \in *], [*(*, *)], [\mathrm{ReflRel}(*, *)], [\mathrm{SymRel}(*, *)], [\mathrm{TransRel}(*, *)], [\mathrm{EqRel}(*, *)],$
$[[* \in *]_*], [\mathrm{Partition}(*, *)];$
Preassociative
$[* \cdot *], [* \cdot_0 *], [* * *], [* *_{\mathrm{f}} *], [* * **];$
Preassociative
$[* + *], [* +_0 *], [* +_1 *], [* - *], [* -_0 *], [* -_1 *], [* + *], [* - *], [* +_{\mathrm{f}} *], [* -_{\mathrm{f}} *],$
$[* + +*], [\mathrm{R}(*) - -\mathrm{R}(*)];$
Preassociative
$[| * |], [\mathrm{if}(*, *, *)];$
Preassociative
$[* = *], [* \neq *], [* <= *], [* < *], [* =_{\mathrm{f}} *], [* <_{\mathrm{f}} *], [\mathrm{SF}(*, *)], [* == *], [* << *],$
$[* <<== *];$
Preassociative
$[* \cup \{*\}], [* \cup *], [* \backslash \{*\}];$
Postassociative
$[* \therefore *], [* \underline{\therefore} *], [* \underline{::} *], [* \underline{+2*} *], [* :: *], [* +2* *];$
Postassociative
$[*, *];$
Preassociative
$[* \stackrel{\mathrm{B}}{\approx} *], [* \stackrel{\mathrm{D}}{\approx} *], [* \stackrel{\mathrm{C}}{\approx} *], [* \stackrel{\mathrm{P}}{\approx} *], [* \approx *], [* = *], [* \stackrel{+}{\to} *], [* \stackrel{\mathrm{t}}{=} *], [* \stackrel{\mathrm{t}^*}{=} *], [* \stackrel{\mathrm{r}}{=} *],$
$[* \in_{\mathrm{t}} *], [* \subseteq_{\mathrm{T}} *], [* \stackrel{\mathrm{T}}{=} *], [* \stackrel{\mathrm{s}}{=} *], [* \text{ free in } *], [* \text{ free in}^* *], [* \text{ free for } * \text{ in } *],$
$[* \text{ free for}^* * \text{ in } *], [* \in_{\mathrm{c}} *], [* < *], [* <' *], [* \leq' *], [* = *], [* \neq *], [*^{\mathrm{var}}],$
$[* \#^0 *], [* \#^1 *], [* \#^* *], [* == *], [* \subseteq *];$
Preassociative
$[\neg *], [\dot{\neg} *], [* \notin *], [* \neq *];$
Preassociative
$[* \wedge *], [* \ddot{\wedge} *], [* \tilde{\wedge} *], [* \wedge_{\mathrm{c}} *], [* \dot{\wedge} *];$
Preassociative
$[* \vee *], [* \parallel *], [* \ddot{\vee} *];$
Postassociative
$[* \dot{\vee} *];$
Preassociative
$[\exists *: *], [\forall *: *], [\forall_{\mathrm{obj}} *: *];$
Postassociative
$[* \ddot{\Rightarrow} *], [* \Rightarrow *], [* \Leftrightarrow *], [* \dot{\Leftrightarrow} *];$
Preassociative

$[\{\mathrm{ph} \in * \mid *\}]$;
Postassociative
$[* : *], [* \text{ spy } *], [*!*]$;
Preassociative
$[* \left\{ \begin{array}{c} * \\ * \end{array} \right.]$;
Preassociative
$[\lambda * .*], [\Lambda * .*], [\Lambda *], [\textbf{if } * \textbf{ then } * \textbf{ else } *], [\textbf{let } * = * \textbf{ in } *], [\textbf{let } * \stackrel{..}{=} * \text{ in } *]$;
Preassociative
$[*\#*]$;
Preassociative
$[*^{\mathrm{I}}], [*^{\rhd}], [*^{\mathrm{V}}], [*^{+}], [*^{-}], [*^{*}]$;
Preassociative
$[* @ *], [* \rhd *], [* \blacktriangleright *], [* \gg *], [* \trianglerighteq *]$;
Postassociative
$[* \vdash *], [* \Vdash *], [* \text{ i.e. } *]$;
Preassociative
$[\forall *: *], [\Pi *: *]$;
Postassociative
$[* \oplus *]$;
Postassociative
$[*; *]$;
Preassociative
$[* \text{ proves } *]$;
Preassociative
$[* \textbf{ proof of } * : *], [\text{Line} * : * \gg *; *], [\text{Last line} * \gg * \square]$,
$[\text{Line} * : \text{Premise} \gg *; *], [\text{Line} * : \text{Side-condition} \gg *; *], [\text{Arbitrary} \gg *; *]$,
$[\text{Local} \gg * = *; *], [\text{Begin} *; * : \text{End}; *], [\text{Last block line} * \gg * ;]$,
$[\text{Arbitrary} \gg *; *]$;
Postassociative
$[* \mid *]$;
Postassociative
$[* , *], [*[*]*]$;
Preassociative
$[*\&*]$;
Preassociative
$[*\backslash\backslash*], [* \text{ linebreak[4] } *], [*\backslash\backslash*];]$

A Pyk definitioner

$[(\cdots) \stackrel{\text{pyk}}{\to}$ "cdots"$]$

$[$`Objekt-var` $\stackrel{\text{pyk}}{\to}$ "object-var"$]$

$[$`Ex-var` $\stackrel{\text{pyk}}{\to}$ "ex-var"$]$

$[$`Ph-var` $\stackrel{\text{pyk}}{\to}$ "ph-var"$]$

[$\texttt{Værdi} \overset{\text{pyk}}{\rightarrow}$ "vaerdi"]

[$\texttt{Variabel} \overset{\text{pyk}}{\rightarrow}$ "variabel"]

[$\mathrm{Op}(*) \overset{\text{pyk}}{\rightarrow}$ "op " end op"]

[$\mathrm{Op}(*,*) \overset{\text{pyk}}{\rightarrow}$ "op2 " comma " end op2"]

[$* \stackrel{..}{==} * \overset{\text{pyk}}{\rightarrow}$ "define-equal " comma " end equal"]

[$\mathrm{ContainsEmpty}(*) \overset{\text{pyk}}{\rightarrow}$ "contains-empty " end empty"]

[$\mathrm{Dedu}(*,*) \overset{\text{pyk}}{\rightarrow}$ "1deduction " conclude " end 1deduction"]

[$\mathrm{Dedu}_0(*,*) \overset{\text{pyk}}{\rightarrow}$ "1deduction zero " conclude " end 1deduction"]

[$\mathrm{Dedu}_s(*,*,*) \overset{\text{pyk}}{\rightarrow}$ "1deduction side " conclude " condition " end 1deduction"]

[$\mathrm{Dedu}_1(*,*,*) \overset{\text{pyk}}{\rightarrow}$ "1deduction one " conclude " condition " end 1deduction"]

[$\mathrm{Dedu}_2(*,*,*) \overset{\text{pyk}}{\rightarrow}$ "1deduction two " conclude " condition " end 1deduction"]

[$\mathrm{Dedu}_3(*,*,*,*) \overset{\text{pyk}}{\rightarrow}$ "1deduction three " conclude " condition " bound " end 1deduction"]

[$\mathrm{Dedu}_4(*,*,*,*) \overset{\text{pyk}}{\rightarrow}$ "1deduction four " conclude " condition " bound " end 1deduction"]

[$\mathrm{Dedu}_4^*(*,*,*,*) \overset{\text{pyk}}{\rightarrow}$ "1deduction four star " conclude " condition " bound " end 1deduction"]

[$\mathrm{Dedu}_5(*,*,*) \overset{\text{pyk}}{\rightarrow}$ "1deduction five " condition " bound " end 1deduction"]

[$\mathrm{Dedu}_6(*,*,*,*) \overset{\text{pyk}}{\rightarrow}$ "1deduction six " conclude " exception " bound " end 1deduction"]

[$\mathrm{Dedu}_6^*(*,*,*,*) \overset{\text{pyk}}{\rightarrow}$ "1deduction six star " conclude " exception " bound " end 1deduction"]

[$\mathrm{Dedu}_7(*) \overset{\text{pyk}}{\rightarrow}$ "1deduction seven " end 1deduction"]

[$\mathrm{Dedu}_8(*,*) \overset{\text{pyk}}{\rightarrow}$ "1deduction eight " bound " end 1deduction"]

[$\mathrm{Dedu}_8^*(*,*) \overset{\text{pyk}}{\rightarrow}$ "1deduction eight star " bound " end 1deduction"]

[$\mathrm{Ex}_1 \overset{\text{pyk}}{\rightarrow}$ "ex1"]

[$\mathrm{Ex}_2 \overset{\text{pyk}}{\rightarrow}$ "ex2"]

[$\mathrm{Ex3} \overset{\text{pyk}}{\rightarrow}$ "ex3"]

[$\mathrm{Ex}_{10} \overset{\text{pyk}}{\rightarrow}$ "ex10"]

[$\mathrm{Ex}_{20} \overset{\text{pyk}}{\rightarrow}$ "ex20"]

[$*_{\mathrm{Ex}} \overset{\text{pyk}}{\rightarrow}$ "existential var " end var"]

[$*^{\mathrm{Ex}} \overset{\text{pyk}}{\rightarrow}$ "" is existential var"]

[$\langle * \equiv * \mid * := = * \rangle_{\mathrm{Ex}} \overset{\text{pyk}}{\rightarrow}$ "exist-sub " is " where " is " end sub"]

[$\langle * \equiv^0 * \mid * := = * \rangle_{\mathrm{Ex}} \overset{\text{pyk}}{\rightarrow}$ "exist-sub0 " is " where " is " end sub"]

[$\langle * \equiv^1 * \mid * := = * \rangle_{\mathrm{Ex}} \overset{\text{pyk}}{\rightarrow}$ "exist-sub1 " is " where " is " end sub"]

[$\langle * \equiv^* * \mid * := = * \rangle_{\mathrm{Ex}} \overset{\text{pyk}}{\rightarrow}$ "exist-sub∗ " is " where " is " end sub"]

[$ph_1 \stackrel{\text{pyk}}{\to}$ "placeholder-var1"]
[$ph_2 \stackrel{\text{pyk}}{\to}$ "placeholder-var2"]
[$ph_3 \stackrel{\text{pyk}}{\to}$ "placeholder-var3"]
[$*_{\text{Ph}} \stackrel{\text{pyk}}{\to}$ "placeholder-var " end var"]
[$*^{\text{Ph}} \stackrel{\text{pyk}}{\to}$ "" is placeholder-var"]
[$\langle * \equiv * \mid * := = * \rangle_{\text{Ph}} \stackrel{\text{pyk}}{\to}$ "ph-sub " is " where " is " end sub"]
[$\langle * \equiv^0 * \mid * := = * \rangle_{\text{Ph}} \stackrel{\text{pyk}}{\to}$ "ph-sub0 " is " where " is " end sub"]
[$\langle * \equiv^1 * \mid * := = * \rangle_{\text{Ph}} \stackrel{\text{pyk}}{\to}$ "ph-sub1 " is " where " is " end sub"]
[$\langle * \equiv^* * \mid * := = * \rangle_{\text{Ph}} \stackrel{\text{pyk}}{\to}$ "ph-sub∗ " is " where " is " end sub"]
[$\mathsf{bs} \stackrel{\text{pyk}}{\to}$ "var big set"]
[$\mathsf{OBS} \stackrel{\text{pyk}}{\to}$ "object big set"]
[$\mathcal{BS} \stackrel{\text{pyk}}{\to}$ "meta big set"]
[$\emptyset \stackrel{\text{pyk}}{\to}$ "zermelo empty set"]
[ZFsub $\stackrel{\text{pyk}}{\to}$ "system Q"]
[MP $\stackrel{\text{pyk}}{\to}$ "1rule mp"]
[Gen $\stackrel{\text{pyk}}{\to}$ "1rule gen"]
[Repetition $\stackrel{\text{pyk}}{\to}$ "1rule repetition"]
[Neg $\stackrel{\text{pyk}}{\to}$ "1rule ad absurdum"]
[Ded $\stackrel{\text{pyk}}{\to}$ "1rule deduction"]
[ExistIntro $\stackrel{\text{pyk}}{\to}$ "1rule exist intro"]
[Extensionality $\stackrel{\text{pyk}}{\to}$ "axiom extensionality"]
[Ødef $\stackrel{\text{pyk}}{\to}$ "axiom empty set"]
[PairDef $\stackrel{\text{pyk}}{\to}$ "axiom pair definition"]
[UnionDef $\stackrel{\text{pyk}}{\to}$ "axiom union definition"]
[PowerDef $\stackrel{\text{pyk}}{\to}$ "axiom power definition"]
[SeparationDef $\stackrel{\text{pyk}}{\to}$ "axiom separation definition"]
[AddDoubleNeg $\stackrel{\text{pyk}}{\to}$ "prop lemma add double neg"]
[RemoveDoubleNeg $\stackrel{\text{pyk}}{\to}$ "prop lemma remove double neg"]
[AndCommutativity $\stackrel{\text{pyk}}{\to}$ "prop lemma and commutativity"]
[AutoImply $\stackrel{\text{pyk}}{\to}$ "prop lemma auto imply"]
[Contrapositive $\stackrel{\text{pyk}}{\to}$ "prop lemma contrapositive"]
[FirstConjunct $\stackrel{\text{pyk}}{\to}$ "prop lemma first conjunct"]
[SecondConjunct $\stackrel{\text{pyk}}{\to}$ "prop lemma second conjunct"]
[FromContradiction $\stackrel{\text{pyk}}{\to}$ "prop lemma from contradiction"]
[FromDisjuncts $\stackrel{\text{pyk}}{\to}$ "prop lemma from disjuncts"]

[IffCommutativity $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma iff commutativity”]

[IffFirst $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma iff first”]

[IffSecond $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma iff second”]

[ImplyTransitivity $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma imply transitivity”]

[JoinConjuncts $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma join conjuncts”]

[MP2 $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma mp2”]

[MP3 $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma mp3”]

[MP4 $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma mp4”]

[MP5 $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma mp5”]

[MT $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma mt”]

[NegativeMT $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma negative mt”]

[Technicality $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma technicality”]

[Weakening $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma weakening”]

[WeakenOr1 $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma weaken or first”]

[WeakenOr2 $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma weaken or second”]

[Formula2Pair $\stackrel{\text{pyk}}{\rightarrow}$ “lemma formula2pair”]

[Pair2Formula $\stackrel{\text{pyk}}{\rightarrow}$ “lemma pair2formula”]

[Formula2Union $\stackrel{\text{pyk}}{\rightarrow}$ “lemma formula2union”]

[Union2Formula $\stackrel{\text{pyk}}{\rightarrow}$ “lemma union2formula”]

[Formula2Sep $\stackrel{\text{pyk}}{\rightarrow}$ “lemma formula2separation”]

[Sep2Formula $\stackrel{\text{pyk}}{\rightarrow}$ “lemma separation2formula”]

[SubsetInPower $\stackrel{\text{pyk}}{\rightarrow}$ “lemma subset in power set”]

[HelperPowerIsSub $\stackrel{\text{pyk}}{\rightarrow}$ “lemma power set is subset0”]

[PowerIsSub $\stackrel{\text{pyk}}{\rightarrow}$ “lemma power set is subset”]

[(Switch)HelperPowerIsSub $\stackrel{\text{pyk}}{\rightarrow}$ “lemma power set is subset0-switch”]

[(Switch)PowerIsSub $\stackrel{\text{pyk}}{\rightarrow}$ “lemma power set is subset-switch”]

[ToSetEquality $\stackrel{\text{pyk}}{\rightarrow}$ “lemma set equality suff condition”]

[HelperToSetEquality(t) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma set equality suff condition(t)0”]

[ToSetEquality(t) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma set equality suff condition(t)”]

[HelperFromSetEquality $\stackrel{\text{pyk}}{\rightarrow}$ “lemma set equality skip quantifier”]

[FromSetEquality $\stackrel{\text{pyk}}{\rightarrow}$ “lemma set equality nec condition”]

[HelperReflexivity $\stackrel{\text{pyk}}{\rightarrow}$ “lemma reflexivity0”]

[Reflexivity $\stackrel{\text{pyk}}{\rightarrow}$ “lemma reflexivity”]

[HelperSymmetry $\stackrel{\text{pyk}}{\rightarrow}$ “lemma symmetry0”]

[Symmetry $\stackrel{\text{pyk}}{\rightarrow}$ “lemma symmetry”]

[HelperTransitivity $\stackrel{\text{pyk}}{\to}$ “lemma transitivity0”]

[Transitivity $\stackrel{\text{pyk}}{\to}$ “lemma transitivity”]

[ERisReflexive $\stackrel{\text{pyk}}{\to}$ “lemma er is reflexive”]

[ERisSymmetric $\stackrel{\text{pyk}}{\to}$ “lemma er is symmetric”]

[ERisTransitive $\stackrel{\text{pyk}}{\to}$ “lemma er is transitive”]

[ØisSubset $\stackrel{\text{pyk}}{\to}$ “lemma empty set is subset”]

[HelperMemberNotØ $\stackrel{\text{pyk}}{\to}$ “lemma member not empty0”]

[MemberNotØ $\stackrel{\text{pyk}}{\to}$ “lemma member not empty”]

[HelperUniqueØ $\stackrel{\text{pyk}}{\to}$ “lemma unique empty set0”]

[UniqueØ $\stackrel{\text{pyk}}{\to}$ “lemma unique empty set”]

[==Reflexivity $\stackrel{\text{pyk}}{\to}$ “lemma ==Reflexivity”]

[==Symmetry $\stackrel{\text{pyk}}{\to}$ “lemma ==Symmetry”]

[Helper==Transitivity $\stackrel{\text{pyk}}{\to}$ “lemma ==Transitivity0”]

[==Transitivity $\stackrel{\text{pyk}}{\to}$ “lemma ==Transitivity”]

[HelperTransferNotEq $\stackrel{\text{pyk}}{\to}$ “lemma transfer ˜is0”]

[TransferNotEq $\stackrel{\text{pyk}}{\to}$ “lemma transfer ˜is”]

[HelperPairSubset $\stackrel{\text{pyk}}{\to}$ “lemma pair subset0”]

[Helper(2)PairSubset $\stackrel{\text{pyk}}{\to}$ “lemma pair subset1”]

[PairSubset $\stackrel{\text{pyk}}{\to}$ “lemma pair subset”]

[SamePair $\stackrel{\text{pyk}}{\to}$ “lemma same pair”]

[SameSingleton $\stackrel{\text{pyk}}{\to}$ “lemma same singleton”]

[UnionSubset $\stackrel{\text{pyk}}{\to}$ “lemma union subset”]

[SameUnion $\stackrel{\text{pyk}}{\to}$ “lemma same union”]

[SeparationSubset $\stackrel{\text{pyk}}{\to}$ “lemma separation subset”]

[SameSeparation $\stackrel{\text{pyk}}{\to}$ “lemma same separation”]

[SameBinaryUnion $\stackrel{\text{pyk}}{\to}$ “lemma same binary union”]

[IntersectionSubset $\stackrel{\text{pyk}}{\to}$ “lemma intersection subset”]

[SameIntersection $\stackrel{\text{pyk}}{\to}$ “lemma same intersection”]

[AutoMember $\stackrel{\text{pyk}}{\to}$ “lemma auto member”]

[HelperEqSysNotØ $\stackrel{\text{pyk}}{\to}$ “lemma eq-system not empty0”]

[EqSysNotØ $\stackrel{\text{pyk}}{\to}$ “lemma eq-system not empty”]

[HelperEqSubset $\stackrel{\text{pyk}}{\to}$ “lemma eq subset0”]

[EqSubset $\stackrel{\text{pyk}}{\to}$ “lemma eq subset”]

[HelperEqNecessary $\stackrel{\text{pyk}}{\to}$ “lemma equivalence nec condition0”]

[EqNecessary $\stackrel{\text{pyk}}{\to}$ “lemma equivalence nec condition”]

[HelperNoneEqNecessary $\stackrel{\text{pyk}}{\to}$ "lemma none-equivalence nec condition0"]

[Helper(2)NoneEqNecessary $\stackrel{\text{pyk}}{\to}$ "lemma none-equivalence nec condition1"]

[NoneEqNecessary $\stackrel{\text{pyk}}{\to}$ "lemma none-equivalence nec condition"]

[EqClassIsSubset $\stackrel{\text{pyk}}{\to}$ "lemma equivalence class is subset"]

[EqClassesAreDisjoint $\stackrel{\text{pyk}}{\to}$ "lemma equivalence classes are disjoint"]

[AllDisjoint $\stackrel{\text{pyk}}{\to}$ "lemma all disjoint"]

[AllDisjointImply $\stackrel{\text{pyk}}{\to}$ "lemma all disjoint-imply"]

[BSsubset $\stackrel{\text{pyk}}{\to}$ "lemma bs subset union(bs/r)"]

[Union(BS/R)subset $\stackrel{\text{pyk}}{\to}$ "lemma union(bs/r) subset bs"]

[UnionIdentity $\stackrel{\text{pyk}}{\to}$ "lemma union(bs/r) is bs"]

[EqSysIsPartition $\stackrel{\text{pyk}}{\to}$ "theorem eq-system is partition"]

[(ϵ) $\stackrel{\text{pyk}}{\to}$ "var ep"]

[(fx) $\stackrel{\text{pyk}}{\to}$ "var fx"]

[(fy) $\stackrel{\text{pyk}}{\to}$ "var fy"]

[(fz) $\stackrel{\text{pyk}}{\to}$ "var fz"]

[(fv) $\stackrel{\text{pyk}}{\to}$ "var fu"]

[var fv $\stackrel{\text{pyk}}{\to}$ "var fv"]

[(rx) $\stackrel{\text{pyk}}{\to}$ "var rx"]

[(ry) $\stackrel{\text{pyk}}{\to}$ "var ry"]

[(rz) $\stackrel{\text{pyk}}{\to}$ "var rz"]

[(ru) $\stackrel{\text{pyk}}{\to}$ "var ru"]

[ϵ $\stackrel{\text{pyk}}{\to}$ "meta ep"]

[FX $\stackrel{\text{pyk}}{\to}$ "meta fx"]

[FY $\stackrel{\text{pyk}}{\to}$ "meta fy"]

[FZ $\stackrel{\text{pyk}}{\to}$ "meta fz"]

[FU $\stackrel{\text{pyk}}{\to}$ "meta fu"]

[FV $\stackrel{\text{pyk}}{\to}$ "meta fv"]

[RX $\stackrel{\text{pyk}}{\to}$ "meta rx"]

[RY $\stackrel{\text{pyk}}{\to}$ "meta ry"]

[RZ $\stackrel{\text{pyk}}{\to}$ "meta rz"]

[RU $\stackrel{\text{pyk}}{\to}$ "meta ru"]

[0 $\stackrel{\text{pyk}}{\to}$ "0"]

[1 $\stackrel{\text{pyk}}{\to}$ "1"]

[(-1) $\stackrel{\text{pyk}}{\to}$ "(-1)"]

[2 $\stackrel{\text{pyk}}{\to}$ "2"]

[1/2 $\stackrel{\text{pyk}}{\to}$ “1/2”]
[0f $\stackrel{\text{pyk}}{\to}$ “0f”]
[1f $\stackrel{\text{pyk}}{\to}$ “1f”]
[00 $\stackrel{\text{pyk}}{\to}$ “00”]
[01 $\stackrel{\text{pyk}}{\to}$ “01”]
[leqReflexivity $\stackrel{\text{pyk}}{\to}$ “axiom leqReflexivity”]
[leqAntisymmetryAxiom $\stackrel{\text{pyk}}{\to}$ “axiom leqAntisymmetry”]
[leqTransitivityAxiom $\stackrel{\text{pyk}}{\to}$ “axiom leqTransitivity”]
[leqTotality $\stackrel{\text{pyk}}{\to}$ “axiom leqTotality”]
[leqAdditionAxiom $\stackrel{\text{pyk}}{\to}$ “axiom leqAddition”]
[leqMultiplicationAxiom $\stackrel{\text{pyk}}{\to}$ “axiom leqMultiplication”]
[plusAssociativity $\stackrel{\text{pyk}}{\to}$ “axiom plusAssociativity”]
[plusCommutativity $\stackrel{\text{pyk}}{\to}$ “axiom plusCommutativity”]
[Negative $\stackrel{\text{pyk}}{\to}$ “axiom negative”]
[plus0 $\stackrel{\text{pyk}}{\to}$ “axiom plus0”]
[timesAssociativity $\stackrel{\text{pyk}}{\to}$ “axiom timesAssociativity”]
[timesCommutativity $\stackrel{\text{pyk}}{\to}$ “axiom timesCommutativity”]
[ReciprocalAxiom $\stackrel{\text{pyk}}{\to}$ “axiom reciprocal”]
[times1 $\stackrel{\text{pyk}}{\to}$ “axiom times1”]
[Distribution $\stackrel{\text{pyk}}{\to}$ “axiom distribution”]
[0not1 $\stackrel{\text{pyk}}{\to}$ “axiom 0not1”]
[equalityAxiom $\stackrel{\text{pyk}}{\to}$ “axiom equality”]
[eqLeqAxiom $\stackrel{\text{pyk}}{\to}$ “axiom eqLeq”]
[eqAdditionAxiom $\stackrel{\text{pyk}}{\to}$ “axiom eqAddition”]
[eqMultiplicationAxiom $\stackrel{\text{pyk}}{\to}$ “axiom eqMultiplication”]
[SENC1 $\stackrel{\text{pyk}}{\to}$ “lemma set equality nec condition(1)”]
[SENC2 $\stackrel{\text{pyk}}{\to}$ “lemma set equality nec condition(2)”]
[IfThenElse(T) $\stackrel{\text{pyk}}{\to}$ “1rule ifThenElse true”]
[IfThenElse(F) $\stackrel{\text{pyk}}{\to}$ “1rule ifThenElse false”]
[From = f $\stackrel{\text{pyk}}{\to}$ “1rule from=f”]
[To = f $\stackrel{\text{pyk}}{\to}$ “1rule to=f”]
[From < f $\stackrel{\text{pyk}}{\to}$ “1rule from<f”]
[To < f $\stackrel{\text{pyk}}{\to}$ “1rule to<f”]
[PlusF $\stackrel{\text{pyk}}{\to}$ “axiom plusF”]
[TimesF $\stackrel{\text{pyk}}{\to}$ “axiom timesF”]

[MinusF $\stackrel{\text{pyk}}{\rightarrow}$ “axiom minusF”]

[0f $\stackrel{\text{pyk}}{\rightarrow}$ “axiom 0f”]

[1f $\stackrel{\text{pyk}}{\rightarrow}$ “axiom 1f”]

[FromSF $\stackrel{\text{pyk}}{\rightarrow}$ “1rule fromSameF”]

[ToSF $\stackrel{\text{pyk}}{\rightarrow}$ “1rule toSameF”]

[To == XX $\stackrel{\text{pyk}}{\rightarrow}$ “1rule to==XX”]

[From ==$\stackrel{\text{pyk}}{\rightarrow}$ “1rule from==”]

[To ==$\stackrel{\text{pyk}}{\rightarrow}$ “1rule to==”]

[From << XX $\stackrel{\text{pyk}}{\rightarrow}$ “1rule from<<XX”]

[From << (1) $\stackrel{\text{pyk}}{\rightarrow}$ “1rule from<<XX(1)”]

[From << (2) $\stackrel{\text{pyk}}{\rightarrow}$ “1rule from<<XX(2)”]

[to << XX $\stackrel{\text{pyk}}{\rightarrow}$ “1rule to<<XX”]

[From <<$\stackrel{\text{pyk}}{\rightarrow}$ “1rule from<<”]

[To <<$\stackrel{\text{pyk}}{\rightarrow}$ “1rule to<<”]

[FromInR $\stackrel{\text{pyk}}{\rightarrow}$ “1rule fromInR”]

[PlusR $\stackrel{\text{pyk}}{\rightarrow}$ “axiom plusR”]

[TimesR $\stackrel{\text{pyk}}{\rightarrow}$ “axiom timesR”]

[leqAntisymmetry $\stackrel{\text{pyk}}{\rightarrow}$ “lemma leqAntisymmetry”]

[leqTransitivity $\stackrel{\text{pyk}}{\rightarrow}$ “lemma leqTransitivity”]

[leqAddition $\stackrel{\text{pyk}}{\rightarrow}$ “lemma leqAddition”]

[leqMultiplication $\stackrel{\text{pyk}}{\rightarrow}$ “lemma leqMultiplication”]

[Reciprocal $\stackrel{\text{pyk}}{\rightarrow}$ “lemma reciprocal”]

[Equality $\stackrel{\text{pyk}}{\rightarrow}$ “lemma equality”]

[eqLeq $\stackrel{\text{pyk}}{\rightarrow}$ “lemma eqLeq”]

[eqAddition $\stackrel{\text{pyk}}{\rightarrow}$ “lemma eqAddition”]

[eqMultiplication $\stackrel{\text{pyk}}{\rightarrow}$ “lemma eqMultiplication”]

[ToNegatedImply $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma to negated imply”]

[TND $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma tertium non datur”]

[ImplyNegation $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma imply negation”]

[FromNegations $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma from negations”]

[From3Disjuncts $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma from three disjuncts”]

[From2 $*$ 2Disjuncts $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma from two times two disjuncts”]

[NegateDisjunct1 $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma negate first disjunct”]

[NegateDisjunct2 $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma negate second disjunct”]

[ExpandDisjuncts $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma expand disjuncts”] [eqReflexivity $\stackrel{\text{pyk}}{\to}$ "lemma eqReflexivity"]

[eqSymmetry $\stackrel{\text{pyk}}{\to}$ "lemma eqSymmetry"]

[eqTransitivity $\stackrel{\text{pyk}}{\to}$ "lemma eqTransitivity"]

[eqTransitivity4 $\stackrel{\text{pyk}}{\to}$ "lemma eqTransitivity4"]

[eqTransitivity5 $\stackrel{\text{pyk}}{\to}$ "lemma eqTransitivity5"]

[eqTransitivity6 $\stackrel{\text{pyk}}{\to}$ "lemma eqTransitivity6"]

[plus0Left $\stackrel{\text{pyk}}{\to}$ "lemma plus0Left"]

[times1Left $\stackrel{\text{pyk}}{\to}$ "lemma times1Left"]

[lemma eqAdditionLeft $\stackrel{\text{pyk}}{\to}$ "lemma eqAdditionLeft"]

[EqMultiplicationLeft $\stackrel{\text{pyk}}{\to}$ "lemma eqMultiplicationLeft"]

[DistributionOut $\stackrel{\text{pyk}}{\to}$ "lemma distributionOut"]

[Three2twoTerms $\stackrel{\text{pyk}}{\to}$ "lemma three2twoTerms"]

[Three2threeTerms $\stackrel{\text{pyk}}{\to}$ "lemma three2threeTerms"]

[Three2threeFactors $\stackrel{\text{pyk}}{\to}$ "lemma three2twoFactors"]

[AddEquations $\stackrel{\text{pyk}}{\to}$ "lemma addEquations"]

[SubtractEquations $\stackrel{\text{pyk}}{\to}$ "lemma subtractEquations"]

[SubtractEquationsLeft $\stackrel{\text{pyk}}{\to}$ "lemma subtractEquationsLeft"]

[EqNegated $\stackrel{\text{pyk}}{\to}$ "lemma eqNegated"]

[PositiveToRight(Eq) $\stackrel{\text{pyk}}{\to}$ "lemma positiveToRight(Eq)"]

[PositiveToLeft(Eq)(1term) $\stackrel{\text{pyk}}{\to}$ "lemma positiveToLeft(Eq)(1 term)"]

[NegativeToLeft(Eq) $\stackrel{\text{pyk}}{\to}$ "lemma negativeToLeft(Eq)"]

[LessNeq $\stackrel{\text{pyk}}{\to}$ "lemma lessNeq"]

[NeqSymmetry $\stackrel{\text{pyk}}{\to}$ "lemma neqSymmetry"]

[NeqNegated $\stackrel{\text{pyk}}{\to}$ "lemma neqNegated"]

[SubNeqRight $\stackrel{\text{pyk}}{\to}$ "lemma subNeqRight"]

[SubNeqLeft $\stackrel{\text{pyk}}{\to}$ "lemma subNeqLeft"]

[NeqAddition $\stackrel{\text{pyk}}{\to}$ "lemma neqAddition"]

[NeqMultiplication $\stackrel{\text{pyk}}{\to}$ "lemma neqMultiplication"]

[UniqueNegative $\stackrel{\text{pyk}}{\to}$ "lemma uniqueNegative"]

[DoubleMinus $\stackrel{\text{pyk}}{\to}$ "lemma doubleMinus"]

[LeqLessEq $\stackrel{\text{pyk}}{\to}$ "lemma leqLessEq"]

[LessLeq $\stackrel{\text{pyk}}{\to}$ "lemma lessLeq"]

[FromLeqGeq $\stackrel{\text{pyk}}{\to}$ "lemma from leqGeq"]

[subLeqRight $\stackrel{\text{pyk}}{\to}$ "lemma subLeqRight"]

[subLeqLeft $\stackrel{\text{pyk}}{\to}$ "lemma subLeqLeft"]

$[\mathrm{Leq}+1 \stackrel{\mathrm{pyk}}{\to}$ "lemma leqPlus1"]

[PositiveToRight(Leq) $\stackrel{\mathrm{pyk}}{\to}$ "lemma positiveToRight(Leq)"]

[PositiveToRight(Leq)(1term) $\stackrel{\mathrm{pyk}}{\to}$ "lemma positiveToRight(Leq)(1 term)"]

[negativeToLeft(Leq) $\stackrel{\mathrm{pyk}}{\to}$ "lemma negativeToLeft(Leq)"]

[LeqAdditionLeft $\stackrel{\mathrm{pyk}}{\to}$ "lemma leqAdditionLeft"]

[leqSubtraction $\stackrel{\mathrm{pyk}}{\to}$ "lemma leqSubtraction"]

[leqSubtractionLeft $\stackrel{\mathrm{pyk}}{\to}$ "lemma leqSubtractionLeft"]

[thirdGeq $\stackrel{\mathrm{pyk}}{\to}$ "lemma thirdGeq"]

[LeqNegated $\stackrel{\mathrm{pyk}}{\to}$ "lemma leqNegated"]

[AddEquations(Leq) $\stackrel{\mathrm{pyk}}{\to}$ "lemma addEquations(Leq)"]

[ThirdGeqSeries $\stackrel{\mathrm{pyk}}{\to}$ "lemma thirdGeqSeries"]

[LeqNeqLess $\stackrel{\mathrm{pyk}}{\to}$ "lemma leqNeqLess"]

[FromLess $\stackrel{\mathrm{pyk}}{\to}$ "lemma fromLess"]

[ToLess $\stackrel{\mathrm{pyk}}{\to}$ "lemma toLess"]

[fromNotLess $\stackrel{\mathrm{pyk}}{\to}$ "lemma fromNotLess"]

[toNotLess $\stackrel{\mathrm{pyk}}{\to}$ "lemma toNotLess"]

[NegativeLessPositive $\stackrel{\mathrm{pyk}}{\to}$ "lemma negativeLessPositive"]

[leqLessTransitivity $\stackrel{\mathrm{pyk}}{\to}$ "lemma leqLessTransitivity"]

[LessLeqTransitivity $\stackrel{\mathrm{pyk}}{\to}$ "lemma lessLeqTransitivity"]

[LessTransitivity $\stackrel{\mathrm{pyk}}{\to}$ "lemma lessTransitivity"]

[LessTotality $\stackrel{\mathrm{pyk}}{\to}$ "lemma lessTotality"]

[SubLessRight $\stackrel{\mathrm{pyk}}{\to}$ "lemma subLessRight"]

[SubLessLeft $\stackrel{\mathrm{pyk}}{\to}$ "lemma subLessLeft"]

[LessAddition $\stackrel{\mathrm{pyk}}{\to}$ "lemma lessAddition"]

[LessAdditionLeft $\stackrel{\mathrm{pyk}}{\to}$ "lemma lessAdditionLeft"]

[LessMultiplication $\stackrel{\mathrm{pyk}}{\to}$ "lemma lessMultiplication"]

[LessMultiplicationLeft $\stackrel{\mathrm{pyk}}{\to}$ "lemma lessMultiplicationLeft"]

[LessDivision $\stackrel{\mathrm{pyk}}{\to}$ "lemma lessDivision"]

[AddEquations(Less) $\stackrel{\mathrm{pyk}}{\to}$ "lemma addEquations(Less)"]

[LessNegated $\stackrel{\mathrm{pyk}}{\to}$ "lemma lessNegated"]

[PositiveNegated $\stackrel{\mathrm{pyk}}{\to}$ "lemma positiveNegated"]

[NonpositiveNegated $\stackrel{\mathrm{pyk}}{\to}$ "lemma nonpositiveNegated"]

[NegativeNegated $\stackrel{\mathrm{pyk}}{\to}$ "lemma negativeNegated"]

[NonnegativeNegated $\stackrel{\mathrm{pyk}}{\to}$ "lemma nonnegativeNegated"]

[PositiveHalved $\stackrel{\mathrm{pyk}}{\to}$ "lemma positiveHalved"]

[NonnegativeNumerical $\stackrel{\text{pyk}}{\to}$ “lemma nonnegativeNumerical”]

[NegativeNumerical $\stackrel{\text{pyk}}{\to}$ “lemma negativeNumerical”]

[PositiveNumerical $\stackrel{\text{pyk}}{\to}$ “lemma positiveNumerical”]

[lemma nonpositiveNumerical $\stackrel{\text{pyk}}{\to}$ “lemma nonpositiveNumerical”]

[$|0| = 0$ $\stackrel{\text{pyk}}{\to}$ “lemma |0|=0”]

[$0 <= |x|$ $\stackrel{\text{pyk}}{\to}$ “lemma 0<=|x|”]

[SameNumerical $\stackrel{\text{pyk}}{\to}$ “lemma sameNumerical”]

[SignNumerical(+) $\stackrel{\text{pyk}}{\to}$ “lemma signNumerical(+)”]

[SignNumerical $\stackrel{\text{pyk}}{\to}$ “lemma signNumerical”]

[NumericalDifference $\stackrel{\text{pyk}}{\to}$ “lemma numericalDifference”]

[SplitNumericalSumHelper $\stackrel{\text{pyk}}{\to}$ “lemma splitNumericalSumHelper”]

[splitNumericalSum(++) $\stackrel{\text{pyk}}{\to}$ “lemma splitNumericalSum(++)”]

[splitNumericalSum($--$) $\stackrel{\text{pyk}}{\to}$ “lemma splitNumericalSum(--)”]

[splitNumericalSum($+-$ small) $\stackrel{\text{pyk}}{\to}$ “lemma splitNumericalSum(+-, smallNegative)”]

[splitNumericalSum($+-$ big) $\stackrel{\text{pyk}}{\to}$ “lemma splitNumericalSum(+-, bigNegative)”]

[splitNumericalSum($+-$) $\stackrel{\text{pyk}}{\to}$ “lemma splitNumericalSum(+-)”]

[splitNumericalSum($-+$) $\stackrel{\text{pyk}}{\to}$ “lemma splitNumericalSum(-+)”]

[splitNumericalSum $\stackrel{\text{pyk}}{\to}$ “lemma splitNumericalSum”]

[insertMiddleTerm(Numerical) $\stackrel{\text{pyk}}{\to}$ “lemma insertMiddleTerm(Numerical)”]

[$x + y = z$Backwards $\stackrel{\text{pyk}}{\to}$ “lemma x+y=zBackwards”]

[$x * y = z$Backwards $\stackrel{\text{pyk}}{\to}$ “lemma x*y=zBackwards”]

[$x = x + (y - y)$ $\stackrel{\text{pyk}}{\to}$ “lemma x=x+(y-y)”]

[$x = x + y - y$ $\stackrel{\text{pyk}}{\to}$ “lemma x=x+y-y”]

[$\stackrel{\text{pyk}}{\to}$ “lemma x=x*y*(1/y)”]

[insertMiddleTerm(Sum) $\stackrel{\text{pyk}}{\to}$ “lemma insertMiddleTerm(Sum)”]

[insertMiddleTerm(Difference) $\stackrel{\text{pyk}}{\to}$ “lemma insertMiddleTerm(Difference)”]

[$x * 0 + x = x$ $\stackrel{\text{pyk}}{\to}$ “lemma x*0+x=x”]

[$x * 0 = 0$ $\stackrel{\text{pyk}}{\to}$ “lemma x*0=0”]

[$(-1) * (-1) + (-1) * 1 = 0$ $\stackrel{\text{pyk}}{\to}$ “lemma (-1)*(-1)+(-1)*1=0”]

[$(-1) * (-1) = 1$ $\stackrel{\text{pyk}}{\to}$ “lemma (-1)*(-1)=1”]

[$0 < 1$Helper $\stackrel{\text{pyk}}{\to}$ “lemma 0<1Helper”]

[$0 < 1$ $\stackrel{\text{pyk}}{\to}$ “lemma 0<1”]

[$0 < 2$ $\stackrel{\text{pyk}}{\to}$ “lemma 0<2”]

[$0 < 1/2$ $\stackrel{\text{pyk}}{\to}$ “lemma 0<1/2”]

[TwoWholes $\stackrel{\text{pyk}}{\rightarrow}$ “lemma x+x=2*x”]

[TwoHalves $\stackrel{\text{pyk}}{\rightarrow}$ “lemma (1/2)x+(1/2)x=x”]

[$-\text{x} - \text{y} = -(\text{x} + \text{y})$ $\stackrel{\text{pyk}}{\rightarrow}$ “lemma -x-y=-(x+y)”]

[MinusNegated $\stackrel{\text{pyk}}{\rightarrow}$ “lemma minusNegated”]

[Times(-1) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma times(-1)”]

[Times(-1)Left $\stackrel{\text{pyk}}{\rightarrow}$ “lemma times(-1)Left”]

[$-0 = 0$ $\stackrel{\text{pyk}}{\rightarrow}$ “lemma -0=0”]

[SFsymmetry $\stackrel{\text{pyk}}{\rightarrow}$ “lemma sameFsymmetry”]

[SFtransitivity $\stackrel{\text{pyk}}{\rightarrow}$ “lemma sameFtransitivity”]

[= fToSameF $\stackrel{\text{pyk}}{\rightarrow}$ “lemma =f to sameF”]

[PlusF(Sym) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma plusF(Sym)”]

[TimesF(Sym) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma timesF(Sym)”]

[f2R(Plus) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma f2R(Plus)”]

[f2R(Times) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma f2R(Times)”]

[PlusR(Sym) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma plusR(Sym)”]

[TimesR(Sym) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma timesR(Sym)”]

[LessLeq(R) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma lessLeq(R)”]

[eqLeq(R) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma eqLeq(R)”]

[SubLessRight(R) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma subLessRight(R)”]

[SubLessLeft(R) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma subLessLeft(R)”]

[<< TransitivityHelper(Q) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma <<TransitivityHelper(Q)”]

[<< Transitivity $\stackrel{\text{pyk}}{\rightarrow}$ “lemma <<Transitivity”]

[<<== Reflexivity $\stackrel{\text{pyk}}{\rightarrow}$ “lemma <<==Reflexivity”]

[<<== AntisymmetryHelper(Q) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma <<==AntisymmetryHelper(Q)”]

[<<== Antisymmetry $\stackrel{\text{pyk}}{\rightarrow}$ “lemma <<==Antisymmetry”]

[<<== Transitivity $\stackrel{\text{pyk}}{\rightarrow}$ “lemma <<==Transitivity”]

[Plus0f $\stackrel{\text{pyk}}{\rightarrow}$ “lemma plus0f”]

[Plus00 $\stackrel{\text{pyk}}{\rightarrow}$ “lemma plus00”]

[== Addition $\stackrel{\text{pyk}}{\rightarrow}$ “lemma ==Addition”]

[== AdditionLeft $\stackrel{\text{pyk}}{\rightarrow}$ “lemma ==AdditionLeft”]

[<< Addition $\stackrel{\text{pyk}}{\rightarrow}$ “lemma <<Addition”]

[<<== Addition $\stackrel{\text{pyk}}{\rightarrow}$ “lemma <<==Addition”]

[PlusAssociativity(F) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma plusAssociativity(F)”]

[PlusAssociativity(R) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma plusAssociativity(R)”]

[Negative(R) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma negative(R)”]

[PlusCommutativity(F) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma plusCommutativity(F)”]

[PlusCommutativity(R) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma plusCommutativity(R)”]

[TimesAssociativity(F) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma timesAssociativity(F)”]

[TimesAssociativity(R) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma timesAssociativity(R)”]

[Times1f $\stackrel{\text{pyk}}{\rightarrow}$ “lemma times1f”]

[Times01 $\stackrel{\text{pyk}}{\rightarrow}$ “lemma times01”]

[TimesCommutativity(F) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma timesCommutativity(F)”]

[TimesCommutativity(R) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma timesCommutativity(R)”]

[Distribution(F) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma distribution(F)”]

[Distribution(R) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma distribution(R)”]

[$\mathrm{R}(*)$ $\stackrel{\text{pyk}}{\rightarrow}$ “R(")”]

[$--\mathrm{R}(*)$ $\stackrel{\text{pyk}}{\rightarrow}$ “--R(")”]

[$\mathrm{rec}*$ $\stackrel{\text{pyk}}{\rightarrow}$ “1/ "”]

[$*/*$ $\stackrel{\text{pyk}}{\rightarrow}$ “eq-system of " modulo "”]

[$*\cap*$ $\stackrel{\text{pyk}}{\rightarrow}$ “intersection " comma " end intersection”]

[$*[*]$ $\stackrel{\text{pyk}}{\rightarrow}$ “[" ; "]”]

[$\cup*$ $\stackrel{\text{pyk}}{\rightarrow}$ “union " end union”]

[$*\cup*$ $\stackrel{\text{pyk}}{\rightarrow}$ “binary-union " comma " end union”]

[$\mathrm{P}(*)$ $\stackrel{\text{pyk}}{\rightarrow}$ “power " end power”]

[$\{*\}$ $\stackrel{\text{pyk}}{\rightarrow}$ “zermelo singleton " end singleton”]

[$\{*,*\}$ $\stackrel{\text{pyk}}{\rightarrow}$ “zermelo pair " comma " end pair”]

[$\langle *,* \rangle$ $\stackrel{\text{pyk}}{\rightarrow}$ “zermelo ordered pair " comma " end pair”]

[$-*$ $\stackrel{\text{pyk}}{\rightarrow}$ “- "”]

[$-_{\mathrm{f}}*$ $\stackrel{\text{pyk}}{\rightarrow}$ “-f "”]

[$*\in*$ $\stackrel{\text{pyk}}{\rightarrow}$ “" in0 "”]

[$*(*,*)$ $\stackrel{\text{pyk}}{\rightarrow}$ “" is related to " under "”]

[$\mathrm{ReflRel}(*,*)$ $\stackrel{\text{pyk}}{\rightarrow}$ “" is reflexive relation in "”]

[$\mathrm{SymRel}(*,*)$ $\stackrel{\text{pyk}}{\rightarrow}$ “" is symmetric relation in "”]

[$\mathrm{TransRel}(*,*)$ $\stackrel{\text{pyk}}{\rightarrow}$ “" is transitive relation in "”]

[$\mathrm{EqRel}(*,*)$ $\stackrel{\text{pyk}}{\rightarrow}$ “" is equivalence relation in "”]

[$[*\in*]_{*}$ $\stackrel{\text{pyk}}{\rightarrow}$ “equivalence class of " in " modulo "”]

[$\mathrm{Partition}(*,*)$ $\stackrel{\text{pyk}}{\rightarrow}$ “" is partition of "”]

[$***$ $\stackrel{\text{pyk}}{\rightarrow}$ “" * "”]

[$* *_{\mathrm{f}} *$ $\stackrel{\text{pyk}}{\rightarrow}$ “" *f "”]

[$* * ** \stackrel{\text{pyk}}{\rightarrow}$ “" ** "”]

[$* + * \stackrel{\text{pyk}}{\rightarrow}$ “" + "”]

[$* - * \stackrel{\text{pyk}}{\rightarrow}$ “" - "”]

[$* +_{\text{f}} * \stackrel{\text{pyk}}{\rightarrow}$ “" +f "”]

[$* -_{\text{f}} * \stackrel{\text{pyk}}{\rightarrow}$ “" -f "”]

[$* + +* \stackrel{\text{pyk}}{\rightarrow}$ “" ++ "”]

[$\text{R}(*) - -\text{R}(*) \stackrel{\text{pyk}}{\rightarrow}$ “R(") -- R(")”]

[$| * | \stackrel{\text{pyk}}{\rightarrow}$ “| " |”]

[$\text{if}(*, *, *) \stackrel{\text{pyk}}{\rightarrow}$ “if(" , " , ")”]

[$* = * \stackrel{\text{pyk}}{\rightarrow}$ “" = "”]

[$* \neq * \stackrel{\text{pyk}}{\rightarrow}$ “" != "”]

[$* <= * \stackrel{\text{pyk}}{\rightarrow}$ “" <= "”]

[$* < * \stackrel{\text{pyk}}{\rightarrow}$ “" < "”]

[$* =_{\text{f}} * \stackrel{\text{pyk}}{\rightarrow}$ “" =f "”]

[$* <_{\text{f}} * \stackrel{\text{pyk}}{\rightarrow}$ “" <f "”]

[$\text{SF}(*, *) \stackrel{\text{pyk}}{\rightarrow}$ “" sameF "”]

[$* == * \stackrel{\text{pyk}}{\rightarrow}$ “" == "”]

[$* << * \stackrel{\text{pyk}}{\rightarrow}$ “" << "”]

[$* <<== * \stackrel{\text{pyk}}{\rightarrow}$ “" <<== "”]

[$* {=}{=} * \stackrel{\text{pyk}}{\rightarrow}$ “" zermelo is "”]

[$* \subseteq * \stackrel{\text{pyk}}{\rightarrow}$ “" is subset of "”]

[$\dot{\neg} * \stackrel{\text{pyk}}{\rightarrow}$ “not0 "”]

[$* \notin * \stackrel{\text{pyk}}{\rightarrow}$ “" zermelo ~in "”]

[$* \neq * \stackrel{\text{pyk}}{\rightarrow}$ “" zermelo ~is "”]

[$* \dot{\wedge} * \stackrel{\text{pyk}}{\rightarrow}$ “" and0 "”]

[$* \dot{\vee} * \stackrel{\text{pyk}}{\rightarrow}$ “" or0 "”]

[$* \dot{\Leftrightarrow} * \stackrel{\text{pyk}}{\rightarrow}$ “" iff "”]

[$\{\text{ph} \in * \mid *\} \stackrel{\text{pyk}}{\rightarrow}$ “the set of ph in " such that " end set”]

[$\text{am} \stackrel{\text{pyk}}{\rightarrow}$ “am”]

B TEX definitioner

[am $\stackrel{\text{tex}}{\to}$ “am”]

[$(\cdots)$ $\stackrel{\text{tex}}{\to}$ “(\cdots{})”]

[`Objekt-var` $\stackrel{\text{tex}}{\to}$ “\texttt{Objekt-var}”]

[`Ex-var` $\stackrel{\text{tex}}{\to}$ “\texttt{Ex-var}”]

[`Ph-var` $\stackrel{\text{tex}}{\to}$ “\texttt{Ph-var}”]

[`Værdi` $\stackrel{\text{tex}}{\to}$ “\texttt{V\ae{}rdi}”]

[`Variabel` $\stackrel{\text{tex}}{\to}$ “\texttt{Variabel}”]

[Op(x) $\stackrel{\text{tex}}{\to}$ “Op(#1.
)”]

[Op(x, y) $\stackrel{\text{tex}}{\to}$ “Op(#1.
, #2.
)”]

[$\mathsf{x} \mathrel{\ddot{==}} \mathsf{y}$ $\stackrel{\text{tex}}{\to}$ “#1.
\mathrel {\ddot {==}} #2.”]

[ContainsEmpty(x) $\stackrel{\text{tex}}{\to}$ “ContainsEmpty(#1.
)”]

[Dedu(x, y) $\stackrel{\text{tex}}{\to}$ “
Dedu(#1.
, #2.
)”]

[$\text{Dedu}_0(\mathsf{x}, \mathsf{y})$ $\stackrel{\text{tex}}{\to}$ “
Dedu_0(#1.
, #2.
)”]

[$\text{Dedu}_{\text{s}}(\mathsf{x}, \mathsf{y}, \mathsf{z})$ $\stackrel{\text{tex}}{\to}$ “Dedu_{s}(#1.
, #2.
, #3.
)”]

[$\text{Dedu}_1(\mathsf{x}, \mathsf{y}, \mathsf{z})$ $\stackrel{\text{tex}}{\to}$ “
Dedu_1(#1.
, #2.
, #3.
)”]

[$\text{Dedu}_2(\mathsf{x},\mathsf{y},\mathsf{z}) \stackrel{\text{tex}}{\to}$ "
Dedu_2(#1.
, #2.
, #3.
)"]

[$\text{Dedu}_3(\mathsf{x},\mathsf{y},\mathsf{z},\mathsf{u}) \stackrel{\text{tex}}{\to}$ "
Dedu_3(#1.
, #2.
, #3.
, #4.
)"]

[$\text{Dedu}_4(\mathsf{x},\mathsf{y},\mathsf{z},\mathsf{u}) \stackrel{\text{tex}}{\to}$ "
Dedu_4(#1.
, #2.
, #3.
, #4.
)"]

[$\text{Dedu}_4^*(\mathsf{x},\mathsf{y},\mathsf{z},\mathsf{u}) \stackrel{\text{tex}}{\to}$ "
Dedu_4ˆ∗(#1.
, #2.
, #3.
, #4.
)"]

[$\text{Dedu}_5(\mathsf{x},\mathsf{y},\mathsf{z}) \stackrel{\text{tex}}{\to}$ "
Dedu_5(#1.
, #2.
, #3.
)"]

[$\text{Dedu}_6(\mathsf{p},\mathsf{c},\mathsf{e},\mathsf{b}) \stackrel{\text{tex}}{\to}$ "
Dedu_6(#1.
, #2.
, #3.
, #4.
)"]

[$\text{Dedu}_6^*(\mathsf{p},\mathsf{c},\mathsf{e},\mathsf{b}) \stackrel{\text{tex}}{\to}$ "
Dedu_6ˆ∗(#1.
, #2.
, #3.
, #4.
)"]

[$\text{Dedu}_7(\mathsf{p}) \stackrel{\text{tex}}{\to}$ "
Dedu_7(#1.

)”]

[$\mathrm{Dedu}_8(\mathsf{p},\mathsf{b}) \stackrel{\text{tex}}{\to}$ “
Dedu_8(#1.
, #2.
)”]

[$\mathrm{Dedu}_8^*(\mathsf{p},\mathsf{b}) \stackrel{\text{tex}}{\to}$ “
Dedu_8ˆ∗(#1.
, #2.
)”]

[$\mathrm{Ex}_1 \stackrel{\text{tex}}{\to}$ “Ex_{1}”]

[$\mathrm{Ex}_2 \stackrel{\text{tex}}{\to}$ “Ex_{2}”]

[$\mathrm{Ex}_{10} \stackrel{\text{tex}}{\to}$ “Ex_{10}”]

[$\mathrm{Ex}_{20} \stackrel{\text{tex}}{\to}$ “Ex_{20}”]

[$\mathsf{x}_{\mathrm{Ex}} \stackrel{\text{tex}}{\to}$ “#1.
_{Ex}”]

[$\mathsf{x}^{\mathrm{Ex}} \stackrel{\text{tex}}{\to}$ “#1.
ˆ{Ex}”]

[$\langle \mathsf{x} {\equiv} \mathsf{y} | \mathsf{z} {:==} \mathsf{u} \rangle_{\mathrm{Ex}} \stackrel{\text{tex}}{\to}$ “\langle #1.
{\equiv} #2.
| #3.
{:==} #4.
\rangle_{Ex} ”]

[$\langle \mathsf{x} {\equiv}^0 \mathsf{y} | \mathsf{z} {:==} \mathsf{u} \rangle_{\mathrm{Ex}} \stackrel{\text{tex}}{\to}$ “\langle #1.
{\equiv}ˆ0 #2.
| #3.
{:==} #4.
\rangle_{Ex} ”]

[$\langle \mathsf{x} {\equiv}^1 \mathsf{y} | \mathsf{z} {:==} \mathsf{u} \rangle_{\mathrm{Ex}} \stackrel{\text{tex}}{\to}$ “\langle #1.
{\equiv}ˆ1 #2.
| #3.
{:==} #4.
\rangle_{Ex} ”]

[$\langle \mathsf{x} {\equiv}^* \mathsf{y} | \mathsf{z} {:==} \mathsf{u} \rangle_{\mathrm{Ex}} \stackrel{\text{tex}}{\to}$ “\langle #1.
{\equiv}ˆ∗ #2.
| #3.
{:==} #4.
\rangle_{Ex} ”]

[$\mathrm{ph}_1 \overset{\text{tex}}{\to}$ “ph_{1}”]

[$\mathrm{ph}_2 \overset{\text{tex}}{\to}$ “ph_{2}”]

[$\mathrm{ph}_3 \overset{\text{tex}}{\to}$ “ph_{3}”]

[$\mathsf{x}_{\mathrm{Ph}} \overset{\text{tex}}{\to}$ “#1.
_{Ph} ”]

[$\mathsf{x}^{\mathrm{Ph}} \overset{\text{tex}}{\to}$ “#1.
^{Ph}”]

[$\langle \mathsf{x}{\equiv}\mathsf{y}|\mathsf{z}{:==}\mathsf{u}\rangle_{\mathrm{Ph}} \overset{\text{tex}}{\to}$ “\langle #1.
{\equiv} #2.
| #3.
{:==} #4.
\rangle_{Ph} ”]

[$\langle \mathsf{x}{\equiv}^0\mathsf{y}|\mathsf{z}{:==}\mathsf{u}\rangle_{\mathrm{Ph}} \overset{\text{tex}}{\to}$ “\langle #1.
{\equiv}^0 #2.
| #3.
{:==} #4.
\rangle_{Ph} ”]

[$\langle \mathsf{x}{\equiv}^1\mathsf{y}|\mathsf{z}{:==}\mathsf{u}\rangle_{\mathrm{Ph}} \overset{\text{tex}}{\to}$ “\langle #1.
{\equiv}^1 #2.
| #3.
{:==} #4.
\rangle_{Ph} ”]

[$\langle \mathsf{x}{\equiv}^*\mathsf{y}|\mathsf{z}{:==}\mathsf{u}\rangle_{\mathrm{Ph}} \overset{\text{tex}}{\to}$ “\langle #1.
{\equiv}^* #2.
| #3.
{:==} #4.
\rangle_{Ph} ”]

[$\mathsf{bs} \overset{\text{tex}}{\to}$ “\mathsf {bs}”]

[$\mathsf{OBS} \overset{\text{tex}}{\to}$ “ \mathsf {OBS}”]

[$\mathcal{BS} \overset{\text{tex}}{\to}$ “{\cal BS}”]

[$\mathrm{\O} \overset{\text{tex}}{\to}$ “\mathrm{\O}”]

[ZFsub $\overset{\text{tex}}{\to}$ “ZFsub”]

[MP $\overset{\text{tex}}{\to}$ “MP”]

[Gen $\overset{\text{tex}}{\to}$ “Gen”]

[Repetition $\stackrel{\text{tex}}{\to}$ "Repetition"]

[Neg $\stackrel{\text{tex}}{\to}$ "Neg"]

[Ded $\stackrel{\text{tex}}{\to}$ "Ded"]

[ExistIntro $\stackrel{\text{tex}}{\to}$ "ExistIntro"]

[Extensionality $\stackrel{\text{tex}}{\to}$ "Extensionality"]

[Ødef $\stackrel{\text{tex}}{\to}$ "\O{}def"]

[PairDef $\stackrel{\text{tex}}{\to}$ "PairDef"]

[UnionDef $\stackrel{\text{tex}}{\to}$ "UnionDef"]

[PowerDef $\stackrel{\text{tex}}{\to}$ "PowerDef"]

[SeparationDef $\stackrel{\text{tex}}{\to}$ "SeparationDef"]

[AddDoubleNeg $\stackrel{\text{tex}}{\to}$ "AddDoubleNeg"]

[RemoveDoubleNeg $\stackrel{\text{tex}}{\to}$ "RemoveDoubleNeg"]

[AndCommutativity $\stackrel{\text{tex}}{\to}$ "AndCommutativity"]

[AutoImply $\stackrel{\text{tex}}{\to}$ "AutoImply"]

[Contrapositive $\stackrel{\text{tex}}{\to}$ "Contrapositive"]

[FirstConjunct $\stackrel{\text{tex}}{\to}$ "FirstConjunct"]

[SecondConjunct $\stackrel{\text{tex}}{\to}$ "SecondConjunct"]

[FromContradiction $\stackrel{\text{tex}}{\to}$ "FromContradiction"]

[FromDisjuncts $\stackrel{\text{tex}}{\to}$ "FromDisjuncts"]

[IffCommutativity $\stackrel{\text{tex}}{\to}$ "IffCommutativity"]

[IffFirst $\stackrel{\text{tex}}{\to}$ "IffFirst"]

[IffSecond $\stackrel{\text{tex}}{\to}$ "IffSecond"]

[ImplyTransitivity $\stackrel{\text{tex}}{\to}$ "ImplyTransitivity"]

[JoinConjuncts $\stackrel{\text{tex}}{\to}$ "JoinConjuncts"]

[MP2 $\stackrel{\text{tex}}{\to}$ "MP2"]

[MP3 $\overset{\text{tex}}{\to}$ "MP3"]

[MP4 $\overset{\text{tex}}{\to}$ "MP4"]

[MP5 $\overset{\text{tex}}{\to}$ "MP5"]

[MT $\overset{\text{tex}}{\to}$ "MT"]

[NegativeMT $\overset{\text{tex}}{\to}$ "NegativeMT"]

[Technicality $\overset{\text{tex}}{\to}$ "Technicality"]

[Weakening $\overset{\text{tex}}{\to}$ "Weakening"]

[WeakenOr1 $\overset{\text{tex}}{\to}$ "WeakenOr1"]

[WeakenOr2 $\overset{\text{tex}}{\to}$ "WeakenOr2"]

[Pair2Formula $\overset{\text{tex}}{\to}$ "Pair2Formula"]

[Formula2Pair $\overset{\text{tex}}{\to}$ "Formula2Pair"]

[Union2Formula $\overset{\text{tex}}{\to}$ "Union2Formula"]

[Formula2Union $\overset{\text{tex}}{\to}$ "Formula2Union"]

[Sep2Formula $\overset{\text{tex}}{\to}$ "Sep2Formula"]

[Formula2Sep $\overset{\text{tex}}{\to}$ "Formula2Sep"]

[SubsetInPower $\overset{\text{tex}}{\to}$ "SubsetInPower"]

[HelperPowerIsSub $\overset{\text{tex}}{\to}$ "HelperPowerIsSub"]

[PowerIsSub $\overset{\text{tex}}{\to}$ "PowerIsSub"]

[(Switch)HelperPowerIsSub $\overset{\text{tex}}{\to}$ "(Switch)HelperPowerIsSub"]

[(Switch)PowerIsSub $\overset{\text{tex}}{\to}$ "(Switch)PowerIsSub"]

[ToSetEquality $\overset{\text{tex}}{\to}$ "ToSetEquality"]

[HelperToSetEquality(t) $\overset{\text{tex}}{\to}$ "HelperToSetEquality(t)"]

[ToSetEquality(t) $\overset{\text{tex}}{\to}$ "ToSetEquality(t)"]

[HelperFromSetEquality $\overset{\text{tex}}{\to}$ "HelperFromSetEquality"]

[FromSetEquality $\overset{\text{tex}}{\to}$ "FromSetEquality"]

[HelperReflexivity $\overset{\text{tex}}{\rightarrow}$ “HelperReflexivity”]

[Reflexivity $\overset{\text{tex}}{\rightarrow}$ “Reflexivity”]

[HelperSymmetry $\overset{\text{tex}}{\rightarrow}$ “HelperSymmetry”]

[Symmetry $\overset{\text{tex}}{\rightarrow}$ “Symmetry”]

[HelperTransitivity $\overset{\text{tex}}{\rightarrow}$ “HelperTransitivity”]

[Transitivity $\overset{\text{tex}}{\rightarrow}$ “Transitivity”],

[ERisReflexive $\overset{\text{tex}}{\rightarrow}$ “ERisReflexive”]

[ERisSymmetric $\overset{\text{tex}}{\rightarrow}$ “ERisSymmetric”]

[ERisTransitive $\overset{\text{tex}}{\rightarrow}$ “ERisTransitive”]

[ØisSubset $\overset{\text{tex}}{\rightarrow}$ “\O{}isSubset”]

[HelperMemberNotØ $\overset{\text{tex}}{\rightarrow}$ “HelperMemberNot\O{}”]

[MemberNotØ $\overset{\text{tex}}{\rightarrow}$ “MemberNot\O{}”]

[HelperUniqueØ $\overset{\text{tex}}{\rightarrow}$ “HelperUnique\O{}”]

[UniqueØ $\overset{\text{tex}}{\rightarrow}$ “Unique\O{}”]

[==Reflexivity $\overset{\text{tex}}{\rightarrow}$ “==\!{}Reflexivity”]

[==Symmetry $\overset{\text{tex}}{\rightarrow}$ “==\!{}Symmetry”]

[Helper==Transitivity $\overset{\text{tex}}{\rightarrow}$ “Helper\!{}==\!{}Transitivity”]

[==Transitivity $\overset{\text{tex}}{\rightarrow}$ “\!{}==\!{}Transitivity”]

[HelperTransferNotEq $\overset{\text{tex}}{\rightarrow}$ “HelperTransferNotEq”]

[TransferNotEq $\overset{\text{tex}}{\rightarrow}$ “TransferNotEq”]

[HelperPairSubset $\overset{\text{tex}}{\rightarrow}$ “HelperPairSubset”]

[Helper(2)PairSubset $\overset{\text{tex}}{\rightarrow}$ “Helper(2)PairSubset”]

[PairSubset $\overset{\text{tex}}{\rightarrow}$ “PairSubset”]

[SamePair $\overset{\text{tex}}{\rightarrow}$ “SamePair”]

[SameSingleton $\overset{\text{tex}}{\rightarrow}$ “SameSingleton”]

[UnionSubset $\stackrel{\text{tex}}{\to}$ "UnionSubset"]

[SameUnion $\stackrel{\text{tex}}{\to}$ "SameUnion"]

[SeparationSubset $\stackrel{\text{tex}}{\to}$ "SeparationSubset"]

[SameSeparation $\stackrel{\text{tex}}{\to}$ "SameSeparation"]

[SameBinaryUnion $\stackrel{\text{tex}}{\to}$ "SameBinaryUnion"]

[IntersectionSubset $\stackrel{\text{tex}}{\to}$ "IntersectionSubset"]

[SameIntersection $\stackrel{\text{tex}}{\to}$ "SameIntersection"]

[AutoMember $\stackrel{\text{tex}}{\to}$ "AutoMember"]

[HelperEqSysNotØ $\stackrel{\text{tex}}{\to}$ "HelperEqSysNot\O{}"]

[EqSysNotØ $\stackrel{\text{tex}}{\to}$ "EqSysNot\O{}"]

[HelperEqSubset $\stackrel{\text{tex}}{\to}$ "HelperEqSubset"]

[EqSubset $\stackrel{\text{tex}}{\to}$ "EqSubset"]

[EqNecessary $\stackrel{\text{tex}}{\to}$ "EqNecessary"]

[HelperEqNecessary $\stackrel{\text{tex}}{\to}$ "HelperEqNecessary"]

[HelperNoneEqNecessary $\stackrel{\text{tex}}{\to}$ "HelperNoneEqNecessary"]

[Helper(2)NoneEqNecessary $\stackrel{\text{tex}}{\to}$ "Helper(2)NoneEqNecessary"]

[NoneEqNecessary $\stackrel{\text{tex}}{\to}$ "NoneEqNecessary"]

[EqClassIsSubset $\stackrel{\text{tex}}{\to}$ "EqClassIsSubset"]

[EqClassesAreDisjoint $\stackrel{\text{tex}}{\to}$ "EqClassesAreDisjoint"]

[AllDisjoint $\stackrel{\text{tex}}{\to}$ "AllDisjoint"]

[AllDisjointImply $\stackrel{\text{tex}}{\to}$ "AllDisjointImply"]

[BSsubset $\stackrel{\text{tex}}{\to}$ "BSsubset"]

[Union(BS/R)subset $\stackrel{\text{tex}}{\to}$ "Union(BS/R)subset"]

[UnionIdentity $\stackrel{\text{tex}}{\to}$ "UnionIdentity"]

[EqSysIsPartition $\stackrel{\text{tex}}{\to}$ "EqSysIsPartition"]

[$\mathsf{x}/\mathsf{y} \stackrel{\text{tex}}{\to}$ "#1.
/ #2."]

[$\mathsf{x} \cap \mathsf{y} \stackrel{\text{tex}}{\to}$ "#1.
\cap #2."]

[$\cup \mathsf{x} \stackrel{\text{tex}}{\to}$ "\cup #1."]

[$\mathsf{x} \cup \mathsf{y} \stackrel{\text{tex}}{\to}$ "#1.
\mathrel{\cup} #2."]

[$\mathrm{P}(\mathsf{x}) \stackrel{\text{tex}}{\to}$ "P(#1.
)"]

[$\{\mathsf{x}\} \stackrel{\text{tex}}{\to}$ "\{#1.
\}"]

[$\{\mathsf{x}, \mathsf{y}\} \stackrel{\text{tex}}{\to}$ "\{#1.
, #2.
\}"]

[$\langle \mathsf{x}, \mathsf{y} \rangle \stackrel{\text{tex}}{\to}$ "\langle #1.
, #2.
\rangle"],

[$\mathsf{x} \in \mathsf{y} \stackrel{\text{tex}}{\to}$ "#1.
\mathrel{\in} #2."]

[$\mathsf{z}(\mathsf{x}, \mathsf{y}) \stackrel{\text{tex}}{\to}$ "#3.
(#1.
, #2.
)"]

[$\mathrm{ReflRel}(\mathsf{r}, \mathsf{x}) \stackrel{\text{tex}}{\to}$ "ReflRel(#1.
, #2.
)"]

[$\mathrm{SymRel}(\mathsf{r}, \mathsf{x}) \stackrel{\text{tex}}{\to}$ "SymRel(#1.
, #2.
)"]

[$\mathrm{TransRel}(\mathsf{r}, \mathsf{x}) \stackrel{\text{tex}}{\to}$ "TransRel(#1.
, #2.
)"]

[$\mathrm{EqRel}(\mathsf{r}, \mathsf{x}) \stackrel{\text{tex}}{\to}$ "EqRel(#1.
, #2.
)"]

$[[\mathsf{x} \in \mathsf{bs}]_{\mathsf{r}} \stackrel{\text{tex}}{\to}$ "[#1.
\mathrel{\in} #2.
]_{#3.
}"]

$[\text{Partition}(\mathsf{x}, \mathsf{y}) \stackrel{\text{tex}}{\to}$ "Partition(#1.
, #2.
)"]

$[\mathsf{x}{=}{=}\mathsf{y} \stackrel{\text{tex}}{\to}$ "#1.
\!\mathrel{==}\! #2."]

$[\mathsf{x} \subseteq \mathsf{y} \stackrel{\text{tex}}{\to}$ "#1.
\mathrel{\subseteq} #2."]

$[\dot{\neg}\, \mathsf{x} \stackrel{\text{tex}}{\to}$ "\dot{\neg}\, #1."]

$[\mathsf{x} \notin \mathsf{y} \stackrel{\text{tex}}{\to}$ "#1.
\mathrel{\notin} #2."]

$[\mathsf{x} \neq \mathsf{y} \stackrel{\text{tex}}{\to}$ "#1.
\mathrel{\neq} #2."]

$[\mathsf{x} \mathrel{\dot{\wedge}} \mathsf{y} \stackrel{\text{tex}}{\to}$ "#1.
\mathrel{\dot{\wedge}} #2."]

$[\mathsf{x} \mathrel{\dot{\vee}} \mathsf{y} \stackrel{\text{tex}}{\to}$ "#1.
\mathrel{\dot{\vee}} #2."]

$[\mathsf{x} \mathrel{\dot{\Leftrightarrow}} \mathsf{y} \stackrel{\text{tex}}{\to}$ "#1.
\mathrel{\dot{\Leftrightarrow}} #2."]

$[\{\text{ph} \in \mathsf{x} \mid \mathsf{a}\} \stackrel{\text{tex}}{\to}$ " \{ ph \mathrel{\in} #1.
\mid #2.
\}"]

————- RRRRRRRRRRRRRR ————-
(*** aksiomer ***)
$[\text{leqReflexivity} \stackrel{\text{stmt}}{\to} \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \underline{\mathsf{x}} <= \underline{\mathsf{x}}][\text{leqReflexivity} \stackrel{\text{proof}}{\to} \text{Rule tactic}]$
$[\text{leqAntisymmetryAxiom} \stackrel{\text{stmt}}{\to} \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{y}} <= \underline{\mathsf{x}} \Rightarrow \underline{\mathsf{x}} = \underline{\mathsf{y}}][\text{leqAntisymmetryAxiom} \stackrel{\text{proof}}{\to} \text{Rule tactic}]$
$[\text{leqTransitivityAxiom} \stackrel{\text{stmt}}{\to} \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \forall \underline{\mathsf{z}}\colon \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{y}} <= \underline{\mathsf{z}} \Rightarrow \underline{\mathsf{x}} <= \underline{\mathsf{z}}][\text{leqTransitivityAxiom} \stackrel{\text{proof}}{\to} \text{Rule tactic}]$
$[\text{leqTotality} \stackrel{\text{stmt}}{\to} \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \dot{\neg}\, \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{y}} <= \underline{\mathsf{x}}][\text{leqTotality} \stackrel{\text{proof}}{\to} \text{Rule tactic}]$

[leqAdditionAxiom $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} <= \underline{y} \Rightarrow \underline{x} + \underline{z} <= \underline{y} + \underline{z}$][leqAdditionAxiom $\overset{\text{proof}}{\rightarrow}$ Rule tactic]

[leqMultiplicationAxiom $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon 0 <= \underline{z} \Rightarrow \underline{x} <= \underline{y} \Rightarrow \underline{x} * \underline{z} <= \underline{y} * \underline{z}$][leqMultiplicationAxiom $\overset{\text{proof}}{\rightarrow}$ Rule tactic]

[plusAssociativity $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} + \underline{y} + \underline{z} = \underline{x} + \underline{y} + \underline{z}$][plusAssociativity $\overset{\text{proof}}{\rightarrow}$ Rule tactic]

[plusCommutativity $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} + \underline{y} = \underline{y} + \underline{x}$][plusCommutativity $\overset{\text{proof}}{\rightarrow}$ Rule tactic]

[Negative $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \underline{x} + -\underline{x} = 0$][Negative $\overset{\text{proof}}{\rightarrow}$ Rule tactic]

[plus0 $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \underline{x} + 0 = \underline{x}$][plus0 $\overset{\text{proof}}{\rightarrow}$ Rule tactic]

[timesAssociativity $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} * \underline{y} * \underline{z} = \underline{x} * \underline{y} * \underline{z}$][timesAssociativity $\overset{\text{proof}}{\rightarrow}$ Rule tactic]

[timesCommutativity $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} * \underline{y} = \underline{y} * \underline{x}$][timesCommutativity $\overset{\text{proof}}{\rightarrow}$ Rule tactic]

[ReciprocalAxiom $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \dot{\neg}\, \underline{x} = 0 \Rightarrow \underline{x} * \mathrm{rec}\underline{x} = 1$][ReciprocalAxiom $\overset{\text{proof}}{\rightarrow}$ Rule tactic]

[times1 $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \underline{x} * 1 = \underline{x}$][times1 $\overset{\text{proof}}{\rightarrow}$ Rule tactic]

[Distribution $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} * \underline{y} + \underline{z} = \underline{x} * \underline{y} + \underline{x} * \underline{z}$][Distribution $\overset{\text{proof}}{\rightarrow}$ Rule tactic]

[0not1 $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \dot{\neg}\, 0 = 1$][0not1 $\overset{\text{proof}}{\rightarrow}$ Rule tactic]

[equalityAxiom $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} \Rightarrow \underline{x} = \underline{z} \Rightarrow \underline{y} = \underline{z}$][equalityAxiom $\overset{\text{proof}}{\rightarrow}$ Rule tactic]

[eqLeqAxiom $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} = \underline{y} \Rightarrow \underline{x} <= \underline{y}$][eqLeqAxiom $\overset{\text{proof}}{\rightarrow}$ Rule tactic]

[eqAdditionAxiom $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} \Rightarrow \underline{x} + \underline{z} = \underline{y} + \underline{z}$][eqAdditionAxiom $\overset{\text{proof}}{\rightarrow}$ Rule tactic]

[eqMultiplicationAxiom $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} \Rightarrow \underline{x} * \underline{z} = \underline{y} * \underline{z}$][eqMultiplicationAxiom $\overset{\text{proof}}{\rightarrow}$ Rule tactic]

(*** XX snydeaksiomer ***)

[==Reflexivity $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{(rx)}\colon \underline{(rx)} == \underline{(rx)}$][==Reflexivity $\overset{\text{proof}}{\rightarrow}$ Rule tactic]

[==Symmetry $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{(rx)}\colon \forall \underline{(ry)}\colon \underline{(rx)} == \underline{(ry)} \vdash \underline{(ry)} == \underline{(rx)}$][==Symmetry $\overset{\text{proof}}{\rightarrow}$ Rule tactic]

[==Transitivity $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{(rx)}\colon \forall \underline{(ry)}\colon \forall \underline{(rz)}\colon \underline{(rx)} == \underline{(ry)} \vdash \underline{(ry)} == \underline{(rz)} \vdash \underline{(rx)} == \underline{(rz)}$][==Transitivity $\overset{\text{proof}}{\rightarrow}$ Rule tactic]

XX ikke 100procent identisk med originalen fra equivalence-relations
$[\mathrm{SENC1} \stackrel{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{(\mathrm{fx})} \colon \forall \underline{(\mathrm{rx})} \colon \forall \underline{(\mathrm{ry})} \colon \underline{(\mathrm{rx})} == \underline{(\mathrm{ry})} \vdash \underline{(\mathrm{fx})} \in \underline{(\mathrm{rx})} \vdash \underline{(\mathrm{fx})} \in \underline{(\mathrm{ry})}][\mathrm{SENC1} \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$
XX boer bevises ud fra nummer 1
$[\mathrm{SENC2} \stackrel{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{(\mathrm{fx})} \colon \forall \underline{(\mathrm{rx})} \colon \forall \underline{(\mathrm{ry})} \colon \underline{(\mathrm{rx})} == \underline{(\mathrm{ry})} \vdash \underline{(\mathrm{fx})} \in \underline{(\mathrm{ry})} \vdash \underline{(\mathrm{fx})} \in \underline{(\mathrm{rx})}][\mathrm{SENC2} \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[\mathrm{IfThenElse(T)} \stackrel{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \underline{\mathsf{a}} \vdash \mathrm{if}(\underline{\mathsf{a}}, \underline{\mathsf{x}}, \underline{\mathsf{y}}) = \underline{\mathsf{x}}][\mathrm{IfThenElse(T)} \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[\mathrm{IfThenElse(F)} \stackrel{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \dot\neg\, \underline{\mathsf{a}} \vdash \mathrm{if}(\underline{\mathsf{a}}, \underline{\mathsf{x}}, \underline{\mathsf{y}}) = \underline{\mathsf{y}}][\mathrm{IfThenElse(F)} \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[\mathrm{FromSF} \stackrel{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}} \colon \forall \underline{(\epsilon)} \colon \forall \underline{(\mathrm{fx})} \colon \forall \underline{(\mathrm{fy})} \colon \mathrm{SF}(\underline{(\mathrm{fx})}, \underline{(\mathrm{fy})}) \vdash \dot\neg\, 0 <= \underline{(\epsilon)} \Rightarrow \dot\neg\,\dot\neg\, 0 = \underline{(\epsilon)} \vdash \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \dot\neg\, \mathrm{if}(0 <= \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) <= \underline{(\epsilon)} \Rightarrow \dot\neg\,\dot\neg\, \mathrm{if}(0 <= \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) = \underline{(\epsilon)}][\mathrm{FromSF} \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[\mathrm{ToSF} \stackrel{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}} \colon \forall \underline{(\epsilon)} \colon \forall \underline{(\mathrm{fx})} \colon \forall \underline{(\mathrm{fy})} \colon \dot\neg\, 0 <= \underline{(\epsilon)} \Rightarrow \dot\neg\,\dot\neg\, 0 = \underline{(\epsilon)} \Rightarrow \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \dot\neg\, \mathrm{if}(0 <= \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) <= \underline{(\epsilon)} \Rightarrow \dot\neg\,\dot\neg\, \mathrm{if}(0 <= \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) = \underline{(\epsilon)} \vdash \mathrm{SF}(\underline{(\mathrm{fx})}, \underline{(\mathrm{fy})})][\mathrm{ToSF} \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[\mathrm{From} = \mathrm{f} \stackrel{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}} \colon \forall \underline{(\mathrm{fx})} \colon \forall \underline{(\mathrm{fy})} \colon \underline{(\mathrm{fx})} =_{\mathrm{f}} \underline{(\mathrm{fy})} \vdash \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fy})}[\underline{\mathsf{m}}]][\mathrm{From} = \mathrm{f} \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$
XX hm... det er nok med bare 1 meta m XX loesning: objektkvantor
$[\mathrm{To} = \mathrm{f} \stackrel{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}} \colon \forall \underline{(\mathrm{fx})} \colon \forall \underline{(\mathrm{fy})} \colon \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] \vdash \underline{(\mathrm{fx})} =_{\mathrm{f}} \underline{(\mathrm{fy})}][\mathrm{To} = \mathrm{f} \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[\mathrm{From} < \mathrm{f} \stackrel{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}} \colon \forall \underline{(\epsilon)} \colon \forall \underline{(\mathrm{fx})} \colon \forall \underline{(\mathrm{fy})} \colon \underline{(\mathrm{fx})} <_{\mathrm{f}} \underline{(\mathrm{fy})} \vdash \dot\neg\, 0 <= \underline{(\epsilon)} \Rightarrow \dot\neg\,\dot\neg\, 0 = \underline{(\epsilon)} \vdash \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)}][\mathrm{From} < \mathrm{f} \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[\mathrm{To} < \mathrm{f} \stackrel{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}} \colon \forall \underline{(\epsilon)} \colon \forall \underline{(\mathrm{fx})} \colon \forall \underline{(\mathrm{fy})} \colon \dot\neg\, 0 <= \underline{(\epsilon)} \Rightarrow \dot\neg\,\dot\neg\, 0 = \underline{(\epsilon)} \Rightarrow \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \vdash \underline{(\mathrm{fx})} <_{\mathrm{f}} \underline{(\mathrm{fy})}][\mathrm{To} < \mathrm{f} \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[\mathrm{PlusF} \stackrel{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}} \colon \forall \underline{(\mathrm{fx})} \colon \forall \underline{(\mathrm{fy})} \colon \underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + \underline{(\mathrm{fy})}[\underline{\mathsf{m}}]][\mathrm{PlusF} \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[\mathrm{MinusF} \stackrel{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}} \colon \forall \underline{(\mathrm{fx})} \colon -_{\mathrm{f}} \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] = -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}]][\mathrm{MinusF} \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[\mathrm{TimesF} \stackrel{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}} \colon \forall \underline{(\mathrm{fx})} \colon \forall \underline{(\mathrm{fy})} \colon \underline{(\mathrm{fx})} *_{\mathrm{f}} \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fy})}[\underline{\mathsf{m}}]][\mathrm{TimesF} \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[0\mathrm{f} \stackrel{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}} \colon 0\mathrm{f}[\underline{\mathsf{m}}] = 0][0\mathrm{f} \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[1\mathrm{f} \stackrel{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}} \colon 1\mathrm{f}[\underline{\mathsf{m}}] = 1][1\mathrm{f} \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[\mathrm{To} == \mathrm{XX} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \forall \underline{(\mathrm{rx})}\colon \forall \underline{(\mathrm{ry})}\colon \underline{(\mathrm{fx})} \in \underline{(\mathrm{rx})} \Rightarrow \underline{(\mathrm{fy})} \in \underline{(\mathrm{ry})} \Rightarrow \mathrm{SF}(\underline{(\mathrm{fx})}, \underline{(\mathrm{fy})}) \vdash \underline{(\mathrm{rx})} == \underline{(\mathrm{ry})}][\mathrm{To} == \mathrm{XX} \stackrel{\mathrm{proof}}{\rightarrow} \mathrm{Rule\ tactic}]$

$[\mathrm{From} == \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \mathrm{R}(\underline{(\mathrm{fx})}) == \mathrm{R}(\underline{(\mathrm{fy})}) \vdash \mathrm{SF}(\underline{(\mathrm{fx})}, \underline{(\mathrm{fy})})][\mathrm{From} == \stackrel{\mathrm{proof}}{\rightarrow} \mathrm{Rule\ tactic}]$

$[\mathrm{To} == \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \mathrm{SF}(\underline{(\mathrm{fx})}, \underline{(\mathrm{fy})}) \vdash \mathrm{R}(\underline{(\mathrm{fx})}) == \mathrm{R}(\underline{(\mathrm{fy})})][\mathrm{To} == \stackrel{\mathrm{proof}}{\rightarrow} \mathrm{Rule\ tactic}]$

$[\mathrm{From} << \mathrm{XX} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}}\colon \forall \underline{(\epsilon)}\colon \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \forall \underline{(\mathrm{rx})}\colon \forall \underline{(\mathrm{ry})}\colon \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})} \vdash \underline{(\mathrm{fx})} \in \underline{(\mathrm{rx})} \vdash \underline{(\mathrm{fy})} \in \underline{(\mathrm{ry})} \vdash \dot{\neg} 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{(\epsilon)} \vdash \mathsf{a}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)}][\mathrm{From} << \mathrm{XX} \stackrel{\mathrm{proof}}{\rightarrow} \mathrm{Rule\ tactic}]$

$[\mathrm{From} << (1) \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{(\mathrm{rx})}\colon \forall \underline{(\mathrm{ry})}\colon \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})} \vdash \mathsf{j}_{\mathrm{Ex}} \in \underline{(\mathrm{rx})}][\mathrm{From} << (1) \stackrel{\mathrm{proof}}{\rightarrow} \mathrm{Rule\ tactic}]$

$[\mathrm{From} << (2) \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{(\mathrm{rx})}\colon \forall \underline{(\mathrm{ry})}\colon \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})} \vdash \mathsf{t}_{\mathrm{Ex}} \in \underline{(\mathrm{ry})}][\mathrm{From} << (2) \stackrel{\mathrm{proof}}{\rightarrow} \mathrm{Rule\ tactic}]$

$[\mathrm{to} << \mathrm{XX} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}}\colon \forall \underline{\mathsf{n}}\colon \forall \underline{(\epsilon)}\colon \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \forall \underline{(\mathrm{rx})}\colon \forall \underline{(\mathrm{ry})}\colon \underline{(\mathrm{fx})} \in \underline{(\mathrm{rx})} \Rightarrow \underline{(\mathrm{fy})} \in \underline{(\mathrm{ry})} \Rightarrow \dot{\neg} 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{(\epsilon)} \Rightarrow \mathsf{a}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \vdash \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})}][\mathrm{to} << \mathrm{XX} \stackrel{\mathrm{proof}}{\rightarrow} \mathrm{Rule\ tactic}]$

$[\mathrm{From} << \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \mathrm{R}(\underline{(\mathrm{fx})}) << \mathrm{R}(\underline{(\mathrm{fy})}) \vdash \underline{(\mathrm{fx})} <_{\mathrm{f}} \underline{(\mathrm{fy})}][\mathrm{From} << \stackrel{\mathrm{proof}}{\rightarrow} \mathrm{Rule\ tactic}]$

$[\mathrm{To} << \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \underline{(\mathrm{fx})} <_{\mathrm{f}} \underline{(\mathrm{fy})} \vdash \mathrm{R}(\underline{(\mathrm{fx})}) << \mathrm{R}(\underline{(\mathrm{fy})})][\mathrm{To} << \stackrel{\mathrm{proof}}{\rightarrow} \mathrm{Rule\ tactic}]$

$[\mathrm{FromInR} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \underline{(\mathrm{fx})} \in \mathrm{R}(\underline{(\mathrm{fy})}) \vdash \mathrm{SF}(\underline{(\mathrm{fx})}, \underline{(\mathrm{fy})})][\mathrm{FromInR} \stackrel{\mathrm{proof}}{\rightarrow} \mathrm{Rule\ tactic}]$

$[\mathrm{PlusR} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}(\underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})})][\mathrm{PlusR} \stackrel{\mathrm{proof}}{\rightarrow} \mathrm{Rule\ tactic}]$

$[\mathrm{TimesR} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \mathrm{R}(\underline{(\mathrm{fx})}) * * \mathrm{R}(\underline{(\mathrm{fy})}) == \mathrm{R}(\underline{(\mathrm{fx})} *_{\mathrm{f}} \underline{(\mathrm{fy})})][\mathrm{TimesR} \stackrel{\mathrm{proof}}{\rightarrow} \mathrm{Rule\ tactic}]$

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$[\epsilon \stackrel{\mathrm{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil[\epsilon \ddot{=} \underline{(\epsilon)}]\rceil)]$

$[\mathrm{FX} \stackrel{\mathrm{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil[\mathrm{FX} \ddot{=} \underline{(\mathrm{fx})}]\rceil)]$

$[\mathrm{FY} \stackrel{\mathrm{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil[\mathrm{FY} \ddot{=} \underline{(\mathrm{fy})}]\rceil)]$

$[\mathrm{FZ} \stackrel{\mathrm{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil[\mathrm{FZ} \ddot{=} \underline{(\mathrm{fz})}]\rceil)]$

$[\mathrm{FU} \stackrel{\mathrm{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil[\mathrm{FU} \ddot{=} \underline{(\mathrm{fv})}]\rceil)]$

$[\mathrm{FV} \stackrel{\mathrm{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil[\mathrm{FV} \ddot{=} \underline{\mathrm{var\ fv}}]\rceil)]$

$[\mathrm{RX} \stackrel{\mathrm{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil[\mathrm{RX} \ddot{=} \underline{(\mathrm{rx})}]\rceil)]$

$[\mathrm{RY} \stackrel{\mathrm{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil[\mathrm{RY} \ddot{=} \underline{(\mathrm{ry})}]\rceil)]$

$[\mathrm{RZ} \stackrel{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathrm{RZ} \ddot{=} \underline{(\mathrm{rz})}]\rceil)]$
$[\mathrm{RU} \stackrel{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathrm{RU} \ddot{=} \underline{(\mathrm{ru})}]\rceil)]$
$[\mathrm{Ex3} \stackrel{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathrm{Ex3} \ddot{=} \mathsf{c}_{\mathrm{Ex}}]\rceil)]$
$[\mathsf{x} <<== \mathsf{y} \stackrel{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} <<== \mathsf{y} \ddot{=} \mathsf{x} << \mathsf{y} \dot{\vee} \mathsf{x} == \mathsf{y}]\rceil)]$
$[(-1) \stackrel{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[(-1) \ddot{=} -1]\rceil)]$
$[2 \stackrel{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[2 \ddot{=} (1+1)]\rceil)]$
$[1/2 \stackrel{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[1/2 \ddot{=} \mathrm{rec2}]\rceil)]$
$[\mathsf{x} < \mathsf{y} \stackrel{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} < \mathsf{y} \ddot{=} \mathsf{x} <= \mathsf{y} \dot{\wedge} \mathsf{x} \neq \mathsf{y}]\rceil)]$
$[\mathsf{x} \neq \mathsf{y} \stackrel{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \neq \mathsf{y} \ddot{=} \dot{\neg} \mathsf{x} = \mathsf{y}]\rceil)]$
$[\mathsf{x} - \mathsf{y} \stackrel{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} - \mathsf{y} \ddot{=} \mathsf{x} + (-\mathsf{y})]\rceil)]$
$[|\mathsf{x}| \stackrel{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[|\mathsf{x}| \ddot{=} \mathrm{if}(0 <= \mathsf{x}, \mathsf{x}, -\mathsf{x})]\rceil)]$
$[00 \stackrel{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[00 \ddot{=} \mathrm{R}(0\mathrm{f})]\rceil)]$
$[01 \stackrel{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[01 \ddot{=} \mathrm{R}(1\mathrm{f})]\rceil)]$
$[\mathrm{R}((\mathrm{fx})) + +\mathrm{R}((\mathrm{fy})) \stackrel{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathrm{R}((\mathrm{fx})) + +\mathrm{R}((\mathrm{fy})) \ddot{=} \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}))]\rceil)]$
$[- - \mathrm{R}((\mathrm{fx})) \stackrel{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[- - \mathrm{R}((\mathrm{fx})) \ddot{=} \mathrm{R}(-_{\mathrm{f}}(\mathrm{fx}))]\rceil)]$ XX noedvendig?
$[\mathrm{R}((\mathrm{fx})) - -\mathrm{R}((\mathrm{fy})) \stackrel{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathrm{R}((\mathrm{fx})) - -\mathrm{R}((\mathrm{fy})) \ddot{=} \mathrm{R}((\mathrm{fx})) + +\mathrm{R}(-_{\mathrm{f}}(\mathrm{fy}))]\rceil)]$ XX noedvendigt med $[\mathrm{R}(\ ) - \mathrm{R}(\ )]$ konstruktionen?
(*** REGELLEMMAER ***)

$[\mathrm{leqTransitivity} \stackrel{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \forall \underline{\mathsf{z}} \colon \underline{\mathsf{x}} <= \underline{\mathsf{y}} \vdash \underline{\mathsf{y}} <= \underline{\mathsf{z}} \vdash \underline{\mathsf{x}} <= \underline{\mathsf{z}}]$

$[\mathrm{leqTransitivity} \stackrel{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \forall \underline{\mathsf{z}} \colon \underline{\mathsf{x}} <= \underline{\mathsf{y}} \vdash \underline{\mathsf{y}} <= \underline{\mathsf{z}} \vdash \mathrm{leqTransitivityAxiom} \gg \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{y}} <= \underline{\mathsf{z}} \Rightarrow \underline{\mathsf{x}} <= \underline{\mathsf{z}}; \mathrm{MP2} \rhd \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{y}} <= \underline{\mathsf{z}} \Rightarrow \underline{\mathsf{x}} <= \underline{\mathsf{z}} \rhd \underline{\mathsf{x}} <= \underline{\mathsf{y}} \rhd \underline{\mathsf{y}} <= \underline{\mathsf{z}} \gg \underline{\mathsf{x}} <= \underline{\mathsf{z}} \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{leqAntisymmetry} \stackrel{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \underline{\mathsf{x}} <= \underline{\mathsf{y}} \vdash \underline{\mathsf{y}} <= \underline{\mathsf{x}} \vdash \underline{\mathsf{x}} = \underline{\mathsf{y}}]$

$[\mathrm{leqAntisymmetry} \stackrel{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \underline{\mathsf{x}} <= \underline{\mathsf{y}} \vdash \underline{\mathsf{y}} <= \underline{\mathsf{x}} \vdash \mathrm{leqAntisymmetryAxiom} \gg \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{y}} <= \underline{\mathsf{x}} \Rightarrow \underline{\mathsf{x}} = \underline{\mathsf{y}}; \mathrm{MP2} \rhd \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{y}} <= \underline{\mathsf{x}} \Rightarrow \underline{\mathsf{x}} = \underline{\mathsf{y}} \rhd \underline{\mathsf{x}} <= \underline{\mathsf{y}} \rhd \underline{\mathsf{y}} <= \underline{\mathsf{x}} \gg \underline{\mathsf{x}} = \underline{\mathsf{y}} \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{leqAddition} \stackrel{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \forall \underline{\mathsf{z}} \colon \underline{\mathsf{x}} <= \underline{\mathsf{y}} \vdash \underline{\mathsf{x}} + \underline{\mathsf{z}} <= \underline{\mathsf{y}} + \underline{\mathsf{z}}]$

$[\mathrm{leqAddition} \stackrel{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \forall \underline{\mathsf{z}} \colon \underline{\mathsf{x}} <= \underline{\mathsf{y}} \vdash \mathrm{leqAdditionAxiom} \gg \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{x}} + \underline{\mathsf{z}} <= \underline{\mathsf{y}} + \underline{\mathsf{z}}; \mathrm{MP} \rhd \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{x}} + \underline{\mathsf{z}} <= \underline{\mathsf{y}} + \underline{\mathsf{z}} \rhd \underline{\mathsf{x}} <= \underline{\mathsf{y}} \gg \underline{\mathsf{x}} + \underline{\mathsf{z}} <= \underline{\mathsf{y}} + \underline{\mathsf{z}} \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{leqMultiplication} \stackrel{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \forall \underline{\mathsf{z}} \colon 0 <= \underline{\mathsf{z}} \vdash \underline{\mathsf{x}} <= \underline{\mathsf{y}} \vdash \underline{\mathsf{x}} * \underline{\mathsf{z}} <= \underline{\mathsf{y}} * \underline{\mathsf{z}}]$

$[\mathrm{leqMultiplication} \stackrel{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \forall \underline{\mathsf{z}} \colon 0 <= \underline{\mathsf{z}} \vdash \underline{\mathsf{x}} <= \underline{\mathsf{y}} \vdash \mathrm{leqMultiplicationAxiom} \gg 0 <= \underline{\mathsf{z}} \Rightarrow \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{x}} * \underline{\mathsf{z}} <= \underline{\mathsf{y}} * \underline{\mathsf{z}}; \mathrm{MP2} \rhd 0 <= \underline{\mathsf{z}} \Rightarrow \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{x}} * \underline{\mathsf{z}} <= \underline{\mathsf{y}} * \underline{\mathsf{z}} \rhd 0 <= \underline{\mathsf{z}} \rhd \underline{\mathsf{x}} <= \underline{\mathsf{y}} \gg \underline{\mathsf{x}} * \underline{\mathsf{z}} <= \underline{\mathsf{y}} * \underline{\mathsf{z}} \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{Reciprocal} \stackrel{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}} \colon \dot{\neg} \underline{\mathsf{x}} = 0 \vdash \underline{\mathsf{x}} * \mathrm{rec}\underline{\mathsf{x}} = 1]$

$[\mathrm{Reciprocal} \stackrel{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}} \colon \dot{\neg} \underline{\mathsf{x}} = 0 \vdash \mathrm{ReciprocalAxiom} \gg \dot{\neg} \underline{\mathsf{x}} = 0 \Rightarrow \underline{\mathsf{x}} * \mathrm{rec}\underline{\mathsf{x}} = 1; \mathrm{MP} \rhd \dot{\neg} \underline{\mathsf{x}} = 0 \Rightarrow \underline{\mathsf{x}} * \mathrm{rec}\underline{\mathsf{x}} = 1 \rhd \dot{\neg} \underline{\mathsf{x}} = 0 \gg \underline{\mathsf{x}} * \mathrm{rec}\underline{\mathsf{x}} = 1 \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{Equality} \stackrel{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \forall \underline{\mathsf{z}} \colon \underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \underline{\mathsf{x}} = \underline{\mathsf{z}} \vdash \underline{\mathsf{y}} = \underline{\mathsf{z}}]$

[Equality $\stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} \vdash \underline{x} = \underline{z} \vdash \text{equalityAxiom} \gg \underline{x} = \underline{y} \Rightarrow \underline{x} = \underline{z} \Rightarrow \underline{y} = \underline{z}; \text{MP2} \rhd \underline{x} = \underline{y} \Rightarrow \underline{x} = \underline{z} \Rightarrow \underline{y} = \underline{z} \rhd \underline{x} = \underline{y} \rhd \underline{x} = \underline{z} \gg \underline{y} = \underline{z}\rceil, \mathrm{p}_0, \mathsf{c})$]
[eqLeq $\stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} = \underline{y} \vdash \underline{x} <= \underline{y}$]
[eqLeq $\stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} = \underline{y} \vdash \text{eqLeqAxiom} \gg \underline{x} = \underline{y} \Rightarrow \underline{x} <= \underline{y}; \text{MP} \rhd \underline{x} = \underline{y} \Rightarrow \underline{x} <= \underline{y} \rhd \underline{x} = \underline{y} \gg \underline{x} <= \underline{y}\rceil, \mathrm{p}_0, \mathsf{c})$]
[eqAddition $\stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} \vdash \underline{x} + \underline{z} = \underline{y} + \underline{z}$]
[eqAddition $\stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} \vdash \text{eqAdditionAxiom} \gg \underline{x} = \underline{y} \Rightarrow \underline{x} + \underline{z} = \underline{y} + \underline{z}; \text{MP} \rhd \underline{x} = \underline{y} \Rightarrow \underline{x} + \underline{z} = \underline{y} + \underline{z} \rhd \underline{x} = \underline{y} \gg \underline{x} + \underline{z} = \underline{y} + \underline{z}\rceil, \mathrm{p}_0, \mathsf{c})$]
[eqMultiplication $\stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} \vdash \underline{x} * \underline{z} = \underline{y} * \underline{z}$]
[eqMultiplication $\stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} \vdash \text{eqMultiplicationAxiom} \gg \underline{x} = \underline{y} \Rightarrow \underline{x} * \underline{z} = \underline{y} * \underline{z}; \text{MP} \rhd \underline{x} = \underline{y} \Rightarrow \underline{x} * \underline{z} = \underline{y} * \underline{z} \rhd \underline{x} = \underline{y} \gg \underline{x} * \underline{z} = \underline{y} * \underline{z}\rceil, \mathrm{p}_0, \mathsf{c})$]
(*** UDSAGNSLOGIK ***)
[ToNegatedImply $\stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \vdash \dot{\neg}\, \underline{b} \vdash \dot{\neg}\, \underline{a} \Rightarrow \underline{b}$]
[ToNegatedImply $\stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \vdash \dot{\neg}\, \underline{b} \vdash \dot{\neg}\,\dot{\neg}\, \underline{a} \Rightarrow \underline{b} \vdash \text{RemoveDoubleNeg} \rhd \dot{\neg}\,\dot{\neg}\, \underline{a} \Rightarrow \underline{b} \gg \underline{a} \Rightarrow \underline{b}; \text{MP} \rhd \underline{a} \Rightarrow \underline{b} \rhd \underline{a} \gg \underline{b}; \text{FromContradiction} \rhd \underline{b} \rhd \dot{\neg}\, \underline{b} \gg \dot{\neg}\, \underline{a} \Rightarrow \underline{b}; \forall \underline{a}\colon \forall \underline{b}\colon \text{Ded} \rhd \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \vdash \dot{\neg}\, \underline{b} \vdash \dot{\neg}\,\dot{\neg}\, \underline{a} \Rightarrow \underline{b} \vdash \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \gg \underline{a} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \dot{\neg}\,\dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\, \underline{a} \Rightarrow \underline{b}; \underline{a} \vdash \dot{\neg}\, \underline{b} \vdash \text{MP2} \rhd \underline{a} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \dot{\neg}\,\dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \rhd \underline{a} \rhd \dot{\neg}\, \underline{b} \gg \dot{\neg}\,\dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\, \underline{a} \Rightarrow \underline{b}; \text{AutoImply} \gg \dot{\neg}\,\dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\,\dot{\neg}\, \underline{a} \Rightarrow \underline{b}; \text{Neg} \rhd \dot{\neg}\,\dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \rhd \dot{\neg}\,\dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\,\dot{\neg}\, \underline{a} \Rightarrow \underline{b} \gg \dot{\neg}\, \underline{a} \Rightarrow \underline{b}\rceil, \mathrm{p}_0, \mathsf{c})$]
[TND $\stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{a}\colon \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{a}$]
[TND $\stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{a}\colon \text{AutoImply} \gg \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{a}; \text{Repetition} \rhd \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{a} \gg \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{a}\rceil, \mathrm{p}_0, \mathsf{c})$]
[FromNegations $\stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \Rightarrow \underline{b} \vdash \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \vdash \underline{b}$]
[FromNegations $\stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \Rightarrow \underline{b} \vdash \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \vdash \text{TND} \gg \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{a}; \text{FromDisjuncts} \rhd \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{a} \rhd \underline{a} \Rightarrow \underline{b} \rhd \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \gg \underline{b}\rceil, \mathrm{p}_0, \mathsf{c})$]
[ImplyNegation $\stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{a}\colon \underline{a} \Rightarrow \dot{\neg}\, \underline{a} \vdash \dot{\neg}\, \underline{a}$]
[ImplyNegation $\stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{a}\colon \underline{a} \Rightarrow \dot{\neg}\, \underline{a} \vdash \text{AutoImply} \gg \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{a}; \text{TND} \gg \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{a}; \text{FromDisjuncts} \rhd \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{a} \rhd \underline{a} \Rightarrow \dot{\neg}\, \underline{a} \rhd \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{a} \gg \dot{\neg}\, \underline{a}\rceil, \mathrm{p}_0, \mathsf{c})$]
[From3Disjuncts $\stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{d}\colon \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{c} \vdash \underline{a} \Rightarrow \underline{d} \vdash \underline{b} \Rightarrow \underline{d} \vdash \underline{c} \Rightarrow \underline{d} \vdash \underline{d}$]
[From3Disjuncts $\stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{d}\colon \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{c} \vdash \underline{b} \Rightarrow \underline{d} \vdash \underline{c} \Rightarrow \underline{d} \vdash \dot{\neg}\, \underline{a} \vdash \text{Repetition} \rhd \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{c} \gg \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{c}; \text{MP} \rhd \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{c} \rhd \dot{\neg}\, \underline{a} \gg \dot{\neg}\, \underline{b} \Rightarrow \underline{c}; \text{FromDisjuncts} \rhd \dot{\neg}\, \underline{b} \Rightarrow \underline{c} \rhd \underline{b} \Rightarrow \underline{d} \rhd \underline{c} \Rightarrow \underline{d} \gg \underline{d}; \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{d}\colon \text{Ded} \rhd \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{d}\colon \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{c} \vdash \underline{b} \Rightarrow \underline{d} \vdash \underline{c} \Rightarrow \underline{d} \vdash \dot{\neg}\, \underline{a} \vdash \underline{d} \gg \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{b} \Rightarrow \underline{d} \Rightarrow \underline{c} \Rightarrow \underline{d} \Rightarrow \dot{\neg}\, \underline{a} \Rightarrow \underline{d}; \text{AutoImply} \gg \underline{a} \Rightarrow \underline{d} \Rightarrow \underline{a} \Rightarrow \underline{d}; \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{c} \vdash \underline{a} \Rightarrow \underline{d} \vdash \underline{b} \Rightarrow \underline{d} \vdash \underline{c} \Rightarrow \underline{d} \vdash \text{MP3} \rhd \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{b} \Rightarrow \underline{d} \Rightarrow$

$\underline{c} \Rightarrow \underline{d} \Rightarrow \dot{\neg}\,\underline{a} \Rightarrow \underline{d} \triangleright \dot{\neg}\,\underline{a} \Rightarrow \dot{\neg}\,\underline{b} \Rightarrow \underline{c} \triangleright \underline{b} \Rightarrow \underline{d} \triangleright \underline{c} \Rightarrow \underline{d} \gg \dot{\neg}\,\underline{a} \Rightarrow \underline{d}; \mathrm{MP} \triangleright \underline{a} \Rightarrow \underline{d} \Rightarrow \underline{a} \Rightarrow \underline{d} \triangleright \underline{a} \Rightarrow \underline{d} \gg \underline{a} \Rightarrow \underline{d}; \mathrm{FromNegations} \triangleright \underline{a} \Rightarrow \underline{d} \triangleright \dot{\neg}\,\underline{a} \Rightarrow \underline{d} \gg \underline{d}\rceil, p_0, \mathsf{c})]$

$[\mathrm{NegateDisjunct1} \overset{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \vdash \dot{\neg}\,\underline{a} \vdash \underline{b}]$

$[\mathrm{NegateDisjunct1} \overset{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \vdash \dot{\neg}\,\underline{a} \vdash \mathrm{Repetition} \triangleright \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \gg \dot{\neg}\,\underline{a} \Rightarrow \underline{b}; \mathrm{MP} \triangleright \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \triangleright \dot{\neg}\,\underline{a} \gg \underline{b}\rceil, p_0, \mathsf{c})]$

$[\mathrm{NegateDisjunct2} \overset{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \vdash \dot{\neg}\,\underline{b} \vdash \underline{a}]$

$[\mathrm{NegateDisjunct2} \overset{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \vdash \dot{\neg}\,\underline{b} \vdash \mathrm{Repetition} \triangleright \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \gg \dot{\neg}\,\underline{a} \Rightarrow \underline{b}; \mathrm{NegativeMT} \triangleright \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \triangleright \dot{\neg}\,\underline{b} \gg \underline{a}\rceil, p_0, \mathsf{c})]$
(***)

$[\mathrm{ExpandDisjuncts} \overset{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{d}\colon \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \vdash \dot{\neg}\,\underline{c} \Rightarrow \underline{d} \vdash \dot{\neg}\,\underline{b} \Rightarrow \dot{\neg}\,\underline{d} \Rightarrow \dot{\neg}\,\underline{a} \Rightarrow \dot{\neg}\,\underline{c}]$

$[\mathrm{ExpandDisjuncts} \overset{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{d}\colon \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \vdash \dot{\neg}\,\underline{c} \Rightarrow \underline{d} \vdash \dot{\neg}\,\underline{b} \vdash \dot{\neg}\,\underline{d} \vdash \mathrm{NegateDisjunct2} \triangleright \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \triangleright \dot{\neg}\,\underline{b} \gg \underline{a}; \mathrm{NegateDisjunct2} \triangleright \dot{\neg}\,\underline{c} \Rightarrow \underline{d} \triangleright \dot{\neg}\,\underline{d} \gg \underline{c}; \mathrm{JoinConjuncts} \triangleright \underline{a} \triangleright \underline{c} \gg \dot{\neg}\,\underline{a} \Rightarrow \dot{\neg}\,\underline{c}; \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{d}\colon \mathrm{Ded} \triangleright \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{d}\colon \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \vdash \dot{\neg}\,\underline{c} \Rightarrow \underline{d} \vdash \dot{\neg}\,\underline{b} \vdash \dot{\neg}\,\underline{d} \vdash \dot{\neg}\,\underline{a} \Rightarrow \dot{\neg}\,\underline{c} \gg \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\,\underline{c} \Rightarrow \underline{d} \Rightarrow \dot{\neg}\,\underline{b} \Rightarrow \dot{\neg}\,\underline{d} \Rightarrow \dot{\neg}\,\underline{a} \Rightarrow \dot{\neg}\,\underline{c}; \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \vdash \dot{\neg}\,\underline{c} \Rightarrow \underline{d} \vdash \mathrm{MP2} \triangleright \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\,\underline{c} \Rightarrow \underline{d} \Rightarrow \dot{\neg}\,\underline{b} \Rightarrow \dot{\neg}\,\underline{d} \Rightarrow \dot{\neg}\,\underline{a} \Rightarrow \dot{\neg}\,\underline{c} \triangleright \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \triangleright \dot{\neg}\,\underline{c} \Rightarrow \underline{d} \gg \dot{\neg}\,\underline{b} \Rightarrow \dot{\neg}\,\underline{d} \Rightarrow \dot{\neg}\,\underline{a} \Rightarrow \dot{\neg}\,\underline{c}; \mathrm{Repetition} \triangleright \dot{\neg}\,\underline{b} \Rightarrow \dot{\neg}\,\underline{d} \Rightarrow \dot{\neg}\,\underline{a} \Rightarrow \dot{\neg}\,\underline{c} \gg \dot{\neg}\,\underline{b} \Rightarrow \dot{\neg}\,\underline{d} \Rightarrow \dot{\neg}\,\underline{a} \Rightarrow \dot{\neg}\,\underline{c}\rceil, p_0, \mathsf{c})]$

$[\mathrm{From2} * \mathrm{2Disjuncts} \overset{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{d}\colon \forall \underline{e}\colon \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \vdash \dot{\neg}\,\underline{c} \Rightarrow \underline{d} \vdash \underline{a} \Rightarrow \underline{c} \Rightarrow \underline{e} \vdash \underline{a} \Rightarrow \underline{d} \Rightarrow \underline{e} \vdash \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{e} \vdash \underline{b} \Rightarrow \underline{d} \Rightarrow \underline{e} \vdash \underline{e}]$

$[\mathrm{From2} * \mathrm{2Disjuncts} \overset{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{d}\colon \forall \underline{e}\colon \dot{\neg}\,\underline{c} \Rightarrow \underline{d} \vdash \underline{a} \Rightarrow \underline{c} \Rightarrow \underline{e} \vdash \underline{a} \Rightarrow \underline{d} \Rightarrow \underline{e} \vdash \underline{a} \vdash \mathrm{MP} \triangleright \underline{a} \Rightarrow \underline{c} \Rightarrow \underline{e} \triangleright \underline{a} \gg \underline{c} \Rightarrow \underline{e}; \mathrm{MP} \triangleright \underline{a} \Rightarrow \underline{d} \Rightarrow \underline{e} \triangleright \underline{a} \gg \underline{d} \Rightarrow \underline{e}; \mathrm{FromDisjuncts} \triangleright \dot{\neg}\,\underline{c} \Rightarrow \underline{d} \triangleright \underline{c} \Rightarrow \underline{e} \triangleright \underline{d} \Rightarrow \underline{e} \gg \underline{e}; \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{d}\colon \forall \underline{e}\colon \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \vdash \dot{\neg}\,\underline{c} \Rightarrow \underline{d} \vdash \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{e} \vdash \underline{b} \Rightarrow \underline{d} \Rightarrow \underline{e} \vdash \dot{\neg}\,\underline{a} \vdash \mathrm{NegateDisjunct1} \triangleright \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \triangleright \dot{\neg}\,\underline{a} \gg \underline{b}; \mathrm{MP} \triangleright \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{e} \triangleright \underline{b} \gg \underline{c} \Rightarrow \underline{e}; \mathrm{MP} \triangleright \underline{b} \Rightarrow \underline{d} \Rightarrow \underline{e} \triangleright \underline{b} \gg \underline{d} \Rightarrow \underline{e}; \mathrm{FromDisjuncts} \triangleright \dot{\neg}\,\underline{c} \Rightarrow \underline{d} \triangleright \underline{c} \Rightarrow \underline{e} \triangleright \underline{d} \Rightarrow \underline{e} \gg \underline{e}; \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{d}\colon \forall \underline{e}\colon \mathrm{Ded} \triangleright \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{d}\colon \forall \underline{e}\colon \dot{\neg}\,\underline{c} \Rightarrow \underline{d} \vdash \underline{a} \Rightarrow \underline{c} \Rightarrow \underline{e} \vdash \underline{a} \Rightarrow \underline{d} \Rightarrow \underline{e} \vdash \underline{a} \vdash \underline{e} \gg \dot{\neg}\,\underline{c} \Rightarrow \underline{d} \Rightarrow \underline{a} \Rightarrow \underline{c} \Rightarrow \underline{e} \Rightarrow \underline{a} \Rightarrow \underline{d} \Rightarrow \underline{e} \Rightarrow \underline{a} \Rightarrow \underline{e}; \mathrm{Ded} \triangleright \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{d}\colon \forall \underline{e}\colon \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \vdash \dot{\neg}\,\underline{c} \Rightarrow \underline{d} \vdash \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{e} \vdash \underline{b} \Rightarrow \underline{d} \Rightarrow \underline{e} \vdash \dot{\neg}\,\underline{a} \vdash \underline{e} \gg \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\,\underline{c} \Rightarrow \underline{d} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{e} \Rightarrow \underline{b} \Rightarrow \underline{d} \Rightarrow \underline{e} \Rightarrow \dot{\neg}\,\underline{a} \Rightarrow \underline{e}; \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \vdash \dot{\neg}\,\underline{c} \Rightarrow \underline{d} \vdash \underline{a} \Rightarrow \underline{c} \Rightarrow \underline{e} \vdash \underline{a} \Rightarrow \underline{d} \Rightarrow \underline{e} \vdash \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{e} \vdash \underline{b} \Rightarrow \underline{d} \Rightarrow \underline{e} \vdash \mathrm{MP3} \triangleright \dot{\neg}\,\underline{c} \Rightarrow \underline{d} \Rightarrow \underline{a} \Rightarrow \underline{c} \Rightarrow \underline{e} \Rightarrow \underline{a} \Rightarrow \underline{d} \Rightarrow \underline{e} \Rightarrow \underline{a} \Rightarrow \underline{e} \triangleright \dot{\neg}\,\underline{c} \Rightarrow \underline{d} \triangleright \underline{a} \Rightarrow \underline{c} \Rightarrow \underline{e} \triangleright \underline{a} \Rightarrow \underline{d} \Rightarrow \underline{e} \gg \underline{a} \Rightarrow \underline{e}; \mathrm{MP4} \triangleright \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\,\underline{c} \Rightarrow \underline{d} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{e} \Rightarrow \underline{b} \Rightarrow \underline{d} \Rightarrow \underline{e} \Rightarrow \dot{\neg}\,\underline{a} \Rightarrow \underline{e} \triangleright \dot{\neg}\,\underline{a} \Rightarrow \underline{b} \triangleright \dot{\neg}\,\underline{c} \Rightarrow \underline{d} \triangleright \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{e} \triangleright \underline{b} \Rightarrow \underline{d} \Rightarrow \underline{e} \gg \dot{\neg}\,\underline{a} \Rightarrow \underline{e}; \mathrm{FromNegations} \triangleright \underline{a} \Rightarrow \underline{e} \triangleright \dot{\neg}\,\underline{a} \Rightarrow \underline{e} \gg \underline{e}\rceil, p_0, \mathsf{c})]$
(*** EQUALITY ***)

$[\mathrm{eqReflexivity} \overset{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{x}\colon \underline{x} = \underline{x}]$

$[\mathrm{eqReflexivity} \overset{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x}\colon \mathrm{leqReflexivity} \gg \underline{x} <= \underline{x}; \mathrm{leqAntisymmetry} \triangleright \underline{x} <= \underline{x} \triangleright \underline{x} <= \underline{x} \gg \underline{x} = \underline{x}\rceil, p_0, \mathsf{c})]$

$[\mathrm{eqSymmetry} \overset{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} = \underline{y} \vdash \underline{y} = \underline{x}]$

$[\mathrm{eqSymmetry} \overset{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} = \underline{y} \vdash \mathrm{eqReflexivity} \gg \underline{x} = \underline{x}; \mathrm{Equality} \triangleright \underline{x} = \underline{y} \triangleright \underline{x} = \underline{x} \gg \underline{y} = \underline{x}\rceil, p_0, \mathsf{c})]$

$[\mathrm{eqTransitivity} \overset{\text{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} \vdash \underline{y} = \underline{z} \vdash \underline{x} = \underline{z}]$

$[\text{eqTransitivity} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} \vdash \underline{y} = \underline{z} \vdash \text{eqSymmetry} \rhd \underline{x} = \underline{y} \gg \underline{y} = \underline{x}; \text{Equality} \rhd \underline{y} = \underline{x} \rhd \underline{y} = \underline{z} \gg \underline{x} = \underline{z}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\text{eqTransitivity4} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \forall \underline{u}\colon \underline{x} = \underline{y} \vdash \underline{y} = \underline{z} \vdash \underline{z} = \underline{u} \vdash \underline{x} = \underline{u}]$

$[\text{eqTransitivity4} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \forall \underline{u}\colon \underline{x} = \underline{y} \vdash \underline{y} = \underline{z} \vdash \underline{z} = \underline{u} \vdash \text{eqTransitivity} \rhd \underline{x} = \underline{y} \rhd \underline{y} = \underline{z} \gg \underline{x} = \underline{z}; \text{eqTransitivity} \rhd \underline{x} = \underline{z} \rhd \underline{z} = \underline{u} \gg \underline{x} = \underline{u}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\text{eqTransitivity5} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \forall \underline{u}\colon \forall \underline{v}\colon \underline{x} = \underline{y} \vdash \underline{y} = \underline{z} \vdash \underline{z} = \underline{u} \vdash \underline{u} = \underline{v} \vdash \underline{x} = \underline{v}]$

$[\text{eqTransitivity5} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \forall \underline{u}\colon \forall \underline{v}\colon \underline{x} = \underline{y} \vdash \underline{y} = \underline{z} \vdash \underline{z} = \underline{u} \vdash \underline{u} = \underline{v} \vdash \text{eqTransitivity4} \rhd \underline{x} = \underline{y} \rhd \underline{y} = \underline{z} \rhd \underline{z} = \underline{u} \gg \underline{x} = \underline{u}; \text{eqTransitivity} \rhd \underline{x} = \underline{u} \rhd \underline{u} = \underline{v} \gg \underline{x} = \underline{v}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\text{eqTransitivity6} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \forall \underline{u}\colon \forall \underline{v}\colon \forall \underline{w}\colon \underline{x} = \underline{y} \vdash \underline{y} = \underline{z} \vdash \underline{z} = \underline{u} \vdash \underline{u} = \underline{v} \vdash \underline{v} = \underline{w} \vdash \underline{x} = \underline{w}]$

$[\text{eqTransitivity6} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \forall \underline{u}\colon \forall \underline{v}\colon \forall \underline{w}\colon \underline{x} = \underline{y} \vdash \underline{y} = \underline{z} \vdash \underline{z} = \underline{u} \vdash \underline{u} = \underline{v} \vdash \underline{v} = \underline{w} \vdash \text{eqTransitivity5} \rhd \underline{x} = \underline{y} \rhd \underline{y} = \underline{z} \rhd \underline{z} = \underline{u} \rhd \underline{u} = \underline{v} \gg \underline{x} = \underline{v}; \text{eqTransitivity} \rhd \underline{x} = \underline{v} \rhd \underline{v} = \underline{w} \gg \underline{x} = \underline{w}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\text{plus0Left} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{x}\colon 0 + \underline{x} = \underline{x}]$

$[\text{plus0Left} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \text{plus0} \gg \underline{x} + 0 = \underline{x}; \text{plusCommutativity} \gg 0 + \underline{x} = \underline{x} + 0; \text{eqTransitivity} \rhd 0 + \underline{x} = \underline{x} + 0 \rhd \underline{x} + 0 = \underline{x} \gg 0 + \underline{x} = \underline{x}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\text{times1Left} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{x}\colon 1 * \underline{x} = \underline{x}]$

$[\text{times1Left} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \text{times1} \gg \underline{x} * 1 = \underline{x}; \text{timesCommutativity} \gg 1 * \underline{x} = \underline{x} * 1; \text{eqTransitivity} \rhd 1 * \underline{x} = \underline{x} * 1 \rhd \underline{x} * 1 = \underline{x} \gg 1 * \underline{x} = \underline{x}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\text{lemma eqAdditionLeft} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} \vdash \underline{z} + \underline{x} = \underline{z} + \underline{y}]$

$[\text{lemma eqAdditionLeft} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} \vdash \text{eqAddition} \rhd \underline{x} = \underline{y} \gg \underline{x} + \underline{z} = \underline{y} + \underline{z}; \text{plusCommutativity} \gg \underline{z} + \underline{x} = \underline{x} + \underline{z}; \text{plusCommutativity} \gg \underline{y} + \underline{z} = \underline{z} + \underline{y}; \text{eqTransitivity4} \rhd \underline{z} + \underline{x} = \underline{x} + \underline{z} \rhd \underline{x} + \underline{z} = \underline{y} + \underline{z} \rhd \underline{y} + \underline{z} = \underline{z} + \underline{y} \gg \underline{z} + \underline{x} = \underline{z} + \underline{y}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\text{EqMultiplicationLeft} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} \vdash \underline{z} * \underline{x} = \underline{z} * \underline{y}]$

$[\text{EqMultiplicationLeft} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} \vdash \text{eqMultiplication} \rhd \underline{x} = \underline{y} \gg \underline{x} * \underline{z} = \underline{y} * \underline{z}; \text{timesCommutativity} \gg \underline{z} * \underline{x} = \underline{x} * \underline{z}; \text{timesCommutativity} \gg \underline{y} * \underline{z} = \underline{z} * \underline{y}; \text{eqTransitivity4} \rhd \underline{z} * \underline{x} = \underline{x} * \underline{z} \rhd \underline{x} * \underline{z} = \underline{y} * \underline{z} \rhd \underline{y} * \underline{z} = \underline{z} * \underline{y} \gg \underline{z} * \underline{x} = \underline{z} * \underline{y}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\text{DistributionOut} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} * \underline{y} + \underline{x} * \underline{z} = \underline{x} * \underline{y} + \underline{z}]$

$[\text{DistributionOut} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \text{Distribution} \gg \underline{x} * \underline{y} + \underline{z} = \underline{x} * \underline{y} + \underline{x} * \underline{z}; \text{eqSymmetry} \rhd \underline{x} * \underline{y} + \underline{z} = \underline{x} * \underline{y} + \underline{x} * \underline{z} \gg \underline{x} * \underline{y} + \underline{x} * \underline{z} = \underline{x} * \underline{y} + \underline{z}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\text{Three2twoTerms} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \forall \underline{u}\colon \underline{y} + \underline{z} = \underline{u} \vdash \underline{x} + \underline{y} + \underline{z} = \underline{x} + \underline{u}]$

$[\text{Three2twoTerms} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \forall \underline{u}\colon \underline{y} + \underline{z} = \underline{u} \vdash \text{lemma eqAdditionLeft} \rhd \underline{y} + \underline{z} = \underline{u} \gg \underline{x} + \underline{y} + \underline{z} = \underline{x} + \underline{u}; \text{plusAssociativity} \gg$

$\underline{x}+\underline{y}+\underline{z}=\underline{x}+\underline{y}+\underline{z};\text{eqTransitivity}\rhd\underline{x}+\underline{y}+\underline{z}=\underline{x}+\underline{y}+\underline{z}\rhd\underline{x}+\underline{y}+\underline{z}=\underline{x}+\underline{u}\gg\underline{x}+\underline{y}+\underline{z}=\underline{x}+\underline{u}\rceil,\mathrm{p}_0,\mathsf{c})]$

$[\text{Three2threeTerms}\overset{\text{stmt}}{\rightarrow}\text{ZFsub}\vdash\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{z}\colon\underline{x}+\underline{y}+\underline{z}=\underline{x}+\underline{z}+\underline{y}]$

$[\text{Three2threeTerms}\overset{\text{proof}}{\rightarrow}\lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil\text{ZFsub}\vdash\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{z}\colon\text{plusCommutativity}\gg\underline{y}+\underline{z}=\underline{z}+\underline{y};\text{Three2twoTerms}\rhd\underline{y}+\underline{z}=\underline{z}+\underline{y}\gg\underline{x}+\underline{y}+\underline{z}=\underline{x}+\underline{z}+\underline{y};\text{plusAssociativity}\gg\underline{x}+\underline{z}+\underline{y}=\underline{x}+\underline{z}+\underline{y};\text{eqSymmetry}\rhd\underline{x}+\underline{z}+\underline{y}=\underline{x}+\underline{z}+\underline{y}\gg\underline{x}+\underline{z}+\underline{y}=\underline{x}+\underline{z}+\underline{y};\text{eqTransitivity}\rhd\underline{x}+\underline{y}+\underline{z}=\underline{x}+\underline{z}+\underline{y}\rhd\underline{x}+\underline{z}+\underline{y}=\underline{x}+\underline{z}+\underline{y}\gg\underline{x}+\underline{y}+\underline{z}=\underline{x}+\underline{z}+\underline{y}\rceil,\mathrm{p}_0,\mathsf{c})]$

$[\text{Three2threeFactors}\overset{\text{stmt}}{\rightarrow}\text{ZFsub}\vdash\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{z}\colon\forall\underline{u}\colon\underline{y}*\underline{z}=\underline{u}\vdash\underline{x}*\underline{y}*\underline{z}=\underline{x}*\underline{u}]$

$[\text{Three2threeFactors}\overset{\text{proof}}{\rightarrow}\lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil\text{ZFsub}\vdash\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{z}\colon\forall\underline{u}\colon\underline{y}*\underline{z}=\underline{u}\vdash\text{EqMultiplicationLeft}\rhd\underline{y}*\underline{z}=\underline{u}\gg\underline{x}*\underline{y}*\underline{z}=\underline{x}*\underline{u};\text{timesAssociativity}\gg\underline{x}*\underline{y}*\underline{z}=\underline{x}*\underline{y}*\underline{z};\text{eqTransitivity}\rhd\underline{x}*\underline{y}*\underline{z}=\underline{x}*\underline{y}*\underline{z}\rhd\underline{x}*\underline{y}*\underline{z}=\underline{x}*\underline{u}\gg\underline{x}*\underline{y}*\underline{z}=\underline{x}*\underline{u}\rceil,\mathrm{p}_0,\mathsf{c})]$

$[\text{AddEquations}\overset{\text{stmt}}{\rightarrow}\text{ZFsub}\vdash\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{z}\colon\forall\underline{u}\colon\underline{x}=\underline{y}\vdash\underline{z}=\underline{u}\vdash\underline{x}+\underline{z}=\underline{y}+\underline{u}]$

$[\text{AddEquations}\overset{\text{proof}}{\rightarrow}\lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil\text{ZFsub}\vdash\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{z}\colon\forall\underline{u}\colon\underline{x}=\underline{y}\vdash\underline{z}=\underline{u}\vdash\text{eqAddition}\rhd\underline{x}=\underline{y}\gg\underline{x}+\underline{z}=\underline{y}+\underline{z};\text{lemma eqAdditionLeft}\rhd\underline{z}=\underline{u}\gg\underline{y}+\underline{z}=\underline{y}+\underline{u};\text{eqTransitivity}\rhd\underline{x}+\underline{z}=\underline{y}+\underline{z}\rhd\underline{y}+\underline{z}=\underline{y}+\underline{u}\gg\underline{x}+\underline{z}=\underline{y}+\underline{u}\rceil,\mathrm{p}_0,\mathsf{c})]$

$[\text{SubtractEquations}\overset{\text{stmt}}{\rightarrow}\text{ZFsub}\vdash\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{z}\colon\forall\underline{u}\colon\underline{x}+\underline{z}=\underline{y}+\underline{u}\vdash\underline{z}=\underline{u}\vdash\underline{x}=\underline{y}]$

$[\text{SubtractEquations}\overset{\text{proof}}{\rightarrow}\lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil\text{ZFsub}\vdash\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{z}\colon\forall\underline{u}\colon\underline{x}+\underline{z}=\underline{y}+\underline{u}\vdash\underline{z}=\underline{u}\vdash\text{eqAddition}\rhd\underline{x}+\underline{z}=\underline{y}+\underline{u}\gg\underline{x}+\underline{z}+-\underline{z}=\underline{y}+\underline{u}+-\underline{z};\text{plus0Left}\gg0+\underline{z}=\underline{z};\text{eqTransitivity}\rhd0+\underline{z}=\underline{z}\rhd\underline{z}=\underline{u}\gg0+\underline{z}=\underline{u};\text{PositiveToRight(Eq)}\rhd0+\underline{z}=\underline{u}\gg0=\underline{u}+-\underline{z};\text{eqSymmetry}\rhd0=\underline{u}+-\underline{z}\gg\underline{u}+-\underline{z}=0;\text{lemma eqAdditionLeft}\rhd\underline{u}+-\underline{z}=0\gg\underline{y}+\underline{u}+-\underline{z}=\underline{y}+0;\text{plusAssociativity}\gg\underline{y}+\underline{u}+-\underline{z}=\underline{y}+\underline{u}+-\underline{z};\text{plus0}\gg\underline{y}+0=\underline{y};\text{eqTransitivity4}\rhd\underline{y}+\underline{u}+-\underline{z}=\underline{y}+\underline{u}+-\underline{z}\rhd\underline{y}+\underline{u}+-\underline{z}=\underline{y}+0\rhd\underline{y}+0=\underline{y}\gg\underline{y}+\underline{u}+-\underline{z}=\underline{y};\mathrm{x}=\mathrm{x}+\mathrm{y}-\mathrm{y}\gg\underline{x}=\underline{x}+\underline{z}+-\underline{z};\text{eqTransitivity4}\rhd\underline{x}=\underline{x}+\underline{z}+-\underline{z}\rhd\underline{x}+\underline{z}+-\underline{z}=\underline{y}+\underline{u}+-\underline{z}\rhd\underline{y}+\underline{u}+-\underline{z}=\underline{y}\gg\underline{x}=\underline{y}\rceil,\mathrm{p}_0,\mathsf{c})]$

$[\text{SubtractEquationsLeft}\overset{\text{stmt}}{\rightarrow}\text{ZFsub}\vdash\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{z}\colon\forall\underline{u}\colon\underline{x}+\underline{z}=\underline{y}+\underline{u}\vdash\underline{x}=\underline{y}\vdash\underline{z}=\underline{u}]$

$[\text{SubtractEquationsLeft}\overset{\text{proof}}{\rightarrow}\lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil\text{ZFsub}\vdash\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{z}\colon\forall\underline{u}\colon\underline{x}+\underline{z}=\underline{y}+\underline{u}\vdash\underline{x}=\underline{y}\vdash\text{plusCommutativity}\gg\underline{z}+\underline{x}=\underline{x}+\underline{z};\text{plusCommutativity}\gg\underline{y}+\underline{u}=\underline{u}+\underline{y};\text{eqTransitivity4}\rhd\underline{z}+\underline{x}=\underline{x}+\underline{z}\rhd\underline{x}+\underline{z}=\underline{y}+\underline{u}\rhd\underline{y}+\underline{u}=\underline{u}+\underline{y}\gg\underline{z}+\underline{x}=\underline{u}+\underline{y};\text{SubtractEquations}\rhd\underline{z}+\underline{x}=\underline{u}+\underline{y}\rhd\underline{x}=\underline{y}\gg\underline{z}=\underline{u}\rceil,\mathrm{p}_0,\mathsf{c})]$

$[\text{EqNegated}\overset{\text{stmt}}{\rightarrow}\text{ZFsub}\vdash\forall\underline{x}\colon\forall\underline{y}\colon\underline{x}=\underline{y}\vdash-\underline{x}=-\underline{y}]$

$[\text{EqNegated}\overset{\text{proof}}{\rightarrow}\lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil\text{ZFsub}\vdash\forall\underline{x}\colon\forall\underline{y}\colon\underline{x}=\underline{y}\vdash\text{Negative}\gg\underline{x}+-\underline{x}=0;\text{Negative}\gg\underline{y}+-\underline{y}=0;\text{eqSymmetry}\rhd\underline{y}+-\underline{y}=0\gg0=\underline{y}+-\underline{y};\text{eqTransitivity}\rhd\underline{x}+-\underline{x}=0\rhd0=\underline{y}+-\underline{y}\gg\underline{x}+-\underline{x}=\underline{y}+-\underline{y};\text{SubtractEquationsLeft}\rhd\underline{x}+-\underline{x}=\underline{y}+-\underline{y}\rhd\underline{x}=\underline{y}\gg-\underline{x}=-\underline{y}\rceil,\mathrm{p}_0,\mathsf{c})]$

$[\text{PositiveToRight(Eq)}\overset{\text{stmt}}{\rightarrow}\text{ZFsub}\vdash\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{z}\colon\underline{x}+\underline{y}=\underline{z}\vdash\underline{x}=\underline{z}+-\underline{y}]$

$[\text{PositiveToRight(Eq)}\overset{\text{proof}}{\rightarrow}\lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil\text{ZFsub}\vdash\forall\underline{x}\colon\forall\underline{y}\colon\forall\underline{z}\colon\underline{x}+\underline{y}=\underline{z}\vdash\text{eqAddition}\rhd\underline{x}+\underline{y}=\underline{z}\gg\underline{x}+\underline{y}+-\underline{y}=\underline{z}+-\underline{y};\mathrm{x}=\mathrm{x}+\mathrm{y}-\mathrm{y}\gg\underline{x}=\underline{x}+\underline{y}+-\underline{y};\text{eqTransitivity}\rhd\underline{x}=\underline{x}+\underline{y}+-\underline{y}\rhd\underline{x}+\underline{y}+-\underline{y}=\underline{z}+-\underline{y}\gg\underline{x}=\underline{z}+-\underline{y}\rceil,\mathrm{p}_0,\mathsf{c})]$

[PositiveToLeft(Eq)(1term) $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} = \underline{y} \vdash \underline{x} + -\underline{y} = 0$]

[PositiveToLeft(Eq)(1term) $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} = \underline{y} \vdash$ eqAddition $\rhd \underline{x} = \underline{y} \gg \underline{x} + -\underline{y} = \underline{y} + -\underline{y};$ Negative $\gg \underline{y} + -\underline{y} = 0;$ eqTransitivity $\rhd \underline{x} + -\underline{y} = \underline{y} + -\underline{y} \rhd \underline{y} + -\underline{y} = 0 \gg \underline{x} + -\underline{y} = 0\rceil, \mathrm{p}_0, c)$]

[PositiveToRight(Leq)(1term) $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{y}\colon \forall \underline{z}\colon \underline{y} <= \underline{z} \vdash 0 <= \underline{z} + -\underline{y}$]

[PositiveToRight(Leq)(1term) $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{y}\colon \forall \underline{z}\colon \underline{y} <= \underline{z} \vdash$ plus0Left $\gg 0 + \underline{y} = \underline{y};$ eqSymmetry $\rhd 0 + \underline{y} = \underline{y} \gg \underline{y} = 0 + \underline{y};$ subLeqLeft $\rhd \underline{y} = 0 + \underline{y} \rhd \underline{y} <= \underline{z} \gg 0 + \underline{y} <= \underline{z};$ PositiveToRight(Leq) $\rhd 0 + \underline{y} <= \underline{z} \gg 0 <= \underline{z} + -\underline{y}\rceil, \mathrm{p}_0, c)$]

[NegativeToLeft(Eq) $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} + -\underline{z} \vdash \underline{x} + \underline{z} = \underline{y}$]

[NegativeToLeft(Eq) $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} + -\underline{z} \vdash$ eqAddition $\rhd \underline{x} = \underline{y} + -\underline{z} \gg \underline{x} + \underline{z} = \underline{y} + -\underline{z} + \underline{z};$ Three2threeTerms $\gg \underline{y} + -\underline{z} + \underline{z} = \underline{y} + \underline{z} + -\underline{z};$ $x = x + y - y \gg \underline{y} = \underline{y} + \underline{z} + -\underline{z};$ eqSymmetry $\rhd \underline{y} = \underline{y} + \underline{z} + -\underline{z} \gg \underline{y} + \underline{z} + -\underline{z} = \underline{y};$ eqTransitivity4 $\rhd \underline{x} + \underline{z} = \underline{y} + -\underline{z} + \underline{z} \rhd \underline{y} + -\underline{z} + \underline{z} = \underline{y} + \underline{z} + -\underline{z} \rhd \underline{y} + \underline{z} + -\underline{z} = \underline{y} \gg \underline{x} + \underline{z} = \underline{y}\rceil, \mathrm{p}_0, c)$]
(*** NO EQUALITY ***)

[LessNeq $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \dot\neg\, \underline{x} <= \underline{y} \Rightarrow \dot\neg\dot\neg\, \underline{x} = \underline{y} \vdash \dot\neg\, \underline{x} = \underline{y}$]

[LessNeq $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \dot\neg\, \underline{x} <= \underline{y} \Rightarrow \dot\neg\dot\neg\, \underline{x} = \underline{y} \vdash$ Repetition $\rhd \dot\neg\, \underline{x} <= \underline{y} \Rightarrow \dot\neg\dot\neg\, \underline{x} = \underline{y} \gg \dot\neg\, \underline{x} <= \underline{y} \Rightarrow \dot\neg\dot\neg\, \underline{x} = \underline{y};$ SecondConjunct $\rhd \dot\neg\, \underline{x} <= \underline{y} \Rightarrow \dot\neg\dot\neg\, \underline{x} = \underline{y} \gg \dot\neg\, \underline{x} = \underline{y}\rceil, \mathrm{p}_0, c)$]

[NeqSymmetry $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \dot\neg\, \underline{x} = \underline{y} \vdash \dot\neg\, \underline{y} = \underline{x}$]

[NeqSymmetry $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \underline{y} = \underline{x} \vdash$ eqSymmetry $\rhd \underline{y} = \underline{x} \gg \underline{x} = \underline{y};$ $\forall \underline{x}\colon \forall \underline{y}\colon$ Ded $\rhd \forall \underline{x}\colon \forall \underline{y}\colon \underline{y} = \underline{x} \vdash \underline{x} = \underline{y} \gg \underline{y} = \underline{x} \Rightarrow \underline{x} = \underline{y};$ $\dot\neg\, \underline{x} = \underline{y} \vdash$ MT $\rhd \underline{y} = \underline{x} \Rightarrow \underline{x} = \underline{y} \rhd \dot\neg\, \underline{x} = \underline{y} \gg \dot\neg\, \underline{y} = \underline{x}\rceil, \mathrm{p}_0, c)$]

[NeqNegated $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \dot\neg\, \underline{x} = \underline{y} \vdash \dot\neg\, -\underline{x} = -\underline{y}$]

[NeqNegated $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \dot\neg\, \underline{x} = \underline{y} \vdash -\underline{x} = -\underline{y} \vdash$ EqNegated $\rhd -\underline{x} = -\underline{y} \gg - - \underline{x} = - - \underline{y};$ DoubleMinus $\gg - - \underline{x} = \underline{x};$ eqSymmetry $\rhd - - \underline{x} = \underline{x} \gg \underline{x} = - - \underline{x};$ DoubleMinus $\gg - - \underline{y} = \underline{y};$ eqTransitivity4 $\rhd \underline{x} = - - \underline{x} \rhd - - \underline{x} = - - \underline{y} \rhd - - \underline{y} = \underline{y} \gg \underline{x} = \underline{y};$ FromContradiction $\rhd \underline{x} = \underline{y} \rhd \dot\neg\, \underline{x} = \underline{y} \gg \dot\neg\, -\underline{x} = -\underline{y};$ $\forall \underline{x}\colon \forall \underline{y}\colon$ Ded $\rhd \forall \underline{x}\colon \forall \underline{y}\colon \dot\neg\, \underline{x} = \underline{y} \vdash -\underline{x} = -\underline{y} \vdash \dot\neg\, -\underline{x} = -\underline{y} \gg \dot\neg\, \underline{x} = \underline{y} \Rightarrow -\underline{x} = -\underline{y} \Rightarrow \dot\neg\, -\underline{x} = -\underline{y};$ $\dot\neg\, \underline{x} = \underline{y} \vdash$ MP $\rhd \dot\neg\, \underline{x} = \underline{y} \Rightarrow -\underline{x} = -\underline{y} \Rightarrow \dot\neg\, -\underline{x} = -\underline{y} \rhd \dot\neg\, \underline{x} = \underline{y} \gg -\underline{x} = -\underline{y} \Rightarrow \dot\neg\, -\underline{x} = -\underline{y};$ ImplyNegation $\rhd -\underline{x} = -\underline{y} \Rightarrow \dot\neg\, -\underline{x} = -\underline{y} \gg \dot\neg\, -\underline{x} = -\underline{y}\rceil, \mathrm{p}_0, c)$]

[SubNeqRight $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} \vdash \dot\neg\, \underline{z} = \underline{x} \vdash \dot\neg\, \underline{z} = \underline{y}$]

[SubNeqRight $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} \vdash \dot\neg\, \underline{z} = \underline{x} \vdash$ NeqSymmetry $\rhd \dot\neg\, \underline{z} = \underline{x} \gg \dot\neg\, \underline{x} = \underline{z};$ SubNeqLeft $\rhd \underline{x} = \underline{y} \rhd \dot\neg\, \underline{x} = \underline{z} \gg \dot\neg\, \underline{y} = \underline{z};$ NeqSymmetry $\rhd \dot\neg\, \underline{y} = \underline{z} \gg \dot\neg\, \underline{z} = \underline{y}\rceil, \mathrm{p}_0, c)$]

[SubNeqLeft $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} \vdash \dot\neg\, \underline{x} = \underline{z} \vdash \dot\neg\, \underline{y} = \underline{z}$]

[SubNeqLeft $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} \vdash \dot\neg\, \underline{x} = \underline{z} \vdash$ equalityAxiom $\gg \underline{y} = \underline{x} \Rightarrow \underline{y} = \underline{z} \Rightarrow \underline{x} = \underline{z};$ eqSymmetry $\rhd \underline{x} = \underline{y} \gg \underline{y} = \underline{x};$ MP $\rhd$

$\underline{y} = \underline{x} \Rightarrow \underline{y} = \underline{z} \Rightarrow \underline{x} = \underline{z} \rhd \underline{y} = \underline{x} \gg \underline{y} = \underline{z} \Rightarrow \underline{x} = \underline{z}; \mathrm{Contrapositive} \rhd \underline{y} = \underline{z} \Rightarrow \underline{x} = \underline{z} \gg \dot{\neg}\,\underline{x} = \underline{z} \Rightarrow \dot{\neg}\,\underline{y} = \underline{z}; \mathrm{MP} \rhd \dot{\neg}\,\underline{x} = \underline{z} \Rightarrow \dot{\neg}\,\underline{y} = \underline{z} \rhd \dot{\neg}\,\underline{x} = \underline{z} \gg \dot{\neg}\,\underline{y} = \underline{z}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{NeqAddition} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{x} \colon \forall \underline{y} \colon \forall \underline{z} \colon \dot{\neg}\,\underline{x} = \underline{y} \vdash \dot{\neg}\,\underline{x} + \underline{z} = \underline{y} + \underline{z}]$

$[\mathrm{NeqAddition} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x} \colon \forall \underline{y} \colon \forall \underline{z} \colon \dot{\neg}\,\underline{x} = \underline{y} \vdash \underline{x} + \underline{z} = \underline{y} + \underline{z} \vdash \mathrm{eqReflexivity} \gg \underline{z} = \underline{z}; \mathrm{SubtractEquations} \rhd \underline{x} + \underline{z} = \underline{y} + \underline{z} \rhd \underline{z} = \underline{z} \gg \underline{x} = \underline{y}; \mathrm{FromContradiction} \rhd \underline{x} = \underline{y} \rhd \dot{\neg}\,\underline{x} = \underline{y} \gg \dot{\neg}\,\underline{x} + \underline{z} = \underline{y} + \underline{z}; \forall \underline{x} \colon \forall \underline{y} \colon \forall \underline{z} \colon \mathrm{Ded} \rhd \forall \underline{x} \colon \forall \underline{y} \colon \forall \underline{z} \colon \dot{\neg}\,\underline{x} = \underline{y} \vdash \underline{x} + \underline{z} = \underline{y} + \underline{z} \vdash \dot{\neg}\,\underline{x} + \underline{z} = \underline{y} + \underline{z} \gg \dot{\neg}\,\underline{x} = \underline{y} \Rightarrow \underline{x} + \underline{z} = \underline{y} + \underline{z} \Rightarrow \dot{\neg}\,\underline{x} + \underline{z} = \underline{y} + \underline{z}; \dot{\neg}\,\underline{x} = \underline{y} \vdash \mathrm{MP} \rhd \dot{\neg}\,\underline{x} = \underline{y} \Rightarrow \underline{x} + \underline{z} = \underline{y} + \underline{z} \Rightarrow \dot{\neg}\,\underline{x} + \underline{z} = \underline{y} + \underline{z} \rhd \dot{\neg}\,\underline{x} = \underline{y} \gg \underline{x} + \underline{z} = \underline{y} + \underline{z} \Rightarrow \dot{\neg}\,\underline{x} + \underline{z} = \underline{y} + \underline{z}; \mathrm{ImplyNegation} \rhd \underline{x} + \underline{z} = \underline{y} + \underline{z} \Rightarrow \dot{\neg}\,\underline{x} + \underline{z} = \underline{y} + \underline{z} \gg \dot{\neg}\,\underline{x} + \underline{z} = \underline{y} + \underline{z}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{NeqMultiplication} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{x} \colon \forall \underline{y} \colon \forall \underline{z} \colon \dot{\neg}\,\underline{z} = 0 \vdash \dot{\neg}\,\underline{x} = \underline{y} \vdash \dot{\neg}\,\underline{x} * \underline{z} = \underline{y} * \underline{z}]$

$[\mathrm{NeqMultiplication} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x} \colon \forall \underline{y} \colon \forall \underline{z} \colon \dot{\neg}\,\underline{z} = 0 \vdash \dot{\neg}\,\underline{x} = \underline{y} \vdash \underline{x} * \underline{z} = \underline{y} * \underline{z} \vdash \rhd \dot{\neg}\,\underline{z} = 0 \gg \underline{x} = \underline{x} * \underline{z} * \mathrm{rec}\underline{z}; \mathrm{eqMultiplication} \rhd \underline{x} * \underline{z} = \underline{y} * \underline{z} \gg \underline{x} * \underline{z} * \mathrm{rec}\underline{z} = \underline{y} * \underline{z} * \mathrm{rec}\underline{z}; \rhd \dot{\neg}\,\underline{z} = 0 \gg \underline{y} = \underline{y} * \underline{z} * \mathrm{rec}\underline{z}; \mathrm{eqSymmetry} \rhd \underline{y} = \underline{y} * \underline{z} * \mathrm{rec}\underline{z} \gg \underline{y} * \underline{z} * \mathrm{rec}\underline{z} = \underline{y}; \mathrm{eqTransitivity4} \rhd \underline{x} = \underline{x} * \underline{z} * \mathrm{rec}\underline{z} \rhd \underline{x} * \underline{z} * \mathrm{rec}\underline{z} = \underline{y} * \underline{z} * \mathrm{rec}\underline{z} \rhd \underline{y} * \underline{z} * \mathrm{rec}\underline{z} = \underline{y} \gg \underline{x} = \underline{y}; \mathrm{FromContradiction} \rhd \underline{x} = \underline{y} \rhd \dot{\neg}\,\underline{x} = \underline{y} \gg \dot{\neg}\,\underline{x} * \underline{z} = \underline{y} * \underline{z}; \forall \underline{x} \colon \forall \underline{y} \colon \forall \underline{z} \colon \mathrm{Ded} \rhd \forall \underline{x} \colon \forall \underline{y} \colon \forall \underline{z} \colon \dot{\neg}\,\underline{z} = 0 \vdash \dot{\neg}\,\underline{x} = \underline{y} \vdash \underline{x} * \underline{z} = \underline{y} * \underline{z} \vdash \dot{\neg}\,\underline{x} * \underline{z} = \underline{y} * \underline{z} \gg \dot{\neg}\,\underline{z} = 0 \Rightarrow \dot{\neg}\,\underline{x} = \underline{y} \Rightarrow \underline{x} * \underline{z} = \underline{y} * \underline{z} \Rightarrow \dot{\neg}\,\underline{x} * \underline{z} = \underline{y} * \underline{z}; \dot{\neg}\,\underline{z} = 0 \vdash \dot{\neg}\,\underline{x} = \underline{y} \vdash \mathrm{MP2} \rhd \dot{\neg}\,\underline{z} = 0 \Rightarrow \dot{\neg}\,\underline{x} = \underline{y} \Rightarrow \underline{x} * \underline{z} = \underline{y} * \underline{z} \Rightarrow \dot{\neg}\,\underline{x} * \underline{z} = \underline{y} * \underline{z} \rhd \dot{\neg}\,\underline{z} = 0 \rhd \dot{\neg}\,\underline{x} = \underline{y} \gg \underline{x} * \underline{z} = \underline{y} * \underline{z} \Rightarrow \dot{\neg}\,\underline{x} * \underline{z} = \underline{y} * \underline{z}; \mathrm{ImplyNegation} \rhd \underline{x} * \underline{z} = \underline{y} * \underline{z} \Rightarrow \dot{\neg}\,\underline{x} * \underline{z} = \underline{y} * \underline{z} \gg \dot{\neg}\,\underline{x} * \underline{z} = \underline{y} * \underline{z}\rceil, \mathrm{p}_0, \mathsf{c})]$
(*** NEGATIVE ***)

$[\mathrm{UniqueNegative} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{x} \colon \forall \underline{y} \colon \forall \underline{z} \colon \underline{x} + \underline{y} = 0 \vdash \underline{x} + \underline{z} = 0 \vdash \underline{y} = \underline{z}]$

$[\mathrm{UniqueNegative} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x} \colon \forall \underline{y} \colon \forall \underline{z} \colon \underline{x} + \underline{y} = 0 \vdash \underline{x} + \underline{z} = 0 \vdash \mathrm{plusCommutativity} \gg \underline{y} + \underline{x} = \underline{x} + \underline{y}; \mathrm{eqTransitivity} \rhd \underline{y} + \underline{x} = \underline{x} + \underline{y} \rhd \underline{x} + \underline{y} = 0 \gg \underline{y} + \underline{x} = 0; \mathrm{PositiveToRight(Eq)} \rhd \underline{y} + \underline{x} = 0 \gg \underline{y} = 0 + -\underline{x}; \mathrm{plusCommutativity} \gg \underline{z} + \underline{x} = \underline{x} + \underline{z}; \mathrm{eqTransitivity} \rhd \underline{z} + \underline{x} = \underline{x} + \underline{z} \rhd \underline{x} + \underline{z} = 0 \gg \underline{z} + \underline{x} = 0; \mathrm{PositiveToRight(Eq)} \rhd \underline{z} + \underline{x} = 0 \gg \underline{z} = 0 + -\underline{x}; \mathrm{eqSymmetry} \rhd \underline{z} = 0 + -\underline{x} \gg 0 + -\underline{x} = \underline{z}; \mathrm{eqTransitivity} \rhd \underline{y} = 0 + -\underline{x} \rhd 0 + -\underline{x} = \underline{z} \gg \underline{y} = \underline{z}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{DoubleMinus} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{x} \colon - - \underline{x} = \underline{x}]$

$[\mathrm{DoubleMinus} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x} \colon \mathrm{Negative} \gg -\underline{x} + - - \underline{x} = 0; \mathrm{x} + \mathrm{y} = \mathrm{zBackwards} \rhd -\underline{x} + - - \underline{x} = 0 \gg 0 = - - \underline{x} + -\underline{x}; \mathrm{NegativeToLeft(Eq)} \rhd 0 = - - \underline{x} + -\underline{x} \gg 0 + \underline{x} = - - \underline{x}; \mathrm{plus0Left} \gg 0 + \underline{x} = \underline{x}; \mathrm{Equality} \rhd 0 + \underline{x} = - - \underline{x} \rhd 0 + \underline{x} = \underline{x} \gg - - \underline{x} = \underline{x}\rceil, \mathrm{p}_0, \mathsf{c})]$
(*** LEQ, nummer 1 af 2 ***)

$[\mathrm{LeqLessEq} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{x} \colon \forall \underline{y} \colon \underline{x} <= \underline{y} \vdash \dot{\neg}\,\dot{\neg}\,\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\,\dot{\neg}\,\underline{x} = \underline{y} \Rightarrow \underline{x} = \underline{y}]$

$[\mathrm{LeqLessEq} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x} \colon \forall \underline{y} \colon \underline{x} <= \underline{y} \vdash \dot{\neg}\,\dot{\neg}\,\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\,\dot{\neg}\,\underline{x} = \underline{y} \vdash \mathrm{fromNotLess} \rhd \dot{\neg}\,\dot{\neg}\,\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\,\dot{\neg}\,\underline{x} = \underline{y} \gg \underline{y} <= \underline{x}; \mathrm{leqAntisymmetry} \rhd \underline{x} <= \underline{y} \rhd \underline{y} <= \underline{x} \gg \underline{x} = \underline{y}; \forall \underline{x} \colon \forall \underline{y} \colon \mathrm{Ded} \rhd \forall \underline{x} \colon \forall \underline{y} \colon \underline{x} <= \underline{y} \vdash \dot{\neg}\,\dot{\neg}\,\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\,\dot{\neg}\,\underline{x} = \underline{y} \vdash \underline{x} = \underline{y} \gg \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\,\dot{\neg}\,\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\,\dot{\neg}\,\underline{x} = \underline{y} \Rightarrow \underline{x} = \underline{y}; \underline{x} <= \underline{y} \vdash \mathrm{MP} \rhd \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\,\dot{\neg}\,\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\,\dot{\neg}\,\underline{x} = \underline{y} \Rightarrow \underline{x} = \underline{y} \rhd \underline{x} <= \underline{y} \gg \dot{\neg}\,\dot{\neg}\,\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\,\dot{\neg}\,\underline{x} = \underline{y} \Rightarrow \underline{x} = \underline{y}; \mathrm{Repetition} \rhd \dot{\neg}\,\dot{\neg}\,\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\,\dot{\neg}\,\underline{x} = \underline{y} \Rightarrow \underline{x} = \underline{y} \gg \dot{\neg}\,\dot{\neg}\,\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\,\dot{\neg}\,\underline{x} = \underline{y} \Rightarrow \underline{x} = \underline{y}\rceil, \mathrm{p}_0, \mathsf{c})]$

[LessLeq $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \dot\neg \underline{x} <= \underline{y} \Rightarrow \dot\neg\dot\neg \underline{x} = \underline{y} \vdash \underline{x} <= \underline{y}$]

[LessLeq $\overset{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \dot\neg \underline{x} <= \underline{y} \Rightarrow \dot\neg\dot\neg \underline{x} = \underline{y} \vdash$ Repetition $\rhd \dot\neg \underline{x} <= \underline{y} \Rightarrow \dot\neg\dot\neg \underline{x} = \underline{y} \gg \dot\neg \underline{x} <= \underline{y} \Rightarrow \dot\neg\dot\neg \underline{x} = \underline{y}$; FirstConjunct $\rhd \dot\neg \underline{x} <= \underline{y} \Rightarrow \dot\neg\dot\neg \underline{x} = \underline{y} \gg \underline{x} <= \underline{y}\rceil, p_0, c)$]

[FromLeqGeq $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{a}\colon \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} <= \underline{y} \Rightarrow \underline{a} \vdash \underline{y} <= \underline{x} \Rightarrow \underline{a} \vdash \underline{a}$]

[FromLeqGeq $\overset{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{a}\colon \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} <= \underline{y} \Rightarrow \underline{a} \vdash \underline{y} <= \underline{x} \Rightarrow \underline{a} \vdash$ leqTotality $\gg \dot\neg \underline{x} <= \underline{y} \Rightarrow \underline{y} <= \underline{x}$; FromDisjuncts $\rhd \dot\neg \underline{x} <= \underline{y} \Rightarrow \underline{y} <= \underline{x} \rhd \underline{x} <= \underline{y} \Rightarrow \underline{a} \rhd \underline{y} <= \underline{x} \Rightarrow \underline{a} \gg \underline{a}\rceil, p_0, c)$]

[subLeqRight $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} \vdash \underline{z} <= \underline{x} \vdash \underline{z} <= \underline{y}$]

[subLeqRight $\overset{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} \vdash \underline{z} <= \underline{x} \vdash$ eqLeq $\rhd \underline{x} = \underline{y} \gg \underline{x} <= \underline{y}$; leqTransitivity $\rhd \underline{z} <= \underline{x} \rhd \underline{x} <= \underline{y} \gg \underline{z} <= \underline{y}\rceil, p_0, c)$]

[subLeqLeft $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} \vdash \underline{x} <= \underline{z} \vdash \underline{y} <= \underline{z}$]

[subLeqLeft $\overset{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} \vdash \underline{x} <= \underline{z} \vdash$ eqSymmetry $\rhd \underline{x} = \underline{y} \gg \underline{y} = \underline{x}$; eqLeq $\rhd \underline{y} = \underline{x} \gg \underline{y} <= \underline{x}$; leqTransitivity $\rhd \underline{y} <= \underline{x} \rhd \underline{x} <= \underline{z} \gg \underline{y} <= \underline{z}\rceil, p_0, c)$]

[Leq + 1 $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} <= \underline{y} \vdash \dot\neg \underline{x} <= \underline{y} + 1 \Rightarrow \dot\neg\dot\neg \underline{x} = \underline{y} + 1$]

[Leq + 1 $\overset{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} <= \underline{y} \vdash 0 < 1 \gg \dot\neg 0 <= 1 \Rightarrow \dot\neg\dot\neg 0 = 1$; LessAdditionLeft $\rhd \dot\neg 0 <= 1 \Rightarrow \dot\neg\dot\neg 0 = 1 \gg \dot\neg \underline{y} + 0 <= \underline{y} + 1 \Rightarrow \dot\neg\dot\neg \underline{y} + 0 = \underline{y} + 1$; plus0 $\gg \underline{y} + 0 = \underline{y}$; SubLessLeft $\rhd \underline{y} + 0 = \underline{y} \rhd \dot\neg \underline{y} + 0 <= \underline{y} + 1 \Rightarrow \dot\neg\dot\neg \underline{y} + 0 = \underline{y} + 1 \gg \dot\neg \underline{y} <= \underline{y} + 1 \Rightarrow \dot\neg\dot\neg \underline{y} = \underline{y} + 1$; leqLessTransitivity $\rhd \underline{x} <= \underline{y} \rhd \dot\neg \underline{y} <= \underline{y} + 1 \Rightarrow \dot\neg\dot\neg \underline{y} = \underline{y} + 1 \gg \dot\neg \underline{x} <= \underline{y} + 1 \Rightarrow \dot\neg\dot\neg \underline{x} = \underline{y} + 1\rceil, p_0, c)$]

[PositiveToRight(Leq) $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} + \underline{y} <= \underline{z} \vdash \underline{x} <= \underline{z} + -\underline{y}$]

[PositiveToRight(Leq) $\overset{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} + \underline{y} <= \underline{z} \vdash$ leqAddition $\rhd \underline{x} + \underline{y} <= \underline{z} \gg \underline{x} + \underline{y} + -\underline{y} <= \underline{z} + -\underline{y}$; $\mathrm{x} = \mathrm{x} + \mathrm{y} - \mathrm{y} \gg \underline{x} = \underline{x} + \underline{y} + -\underline{y}$; eqSymmetry $\rhd \underline{x} = \underline{x} + \underline{y} + -\underline{y} \gg \underline{x} + \underline{y} + -\underline{y} = \underline{x}$; subLeqLeft $\rhd \underline{x} + \underline{y} + -\underline{y} = \underline{x} \rhd \underline{x} + \underline{y} + -\underline{y} <= \underline{z} + -\underline{y} \gg \underline{x} <= \underline{z} + -\underline{y}\rceil, p_0, c)$]

[LeqAdditionLeft $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} <= \underline{y} \vdash \underline{z} + \underline{x} <= \underline{z} + \underline{y}$]

[LeqAdditionLeft $\overset{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} <= \underline{y} \vdash$ leqAddition $\rhd \underline{x} <= \underline{y} \gg \underline{x} + \underline{z} <= \underline{y} + \underline{z}$; plusCommutativity $\gg \underline{x} + \underline{z} = \underline{z} + \underline{x}$; plusCommutativity $\gg \underline{y} + \underline{z} = \underline{z} + \underline{y}$; subLeqLeft $\rhd \underline{x} + \underline{z} = \underline{z} + \underline{x} \rhd \underline{x} + \underline{z} <= \underline{y} + \underline{z} \gg \underline{z} + \underline{x} <= \underline{y} + \underline{z}$; subLeqRight $\rhd \underline{y} + \underline{z} = \underline{z} + \underline{y} \rhd \underline{z} + \underline{x} <= \underline{y} + \underline{z} \gg \underline{z} + \underline{x} <= \underline{z} + \underline{y}\rceil, p_0, c)$]

[leqSubtraction $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} + \underline{z} <= \underline{y} + \underline{z} \vdash \underline{x} <= \underline{y}$]

[leqSubtraction $\overset{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} + \underline{z} <= \underline{y} + \underline{z} \vdash$ leqAddition $\rhd \underline{x} + \underline{z} <= \underline{y} + \underline{z} \gg \underline{x} + \underline{z} + -\underline{z} <= \underline{y} + \underline{z} + -\underline{z}$; $\mathrm{x} = \mathrm{x} + \mathrm{y} - \mathrm{y} \gg \underline{x} = \underline{x} + \underline{z} + -\underline{z}$; eqSymmetry $\rhd \underline{x} = \underline{x} + \underline{z} + -\underline{z} \gg \underline{x} + \underline{z} + -\underline{z} = \underline{x}$; $\mathrm{x} = \mathrm{x} + \mathrm{y} - \mathrm{y} \gg \underline{y} = \underline{y} + \underline{z} + -\underline{z}$; eqSymmetry $\rhd \underline{y} = \underline{y} + \underline{z} + -\underline{z} \gg \underline{y} + \underline{z} + -\underline{z} = \underline{y}$; subLeqLeft $\rhd \underline{x} + \underline{z} + -\underline{z} = \underline{x} \rhd \underline{x} + \underline{z} + -\underline{z} <= \underline{y} + \underline{z} + -\underline{z} \gg \underline{x} <= \underline{y} + \underline{z} + -\underline{z}$; subLeqRight $\rhd \underline{y} + \underline{z} + -\underline{z} = \underline{y} \rhd \underline{x} <= \underline{y} + \underline{z} + -\underline{z} \gg \underline{x} <= \underline{y}\rceil, p_0, c)$]

[leqSubtractionLeft $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{z} + \underline{x} <= \underline{z} + \underline{y} \vdash \underline{x} <= \underline{y}$]

$[\text{leqSubtractionLeft} \stackrel{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{z} + \underline{x} <= \underline{z} + \underline{y} \vdash \text{plusCommutativity} \gg \underline{z} + \underline{x} = \underline{x} + \underline{z}; \text{plusCommutativity} \gg \underline{z} + \underline{y} = \underline{y} + \underline{z}; \text{subLeqLeft} \rhd \underline{z} + \underline{x} = \underline{x} + \underline{z} \rhd \underline{z} + \underline{x} <= \underline{z} + \underline{y} \gg \underline{x} + \underline{z} <= \underline{z} + \underline{y}; \text{subLeqRight} \rhd \underline{z} + \underline{y} = \underline{y} + \underline{z} \rhd \underline{x} + \underline{z} <= \underline{z} + \underline{y} \gg \underline{x} + \underline{z} <= \underline{y} + \underline{z}; \text{leqSubtraction} \rhd \underline{x} + \underline{z} <= \underline{y} + \underline{z} \gg \underline{x} <= \underline{y}\rceil, p_0, c)]$

$[\text{thirdGeq} \stackrel{\text{stmt}}{\to} \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \dot{\neg}\, \underline{x} <= c_{\text{Ex}} \Rightarrow \dot{\neg}\, \underline{y} <= c_{\text{Ex}}]$

$[\text{thirdGeq} \stackrel{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} <= \underline{y} \vdash \text{leqReflexivity} \gg \underline{y} <= \underline{y}; \text{JoinConjuncts} \rhd \underline{x} <= \underline{y} \rhd \underline{y} <= \underline{y} \gg \dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\, \underline{y} <= \underline{y}; \text{ExistIntro} @ c_{\text{Ex}} @ \underline{y} \rhd \dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\, \underline{y} <= \underline{y} \gg \dot{\neg}\, \underline{x} <= c_{\text{Ex}} \Rightarrow \dot{\neg}\, \underline{y} <= c_{\text{Ex}}; \forall \underline{x}\colon \forall \underline{y}\colon \underline{y} <= \underline{x} \vdash \text{leqReflexivity} \gg \underline{x} <= \underline{x}; \text{JoinConjuncts} \rhd \underline{x} <= \underline{x} \rhd \underline{y} <= \underline{x} \gg \dot{\neg}\, \underline{x} <= \underline{x} \Rightarrow \dot{\neg}\, \underline{y} <= \underline{x}; \text{ExistIntro} @ c_{\text{Ex}} @ \underline{x} \rhd \dot{\neg}\, \underline{x} <= \underline{x} \Rightarrow \dot{\neg}\, \underline{y} <= \underline{x} \gg \dot{\neg}\, \underline{x} <= c_{\text{Ex}} \Rightarrow \dot{\neg}\, \underline{y} <= c_{\text{Ex}}; \forall \underline{x}\colon \forall \underline{y}\colon \text{Ded} \rhd \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} <= \underline{y} \vdash \dot{\neg}\, \underline{x} <= c_{\text{Ex}} \Rightarrow \dot{\neg}\, \underline{y} <= c_{\text{Ex}} \gg \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\, \underline{x} <= c_{\text{Ex}} \Rightarrow \dot{\neg}\, \underline{y} <= c_{\text{Ex}}; \text{Ded} \rhd \forall \underline{x}\colon \forall \underline{y}\colon \underline{y} <= \underline{x} \vdash \dot{\neg}\, \underline{x} <= c_{\text{Ex}} \Rightarrow \dot{\neg}\, \underline{y} <= c_{\text{Ex}} \gg \underline{y} <= \underline{x} \Rightarrow \dot{\neg}\, \underline{x} <= c_{\text{Ex}} \Rightarrow \dot{\neg}\, \underline{y} <= c_{\text{Ex}}; \text{leqTotality} \gg \dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \underline{y} <= \underline{x}; \text{FromDisjuncts} \rhd \dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \underline{y} <= \underline{x} \rhd \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\, \underline{x} <= c_{\text{Ex}} \Rightarrow \dot{\neg}\, \underline{y} <= c_{\text{Ex}} \rhd \underline{y} <= \underline{x} \Rightarrow \dot{\neg}\, \underline{x} <= c_{\text{Ex}} \Rightarrow \dot{\neg}\, \underline{y} <= c_{\text{Ex}} \gg \dot{\neg}\, \underline{x} <= c_{\text{Ex}} \Rightarrow \dot{\neg}\, \underline{y} <= c_{\text{Ex}}\rceil, p_0, c)]$

$[\text{LeqNegated} \stackrel{\text{stmt}}{\to} \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} <= \underline{y} \vdash -\underline{y} <= -\underline{x}]$

$[\text{LeqNegated} \stackrel{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} <= \underline{y} \vdash \text{leqAddition} \rhd \underline{x} <= \underline{y} \gg \underline{x} + -\underline{x} <= \underline{y} + -\underline{x}; \text{Negative} \gg \underline{x} + -\underline{x} = 0; \text{subLeqLeft} \rhd \underline{x} + -\underline{x} = 0 \rhd \underline{x} + -\underline{x} <= \underline{y} + -\underline{x} \gg 0 <= \underline{y} + -\underline{x}; \text{plusCommutativity} \gg \underline{y} + -\underline{x} = -\underline{x} + \underline{y}; \text{subLeqRight} \rhd \underline{y} + -\underline{x} = -\underline{x} + \underline{y} \rhd 0 <= \underline{y} + -\underline{x} \gg 0 <= -\underline{x} + \underline{y}; \text{leqAddition} \rhd 0 <= -\underline{x} + \underline{y} \gg 0 + -\underline{y} <= -\underline{x} + \underline{y} + -\underline{y}; \text{plus0Left} \gg 0 + -\underline{y} = -\underline{y}; \text{x = x + y − y} \gg -\underline{x} = -\underline{x} + \underline{y} + -\underline{y}; \text{eqSymmetry} \rhd -\underline{x} = -\underline{x} + \underline{y} + -\underline{y} \gg -\underline{x} + \underline{y} + -\underline{y} = -\underline{x}; \text{subLeqLeft} \rhd 0 + -\underline{y} = -\underline{y} \rhd 0 + -\underline{y} <= -\underline{x} + \underline{y} + -\underline{y} \gg -\underline{y} <= -\underline{x} + \underline{y} + -\underline{y}; \text{subLeqRight} \rhd -\underline{x} + \underline{y} + -\underline{y} = -\underline{x} \rhd -\underline{y} <= -\underline{x} + \underline{y} + -\underline{y} \gg -\underline{y} <= -\underline{x}\rceil, p_0, c)]$

$[\text{AddEquations(Leq)} \stackrel{\text{stmt}}{\to} \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \forall \underline{u}\colon \underline{x} <= \underline{y} \vdash \underline{z} <= \underline{u} \vdash \underline{x} + \underline{z} <= \underline{y} + \underline{u}]$

$[\text{AddEquations(Leq)} \stackrel{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \forall \underline{u}\colon \underline{x} <= \underline{y} \vdash \underline{z} <= \underline{u} \vdash \text{leqAddition} \rhd \underline{x} <= \underline{y} \gg \underline{x} + \underline{z} <= \underline{y} + \underline{z}; \text{LeqAdditionLeft} \rhd \underline{z} <= \underline{u} \gg \underline{y} + \underline{z} <= \underline{y} + \underline{u}; \text{leqTransitivity} \rhd \underline{x} + \underline{z} <= \underline{y} + \underline{z} \rhd \underline{y} + \underline{z} <= \underline{y} + \underline{u} \gg \underline{x} + \underline{z} <= \underline{y} + \underline{u}\rceil, p_0, c)]$
(*** LESS ***)

$[\text{LeqNeqLess} \stackrel{\text{stmt}}{\to} \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} <= \underline{y} \vdash \dot{\neg}\, \underline{x} = \underline{y} \vdash \dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\, \dot{\neg}\, \underline{x} = \underline{y}]$

$[\text{LeqNeqLess} \stackrel{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} <= \underline{y} \vdash \dot{\neg}\, \underline{x} = \underline{y} \vdash \text{JoinConjuncts} \rhd \underline{x} <= \underline{y} \rhd \dot{\neg}\, \underline{x} = \underline{y} \gg \dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\, \dot{\neg}\, \underline{x} = \underline{y}; \text{Repetition} \rhd \dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\, \dot{\neg}\, \underline{x} = \underline{y} \gg \dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\, \dot{\neg}\, \underline{x} = \underline{y}\rceil, p_0, c)]$

$[\text{FromLess} \stackrel{\text{stmt}}{\to} \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\, \dot{\neg}\, \underline{x} = \underline{y} \vdash \dot{\neg}\, \underline{y} <= \underline{x}]$

$[\text{FromLess} \stackrel{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \underline{y} <= \underline{x} \vdash \text{toNotLess} \rhd \underline{y} <= \underline{x} \gg \dot{\neg}\, \dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\, \dot{\neg}\, \underline{x} = \underline{y}; \forall \underline{x}\colon \forall \underline{y}\colon \text{Ded} \rhd \forall \underline{x}\colon \forall \underline{y}\colon \underline{y} <= \underline{x} \vdash \dot{\neg}\, \dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\, \dot{\neg}\, \underline{x} = \underline{y} \gg \underline{y} <= \underline{x} \Rightarrow \dot{\neg}\, \dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\, \dot{\neg}\, \underline{x} = \underline{y}; \dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\, \dot{\neg}\, \underline{x} = \underline{y} \vdash \text{AddDoubleNeg} \rhd \dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\, \dot{\neg}\, \underline{x} = \underline{y} \gg \dot{\neg}\, \dot{\neg}\, \dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\, \dot{\neg}\, \underline{x} =$

$\underline{y}; \mathrm{MT} \rhd \underline{y} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \rhd \dot{\neg}\dot{\neg}\dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \gg \dot{\neg}\underline{y} <= \underline{x}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{ToLess} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \dot{\neg}\underline{x} <= \underline{y} \vdash \dot{\neg}\underline{y} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{x}]$

$[\mathrm{ToLess} \overset{\mathrm{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil\mathrm{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \dot{\neg}\dot{\neg}\underline{y} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{x} \vdash \mathrm{fromNotLess} \rhd \dot{\neg}\dot{\neg}\underline{y} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{x} \gg \underline{x} <= \underline{y}; \forall\underline{x}\colon \forall\underline{y}\colon \mathrm{Ded} \rhd \forall\underline{x}\colon \forall\underline{y}\colon \dot{\neg}\dot{\neg}\underline{y} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{x} \vdash \underline{x} <= \underline{y} \gg \dot{\neg}\dot{\neg}\underline{y} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{x} \Rightarrow \underline{x} <= \underline{y}; \dot{\neg}\underline{x} <= \underline{y} \vdash \mathrm{NegativeMT} \rhd \dot{\neg}\dot{\neg}\underline{y} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{x} \Rightarrow \underline{x} <= \underline{y} \rhd \dot{\neg}\underline{x} <= \underline{y} \gg \dot{\neg}\underline{y} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{x}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{fromNotLess} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \dot{\neg}\dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \vdash \underline{y} <= \underline{x}]$

$[\mathrm{fromNotLess} \overset{\mathrm{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil\mathrm{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \dot{\neg}\dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \vdash \underline{x} <= \underline{y} \vdash \mathrm{Repetition} \rhd \dot{\neg}\dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \gg \dot{\neg}\dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y}; \mathrm{RemoveDoubleNeg} \rhd \dot{\neg}\dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \gg \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y}; \mathrm{MP} \rhd \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \rhd \underline{x} <= \underline{y} \gg \dot{\neg}\dot{\neg}\underline{x} = \underline{y}; \mathrm{RemoveDoubleNeg} \rhd \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \gg \underline{x} = \underline{y}; \mathrm{eqSymmetry} \rhd \underline{x} = \underline{y} \gg \underline{y} = \underline{x}; \mathrm{eqLeq} \rhd \underline{y} = \underline{x} \gg \underline{y} <= \underline{x}; \forall\underline{x}\colon \forall\underline{y}\colon \mathrm{Ded} \rhd \forall\underline{x}\colon \forall\underline{y}\colon \dot{\neg}\dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \vdash \underline{x} <= \underline{y} \vdash \underline{y} <= \underline{x} \gg \dot{\neg}\dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \Rightarrow \underline{x} <= \underline{y} \Rightarrow \underline{y} <= \underline{x}; \dot{\neg}\dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \vdash \mathrm{MP} \rhd \dot{\neg}\dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \Rightarrow \underline{x} <= \underline{y} \Rightarrow \underline{y} <= \underline{x} \rhd \dot{\neg}\dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \gg \underline{x} <= \underline{y} \Rightarrow \underline{y} <= \underline{x}; \mathrm{AutoImply} \gg \underline{y} <= \underline{x} \Rightarrow \underline{y} <= \underline{x}; \mathrm{leqTotality} \gg \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \underline{y} <= \underline{x}; \mathrm{FromDisjuncts} \rhd \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \underline{y} <= \underline{x} \rhd \underline{x} <= \underline{y} \Rightarrow \underline{y} <= \underline{x} \rhd \underline{y} <= \underline{x} \Rightarrow \underline{y} <= \underline{x} \gg \underline{y} <= \underline{x}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{toNotLess} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \underline{x} <= \underline{y} \vdash \dot{\neg}\dot{\neg}\underline{y} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{x}]$

$[\mathrm{toNotLess} \overset{\mathrm{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil\mathrm{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \underline{x} <= \underline{y} \vdash \underline{y} <= \underline{x} \vdash \mathrm{leqAntisymmetry} \rhd \underline{y} <= \underline{x} \rhd \underline{x} <= \underline{y} \gg \underline{y} = \underline{x}; \mathrm{AddDoubleNeg} \rhd \underline{y} = \underline{x} \gg \dot{\neg}\dot{\neg}\underline{y} = \underline{x}; \forall\underline{x}\colon \forall\underline{y}\colon \mathrm{Ded} \rhd \forall\underline{x}\colon \forall\underline{y}\colon \underline{x} <= \underline{y} \vdash \underline{y} <= \underline{x} \vdash \dot{\neg}\dot{\neg}\underline{y} = \underline{x} \gg \underline{x} <= \underline{y} \Rightarrow \underline{y} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{x}; \underline{x} <= \underline{y} \vdash \mathrm{MP} \rhd \underline{x} <= \underline{y} \Rightarrow \underline{y} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{x} \rhd \underline{x} <= \underline{y} \gg \underline{y} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{x}; \mathrm{AddDoubleNeg} \rhd \underline{y} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{x} \gg \dot{\neg}\dot{\neg}\underline{y} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{x}; \mathrm{Repetition} \rhd \dot{\neg}\dot{\neg}\underline{y} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{x} \gg \dot{\neg}\dot{\neg}\underline{y} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{x}; \mathrm{Repetition} \rhd \dot{\neg}\dot{\neg}\underline{y} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{x} \gg \dot{\neg}\dot{\neg}\underline{y} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{x}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{LessAddition} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \vdash \dot{\neg}\underline{x} + \underline{z} <= \underline{y} + \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} + \underline{z} = \underline{y} + \underline{z}]$

$[\mathrm{LessAddition} \overset{\mathrm{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil\mathrm{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \vdash \mathrm{LessLeq} \rhd \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \gg \underline{x} <= \underline{y}; \mathrm{leqAddition} \rhd \underline{x} <= \underline{y} \gg \underline{x} + \underline{z} <= \underline{y} + \underline{z}; \mathrm{LessNeq} \rhd \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \gg \dot{\neg}\underline{x} = \underline{y}; \mathrm{NeqAddition} \rhd \dot{\neg}\underline{x} = \underline{y} \gg \dot{\neg}\underline{x} + \underline{z} = \underline{y} + \underline{z}; \mathrm{JoinConjuncts} \rhd \underline{x} + \underline{z} <= \underline{y} + \underline{z} \rhd \dot{\neg}\underline{x} + \underline{z} = \underline{y} + \underline{z} \gg \dot{\neg}\underline{x} + \underline{z} <= \underline{y} + \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} + \underline{z} = \underline{y} + \underline{z}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{LessAdditionLeft} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \vdash \dot{\neg}\underline{z} + \underline{x} <= \underline{z} + \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{z} + \underline{x} = \underline{z} + \underline{y}]$

$[\mathrm{LessAdditionLeft} \overset{\mathrm{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil\mathrm{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \vdash \mathrm{LessAddition} \rhd \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \gg \dot{\neg}\underline{x} + \underline{z} <= \underline{y} + \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} + \underline{z} = \underline{y} + \underline{z}; \mathrm{plusCommutativity} \gg \underline{x} + \underline{z} = \underline{z} + \underline{x}; \mathrm{SubLessLeft} \rhd \underline{x} + \underline{z} = \underline{z} + \underline{x} \rhd \dot{\neg}\underline{x} + \underline{z} <= \underline{y} + \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} + \underline{z} = \underline{y} + \underline{z} \gg \dot{\neg}\underline{z} + \underline{x} <= \underline{y} + \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{z} + \underline{x} = \underline{y} + \underline{z}; \mathrm{plusCommutativity} \gg \underline{y} + \underline{z} = \underline{z} + \underline{y}; \mathrm{SubLessRight} \rhd \underline{y} + \underline{z} = \underline{z} + \underline{y} \rhd \dot{\neg}\underline{z} + \underline{x} <=$

$\underline{y} + \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{z} + \underline{x} = \underline{y} + \underline{z} \gg \dot{\neg}\underline{z} + \underline{x} <= \underline{z} + \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{z} + \underline{x} = \underline{z} + \underline{y}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{LessMultiplication} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \dot{\neg}0 <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg}0 = \underline{z} \vdash \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \vdash \dot{\neg}\underline{x} * \underline{z} <= \underline{y} * \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} * \underline{z} = \underline{y} * \underline{z}]$

$[\mathrm{LessMultiplication} \stackrel{\mathrm{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil\mathrm{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \dot{\neg}0 <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg}0 = \underline{z} \vdash \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \vdash \mathrm{LessLeq} \rhd \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \gg \underline{x} <= \underline{y}; \mathrm{LessLeq} \rhd \dot{\neg}0 <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg}0 = \underline{z} \gg 0 <= \underline{z}; \mathrm{leqMultiplication} \rhd 0 <= \underline{z} \rhd \underline{x} <= \underline{y} \gg \underline{x} * \underline{z} <= \underline{y} * \underline{z}; \mathrm{LessNeq} \rhd \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \gg \dot{\neg}\underline{x} = \underline{y}; \mathrm{LessNeq} \rhd \dot{\neg}0 <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg}0 = \underline{z} \gg \dot{\neg}0 = \underline{z}; \mathrm{NeqSymmetry} \rhd \dot{\neg}0 = \underline{z} \gg \dot{\neg}\underline{z} = 0; \mathrm{NeqMultiplication} \rhd \dot{\neg}\underline{z} = 0 \rhd \dot{\neg}\underline{x} = \underline{y} \gg \dot{\neg}\underline{x} * \underline{z} = \underline{y} * \underline{z}; \mathrm{LeqNeqLess} \rhd \underline{x} * \underline{z} <= \underline{y} * \underline{z} \rhd \dot{\neg}\underline{x} * \underline{z} = \underline{y} * \underline{z} \gg \dot{\neg}\underline{x} * \underline{z} <= \underline{y} * \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} * \underline{z} = \underline{y} * \underline{z}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{LessMultiplicationLeft} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \dot{\neg}0 <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg}0 = \underline{z} \vdash \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \vdash \dot{\neg}\underline{z} * \underline{x} <= \underline{z} * \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{z} * \underline{x} = \underline{z} * \underline{y}]$

$[\mathrm{LessMultiplicationLeft} \stackrel{\mathrm{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil\mathrm{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \dot{\neg}0 <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg}0 = \underline{z} \vdash \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \vdash \mathrm{LessMultiplication} \rhd \dot{\neg}0 <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg}0 = \underline{z} \rhd \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \gg \dot{\neg}\underline{x} * \underline{z} <= \underline{y} * \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} * \underline{z} = \underline{y} * \underline{z}; \mathrm{timesCommutativity} \gg \underline{x} * \underline{z} = \underline{z} * \underline{x}; \mathrm{timesCommutativity} \gg \underline{y} * \underline{z} = \underline{z} * \underline{y}; \mathrm{SubLessLeft} \rhd \underline{x} * \underline{z} = \underline{z} * \underline{x} \rhd \dot{\neg}\underline{x} * \underline{z} <= \underline{y} * \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} * \underline{z} = \underline{y} * \underline{z} \gg \dot{\neg}\underline{z} * \underline{x} <= \underline{y} * \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{z} * \underline{x} = \underline{y} * \underline{z}; \mathrm{SubLessRight} \rhd \underline{y} * \underline{z} = \underline{z} * \underline{y} \rhd \dot{\neg}\underline{z} * \underline{x} <= \underline{y} * \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{z} * \underline{x} = \underline{y} * \underline{z} \gg \dot{\neg}\underline{z} * \underline{x} <= \underline{z} * \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{z} * \underline{x} = \underline{z} * \underline{y}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{LessDivision} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon 0 <= \underline{z} \vdash \dot{\neg}\underline{x} * \underline{z} <= \underline{y} * \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} * \underline{z} = \underline{y} * \underline{z} \vdash \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y}]$

$[\mathrm{LessDivision} \stackrel{\mathrm{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil\mathrm{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon 0 <= \underline{z} \vdash \dot{\neg}\underline{x} * \underline{z} <= \underline{y} * \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} * \underline{z} = \underline{y} * \underline{z} \vdash \mathrm{FromLess} \rhd \dot{\neg}\underline{x} * \underline{z} <= \underline{y} * \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} * \underline{z} = \underline{y} * \underline{z} \gg \dot{\neg}\underline{y} * \underline{z} <= \underline{x} * \underline{z}; \mathrm{leqMultiplicationAxiom} \gg 0 <= \underline{z} \Rightarrow \underline{y} <= \underline{x} \Rightarrow \underline{y} * \underline{z} <= \underline{x} * \underline{z}; \mathrm{MP} \rhd 0 <= \underline{z} \Rightarrow \underline{y} <= \underline{x} \Rightarrow \underline{y} * \underline{z} <= \underline{x} * \underline{z} \rhd 0 <= \underline{z} \gg \underline{y} <= \underline{x} \Rightarrow \underline{y} * \underline{z} <= \underline{x} * \underline{z}; \mathrm{Contrapositive} \rhd \underline{y} <= \underline{x} \Rightarrow \underline{y} * \underline{z} <= \underline{x} * \underline{z} \gg \dot{\neg}\underline{y} * \underline{z} <= \underline{x} * \underline{z} \Rightarrow \dot{\neg}\underline{y} <= \underline{x}; \mathrm{MP} \rhd \dot{\neg}\underline{y} * \underline{z} <= \underline{x} * \underline{z} \Rightarrow \dot{\neg}\underline{y} <= \underline{x} \rhd \dot{\neg}\underline{y} * \underline{z} <= \underline{x} * \underline{z} \gg \dot{\neg}\underline{y} <= \underline{x}; \mathrm{ToLess} \rhd \dot{\neg}\underline{y} <= \underline{x} \gg \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{AddEquations(Less)} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \forall\underline{u}\colon \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \vdash \dot{\neg}\underline{z} <= \underline{u} \Rightarrow \dot{\neg}\dot{\neg}\underline{z} = \underline{u} \vdash \dot{\neg}\underline{x} + \underline{z} <= \underline{y} + \underline{u} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} + \underline{z} = \underline{y} + \underline{u}]$

$[\mathrm{AddEquations(Less)} \stackrel{\mathrm{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil\mathrm{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \forall\underline{u}\colon \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \vdash \dot{\neg}\underline{z} <= \underline{u} \Rightarrow \dot{\neg}\dot{\neg}\underline{z} = \underline{u} \vdash \mathrm{LessAddition} \rhd \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \gg \dot{\neg}\underline{x} + \underline{z} <= \underline{y} + \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} + \underline{z} = \underline{y} + \underline{z}; \mathrm{LessAdditionLeft} \rhd \dot{\neg}\underline{z} <= \underline{u} \Rightarrow \dot{\neg}\dot{\neg}\underline{z} = \underline{u} \gg \dot{\neg}\underline{y} + \underline{z} <= \underline{y} + \underline{u} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} + \underline{z} = \underline{y} + \underline{u}; \mathrm{LessTransitivity} \rhd \dot{\neg}\underline{x} + \underline{z} <= \underline{y} + \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} + \underline{z} = \underline{y} + \underline{z} \rhd \dot{\neg}\underline{y} + \underline{z} <= \underline{y} + \underline{u} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} + \underline{z} = \underline{y} + \underline{u} \gg \dot{\neg}\underline{x} + \underline{z} <= \underline{y} + \underline{u} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} + \underline{z} = \underline{y} + \underline{u}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{leqLessTransitivity} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \underline{x} <= \underline{y} \vdash \dot{\neg}\underline{y} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{z} \vdash \dot{\neg}\underline{x} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{z}]$

$[\mathrm{leqLessTransitivity} \stackrel{\mathrm{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil\mathrm{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \underline{x} <= \underline{y} \vdash \dot{\neg}\underline{y} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{z} \vdash \underline{x} = \underline{z} \vdash \mathrm{FirstConjunct} \rhd \dot{\neg}\underline{y} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{z} \gg \underline{y} <= \underline{z}; \mathrm{SecondConjunct} \rhd \dot{\neg}\underline{y} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{z} \gg \dot{\neg}\underline{y} = \underline{z}; \mathrm{subLeqLeft} \rhd \underline{x} = \underline{z} \rhd \underline{x} <= \underline{y} \gg \underline{z} <= \underline{y}; \mathrm{leqAntisymmetry} \rhd \underline{y} <= \underline{z} \rhd \underline{z} <= \underline{y} \gg \underline{y} = \underline{z}; \mathrm{FromContradiction} \rhd \underline{y} = \underline{z} \rhd \dot{\neg}\underline{y} = \underline{z} \gg \dot{\neg}\underline{x} = \underline{z}; \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \mathrm{Ded} \rhd \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \underline{x} <= \underline{y} \vdash \dot{\neg}\underline{y} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{z} \vdash \underline{x} = \underline{z} \vdash \dot{\neg}\underline{x} = \underline{z} \gg \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\underline{y} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{z} \Rightarrow \underline{x} = \underline{z} \Rightarrow$

$\dot{\neg}\underline{x} = \underline{z}; \underline{x} <= \underline{y} \vdash \dot{\neg}\underline{y} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{z} \vdash \mathrm{MP2} \rhd \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\underline{y} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{z} \Rightarrow \underline{x} = \underline{z} \Rightarrow \dot{\neg}\underline{x} = \underline{z} \rhd \underline{x} <= \underline{y} \rhd \dot{\neg}\underline{y} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{z} \gg \underline{x} = \underline{z} \Rightarrow \dot{\neg}\underline{x} = \underline{z}; \mathrm{ImplyNegation} \rhd \underline{x} = \underline{z} \Rightarrow \dot{\neg}\underline{x} = \underline{z} \gg \dot{\neg}\underline{x} = \underline{z}; \mathrm{FirstConjunct} \rhd \dot{\neg}\underline{y} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{z} \gg \underline{y} <= \underline{z}; \mathrm{leqTransitivity} \rhd \underline{x} <= \underline{y} \rhd \underline{y} <= \underline{z} \gg \underline{x} <= \underline{z}; \mathrm{JoinConjuncts} \rhd \underline{x} <= \underline{z} \rhd \dot{\neg}\underline{x} = \underline{z} \gg \dot{\neg}\underline{x} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{z}\rceil, p_0, \mathsf{c})]$

$[\mathrm{LessLeqTransitivity} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \vdash \underline{y} <= \underline{z} \vdash \dot{\neg}\underline{x} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{z}]$

$[\mathrm{LessLeqTransitivity} \overset{\mathrm{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \vdash \underline{y} <= \underline{z} \vdash \underline{z} = \underline{x} \vdash \mathrm{FirstConjunct} \rhd \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \gg \underline{x} <= \underline{y}; \mathrm{SecondConjunct} \rhd \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \gg \dot{\neg}\underline{x} = \underline{y}; \mathrm{subLeqRight} \rhd \underline{z} = \underline{x} \rhd \underline{y} <= \underline{z} \gg \underline{y} <= \underline{x}; \mathrm{leqAntisymmetry} \rhd \underline{x} <= \underline{y} \rhd \underline{y} <= \underline{x} \gg \underline{x} = \underline{y}; \mathrm{FromContradiction} \rhd \underline{x} = \underline{y} \rhd \dot{\neg}\underline{x} = \underline{y} \gg \dot{\neg}\underline{z} = \underline{x}; \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \mathrm{Ded} \rhd \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \vdash \underline{y} <= \underline{z} \vdash \underline{z} = \underline{x} \vdash \dot{\neg}\underline{z} = \underline{x} \gg \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \Rightarrow \underline{y} <= \underline{z} \Rightarrow \underline{z} = \underline{x} \Rightarrow \dot{\neg}\underline{z} = \underline{x}; \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \vdash \underline{y} <= \underline{z} \vdash \mathrm{MP2} \rhd \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \Rightarrow \underline{y} <= \underline{z} \Rightarrow \underline{z} = \underline{x} \Rightarrow \dot{\neg}\underline{z} = \underline{x} \rhd \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \rhd \underline{y} <= \underline{z} \gg \underline{z} = \underline{x} \Rightarrow \dot{\neg}\underline{z} = \underline{x}; \mathrm{ImplyNegation} \rhd \underline{z} = \underline{x} \Rightarrow \dot{\neg}\underline{z} = \underline{x} \gg \dot{\neg}\underline{z} = \underline{x}; \mathrm{NeqSymmetry} \rhd \dot{\neg}\underline{z} = \underline{x} \gg \dot{\neg}\underline{x} = \underline{z}; \mathrm{FirstConjunct} \rhd \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \gg \underline{x} <= \underline{y}; \mathrm{leqTransitivity} \rhd \underline{x} <= \underline{y} \rhd \underline{y} <= \underline{z} \gg \underline{x} <= \underline{z}; \mathrm{JoinConjuncts} \rhd \underline{x} <= \underline{z} \rhd \dot{\neg}\underline{x} = \underline{z} \gg \dot{\neg}\underline{x} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{z}\rceil, p_0, \mathsf{c})]$

$[\mathrm{LessTransitivity} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \vdash \dot{\neg}\underline{y} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{z} \vdash \dot{\neg}\underline{x} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{z}]$

$[\mathrm{LessTransitivity} \overset{\mathrm{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \vdash \dot{\neg}\underline{y} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{z} \vdash \mathrm{FirstConjunct} \rhd \dot{\neg}\underline{y} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{z} \gg \underline{y} <= \underline{z}; \mathrm{LessLeqTransitivity} \rhd \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \rhd \underline{y} <= \underline{z} \gg \dot{\neg}\underline{x} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{z}\rceil, p_0, \mathsf{c})]$

$[\mathrm{LessTotality} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \dot{\neg}\dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \Rightarrow \dot{\neg}\underline{x} = \underline{y} \Rightarrow \dot{\neg}\underline{y} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{x}]$

$[\mathrm{LessTotality} \overset{\mathrm{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \dot{\neg}\dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \vdash \dot{\neg}\underline{x} = \underline{y} \vdash \mathrm{fromNotLess} \rhd \dot{\neg}\dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \gg \underline{y} <= \underline{x}; \mathrm{NeqSymmetry} \rhd \dot{\neg}\underline{x} = \underline{y} \gg \dot{\neg}\underline{y} = \underline{x}; \mathrm{LeqNeqLess} \rhd \underline{y} <= \underline{x} \rhd \dot{\neg}\underline{y} = \underline{x} \gg \dot{\neg}\underline{y} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{x}; \forall\underline{x}\colon \forall\underline{y}\colon \mathrm{Ded} \rhd \forall\underline{x}\colon \forall\underline{y}\colon \dot{\neg}\dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \vdash \dot{\neg}\underline{x} = \underline{y} \vdash \dot{\neg}\underline{y} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{x} \gg \dot{\neg}\dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \Rightarrow \dot{\neg}\underline{x} = \underline{y} \Rightarrow \dot{\neg}\underline{y} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{x}; \mathrm{Repetition} \rhd \dot{\neg}\dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \Rightarrow \dot{\neg}\underline{x} = \underline{y} \Rightarrow \dot{\neg}\underline{y} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{x} \gg \dot{\neg}\dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \Rightarrow \dot{\neg}\underline{x} = \underline{y} \Rightarrow \dot{\neg}\underline{y} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\underline{y} = \underline{x}\rceil, p_0, \mathsf{c})]$

$[\mathrm{SubLessRight} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \underline{x} = \underline{y} \vdash \dot{\neg}\underline{z} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\underline{z} = \underline{x} \vdash \dot{\neg}\underline{z} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{z} = \underline{y}]$

$[\mathrm{SubLessRight} \overset{\mathrm{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \underline{x} = \underline{y} \vdash \dot{\neg}\underline{z} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\underline{z} = \underline{x} \vdash \mathrm{Repetition} \rhd \dot{\neg}\underline{z} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\underline{z} = \underline{x} \gg \dot{\neg}\underline{z} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\underline{z} = \underline{x}; \mathrm{FirstConjunct} \rhd \dot{\neg}\underline{z} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\underline{z} = \underline{x} \gg \underline{z} <= \underline{x}; \mathrm{subLeqRight} \rhd \underline{x} = \underline{y} \rhd \underline{z} <= \underline{x} \gg \underline{z} <= \underline{y}; \mathrm{SecondConjunct} \rhd \dot{\neg}\underline{z} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\underline{z} = \underline{x} \gg \dot{\neg}\underline{z} = \underline{x}; \mathrm{SubNeqRight} \rhd \underline{x} = \underline{y} \rhd \dot{\neg}\underline{z} = \underline{x} \gg \dot{\neg}\underline{z} = \underline{y}; \mathrm{JoinConjuncts} \rhd \underline{z} <= \underline{y} \rhd \dot{\neg}\underline{z} = \underline{y} \gg \dot{\neg}\underline{z} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{z} = \underline{y}\rceil, p_0, \mathsf{c})]$

$[\text{SubLessLeft} \overset{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \forall \underline{\mathsf{z}}\colon \underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \dot{\neg} \underline{\mathsf{x}} <= \underline{\mathsf{z}} \Rightarrow \dot{\neg}\dot{\neg} \underline{\mathsf{x}} = \underline{\mathsf{z}} \vdash \dot{\neg} \underline{\mathsf{y}} <= \underline{\mathsf{z}} \Rightarrow \dot{\neg}\dot{\neg} \underline{\mathsf{y}} = \underline{\mathsf{z}}]$

$[\text{SubLessLeft} \overset{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \forall \underline{\mathsf{z}}\colon \underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \dot{\neg} \underline{\mathsf{x}} <= \underline{\mathsf{z}} \Rightarrow \dot{\neg}\dot{\neg} \underline{\mathsf{x}} = \underline{\mathsf{z}} \vdash \text{Repetition} \rhd \dot{\neg} \underline{\mathsf{x}} <= \underline{\mathsf{z}} \Rightarrow \dot{\neg}\dot{\neg} \underline{\mathsf{x}} = \underline{\mathsf{z}} \gg \dot{\neg} \underline{\mathsf{x}} <= \underline{\mathsf{z}} \Rightarrow \dot{\neg}\dot{\neg} \underline{\mathsf{x}} = \underline{\mathsf{z}}; \text{FirstConjunct} \rhd \dot{\neg} \underline{\mathsf{x}} <= \underline{\mathsf{z}} \Rightarrow \dot{\neg}\dot{\neg} \underline{\mathsf{x}} = \underline{\mathsf{z}} \gg \underline{\mathsf{x}} <= \underline{\mathsf{z}}; \text{subLeqLeft} \rhd \underline{\mathsf{x}} = \underline{\mathsf{y}} \rhd \underline{\mathsf{x}} <= \underline{\mathsf{z}} \gg \underline{\mathsf{y}} <= \underline{\mathsf{z}}; \text{SecondConjunct} \rhd \dot{\neg} \underline{\mathsf{x}} <= \underline{\mathsf{z}} \Rightarrow \dot{\neg}\dot{\neg} \underline{\mathsf{x}} = \underline{\mathsf{z}} \gg \dot{\neg} \underline{\mathsf{x}} = \underline{\mathsf{z}}; \text{SubNeqLeft} \rhd \underline{\mathsf{x}} = \underline{\mathsf{y}} \rhd \dot{\neg} \underline{\mathsf{x}} = \underline{\mathsf{z}} \gg \dot{\neg} \underline{\mathsf{y}} = \underline{\mathsf{z}}; \text{JoinConjuncts} \rhd \underline{\mathsf{y}} <= \underline{\mathsf{z}} \rhd \dot{\neg} \underline{\mathsf{y}} = \underline{\mathsf{z}} \gg \dot{\neg} \underline{\mathsf{y}} <= \underline{\mathsf{z}} \Rightarrow \dot{\neg}\dot{\neg} \underline{\mathsf{y}} = \underline{\mathsf{z}} \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\text{NegativeLessPositive} \overset{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \dot{\neg} 0 <= \underline{\mathsf{x}} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{\mathsf{x}} \vdash \dot{\neg} -\underline{\mathsf{x}} <= \underline{\mathsf{x}} \Rightarrow \dot{\neg}\dot{\neg} -\underline{\mathsf{x}} = \underline{\mathsf{x}}]$

$[\text{NegativeLessPositive} \overset{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \dot{\neg} 0 <= \underline{\mathsf{x}} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{\mathsf{x}} \vdash \text{FirstConjunct} \rhd \dot{\neg} 0 <= \underline{\mathsf{x}} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{\mathsf{x}} \gg 0 <= \underline{\mathsf{x}}; \text{leqAddition} \rhd 0 <= \underline{\mathsf{x}} \gg 0 + -\underline{\mathsf{x}} <= \underline{\mathsf{x}} + -\underline{\mathsf{x}}; \text{plus0Left} \gg 0 + -\underline{\mathsf{x}} = -\underline{\mathsf{x}}; \text{Negative} \gg \underline{\mathsf{x}} + -\underline{\mathsf{x}} = 0; \text{subLeqLeft} \rhd 0 + -\underline{\mathsf{x}} = -\underline{\mathsf{x}} \rhd 0 + -\underline{\mathsf{x}} <= \underline{\mathsf{x}} + -\underline{\mathsf{x}} \gg -\underline{\mathsf{x}} <= \underline{\mathsf{x}} + -\underline{\mathsf{x}}; \text{subLeqRight} \rhd \underline{\mathsf{x}} + -\underline{\mathsf{x}} = 0 \rhd -\underline{\mathsf{x}} <= \underline{\mathsf{x}} + -\underline{\mathsf{x}} \gg -\underline{\mathsf{x}} <= 0; \text{leqLessTransitivity} \rhd -\underline{\mathsf{x}} <= 0 \rhd \dot{\neg} 0 <= \underline{\mathsf{x}} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{\mathsf{x}} \gg \dot{\neg} -\underline{\mathsf{x}} <= \underline{\mathsf{x}} \Rightarrow \dot{\neg}\dot{\neg} -\underline{\mathsf{x}} = \underline{\mathsf{x}} \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\text{LessNegated} \overset{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \dot{\neg} \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg} \underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \dot{\neg} -\underline{\mathsf{y}} <= -\underline{\mathsf{x}} \Rightarrow \dot{\neg}\dot{\neg} -\underline{\mathsf{y}} = -\underline{\mathsf{x}}]$

$[\text{LessNegated} \overset{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \dot{\neg} \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg} \underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \text{LessLeq} \rhd \dot{\neg} \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg} \underline{\mathsf{x}} = \underline{\mathsf{y}} \gg \underline{\mathsf{x}} <= \underline{\mathsf{y}}; \text{LeqNegated} \rhd \underline{\mathsf{x}} <= \underline{\mathsf{y}} \gg -\underline{\mathsf{y}} <= -\underline{\mathsf{x}}; \text{LessNeq} \rhd \dot{\neg} \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg} \underline{\mathsf{x}} = \underline{\mathsf{y}} \gg \dot{\neg} \underline{\mathsf{x}} = \underline{\mathsf{y}}; \text{NeqNegated} \rhd \dot{\neg} \underline{\mathsf{x}} = \underline{\mathsf{y}} \gg \dot{\neg} -\underline{\mathsf{x}} = -\underline{\mathsf{y}}; \text{NeqSymmetry} \rhd \dot{\neg} -\underline{\mathsf{x}} = -\underline{\mathsf{y}} \gg \dot{\neg} -\underline{\mathsf{y}} = -\underline{\mathsf{x}}; \text{LeqNeqLess} \rhd -\underline{\mathsf{y}} <= -\underline{\mathsf{x}} \rhd \dot{\neg} -\underline{\mathsf{y}} = -\underline{\mathsf{x}} \gg \dot{\neg} -\underline{\mathsf{y}} <= -\underline{\mathsf{x}} \Rightarrow \dot{\neg}\dot{\neg} -\underline{\mathsf{y}} = -\underline{\mathsf{x}} \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\text{PositiveNegated} \overset{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \dot{\neg} 0 <= \underline{\mathsf{x}} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{\mathsf{x}} \vdash \dot{\neg} -\underline{\mathsf{x}} <= 0 \Rightarrow \dot{\neg}\dot{\neg} -\underline{\mathsf{x}} = 0]$

$[\text{PositiveNegated} \overset{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \dot{\neg} 0 <= \underline{\mathsf{x}} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{\mathsf{x}} \vdash \text{LessNegated} \rhd \dot{\neg} 0 <= \underline{\mathsf{x}} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{\mathsf{x}} \gg \dot{\neg} -\underline{\mathsf{x}} <= -0 \Rightarrow \dot{\neg}\dot{\neg} -\underline{\mathsf{x}} = -0; -0 = 0 \gg -0 = 0; \text{SubLessRight} \rhd -0 = 0 \rhd \dot{\neg} -\underline{\mathsf{x}} <= -0 \Rightarrow \dot{\neg}\dot{\neg} -\underline{\mathsf{x}} = -0 \gg \dot{\neg} -\underline{\mathsf{x}} <= 0 \Rightarrow \dot{\neg}\dot{\neg} -\underline{\mathsf{x}} = 0 \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\text{NonpositiveNegated} \overset{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \underline{\mathsf{x}} <= 0 \vdash 0 <= -\underline{\mathsf{x}}]$

$[\text{NonpositiveNegated} \overset{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \underline{\mathsf{x}} <= 0 \vdash \text{LeqNegated} \rhd \underline{\mathsf{x}} <= 0 \gg -0 <= -\underline{\mathsf{x}}; -0 = 0 \gg -0 = 0; \text{subLeqLeft} \rhd -0 = 0 \rhd -0 <= -\underline{\mathsf{x}} \gg 0 <= -\underline{\mathsf{x}} \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\text{NegativeNegated} \overset{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \dot{\neg} \underline{\mathsf{x}} <= 0 \Rightarrow \dot{\neg}\dot{\neg} \underline{\mathsf{x}} = 0 \vdash \dot{\neg} 0 <= -\underline{\mathsf{x}} \Rightarrow \dot{\neg}\dot{\neg} 0 = -\underline{\mathsf{x}}]$

$[\text{NegativeNegated} \overset{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \dot{\neg} \underline{\mathsf{x}} <= 0 \Rightarrow \dot{\neg}\dot{\neg} \underline{\mathsf{x}} = 0 \vdash \text{LessNegated} \rhd \dot{\neg} \underline{\mathsf{x}} <= 0 \Rightarrow \dot{\neg}\dot{\neg} \underline{\mathsf{x}} = 0 \gg \dot{\neg} -0 <= -\underline{\mathsf{x}} \Rightarrow \dot{\neg}\dot{\neg} -0 = -\underline{\mathsf{x}}; -0 = 0 \gg -0 = 0; \text{SubLessLeft} \rhd -0 = 0 \rhd \dot{\neg} -0 <= -\underline{\mathsf{x}} \Rightarrow \dot{\neg}\dot{\neg} -0 = -\underline{\mathsf{x}} \gg \dot{\neg} 0 <= -\underline{\mathsf{x}} \Rightarrow \dot{\neg}\dot{\neg} 0 = -\underline{\mathsf{x}} \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\text{NonnegativeNegated} \overset{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon 0 <= \underline{\mathsf{x}} \vdash -\underline{\mathsf{x}} <= 0]$

$[\text{NonnegativeNegated} \overset{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon 0 <= \underline{\mathsf{x}} \vdash \text{LeqNegated} \rhd 0 <= \underline{\mathsf{x}} \gg -\underline{\mathsf{x}} <= -0; -0 = 0 \gg -0 = 0; \text{subLeqRight} \rhd -0 = 0 \rhd -\underline{\mathsf{x}} <= -0 \gg -\underline{\mathsf{x}} <= 0 \rceil, \mathrm{p}_0, \mathsf{c})]$

[PositiveHalved $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall\underline{x}\colon \dot\neg 0 <= \underline{x} \Rightarrow \dot\neg\dot\neg 0 = \underline{x} \vdash \dot\neg 0 <= \mathrm{rec1}+1*\underline{x} \Rightarrow \dot\neg\dot\neg 0 = \mathrm{rec1}+1*\underline{x}$]

[PositiveHalved $\overset{\text{proof}}{\rightarrow}$ $\lambda\mathsf{c}.\lambda\mathrm{x}.\mathcal{P}(\lceil$ZFsub $\vdash \forall\underline{x}\colon \dot\neg 0 <= \underline{x} \Rightarrow \dot\neg\dot\neg 0 = \underline{x} \vdash 0 < 1/2 \gg \dot\neg 0 <= \mathrm{rec1}+1 \Rightarrow \dot\neg\dot\neg 0 = \mathrm{rec1}+1;$ LessMultiplicationLeft $\rhd \dot\neg 0 <= \mathrm{rec1}+1 \Rightarrow \dot\neg\dot\neg 0 = \mathrm{rec1}+1 \rhd \dot\neg 0 <= \underline{x} \Rightarrow \dot\neg\dot\neg 0 = \underline{x} \gg \dot\neg \mathrm{rec1}+1*0 <= \mathrm{rec1}+1*\underline{x} \Rightarrow \dot\neg\dot\neg \mathrm{rec1}+1*0 = \mathrm{rec1}+1*\underline{x}; \mathrm{x}*0 = 0 \gg \mathrm{rec1}+1*0 = 0;$ SubLessLeft $\rhd \mathrm{rec1}+1*0 = 0 \rhd \dot\neg \mathrm{rec1}+1*0 <= \mathrm{rec1}+1*\underline{x} \Rightarrow \dot\neg\dot\neg \mathrm{rec1}+1*0 = \mathrm{rec1}+1*\underline{x} \gg \dot\neg 0 <= \mathrm{rec1}+1*\underline{x} \Rightarrow \dot\neg\dot\neg 0 = \mathrm{rec1}+1*\underline{x}\rceil, \mathrm{p}_0, \mathsf{c})$]

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[NonnegativeNumerical $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall\underline{x}\colon 0 <= \underline{x} \vdash \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \underline{x}$]

[NonnegativeNumerical $\overset{\text{proof}}{\rightarrow}$ $\lambda\mathsf{c}.\lambda\mathrm{x}.\mathcal{P}(\lceil$ZFsub $\vdash \forall\underline{x}\colon 0 <= \underline{x} \vdash$ IfThenElse(T) $\rhd 0 <= \underline{x} \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \underline{x};$ Repetition $\rhd \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \underline{x} \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \underline{x}\rceil, \mathrm{p}_0, \mathsf{c})$]

[PositiveNumerical $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall\underline{x}\colon \dot\neg 0 <= \underline{x} \Rightarrow \dot\neg\dot\neg 0 = \underline{x} \vdash \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \underline{x}$]

[PositiveNumerical $\overset{\text{proof}}{\rightarrow}$ $\lambda\mathsf{c}.\lambda\mathrm{x}.\mathcal{P}(\lceil$ZFsub $\vdash \forall\underline{x}\colon \dot\neg 0 <= \underline{x} \Rightarrow \dot\neg\dot\neg 0 = \underline{x} \vdash$ LessLeq $\rhd \dot\neg 0 <= \underline{x} \Rightarrow \dot\neg\dot\neg 0 = \underline{x} \gg 0 <= \underline{x};$ NonnegativeNumerical $\rhd 0 <= \underline{x} \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \underline{x}\rceil, \mathrm{p}_0, \mathsf{c})$]

[NegativeNumerical $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall\underline{x}\colon \dot\neg \underline{x} <= 0 \Rightarrow \dot\neg\dot\neg \underline{x} = 0 \vdash \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = -\underline{x}$]

[NegativeNumerical $\overset{\text{proof}}{\rightarrow}$ $\lambda\mathsf{c}.\lambda\mathrm{x}.\mathcal{P}(\lceil$ZFsub $\vdash \forall\underline{x}\colon \dot\neg \underline{x} <= 0 \Rightarrow \dot\neg\dot\neg \underline{x} = 0 \vdash$ FromLess $\rhd \dot\neg \underline{x} <= 0 \Rightarrow \dot\neg\dot\neg \underline{x} = 0 \gg \dot\neg 0 <= \underline{x};$ IfThenElse(F) $\rhd \dot\neg 0 <= \underline{x} \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = -\underline{x};$ Repetition $\rhd \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = -\underline{x} \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = -\underline{x}\rceil, \mathrm{p}_0, \mathsf{c})$]

[lemma nonpositiveNumerical $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall\underline{x}\colon \underline{x} <= 0 \vdash \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = -\underline{x}$]

[lemma nonpositiveNumerical $\overset{\text{proof}}{\rightarrow}$ $\lambda\mathsf{c}.\lambda\mathrm{x}.\mathcal{P}(\lceil$ZFsub $\vdash \forall\underline{x}\colon \dot\neg \underline{x} <= 0 \Rightarrow \dot\neg\dot\neg \underline{x} = 0 \vdash$ NegativeNumerical $\rhd \dot\neg \underline{x} <= 0 \Rightarrow \dot\neg\dot\neg \underline{x} = 0 \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = -\underline{x}; \forall\underline{x}\colon \underline{x} = 0 \vdash$ eqSymmetry $\rhd \underline{x} = 0 \gg 0 = \underline{x};$ eqLeq $\rhd 0 = \underline{x} \gg 0 <= \underline{x};$ NonnegativeNumerical $\rhd 0 <= \underline{x} \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \underline{x}; -0 = 0 \gg -0 = 0;$ eqSymmetry $\rhd -0 = 0 \gg 0 = -0;$ EqNegated $\rhd 0 = \underline{x} \gg -0 = -\underline{x};$ eqTransitivity5 $\rhd \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \underline{x} \rhd \underline{x} = 0 \rhd 0 = -0 \rhd -0 = -\underline{x} \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = -\underline{x}; \forall\underline{x}\colon$ Ded $\rhd \forall\underline{x}\colon \dot\neg \underline{x} <= 0 \Rightarrow \dot\neg\dot\neg \underline{x} = 0 \vdash \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = -\underline{x} \gg \dot\neg \underline{x} <= 0 \Rightarrow \dot\neg\dot\neg \underline{x} = 0 \Rightarrow \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = -\underline{x};$ Ded $\rhd \forall\underline{x}\colon \underline{x} = 0 \vdash \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = -\underline{x} \gg \underline{x} = 0 \Rightarrow \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = -\underline{x}; \underline{x} <= 0 \vdash$ LeqLessEq $\rhd \underline{x} <= 0 \gg \dot\neg\dot\neg \underline{x} <= 0 \Rightarrow \dot\neg\dot\neg \underline{x} = 0 \Rightarrow \underline{x} = 0;$ FromDisjuncts $\rhd \dot\neg\dot\neg \underline{x} <= 0 \Rightarrow \dot\neg\dot\neg \underline{x} = 0 \Rightarrow \underline{x} = 0 \rhd \dot\neg \underline{x} <= 0 \Rightarrow \dot\neg\dot\neg \underline{x} = 0 \Rightarrow \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = -\underline{x} \rhd \underline{x} = 0 \Rightarrow \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = -\underline{x} \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = -\underline{x}\rceil, \mathrm{p}_0, \mathsf{c})$]

[$|0| = 0 \overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \mathrm{if}(0 <= 0, 0, -0) = 0$]

[$|0| = 0 \overset{\text{proof}}{\rightarrow}$ $\lambda\mathsf{c}.\lambda\mathrm{x}.\mathcal{P}(\lceil$ZFsub $\vdash$ leqReflexivity $\gg 0 <= 0;$ NonnegativeNumerical $\rhd 0 <= 0 \gg \mathrm{if}(0 <= 0, 0, -0) = 0\rceil, \mathrm{p}_0, \mathsf{c})$]

[$0 <= |\mathrm{x}| \overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall\underline{x}\colon 0 <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x})$]

$[0 <= |\mathrm{x}| \stackrel{\text{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall\underline{x}\colon 0 <= \underline{x} \vdash \text{NonnegativeNumerical} \rhd 0 <= \underline{x} \gg \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \underline{x}; \text{eqSymmetry} \rhd \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \underline{x} \gg \underline{x} = \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}); \text{subLeqRight} \rhd \underline{x} = \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \rhd 0 <= \underline{x} \gg 0 <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}); \forall\underline{x}\colon \dot\neg 0 <= \underline{x} \vdash \text{ToLess} \rhd \dot\neg 0 <= \underline{x} \gg \dot\neg \underline{x} <= 0 \Rightarrow \dot\neg\dot\neg \underline{x} = 0; \text{NegativeNumerical} \rhd \dot\neg \underline{x} <= 0 \Rightarrow \dot\neg\dot\neg \underline{x} = 0 \gg \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = -\underline{x}; \text{eqSymmetry} \rhd \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = -\underline{x} \gg -\underline{x} = \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}); \text{NegativeNegated} \rhd \dot\neg \underline{x} <= 0 \Rightarrow \dot\neg\dot\neg \underline{x} = 0 \gg \dot\neg 0 <= -\underline{x} \Rightarrow \dot\neg\dot\neg 0 = -\underline{x}; \text{LessLeq} \rhd \dot\neg 0 <= -\underline{x} \Rightarrow \dot\neg\dot\neg 0 = -\underline{x} \gg 0 <= -\underline{x}; \text{subLeqRight} \rhd -\underline{x} = \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \rhd 0 <= -\underline{x} \gg 0 <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}); \forall\underline{x}\colon \text{Ded} \rhd \forall\underline{x}\colon 0 <= \underline{x} \vdash 0 <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \gg 0 <= \underline{x} \Rightarrow 0 <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}); \text{Ded} \rhd \forall\underline{x}\colon \dot\neg 0 <= \underline{x} \vdash 0 <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \gg \dot\neg 0 <= \underline{x} \Rightarrow 0 <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}); \text{FromNegations} \rhd 0 <= \underline{x} \Rightarrow 0 <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \rhd \dot\neg 0 <= \underline{x} \Rightarrow 0 <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \gg 0 <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x})\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\text{SameNumerical} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \underline{x} = \underline{y} \vdash \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y})]$

$[\text{SameNumerical} \stackrel{\text{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon 0 <= \underline{x} \vdash \underline{x} = \underline{y} \vdash \text{NonnegativeNumerical} \rhd 0 <= \underline{x} \gg \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \underline{x}; \text{subLeqRight} \rhd \underline{x} = \underline{y} \rhd 0 <= \underline{x} \gg 0 <= \underline{y}; \text{NonnegativeNumerical} \rhd 0 <= \underline{y} \gg \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = \underline{y}; \text{eqSymmetry} \rhd \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = \underline{y} \gg \underline{y} = \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \text{eqTransitivity4} \rhd \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \underline{x} \rhd \underline{x} = \underline{y} \rhd \underline{y} = \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \forall\underline{x}\colon \forall\underline{y}\colon \dot\neg 0 <= \underline{x} \vdash \underline{x} = \underline{y} \vdash \text{ToLess} \rhd \dot\neg 0 <= \underline{x} \gg \dot\neg \underline{x} <= 0 \Rightarrow \dot\neg\dot\neg \underline{x} = 0; \text{NegativeNumerical} \rhd \dot\neg \underline{x} <= 0 \Rightarrow \dot\neg\dot\neg \underline{x} = 0 \gg \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = -\underline{x}; \text{SubLessLeft} \rhd \underline{x} = \underline{y} \rhd \dot\neg \underline{x} <= 0 \Rightarrow \dot\neg\dot\neg \underline{x} = 0 \gg \dot\neg \underline{y} <= 0 \Rightarrow \dot\neg\dot\neg \underline{y} = 0; \text{NegativeNumerical} \rhd \dot\neg \underline{y} <= 0 \Rightarrow \dot\neg\dot\neg \underline{y} = 0 \gg \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = -\underline{y}; \text{eqSymmetry} \rhd \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = -\underline{y} \gg -\underline{y} = \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \text{EqNegated} \rhd \underline{x} = \underline{y} \gg -\underline{x} = -\underline{y}; \text{eqTransitivity4} \rhd \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = -\underline{x} \rhd -\underline{x} = -\underline{y} \rhd -\underline{y} = \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \forall\underline{x}\colon \forall\underline{y}\colon \underline{x} = \underline{y} \vdash \text{Ded} \rhd \forall\underline{x}\colon \forall\underline{y}\colon 0 <= \underline{x} \vdash \underline{x} = \underline{y} \vdash \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg 0 <= \underline{x} \Rightarrow \underline{x} = \underline{y} \Rightarrow \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \text{Ded} \rhd \forall\underline{x}\colon \forall\underline{y}\colon \dot\neg 0 <= \underline{x} \vdash \underline{x} = \underline{y} \vdash \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \dot\neg 0 <= \underline{x} \Rightarrow \underline{x} = \underline{y} \Rightarrow \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \text{FromNegations} \rhd 0 <= \underline{x} \Rightarrow \underline{x} = \underline{y} \Rightarrow \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \rhd \dot\neg 0 <= \underline{x} \Rightarrow \underline{x} = \underline{y} \Rightarrow \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \underline{x} = \underline{y} \Rightarrow \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \text{MP} \rhd \underline{x} = \underline{y} \Rightarrow \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \rhd \underline{x} = \underline{y} \gg \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y})\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\text{SignNumerical}(+) \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall\underline{x}\colon \dot\neg 0 <= \underline{x} \Rightarrow \dot\neg\dot\neg 0 = \underline{x} \vdash \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \text{if}(0 <= -\underline{x}, -\underline{x}, - - \underline{x})]$

$[\text{SignNumerical}(+) \stackrel{\text{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall\underline{x}\colon \dot\neg 0 <= \underline{x} \Rightarrow \dot\neg\dot\neg 0 = \underline{x} \vdash \text{PositiveNumerical} \rhd \dot\neg 0 <= \underline{x} \Rightarrow \dot\neg\dot\neg 0 = \underline{x} \gg \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \underline{x}; \text{PositiveNegated} \rhd \dot\neg 0 <= \underline{x} \Rightarrow \dot\neg\dot\neg 0 = \underline{x} \gg \dot\neg - \underline{x} <= 0 \Rightarrow \dot\neg\dot\neg - \underline{x} = 0; \text{NegativeNumerical} \rhd \dot\neg - \underline{x} <= 0 \Rightarrow \dot\neg\dot\neg - \underline{x} = 0 \gg \text{if}(0 <= -\underline{x}, -\underline{x}, - - \underline{x}) = - - \underline{x}; \text{DoubleMinus} \gg - - \underline{x} = \underline{x}; \text{eqTransitivity} \rhd \text{if}(0 <= -\underline{x}, -\underline{x}, - - \underline{x}) = - - \underline{x} \rhd - - \underline{x} = \underline{x} \gg \text{if}(0 <= -\underline{x}, -\underline{x}, - - \underline{x}) = \underline{x}; \text{eqSymmetry} \rhd \text{if}(0 <= -\underline{x}, -\underline{x}, - - \underline{x}) = \underline{x} \gg \underline{x} = \text{if}(0 <= -\underline{x}, -\underline{x}, - - \underline{x}); \text{eqTransitivity} \rhd \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \underline{x} \rhd \underline{x} =$

$\mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x})\rceil, p_0, \mathsf{c})]$

$[\mathrm{SignNumerical} \stackrel{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{x}\colon \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x})]$

$[\mathrm{SignNumerical} \stackrel{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x}\colon \dot\neg \underline{x} <= 0 \Rightarrow \dot\neg\dot\neg \underline{x} = 0 \vdash \mathrm{NegativeNegated} \rhd \dot\neg \underline{x} <= 0 \Rightarrow \dot\neg\dot\neg \underline{x} = 0 \gg \dot\neg 0 <= -\underline{x} \Rightarrow \dot\neg\dot\neg 0 = -\underline{x}; \mathrm{SignNumerical}(+) \rhd \dot\neg 0 <= -\underline{x} \Rightarrow \dot\neg\dot\neg 0 = -\underline{x} \gg \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) = \mathrm{if}(0 <= --\underline{x}, --\underline{x}, ---\underline{x}); \mathrm{DoubleMinus} \gg --\underline{x} = \underline{x}; \mathrm{SameNumerical} \rhd --\underline{x} = \underline{x} \gg \mathrm{if}(0 <= --\underline{x}, --\underline{x}, ---\underline{x}) = \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}); \mathrm{eqTransitivity} \rhd \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) = \mathrm{if}(0 <= --\underline{x}, --\underline{x}, ---\underline{x}) \rhd \mathrm{if}(0 <= --\underline{x}, --\underline{x}, ---\underline{x}) = \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \gg \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) = \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}); \mathrm{eqSymmetry} \rhd \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) = \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}); \forall \underline{x}\colon \underline{x} = 0 \vdash \mathrm{EqNegated} \rhd \underline{x} = 0 \gg -\underline{x} = -0; -0 = 0 \gg -0 = 0; \mathrm{eqSymmetry} \rhd \underline{x} = 0 \gg 0 = \underline{x}; \mathrm{eqTransitivity4} \rhd -\underline{x} = -0 \rhd -0 = 0 \rhd 0 = \underline{x} \gg -\underline{x} = \underline{x}; \mathrm{eqSymmetry} \rhd -\underline{x} = \underline{x} \gg \underline{x} = -\underline{x}; \mathrm{SameNumerical} \rhd \underline{x} = -\underline{x} \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}); \forall \underline{x}\colon \dot\neg 0 <= \underline{x} \Rightarrow \dot\neg\dot\neg 0 = \underline{x} \vdash \mathrm{SignNumerical}(+) \rhd \dot\neg 0 <= \underline{x} \Rightarrow \dot\neg\dot\neg 0 = \underline{x} \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}); \forall \underline{x}\colon \mathrm{Ded} \rhd \forall \underline{x}\colon \dot\neg \underline{x} <= 0 \Rightarrow \dot\neg\dot\neg \underline{x} = 0 \vdash \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) \gg \dot\neg \underline{x} <= 0 \Rightarrow \dot\neg\dot\neg \underline{x} = 0 \Rightarrow \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}); \mathrm{Ded} \rhd \forall \underline{x}\colon \underline{x} = 0 \vdash \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) \gg \underline{x} = 0 \Rightarrow \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}); \mathrm{Ded} \rhd \forall \underline{x}\colon \dot\neg 0 <= \underline{x} \Rightarrow \dot\neg\dot\neg 0 = \underline{x} \vdash \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) \gg \dot\neg 0 <= \underline{x} \Rightarrow \dot\neg\dot\neg 0 = \underline{x} \Rightarrow \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}); \mathrm{LessTotality} \gg \dot\neg\dot\neg \underline{x} <= 0 \Rightarrow \dot\neg\dot\neg \underline{x} = 0 \Rightarrow \dot\neg \underline{x} = 0 \Rightarrow \dot\neg 0 <= \underline{x} \Rightarrow \dot\neg\dot\neg 0 = \underline{x}; \mathrm{From3Disjuncts} \rhd \dot\neg\dot\neg \underline{x} <= 0 \Rightarrow \dot\neg\dot\neg \underline{x} = 0 \Rightarrow \dot\neg \underline{x} = 0 \Rightarrow \dot\neg 0 <= \underline{x} \Rightarrow \dot\neg\dot\neg 0 = \underline{x} \rhd \dot\neg \underline{x} <= 0 \Rightarrow \dot\neg\dot\neg \underline{x} = 0 \Rightarrow \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) \rhd \underline{x} = 0 \Rightarrow \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) \rhd \dot\neg 0 <= \underline{x} \Rightarrow \dot\neg\dot\neg 0 = \underline{x} \Rightarrow \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x})\rceil, p_0, \mathsf{c})]$

$[\mathrm{NumericalDifference} \stackrel{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \mathrm{if}(0 <= \underline{x} + -\underline{y}, \underline{x} + -\underline{y}, -\underline{x} + -\underline{y}) = \mathrm{if}(0 <= \underline{y} + -\underline{x}, \underline{y} + -\underline{x}, -\underline{y} + -\underline{x})]$

$[\mathrm{NumericalDifference} \stackrel{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \mathrm{SignNumerical} \gg \mathrm{if}(0 <= \underline{x} + -\underline{y}, \underline{x} + -\underline{y}, -\underline{x} + -\underline{y}) = \mathrm{if}(0 <= -\underline{x} + -\underline{y}, -\underline{x} + -\underline{y}, --\underline{x} + -\underline{y}); \mathrm{MinusNegated} \gg -\underline{x} + -\underline{y} = \underline{y} + -\underline{x}; \mathrm{SameNumerical} \rhd -\underline{x} + -\underline{y} = \underline{y} + -\underline{x} \gg \mathrm{if}(0 <= -\underline{x} + -\underline{y}, -\underline{x} + -\underline{y}, --\underline{x} + -\underline{y}) = \mathrm{if}(0 <= \underline{y} + -\underline{x}, \underline{y} + -\underline{x}, -\underline{y} + -\underline{x}); \mathrm{eqTransitivity} \rhd \mathrm{if}(0 <= \underline{x} + -\underline{y}, \underline{x} + -\underline{y}, -\underline{x} + -\underline{y}) = \mathrm{if}(0 <= -\underline{x} + -\underline{y}, -\underline{x} + -\underline{y}, --\underline{x} + -\underline{y}) \rhd \mathrm{if}(0 <= -\underline{x} + -\underline{y}, -\underline{x} + -\underline{y}, --\underline{x} + -\underline{y}) = \mathrm{if}(0 <= \underline{y} + -\underline{x}, \underline{y} + -\underline{x}, -\underline{y} + -\underline{x}) \gg \mathrm{if}(0 <= \underline{x} + -\underline{y}, \underline{x} + -\underline{y}, -\underline{x} + -\underline{y}) = \mathrm{if}(0 <= \underline{y} + -\underline{x}, \underline{y} + -\underline{x}, -\underline{y} + -\underline{x})\rceil, p_0, \mathsf{c})]$

$[\mathrm{SplitNumericalSumHelper} \stackrel{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \mathrm{if}(0 <= -\underline{x} + -\underline{y}, -\underline{x} + -\underline{y}, --\underline{x} + -\underline{y}) <= \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) + \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y}) \vdash \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})]$

$[\mathrm{SplitNumericalSumHelper} \stackrel{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \mathrm{if}(0 <= -\underline{x} + -\underline{y}, -\underline{x} + -\underline{y}, --\underline{x} + -\underline{y}) <= \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) + \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y}) \vdash \mathrm{SignNumerical} \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}); \mathrm{SignNumerical} \gg \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y}); \mathrm{AddEquations} \rhd \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) \rhd \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y}) \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) + \mathrm{if}(0 <=$

$-\underline{y}, -\underline{y}, - - \underline{y})$; eqSymmetry $\triangleright \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, - - \underline{x}) + \mathrm{if}(0 <= -\underline{y}, -\underline{y}, - - \underline{y}) \gg \mathrm{if}(0 <= -\underline{x}, -\underline{x}, - - \underline{x}) + \mathrm{if}(0 <= -\underline{y}, -\underline{y}, - - \underline{y}) = \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})$; $-\mathrm{x} - \mathrm{y} = -(\mathrm{x} + \mathrm{y}) \gg -\underline{x} + -\underline{y} = -\underline{x} + \underline{y}$; SameNumerical $\triangleright -\underline{x} + -\underline{y} = -\underline{x} + \underline{y} \gg \mathrm{if}(0 <= -\underline{x} + -\underline{y}, -\underline{x} + -\underline{y}, - - \underline{x} + -\underline{y}) = \mathrm{if}(0 <= -\underline{x} + \underline{y}, -\underline{x} + \underline{y}, - - \underline{x} + \underline{y})$; SignNumerical $\gg \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) = \mathrm{if}(0 <= -\underline{x} + \underline{y}, -\underline{x} + \underline{y}, - - \underline{x} + \underline{y})$; eqSymmetry $\triangleright \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) = \mathrm{if}(0 <= -\underline{x} + \underline{y}, -\underline{x} + \underline{y}, - - \underline{x} + \underline{y}) \gg \mathrm{if}(0 <= -\underline{x} + \underline{y}, -\underline{x} + \underline{y}, - - \underline{x} + \underline{y}) = \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y})$; eqTransitivity $\triangleright \mathrm{if}(0 <= -\underline{x} + -\underline{y}, -\underline{x} + -\underline{y}, - - \underline{x} + -\underline{y}) = \mathrm{if}(0 <= -\underline{x} + \underline{y}, -\underline{x} + \underline{y}, - - \underline{x} + \underline{y}) \triangleright \mathrm{if}(0 <= -\underline{x} + \underline{y}, -\underline{x} + \underline{y}, - - \underline{x} + \underline{y}) = \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) \gg \mathrm{if}(0 <= -\underline{x} + -\underline{y}, -\underline{x} + -\underline{y}, - - \underline{x} + -\underline{y}) = \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y})$; subLeqRight $\triangleright \mathrm{if}(0 <= -\underline{x}, -\underline{x}, - - \underline{x}) + \mathrm{if}(0 <= -\underline{y}, -\underline{y}, - - \underline{y}) = \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \triangleright \mathrm{if}(0 <= -\underline{x} + -\underline{y}, -\underline{x} + -\underline{y}, - - \underline{x} + -\underline{y}) <= \mathrm{if}(0 <= -\underline{x}, -\underline{x}, - - \underline{x}) + \mathrm{if}(0 <= -\underline{y}, -\underline{y}, - - \underline{y}) \gg \mathrm{if}(0 <= -\underline{x} + -\underline{y}, -\underline{x} + -\underline{y}, - - \underline{x} + -\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})$; subLeqLeft $\triangleright \mathrm{if}(0 <= -\underline{x} + -\underline{y}, -\underline{x} + -\underline{y}, - - \underline{x} + -\underline{y}) = \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) \triangleright \mathrm{if}(0 <= -\underline{x} + -\underline{y}, -\underline{x} + -\underline{y}, - - \underline{x} + -\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})\rceil, \mathrm{p}_0, \mathsf{c})]$

[splitNumericalSum(++) $\overset{\mathrm{stmt}}{\to}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon 0 <= \underline{x} \vdash 0 <= \underline{y} \vdash \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})]$

[splitNumericalSum(++) $\overset{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon 0 <= \underline{x} \vdash 0 <= \underline{y} \vdash$ AddEquations(Leq) $\triangleright 0 <= \underline{x} \triangleright 0 <= \underline{y} \gg 0 + 0 <= \underline{x} + \underline{y}$; plus0 $\gg 0 + 0 = 0$; subLeqLeft $\triangleright 0 + 0 = 0 \triangleright 0 + 0 <= \underline{x} + \underline{y} \gg 0 <= \underline{x} + \underline{y}$; NonnegativeNumerical $\triangleright 0 <= \underline{x} + \underline{y} \gg \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) = \underline{x} + \underline{y}$; NonnegativeNumerical $\triangleright 0 <= \underline{x} \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \underline{x}$; NonnegativeNumerical $\triangleright 0 <= \underline{y} \gg \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = \underline{y}$; AddEquations $\triangleright \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \underline{x} \triangleright \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = \underline{y} \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = \underline{x} + \underline{y}$; eqSymmetry $\triangleright \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = \underline{x} + \underline{y} \gg \underline{x} + \underline{y} = \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})$; eqTransitivity $\triangleright \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) = \underline{x} + \underline{y} \triangleright \underline{x} + \underline{y} = \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) = \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})$; eqLeq $\triangleright \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) = \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})\rceil, \mathrm{p}_0, \mathsf{c})]$

[splitNumericalSum(−−) $\overset{\mathrm{stmt}}{\to}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} <= 0 \vdash \underline{y} <= 0 \vdash \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})]$

[splitNumericalSum(−−) $\overset{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} <= 0 \vdash \underline{y} <= 0 \vdash$ NonpositiveNegated $\triangleright \underline{x} <= 0 \gg 0 <= -\underline{x}$; NonpositiveNegated $\triangleright \underline{y} <= 0 \gg 0 <= -\underline{y}$; splitNumericalSum(++) $\triangleright 0 <= -\underline{x} \triangleright 0 <= -\underline{y} \gg \mathrm{if}(0 <= -\underline{x} + -\underline{y}, -\underline{x} + -\underline{y}, - - \underline{x} + -\underline{y}) <= \mathrm{if}(0 <= -\underline{x}, -\underline{x}, - - \underline{x}) + \mathrm{if}(0 <= -\underline{y}, -\underline{y}, - - \underline{y})$; SplitNumericalSumHelper $\triangleright \mathrm{if}(0 <= -\underline{x} + -\underline{y}, -\underline{x} + -\underline{y}, - - \underline{x} + -\underline{y}) <= \mathrm{if}(0 <= -\underline{x}, -\underline{x}, - - \underline{x}) + \mathrm{if}(0 <= -\underline{y}, -\underline{y}, - - \underline{y}) \gg \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})\rceil, \mathrm{p}_0, \mathsf{c})]$

[splitNumericalSum(+ − small) $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon 0 <= \underline{x} \vdash \underline{y} <= 0 \vdash \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \vdash \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x})$]

[splitNumericalSum(+ − small) $\stackrel{\text{proof}}{\rightarrow}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon 0 <= \underline{x} \vdash \underline{y} <= 0 \vdash \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \vdash$ LeqAdditionLeft $\rhd \underline{y} <= 0 \gg \underline{x} + \underline{y} <= \underline{x} + 0$; plus0 $\gg \underline{x} + 0 = \underline{x}$; subLeqRight $\rhd \underline{x} + 0 = \underline{x} \rhd \underline{x} + \underline{y} <= \underline{x} + 0 \gg \underline{x} + \underline{y} <= \underline{x}$; PositiveToRight(Leq)(1term) $\rhd \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \gg 0 <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + -\mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})$; lemma nonpositiveNumerical $\rhd \underline{y} <= 0 \gg \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = -\underline{y}$; EqNegated $\rhd \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = -\underline{y} \gg -\mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = --\underline{y}$; DoubleMinus $\gg --\underline{y} = \underline{y}$; eqTransitivity $\rhd -\mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = --\underline{y} \rhd --\underline{y} = \underline{y} \gg -\mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = \underline{y}$; NonnegativeNumerical $\rhd 0 <= \underline{x} \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \underline{x}$; AddEquations $\rhd \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \underline{x} \rhd -\mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = \underline{y} \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + -\mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = \underline{x} + \underline{y}$; subLeqRight $\rhd \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + -\mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = \underline{x} + \underline{y} \rhd 0 <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + -\mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg 0 <= \underline{x} + \underline{y}$; NonnegativeNumerical $\rhd 0 <= \underline{x} + \underline{y} \gg \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) = \underline{x} + \underline{y}$; eqSymmetry $\rhd \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) = \underline{x} + \underline{y} \gg \underline{x} + \underline{y} = \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y})$; eqSymmetry $\rhd \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \underline{x} \gg \underline{x} = \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x})$; subLeqLeft $\rhd \underline{x} + \underline{y} = \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) \rhd \underline{x} + \underline{y} <= \underline{x} \gg \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) <= \underline{x}$; subLeqRight $\rhd \underline{x} = \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \rhd \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) <= \underline{x} \gg \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x})\rceil, \mathrm{p}_0, \mathsf{c})$]

[splitNumericalSum(+ − big) $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon 0 <= \underline{x} \vdash \underline{y} <= 0 \vdash \dot{\neg}\,\mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) <= \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \Rightarrow \dot{\neg}\dot{\neg}\,\mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \vdash \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) <= \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})$]

[splitNumericalSum(+ − big) $\stackrel{\text{proof}}{\rightarrow}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon 0 <= \underline{x} \vdash \underline{y} <= 0 \vdash \dot{\neg}\,\mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) <= \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \Rightarrow \dot{\neg}\dot{\neg}\,\mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \vdash$ NonnegativeNegated $\rhd 0 <= \underline{x} \gg -\underline{x} <= 0$; NonpositiveNegated $\rhd \underline{y} <= 0 \gg 0 <= -\underline{y}$; SignNumerical $\gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x})$; SubLessLeft $\rhd \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) \rhd \dot{\neg}\,\mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) <= \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \Rightarrow \dot{\neg}\dot{\neg}\,\mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \dot{\neg}\,\mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) <= \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \Rightarrow \dot{\neg}\dot{\neg}\,\mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})$; SignNumerical $\gg \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y})$; SubLessRight $\rhd \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y}) \rhd \dot{\neg}\,\mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) <= \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \Rightarrow \dot{\neg}\dot{\neg}\,\mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \dot{\neg}\,\mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) <= \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y}) \Rightarrow \dot{\neg}\dot{\neg}\,\mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) = \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y})$; LessLeq $\rhd \dot{\neg}\,\mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) <= \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y}) \Rightarrow \dot{\neg}\dot{\neg}\,\mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) = \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y}) \gg \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) <= \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y})$; splitNumericalSum(+−small) $\rhd 0 <= -\underline{y} \rhd -\underline{x} <= 0 \rhd \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) <= \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y}) \gg \mathrm{if}(0 <= -\underline{y} + -\underline{x}, -\underline{y} + -\underline{x}, --\underline{y} + -\underline{x}) <= \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y})$; SignNumerical $\gg \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) = \mathrm{if}(0 <= -\underline{x} + \underline{y}, -\underline{x} + \underline{y}, --\underline{x} + \underline{y})$; $-\mathrm{x} - \mathrm{y} = -(\mathrm{x} + \mathrm{y}) \gg -\underline{x} + -\underline{y} = -\underline{x} + \underline{y}$; plusCommutativity $\gg -\underline{x} + -\underline{y} = -\underline{y} + -\underline{x}$; Equality $\rhd -\underline{x} + -\underline{y} = -\underline{x} + \underline{y} \rhd -\underline{x} + -\underline{y} = -\underline{y} + -\underline{x} \gg -\underline{x} + \underline{y} =$

$-\underline{y}+-\underline{x}$; SameNumerical $\rhd -\underline{x}+\underline{y} = -\underline{y}+-\underline{x} \gg \mathrm{if}(0 <= -\underline{x}+\underline{y}, -\underline{x}+\underline{y}, --\underline{x}+\underline{y}) = \mathrm{if}(0 <= -\underline{y}+-\underline{x}, -\underline{y}+-\underline{x}, --\underline{y}+-\underline{x})$; eqTransitivity $\rhd \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) = \mathrm{if}(0 <= -\underline{x}+\underline{y}, -\underline{x}+\underline{y}, --\underline{x}+\underline{y}) \rhd \mathrm{if}(0 <= -\underline{x}+\underline{y}, -\underline{x}+\underline{y}, --\underline{x}+\underline{y}) = \mathrm{if}(0 <= -\underline{y}+-\underline{x}, -\underline{y}+-\underline{x}, --\underline{y}+-\underline{x}) \gg \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) = \mathrm{if}(0 <= -\underline{y}+-\underline{x}, -\underline{y}+-\underline{x}, --\underline{y}+-\underline{x})$; eqSymmetry $\rhd \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) = \mathrm{if}(0 <= -\underline{y}+-\underline{x}, -\underline{y}+-\underline{x}, --\underline{y}+-\underline{x}) \gg \mathrm{if}(0 <= -\underline{y}+-\underline{x}, -\underline{y}+-\underline{x}, --\underline{y}+-\underline{x}) = \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y})$; eqSymmetry $\rhd \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y}) \gg \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y}) = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})$; subLeqLeft $\rhd \mathrm{if}(0 <= -\underline{y}+-\underline{x}, -\underline{y}+-\underline{x}, --\underline{y}+-\underline{x}) = \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) \rhd \mathrm{if}(0 <= -\underline{y}+-\underline{x}, -\underline{y}+-\underline{x}, --\underline{y}+-\underline{x}) <= \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y}) \gg \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y})$; subLeqRight $\rhd \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y}) = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \rhd \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y}) \gg \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})\rceil, p_0, \mathsf{c})]$

[splitNumericalSum(+−) $\overset{\text{stmt}}{\to}$ ZFsub $\vdash \forall\underline{x}\colon \forall\underline{y}\colon 0 <= \underline{x} \vdash \underline{y} <= 0 \vdash \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})$]

[splitNumericalSum(+−) $\overset{\text{proof}}{\to} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil$ZFsub $\vdash \forall\underline{x}\colon \forall\underline{y}\colon \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \vdash 0 <= \underline{x} \vdash \underline{y} <= 0 \vdash$ splitNumericalSum(+ − small) $\rhd 0 <= \underline{x} \rhd \underline{y} <= 0 \rhd \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \gg \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}); 0 <= |\mathrm{x}| \gg 0 <= \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})$; LeqAdditionLeft $\rhd 0 <= \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + 0 <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})$; plus0 $\gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + 0 = \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x})$; subLeqLeft $\rhd \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + 0 = \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \rhd \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + 0 <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})$; leqTransitivity $\rhd \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \rhd \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \forall\underline{x}\colon \forall\underline{y}\colon \dot{\neg}\, \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \vdash 0 <= \underline{x} \vdash \underline{y} <= 0 \vdash$ ToLess $\rhd \dot{\neg}\, \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \gg \dot{\neg}\, \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) <= \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \Rightarrow \dot{\neg}\dot{\neg}\, \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})$; splitNumericalSum(+ − big) $\rhd 0 <= \underline{x} \rhd \underline{y} <= 0 \rhd \dot{\neg}\, \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) <= \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \Rightarrow \dot{\neg}\dot{\neg}\, \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); 0 <= |\mathrm{x}| \gg 0 <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x})$; leqAddition $\rhd 0 <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \gg 0 + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})$; plus0Left $\gg 0 + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})$; subLeqLeft $\rhd 0 + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \rhd 0 + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})$; leqTransitivity $\rhd \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \rhd \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \forall\underline{x}\colon \forall\underline{y}\colon$ Ded $\rhd \forall\underline{x}\colon \forall\underline{y}\colon \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \vdash 0 <= \underline{x} \vdash \underline{y} <= 0 \vdash \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \Rightarrow 0 <= \underline{x} \Rightarrow \underline{y} <= 0 \Rightarrow \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})$; Ded $\rhd \forall\underline{x}\colon \forall\underline{y}\colon \dot{\neg}\, \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \vdash 0 <= \underline{x} \vdash \underline{y} <= 0 \vdash \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <=$

$\underline{y}, \underline{y}, -\underline{y}) \gg \dot{\neg}\, \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \Rightarrow 0 <= \underline{x} \Rightarrow \underline{y} <= 0 \Rightarrow \text{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); 0 <= \underline{x} \vdash \underline{y} <= 0 \vdash$ FromNegations $\rhd \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \Rightarrow 0 <= \underline{x} \Rightarrow \underline{y} <= 0 \Rightarrow \text{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \rhd \dot{\neg}\, \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \Rightarrow 0 <= \underline{x} \Rightarrow \underline{y} <= 0 \Rightarrow \text{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg 0 <= \underline{x} \Rightarrow \underline{y} <= 0 \Rightarrow \text{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y});$ MP2 $\rhd 0 <= \underline{x} \Rightarrow \underline{y} <= 0 \Rightarrow \text{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \rhd 0 <= \underline{x} \rhd \underline{y} <= 0 \gg \text{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y})\rceil, p_0, \mathsf{c})]$

[splitNumericalSum(−+) $\overset{\text{stmt}}{\to}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} <= 0 \vdash 0 <= \underline{y} \vdash \text{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y})]$

[splitNumericalSum(−+) $\overset{\text{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} <= 0 \vdash 0 <= \underline{y} \vdash$ NonpositiveNegated $\rhd \underline{x} <= 0 \gg 0 <= -\underline{x};$ NonnegativeNegated $\rhd 0 <= \underline{y} \gg -\underline{y} <= 0;$ splitNumericalSum(+−) $\rhd 0 <= -\underline{x} \rhd -\underline{y} <= 0 \gg \text{if}(0 <= -\underline{x} + -\underline{y}, -\underline{x} + -\underline{y}, - - \underline{x} + -\underline{y}) <= \text{if}(0 <= -\underline{x}, -\underline{x}, - - \underline{x}) + \text{if}(0 <= -\underline{y}, -\underline{y}, - - \underline{y});$ SplitNumericalSumHelper $\rhd \text{if}(0 <= -\underline{x} + -\underline{y}, -\underline{x} + -\underline{y}, - - \underline{x} + -\underline{y}) <= \text{if}(0 <= -\underline{x}, -\underline{x}, - - \underline{x}) + \text{if}(0 <= -\underline{y}, -\underline{y}, - - \underline{y}) \gg \text{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y})\rceil, p_0, \mathsf{c})]$

[splitNumericalSum $\overset{\text{stmt}}{\to}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \text{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y})]$

[splitNumericalSum $\overset{\text{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon 0 <= \underline{x} \vdash 0 <= \underline{y} \vdash$ splitNumericalSum(++) $\rhd 0 <= \underline{x} \rhd 0 <= \underline{y} \gg \text{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \forall \underline{x}\colon \forall \underline{y}\colon 0 <= \underline{x} \vdash \underline{y} <= 0 \vdash$ splitNumericalSum(+−) $\rhd 0 <= \underline{x} \rhd \underline{y} <= 0 \gg \text{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} <= 0 \vdash 0 <= \underline{y} \vdash$ splitNumericalSum(−+) $\rhd \underline{x} <= 0 \rhd 0 <= \underline{y} \gg \text{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} <= 0 \vdash \underline{y} <= 0 \vdash$ splitNumericalSum(−−) $\rhd \underline{x} <= 0 \rhd \underline{y} <= 0 \gg \text{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \forall \underline{x}\colon \forall \underline{y}\colon$ Ded $\rhd \forall \underline{x}\colon \forall \underline{y}\colon 0 <= \underline{x} \vdash 0 <= \underline{y} \vdash \text{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg 0 <= \underline{x} \Rightarrow 0 <= \underline{y} \Rightarrow \text{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y});$ Ded $\rhd \forall \underline{x}\colon \forall \underline{y}\colon 0 <= \underline{x} \vdash \underline{y} <= 0 \vdash \text{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg 0 <= \underline{x} \Rightarrow \underline{y} <= 0 \Rightarrow \text{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y});$ Ded $\rhd \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} <= 0 \vdash 0 <= \underline{y} \vdash \text{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \underline{x} <= 0 \Rightarrow 0 <= \underline{y} \Rightarrow \text{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y});$ Ded $\rhd \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} <= 0 \vdash \underline{y} <= 0 \vdash \text{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \underline{x} <= 0 \Rightarrow \underline{y} <= 0 \Rightarrow \text{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y});$ FromLeqGeq $\rhd 0 <= \underline{x} \Rightarrow 0 <= \underline{y} \Rightarrow \text{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \rhd \underline{x} <= 0 \Rightarrow 0 <= \underline{y} \Rightarrow \text{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg 0 <= \underline{y} \Rightarrow \text{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \text{if}(0 <= \underline{y}, \underline{y}, -\underline{y});$ FromLeqGeq $\rhd 0 <= \underline{x} \Rightarrow \underline{y} <= 0 \Rightarrow \text{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <=$

$\mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \rhd \underline{x} <= 0 \Rightarrow \underline{y} <= 0 \Rightarrow \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \underline{y} <= 0 \Rightarrow \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \mathrm{FromLeqGeq} \rhd 0 <= \underline{y} \Rightarrow \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \rhd \underline{y} <= 0 \Rightarrow \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})\rceil, p_0, \mathsf{c})]$

$[\mathrm{insertMiddleTerm(Numerical)} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}+-\underline{z}, \underline{x}+-\underline{z}, -\underline{x}+-\underline{z}) + \mathrm{if}(0 <= \underline{z}+\underline{y}, \underline{z}+\underline{y}, -\underline{z}+\underline{y})]$

$[\mathrm{insertMiddleTerm(Numerical)} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \mathrm{splitNumericalSum} \gg \mathrm{if}(0 <= \underline{x}+-\underline{z}+\underline{z}+\underline{y}, \underline{x}+-\underline{z}+\underline{z}+\underline{y}, -\underline{x}+-\underline{z}+\underline{z}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}+-\underline{z}, \underline{x}+-\underline{z}, -\underline{x}+-\underline{z}) + \mathrm{if}(0 <= \underline{z}+\underline{y}, \underline{z}+\underline{y}, -\underline{z}+\underline{y}); \mathrm{insertMiddleTerm(Sum)} \gg \underline{x}+\underline{y} = \underline{x}+-\underline{z}+\underline{z}+\underline{y}; \mathrm{SameNumerical} \rhd \underline{x}+\underline{y} = \underline{x}+-\underline{z}+\underline{z}+\underline{y} \gg \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) = \mathrm{if}(0 <= \underline{x}+-\underline{z}+\underline{z}+\underline{y}, \underline{x}+-\underline{z}+\underline{z}+\underline{y}, -\underline{x}+-\underline{z}+\underline{z}+\underline{y}); \mathrm{eqSymmetry} \rhd \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) = \mathrm{if}(0 <= \underline{x}+-\underline{z}+\underline{z}+\underline{y}, \underline{x}+-\underline{z}+\underline{z}+\underline{y}, -\underline{x}+-\underline{z}+\underline{z}+\underline{y}) \gg \mathrm{if}(0 <= \underline{x}+-\underline{z}+\underline{z}+\underline{y}, \underline{x}+-\underline{z}+\underline{z}+\underline{y}, -\underline{x}+-\underline{z}+\underline{z}+\underline{y}) = \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}); \mathrm{subLeqLeft} \rhd \mathrm{if}(0 <= \underline{x}+-\underline{z}+\underline{z}+\underline{y}, \underline{x}+-\underline{z}+\underline{z}+\underline{y}, -\underline{x}+-\underline{z}+\underline{z}+\underline{y}) = \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) \rhd \mathrm{if}(0 <= \underline{x}+-\underline{z}+\underline{z}+\underline{y}, \underline{x}+-\underline{z}+\underline{z}+\underline{y}, -\underline{x}+-\underline{z}+\underline{z}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}+-\underline{z}, \underline{x}+-\underline{z}, -\underline{x}+-\underline{z}) + \mathrm{if}(0 <= \underline{z}+\underline{y}, \underline{z}+\underline{y}, -\underline{z}+\underline{y}) \gg \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}+-\underline{z}, \underline{x}+-\underline{z}, -\underline{x}+-\underline{z}) + \mathrm{if}(0 <= \underline{z}+\underline{y}, \underline{z}+\underline{y}, -\underline{z}+\underline{y})\rceil, p_0, \mathsf{c})]$

(*** REGNESTYKKER ***)

$[\mathrm{x} + \mathrm{y} = \mathrm{zBackwards} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} + \underline{y} = \underline{z} \vdash \underline{z} = \underline{y} + \underline{x}]$

$[\mathrm{x} + \mathrm{y} = \mathrm{zBackwards} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} + \underline{y} = \underline{z} \vdash \mathrm{plusCommutativity} \gg \underline{x} + \underline{y} = \underline{y} + \underline{x}; \mathrm{Equality} \rhd \underline{x} + \underline{y} = \underline{z} \gg \underline{z} = \underline{y} + \underline{x}\rceil, p_0, \mathsf{c})]$

$[\mathrm{x} * \mathrm{y} = \mathrm{zBackwards} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} * \underline{y} = \underline{z} \vdash \underline{z} = \underline{y} * \underline{x}]$

$[\mathrm{x} * \mathrm{y} = \mathrm{zBackwards} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} * \underline{y} = \underline{z} \vdash \mathrm{timesCommutativity} \gg \underline{x} * \underline{y} = \underline{y} * \underline{x}; \mathrm{Equality} \rhd \underline{x} * \underline{y} = \underline{z} \gg \underline{z} = \underline{y} * \underline{x}\rceil, p_0, \mathsf{c})]$

$[\mathrm{x} = \mathrm{x} + (\mathrm{y} - \mathrm{y}) \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} = \underline{x} + \underline{y} + -\underline{y}]$

$[\mathrm{x} = \mathrm{x} + (\mathrm{y} - \mathrm{y}) \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \mathrm{plus0} \gg \underline{x} + 0 = \underline{x}; \mathrm{Negative} \gg \underline{y} + -\underline{y} = 0; \mathrm{eqSymmetry} \rhd \underline{y} + -\underline{y} = 0 \gg 0 = \underline{y} + -\underline{y}; \mathrm{lemma\ eqAdditionLeft} \rhd 0 = \underline{y} + -\underline{y} \gg \underline{x} + 0 = \underline{x} + \underline{y} + -\underline{y}; \mathrm{Equality} \rhd \underline{x} + 0 = \underline{x} \rhd \underline{x} + 0 = \underline{x} + \underline{y} + -\underline{y} \gg \underline{x} = \underline{x} + \underline{y} + -\underline{y}\rceil, p_0, \mathsf{c})]$

$[\mathrm{x} = \mathrm{x} + \mathrm{y} - \mathrm{y} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} = \underline{x} + \underline{y} + -\underline{y}]$

$[\mathrm{x} = \mathrm{x} + \mathrm{y} - \mathrm{y} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \mathrm{x} = \mathrm{x} + (\mathrm{y} - \mathrm{y}) \gg \underline{x} = \underline{x} + \underline{y} + -\underline{y}; \mathrm{plusAssociativity} \gg \underline{x} + \underline{y} + -\underline{y} = \underline{x} + \underline{y} + -\underline{y}; \mathrm{eqSymmetry} \rhd \underline{x} + \underline{y} + -\underline{y} = \underline{x} + \underline{y} + -\underline{y} \gg \underline{x} + \underline{y} + -\underline{y} = \underline{x} + \underline{y} + -\underline{y}; \mathrm{eqTransitivity} \rhd \underline{x} = \underline{x} + \underline{y} + -\underline{y} \rhd \underline{x} + \underline{y} + -\underline{y} = \underline{x} + \underline{y} + -\underline{y} \gg \underline{x} = \underline{x} + \underline{y} + -\underline{y}\rceil, p_0, \mathsf{c})]$

$[\stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \dot{\neg}\, \underline{y} = 0 \vdash \underline{x} = \underline{x} * \underline{y} * \mathrm{rec}\underline{y}]$

$[\stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \dot{\neg}\, \underline{y} = 0 \vdash \mathrm{times1} \gg \underline{x} * 1 = \underline{x}; \mathrm{Reciprocal} \rhd \dot{\neg}\, \underline{y} = 0 \gg \underline{y} * \mathrm{rec}\underline{y} = 1; \mathrm{Three2threeFactors} \rhd \underline{y} * \mathrm{rec}\underline{y} = 1 \gg \underline{x} * \underline{y} * \mathrm{rec}\underline{y} = \underline{x} * 1; \mathrm{eqTransitivity} \rhd \underline{x} * \underline{y} * \mathrm{rec}\underline{y} = \underline{x} * 1 \rhd \underline{x} * 1 = \underline{x} \gg \underline{x} * \underline{y} * \mathrm{rec}\underline{y} =$

$\underline{x}; \text{eqSymmetry} \rhd \underline{x} * \underline{y} * \text{rec}\underline{y} = \underline{x} \gg \underline{x} = \underline{x} * \underline{y} * \text{rec}\underline{y}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\text{insertMiddleTerm(Sum)} \overset{\text{stmt}}{\to} \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} + \underline{y} = \underline{x} + -\underline{z} + \underline{z} + \underline{y}]$

$[\text{insertMiddleTerm(Sum)} \overset{\text{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \mathrm{x} = \mathrm{x} + \mathrm{y} - \mathrm{y} \gg \underline{x} = \underline{x} + \underline{z} + -\underline{z}; \text{Three2threeTerms} \gg \underline{x} + \underline{z} + -\underline{z} = \underline{x} + -\underline{z} + \underline{z}; \text{eqTransitivity} \rhd \underline{x} = \underline{x} + \underline{z} + -\underline{z} \rhd \underline{x} + \underline{z} + -\underline{z} = \underline{x} + -\underline{z} + \underline{z} \gg \underline{x} = \underline{x} + -\underline{z} + \underline{z}; \text{eqAddition} \rhd \underline{x} = \underline{x} + -\underline{z} + \underline{z} \gg \underline{x} + \underline{y} = \underline{x} + -\underline{z} + \underline{z} + \underline{y}; \text{plusAssociativity} \gg \underline{x} + -\underline{z} + \underline{z} + \underline{y} = \underline{x} + -\underline{z} + \underline{z} + \underline{y}; \text{eqTransitivity} \rhd \underline{x} + \underline{y} = \underline{x} + -\underline{z} + \underline{z} + \underline{y} \rhd \underline{x} + -\underline{z} + \underline{z} + \underline{y} = \underline{x} + -\underline{z} + \underline{z} + \underline{y} \gg \underline{x} + \underline{y} = \underline{x} + -\underline{z} + \underline{z} + \underline{y}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\text{insertMiddleTerm(Difference)} \overset{\text{stmt}}{\to} \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} + -\underline{y} = \underline{x} + \underline{z} + -\underline{y} + \underline{z}]$

$[\text{insertMiddleTerm(Difference)} \overset{\text{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \text{insertMiddleTerm(Sum)} \gg \underline{x} + -\underline{y} = \underline{x} + --\underline{z} + -\underline{z} + -\underline{y}; \text{DoubleMinus} \gg --\underline{z} = \underline{z}; \text{lemma eqAdditionLeft} \rhd --\underline{z} = \underline{z} \gg \underline{x} + --\underline{z} = \underline{x} + \underline{z}; \text{plusCommutativity} \gg -\underline{z} + -\underline{y} = -\underline{y} + -\underline{z}; -\mathrm{x} - \mathrm{y} = -(\mathrm{x} + \mathrm{y}) \gg -\underline{y} + -\underline{z} = -\underline{y} + \underline{z}; \text{eqTransitivity} \rhd -\underline{z} + -\underline{y} = -\underline{y} + -\underline{z} \rhd -\underline{y} + -\underline{z} = -\underline{y} + \underline{z} \gg -\underline{z} + -\underline{y} = -\underline{y} + \underline{z}; \text{AddEquations} \rhd \underline{x} + --\underline{z} = \underline{x} + \underline{z} \rhd -\underline{z} + -\underline{y} = -\underline{y} + \underline{z} \gg \underline{x} + --\underline{z} + -\underline{z} + -\underline{y} = \underline{x} + \underline{z} + -\underline{y} + \underline{z}; \text{eqTransitivity} \rhd \underline{x} + -\underline{y} = \underline{x} + --\underline{z} + -\underline{z} + -\underline{y} \rhd \underline{x} + --\underline{z} + -\underline{z} + -\underline{y} = \underline{x} + \underline{z} + -\underline{y} + \underline{z} \gg \underline{x} + -\underline{y} = \underline{x} + \underline{z} + -\underline{y} + \underline{z}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{x} * 0 + \mathrm{x} = \mathrm{x} \overset{\text{stmt}}{\to} \text{ZFsub} \vdash \forall \underline{x}\colon \underline{x} * 0 + \underline{x} = \underline{x}]$

$[\mathrm{x} * 0 + \mathrm{x} = \mathrm{x} \overset{\text{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \text{times1} \gg \underline{x} * 1 = \underline{x}; \text{eqSymmetry} \rhd \underline{x} * 1 = \underline{x} \gg \underline{x} = \underline{x} * 1; \text{lemma eqAdditionLeft} \rhd \underline{x} = \underline{x} * 1 \gg \underline{x} * 0 + \underline{x} = \underline{x} * 0 + \underline{x} * 1; \text{Distribution} \gg \underline{x} * 0 + 1 = \underline{x} * 0 + \underline{x} * 1; \text{eqSymmetry} \rhd \underline{x} * 0 + 1 = \underline{x} * 0 + \underline{x} * 1 \gg \underline{x} * 0 + \underline{x} * 1 = \underline{x} * 0 + 1; \text{plus0Left} \gg 0 + 1 = 1; \text{EqMultiplicationLeft} \rhd 0 + 1 = 1 \gg \underline{x} * 0 + 1 = \underline{x} * 1; \text{eqTransitivity5} \rhd \underline{x} * 0 + \underline{x} = \underline{x} * 0 + \underline{x} * 1 \rhd \underline{x} * 0 + \underline{x} * 1 = \underline{x} * 0 + 1 \rhd \underline{x} * 0 + 1 = \underline{x} * 1 \rhd \underline{x} * 1 = \underline{x} \gg \underline{x} * 0 + \underline{x} = \underline{x}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{x} * 0 = 0 \overset{\text{stmt}}{\to} \text{ZFsub} \vdash \forall \underline{x}\colon \underline{x} * 0 = 0]$

$[\mathrm{x} * 0 = 0 \overset{\text{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \mathrm{x} = \mathrm{x} + (\mathrm{y} - \mathrm{y}) \gg \underline{x} * 0 = \underline{x} * 0 + \underline{x} + -\underline{x}; \text{plusAssociativity} \gg \underline{x} * 0 + \underline{x} + -\underline{x} = \underline{x} * 0 + \underline{x} + -\underline{x}; \text{eqSymmetry} \rhd \underline{x} * 0 + \underline{x} + -\underline{x} = \underline{x} * 0 + \underline{x} + -\underline{x} \gg \underline{x} * 0 + \underline{x} + -\underline{x} = \underline{x} * 0 + \underline{x} + -\underline{x}; \mathrm{x} * 0 + \mathrm{x} = \mathrm{x} \gg \underline{x} * 0 + \underline{x} = \underline{x}; \text{eqAddition} \rhd \underline{x} * 0 + \underline{x} = \underline{x} \gg \underline{x} * 0 + \underline{x} + -\underline{x} = \underline{x} + -\underline{x}; \text{Negative} \gg \underline{x} + -\underline{x} = 0; \text{eqTransitivity5} \rhd \underline{x} * 0 = \underline{x} * 0 + \underline{x} + -\underline{x} \rhd \underline{x} * 0 + \underline{x} + -\underline{x} = \underline{x} * 0 + \underline{x} + -\underline{x} \rhd \underline{x} * 0 + \underline{x} + -\underline{x} = \underline{x} + -\underline{x} \rhd \underline{x} + -\underline{x} = 0 \gg \underline{x} * 0 = 0\rceil, \mathrm{p}_0, \mathsf{c})]$

$[(-1) * (-1) + (-1) * 1 = 0 \overset{\text{stmt}}{\to} \text{ZFsub} \vdash -1 * -1 + -1 * 1 = 0]$

$[(-1) * (-1) + (-1) * 1 = 0 \overset{\text{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \text{DistributionOut} \gg -1 * -1 + -1 * 1 = -1 * -1 + 1; \text{Negative} \gg 1 + -1 = 0; \text{plusCommutativity} \gg -1 + 1 = 1 + -1; \text{eqTransitivity} \rhd -1 + 1 = 1 + -1 \rhd 1 + -1 = 0 \gg -1 + 1 = 0; \text{EqMultiplicationLeft} \rhd -1 + 1 = 0 \gg -1 * -1 + 1 = -1 * 0; \mathrm{x} * 0 = 0 \gg -1 * 0 = 0; \text{eqTransitivity4} \rhd -1 * -1 + -1 * 1 = -1 * -1 + 1 \rhd -1 * -1 + 1 = -1 * 0 \rhd -1 * 0 = 0 \gg -1 * -1 + -1 * 1 = 0\rceil, \mathrm{p}_0, \mathsf{c})]$

$[(-1) * (-1) = 1 \overset{\text{stmt}}{\to} \text{ZFsub} \vdash -1 * -1 = 1]$

$[(-1) * (-1) = 1 \overset{\text{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \mathrm{x} = \mathrm{x} + (\mathrm{y} - \mathrm{y}) \gg -1 * -1 = -1 * -1 + 1 + -1; \text{times1} \gg -1 * 1 = -1; \text{eqSymmetry} \rhd -1 * 1 = -1 \gg -1 = -1 * 1; \text{lemma eqAdditionLeft} \rhd -1 = -1 * 1 \gg 1 + -1 =$

$1 + -1 * 1$; lemma eqAdditionLeft $\triangleright 1 + -1 = 1 + -1 * 1 \gg -1 * -1 + 1 + -1 =$
$-1 * -1 + 1 + -1 * 1$; plusCommutativity $\gg 1 + -1 * 1 =$
$-1*1+1$; lemma eqAdditionLeft $\triangleright 1+-1*1 = -1*1+1 \gg -1*-1+1+-1*1 =$
$-1 * -1 + -1 * 1 + 1$; plusAssociativity $\gg -1 * -1 + -1 * 1 + 1 =$
$-1*-1+-1*1+1$; eqSymmetry $\triangleright -1*-1+-1*1+1 = -1*-1+-1*1+1 \gg$
$-1 * -1 + -1 * 1 + 1 = -1 * -1 + -1 * 1 + 1; (-1) * (-1) + (-1) * 1 = 0 \gg$
$-1 * -1 + -1 * 1 = 0$; eqAddition $\triangleright -1 * -1 + -1 * 1 = 0 \gg$
$-1 * -1 + -1 * 1 + 1 = 0 + 1$; plus0Left $\gg 0 + 1 = 1$; eqTransitivity5 $\triangleright -1 * -1 =$
$-1*-1+1+-1 \triangleright -1*-1+1+-1 = -1*-1+1+-1*1 \triangleright -1*-1+1+-1*1 =$
$-1 * -1 + -1 * 1 + 1 \triangleright -1 * -1 + -1 * 1 + 1 = -1 * -1 + -1 * 1 + 1 \gg$
$-1 * -1 = -1 * -1 + -1 * 1 + 1$; eqTransitivity4 $\triangleright -1 * -1 =$
$-1*-1+-1*1+1 \triangleright -1*-1+-1*1+1 = 0+1 \triangleright 0+1 = 1 \gg -1*-1 = 1\rceil, p_0, \mathsf{c})]$

[$0 <$ 1Helper $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash 1 <= 0 \Rightarrow 0 <= 1$]

[$0 <$ 1Helper $\stackrel{\text{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil$ZFsub $\vdash 1 <= 0 \vdash$ leqAddition $\triangleright 1 <= 0 \gg$
$1+-1 <= 0+-1$; Negative $\gg 1+-1 = 0$; subLeqLeft $\triangleright 1+-1 = 0 \triangleright 1+-1 <=$
$0 + -1 \gg 0 <= 0 + -1$; plus0Left $\gg 0 + -1 = -1$; subLeqRight $\triangleright 0 + -1 =$
$-1 \triangleright 0 <= 0 + -1 \gg 0 <= -1$; leqMultiplication $\triangleright 0 <= -1 \triangleright 0 <= -1 \gg$
$0 * -1 <= -1 * -1; x * 0 = 0 \gg -1 * 0 = 0$; timesCommutativity $\gg 0 * -1 =$
$-1 * 0$; eqTransitivity $\triangleright 0 * -1 = -1 * 0 \triangleright -1 * 0 = 0 \gg 0 * -1 =$
0; subLeqLeft $\triangleright 0 * -1 = 0 \triangleright 0 * -1 <= -1 * -1 \gg 0 <=$
$-1 * -1; (-1) * (-1) = 1 \gg -1 * -1 = 1$; subLeqRight $\triangleright -1 * -1 = 1 \triangleright 0 <=$
$-1 * -1 \gg 0 <= 1$; Ded $\triangleright 1 <= 0 \vdash 0 <= 1 \gg 1 <= 0 \Rightarrow 0 <= 1\rceil, p_0, \mathsf{c})]$

[$0 < 1 \stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \dot{\neg} 0 <= 1 \Rightarrow \dot{\neg}\dot{\neg} 0 = 1$]

[$0 < 1 \stackrel{\text{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil$ZFsub $\vdash$ leqTotality $\gg \dot{\neg} 0 <= 1 \Rightarrow 1 <=$
0; AutoImply $\gg 0 <= 1 \Rightarrow 0 <= 1$; $0 <$ 1Helper $\gg 1 <= 0 \Rightarrow 0 <=$
1; FromDisjuncts $\triangleright \dot{\neg} 0 <= 1 \Rightarrow 1 <= 0 \triangleright 0 <= 1 \Rightarrow 0 <= 1 \triangleright 1 <= 0 \Rightarrow 0 <=$
$1 \gg 0 <= 1$; 0not1 $\gg \dot{\neg} 0 = 1$; JoinConjuncts $\triangleright 0 <= 1 \triangleright \dot{\neg} 0 = 1 \gg \dot{\neg} 0 <=$
$1 \Rightarrow \dot{\neg}\dot{\neg} 0 = 1\rceil, p_0, \mathsf{c})]$

[$0 < 2 \stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \dot{\neg} 0 <= 1 + 1 \Rightarrow \dot{\neg}\dot{\neg} 0 = 1 + 1$]

[$0 < 2 \stackrel{\text{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil$ZFsub $\vdash 0 < 1 \gg \dot{\neg} 0 <= 1 \Rightarrow \dot{\neg}\dot{\neg} 0 =$
1; LessAddition $\triangleright \dot{\neg} 0 <= 1 \Rightarrow \dot{\neg}\dot{\neg} 0 = 1 \gg \dot{\neg} 0 + 1 <= 1 + 1 \Rightarrow \dot{\neg}\dot{\neg} 0 + 1 =$
$1 + 1$; plus0Left $\gg 0 + 1 = 1$; SubLessLeft $\triangleright 0 + 1 = 1 \triangleright \dot{\neg} 0 + 1 <= 1 + 1 \Rightarrow$
$\dot{\neg}\dot{\neg} 0+1 = 1+1 \gg \dot{\neg} 1 <= 1+1 \Rightarrow \dot{\neg}\dot{\neg} 1 = 1+1$; LessTransitivity $\triangleright \dot{\neg} 0 <= 1 \Rightarrow$
$\dot{\neg}\dot{\neg} 0 = 1 \triangleright \dot{\neg} 1 <= 1+1 \Rightarrow \dot{\neg}\dot{\neg} 1 = 1+1 \gg \dot{\neg} 0 <= 1+1 \Rightarrow \dot{\neg}\dot{\neg} 0 = 1+1\rceil, p_0, \mathsf{c})]$

[$0 < 1/2 \stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \dot{\neg} 0 <=$ rec1 $+ 1 \Rightarrow \dot{\neg}\dot{\neg} 0 =$ rec1 $+ 1$]

[$0 < 1/2 \stackrel{\text{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil$ZFsub $\vdash 0 < 2 \gg \dot{\neg} 0 <= 1 + 1 \Rightarrow \dot{\neg}\dot{\neg} 0 =$
$1 + 1$; FirstConjunct $\triangleright \dot{\neg} 0 <= 1 + 1 \Rightarrow \dot{\neg}\dot{\neg} 0 = 1 + 1 \gg 0 <=$
$1 + 1$; SecondConjunct $\triangleright \dot{\neg} 0 <= 1 + 1 \Rightarrow \dot{\neg}\dot{\neg} 0 = 1 + 1 \gg \dot{\neg} 0 =$
$1 + 1$; NeqSymmetry $\triangleright \dot{\neg} 0 = 1 + 1 \gg \dot{\neg} 1 + 1 = 0; 0 < 1 \gg \dot{\neg} 0 <= 1 \Rightarrow \dot{\neg}\dot{\neg} 0 =$
$1; x * 0 = 0 \gg 1 + 1 * 0 = 0; x * y =$ zBackwards $\triangleright 1 + 1 * 0 = 0 \gg 0 =$
$0 * 1 + 1$; SubLessLeft $\triangleright 0 = 0 * 1 + 1 \triangleright \dot{\neg} 0 <= 1 \Rightarrow \dot{\neg}\dot{\neg} 0 = 1 \gg \dot{\neg} 0 * 1 + 1 <=$
$1 \Rightarrow \dot{\neg}\dot{\neg} 0 * 1 + 1 = 1$; Reciprocal $\triangleright \dot{\neg} 1 + 1 = 0 \gg 1 + 1 *$ rec1 $+ 1 = 1; x * y =$
zBackwards $\triangleright 1 + 1 *$ rec1 $+ 1 = 1 \gg 1 =$ rec1 $+ 1 * 1 + 1$; SubLessRight $\triangleright 1 =$
rec1 $+ 1 * 1 + 1 \triangleright \dot{\neg} 0 * 1 + 1 <= 1 \Rightarrow \dot{\neg}\dot{\neg} 0 * 1 + 1 = 1 \gg \dot{\neg} 0 * 1 + 1 <=$

$\mathrm{rec1} + 1 * 1 + 1 \Rightarrow \dot{\neg}\dot{\neg} 0 * 1 + 1 = \mathrm{rec1} + 1 * 1 + 1; \mathrm{LessDivision} \rhd 0 <= 1 + 1 \rhd \dot{\neg} 0 * 1 + 1 <= \mathrm{rec1} + 1 * 1 + 1 \Rightarrow \dot{\neg}\dot{\neg} 0 * 1 + 1 = \mathrm{rec1} + 1 * 1 + 1 \gg \dot{\neg} 0 <= \mathrm{rec1} + 1 \Rightarrow \dot{\neg}\dot{\neg} 0 = \mathrm{rec1} + 1\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{TwoWholes} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{x}\colon \underline{x} + \underline{x} = 1 + 1 * \underline{x}]$

$[\mathrm{TwoWholes} \overset{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x}\colon \mathrm{times1} \gg \underline{x} * 1 = \underline{x}; \mathrm{eqSymmetry} \gg \underline{x} = \underline{x} * 1; \mathrm{lemma\ eqAdditionLeft} \rhd \underline{x} = \underline{x} * 1 \gg \underline{x} + \underline{x} = \underline{x} + \underline{x} * 1; \mathrm{eqAddition} \rhd \underline{x} = \underline{x} * 1 \gg \underline{x} + \underline{x} * 1 = \underline{x} * 1 + \underline{x} * 1; \mathrm{eqTransitivity} \rhd \underline{x} + \underline{x} = \underline{x} + \underline{x} * 1 \rhd \underline{x} + \underline{x} * 1 = \underline{x} * 1 + \underline{x} * 1 \gg \underline{x} + \underline{x} = \underline{x} * 1 + \underline{x} * 1; \mathrm{DistributionOut} \gg \underline{x} * 1 + \underline{x} * 1 = \underline{x} * 1 + 1; \mathrm{Repetition} \rhd \underline{x} * 1 + \underline{x} * 1 = \underline{x} * 1 + 1 \gg \underline{x} * 1 + \underline{x} * 1 = \underline{x} * 1 + 1; \mathrm{timesCommutativity} \gg \underline{x} * 1 + 1 = 1 + 1 * \underline{x}; \mathrm{eqTransitivity4} \rhd \underline{x} + \underline{x} = \underline{x} * 1 + \underline{x} * 1 \rhd \underline{x} * 1 + \underline{x} * 1 = \underline{x} * 1 + 1 \rhd \underline{x} * 1 + 1 = 1 + 1 * \underline{x} \gg \underline{x} + \underline{x} = 1 + 1 * \underline{x}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{TwoHalves} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{x}\colon \mathrm{rec1} + 1 * \underline{x} + \mathrm{rec1} + 1 * \underline{x} = \underline{x}]$

$[\mathrm{TwoHalves} \overset{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x}\colon 0 < 2 \gg \dot{\neg} 0 <= 1 + 1 \Rightarrow \dot{\neg}\dot{\neg} 0 = 1 + 1; \mathrm{LessNeq} \rhd \dot{\neg} 0 <= 1 + 1 \Rightarrow \dot{\neg}\dot{\neg} 0 = 1 + 1 \gg \dot{\neg} 0 = 1 + 1; \mathrm{NeqSymmetry} \rhd \dot{\neg} 0 = 1 + 1 \gg \dot{\neg} 1 + 1 = 0; \mathrm{TwoWholes} \gg \mathrm{rec1} + 1 * \underline{x} + \mathrm{rec1} + 1 * \underline{x} = 1 + 1 * \mathrm{rec1} + 1 * \underline{x}; \mathrm{timesAssociativity} \gg 1 + 1 * \mathrm{rec1} + 1 * \underline{x} = 1 + 1 * \mathrm{rec1} + 1 * \underline{x}; \mathrm{eqSymmetry} \rhd 1 + 1 * \mathrm{rec1} + 1 * \underline{x} = 1 + 1 * \mathrm{rec1} + 1 * \underline{x} \gg 1 + 1 * \mathrm{rec1} + 1 * \underline{x} = 1 + 1 * \mathrm{rec1} + 1 * \underline{x}; \mathrm{Reciprocal} \rhd \dot{\neg} 1 + 1 = 0 \gg 1 + 1 * \mathrm{rec1} + 1 = 1; \mathrm{eqMultiplication} \rhd 1 + 1 * \mathrm{rec1} + 1 = 1 \gg 1 + 1 * \mathrm{rec1} + 1 * \underline{x} = 1 * \underline{x}; \mathrm{times1Left} \gg 1 * \underline{x} = \underline{x}; \mathrm{eqTransitivity5} \rhd \mathrm{rec1} + 1 * \underline{x} + \mathrm{rec1} + 1 * \underline{x} = 1 + 1 * \mathrm{rec1} + 1 * \underline{x} \rhd 1 + 1 * \mathrm{rec1} + 1 * \underline{x} = 1 + 1 * \mathrm{rec1} + 1 * \underline{x} \rhd 1 + 1 * \mathrm{rec1} + 1 * \underline{x} = 1 * \underline{x} \rhd 1 * \underline{x} = \underline{x} \gg \mathrm{rec1} + 1 * \underline{x} + \mathrm{rec1} + 1 * \underline{x} = \underline{x}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{Times}(-1) \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{x}\colon \underline{x} * -1 = -\underline{x}][\mathrm{Times}(-1) \overset{\mathrm{proof}}{\rightarrow} \mathrm{Rule\ tactic}]$

$[\mathrm{Times}(-1) \overset{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x}\colon \mathrm{Negative} \gg 1 + -1 = 0; \mathrm{plusCommutativity} \gg -1 + 1 = 1 + -1; \mathrm{eqTransitivity} \rhd -1 + 1 = 1 + -1 \rhd 1 + -1 = 0 \gg -1 + 1 = 0; \mathrm{EqMultiplicationLeft} \rhd -1 + 1 = 0 \gg \underline{x} * -1 + 1 = \underline{x} * 0; \mathrm{x} * 0 = 0 \gg \underline{x} * 0 = 0; \mathrm{eqTransitivity} \rhd \underline{x} * -1 + 1 = \underline{x} * 0 \rhd \underline{x} * 0 = 0 \gg \underline{x} * -1 + 1 = 0; \mathrm{Distribution} \gg \underline{x} * -1 + 1 = \underline{x} * -1 + \underline{x} * 1; \mathrm{eqSymmetry} \rhd \underline{x} * -1 + 1 = \underline{x} * -1 + \underline{x} * 1 \gg \underline{x} * -1 + \underline{x} * 1 = \underline{x} * -1 + 1; \mathrm{eqTransitivity} \rhd \underline{x} * -1 + \underline{x} * 1 = \underline{x} * -1 + 1 \rhd \underline{x} * -1 + 1 = 0 \gg \underline{x} * -1 + \underline{x} * 1 = 0; \mathrm{PositiveToRight(Eq)} \rhd \underline{x} * -1 + \underline{x} * 1 = 0 \gg \underline{x} * -1 = 0 + -\underline{x} * 1; \mathrm{plus0Left} \gg 0 + -\underline{x} * 1 = -\underline{x} * 1; \mathrm{eqTransitivity} \rhd \underline{x} * -1 = 0 + -\underline{x} * 1 \rhd 0 + -\underline{x} * 1 = -\underline{x} * 1 \gg \underline{x} * -1 = -\underline{x} * 1; \mathrm{times1} \gg \underline{x} * 1 = \underline{x}; \mathrm{EqNegated} \rhd \underline{x} * 1 = \underline{x} \gg -\underline{x} * 1 = -\underline{x}; \mathrm{eqTransitivity} \rhd \underline{x} * -1 = -\underline{x} * 1 \rhd -\underline{x} * 1 = -\underline{x} \gg \underline{x} * -1 = -\underline{x}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{Times}(-1)\mathrm{Left} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{x}\colon -1 * \underline{x} = -\underline{x}]$

$[\mathrm{Times}(-1)\mathrm{Left} \overset{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x}\colon \mathrm{Times}(-1) \gg \underline{x} * -1 = -\underline{x}; \mathrm{timesCommutativity} \gg -1 * \underline{x} = \underline{x} * -1; \mathrm{eqTransitivity} \rhd -1 * \underline{x} = \underline{x} * -1 \rhd \underline{x} * -1 = -\underline{x} \gg -1 * \underline{x} = -\underline{x}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[-\mathrm{x} - \mathrm{y} = -(\mathrm{x} + \mathrm{y}) \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon -\underline{x} + -\underline{y} = -\underline{x} + \underline{y}]$

$[-\mathrm{x} - \mathrm{y} = -(\mathrm{x} + \mathrm{y}) \overset{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \mathrm{Times}(-1)\mathrm{Left} \gg -1 * \underline{x} = -\underline{x}; \mathrm{Times}(-1)\mathrm{Left} \gg -1 * \underline{y} = -\underline{y}; \mathrm{AddEquations} \rhd -1 * \underline{x} = -\underline{x} \rhd -1 * \underline{y} = -\underline{y} \gg -1 * \underline{x} + -1 * \underline{y} = -\underline{x} + -\underline{y}; \mathrm{eqSymmetry} \rhd -1 * \underline{x} + -1 * \underline{y} = -\underline{x} + -\underline{y} \gg -\underline{x} + -\underline{y} = -1 * \underline{x} + -1 * \underline{y}; \mathrm{DistributionOut} \gg -1 * \underline{x} + -1 * \underline{y} = -1 * \underline{x} + \underline{y}; \mathrm{Times}(-1)\mathrm{Left} \gg$

$-1*\underline{x}+\underline{y} = -\underline{x}+\underline{y}$; eqTransitivity4 $\rhd -\underline{x}+-\underline{y} = -1*\underline{x}+-1*\underline{y} \rhd -1*\underline{x}+-1*\underline{y} = -1*\underline{x}+\underline{y} \rhd -1*\underline{x}+\underline{y} = -\underline{x}+\underline{y} \gg -\underline{x}+-\underline{y} = -\underline{x}+\underline{y}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\text{MinusNegated} \overset{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon -\underline{x}+-\underline{y} = \underline{y}+-\underline{x}]$

$[\text{MinusNegated} \overset{\text{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil\text{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \text{DoubleMinus} \gg --\underline{y} = \underline{y}; \text{eqAddition} \rhd --\underline{y} = \underline{y} \gg --\underline{y}+-\underline{x} = \underline{y}+-\underline{x}; \text{eqSymmetry} \rhd --\underline{y}+-\underline{x} = \underline{y}+-\underline{x} \gg \underline{y}+-\underline{x} = --\underline{y}+-\underline{x}; -\mathrm{x}-\mathrm{y} = -(\mathrm{x}+\mathrm{y}) \gg --\underline{y}+-\underline{x} = --\underline{y}+\underline{x}; \text{plusCommutativity} \gg -\underline{y}+\underline{x} = \underline{x}+-\underline{y}; \text{EqNegated} \rhd -\underline{y}+\underline{x} = \underline{x}+-\underline{y} \gg --\underline{y}+\underline{x} = -\underline{x}+-\underline{y}; \text{eqTransitivity4} \rhd \underline{y}+-\underline{x} = --\underline{y}+-\underline{x} \rhd --\underline{y}+-\underline{x} = --\underline{y}+\underline{x} \rhd --\underline{y}+\underline{x} = -\underline{x}+-\underline{y} \gg \underline{y}+-\underline{x} = -\underline{x}+-\underline{y}; \text{eqSymmetry} \rhd \underline{y}+-\underline{x} = -\underline{x}+-\underline{y} \gg -\underline{x}+-\underline{y} = \underline{y}+-\underline{x}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[-0 = 0 \overset{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash -0 = 0]$

$[-0 = 0 \overset{\text{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil\text{ZFsub} \vdash \text{Negative} \gg 0+-0 = 0; \text{plus0} \gg 0+0 = 0; \text{UniqueNegative} \rhd 0+-0 = 0 \rhd 0+0 = 0 \gg -0 = 0\rceil, \mathrm{p}_0, \mathsf{c})]$
(*** LEQ, nummer 2 af 2 ***)

$[\text{negativeToLeft(Leq)} \overset{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \underline{x} <= \underline{y}+-\underline{z} \vdash \underline{x}+\underline{z} <= \underline{y}]$

$[\text{negativeToLeft(Leq)} \overset{\text{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil\text{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \underline{x} <= \underline{y}+-\underline{z} \vdash \text{leqAddition} \rhd \underline{x} <= \underline{y}+-\underline{z} \gg \underline{x}+\underline{z} <= \underline{y}+-\underline{z}+\underline{z}; \mathrm{x} = \mathrm{x}+\mathrm{y}-\mathrm{y} \gg \underline{y} = \underline{y}+\underline{z}+-\underline{z}; \text{Three2threeTerms} \gg \underline{y}+\underline{z}+-\underline{z} = \underline{y}+-\underline{z}+\underline{z}; \text{eqTransitivity} \rhd \underline{y} = \underline{y}+\underline{z}+-\underline{z} \rhd \underline{y}+\underline{z}+-\underline{z} = \underline{y}+-\underline{z}+\underline{z} \gg \underline{y} = \underline{y}+-\underline{z}+\underline{z}; \text{eqSymmetry} \rhd \underline{y} = \underline{y}+-\underline{z}+\underline{z} \gg \underline{y}+-\underline{z}+\underline{z} = \underline{y}; \text{subLeqRight} \rhd \underline{y}+-\underline{z}+\underline{z} = \underline{y} \rhd \underline{x}+\underline{z} <= \underline{y}+-\underline{z}+\underline{z} \gg \underline{x}+\underline{z} <= \underline{y}\rceil, \mathrm{p}_0, \mathsf{c})]$
(*** SAME-F ***)

$[\text{SFsymmetry} \overset{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall\underline{(\epsilon)}\colon \forall\underline{m}\colon \forall\underline{(fx)}\colon \forall\underline{(fy)}\colon \text{SF}(\underline{(fx)}, \underline{(fy)}) \vdash \text{SF}(\underline{(fy)}, \underline{(fx)})]$

$[\text{SFsymmetry} \overset{\text{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil\text{ZFsub} \vdash \forall\underline{(\epsilon)}\colon \forall\underline{m}\colon \forall\underline{(fx)}\colon \forall\underline{(fy)}\colon \text{SF}(\underline{(fx)}, \underline{(fy)}) \vdash \dot{\neg}\, 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}\, 0 = \underline{(\epsilon)} \vdash \mathsf{c}_{\text{Ex}} <= \underline{m} \vdash \text{FromSF} \rhd \text{SF}(\underline{(fx)}, \underline{(fy)}) \rhd \dot{\neg}\, 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}\, 0 = \underline{(\epsilon)} \gg \mathsf{c}_{\text{Ex}} <= \underline{m} \Rightarrow \dot{\neg}\,\text{if}(0 <= \underline{(fx)}[\underline{m}]+-\underline{(fy)}[\underline{m}], \underline{(fx)}[\underline{m}]+-\underline{(fy)}[\underline{m}], -\underline{(fx)}[\underline{m}]+-\underline{(fy)}[\underline{m}]) <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}\,\text{if}(0 <= \underline{(fx)}[\underline{m}]+-\underline{(fy)}[\underline{m}], \underline{(fx)}[\underline{m}]+-\underline{(fy)}[\underline{m}], -\underline{(fx)}[\underline{m}]+-\underline{(fy)}[\underline{m}]) = \underline{(\epsilon)}; \text{MP} \rhd \mathsf{c}_{\text{Ex}} <= \underline{m} \Rightarrow \dot{\neg}\,\text{if}(0 <= \underline{(fx)}[\underline{m}]+-\underline{(fy)}[\underline{m}], \underline{(fx)}[\underline{m}]+-\underline{(fy)}[\underline{m}], -\underline{(fx)}[\underline{m}]+-\underline{(fy)}[\underline{m}]) <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}\,\text{if}(0 <= \underline{(fx)}[\underline{m}]+-\underline{(fy)}[\underline{m}], \underline{(fx)}[\underline{m}]+-\underline{(fy)}[\underline{m}], -\underline{(fx)}[\underline{m}]+-\underline{(fy)}[\underline{m}]) = \underline{(\epsilon)} \rhd \mathsf{c}_{\text{Ex}} <= \underline{m} \gg \dot{\neg}\,\text{if}(0 <= \underline{(fx)}[\underline{m}]+-\underline{(fy)}[\underline{m}], \underline{(fx)}[\underline{m}]+-\underline{(fy)}[\underline{m}], -\underline{(fx)}[\underline{m}]+-\underline{(fy)}[\underline{m}]) <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}\,\text{if}(0 <= \underline{(fx)}[\underline{m}]+-\underline{(fy)}[\underline{m}], \underline{(fx)}[\underline{m}]+-\underline{(fy)}[\underline{m}], -\underline{(fx)}[\underline{m}]+-\underline{(fy)}[\underline{m}]) = \underline{(\epsilon)}; \text{NumericalDifference} \gg \text{if}(0 <= \underline{(fx)}[\underline{m}]+-\underline{(fy)}[\underline{m}], \underline{(fx)}[\underline{m}]+-\underline{(fy)}[\underline{m}], -\underline{(fx)}[\underline{m}]+-\underline{(fy)}[\underline{m}]) = \text{if}(0 <= \underline{(fy)}[\underline{m}]+-\underline{(fx)}[\underline{m}], \underline{(fy)}[\underline{m}]+-\underline{(fx)}[\underline{m}], -\underline{(fy)}[\underline{m}]+-\underline{(fx)}[\underline{m}]); \text{SubLessLeft} \rhd \text{if}(0 <= \underline{(fx)}[\underline{m}]+-\underline{(fy)}[\underline{m}], \underline{(fx)}[\underline{m}]+-\underline{(fy)}[\underline{m}], -\underline{(fx)}[\underline{m}]+-\underline{(fy)}[\underline{m}]) = \text{if}(0 <= \underline{(fy)}[\underline{m}]+-\underline{(fx)}[\underline{m}], \underline{(fy)}[\underline{m}]+-\underline{(fx)}[\underline{m}], -\underline{(fy)}[\underline{m}]+-\underline{(fx)}[\underline{m}]) \rhd \dot{\neg}\,\text{if}(0 <= \underline{(fx)}[\underline{m}]+-\underline{(fy)}[\underline{m}], \underline{(fx)}[\underline{m}]+-\underline{(fy)}[\underline{m}], -\underline{(fx)}[\underline{m}]+-\underline{(fy)}[\underline{m}]) <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}\,\text{if}(0 <= \underline{(fx)}[\underline{m}]+-\underline{(fy)}[\underline{m}], \underline{(fx)}[\underline{m}]+-\underline{(fy)}[\underline{m}], -\underline{(fx)}[\underline{m}]+-\underline{(fy)}[\underline{m}]) = \underline{(\epsilon)} \gg \dot{\neg}\,\text{if}(0 <= \underline{(fy)}[\underline{m}]+-\underline{(fx)}[\underline{m}], \underline{(fy)}[\underline{m}]+-\underline{(fx)}[\underline{m}], -\underline{(fy)}[\underline{m}]+-\underline{(fx)}[\underline{m}]) <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}\,\text{if}(0 <= \underline{(fy)}[\underline{m}]+-\underline{(fx)}[\underline{m}], \underline{(fy)}[\underline{m}]+-\underline{(fx)}[\underline{m}], -\underline{(fy)}[\underline{m}]+-\underline{(fx)}[\underline{m}]) = \underline{(\epsilon)}; \forall\underline{(\epsilon)}\colon \forall\underline{m}\colon \forall\underline{(fx)}\colon \forall\underline{(fy)}\colon \text{Ded} \rhd \forall\underline{(\epsilon)}\colon \forall\underline{m}\colon \forall\underline{(fx)}\colon \forall\underline{(fy)}\colon \text{SF}(\underline{(fx)}, \underline{(fy)}) \vdash \dot{\neg}\, 0 <=$

$\underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{(\epsilon)} \vdash \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \vdash \dot{\neg} \mathrm{if}(0 <=$
$\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}]) <= \underline{(\epsilon)} \Rightarrow$
$\dot{\neg}\dot{\neg} \mathrm{if}(0 <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}]) =$
$\underline{(\epsilon)} \gg \mathrm{SF}(\underline{(\mathrm{fx})}, \underline{(\mathrm{fy})}) \Rightarrow \dot{\neg} 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{(\epsilon)} \Rightarrow \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \dot{\neg} \mathrm{if}(0 <=$
$\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}]) <= \underline{(\epsilon)} \Rightarrow$
$\dot{\neg}\dot{\neg} \mathrm{if}(0 <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}]) =$
$\underline{(\epsilon)}; \mathrm{SF}(\underline{(\mathrm{fx})}, \underline{(\mathrm{fy})}) \vdash \mathrm{MP} \rhd \mathrm{SF}(\underline{(\mathrm{fx})}, \underline{(\mathrm{fy})}) \Rightarrow \dot{\neg} 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{(\epsilon)} \Rightarrow \mathsf{c}_{\mathrm{Ex}} <=$
$\underline{\mathsf{m}} \Rightarrow \dot{\neg} \mathrm{if}(0 <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}]) <=$
$\underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} \mathrm{if}(0 <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}]) =$
$\underline{(\epsilon)} \rhd \mathrm{SF}(\underline{(\mathrm{fx})}, \underline{(\mathrm{fy})}) \gg \dot{\neg} 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{(\epsilon)} \Rightarrow \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \dot{\neg} \mathrm{if}(0 <=$
$\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}]) <= \underline{(\epsilon)} \Rightarrow$
$\dot{\neg}\dot{\neg} \mathrm{if}(0 <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}]) =$
$\underline{(\epsilon)}; \mathrm{ToSF} \rhd \dot{\neg} 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{(\epsilon)} \Rightarrow \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \dot{\neg} \mathrm{if}(0 <=$
$\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}]) <= \underline{(\epsilon)} \Rightarrow$
$\dot{\neg}\dot{\neg} \mathrm{if}(0 <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}]) =$
$\underline{(\epsilon)} \gg \mathrm{SF}(\underline{(\mathrm{fy})}, \underline{(\mathrm{fx})})\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{PlusF(Sym)} \overset{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}}\colon \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})}[\underline{\mathsf{m}}]]$

$[\mathrm{PlusF(Sym)} \overset{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}}\colon \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \mathrm{PlusF} \gg \underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] =$
$\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + \underline{(\mathrm{fy})}[\underline{\mathsf{m}}]; \mathrm{eqSymmetry} \rhd \underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] \gg$
$\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})}[\underline{\mathsf{m}}]\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{TimesF(Sym)} \overset{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}}\colon \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})} *_{\mathrm{f}} \underline{(\mathrm{fy})}[\underline{\mathsf{m}}]]$

$[\mathrm{TimesF(Sym)} \overset{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}}\colon \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \mathrm{TimesF} \gg$
$\underline{(\mathrm{fx})} *_{\mathrm{f}} \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fy})}[\underline{\mathsf{m}}]; \mathrm{eqSymmetry} \rhd \underline{(\mathrm{fx})} *_{\mathrm{f}} \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] =$
$\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] \gg \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})} *_{\mathrm{f}} \underline{(\mathrm{fy})}[\underline{\mathsf{m}}]\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{Plus0f} \overset{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}}\colon \forall \underline{(\mathrm{fx})}\colon \underline{(\mathrm{fx})} +_{\mathrm{f}} 0\mathrm{f} =_{\mathrm{f}} \underline{(\mathrm{fx})}]$

$[\mathrm{Plus0f} \overset{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}}\colon \forall \underline{(\mathrm{fx})}\colon \mathrm{PlusF} \gg \underline{(\mathrm{fx})} +_{\mathrm{f}} 0\mathrm{f}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] +$
$0\mathrm{f}[\underline{\mathsf{m}}]; 0\mathrm{f} \gg 0\mathrm{f}[\underline{\mathsf{m}}] = 0; \mathrm{lemma\ eqAdditionLeft} \rhd 0\mathrm{f}[\underline{\mathsf{m}}] = 0 \gg \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + 0\mathrm{f}[\underline{\mathsf{m}}] =$
$\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + 0; \mathrm{plus0} \gg \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + 0 = \underline{(\mathrm{fx})}[\underline{\mathsf{m}}]; \mathrm{eqTransitivity4} \rhd \underline{(\mathrm{fx})} +_{\mathrm{f}} 0\mathrm{f}[\underline{\mathsf{m}}] =$
$\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + 0\mathrm{f}[\underline{\mathsf{m}}] \rhd \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + 0\mathrm{f}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + 0 \rhd \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + 0 = \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] \gg$
$\underline{(\mathrm{fx})} +_{\mathrm{f}} 0\mathrm{f}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})}[\underline{\mathsf{m}}]; \mathrm{To} = \mathrm{f} \rhd \underline{(\mathrm{fx})} +_{\mathrm{f}} 0\mathrm{f}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] \gg \underline{(\mathrm{fx})} +_{\mathrm{f}} 0\mathrm{f} =_{\mathrm{f}} \underline{(\mathrm{fx})}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{SFtransitivity} \overset{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}}\colon \forall \underline{(\epsilon)}\colon \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \forall \underline{(\mathrm{fz})}\colon \mathrm{SF}(\underline{(\mathrm{fx})}, \underline{(\mathrm{fy})}) \vdash$
$\mathrm{SF}(\underline{(\mathrm{fy})}, \underline{(\mathrm{fz})}) \vdash \mathrm{SF}(\underline{(\mathrm{fx})}, \underline{(\mathrm{fz})})]$

$[\mathrm{SFtransitivity} \overset{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash$
$\forall \underline{\mathsf{m}}\colon \forall \underline{(\epsilon)}\colon \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \forall \underline{(\mathrm{fz})}\colon \mathrm{SF}(\underline{(\mathrm{fx})}, \underline{(\mathrm{fy})}) \vdash \mathrm{SF}(\underline{(\mathrm{fy})}, \underline{(\mathrm{fz})}) \vdash \dot{\neg} 0 <= \underline{(\epsilon)} \Rightarrow$
$\dot{\neg}\dot{\neg} 0 = \underline{(\epsilon)} \vdash \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \vdash \mathrm{PositiveHalved} \rhd \dot{\neg} 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{(\epsilon)} \gg$
$\dot{\neg} 0 <= \mathrm{rec1} + 1 * \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \mathrm{rec1} + 1 * \underline{(\epsilon)}; \mathrm{FromSF} \rhd \mathrm{SF}(\underline{(\mathrm{fx})}, \underline{(\mathrm{fy})}) \rhd \dot{\neg} 0 <=$
$\mathrm{rec1} + 1 * \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \mathrm{rec1} + 1 * \underline{(\epsilon)} \gg \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \dot{\neg} \mathrm{if}(0 <=$
$\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) <= \mathrm{rec1} + 1 * \underline{(\epsilon)} \Rightarrow$
$\dot{\neg}\dot{\neg} \mathrm{if}(0 <= \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) =$
$\mathrm{rec1} + 1 * \underline{(\epsilon)}; \mathrm{MP} \rhd \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \dot{\neg} \mathrm{if}(0 <=$
$\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) <= \mathrm{rec1} + 1 * \underline{(\epsilon)} \Rightarrow$
$\dot{\neg}\dot{\neg} \mathrm{if}(0 <= \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) =$

$\mathrm{rec1} + 1 * \underline{(\epsilon)} \rhd \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \gg \dot{\neg}\,\mathrm{if}(0 <=$
$\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) <= \mathrm{rec1} + 1 * \underline{(\epsilon)} \Rightarrow$
$\dot{\neg}\dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) =$
$\mathrm{rec1} + 1 * \underline{(\epsilon)}; \mathrm{FromSF} \rhd \mathrm{SF}(\underline{(\mathrm{fy})}, \underline{(\mathrm{fz})}) \rhd \dot{\neg}\,0 <= \mathrm{rec1} + 1 * \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}\,0 =$
$\mathrm{rec1} + 1 * \underline{(\epsilon)} \gg \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \dot{\neg}\,\mathrm{if}(0 <=$
$\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}]) <= \mathrm{rec1} + 1 * \underline{(\epsilon)} \Rightarrow$
$\dot{\neg}\dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}]) =$
$\mathrm{rec1} + 1 * \underline{(\epsilon)}; \mathrm{MP} \rhd \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \dot{\neg}\,\mathrm{if}(0 <=$
$\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}]) <= \mathrm{rec1} + 1 * \underline{(\epsilon)} \Rightarrow$
$\dot{\neg}\dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}]) =$
$\mathrm{rec1} + 1 * \underline{(\epsilon)} \rhd \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \gg \dot{\neg}\,\mathrm{if}(0 <=$
$\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}]) <= \mathrm{rec1} + 1 * \underline{(\epsilon)} \Rightarrow$
$\dot{\neg}\dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}]) =$
$\mathrm{rec1} + 1 * \underline{(\epsilon)}; \mathrm{AddEquations}(\mathrm{Less}) \rhd \dot{\neg}\,\mathrm{if}(0 <=$
$\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) <= \mathrm{rec1} + 1 * \underline{(\epsilon)} \Rightarrow$
$\dot{\neg}\dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) =$
$\mathrm{rec1} + 1 * \underline{(\epsilon)} \rhd \dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] +$
$-\underline{(\mathrm{fz})}[\underline{\mathsf{m}}]) <= \mathrm{rec1} + 1 * \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}\,\mathrm{if}(0 <=$
$\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}]) = \mathrm{rec1} + 1 * \underline{(\epsilon)} \gg$
$\dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) + \mathrm{if}(0 <=$
$\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}]) <=$
$\mathrm{rec1} + 1 * \underline{(\epsilon)} + \mathrm{rec1} + 1 * \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}\,\mathrm{if}(0 <=$
$\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) + \mathrm{if}(0 <=$
$\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}]) =$
$\mathrm{rec1} + 1 * \underline{(\epsilon)} + \mathrm{rec1} + 1 * \underline{(\epsilon)}; \mathrm{TwoHalves} \gg \mathrm{rec1} + 1 * \underline{(\epsilon)} + \mathrm{rec1} + 1 * \underline{(\epsilon)} =$
$\underline{(\epsilon)}; \mathrm{SubLessRight} \rhd \mathrm{rec1} + 1 * \underline{(\epsilon)} + \mathrm{rec1} + 1 * \underline{(\epsilon)} = \underline{(\epsilon)} \rhd \dot{\neg}\,\mathrm{if}(0 <=$
$\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) + \mathrm{if}(0 <=$
$\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}]) <=$
$\mathrm{rec1} + 1 * \underline{(\epsilon)} + \mathrm{rec1} + 1 * \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}\,\mathrm{if}(0 <=$
$\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) + \mathrm{if}(0 <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] +$
$-\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}]) = \mathrm{rec1} + 1 * \underline{(\epsilon)} + \mathrm{rec1} + 1 * \underline{(\epsilon)} \gg$
$\dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) + \mathrm{if}(0 <=$
$\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}]) <= \underline{(\epsilon)} \Rightarrow$
$\dot{\neg}\dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) + \mathrm{if}(0 <=$
$\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}]) =$
$\underline{(\epsilon)}; \mathrm{insertMiddleTerm}(\mathrm{Numerical}) \gg \mathrm{if}(0 <=$
$\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}]) <= \mathrm{if}(0 <=$
$\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) + \mathrm{if}(0 <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] +$
$-\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}]); \mathrm{leqLessTransitivity} \rhd \mathrm{if}(0 <=$
$\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}]) <= \mathrm{if}(0 <=$
$\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) + \mathrm{if}(0 <=$
$\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}]) \rhd \dot{\neg}\,\mathrm{if}(0 <=$
$\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) + \mathrm{if}(0 <=$
$\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}]) <= \underline{(\epsilon)} \Rightarrow$
$\dot{\neg}\dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) + \mathrm{if}(0 <=$
$\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}]) = \underline{(\epsilon)} \gg \dot{\neg}\,\mathrm{if}(0 <=$

(fx)[m] + −(fz)[m], (fx)[m] + −(fz)[m], −(fx)[m] + −(fz)[m]) <= (ϵ) ⇒ ¬̇¬̇if(0 <= (fx)[m] + −(fz)[m], (fx)[m] + −(fz)[m], −(fx)[m] + −(fz)[m]) = (ϵ); ∀m: ∀(ϵ): ∀(fx): ∀(fy): ∀(fz): Ded ▷ ∀m: ∀(ϵ): ∀(fx): ∀(fy): ∀(fz): SF((fx), (fy)) ⊢ SF((fy), (fz)) ⊢ ¬̇0 <= (ϵ) ⇒ ¬̇¬̇0 = (ϵ) ⊢ c_{Ex} <= m ⊢ ¬̇if(0 <= (fx)[m] + −(fz)[m], (fx)[m] + −(fz)[m], −(fx)[m] + −(fz)[m]) <= (ϵ) ⇒ ¬̇¬̇if(0 <= (fx)[m] + −(fz)[m], (fx)[m] + −(fz)[m], −(fx)[m] + −(fz)[m]) = (ϵ) ≫ SF((fx), (fy)) ⇒ SF((fy), (fz)) ⇒ ¬̇0 <= (ϵ) ⇒ ¬̇¬̇0 = (ϵ) ⇒ c_{Ex} <= m ⇒ ¬̇if(0 <= (fx)[m] + −(fz)[m], (fx)[m] + −(fz)[m], −(fx)[m] + −(fz)[m]) <= (ϵ) ⇒ ¬̇¬̇if(0 <= (fx)[m] + −(fz)[m], (fx)[m] + −(fz)[m], −(fx)[m] + −(fz)[m]) = (ϵ); SF((fx), (fy)) ⊢ SF((fy), (fz)) ⊢ MP2 ▷ SF((fx), (fy)) ⇒ SF((fy), (fz)) ⇒ ¬̇0 <= (ϵ) ⇒ ¬̇¬̇0 = (ϵ) ⇒ c_{Ex} <= m ⇒ ¬̇if(0 <= (fx)[m] + −(fz)[m], (fx)[m] + −(fz)[m], −(fx)[m] + −(fz)[m]) <= (ϵ) ⇒ ¬̇¬̇if(0 <= (fx)[m] + −(fz)[m], (fx)[m] + −(fz)[m], −(fx)[m] + −(fz)[m]) = (ϵ) ▷ SF((fx), (fy)) ▷ SF((fy), (fz)) ≫ ¬̇0 <= (ϵ) ⇒ ¬̇¬̇0 = (ϵ) ⇒ c_{Ex} <= m ⇒ ¬̇if(0 <= (fx)[m] + −(fz)[m], (fx)[m] + −(fz)[m], −(fx)[m] + −(fz)[m]) <= (ϵ) ⇒ ¬̇¬̇if(0 <= (fx)[m] + −(fz)[m], (fx)[m] + −(fz)[m], −(fx)[m] + −(fz)[m]) = (ϵ); ToSF ▷ ¬̇0 <= (ϵ) ⇒ ¬̇¬̇0 = (ϵ) ⇒ c_{Ex} <= m ⇒ ¬̇if(0 <= (fx)[m] + −(fz)[m], (fx)[m] + −(fz)[m], −(fx)[m] + −(fz)[m]) <= (ϵ) ⇒ ¬̇¬̇if(0 <= (fx)[m] + −(fz)[m], (fx)[m] + −(fz)[m], −(fx)[m] + −(fz)[m]) = (ϵ) ≫ SF((fx), (fz))⌉, p_0, c)]

[= fToSameF $\stackrel{\mathrm{stmt}}{\to}$ ZFsub ⊢ ∀(ϵ): ∀m: ∀(fx): ∀(fy): (fx) $=_{\mathrm{f}}$ (fy) ⊢ SF((fx), (fy))]

[= fToSameF $\stackrel{\mathrm{proof}}{\to}$ λc.λx.𝒫(⌈ZFsub ⊢ ∀(ϵ): ∀m: ∀(fx): ∀(fy): (fx) $=_{\mathrm{f}}$ (fy) ⊢ ¬̇0 <= (ϵ) ⇒ ¬̇¬̇0 = (ϵ) ⊢ From = f ▷ (fx) $=_{\mathrm{f}}$ (fy) ≫ (fx)[m] = (fy)[m]; PositiveToLeft(Eq)(1term) ▷ (fx)[m] = (fy)[m] ≫ (fx)[m] + −(fy)[m] = 0; SameNumerical ▷ (fx)[m] + −(fy)[m] = 0 ≫ if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) = if(0 <= 0, 0, −0); |0| = 0 ≫ if(0 <= 0, 0, −0) = 0; eqTransitivity ▷ if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) = if(0 <= 0, 0, −0) ▷ if(0 <= 0, 0, −0) = 0 ≫ if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) = 0; eqSymmetry ▷ if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) = 0 ≫ 0 = if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]); SubLessLeft ▷ 0 = if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) ▷ ¬̇0 <= (ϵ) ⇒ ¬̇¬̇0 = (ϵ) ≫ ¬̇if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) <= (ϵ) ⇒ ¬̇¬̇if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) = (ϵ); Weakening ▷ ¬̇if(0 <= (fx)[m]+−(fy)[m], (fx)[m]+−(fy)[m], −(fx)[m]+−(fy)[m]) <= (ϵ) ⇒ ¬̇¬̇if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) = (ϵ) ≫ 0 <= m ⇒ ¬̇if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) <= (ϵ) ⇒ ¬̇¬̇if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) = (ϵ); ExistIntro @ c_{Ex} @ 0 ▷ 0 <= m ⇒ ¬̇if(0 <= (fx)[m]+−(fy)[m], (fx)[m]+−(fy)[m], −(fx)[m]+−(fy)[m]) <= (ϵ) ⇒ ¬̇¬̇if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) = (ϵ) ≫ c_{Ex} <= m ⇒ ¬̇if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) <=

$(\epsilon) \Rightarrow \dot{\neg}\,\dot{\neg}\,\mathrm{if}(0 <= (\mathrm{fx})[\mathsf{m}] + -(\mathrm{fy})[\mathsf{m}], (\mathrm{fx})[\mathsf{m}] + -(\mathrm{fy})[\mathsf{m}], -(\mathrm{fx})[\mathsf{m}] + -(\mathrm{fy})[\mathsf{m}]) = (\epsilon); \forall(\epsilon)\colon \forall \mathsf{m}\colon \forall(\mathrm{fx})\colon \forall(\mathrm{fy})\colon \mathrm{Ded} \rhd \forall(\epsilon)\colon \forall \mathsf{m}\colon \forall(\mathrm{fx})\colon \forall(\mathrm{fy})\colon (\mathrm{fx}) =_{\mathrm{f}} (\mathrm{fy}) \vdash \dot{\neg}\,0 <= (\epsilon) \Rightarrow \dot{\neg}\,\dot{\neg}\,0 = (\epsilon) \vdash \mathsf{c}_{\mathrm{Ex}} <= \mathsf{m} \Rightarrow \dot{\neg}\,\mathrm{if}(0 <= (\mathrm{fx})[\mathsf{m}] + -(\mathrm{fy})[\mathsf{m}], (\mathrm{fx})[\mathsf{m}] + -(\mathrm{fy})[\mathsf{m}], -(\mathrm{fx})[\mathsf{m}] + -(\mathrm{fy})[\mathsf{m}]) <= (\epsilon) \Rightarrow \dot{\neg}\,\dot{\neg}\,\mathrm{if}(0 <= (\mathrm{fx})[\mathsf{m}] + -(\mathrm{fy})[\mathsf{m}], (\mathrm{fx})[\mathsf{m}] + -(\mathrm{fy})[\mathsf{m}], -(\mathrm{fx})[\mathsf{m}] + -(\mathrm{fy})[\mathsf{m}]) = (\epsilon) \gg (\mathrm{fx}) =_{\mathrm{f}} (\mathrm{fy}) \Rightarrow \dot{\neg}\,0 <= (\epsilon) \Rightarrow \dot{\neg}\,\dot{\neg}\,0 = (\epsilon) \Rightarrow \mathsf{c}_{\mathrm{Ex}} <= \mathsf{m} \Rightarrow \dot{\neg}\,\mathrm{if}(0 <= (\mathrm{fx})[\mathsf{m}] + -(\mathrm{fy})[\mathsf{m}], (\mathrm{fx})[\mathsf{m}] + -(\mathrm{fy})[\mathsf{m}], -(\mathrm{fx})[\mathsf{m}] + -(\mathrm{fy})[\mathsf{m}]) <= (\epsilon) \Rightarrow \dot{\neg}\,\dot{\neg}\,\mathrm{if}(0 <= (\mathrm{fx})[\mathsf{m}] + -(\mathrm{fy})[\mathsf{m}], (\mathrm{fx})[\mathsf{m}] + -(\mathrm{fy})[\mathsf{m}], -(\mathrm{fx})[\mathsf{m}] + -(\mathrm{fy})[\mathsf{m}]) = (\epsilon); (\mathrm{fx}) =_{\mathrm{f}} (\mathrm{fy}) \vdash \mathrm{MP} \rhd (\mathrm{fx}) =_{\mathrm{f}} (\mathrm{fy}) \Rightarrow \dot{\neg}\,0 <= (\epsilon) \Rightarrow \dot{\neg}\,\dot{\neg}\,0 = (\epsilon) \Rightarrow \mathsf{c}_{\mathrm{Ex}} <= \mathsf{m} \Rightarrow \dot{\neg}\,\mathrm{if}(0 <= (\mathrm{fx})[\mathsf{m}] + -(\mathrm{fy})[\mathsf{m}], (\mathrm{fx})[\mathsf{m}] + -(\mathrm{fy})[\mathsf{m}], -(\mathrm{fx})[\mathsf{m}] + -(\mathrm{fy})[\mathsf{m}]) <= (\epsilon) \Rightarrow \dot{\neg}\,\dot{\neg}\,\mathrm{if}(0 <= (\mathrm{fx})[\mathsf{m}] + -(\mathrm{fy})[\mathsf{m}], (\mathrm{fx})[\mathsf{m}] + -(\mathrm{fy})[\mathsf{m}], -(\mathrm{fx})[\mathsf{m}] + -(\mathrm{fy})[\mathsf{m}]) = (\epsilon) \rhd (\mathrm{fx}) =_{\mathrm{f}} (\mathrm{fy}) \gg \dot{\neg}\,0 <= (\epsilon) \Rightarrow \dot{\neg}\,\dot{\neg}\,0 = (\epsilon) \Rightarrow \mathsf{c}_{\mathrm{Ex}} <= \mathsf{m} \Rightarrow \dot{\neg}\,\mathrm{if}(0 <= (\mathrm{fx})[\mathsf{m}] + -(\mathrm{fy})[\mathsf{m}], (\mathrm{fx})[\mathsf{m}] + -(\mathrm{fy})[\mathsf{m}], -(\mathrm{fx})[\mathsf{m}] + -(\mathrm{fy})[\mathsf{m}]) <= (\epsilon) \Rightarrow \dot{\neg}\,\dot{\neg}\,\mathrm{if}(0 <= (\mathrm{fx})[\mathsf{m}] + -(\mathrm{fy})[\mathsf{m}], (\mathrm{fx})[\mathsf{m}] + -(\mathrm{fy})[\mathsf{m}], -(\mathrm{fx})[\mathsf{m}] + -(\mathrm{fy})[\mathsf{m}]) = (\epsilon); \mathrm{ToSF} \rhd \dot{\neg}\,0 <= (\epsilon) \Rightarrow \dot{\neg}\,\dot{\neg}\,0 = (\epsilon) \Rightarrow \mathsf{c}_{\mathrm{Ex}} <= \mathsf{m} \Rightarrow \dot{\neg}\,\mathrm{if}(0 <= (\mathrm{fx})[\mathsf{m}] + -(\mathrm{fy})[\mathsf{m}], (\mathrm{fx})[\mathsf{m}] + -(\mathrm{fy})[\mathsf{m}], -(\mathrm{fx})[\mathsf{m}] + -(\mathrm{fy})[\mathsf{m}]) <= (\epsilon) \Rightarrow \dot{\neg}\,\dot{\neg}\,\mathrm{if}(0 <= (\mathrm{fx})[\mathsf{m}] + -(\mathrm{fy})[\mathsf{m}], (\mathrm{fx})[\mathsf{m}] + -(\mathrm{fy})[\mathsf{m}], -(\mathrm{fx})[\mathsf{m}] + -(\mathrm{fy})[\mathsf{m}]) = (\epsilon) \gg \mathrm{SF}((\mathrm{fx}), (\mathrm{fy}))\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{f2R(Plus)} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall(\mathrm{fx})\colon \forall(\mathrm{fy})\colon \forall(\mathrm{fz})\colon \forall(\mathrm{fv})\colon \forall \mathrm{var\ fv}\colon \mathrm{SF}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}), (\mathrm{fz})) \vdash \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}((\mathrm{fz}))]$

$[\mathrm{f2R(Plus)} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall(\mathrm{fx})\colon \forall(\mathrm{fy})\colon \forall(\mathrm{fz})\colon \forall(\mathrm{fv})\colon \forall \mathrm{var\ fv}\colon \mathrm{SF}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}), (\mathrm{fz})) \vdash (\mathrm{fv}) \in \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) \vdash \mathrm{var\ fv} \in \mathrm{R}((\mathrm{fz})) \vdash \mathrm{FromInR} \rhd (\mathrm{fv}) \in \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) \gg \mathrm{SF}((\mathrm{fv}), (\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})); \mathrm{FromInR} \rhd \mathrm{var\ fv} \in \mathrm{R}((\mathrm{fz})) \gg \mathrm{SF}(\mathrm{var\ fv}, (\mathrm{fz})); \mathrm{SFsymmetry} \rhd \mathrm{SF}(\mathrm{var\ fv}, (\mathrm{fz})) \gg \mathrm{SF}((\mathrm{fz}), \mathrm{var\ fv}); \mathrm{SFtransitivity} \rhd \mathrm{SF}((\mathrm{fv}), (\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) \rhd \mathrm{SF}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}), (\mathrm{fz})) \gg \mathrm{SF}((\mathrm{fv}), (\mathrm{fz})); \mathrm{SFtransitivity} \rhd \mathrm{SF}((\mathrm{fv}), (\mathrm{fz})) \rhd \mathrm{SF}((\mathrm{fz}), \mathrm{var\ fv}) \gg \mathrm{SF}((\mathrm{fv}), \mathrm{var\ fv}); \forall(\mathrm{fx})\colon \forall(\mathrm{fy})\colon \forall(\mathrm{fz})\colon \forall(\mathrm{fv})\colon \forall \mathrm{var\ fv}\colon \mathrm{Ded} \rhd \forall(\mathrm{fx})\colon \forall(\mathrm{fy})\colon \forall(\mathrm{fz})\colon \forall(\mathrm{fv})\colon \forall \mathrm{var\ fv}\colon \mathrm{SF}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}), (\mathrm{fz})) \vdash (\mathrm{fv}) \in \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) \vdash \mathrm{var\ fv} \in \mathrm{R}((\mathrm{fz})) \vdash \mathrm{SF}((\mathrm{fv}), \mathrm{var\ fv}) \gg \mathrm{SF}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}), (\mathrm{fz})) \Rightarrow (\mathrm{fv}) \in \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) \Rightarrow \mathrm{var\ fv} \in \mathrm{R}((\mathrm{fz})) \Rightarrow \mathrm{SF}((\mathrm{fv}), \mathrm{var\ fv}); \mathrm{SF}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}), (\mathrm{fz})) \vdash \mathrm{MP} \rhd \mathrm{SF}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}), (\mathrm{fz})) \Rightarrow (\mathrm{fv}) \in \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) \Rightarrow \mathrm{var\ fv} \in \mathrm{R}((\mathrm{fz})) \Rightarrow \mathrm{SF}((\mathrm{fv}), \mathrm{var\ fv}) \rhd \mathrm{SF}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}), (\mathrm{fz})) \gg (\mathrm{fv}) \in \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) \Rightarrow \mathrm{var\ fv} \in \mathrm{R}((\mathrm{fz})) \Rightarrow \mathrm{SF}((\mathrm{fv}), \mathrm{var\ fv}); \mathrm{To} == \mathrm{XX} \rhd (\mathrm{fv}) \in \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) \Rightarrow \mathrm{var\ fv} \in \mathrm{R}((\mathrm{fz})) \Rightarrow \mathrm{SF}((\mathrm{fv}), \mathrm{var\ fv}) \gg \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}((\mathrm{fz})); \mathrm{PlusR} \gg \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})); ==\mathrm{Transitivity} \rhd \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) \rhd \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}((\mathrm{fz})) \gg \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}((\mathrm{fz}))\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{f2R(Times)} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall(\mathrm{fx})\colon \forall(\mathrm{fy})\colon \forall(\mathrm{fz})\colon \mathrm{SF}((\mathrm{fx}) *_{\mathrm{f}} (\mathrm{fy}), (\mathrm{fz})) \vdash \mathrm{R}((\mathrm{fx})) * *\mathrm{R}((\mathrm{fy})) == \mathrm{R}((\mathrm{fz}))]$

$[\mathrm{f2R(Times)} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall(\mathrm{fx})\colon \forall(\mathrm{fy})\colon \forall(\mathrm{fz})\colon \mathrm{SF}((\mathrm{fx}) *_{\mathrm{f}} (\mathrm{fy}), (\mathrm{fz})) \vdash \mathrm{To} == \rhd \mathrm{SF}((\mathrm{fx}) *_{\mathrm{f}} (\mathrm{fy}), (\mathrm{fz})) \gg \mathrm{R}((\mathrm{fx}) *_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}((\mathrm{fz})); \mathrm{TimesR} \gg \mathrm{R}((\mathrm{fx})) * *\mathrm{R}((\mathrm{fy})) == \mathrm{R}((\mathrm{fx}) *_{\mathrm{f}} (\mathrm{fy})); ==\mathrm{Transitivity} \rhd \mathrm{R}((\mathrm{fx})) * *\mathrm{R}((\mathrm{fy})) == \mathrm{R}((\mathrm{fx}) *_{\mathrm{f}} (\mathrm{fy})) \rhd \mathrm{R}((\mathrm{fx}) *_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}((\mathrm{fz})) \gg \mathrm{R}((\mathrm{fx})) * *\mathrm{R}((\mathrm{fy})) == \mathrm{R}((\mathrm{fz}))\rceil, \mathrm{p}_0, \mathsf{c})]$

(*** R-AFDELINGEN ***)

$[\mathrm{PlusR(Sym)} \overset{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \mathrm{R}(\underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})}) == \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}))]$

$[\mathrm{PlusR(Sym)} \overset{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \mathrm{PlusR} \gg \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}(\underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})}); {==}\mathrm{Symmetry} \rhd \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}(\underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})}) \gg \mathrm{R}(\underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})}) == \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}))\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{TimesR(Sym)} \overset{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \mathrm{R}(\underline{(\mathrm{fx})} *_{\mathrm{f}} \underline{(\mathrm{fy})}) == \mathrm{R}(\underline{(\mathrm{fx})}) * *\mathrm{R}(\underline{(\mathrm{fy})})]$

$[\mathrm{TimesR(Sym)} \overset{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \mathrm{TimesR} \gg \mathrm{R}((\mathrm{fx})) * *\mathrm{R}((\mathrm{fy})) == \mathrm{R}(\underline{(\mathrm{fx})} *_{\mathrm{f}} \underline{(\mathrm{fy})}); {==}\mathrm{Symmetry} \rhd \mathrm{R}(\underline{(\mathrm{fx})}) * *\mathrm{R}(\underline{(\mathrm{fy})}) == \mathrm{R}(\underline{(\mathrm{fx})} *_{\mathrm{f}} (\mathrm{fy})) \gg \mathrm{R}(\underline{(\mathrm{fx})} *_{\mathrm{f}} \underline{(\mathrm{fy})}) == \mathrm{R}(\underline{(\mathrm{fx})}) * *\mathrm{R}(\underline{(\mathrm{fy})})\rceil, \mathrm{p}_0, \mathsf{c})]$

(*** LEQ-R ***)

$[\mathrm{eqLeq(R)} \overset{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{(\mathrm{rx})}\colon \forall \underline{(\mathrm{ry})}\colon \underline{(\mathrm{rx})} == \underline{(\mathrm{ry})} \vdash \dot\neg \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})} \Rightarrow \underline{(\mathrm{rx})} == \underline{(\mathrm{ry})}]$

$[\mathrm{eqLeq(R)} \overset{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{(\mathrm{rx})}\colon \forall \underline{(\mathrm{ry})}\colon \underline{(\mathrm{rx})} == \underline{(\mathrm{ry})} \vdash \mathrm{WeakenOr1} \rhd \underline{(\mathrm{rx})} == \underline{(\mathrm{ry})} \gg \dot\neg \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})} \Rightarrow \underline{(\mathrm{rx})} == \underline{(\mathrm{ry})}; \mathrm{Repetition} \rhd \dot\neg \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})} \Rightarrow \underline{(\mathrm{rx})} == \underline{(\mathrm{ry})} \gg \dot\neg \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})} \Rightarrow \underline{(\mathrm{rx})} == \underline{(\mathrm{ry})}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{ThirdGeqSeries} \overset{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}}\colon \forall \underline{(\epsilon)}\colon \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \forall \underline{(\mathrm{fz})}\colon \forall \underline{(\mathrm{fv})}\colon \forall \underline{(\mathrm{rx})}\colon \forall \underline{(\mathrm{ry})}\colon \forall \underline{(\mathrm{rz})}\colon \forall \underline{(\mathrm{ru})}\colon \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})} \vdash \underline{(\mathrm{rz})} << \underline{(\mathrm{ru})} \vdash \underline{(\mathrm{fx})} \in \underline{(\mathrm{rx})} \vdash \underline{(\mathrm{fy})} \in \underline{(\mathrm{ry})} \vdash \underline{(\mathrm{fz})} \in \underline{(\mathrm{rz})} \vdash \underline{(\mathrm{fv})} \in \underline{(\mathrm{ru})} \vdash \dot\neg 0 <= \underline{(\epsilon)} \Rightarrow \dot\neg \dot\neg 0 = \underline{(\epsilon)} \vdash \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \vdash \dot\neg \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \Rightarrow \dot\neg \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fv})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)}]$

$[\mathrm{ThirdGeqSeries} \overset{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}}\colon \forall \underline{(\epsilon)}\colon \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \forall \underline{(\mathrm{fz})}\colon \forall \underline{(\mathrm{fv})}\colon \forall \underline{(\mathrm{rx})}\colon \forall \underline{(\mathrm{ry})}\colon \forall \underline{(\mathrm{rz})}\colon \forall \underline{(\mathrm{ru})}\colon \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})} \vdash \underline{(\mathrm{rz})} << \underline{(\mathrm{ru})} \vdash \underline{(\mathrm{fx})} \in \underline{(\mathrm{rx})} \vdash \underline{(\mathrm{fy})} \in \underline{(\mathrm{ry})} \vdash \underline{(\mathrm{fz})} \in \underline{(\mathrm{rz})} \vdash \underline{(\mathrm{fv})} \in \underline{(\mathrm{ru})} \vdash \dot\neg 0 <= \underline{(\epsilon)} \Rightarrow \dot\neg \dot\neg 0 = \underline{(\epsilon)} \vdash \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \vdash \mathrm{From} << \mathrm{XX} \rhd \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})} \rhd \underline{(\mathrm{fx})} \in \underline{(\mathrm{rx})} \rhd \underline{(\mathrm{fy})} \in \underline{(\mathrm{ry})} \rhd \dot\neg 0 <= \underline{(\epsilon)} \Rightarrow \dot\neg \dot\neg 0 = \underline{(\epsilon)} \gg \mathsf{a}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)}; \mathrm{From} << \mathrm{XX} \rhd \underline{(\mathrm{rz})} << \underline{(\mathrm{ru})} \rhd \underline{(\mathrm{fz})} \in \underline{(\mathrm{rz})} \rhd \underline{(\mathrm{fv})} \in \underline{(\mathrm{ru})} \rhd \dot\neg 0 <= \underline{(\epsilon)} \Rightarrow \dot\neg \dot\neg 0 = \underline{(\epsilon)} \gg \mathsf{a}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fv})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)}; \mathrm{ExistIntro} \,@\, \mathsf{b}_{\mathrm{Ex}} \,@\, \mathsf{a}_{\mathrm{Ex}} \rhd \mathsf{a}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fv})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \gg \mathsf{b}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fv})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)}; \mathrm{thirdGeq} \gg \dot\neg \mathsf{a}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}} \Rightarrow \dot\neg \mathsf{b}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}}; \mathrm{FirstConjunct} \rhd \dot\neg \mathsf{a}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}} \Rightarrow \dot\neg \mathsf{b}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}} \gg \mathsf{a}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}}; \mathrm{leqTransitivity} \rhd \mathsf{a}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}} \rhd \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \gg \mathsf{a}_{\mathrm{Ex}} <= \underline{\mathsf{m}}; \mathrm{MP} \rhd \mathsf{a}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \rhd \mathsf{a}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \gg \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)}; \mathrm{SecondConjunct} \rhd \dot\neg \mathsf{a}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}} \Rightarrow \dot\neg \mathsf{b}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}} \gg \mathsf{b}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}}; \mathrm{leqTransitivity} \rhd \mathsf{b}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}} \rhd \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \gg \mathsf{b}_{\mathrm{Ex}} <= \underline{\mathsf{m}}; \mathrm{MP} \rhd \mathsf{b}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fv})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \rhd \mathsf{b}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \gg \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fv})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)}; \mathrm{JoinConjuncts} \rhd \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \rhd \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fv})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \gg \dot\neg \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \Rightarrow \dot\neg \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fv})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)}\rceil, \mathrm{p}_0, \mathsf{c})]$

(*** LESS-R ***)

XX lidt grimt med de ekstra variable

$[\mathrm{SubLessLeft(R)} \overset{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{(\epsilon)}\colon \forall \underline{\mathsf{m}}\colon \forall \underline{(\mathrm{fy})}\colon \forall \underline{(\mathrm{fz})}\colon \forall \underline{(\mathrm{rx})}\colon \forall \underline{(\mathrm{ry})}\colon \forall \underline{(\mathrm{rz})}\colon \underline{(\mathrm{rx})} == \underline{(\mathrm{ry})} \vdash \underline{(\mathrm{rx})} << \underline{(\mathrm{rz})} \vdash \underline{(\mathrm{ry})} << \underline{(\mathrm{rz})}]$

$[\mathrm{SubLessLeft(R)} \overset{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{(\epsilon)}\colon \forall \underline{\mathsf{m}}\colon \forall \underline{(\mathrm{fy})}\colon \forall \underline{(\mathrm{fz})}\colon \forall \underline{(\mathrm{rx})}\colon \forall \underline{(\mathrm{ry})}\colon \forall \underline{(\mathrm{rz})}\colon \underline{(\mathrm{rx})} == \underline{(\mathrm{ry})} \vdash \underline{(\mathrm{rx})} << \underline{(\mathrm{rz})} \vdash \underline{(\mathrm{fy})} \in$

$(ry) \vdash (fz) \in (rz) \vdash \dot{\neg} 0 <= (\epsilon) \Rightarrow \dot{\neg}\dot{\neg} 0 = (\epsilon) \vdash \mathrm{SENC2} \rhd (rx) == (ry) \rhd (fy) \in (ry) \gg (fy) \in (rx); \mathrm{From} << \mathrm{XX} \rhd (rx) << (rz) \rhd (fy) \in (rx) \rhd (fz) \in (rz) \rhd \dot{\neg} 0 <= (\epsilon) \Rightarrow \dot{\neg}\dot{\neg} 0 = (\epsilon) \gg \mathsf{a}_{\mathrm{Ex}} <= \mathsf{m} \Rightarrow (fy)[\mathsf{m}] <= (fz)[\mathsf{m}] + -(\epsilon); \forall(\epsilon)\colon \forall \mathsf{m}\colon \forall(fy)\colon \forall(fz)\colon \forall(rx)\colon \forall(ry)\colon \forall(rz)\colon \mathrm{Ded} \rhd \forall(\epsilon)\colon \forall \mathsf{m}\colon \forall(fy)\colon \forall(fz)\colon \forall(rx)\colon \forall(ry)\colon \forall(rz)\colon (rx) == (ry) \vdash (rx) << (rz) \vdash (fy) \in (ry) \vdash (fz) \in (rz) \vdash \dot{\neg} 0 <= (\epsilon) \Rightarrow \dot{\neg}\dot{\neg} 0 = (\epsilon) \vdash \mathsf{a}_{\mathrm{Ex}} <= \mathsf{m} \Rightarrow (fy)[\mathsf{m}] <= (fz)[\mathsf{m}] + -(\epsilon) \gg (rx) == (ry) \Rightarrow (rx) << (rz) \Rightarrow (fy) \in (ry) \Rightarrow (fz) \in (rz) \Rightarrow \dot{\neg} 0 <= (\epsilon) \Rightarrow \dot{\neg}\dot{\neg} 0 = (\epsilon) \Rightarrow \mathsf{a}_{\mathrm{Ex}} <= \mathsf{m} \Rightarrow (fy)[\mathsf{m}] <= (fz)[\mathsf{m}] + -(\epsilon); (rx) == (ry) \vdash (rx) << (rz) \vdash \mathrm{MP2} \rhd (rx) == (ry) \Rightarrow (rx) << (rz) \Rightarrow (fy) \in (ry) \Rightarrow (fz) \in (rz) \Rightarrow \dot{\neg} 0 <= (\epsilon) \Rightarrow \dot{\neg}\dot{\neg} 0 = (\epsilon) \Rightarrow \mathsf{a}_{\mathrm{Ex}} <= \mathsf{m} \Rightarrow (fy)[\mathsf{m}] <= (fz)[\mathsf{m}] + -(\epsilon) \rhd (rx) == (ry) \rhd (rx) << (rz) \gg (fy) \in (ry) \Rightarrow (fz) \in (rz) \Rightarrow \dot{\neg} 0 <= (\epsilon) \Rightarrow \dot{\neg}\dot{\neg} 0 = (\epsilon) \Rightarrow \mathsf{a}_{\mathrm{Ex}} <= \mathsf{m} \Rightarrow (fy)[\mathsf{m}] <= (fz)[\mathsf{m}] + -(\epsilon); \mathrm{to} << \mathrm{XX} \rhd (fy) \in (ry) \Rightarrow (fz) \in (rz) \Rightarrow \dot{\neg} 0 <= (\epsilon) \Rightarrow \dot{\neg}\dot{\neg} 0 = (\epsilon) \Rightarrow \mathsf{a}_{\mathrm{Ex}} <= \mathsf{m} \Rightarrow (fy)[\mathsf{m}] <= (fz)[\mathsf{m}] + -(\epsilon) \gg (ry) << (rz)\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{SubLessRight(R)} \overset{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall(\epsilon)\colon \forall \mathsf{m}\colon \forall(fy)\colon \forall(fz)\colon \forall(rx)\colon \forall(ry)\colon \forall(rz)\colon (rx) == (ry) \vdash (rz) << (rx) \vdash (rz) << (ry)]$

$[\mathrm{SubLessRight(R)} \overset{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall(\epsilon)\colon \forall \mathsf{m}\colon \forall(fy)\colon \forall(fz)\colon \forall(rx)\colon \forall(ry)\colon \forall(rz)\colon (rx) == (ry) \vdash (rz) << (rx) \vdash (fz) \in (rz) \vdash (fy) \in (ry) \vdash \dot{\neg} 0 <= (\epsilon) \Rightarrow \dot{\neg}\dot{\neg} 0 = (\epsilon) \vdash \mathrm{SENC2} \rhd (rx) == (ry) \gg (fy) \in (rx); \mathrm{From} << \mathrm{XX} \rhd (rz) << (rx) \rhd (fz) \in (rz) \rhd (fy) \in (rx) \rhd \dot{\neg} 0 <= (\epsilon) \Rightarrow \dot{\neg}\dot{\neg} 0 = (\epsilon) \gg \mathsf{a}_{\mathrm{Ex}} <= \mathsf{m} \Rightarrow (fz)[\mathsf{m}] <= (fy)[\mathsf{m}] + -(\epsilon); \forall(\epsilon)\colon \forall \mathsf{m}\colon \forall(fy)\colon \forall(fz)\colon \forall(rx)\colon \forall(ry)\colon \forall(rz)\colon \mathrm{Ded} \rhd \forall(\epsilon)\colon \forall \mathsf{m}\colon \forall(fy)\colon \forall(fz)\colon \forall(rx)\colon \forall(ry)\colon \forall(rz)\colon (rx) == (ry) \vdash (rz) << (rx) \vdash (fz) \in (rz) \vdash (fy) \in (ry) \vdash \dot{\neg} 0 <= (\epsilon) \Rightarrow \dot{\neg}\dot{\neg} 0 = (\epsilon) \vdash \mathsf{a}_{\mathrm{Ex}} <= \mathsf{m} \Rightarrow (fz)[\mathsf{m}] <= (fy)[\mathsf{m}] + -(\epsilon) \gg (rx) == (ry) \Rightarrow (rz) << (rx) \Rightarrow (fz) \in (rz) \Rightarrow (fy) \in (ry) \Rightarrow \dot{\neg} 0 <= (\epsilon) \Rightarrow \dot{\neg}\dot{\neg} 0 = (\epsilon) \Rightarrow \mathsf{a}_{\mathrm{Ex}} <= \mathsf{m} \Rightarrow (fz)[\mathsf{m}] <= (fy)[\mathsf{m}] + -(\epsilon); (rx) == (ry) \vdash (rz) << (rx) \vdash \mathrm{MP2} \rhd (rx) == (ry) \Rightarrow (rz) << (rx) \Rightarrow (fz) \in (rz) \Rightarrow (fy) \in (ry) \Rightarrow \dot{\neg} 0 <= (\epsilon) \Rightarrow \dot{\neg}\dot{\neg} 0 = (\epsilon) \Rightarrow \mathsf{a}_{\mathrm{Ex}} <= \mathsf{m} \Rightarrow (fz)[\mathsf{m}] <= (fy)[\mathsf{m}] + -(\epsilon) \rhd (rx) == (ry) \rhd (rz) << (rx) \gg (fz) \in (rz) \Rightarrow (fy) \in (ry) \Rightarrow \dot{\neg} 0 <= (\epsilon) \Rightarrow \dot{\neg}\dot{\neg} 0 = (\epsilon) \Rightarrow \mathsf{a}_{\mathrm{Ex}} <= \mathsf{m} \Rightarrow (fz)[\mathsf{m}] <= (fy)[\mathsf{m}] + -(\epsilon); \mathrm{to} << \mathrm{XX} \rhd (fz) \in (rz) \Rightarrow (fy) \in (ry) \Rightarrow \dot{\neg} 0 <= (\epsilon) \Rightarrow \dot{\neg}\dot{\neg} 0 = (\epsilon) \Rightarrow \mathsf{a}_{\mathrm{Ex}} <= \mathsf{m} \Rightarrow (fz)[\mathsf{m}] <= (fy)[\mathsf{m}] + -(\epsilon) \gg (rz) << (ry)\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{LessLeq(R)} \overset{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall(rx)\colon \forall(ry)\colon (rx) << (ry) \vdash \dot{\neg} (rx) << (ry) \Rightarrow (rx) == (ry)]$

$[\mathrm{LessLeq(R)} \overset{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall(rx)\colon \forall(ry)\colon (rx) << (ry) \vdash \mathrm{WeakenOr2} \rhd (rx) << (ry) \gg \dot{\neg} (rx) << (ry) \Rightarrow (rx) == (ry); \mathrm{Repetition} \rhd \dot{\neg} (rx) << (ry) \Rightarrow (rx) == (ry) \gg \dot{\neg} (rx) << (ry) \Rightarrow (rx) == (ry)\rceil, \mathrm{p}_0, \mathsf{c})]$

$[<< \mathrm{Transitivity} \overset{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \mathsf{m}\colon \forall(\epsilon)\colon \forall(fx)\colon \forall(fz)\colon \forall(rx)\colon \forall(ry)\colon \forall(rz)\colon (rx) << (ry) \vdash (ry) << (rz) \vdash (rx) << (rz)]$

$[<< \mathrm{Transitivity} \overset{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \mathsf{m}\colon \forall(\epsilon)\colon \forall(fx)\colon \forall(fz)\colon \forall(rx)\colon \forall(ry)\colon \forall(rz)\colon (rx) << (ry) \vdash (ry) << (rz) \vdash (fx) \in (rx) \vdash (fz) \in (rz) \vdash \dot{\neg} 0 <= (\epsilon) \Rightarrow \dot{\neg}\dot{\neg} 0 = (\epsilon) \vdash \mathsf{c}_{\mathrm{Ex}} <= \mathsf{m} \vdash \mathrm{From} << (2) \rhd (rx) << (ry) \gg \mathsf{t}_{\mathrm{Ex}} \in (ry); \mathrm{ThirdGeqSeries} \rhd (rx) << (ry) \rhd (ry) <<$

$\underline{(rz)} \rhd \underline{(fx)} \in \underline{(rx)} \rhd \mathsf{t}_{\mathrm{Ex}} \in \underline{(ry)} \rhd \mathsf{t}_{\mathrm{Ex}} \in \underline{(ry)} \rhd \underline{(fz)} \in \underline{(rz)} \rhd \dot{\neg} 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{(\epsilon)} \rhd \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \gg \dot{\neg}\underline{(fx)}[\underline{\mathsf{m}}] <= \mathsf{t}_{\mathrm{Ex}}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \Rightarrow \dot{\neg}\mathsf{t}_{\mathrm{Ex}}[\underline{\mathsf{m}}] <= \underline{(fz)}[\underline{\mathsf{m}}] + -\underline{(\epsilon)}$; $\mathrm{FirstConjunct} \rhd \dot{\neg}\underline{(fx)}[\underline{\mathsf{m}}] <= \mathsf{t}_{\mathrm{Ex}}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \Rightarrow \dot{\neg}\mathsf{t}_{\mathrm{Ex}}[\underline{\mathsf{m}}] <= \underline{(fz)}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \gg \underline{(fx)}[\underline{\mathsf{m}}] <= \mathsf{t}_{\mathrm{Ex}}[\underline{\mathsf{m}}] + -\underline{(\epsilon)}$; $\mathrm{SecondConjunct} \rhd \dot{\neg}\underline{(fx)}[\underline{\mathsf{m}}] <= \mathsf{t}_{\mathrm{Ex}}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \Rightarrow \dot{\neg}\mathsf{t}_{\mathrm{Ex}}[\underline{\mathsf{m}}] <= \underline{(fz)}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \gg \mathsf{t}_{\mathrm{Ex}}[\underline{\mathsf{m}}] <= \underline{(fz)}[\underline{\mathsf{m}}] + -\underline{(\epsilon)}$; $<< \mathrm{TransitivityHelper(Q)} \rhd \underline{(fx)}[\underline{\mathsf{m}}] <= \mathsf{t}_{\mathrm{Ex}}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \rhd \mathsf{t}_{\mathrm{Ex}}[\underline{\mathsf{m}}] <= \underline{(fz)}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \rhd \dot{\neg} 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{(\epsilon)} \gg \underline{(fx)}[\underline{\mathsf{m}}] <= \underline{(fz)}[\underline{\mathsf{m}}] + -\underline{(\epsilon)}$; $\forall \underline{\mathsf{m}} \colon \forall \underline{(\epsilon)} \colon \forall \underline{(fx)} \colon \forall \underline{(fz)} \colon \forall \underline{(rx)} \colon \forall \underline{(ry)} \colon \forall \underline{(rz)} \colon \mathrm{Ded} \rhd \forall \underline{\mathsf{m}} \colon \forall \underline{(\epsilon)} \colon \forall \underline{(fx)} \colon \forall \underline{(fz)} \colon \forall \underline{(rx)} \colon \forall \underline{(ry)} \colon \forall \underline{(rz)} \colon \underline{(rx)} << \underline{(ry)} \vdash \underline{(ry)} << \underline{(rz)} \vdash \underline{(fx)} \in \underline{(rx)} \vdash \underline{(fz)} \in \underline{(rz)} \vdash \dot{\neg} 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{(\epsilon)} \vdash \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \vdash \underline{(fx)}[\underline{\mathsf{m}}] <= \underline{(fz)}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \gg \underline{(rx)} << \underline{(ry)} \Rightarrow \underline{(ry)} << \underline{(rz)} \Rightarrow \underline{(fx)} \in \underline{(rx)} \Rightarrow \underline{(fz)} \in \underline{(rz)} \Rightarrow \dot{\neg} 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{(\epsilon)} \Rightarrow \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(fx)}[\underline{\mathsf{m}}] <= \underline{(fz)}[\underline{\mathsf{m}}] + -\underline{(\epsilon)}$; $\underline{(rx)} << \underline{(ry)} \vdash \underline{(ry)} << \underline{(rz)} \vdash \mathrm{MP2} \rhd \underline{(rx)} << \underline{(ry)} \Rightarrow \underline{(ry)} << \underline{(rz)} \Rightarrow \underline{(fx)} \in \underline{(rx)} \Rightarrow \underline{(fz)} \in \underline{(rz)} \Rightarrow \dot{\neg} 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{(\epsilon)} \Rightarrow \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(fx)}[\underline{\mathsf{m}}] <= \underline{(fz)}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \rhd \underline{(rx)} << \underline{(ry)} \rhd \underline{(ry)} << \underline{(rz)} \gg \underline{(fx)} \in \underline{(rx)} \Rightarrow \underline{(fz)} \in \underline{(rz)} \Rightarrow \dot{\neg} 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{(\epsilon)} \Rightarrow \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(fx)}[\underline{\mathsf{m}}] <= \underline{(fz)}[\underline{\mathsf{m}}] + -\underline{(\epsilon)}$; $\mathrm{ExistIntro} \mathbin{@} \mathsf{a}_{\mathrm{Ex}} \mathbin{@} \mathsf{c}_{\mathrm{Ex}} \rhd \underline{(fx)} \in \underline{(rx)} \Rightarrow \underline{(fz)} \in \underline{(rz)} \Rightarrow \dot{\neg} 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{(\epsilon)} \Rightarrow \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(fx)}[\underline{\mathsf{m}}] <= \underline{(fz)}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \gg \underline{(fx)} \in \underline{(rx)} \Rightarrow \underline{(fz)} \in \underline{(rz)} \Rightarrow \dot{\neg} 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{(\epsilon)} \Rightarrow \mathsf{a}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(fx)}[\underline{\mathsf{m}}] <= \underline{(fz)}[\underline{\mathsf{m}}] + -\underline{(\epsilon)}$; $\mathrm{to} << \mathrm{XX} \rhd \underline{(fx)} \in \underline{(rx)} \Rightarrow \underline{(fz)} \in \underline{(rz)} \Rightarrow \dot{\neg} 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{(\epsilon)} \Rightarrow \mathsf{a}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(fx)}[\underline{\mathsf{m}}] <= \underline{(fz)}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \gg \underline{(rx)} << \underline{(rz)} \rceil, \mathrm{p}_0, \mathsf{c})]$

(*** NUMMER 1 ***)

$[<<== \mathrm{Reflexivity} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{(rx)} \colon \dot{\neg}\underline{(rx)} << \underline{(rx)} \Rightarrow \underline{(rx)} == \underline{(rx)}]$

$[<<== \mathrm{Reflexivity} \overset{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{(rx)} \colon ==\mathrm{Reflexivity} \gg \underline{(rx)} == \underline{(rx)}; \mathrm{eqLeq(R)} \rhd \underline{(rx)} == \underline{(rx)} \gg \dot{\neg}\underline{(rx)} << \underline{(rx)} \Rightarrow \underline{(rx)} == \underline{(rx)} \rceil, \mathrm{p}_0, \mathsf{c})]$

(*** NUMMER 2 ***)

$[<<== \mathrm{AntisymmetryHelper(Q)} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \forall \underline{\mathsf{z}} \colon \dot{\neg} 0 <= \underline{\mathsf{z}} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{\mathsf{z}} \vdash \underline{\mathsf{x}} <= \underline{\mathsf{y}} + -\underline{\mathsf{z}} \vdash \underline{\mathsf{y}} <= \underline{\mathsf{x}} + -\underline{\mathsf{z}} \vdash \underline{\mathsf{a}}]$

$[<<== \mathrm{AntisymmetryHelper(Q)} \overset{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \forall \underline{\mathsf{z}} \colon \dot{\neg} 0 <= \underline{\mathsf{z}} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{\mathsf{z}} \vdash \underline{\mathsf{x}} <= \underline{\mathsf{y}} + -\underline{\mathsf{z}} \vdash \underline{\mathsf{y}} <= \underline{\mathsf{x}} + -\underline{\mathsf{z}} \vdash \mathrm{leqAddition} \rhd \underline{\mathsf{x}} <= \underline{\mathsf{y}} + -\underline{\mathsf{z}} \gg \underline{\mathsf{x}} + \underline{\mathsf{z}} <= \underline{\mathsf{y}} + -\underline{\mathsf{z}} + \underline{\mathsf{z}}; \mathrm{plusAssociativity} \gg \underline{\mathsf{y}} + -\underline{\mathsf{z}} + \underline{\mathsf{z}} = \underline{\mathsf{y}} + -\underline{\mathsf{z}} + \underline{\mathsf{z}}; \mathrm{plusCommutativity} \gg -\underline{\mathsf{z}} + \underline{\mathsf{z}} = \underline{\mathsf{z}} + -\underline{\mathsf{z}}; \mathrm{lemma\ eqAdditionLeft} \rhd -\underline{\mathsf{z}} + \underline{\mathsf{z}} = \underline{\mathsf{z}} + -\underline{\mathsf{z}} \gg \underline{\mathsf{y}} + -\underline{\mathsf{z}} + \underline{\mathsf{z}} = \underline{\mathsf{y}} + \underline{\mathsf{z}} + -\underline{\mathsf{z}}; \mathrm{x} = \mathrm{x} + (\mathrm{y} - \mathrm{y}) \gg \underline{\mathsf{y}} = \underline{\mathsf{y}} + \underline{\mathsf{z}} + -\underline{\mathsf{z}}; \mathrm{eqSymmetry} \rhd \underline{\mathsf{y}} = \underline{\mathsf{y}} + \underline{\mathsf{z}} + -\underline{\mathsf{z}} \gg \underline{\mathsf{y}} + \underline{\mathsf{z}} + -\underline{\mathsf{z}} = \underline{\mathsf{y}}; \mathrm{eqTransitivity4} \rhd \underline{\mathsf{y}} + -\underline{\mathsf{z}} + \underline{\mathsf{z}} = \underline{\mathsf{y}} + -\underline{\mathsf{z}} + \underline{\mathsf{z}} \rhd \underline{\mathsf{y}} + -\underline{\mathsf{z}} + \underline{\mathsf{z}} = \underline{\mathsf{y}} + \underline{\mathsf{z}} + -\underline{\mathsf{z}} \rhd \underline{\mathsf{y}} + \underline{\mathsf{z}} + -\underline{\mathsf{z}} = \underline{\mathsf{y}} \gg \underline{\mathsf{y}} + -\underline{\mathsf{z}} + \underline{\mathsf{z}} = \underline{\mathsf{y}}; \mathrm{subLeqRight} \rhd \underline{\mathsf{y}} + -\underline{\mathsf{z}} + \underline{\mathsf{z}} = \underline{\mathsf{y}} \rhd \underline{\mathsf{x}} + \underline{\mathsf{z}} <= \underline{\mathsf{y}} + -\underline{\mathsf{z}} + \underline{\mathsf{z}} \gg \underline{\mathsf{x}} + \underline{\mathsf{z}} <= \underline{\mathsf{y}}; \mathrm{leqTransitivity} \rhd \underline{\mathsf{x}} + \underline{\mathsf{z}} <= \underline{\mathsf{y}} \rhd \underline{\mathsf{y}} <= \underline{\mathsf{x}} + -\underline{\mathsf{z}} \gg \underline{\mathsf{x}} + \underline{\mathsf{z}} <= \underline{\mathsf{x}} + -\underline{\mathsf{z}}; \mathrm{leqSubtractionLeft} \rhd \underline{\mathsf{x}} + \underline{\mathsf{z}} <= \underline{\mathsf{x}} + -\underline{\mathsf{z}} \gg \underline{\mathsf{z}} <= -\underline{\mathsf{z}}; \mathrm{toNotLess} \rhd \underline{\mathsf{z}} <= -\underline{\mathsf{z}} \gg \dot{\neg}\dot{\neg} -\underline{\mathsf{z}} <= \underline{\mathsf{z}} \Rightarrow \dot{\neg}\dot{\neg} -\underline{\mathsf{z}} = \underline{\mathsf{z}}; \mathrm{NegativeLessPositive} \rhd \dot{\neg} 0 <= \underline{\mathsf{z}} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{\mathsf{z}} \gg \dot{\neg} -\underline{\mathsf{z}} <= \underline{\mathsf{z}} \Rightarrow \dot{\neg}\dot{\neg} -\underline{\mathsf{z}} = \underline{\mathsf{z}}; \mathrm{FromContradiction} \rhd \dot{\neg} -\underline{\mathsf{z}} <= \underline{\mathsf{z}} \Rightarrow \dot{\neg}\dot{\neg} -\underline{\mathsf{z}} = \underline{\mathsf{z}} \rhd \dot{\neg}\dot{\neg} -\underline{\mathsf{z}} <= \underline{\mathsf{z}} \Rightarrow \dot{\neg}\dot{\neg} -\underline{\mathsf{z}} = \underline{\mathsf{z}} \gg \underline{\mathsf{a}} \rceil, \mathrm{p}_0, \mathsf{c})]$

$[<<== \mathrm{Antisymmetry} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{(rx)} \colon \forall \underline{(ry)} \colon \dot{\neg}\underline{(rx)} << \underline{(ry)} \Rightarrow \underline{(rx)} == \underline{(ry)} \vdash \dot{\neg}\underline{(ry)} << \underline{(rx)} \Rightarrow \underline{(ry)} == \underline{(rx)} \vdash \underline{(rx)} == \underline{(ry)}]$

$[<<== \text{Antisymmetry} \overset{\text{proof}}{\to} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall\underline{(rx)}\colon \forall\underline{(ry)}\colon \dot{\neg}\,\underline{(rx)} << \underline{(ry)} \Rightarrow$
$\underline{(rx)} == \underline{(ry)} \vdash \dot{\neg}\,\underline{(ry)} << \underline{(rx)} \Rightarrow \underline{(ry)} == \underline{(rx)} \vdash \text{Repetition} \rhd \dot{\neg}\,\underline{(rx)} << \underline{(ry)} \Rightarrow$
$\underline{(rx)} == \underline{(ry)} \gg \dot{\neg}\,\underline{(rx)} << \underline{(ry)} \Rightarrow \underline{(rx)} == \underline{(ry)}; \text{Repetition} \rhd \dot{\neg}\,\underline{(ry)} << \underline{(rx)} \Rightarrow$
$\underline{(ry)} == \underline{(rx)} \gg \dot{\neg}\,\underline{(ry)} << \underline{(rx)} \Rightarrow \underline{(ry)} == \underline{(rx)}; \text{ExpandDisjuncts} \rhd \dot{\neg}\,\underline{(rx)} <<$
$\underline{(ry)} \Rightarrow \underline{(rx)} == \underline{(ry)} \rhd \dot{\neg}\,\underline{(ry)} << \underline{(rx)} \Rightarrow \underline{(ry)} == \underline{(rx)} \gg \dot{\neg}\,\underline{(rx)} == \underline{(ry)} \Rightarrow$
$\dot{\neg}\,\underline{(ry)} == \underline{(rx)} \Rightarrow \dot{\neg}\,\underline{(rx)} << \underline{(ry)} \Rightarrow \dot{\neg}\,\underline{(ry)} << \underline{(rx)}; \text{AutoImply} \gg \underline{(rx)} ==$
$\underline{(ry)} \Rightarrow \underline{(rx)} == \underline{(ry)}; \forall\underline{(rx)}\colon \forall\underline{(ry)}\colon \underline{(ry)} == \underline{(rx)} \vdash {==}\text{Symmetry} \rhd \underline{(ry)} ==$
$\underline{(rx)} \gg \underline{(rx)} == \underline{(ry)}; \text{Ded} \rhd \forall\underline{(rx)}\colon \forall\underline{(ry)}\colon \underline{(ry)} == \underline{(rx)} \vdash \underline{(rx)} == \underline{(ry)} \gg$
$\underline{(ry)} == \underline{(rx)} \Rightarrow \underline{(rx)} == \underline{(ry)}; \forall\underline{(rx)}\colon \forall\underline{(ry)}\colon \dot{\neg}\,\underline{(rx)} << \underline{(ry)} \Rightarrow \dot{\neg}\,\underline{(ry)} << \underline{(rx)} \vdash$
$\text{FirstConjunct} \rhd \dot{\neg}\,\underline{(rx)} << \underline{(ry)} \Rightarrow \dot{\neg}\,\underline{(ry)} << \underline{(rx)} \gg \underline{(rx)} << \underline{(ry)}; \text{From} <<$
$(1) \rhd \underline{(rx)} << \underline{(ry)} \gg \mathsf{j}_{\text{Ex}} \in \underline{(rx)}; \text{From} << (2) \rhd \underline{(rx)} << \underline{(ry)} \gg \mathsf{t}_{\text{Ex}} \in \underline{(ry)}; 0 <$
$1 \gg \dot{\neg}\,0 <= 1 \Rightarrow \dot{\neg}\,\dot{\neg}\,0 = 1; \text{From} << \text{XX} \rhd \underline{(rx)} << \underline{(ry)} \rhd \mathsf{j}_{\text{Ex}} \in \underline{(rx)} \rhd \mathsf{t}_{\text{Ex}} \in$
$\underline{(ry)} \rhd \dot{\neg}\,0 <= 1 \Rightarrow \dot{\neg}\,\dot{\neg}\,0 = 1 \gg \mathsf{a}_{\text{Ex}} <= \mathsf{c}_{\text{Ex}} \Rightarrow \mathsf{j}_{\text{Ex}}[\mathsf{c}_{\text{Ex}}] <=$
$\mathsf{t}_{\text{Ex}}[\mathsf{c}_{\text{Ex}}] + -1; \text{SecondConjunct} \rhd \dot{\neg}\,\underline{(rx)} << \underline{(ry)} \Rightarrow \dot{\neg}\,\underline{(ry)} << \underline{(rx)} \gg \underline{(ry)} <<$
$\underline{(rx)}; \text{From} << \text{XX} \rhd \underline{(ry)} << \underline{(rx)} \rhd \mathsf{t}_{\text{Ex}} \in \underline{(ry)} \rhd \mathsf{j}_{\text{Ex}} \in \underline{(rx)} \rhd \dot{\neg}\,0 <= 1 \Rightarrow$
$\dot{\neg}\,\dot{\neg}\,0 = 1 \gg \mathsf{a}_{\text{Ex}} <= \mathsf{c}_{\text{Ex}} \Rightarrow \mathsf{t}_{\text{Ex}}[\mathsf{c}_{\text{Ex}}] <=$
$\mathsf{j}_{\text{Ex}}[\mathsf{c}_{\text{Ex}}] + -1; \text{ExistIntro} @ \mathsf{b}_{\text{Ex}} @ \mathsf{a}_{\text{Ex}} \rhd \mathsf{a}_{\text{Ex}} <= \mathsf{c}_{\text{Ex}} \Rightarrow \mathsf{t}_{\text{Ex}}[\mathsf{c}_{\text{Ex}}] <=$
$\mathsf{j}_{\text{Ex}}[\mathsf{c}_{\text{Ex}}] + -1 \gg \mathsf{b}_{\text{Ex}} <= \mathsf{c}_{\text{Ex}} \Rightarrow \mathsf{t}_{\text{Ex}}[\mathsf{c}_{\text{Ex}}] <= \mathsf{j}_{\text{Ex}}[\mathsf{c}_{\text{Ex}}] + -1; \text{thirdGeq} \gg$
$\dot{\neg}\,\mathsf{a}_{\text{Ex}} <= \mathsf{c}_{\text{Ex}} \Rightarrow \dot{\neg}\,\mathsf{b}_{\text{Ex}} <= \mathsf{c}_{\text{Ex}}; \text{FirstConjunct} \rhd \dot{\neg}\,\mathsf{a}_{\text{Ex}} <= \mathsf{c}_{\text{Ex}} \Rightarrow \dot{\neg}\,\mathsf{b}_{\text{Ex}} <=$
$\mathsf{c}_{\text{Ex}} \gg \mathsf{a}_{\text{Ex}} <= \mathsf{c}_{\text{Ex}}; \text{MP} \rhd \mathsf{a}_{\text{Ex}} <= \mathsf{c}_{\text{Ex}} \Rightarrow \mathsf{j}_{\text{Ex}}[\mathsf{c}_{\text{Ex}}] <= \mathsf{t}_{\text{Ex}}[\mathsf{c}_{\text{Ex}}] + -1 \rhd \mathsf{a}_{\text{Ex}} <=$
$\mathsf{c}_{\text{Ex}} \gg \mathsf{j}_{\text{Ex}}[\mathsf{c}_{\text{Ex}}] <= \mathsf{t}_{\text{Ex}}[\mathsf{c}_{\text{Ex}}] + -1; \text{SecondConjunct} \rhd \dot{\neg}\,\mathsf{a}_{\text{Ex}} <= \mathsf{c}_{\text{Ex}} \Rightarrow \dot{\neg}\,\mathsf{b}_{\text{Ex}} <=$
$\mathsf{c}_{\text{Ex}} \gg \mathsf{b}_{\text{Ex}} <= \mathsf{c}_{\text{Ex}}; \text{MP} \rhd \mathsf{b}_{\text{Ex}} <= \mathsf{c}_{\text{Ex}} \Rightarrow \mathsf{t}_{\text{Ex}}[\mathsf{c}_{\text{Ex}}] <= \mathsf{j}_{\text{Ex}}[\mathsf{c}_{\text{Ex}}] + -1 \rhd \mathsf{b}_{\text{Ex}} <=$
$\mathsf{c}_{\text{Ex}} \gg \mathsf{t}_{\text{Ex}}[\mathsf{c}_{\text{Ex}}] <= \mathsf{j}_{\text{Ex}}[\mathsf{c}_{\text{Ex}}] + -1; <<== \text{AntisymmetryHelper(Q)} \rhd \dot{\neg}\,0 <=$
$1 \Rightarrow \dot{\neg}\,\dot{\neg}\,0 = 1 \rhd \mathsf{j}_{\text{Ex}}[\mathsf{c}_{\text{Ex}}] <= \mathsf{t}_{\text{Ex}}[\mathsf{c}_{\text{Ex}}] + -1 \rhd \mathsf{t}_{\text{Ex}}[\mathsf{c}_{\text{Ex}}] <= \mathsf{j}_{\text{Ex}}[\mathsf{c}_{\text{Ex}}] + -1 \gg$
$\underline{(rx)} == \underline{(ry)}; \text{Ded} \rhd \forall\underline{(rx)}\colon \forall\underline{(ry)}\colon \dot{\neg}\,\underline{(rx)} << \underline{(ry)} \Rightarrow \dot{\neg}\,\underline{(ry)} << \underline{(rx)} \vdash \underline{(rx)} ==$
$\underline{(ry)} \gg \dot{\neg}\,\underline{(rx)} << \underline{(ry)} \Rightarrow \dot{\neg}\,\underline{(ry)} << \underline{(rx)} \Rightarrow \underline{(rx)} ==$
$\underline{(ry)}; \text{From3Disjuncts} \rhd \dot{\neg}\,\underline{(rx)} == \underline{(ry)} \Rightarrow \dot{\neg}\,\underline{(ry)} == \underline{(rx)} \Rightarrow \dot{\neg}\,\underline{(rx)} << \underline{(ry)} \Rightarrow$
$\dot{\neg}\,\underline{(ry)} << \underline{(rx)} \rhd \underline{(rx)} == \underline{(ry)} \Rightarrow \underline{(rx)} == \underline{(ry)} \rhd \underline{(ry)} == \underline{(rx)} \Rightarrow \underline{(rx)} ==$
$\underline{(ry)} \rhd \dot{\neg}\,\underline{(rx)} << \underline{(ry)} \Rightarrow \dot{\neg}\,\underline{(ry)} << \underline{(rx)} \Rightarrow \underline{(rx)} == \underline{(ry)} \gg \underline{(rx)} == \underline{(ry)}\rceil, \mathrm{p}_0, \mathsf{c})]$
(*** NUMMER 3 ***)

$[<< \text{TransitivityHelper(Q)} \overset{\text{stmt}}{\to} \text{ZFsub} \vdash \forall\underline{\mathsf{x}}\colon \forall\underline{\mathsf{y}}\colon \forall\underline{\mathsf{z}}\colon \forall\underline{\mathsf{u}}\colon \underline{\mathsf{x}} <= \underline{\mathsf{y}} + -\underline{\mathsf{u}} \vdash \underline{\mathsf{y}} <=$
$\underline{\mathsf{z}} + -\underline{\mathsf{u}} \vdash \dot{\neg}\,0 <= \underline{\mathsf{u}} \Rightarrow \dot{\neg}\,\dot{\neg}\,0 = \underline{\mathsf{u}} \vdash \underline{\mathsf{x}} <= \underline{\mathsf{z}} + -\underline{\mathsf{u}}]$

$[<< \text{TransitivityHelper(Q)} \overset{\text{proof}}{\to} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall\underline{\mathsf{x}}\colon \forall\underline{\mathsf{y}}\colon \forall\underline{\mathsf{z}}\colon \forall\underline{\mathsf{u}}\colon \underline{\mathsf{x}} <=$
$\underline{\mathsf{y}} + -\underline{\mathsf{u}} \vdash \underline{\mathsf{y}} <= \underline{\mathsf{z}} + -\underline{\mathsf{u}} \vdash \dot{\neg}\,0 <= \underline{\mathsf{u}} \Rightarrow \dot{\neg}\,\dot{\neg}\,0 = \underline{\mathsf{u}} \vdash \text{PositiveNegated} \rhd \dot{\neg}\,0 <=$
$\underline{\mathsf{u}} \Rightarrow \dot{\neg}\,\dot{\neg}\,0 = \underline{\mathsf{u}} \gg \dot{\neg}\, -\underline{\mathsf{u}} <= 0 \Rightarrow \dot{\neg}\,\dot{\neg}\, -\underline{\mathsf{u}} = 0; \text{LessAdditionLeft} \rhd \dot{\neg}\, -\underline{\mathsf{u}} <=$
$0 \Rightarrow \dot{\neg}\,\dot{\neg}\, -\underline{\mathsf{u}} = 0 \gg \dot{\neg}\,\underline{\mathsf{y}} + -\underline{\mathsf{u}} <= \underline{\mathsf{y}} + 0 \Rightarrow \dot{\neg}\,\dot{\neg}\,\underline{\mathsf{y}} + -\underline{\mathsf{u}} = \underline{\mathsf{y}} + 0; \text{plus0} \gg \underline{\mathsf{y}} + 0 =$
$\underline{\mathsf{y}}; \text{SubLessRight} \rhd \underline{\mathsf{y}} + 0 = \underline{\mathsf{y}} \rhd \dot{\neg}\,\underline{\mathsf{y}} + -\underline{\mathsf{u}} <= \underline{\mathsf{y}} + 0 \Rightarrow \dot{\neg}\,\dot{\neg}\,\underline{\mathsf{y}} + -\underline{\mathsf{u}} = \underline{\mathsf{y}} + 0 \gg$
$\dot{\neg}\,\underline{\mathsf{y}}+-\underline{\mathsf{u}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\,\dot{\neg}\,\underline{\mathsf{y}}+-\underline{\mathsf{u}} = \underline{\mathsf{y}}; \text{leqLessTransitivity} \rhd \underline{\mathsf{x}} <= \underline{\mathsf{y}}+-\underline{\mathsf{u}} \rhd \dot{\neg}\,\underline{\mathsf{y}}+-\underline{\mathsf{u}} <=$
$\underline{\mathsf{y}} \Rightarrow \dot{\neg}\,\dot{\neg}\,\underline{\mathsf{y}} + -\underline{\mathsf{u}} = \underline{\mathsf{y}} \gg \dot{\neg}\,\underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\,\dot{\neg}\,\underline{\mathsf{x}} = \underline{\mathsf{y}}; \text{LessLeqTransitivity} \rhd \dot{\neg}\,\underline{\mathsf{x}} <=$
$\underline{\mathsf{y}} \Rightarrow \dot{\neg}\,\dot{\neg}\,\underline{\mathsf{x}} = \underline{\mathsf{y}} \rhd \underline{\mathsf{y}} <= \underline{\mathsf{z}} + -\underline{\mathsf{u}} \gg \dot{\neg}\,\underline{\mathsf{x}} <= \underline{\mathsf{z}} + -\underline{\mathsf{u}} \Rightarrow \dot{\neg}\,\dot{\neg}\,\underline{\mathsf{x}} =$
$\underline{\mathsf{z}} + -\underline{\mathsf{u}}; \text{LessLeq} \rhd \dot{\neg}\,\underline{\mathsf{x}} <= \underline{\mathsf{z}} + -\underline{\mathsf{u}} \Rightarrow \dot{\neg}\,\dot{\neg}\,\underline{\mathsf{x}} = \underline{\mathsf{z}} + -\underline{\mathsf{u}} \gg \underline{\mathsf{x}} <= \underline{\mathsf{z}} + -\underline{\mathsf{u}}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[<<== \text{Transitivity} \overset{\text{stmt}}{\to} \text{ZFsub} \vdash \forall\underline{(rx)}\colon \forall\underline{(ry)}\colon \forall\underline{(rz)}\colon \dot{\neg}\,\underline{(rx)} << \underline{(ry)} \Rightarrow \underline{(rx)} ==$
$\underline{(ry)} \vdash \dot{\neg}\,\underline{(ry)} << \underline{(rz)} \Rightarrow \underline{(ry)} == \underline{(rz)} \vdash \dot{\neg}\,\underline{(rx)} << \underline{(rz)} \Rightarrow \underline{(rx)} == \underline{(rz)}]$

$[{<<}{==}\ \text{Transitivity} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{(rx)}\colon \forall \underline{(ry)}\colon \forall \underline{(rz)}\colon \underline{(rx)} << \underline{(ry)} \vdash \underline{(ry)} << \underline{(rz)} \vdash {<<}\ \text{Transitivity} \rhd \underline{(rx)} << \underline{(ry)} \rhd \underline{(ry)} << \underline{(rz)} \gg \underline{(rx)} << \underline{(rz)}; \text{LessLeq}(\mathrm{R}) \rhd \underline{(rx)} << \underline{(rz)} \gg \dot{\neg}\, \underline{(rx)} << \underline{(rz)} \Rightarrow \underline{(rx)} == \underline{(rz)}; \forall \underline{(rx)}\colon \forall \underline{(ry)}\colon \forall \underline{(rz)}\colon \underline{(rx)} << \underline{(ry)} \vdash \underline{(ry)} == \underline{(rz)} \vdash \text{SubLessRight}(\mathrm{R}) \rhd \underline{(ry)} == \underline{(rz)} \rhd \underline{(rx)} << \underline{(ry)} \gg \underline{(rx)} << \underline{(rz)}; \text{LessLeq}(\mathrm{R}) \rhd \underline{(rx)} << \underline{(rz)} \gg \dot{\neg}\, \underline{(rx)} << \underline{(rz)} \Rightarrow \underline{(rx)} == \underline{(rz)}; \forall \underline{(rx)}\colon \forall \underline{(ry)}\colon \forall \underline{(rz)}\colon \underline{(rx)} == \underline{(ry)} \vdash \underline{(ry)} << \underline{(rz)} \vdash {==}\text{Symmetry} \rhd \underline{(rx)} == \underline{(ry)} \gg \underline{(ry)} == \underline{(rx)}; \text{SubLessLeft}(\mathrm{R}) \rhd \underline{(ry)} == \underline{(rx)} \rhd \underline{(ry)} << \underline{(rz)} \gg \underline{(rx)} << \underline{(rz)}; \text{LessLeq}(\mathrm{R}) \rhd \underline{(rx)} << \underline{(rz)} \gg \dot{\neg}\, \underline{(rx)} << \underline{(rz)} \Rightarrow \underline{(rx)} == \underline{(rz)}; \forall \underline{(rx)}\colon \forall \underline{(ry)}\colon \forall \underline{(rz)}\colon \underline{(rx)} == \underline{(ry)} \vdash \underline{(ry)} == \underline{(rz)} \vdash {==}\ \text{Transitivity} \rhd \underline{(rx)} == \underline{(ry)} \rhd \underline{(ry)} == \underline{(rz)} \gg \underline{(rx)} == \underline{(rz)}; \text{eqLeq}(\mathrm{R}) \rhd \underline{(rx)} == \underline{(rz)} \gg \dot{\neg}\, \underline{(rx)} << \underline{(rz)} \Rightarrow \underline{(rx)} == \underline{(rz)}; \forall \underline{(rx)}\colon \forall \underline{(ry)}\colon \forall \underline{(rz)}\colon \dot{\neg}\, \underline{(rx)} << \underline{(ry)} \Rightarrow \underline{(rx)} == \underline{(ry)} \vdash \dot{\neg}\, \underline{(ry)} << \underline{(rz)} \Rightarrow \underline{(ry)} == \underline{(rz)} \vdash \text{Repetition} \rhd \dot{\neg}\, \underline{(rx)} << \underline{(ry)} \Rightarrow \underline{(rx)} == \underline{(ry)} \gg \dot{\neg}\, \underline{(rx)} << \underline{(ry)} \Rightarrow \underline{(rx)} == \underline{(ry)}; \text{Repetition} \rhd \dot{\neg}\, \underline{(ry)} << \underline{(rz)} \Rightarrow \underline{(ry)} == \underline{(rz)} \gg \dot{\neg}\, \underline{(ry)} << \underline{(rz)} \Rightarrow \underline{(ry)} == \underline{(rz)}; \text{Ded} \rhd \forall \underline{(rx)}\colon \forall \underline{(ry)}\colon \forall \underline{(rz)}\colon \underline{(rx)} << \underline{(ry)} \vdash \underline{(ry)} << \underline{(rz)} \vdash \dot{\neg}\, \underline{(rx)} << \underline{(rz)} \Rightarrow \underline{(rx)} == \underline{(rz)} \gg \underline{(rx)} << \underline{(ry)} \Rightarrow \underline{(ry)} << \underline{(rz)} \Rightarrow \dot{\neg}\, \underline{(rx)} << \underline{(rz)} \Rightarrow \underline{(rx)} == \underline{(rz)}; \text{Ded} \rhd \forall \underline{(rx)}\colon \forall \underline{(ry)}\colon \forall \underline{(rz)}\colon \underline{(rx)} << \underline{(ry)} \vdash \underline{(ry)} == \underline{(rz)} \vdash \dot{\neg}\, \underline{(rx)} << \underline{(rz)} \Rightarrow \underline{(rx)} == \underline{(rz)} \gg \underline{(rx)} << \underline{(ry)} \Rightarrow \underline{(ry)} == \underline{(rz)} \Rightarrow \dot{\neg}\, \underline{(rx)} << \underline{(rz)} \Rightarrow \underline{(rx)} == \underline{(rz)}; \text{Ded} \rhd \forall \underline{(rx)}\colon \forall \underline{(ry)}\colon \forall \underline{(rz)}\colon \underline{(rx)} == \underline{(ry)} \vdash \underline{(ry)} << \underline{(rz)} \vdash \dot{\neg}\, \underline{(rx)} << \underline{(rz)} \Rightarrow \underline{(rx)} == \underline{(rz)} \gg \underline{(rx)} == \underline{(ry)} \Rightarrow \underline{(ry)} << \underline{(rz)} \Rightarrow \dot{\neg}\, \underline{(rx)} << \underline{(rz)} \Rightarrow \underline{(rx)} == \underline{(rz)}; \text{Ded} \rhd \forall \underline{(rx)}\colon \forall \underline{(ry)}\colon \forall \underline{(rz)}\colon \underline{(rx)} == \underline{(ry)} \vdash \underline{(ry)} == \underline{(rz)} \vdash \dot{\neg}\, \underline{(rx)} << \underline{(rz)} \Rightarrow \underline{(rx)} == \underline{(rz)} \gg \underline{(rx)} == \underline{(ry)} \Rightarrow \underline{(ry)} == \underline{(rz)} \Rightarrow \dot{\neg}\, \underline{(rx)} << \underline{(rz)} \Rightarrow \underline{(rx)} == \underline{(rz)}; \text{From2} * \text{2Disjuncts} \rhd \dot{\neg}\, \underline{(rx)} << \underline{(ry)} \Rightarrow \underline{(rx)} == \underline{(ry)} \rhd \dot{\neg}\, \underline{(ry)} << \underline{(rz)} \Rightarrow \underline{(ry)} == \underline{(rz)} \rhd \underline{(rx)} << \underline{(ry)} \Rightarrow \underline{(ry)} << \underline{(rz)} \Rightarrow \dot{\neg}\, \underline{(rx)} << \underline{(rz)} \Rightarrow \underline{(rx)} == \underline{(rz)} \rhd \underline{(rx)} << \underline{(ry)} \Rightarrow \underline{(ry)} == \underline{(rz)} \Rightarrow \dot{\neg}\, \underline{(rx)} << \underline{(rz)} \Rightarrow \underline{(rx)} == \underline{(rz)} \rhd \underline{(rx)} == \underline{(ry)} \Rightarrow \underline{(ry)} << \underline{(rz)} \Rightarrow \dot{\neg}\, \underline{(rx)} << \underline{(rz)} \Rightarrow \underline{(rx)} == \underline{(rz)} \rhd \underline{(rx)} == \underline{(ry)} \Rightarrow \underline{(ry)} == \underline{(rz)} \Rightarrow \dot{\neg}\, \underline{(rx)} << \underline{(rz)} \Rightarrow \underline{(rx)} == \underline{(rz)} \gg \dot{\neg}\, \underline{(rx)} << \underline{(rz)} \Rightarrow \underline{(rx)} == \underline{(rz)}\rceil, \mathrm{p}_0, \mathsf{c})]$

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$[{==}\ \text{Addition} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{m}}\colon \forall \underline{(\epsilon)}\colon \forall \underline{(fx)}\colon \forall \underline{(fy)}\colon \forall \underline{(fz)}\colon \mathrm{R}(\underline{(fx)}) == \mathrm{R}(\underline{(fy)}) \vdash \mathrm{R}((fx) +_{\mathrm{f}} (fy)) == \mathrm{R}((fx) +_{\mathrm{f}} (fy))]$

$[{==}\ \text{Addition} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{m}}\colon \forall \underline{(\epsilon)}\colon \forall \underline{(fx)}\colon \forall \underline{(fy)}\colon \forall \underline{(fz)}\colon \mathrm{R}(\underline{(fx)}) == \mathrm{R}(\underline{(fy)}) \vdash \dot{\neg}\, 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\,\dot{\neg}\, 0 = \underline{(\epsilon)} \vdash \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \vdash \text{From} == \rhd \mathrm{R}(\underline{(fx)}) == \mathrm{R}(\underline{(fy)}) \gg \text{SF}(\underline{(fx)}, \underline{(fy)}); \text{FromSF} \rhd \text{SF}(\underline{(fx)}, \underline{(fy)}) \rhd \dot{\neg}\, 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\,\dot{\neg}\, 0 = \underline{(\epsilon)} \gg \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \dot{\neg}\, \text{if}(0 <= \underline{(fx)}[\underline{\mathsf{m}}] + -\underline{(fy)}[\underline{\mathsf{m}}], \underline{(fx)}[\underline{\mathsf{m}}] + -\underline{(fy)}[\underline{\mathsf{m}}], -\underline{(fx)}[\underline{\mathsf{m}}] + -\underline{(fy)}[\underline{\mathsf{m}}]) <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\,\dot{\neg}\, \text{if}(0 <= \underline{(fx)}[\underline{\mathsf{m}}] + -\underline{(fy)}[\underline{\mathsf{m}}], \underline{(fx)}[\underline{\mathsf{m}}] + -\underline{(fy)}[\underline{\mathsf{m}}], -\underline{(fx)}[\underline{\mathsf{m}}] + -\underline{(fy)}[\underline{\mathsf{m}}]) = \underline{(\epsilon)}; \text{MP} \rhd \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \dot{\neg}\, \text{if}(0 <= \underline{(fx)}[\underline{\mathsf{m}}] + -\underline{(fy)}[\underline{\mathsf{m}}], \underline{(fx)}[\underline{\mathsf{m}}] + -\underline{(fy)}[\underline{\mathsf{m}}], -\underline{(fx)}[\underline{\mathsf{m}}] + -\underline{(fy)}[\underline{\mathsf{m}}]) <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\,\dot{\neg}\, \text{if}(0 <= \underline{(fx)}[\underline{\mathsf{m}}] + -\underline{(fy)}[\underline{\mathsf{m}}], \underline{(fx)}[\underline{\mathsf{m}}] + -\underline{(fy)}[\underline{\mathsf{m}}], -\underline{(fx)}[\underline{\mathsf{m}}] + -\underline{(fy)}[\underline{\mathsf{m}}]) = \underline{(\epsilon)} \rhd \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \gg \dot{\neg}\, \text{if}(0 <= \underline{(fx)}[\underline{\mathsf{m}}] + -\underline{(fy)}[\underline{\mathsf{m}}], \underline{(fx)}[\underline{\mathsf{m}}] + -\underline{(fy)}[\underline{\mathsf{m}}], -\underline{(fx)}[\underline{\mathsf{m}}] + -\underline{(fy)}[\underline{\mathsf{m}}]) <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\,\dot{\neg}\, \text{if}(0 <= \underline{(fx)}[\underline{\mathsf{m}}] + -\underline{(fy)}[\underline{\mathsf{m}}], \underline{(fx)}[\underline{\mathsf{m}}] + -\underline{(fy)}[\underline{\mathsf{m}}], -\underline{(fx)}[\underline{\mathsf{m}}] + -\underline{(fy)}[\underline{\mathsf{m}}]) = \underline{(\epsilon)}; \text{insertMiddleTerm}(\text{Difference}) \gg \underline{(fx)}[\underline{\mathsf{m}}] + -\underline{(fy)}[\underline{\mathsf{m}}] = \underline{(fx)}[\underline{\mathsf{m}}] + \underline{(fz)}[\underline{\mathsf{m}}] + -\underline{(fy)}[\underline{\mathsf{m}}] + \underline{(fz)}[\underline{\mathsf{m}}]; \text{PlusF} \gg \underline{(fx)} +_{\mathrm{f}} \underline{(fz)}[\underline{\mathsf{m}}] = \underline{(fx)}[\underline{\mathsf{m}}] + \underline{(fz)}[\underline{\mathsf{m}}]; \text{PlusF} \gg \underline{(fy)} +_{\mathrm{f}} \underline{(fz)}[\underline{\mathsf{m}}] =$

(fy)[m] + (fz)[m]; EqNegated ▷ (fy) $+_f$ (fz)[m] = (fy)[m] + (fz)[m] ≫ −(fy) $+_f$ (fz)[m] = −(fy)[m] + (fz)[m]; AddEquations ▷ (fx) $+_f$ (fz)[m] = (fx)[m] + (fz)[m] ▷ −(fy) $+_f$ (fz)[m] = −(fy)[m] + (fz)[m] ≫ (fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m] = (fx)[m] + (fz)[m] + −(fy)[m] + (fz)[m]; eqSymmetry ▷ (fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m] = (fx)[m] + (fz)[m] + −(fy)[m] + (fz)[m] ≫ (fx)[m] + (fz)[m] + −(fy)[m] + (fz)[m] = (fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m]; eqTransitivity ▷ (fx)[m] + −(fy)[m] = (fx)[m] + (fz)[m] + −(fy)[m] + (fz)[m] ▷ (fx)[m] + (fz)[m] + −(fy)[m] + (fz)[m] = (fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m] ≫ (fx)[m] + −(fy)[m] = (fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m]; SameNumerical ▷ (fx)[m] + −(fy)[m] = (fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m] ≫ if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) = if(0 <= (fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m], (fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m], −(fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m]); SubLessLeft ▷ if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) = if(0 <= (fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m], (fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m], −(fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m]) ▷ $\dot\neg$if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) <= (ε) ⇒ $\dot\neg\dot\neg$if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) = (ε) ≫ $\dot\neg$if(0 <= (fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m], (fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m], −(fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m]) <= (ε) ⇒ $\dot\neg\dot\neg$if(0 <= (fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m], (fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m], −(fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m]) = (ε); ∀m: ∀(ε): ∀(fx): ∀(fy): ∀(fz): Ded ▷ ∀m: ∀(ε): ∀(fx): ∀(fy): ∀(fz): R((fx)) == R((fy)) ⊢ $\dot\neg$0 <= (ε) ⇒ $\dot\neg\dot\neg$0 = (ε) ⊢ c_{Ex} <= m ⊢ $\dot\neg$if(0 <= (fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m], (fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m], −(fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m]) <= (ε) ⇒ $\dot\neg\dot\neg$if(0 <= (fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m], (fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m], −(fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m]) = (ε) ≫ R((fx)) == R((fy)) ⇒ $\dot\neg$0 <= (ε) ⇒ $\dot\neg\dot\neg$0 = (ε) ⇒ c_{Ex} <= m ⇒ $\dot\neg$if(0 <= (fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m], (fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m], −(fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m]) <= (ε) ⇒ $\dot\neg\dot\neg$if(0 <= (fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m], (fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m], −(fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m]) = (ε); R((fx)) == R((fy)) ⊢ MP ▷ R((fx)) == R((fy)) ⇒ $\dot\neg$0 <= (ε) ⇒ $\dot\neg\dot\neg$0 = (ε) ⇒ c_{Ex} <= m ⇒ $\dot\neg$if(0 <= (fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m], (fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m], −(fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m]) <= (ε) ⇒ $\dot\neg\dot\neg$if(0 <= (fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m], (fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m], −(fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m]) = (ε) ▷ R((fx)) == R((fy)) ≫ $\dot\neg$0 <= (ε) ⇒ $\dot\neg\dot\neg$0 = (ε) ⇒ c_{Ex} <= m ⇒ $\dot\neg$if(0 <= (fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m], (fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m], −(fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m]) <= (ε) ⇒ $\dot\neg\dot\neg$if(0 <= (fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m], (fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m], −(fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m]) = (ε); ToSF ▷ $\dot\neg$0 <= (ε) ⇒ $\dot\neg\dot\neg$0 = (ε) ⇒ c_{Ex} <= m ⇒ $\dot\neg$if(0 <= (fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m], (fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m], −(fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m]) <= (ε) ⇒ $\dot\neg\dot\neg$if(0 <= (fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m], (fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m], −(fx) $+_f$ (fz)[m] + −(fy) $+_f$ (fz)[m]) = (ε) ≫ SF((fx) $+_f$ (fz), (fy) $+_f$ (fz)); To == ▷SF((fx) $+_f$ (fz), (fy) $+_f$ (fz)) ≫ R((fx) $+_f$ (fz)) == R((fy) $+_f$ (fz)); PlusR ≫ R((fx) $+_f$ (fy)) == R((fx) $+_f$ (fz)); PlusR ≫ R((fx) $+_f$ (fy)) == R((fy) $+_f$ (fz)); ==Symmetry ▷ R((fx) $+_f$ (fy)) == R((fy) $+_f$ (fz)) ≫ R((fy) $+_f$ (fz)) == R((fx) $+_f$ (fy)); ==Transitivity ▷ R((fx) $+_f$ (fy)) ==

$R((\underline{fx}) +_f (\underline{fz})) \rhd R((\underline{fx}) +_f (\underline{fz})) == R((\underline{fy}) +_f (\underline{fz})) \gg R((fx) +_f (fy)) == R((\underline{fy}) +_f (\underline{fz}))$; $==$ Transitivity $\rhd R((fx) +_f (fy)) == R((\underline{fy}) +_f (\underline{fz})) \rhd R((\underline{fy}) +_f (\underline{fz})) == R((fx) +_f (fy)) \gg R((fx) +_f (fy)) == R((fx) +_f (fy))\rceil, p_0, \mathsf{c})]$

$[==$ AdditionLeft $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall(\underline{fx})\colon \forall(\underline{fy})\colon \forall(\underline{fz})\colon R((\underline{fx})) == R((\underline{fy})) \vdash R((fx) +_f (fy)) == R((fx) +_f (fy))]$

$[==$ AdditionLeft $\stackrel{\text{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil$ZFsub $\vdash \forall(\underline{fx})\colon \forall(\underline{fy})\colon \forall(\underline{fz})\colon R((\underline{fx})) == R((\underline{fy})) \vdash ==$ Addition $\rhd R((\underline{fx})) == R((\underline{fy})) \gg R((fx) +_f (fy)) == R((fx) +_f (fy))$; PlusCommutativity$(R) \gg R((fx) +_f (fy)) == R((fx) +_f (fy))$; PlusCommutativity$(R) \gg R((fx) +_f (fy)) == R((fx) +_f (fy))$; $==$ Transitivity $\rhd R((fx) +_f (fy)) == R((fx) +_f (fy)) \rhd R((fx) +_f (fy)) == R((fx) +_f (fy)) \gg R((fx) +_f (fy)) == R((fx) +_f (fy))$; $==$ Transitivity $\rhd R((fx) +_f (fy)) == R((fx) +_f (fy)) \rhd R((fx) +_f (fy)) == R((fx) +_f (fy)) \gg R((fx) +_f (fy)) == R((fx) +_f (fy))\rceil, p_0, \mathsf{c})]$

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$[<<$ Addition $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall\underline{\mathsf{m}}\colon \forall(\underline{\epsilon})\colon \forall(\underline{fx})\colon \forall(\underline{fy})\colon \forall(\underline{fz})\colon R((\underline{fx})) << R((\underline{fy})) \vdash R((fx) +_f (fy)) << R((fx) +_f (fy))]$

$[<<$ Addition $\stackrel{\text{proof}}{\rightarrow} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil$ZFsub $\vdash \forall\underline{\mathsf{m}}\colon \forall(\underline{\epsilon})\colon \forall(\underline{fx})\colon \forall(\underline{fy})\colon \forall(\underline{fz})\colon R((\underline{fx})) << R((\underline{fy})) \vdash \dot\neg 0 <= (\underline{\epsilon}) \Rightarrow \dot\neg\dot\neg 0 = (\underline{\epsilon}) \vdash \mathsf{c}_{Ex} <= \underline{\mathsf{m}} \vdash$ From $<< \rhd R((\underline{fx})) << R((\underline{fy})) \gg (\underline{fx}) <_f (\underline{fy})$; From $<$ f $\rhd (\underline{fx}) <_f (\underline{fy}) \rhd \dot\neg 0 <= (\underline{\epsilon}) \Rightarrow \dot\neg\dot\neg 0 = (\underline{\epsilon}) \gg \mathsf{c}_{Ex} <= \underline{\mathsf{m}} \Rightarrow (\underline{fx})[\underline{\mathsf{m}}] <= (\underline{fy})[\underline{\mathsf{m}}] + -(\underline{\epsilon})$; MP $\rhd \mathsf{c}_{Ex} <= \underline{\mathsf{m}} \Rightarrow (\underline{fx})[\underline{\mathsf{m}}] <= (\underline{fy})[\underline{\mathsf{m}}] + -(\underline{\epsilon}) \rhd \mathsf{c}_{Ex} <= \underline{\mathsf{m}} \gg (\underline{fx})[\underline{\mathsf{m}}] <= (\underline{fy})[\underline{\mathsf{m}}] + -(\underline{\epsilon})$; leqAddition $\rhd (\underline{fx})[\underline{\mathsf{m}}] <= (\underline{fy})[\underline{\mathsf{m}}] + -(\underline{\epsilon}) \gg (\underline{fx})[\underline{\mathsf{m}}] + (\underline{fz})[\underline{\mathsf{m}}] <= (\underline{fy})[\underline{\mathsf{m}}] + -(\underline{\epsilon}) + (\underline{fz})[\underline{\mathsf{m}}]$; PlusF(Sym) $\gg (\underline{fx})[\underline{\mathsf{m}}] + (\underline{fz})[\underline{\mathsf{m}}] = (\underline{fx}) +_f (\underline{fz})[\underline{\mathsf{m}}]$; subLeqLeft $\rhd (\underline{fx})[\underline{\mathsf{m}}] + (\underline{fz})[\underline{\mathsf{m}}] = (\underline{fx}) +_f (\underline{fz})[\underline{\mathsf{m}}] \rhd (\underline{fx})[\underline{\mathsf{m}}] + (\underline{fz})[\underline{\mathsf{m}}] <= (\underline{fy})[\underline{\mathsf{m}}] + -(\underline{\epsilon}) + (\underline{fz})[\underline{\mathsf{m}}] \gg (\underline{fx}) +_f (\underline{fz})[\underline{\mathsf{m}}] <= (\underline{fy})[\underline{\mathsf{m}}] + -(\underline{\epsilon}) + (\underline{fz})[\underline{\mathsf{m}}]$; Three2threeTerms $\gg (\underline{fy})[\underline{\mathsf{m}}] + -(\underline{\epsilon}) + (\underline{fz})[\underline{\mathsf{m}}] = (\underline{fy})[\underline{\mathsf{m}}] + (\underline{fz})[\underline{\mathsf{m}}] + -(\underline{\epsilon})$; PlusF(Sym) $\gg (\underline{fy})[\underline{\mathsf{m}}] + (\underline{fz})[\underline{\mathsf{m}}] = (\underline{fy}) +_f (\underline{fz})[\underline{\mathsf{m}}]$; eqAddition $\rhd (\underline{fy})[\underline{\mathsf{m}}] + (\underline{fz})[\underline{\mathsf{m}}] = (\underline{fy}) +_f (\underline{fz})[\underline{\mathsf{m}}] \gg (\underline{fy})[\underline{\mathsf{m}}] + (\underline{fz})[\underline{\mathsf{m}}] + -(\underline{\epsilon}) = (\underline{fy}) +_f (\underline{fz})[\underline{\mathsf{m}}] + -(\underline{\epsilon})$; eqTransitivity $\rhd (\underline{fy})[\underline{\mathsf{m}}] + -(\underline{\epsilon}) + (\underline{fz})[\underline{\mathsf{m}}] = (\underline{fy})[\underline{\mathsf{m}}] + (\underline{fz})[\underline{\mathsf{m}}] + -(\underline{\epsilon}) \rhd (\underline{fy})[\underline{\mathsf{m}}] + (\underline{fz})[\underline{\mathsf{m}}] + -(\underline{\epsilon}) = (\underline{fy}) +_f (\underline{fz})[\underline{\mathsf{m}}] + -(\underline{\epsilon}) \gg (\underline{fy})[\underline{\mathsf{m}}] + -(\underline{\epsilon}) + (\underline{fz})[\underline{\mathsf{m}}] = (\underline{fy}) +_f (\underline{fz})[\underline{\mathsf{m}}] + -(\underline{\epsilon})$; subLeqRight $\rhd (\underline{fy})[\underline{\mathsf{m}}] + -(\underline{\epsilon}) + (\underline{fz})[\underline{\mathsf{m}}] = (\underline{fy}) +_f (\underline{fz})[\underline{\mathsf{m}}] + -(\underline{\epsilon}) \rhd (\underline{fx}) +_f (\underline{fz})[\underline{\mathsf{m}}] <= (\underline{fy})[\underline{\mathsf{m}}] + -(\underline{\epsilon}) + (\underline{fz})[\underline{\mathsf{m}}] \gg (\underline{fx}) +_f (\underline{fz})[\underline{\mathsf{m}}] <= (\underline{fy}) +_f (\underline{fz})[\underline{\mathsf{m}}] + -(\underline{\epsilon})$; $\forall\underline{\mathsf{m}}\colon \forall(\underline{\epsilon})\colon \forall(\underline{fx})\colon \forall(\underline{fy})\colon \forall(\underline{fz})\colon$ Ded $\rhd \forall\underline{\mathsf{m}}\colon \forall(\underline{\epsilon})\colon \forall(\underline{fx})\colon \forall(\underline{fy})\colon \forall(\underline{fz})\colon R((\underline{fx})) << R((\underline{fy})) \vdash \dot\neg 0 <= (\underline{\epsilon}) \Rightarrow \dot\neg\dot\neg 0 = (\underline{\epsilon}) \vdash \mathsf{c}_{Ex} <= \underline{\mathsf{m}} \vdash (\underline{fx}) +_f (\underline{fz})[\underline{\mathsf{m}}] <= (\underline{fy}) +_f (\underline{fz})[\underline{\mathsf{m}}] + -(\underline{\epsilon}) \gg R((\underline{fx})) << R((\underline{fy})) \Rightarrow \dot\neg 0 <= (\underline{\epsilon}) \Rightarrow \dot\neg\dot\neg 0 = (\underline{\epsilon}) \Rightarrow \mathsf{c}_{Ex} <= \underline{\mathsf{m}} \Rightarrow (\underline{fx}) +_f (\underline{fz})[\underline{\mathsf{m}}] <= (\underline{fy}) +_f (\underline{fz})[\underline{\mathsf{m}}] + -(\underline{\epsilon})$; $R((\underline{fx})) << R((\underline{fy})) \vdash$ MP $\rhd R((\underline{fx})) << R((\underline{fy})) \Rightarrow \dot\neg 0 <= (\underline{\epsilon}) \Rightarrow \dot\neg\dot\neg 0 = (\underline{\epsilon}) \Rightarrow \mathsf{c}_{Ex} <= \underline{\mathsf{m}} \Rightarrow (\underline{fx}) +_f (\underline{fz})[\underline{\mathsf{m}}] <= (\underline{fy}) +_f (\underline{fz})[\underline{\mathsf{m}}] + -(\underline{\epsilon}) \rhd R((\underline{fx})) << R((\underline{fy})) \gg \dot\neg 0 <= (\underline{\epsilon}) \Rightarrow \dot\neg\dot\neg 0 = (\underline{\epsilon}) \Rightarrow \mathsf{c}_{Ex} <= \underline{\mathsf{m}} \Rightarrow (\underline{fx}) +_f (\underline{fz})[\underline{\mathsf{m}}] <= (\underline{fy}) +_f (\underline{fz})[\underline{\mathsf{m}}] + -(\underline{\epsilon})$; To $<$ f $\rhd \dot\neg 0 <= (\underline{\epsilon}) \Rightarrow \dot\neg\dot\neg 0 = (\underline{\epsilon}) \Rightarrow \mathsf{c}_{Ex} <= \underline{\mathsf{m}} \Rightarrow (\underline{fx}) +_f (\underline{fz})[\underline{\mathsf{m}}] <= (\underline{fy}) +_f (\underline{fz})[\underline{\mathsf{m}}] + -(\underline{\epsilon}) \gg (fx) +_f (fz) <_f (fy) +_f (fz)$; To $<< \rhd (\underline{fx}) +_f (\underline{fz}) <_f (\underline{fy}) +_f (\underline{fz}) \gg R((\underline{fx}) +_f (\underline{fz})) << R((\underline{fy}) +_f (\underline{fz}))$; PlusR(Sym) $\gg R((\underline{fx}) +_f (\underline{fz})) == R((fx) +_f (fy))$; PlusR(Sym) $\gg R((\underline{fy}) +_f (\underline{fz})) == R((fx) +_f (fy))$; SubLessLeft$(R) \rhd R((\underline{fx}) +_f (\underline{fz})) == R((fx) +_f (fy)) \rhd R((\underline{fx}) +_f (\underline{fz})) << R((\underline{fy}) +_f (\underline{fz})) \gg R((fx) +_f (fy)) <<$

R((fy) $+_f$ (fz)); SubLessRight(R) ▷ R((fy) $+_f$ (fz)) == R((fx) $+_f$ (fy)) ▷
R((fx) $+_f$ (fy)) << R((fy) $+_f$ (fz)) ≫ R((fx) $+_f$ (fy)) << R((fx) $+_f$ (fy))⌉, p_0, c)]
(***)

[<<== Addition $\overset{\text{stmt}}{\to}$ ZFsub ⊢ ∀(fx): ∀(fy): ∀(fz): $\dot\neg$ R((fx)) << R((fy)) ⇒
R((fx)) == R((fy)) ⊢ $\dot\neg$ R((fx) $+_f$ (fy)) << R((fx) $+_f$ (fy)) ⇒
R((fx) $+_f$ (fy)) == R((fx) $+_f$ (fy))]

[<<== Addition $\overset{\text{proof}}{\to}$ λc.λx.$\mathcal{P}$(⌈ZFsub ⊢ ∀(fx): ∀(fy): ∀(fz): R((fx)) <<
R((fy)) ⊢<< Addition ▷ R((fx)) << R((fy)) ≫ R((fx) $+_f$ (fy)) <<
R((fx) $+_f$ (fy)); LessLeq(R) ▷ R((fx) $+_f$ (fy)) << R((fx) $+_f$ (fy)) ≫
$\dot\neg$ R((fx) $+_f$ (fy)) << R((fx) $+_f$ (fy)) ⇒ R((fx) $+_f$ (fy)) == R((fx) $+_f$
(fy)); ∀(fx): ∀(fy): ∀(fz): R((fx)) == R((fy)) ⊢== Addition ▷ R((fx)) ==
R((fy)) ≫ R((fx) $+_f$ (fy)) == R((fx) $+_f$ (fy)); eqLeq(R) ▷ R((fx) $+_f$ (fy)) ==
R((fx) $+_f$ (fy)) ≫ $\dot\neg$ R((fx) $+_f$ (fy)) << R((fx) $+_f$ (fy)) ⇒ R((fx) $+_f$ (fy)) ==
R((fx) $+_f$ (fy)); ∀(fx): ∀(fy): ∀(fz): Ded ▷ ∀(fx): ∀(fy): ∀(fz): R((fx)) << R((fy)) ⊢
$\dot\neg$ R((fx) $+_f$ (fy)) << R((fx) $+_f$ (fy)) ⇒ R((fx) $+_f$ (fy)) == R((fx) $+_f$ (fy)) ≫
R((fx)) << R((fy)) ⇒ $\dot\neg$ R((fx) $+_f$ (fy)) << R((fx) $+_f$ (fy)) ⇒
R((fx) $+_f$ (fy)) == R((fx) $+_f$ (fy)); Ded ▷ ∀(fx): ∀(fy): ∀(fz): R((fx)) ==
R((fy)) ⊢ $\dot\neg$ R((fx) $+_f$ (fy)) << R((fx) $+_f$ (fy)) ⇒ R((fx) $+_f$ (fy)) ==
R((fx) $+_f$ (fy)) ≫ R((fx)) == R((fy)) ⇒ $\dot\neg$ R((fx) $+_f$ (fy)) << R((fx) $+_f$ (fy)) ⇒
R((fx) $+_f$ (fy)) == R((fx) $+_f$ (fy)); $\dot\neg$ R((fx)) << R((fy)) ⇒ R((fx)) ==
R((fy)) ⊢ Repetition ▷ $\dot\neg$ R((fx)) << R((fy)) ⇒ R((fx)) == R((fy)) ≫
$\dot\neg$ R((fx)) << R((fy)) ⇒ R((fx)) == R((fy)); FromDisjuncts ▷ $\dot\neg$ R((fx)) <<
R((fy)) ⇒ R((fx)) == R((fy)) ▷ R((fx)) << R((fy)) ⇒ $\dot\neg$ R((fx) $+_f$ (fy)) <<
R((fx) $+_f$ (fy)) ⇒ R((fx) $+_f$ (fy)) == R((fx) $+_f$ (fy)) ▷ R((fx)) == R((fy)) ⇒
$\dot\neg$ R((fx) $+_f$ (fy)) << R((fx) $+_f$ (fy)) ⇒ R((fx) $+_f$ (fy)) == R((fx) $+_f$ (fy)) ≫
$\dot\neg$ R((fx) $+_f$ (fy)) << R((fx) $+_f$ (fy)) ⇒ R((fx) $+_f$ (fy)) == R((fx) $+_f$ (fy))⌉, p_0, c)]
(*** NUMMER 7 ***)

[PlusAssociativity(F) $\overset{\text{stmt}}{\to}$ ZFsub ⊢ ∀$\underline{m}$: ∀(fx): ∀(fy): ∀(fz): (fx) $+_f$ (fy) $+_f$ (fz) $=_f$
(fx) $+_f$ (fy) $+_f$ (fz)]

[PlusAssociativity(F) $\overset{\text{proof}}{\to}$ λc.λx.$\mathcal{P}$(⌈ZFsub ⊢ ∀$\underline{m}$: ∀(fx): ∀(fy): ∀(fz): PlusF ≫
(fx) $+_f$ (fy) $+_f$ (fz)[$\underline{m}$] = (fx) $+_f$ (fy)[$\underline{m}$] + (fz)[$\underline{m}$]; PlusF ≫ (fx) $+_f$ (fy)[$\underline{m}$] =
(fx)[$\underline{m}$] + (fy)[$\underline{m}$]; eqAddition ▷ (fx) $+_f$ (fy)[$\underline{m}$] = (fx)[$\underline{m}$] + (fy)[$\underline{m}$] ≫
(fx) $+_f$ (fy)[$\underline{m}$] + (fz)[$\underline{m}$] = (fx)[$\underline{m}$] + (fy)[$\underline{m}$] + (fz)[$\underline{m}$]; plusAssociativity ≫
(fx)[$\underline{m}$] + (fy)[$\underline{m}$] + (fz)[$\underline{m}$] = (fx)[$\underline{m}$] + (fy)[$\underline{m}$] + (fz)[$\underline{m}$]; PlusF(Sym) ≫
(fy)[$\underline{m}$] + (fz)[$\underline{m}$] = (fy) $+_f$ (fz)[$\underline{m}$]; lemma eqAdditionLeft ▷ (fy)[$\underline{m}$] + (fz)[$\underline{m}$] =
(fy) $+_f$ (fz)[$\underline{m}$] ≫ (fx)[$\underline{m}$] + (fy)[$\underline{m}$] + (fz)[$\underline{m}$] =
(fx)[$\underline{m}$] + (fy) $+_f$ (fz)[$\underline{m}$]; PlusF(Sym) ≫ (fx)[$\underline{m}$] + (fy) $+_f$ (fz)[$\underline{m}$] =
(fx) $+_f$ (fy) $+_f$ (fz)[$\underline{m}$]; eqTransitivity6 ▷ (fx) $+_f$ (fy) $+_f$ (fz)[$\underline{m}$] =
(fx) $+_f$ (fy)[$\underline{m}$] + (fz)[$\underline{m}$] ▷ (fx) $+_f$ (fy)[$\underline{m}$] + (fz)[$\underline{m}$] = (fx)[$\underline{m}$] + (fy)[$\underline{m}$] + (fz)[$\underline{m}$] ▷
(fx)[$\underline{m}$]+(fy)[$\underline{m}$]+(fz)[$\underline{m}$] = (fx)[$\underline{m}$]+(fy)[$\underline{m}$]+(fz)[$\underline{m}$] ▷ (fx)[$\underline{m}$]+(fy)[$\underline{m}$]+(fz)[$\underline{m}$] =
(fx)[$\underline{m}$] + (fy) $+_f$ (fz)[$\underline{m}$] ▷ (fx)[$\underline{m}$] + (fy) $+_f$ (fz)[$\underline{m}$] = (fx) $+_f$ (fy) $+_f$ (fz)[$\underline{m}$] ≫
(fx) $+_f$ (fy) $+_f$ (fz)[$\underline{m}$] = (fx) $+_f$ (fy) $+_f$ (fz)[$\underline{m}$]; To = f ▷ (fx) $+_f$ (fy) $+_f$ (fz)[$\underline{m}$] =
(fx) $+_f$ (fy) $+_f$ (fz)[$\underline{m}$] ≫ (fx) $+_f$ (fy) $+_f$ (fz) $=_f$ (fx) $+_f$ (fy) $+_f$ (fz)⌉, p_0, c)]

$[\mathrm{PlusAssociativity(R)} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{(\mathrm{fx})} \colon \forall \underline{(\mathrm{fy})} \colon \forall \underline{(\mathrm{fz})} \colon \mathrm{R}(\underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})} +_{\mathrm{f}} \underline{(\mathrm{fz})}) == \mathrm{R}(\underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})} +_{\mathrm{f}} \underline{(\mathrm{fz})})]$

$[\mathrm{PlusAssociativity(R)} \overset{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{(\mathrm{fx})} \colon \forall \underline{(\mathrm{fy})} \colon \forall \underline{(\mathrm{fz})} \colon \mathrm{PlusAssociativity(F)} \gg \underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})} +_{\mathrm{f}} \underline{(\mathrm{fz})} =_{\mathrm{f}} \underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})} +_{\mathrm{f}} \underline{(\mathrm{fz})}; = \mathrm{fToSameF} \rhd \underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})} +_{\mathrm{f}} \underline{(\mathrm{fz})} =_{\mathrm{f}} \underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})} +_{\mathrm{f}} \underline{(\mathrm{fz})} \gg \mathrm{SF}(\underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})} +_{\mathrm{f}} \underline{(\mathrm{fz})}, \underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})} +_{\mathrm{f}} \underline{(\mathrm{fz})}); \mathrm{f2R(Plus)} \rhd \mathrm{SF}(\underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})} +_{\mathrm{f}} \underline{(\mathrm{fz})}, \underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})} +_{\mathrm{f}} \underline{(\mathrm{fz})}) \gg \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}(\underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})} +_{\mathrm{f}} \underline{(\mathrm{fz})}); \mathrm{PlusR(Sym)} \gg \mathrm{R}(\underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})} +_{\mathrm{f}} \underline{(\mathrm{fz})}) == \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})); == \mathrm{Transitivity} \rhd \mathrm{R}(\underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})} +_{\mathrm{f}} \underline{(\mathrm{fz})}) == \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) \rhd \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}(\underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})} +_{\mathrm{f}} \underline{(\mathrm{fz})}) \gg \mathrm{R}(\underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})} +_{\mathrm{f}} \underline{(\mathrm{fz})}) == \mathrm{R}(\underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})} +_{\mathrm{f}} \underline{(\mathrm{fz})}) \rceil, \mathrm{p}_0, \mathsf{c})]$
(*** NUMMER 8 ***)

$[\mathrm{Plus00} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{(\mathrm{fx})} \colon \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}(\underline{(\mathrm{fx})})]$

$[\mathrm{Plus00} \overset{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{(\mathrm{fx})} \colon \mathrm{Plus0f} \gg \underline{(\mathrm{fx})} +_{\mathrm{f}} 0\mathrm{f} =_{\mathrm{f}} \underline{(\mathrm{fx})}; = \mathrm{fToSameF} \rhd \underline{(\mathrm{fx})} +_{\mathrm{f}} 0\mathrm{f} =_{\mathrm{f}} \underline{(\mathrm{fx})} \gg \mathrm{SF}(\underline{(\mathrm{fx})} +_{\mathrm{f}} 0\mathrm{f}, \underline{(\mathrm{fx})}); \mathrm{f2R(Plus)} \rhd \mathrm{SF}(\underline{(\mathrm{fx})} +_{\mathrm{f}} 0\mathrm{f}, \underline{(\mathrm{fx})}) \gg \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}(\underline{(\mathrm{fx})}); \mathrm{Repetition} \rhd \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}(\underline{(\mathrm{fx})}) \gg \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}(\underline{(\mathrm{fx})}) \rceil, \mathrm{p}_0, \mathsf{c})]$
(*** NUMMER 9 ***)

$[\mathrm{Negative(R)} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}} \colon \forall \underline{(\mathrm{fx})} \colon \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}(0\mathrm{f})]$

$[\mathrm{Negative(R)} \overset{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}} \colon \forall \underline{(\mathrm{fx})} \colon \mathrm{PlusF} \gg \underline{(\mathrm{fx})} +_{\mathrm{f}} -_{\mathrm{f}} \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -_{\mathrm{f}} \underline{(\mathrm{fx})}[\underline{\mathsf{m}}]; \mathrm{MinusF} \gg -_{\mathrm{f}} \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] = -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}]; \mathrm{lemma\ eqAdditionLeft} \rhd -_{\mathrm{f}} \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] = -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] \gg \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -_{\mathrm{f}} \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}]; \mathrm{Negative} \gg \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] = 0; 0\mathrm{f} \gg 0\mathrm{f}[\underline{\mathsf{m}}] = 0; \mathrm{eqSymmetry} \rhd 0\mathrm{f}[\underline{\mathsf{m}}] = 0 \gg 0 = 0\mathrm{f}[\underline{\mathsf{m}}]; \mathrm{eqTransitivity5} \rhd \underline{(\mathrm{fx})} +_{\mathrm{f}} -_{\mathrm{f}} \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -_{\mathrm{f}} \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] \rhd \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -_{\mathrm{f}} \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] \rhd \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] = 0 \rhd 0 = 0\mathrm{f}[\underline{\mathsf{m}}] \gg \underline{(\mathrm{fx})} +_{\mathrm{f}} -_{\mathrm{f}} \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] = 0\mathrm{f}[\underline{\mathsf{m}}]; \mathrm{To} = \mathrm{f} \rhd \underline{(\mathrm{fx})} +_{\mathrm{f}} -_{\mathrm{f}} \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] = 0\mathrm{f}[\underline{\mathsf{m}}] \gg \underline{(\mathrm{fx})} +_{\mathrm{f}} -_{\mathrm{f}} \underline{(\mathrm{fx})} =_{\mathrm{f}} 0\mathrm{f}; = \mathrm{fToSameF} \rhd \underline{(\mathrm{fx})} +_{\mathrm{f}} -_{\mathrm{f}} \underline{(\mathrm{fx})} =_{\mathrm{f}} 0\mathrm{f} \gg \mathrm{SF}(\underline{(\mathrm{fx})} +_{\mathrm{f}} -_{\mathrm{f}} \underline{(\mathrm{fx})}, 0\mathrm{f}); \mathrm{f2R(Plus)} \rhd \mathrm{SF}(\underline{(\mathrm{fx})} +_{\mathrm{f}} -_{\mathrm{f}} \underline{(\mathrm{fx})}, 0\mathrm{f}) \gg \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}(0\mathrm{f}); \mathrm{Repetition} \rhd \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}(0\mathrm{f}) \gg \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}(0\mathrm{f}) \rceil, \mathrm{p}_0, \mathsf{c})]$
(*** NUMMER 10 ***)

$[\mathrm{PlusCommutativity(F)} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}} \colon \forall \underline{(\mathrm{fx})} \colon \forall \underline{(\mathrm{fy})} \colon \underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})} =_{\mathrm{f}} \underline{(\mathrm{fy})} +_{\mathrm{f}} \underline{(\mathrm{fx})}]$

$[\mathrm{PlusCommutativity(F)} \overset{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}} \colon \forall \underline{(\mathrm{fx})} \colon \forall \underline{(\mathrm{fy})} \colon \mathrm{PlusF} \gg \underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + \underline{(\mathrm{fy})}[\underline{\mathsf{m}}]; \mathrm{plusCommutativity} \gg \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + \underline{(\mathrm{fx})}[\underline{\mathsf{m}}]; \mathrm{PlusF(Sym)} \gg \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fy})} +_{\mathrm{f}} \underline{(\mathrm{fx})}[\underline{\mathsf{m}}]; \mathrm{eqTransitivity4} \rhd \underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] \rhd \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] \rhd \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fy})} +_{\mathrm{f}} \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] \gg \underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fy})} +_{\mathrm{f}} \underline{(\mathrm{fx})}[\underline{\mathsf{m}}]; \mathrm{To} = \mathrm{f} \rhd \underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fy})} +_{\mathrm{f}} \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] \gg \underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})} =_{\mathrm{f}} \underline{(\mathrm{fy})} +_{\mathrm{f}} \underline{(\mathrm{fx})} \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{PlusCommutativity(R)} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{(\mathrm{fx})} \colon \forall \underline{(\mathrm{fy})} \colon \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}))]$

$[\mathrm{PlusCommutativity(R)} \overset{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{(\mathrm{fx})} \colon \forall \underline{(\mathrm{fy})} \colon \mathrm{PlusCommutativity(F)} \gg \underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})} =_{\mathrm{f}} \underline{(\mathrm{fy})} +_{\mathrm{f}} \underline{(\mathrm{fx})}; = \mathrm{fToSameF} \rhd \underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})} =_{\mathrm{f}} \underline{(\mathrm{fy})} +_{\mathrm{f}} \underline{(\mathrm{fx})} \gg \mathrm{SF}(\underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})}, \underline{(\mathrm{fy})} +_{\mathrm{f}}$

$\underline{(fx)}); f2R(Plus) \rhd SF(\underline{(fx)} +_f \underline{(fy)}, \underline{(fy)} +_f \underline{(fx)}) \gg R((fx) +_f (fy)) ==$
$R((fy) +_f (fx)); PlusR(Sym) \gg R((fy) +_f (fx)) == R((fx) +_f (fy)); ==$
$Transitivity \rhd R((fx) +_f (fy)) == R((fy) +_f (fx)) \rhd R(\underline{(fy)} +_f \underline{(fx)}) ==$
$R((fx) +_f (fy)) \gg R((fx) +_f (fy)) == R((fx) +_f (fy))\rceil, p_0, c)]$

(*** NUMMER 11 ***)

$[TimesAssociativity(F) \overset{stmt}{\rightarrow} ZFsub \vdash \forall \underline{m}\colon \forall \underline{(fx)}\colon \forall \underline{(fy)}\colon \forall \underline{(fz)}\colon \underline{(fx)} *_f \underline{(fy)} *_f \underline{(fz)} =_f \underline{(fx)} *_f \underline{(fy)} *_f \underline{(fz)}]$

$[TimesAssociativity(F) \overset{proof}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil ZFsub \vdash \forall \underline{m}\colon \forall \underline{(fx)}\colon \forall \underline{(fy)}\colon \forall \underline{(fz)}\colon TimesF \gg \underline{(fx)} *_f \underline{(fy)} *_f \underline{(fz)}[\underline{m}] = \underline{(fx)} *_f \underline{(fy)}[\underline{m}] * \underline{(fz)}[\underline{m}]; TimesF \gg \underline{(fx)} *_f \underline{(fy)}[\underline{m}] = \underline{(fx)}[\underline{m}] * \underline{(fy)}[\underline{m}]; eqMultiplication \rhd \underline{(fx)} *_f \underline{(fy)}[\underline{m}] = \underline{(fx)}[\underline{m}] * \underline{(fy)}[\underline{m}] \gg \underline{(fx)} *_f \underline{(fy)}[\underline{m}] * \underline{(fz)}[\underline{m}] = \underline{(fx)}[\underline{m}] * \underline{(fy)}[\underline{m}] * \underline{(fz)}[\underline{m}]; timesAssociativity \gg \underline{(fx)}[\underline{m}] * \underline{(fy)}[\underline{m}] * \underline{(fz)}[\underline{m}] = \underline{(fx)}[\underline{m}] * \underline{(fy)}[\underline{m}] * \underline{(fz)}[\underline{m}]; TimesF(Sym) \gg \underline{(fy)}[\underline{m}] * \underline{(fz)}[\underline{m}] = \underline{(fy)} *_f \underline{(fz)}[\underline{m}]; EqMultiplicationLeft \rhd \underline{(fy)}[\underline{m}] * \underline{(fz)}[\underline{m}] = \underline{(fy)} *_f \underline{(fz)}[\underline{m}] \gg \underline{(fx)}[\underline{m}] * \underline{(fy)}[\underline{m}] * \underline{(fz)}[\underline{m}] = \underline{(fx)}[\underline{m}] * \underline{(fy)} *_f \underline{(fz)}[\underline{m}]; TimesF(Sym) \gg \underline{(fx)}[\underline{m}] * \underline{(fy)} *_f \underline{(fz)}[\underline{m}] = \underline{(fx)} *_f \underline{(fy)} *_f \underline{(fz)}[\underline{m}]; eqTransitivity6 \rhd \underline{(fx)} *_f \underline{(fy)} *_f \underline{(fz)}[\underline{m}] = \underline{(fx)} *_f \underline{(fy)}[\underline{m}] * \underline{(fz)}[\underline{m}] \rhd \underline{(fx)} *_f \underline{(fy)}[\underline{m}] * \underline{(fz)}[\underline{m}] = \underline{(fx)}[\underline{m}] * \underline{(fy)}[\underline{m}] * \underline{(fz)}[\underline{m}] \rhd \underline{(fx)}[\underline{m}] * \underline{(fy)}[\underline{m}] * \underline{(fz)}[\underline{m}] = \underline{(fx)}[\underline{m}] * \underline{(fy)}[\underline{m}] * \underline{(fz)}[\underline{m}] \rhd \underline{(fx)}[\underline{m}] * \underline{(fy)}[\underline{m}] * \underline{(fz)}[\underline{m}] = \underline{(fx)}[\underline{m}] * \underline{(fy)} *_f \underline{(fz)}[\underline{m}] \rhd \underline{(fx)}[\underline{m}] * \underline{(fy)} *_f \underline{(fz)}[\underline{m}] = \underline{(fx)} *_f \underline{(fy)} *_f \underline{(fz)}[\underline{m}] \gg \underline{(fx)} *_f \underline{(fy)} *_f \underline{(fz)}[\underline{m}] = \underline{(fx)} *_f \underline{(fy)} *_f \underline{(fz)}[\underline{m}]; To = f \rhd \underline{(fx)} *_f \underline{(fy)} *_f \underline{(fz)}[\underline{m}] = \underline{(fx)} *_f \underline{(fy)} *_f \underline{(fz)}[\underline{m}] \gg \underline{(fx)} *_f \underline{(fy)} *_f \underline{(fz)} =_f \underline{(fx)} *_f \underline{(fy)} *_f \underline{(fz)}\rceil, p_0, c)]$

$[TimesAssociativity(R) \overset{stmt}{\rightarrow} ZFsub \vdash \forall \underline{(fx)}\colon \forall \underline{(fy)}\colon \forall \underline{(fz)}\colon R(\underline{(fx)} *_f \underline{(fy)} *_f \underline{(fz)}) == R(\underline{(fx)} *_f \underline{(fy)} *_f \underline{(fz)})]$

$[TimesAssociativity(R) \overset{proof}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil ZFsub \vdash \forall \underline{(fx)}\colon \forall \underline{(fy)}\colon \forall \underline{(fz)}\colon TimesAssociativity(F) \gg \underline{(fx)} *_f \underline{(fy)} *_f \underline{(fz)} =_f \underline{(fx)} *_f \underline{(fy)} *_f \underline{(fz)}; = fToSameF \rhd \underline{(fx)} *_f \underline{(fy)} *_f \underline{(fz)} =_f \underline{(fx)} *_f \underline{(fy)} *_f \underline{(fz)} \gg SF(\underline{(fx)} *_f \underline{(fy)} *_f \underline{(fz)}, \underline{(fx)} *_f \underline{(fy)} *_f \underline{(fz)}); f2R(Times) \rhd SF(\underline{(fx)} *_f \underline{(fy)} *_f \underline{(fz)}, \underline{(fx)} *_f \underline{(fy)} *_f \underline{(fz)}) \gg R(\underline{(fx)} *_f \underline{(fy)}) * *R(\underline{(fz)}) == R(\underline{(fx)} *_f \underline{(fy)} *_f \underline{(fz)}); TimesR(Sym) \gg R(\underline{(fx)} *_f \underline{(fy)} *_f \underline{(fz)}) == R(\underline{(fx)} *_f \underline{(fy)}) * *R(\underline{(fz)}); == Transitivity \rhd R(\underline{(fx)} *_f \underline{(fy)} *_f \underline{(fz)}) == R(\underline{(fx)} *_f \underline{(fy)}) * *R(\underline{(fz)}) \rhd R(\underline{(fx)} *_f \underline{(fy)}) * *R(\underline{(fz)}) == R(\underline{(fx)} *_f \underline{(fy)} *_f \underline{(fz)}) \gg R(\underline{(fx)} *_f \underline{(fy)} *_f \underline{(fz)}) == R(\underline{(fx)} *_f \underline{(fy)} *_f \underline{(fz)})\rceil, p_0, c)]$

(*** NUMMER 12 ***)

$[Times1f \overset{stmt}{\rightarrow} ZFsub \vdash \forall \underline{m}\colon \forall \underline{(fx)}\colon \underline{(fx)} *_f 1f =_f \underline{(fx)}]$

$[Times1f \overset{proof}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil ZFsub \vdash \forall \underline{m}\colon \forall \underline{(fx)}\colon TimesF \gg \underline{(fx)} *_f 1f[\underline{m}] = \underline{(fx)}[\underline{m}] * 1f[\underline{m}]; 1f \gg 1f[\underline{m}] = 1; EqMultiplicationLeft \rhd 1f[\underline{m}] = 1 \gg \underline{(fx)}[\underline{m}] * 1f[\underline{m}] = \underline{(fx)}[\underline{m}] * 1; times1 \gg \underline{(fx)}[\underline{m}] * 1 = \underline{(fx)}[\underline{m}]; eqTransitivity4 \rhd \underline{(fx)} *_f 1f[\underline{m}] = \underline{(fx)}[\underline{m}] * 1f[\underline{m}] \rhd \underline{(fx)}[\underline{m}] * 1f[\underline{m}] = \underline{(fx)}[\underline{m}] * 1 \rhd \underline{(fx)}[\underline{m}] * 1 = \underline{(fx)}[\underline{m}] \gg \underline{(fx)} *_f 1f[\underline{m}] = \underline{(fx)}[\underline{m}]; To = f \rhd \underline{(fx)} *_f 1f[\underline{m}] = \underline{(fx)}[\underline{m}] \gg \underline{(fx)} *_f 1f =_f \underline{(fx)}\rceil, p_0, c)]$

$[Times01 \overset{stmt}{\rightarrow} ZFsub \vdash \forall \underline{(fx)}\colon R(\underline{(fx)}) * *R(1f) == R(\underline{(fx)})]$

$[Times01 \overset{proof}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil ZFsub \vdash \forall \underline{(fx)}\colon Times1f \gg \underline{(fx)} *_f 1f =_f \underline{(fx)}; = fToSameF \rhd \underline{(fx)} *_f 1f =_f \underline{(fx)} \gg SF(\underline{(fx)} *_f 1f, \underline{(fx)}); f2R(Times) \rhd SF(\underline{(fx)} *_f 1f, \underline{(fx)}) \gg R(\underline{(fx)}) * *R(1f) == R(\underline{(fx)}); Repetition \rhd R(\underline{(fx)}) * *R(1f) ==$

$\mathrm{R}(\underline{(\mathrm{fx})}) \gg \mathrm{R}(\underline{(\mathrm{fx})}) **\mathrm{R}(1\mathrm{f}) == \mathrm{R}(\underline{(\mathrm{fx})})\rceil, \mathrm{p}_0, \mathsf{c})]$
(*** NUMMER 14 ***)

$[\mathrm{TimesCommutativity(F)} \stackrel{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}} \colon \forall \underline{(\mathrm{fx})} \colon \forall \underline{(\mathrm{fy})} \colon \underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fy})} =_\mathrm{f} \underline{(\mathrm{fy})} *_\mathrm{f} \underline{(\mathrm{fx})}]$

$[\mathrm{TimesCommutativity(F)} \stackrel{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}} \colon \forall \underline{(\mathrm{fx})} \colon \forall \underline{(\mathrm{fy})} \colon \mathrm{TimesF} \gg$
$\underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fy})}[\underline{\mathsf{m}}]; \mathrm{timesCommutativity} \gg \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] =$
$\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fx})}[\underline{\mathsf{m}}]; \mathrm{TimesF(Sym)} \gg \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] =$
$\underline{(\mathrm{fy})} *_\mathrm{f} \underline{(\mathrm{fx})}[\underline{\mathsf{m}}]; \mathrm{eqTransitivity4} \rhd \underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] =$
$\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] \rhd \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] \rhd \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] =$
$\underline{(\mathrm{fy})} *_\mathrm{f} \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] \gg \underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fy})} *_\mathrm{f} \underline{(\mathrm{fx})}[\underline{\mathsf{m}}]; \mathrm{To} = \mathrm{f} \rhd \underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] =$
$\underline{(\mathrm{fy})} *_\mathrm{f} \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] \gg \underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fy})} =_\mathrm{f} \underline{(\mathrm{fy})} *_\mathrm{f} \underline{(\mathrm{fx})}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{TimesCommutativity(R)} \stackrel{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{(\mathrm{fx})} \colon \forall \underline{(\mathrm{fy})} \colon \mathrm{R}(\underline{(\mathrm{fx})}) **\mathrm{R}(\underline{(\mathrm{fy})}) ==$
$\mathrm{R}(\underline{(\mathrm{fy})}) **\mathrm{R}(\underline{(\mathrm{fx})})]$

$[\mathrm{TimesCommutativity(R)} \stackrel{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash$
$\forall \underline{(\mathrm{fx})} \colon \forall \underline{(\mathrm{fy})} \colon \mathrm{TimesCommutativity(F)} \gg \underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fy})} =_\mathrm{f} \underline{(\mathrm{fy})} *_\mathrm{f} \underline{(\mathrm{fx})}; =$
$\mathrm{fToSameF} \rhd \underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fy})} =_\mathrm{f} \underline{(\mathrm{fy})} *_\mathrm{f} \underline{(\mathrm{fx})} \gg \mathrm{SF}(\underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fy})}, \underline{(\mathrm{fy})} *_\mathrm{f}$
$\underline{(\mathrm{fx})}); \mathrm{f2R(Times)} \rhd \mathrm{SF}(\underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fy})}, \underline{(\mathrm{fy})} *_\mathrm{f} \underline{(\mathrm{fx})}) \gg \mathrm{R}(\underline{(\mathrm{fx})}) **\mathrm{R}(\underline{(\mathrm{fy})}) ==$
$\mathrm{R}(\underline{(\mathrm{fy})} *_\mathrm{f} \underline{(\mathrm{fx})}); \mathrm{TimesR(Sym)} \gg \mathrm{R}(\underline{(\mathrm{fy})} *_\mathrm{f} \underline{(\mathrm{fx})}) == \mathrm{R}(\underline{(\mathrm{fy})}) **\mathrm{R}(\underline{(\mathrm{fx})}); ==$
$\mathrm{Transitivity} \rhd \mathrm{R}(\underline{(\mathrm{fx})}) **\mathrm{R}(\underline{(\mathrm{fy})}) == \mathrm{R}(\underline{(\mathrm{fy})} *_\mathrm{f} \underline{(\mathrm{fx})}) \rhd \mathrm{R}(\underline{(\mathrm{fy})} *_\mathrm{f} \underline{(\mathrm{fx})}) ==$
$\mathrm{R}(\underline{(\mathrm{fy})}) **\mathrm{R}(\underline{(\mathrm{fx})}) \gg \mathrm{R}(\underline{(\mathrm{fx})}) **\mathrm{R}(\underline{(\mathrm{fy})}) == \mathrm{R}(\underline{(\mathrm{fy})}) **\mathrm{R}(\underline{(\mathrm{fx})})\rceil, \mathrm{p}_0, \mathsf{c})]$
(*** NUMMER 15 ***)

$[\mathrm{Distribution(F)} \stackrel{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{(\mathrm{fx})} \colon \forall \underline{(\mathrm{fy})} \colon \forall \underline{(\mathrm{fz})} \colon \underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fy})} +_\mathrm{f} \underline{(\mathrm{fz})} =_\mathrm{f}$
$\underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fy})} +_\mathrm{f} \underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fz})}]$

$[\mathrm{Distribution(F)} \stackrel{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{(\mathrm{fx})} \colon \forall \underline{(\mathrm{fy})} \colon \forall \underline{(\mathrm{fz})} \colon \mathrm{TimesF} \gg$
$\underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fy})} +_\mathrm{f} \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fy})} +_\mathrm{f} \underline{(\mathrm{fz})}[\underline{\mathsf{m}}]; \mathrm{PlusF} \gg \underline{(\mathrm{fy})} +_\mathrm{f} \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] =$
$\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + \underline{(\mathrm{fz})}[\underline{\mathsf{m}}]; \mathrm{EqMultiplicationLeft} \rhd \underline{(\mathrm{fy})} +_\mathrm{f} \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] \gg$
$\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fy})} +_\mathrm{f} \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + \underline{(\mathrm{fz})}[\underline{\mathsf{m}}]; \mathrm{Distribution} \gg$
$\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fz})}[\underline{\mathsf{m}}]; \mathrm{TimesF(Sym)} \gg$
$\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fy})}[\underline{\mathsf{m}}]; \mathrm{TimesF(Sym)} \gg \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] =$
$\underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fz})}[\underline{\mathsf{m}}]; \mathrm{AddEquations} \rhd \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] \rhd \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] =$
$\underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] \gg \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] =$
$\underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + \underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fz})}[\underline{\mathsf{m}}]; \mathrm{PlusF(Sym)} \gg \underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + \underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] =$
$\underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fy})} +_\mathrm{f} \underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fz})}[\underline{\mathsf{m}}]; \mathrm{eqTransitivity6} \rhd \underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fy})} +_\mathrm{f} \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] =$
$\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fy})} +_\mathrm{f} \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] \rhd \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fy})} +_\mathrm{f} \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] =$
$\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] \rhd \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] =$
$\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] \rhd \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] =$
$\underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + \underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] \rhd \underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + \underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] =$
$\underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fy})} +_\mathrm{f} \underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] \gg \underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fy})} +_\mathrm{f} \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] =$
$\underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fy})} +_\mathrm{f} \underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fz})}[\underline{\mathsf{m}}]; \mathrm{To} = \mathrm{f} \rhd \underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fy})} +_\mathrm{f} \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] =$
$\underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fy})} +_\mathrm{f} \underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] \gg \underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fy})} +_\mathrm{f} \underline{(\mathrm{fz})} =_\mathrm{f} \underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fy})} +_\mathrm{f} \underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fz})}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{Distribution(R)} \stackrel{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{(\mathrm{fx})} \colon \forall \underline{(\mathrm{fy})} \colon \forall \underline{(\mathrm{fz})} \colon \mathrm{R}(\underline{(\mathrm{fx})}) **\mathrm{R}(\underline{(\mathrm{fy})} +_\mathrm{f} \underline{(\mathrm{fz})}) ==$
$\mathrm{R}((\mathrm{fx}) +_\mathrm{f} (\mathrm{fy}))]$

$[\mathrm{Distribution(R)} \stackrel{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{(\mathrm{fx})} \colon \forall \underline{(\mathrm{fy})} \colon \forall \underline{(\mathrm{fz})} \colon \mathrm{Distribution(F)} \gg$
$\underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fy})} +_\mathrm{f} \underline{(\mathrm{fz})} =_\mathrm{f} \underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fy})} +_\mathrm{f} \underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fz})}; = \mathrm{fToSameF} \rhd \underline{(\mathrm{fx})} *_\mathrm{f} \underline{(\mathrm{fy})} +_\mathrm{f} \underline{(\mathrm{fz})} =_\mathrm{f}$

$\underline{(fx)} *_f \underline{(fy)} +_f \underline{(fx)} *_f \underline{(fz)} \gg SF(\underline{(fx)} *_f \underline{(fy)} +_f \underline{(fz)}, \underline{(fx)} *_f \underline{(fy)} +_f \underline{(fx)} *_f \underline{(fz)}); f2R(Times) \rhd SF(\overline{(fx)} *_f \overline{(fy)} +_f \overline{(fz)}, \overline{(fx)} *_f \overline{(fy)} +_f \overline{(fx)} *_f \overline{(fz)}) \gg R(\overline{(fx)}) * *R(\overline{(fy)} +_f \overline{(fz)}) == R(\overline{(fx)} *_f \overline{(fy)} +_f \overline{(fx)} *_f \overline{(fz)}); PlusR(Sym) \gg R(\overline{(fx)} *_f \overline{(fy)} +_f \overline{(fx)} *_f \overline{(fz)}) == R(\overline{(fx)} +_f \overline{(fy)}); == Transitivity \rhd R(\overline{(fx)}) * *R(\overline{(fy)} +_f \overline{(fz)}) == R(\overline{(fx)} *_f \overline{(fy)} +_f \overline{(fx)} *_f \overline{(fz)}) \rhd R(\overline{(fx)} *_f \overline{(fy)} +_f \overline{(fx)} *_f \overline{(fz)}) == R(\overline{(fx)} +_f \overline{(fy)}) \gg R(\overline{(fx)}) * *R(\underline{(fy)} +_f \underline{(fz)}) == R(\overline{(fx)} +_f \overline{(fy)})\rceil, \overline{p_0}, \mathsf{c})]$
(*** tex-definitioner ***)

$[(\epsilon) \overset{\text{tex}}{\to}$ "(\epsilon)"]

$[(fx) \overset{\text{tex}}{\to}$ "(fx)"]

$[(fy) \overset{\text{tex}}{\to}$ "(fy)"]

$[(fz) \overset{\text{tex}}{\to}$ "(fz)"]

$[(fv) \overset{\text{tex}}{\to}$ "(fu)"]

$[(fv) \overset{\text{tex}}{\to}$ "(fv)"]

$[(rx) \overset{\text{tex}}{\to}$ "(rx)"]

$[(ry) \overset{\text{tex}}{\to}$ "(ry)"]

$[(rz) \overset{\text{tex}}{\to}$ "(rz)"]

$[(ru) \overset{\text{tex}}{\to}$ "(ru)"]

$[\epsilon \overset{\text{tex}}{\to}$ "\epsilon"]

$[FX \overset{\text{tex}}{\to}$ "FX"]

$[FY \overset{\text{tex}}{\to}$ "FY"]

$[FZ \overset{\text{tex}}{\to}$ "FZ"]

$[FU \overset{\text{tex}}{\to}$ "FU"]

$[FV \overset{\text{tex}}{\to}$ "FV"]

$[RX \overset{\text{tex}}{\to}$ "RX"]

$[RY \overset{\text{tex}}{\to}$ "RY"]

$[RZ \overset{\text{tex}}{\to}$ "RZ"]

$[RU \overset{\text{tex}}{\to}$ "RU"]

[Ex3 $\stackrel{\text{tex}}{\to}$ “Ex3”]

[0 $\stackrel{\text{tex}}{\to}$ “0”]

[1 $\stackrel{\text{tex}}{\to}$ “1”]

[(-1) $\stackrel{\text{tex}}{\to}$ “(-1)”]

[2 $\stackrel{\text{tex}}{\to}$ “2”]

[1/2 $\stackrel{\text{tex}}{\to}$ “1/2”]

[0f $\stackrel{\text{tex}}{\to}$ “0f”]

[00 $\stackrel{\text{tex}}{\to}$ “00”]

[$\mathsf{x} = \mathsf{y}$ $\stackrel{\text{tex}}{\to}$ “#1.
= #2.”]

[$\mathsf{x} \neq \mathsf{y}$ $\stackrel{\text{tex}}{\to}$ “#1.
\neq #2.”]

[$\mathsf{x} < \mathsf{y}$ $\stackrel{\text{tex}}{\to}$ “#1.
< #2.”]

[$\mathsf{x} <= \mathsf{y}$ $\stackrel{\text{tex}}{\to}$ “#1.
<= #2.”]

[$\mathsf{x} =_{\mathsf{f}} \mathsf{y}$ $\stackrel{\text{tex}}{\to}$ “#1.
=_{f}#2.”]

[$\mathsf{x} <_{\mathsf{f}} \mathsf{y}$ $\stackrel{\text{tex}}{\to}$ “#1.
<_{f}#2.”]

[$\mathrm{SF}(\mathsf{x}, \mathsf{y})$ $\stackrel{\text{tex}}{\to}$ “SF(#1.
, #2.
)”]

[$\mathsf{x} == \mathsf{y}$ $\stackrel{\text{tex}}{\to}$ “#1.
== #2.”]

[$\mathsf{x} << \mathsf{y}$ $\stackrel{\text{tex}}{\to}$ “#1.
<< #2.”]

[$\mathsf{x} <<== \mathsf{y}$ $\stackrel{\text{tex}}{\to}$ “#1.
<<== #2.”]

[x[y] $\stackrel{\text{tex}}{\to}$ “#1.
[#2.
]”]

[−x $\stackrel{\text{tex}}{\to}$ “-#1.”]

[$-_{\mathrm{f}}$x $\stackrel{\text{tex}}{\to}$ “-_{f}#1.”]

[x + y $\stackrel{\text{tex}}{\to}$ “#1.
+#2.”]

[x − y $\stackrel{\text{tex}}{\to}$ “#1.
-#2.”]

[(fx) $+_{\mathrm{f}}$ (fy) $\stackrel{\text{tex}}{\to}$ “#1.
+_{f}#2.”]

[(fx) $-_{\mathrm{f}}$ (fy) $\stackrel{\text{tex}}{\to}$ “#1.
-_{f}#2.”]

[(fx) $*_{\mathrm{f}}$ (fy) $\stackrel{\text{tex}}{\to}$ “#1.
*_{f}#2.”]

[x + +y $\stackrel{\text{tex}}{\to}$ “#1.
++#2.”]

[R((fx)) − −R((fy)) $\stackrel{\text{tex}}{\to}$ “R(#1.
) -- R(#2.
)”]

[x ∗ y $\stackrel{\text{tex}}{\to}$ “#1.
*#2.”]

[x ∗ ∗y $\stackrel{\text{tex}}{\to}$ “#1.
**#2.”]

[leqReflexivity $\stackrel{\text{tex}}{\to}$ “leqReflexivity”]

[recx $\stackrel{\text{tex}}{\to}$ “rec#1.”]

[|x| $\stackrel{\text{tex}}{\to}$ “|#1.|”]

[if(x, y, z) $\stackrel{\text{tex}}{\to}$ “if(#1.
, #2.
, #3.
)”]

[R(x) $\stackrel{\text{tex}}{\to}$ "R(#1.
)"]

[− − R(x) $\stackrel{\text{tex}}{\to}$ "--R(#1.
)"]

[leqAntisymmetryAxiom $\stackrel{\text{tex}}{\to}$ "leqAntisymmetryAxiom"]

[leqTransitivityAxiom $\stackrel{\text{tex}}{\to}$ "leqTransitivityAxiom"]

[leqTotality $\stackrel{\text{tex}}{\to}$ "leqTotality"]

[leqAdditionAxiom $\stackrel{\text{tex}}{\to}$ "leqAdditionAxiom"]

[leqMultiplicationAxiom $\stackrel{\text{tex}}{\to}$ "leqMultiplicationAxiom"]

[plusAssociativity $\stackrel{\text{tex}}{\to}$ "plusAssociativity"]

[plusCommutativity $\stackrel{\text{tex}}{\to}$ "plusCommutativity"]

[Negative $\stackrel{\text{tex}}{\to}$ "Negative"]

[plus0 $\stackrel{\text{tex}}{\to}$ "plus0"]

[timesAssociativity $\stackrel{\text{tex}}{\to}$ "timesAssociativity"]

[timesCommutativity $\stackrel{\text{tex}}{\to}$ "timesCommutativity"]

[ReciprocalAxiom $\stackrel{\text{tex}}{\to}$ "ReciprocalAxiom"]

[times1 $\stackrel{\text{tex}}{\to}$ "times1"]

[plusAssociativity $\stackrel{\text{tex}}{\to}$ "plusAssociativity"]

[plusCommutativity $\stackrel{\text{tex}}{\to}$ "plusCommutativity"]

[Negative $\stackrel{\text{tex}}{\to}$ "Negative"]

[Distribution $\stackrel{\text{tex}}{\to}$ "Distribution"]

[0not1 $\stackrel{\text{tex}}{\to}$ "0not1"]

[equalityAxiom $\stackrel{\text{tex}}{\to}$ "equalityAxiom"]

[eqLeqAxiom $\stackrel{\text{tex}}{\to}$ "eqLeqAxiom"]

[eqAdditionAxiom $\stackrel{\text{tex}}{\to}$ "eqAdditionAxiom"] [eqMultiplicationAxiom $\stackrel{\text{tex}}{\to}$ “eqMultiplicationAxiom”]

[SENC1 $\stackrel{\text{tex}}{\to}$ “SENC1”]

[SENC2 $\stackrel{\text{tex}}{\to}$ “SENC2”]

[IfThenElse(T) $\stackrel{\text{tex}}{\to}$ “IfThenElse(T)”]

[IfThenElse(F) $\stackrel{\text{tex}}{\to}$ “IfThenElse(F)”]

[From = f $\stackrel{\text{tex}}{\to}$ “From=f”]

[To = f $\stackrel{\text{tex}}{\to}$ “To=f”]

[From < f $\stackrel{\text{tex}}{\to}$ “From<f”]

[To < f $\stackrel{\text{tex}}{\to}$ “To<f”]

[PlusF $\stackrel{\text{tex}}{\to}$ “PlusF”]

[MinusF $\stackrel{\text{tex}}{\to}$ “MinusF”]

[TimesF $\stackrel{\text{tex}}{\to}$ “TimesF”]

[0f $\stackrel{\text{tex}}{\to}$ “0f”]

[1f $\stackrel{\text{tex}}{\to}$ “1f”]

[FromSF $\stackrel{\text{tex}}{\to}$ “FromSF”]

[ToSF $\stackrel{\text{tex}}{\to}$ “ToSF”]

[To == XX $\stackrel{\text{tex}}{\to}$ “To==XX”]

[From ==$\stackrel{\text{tex}}{\to}$ “From==”]

[To ==$\stackrel{\text{tex}}{\to}$ “To==”]

[From << XX $\stackrel{\text{tex}}{\to}$ “From<<XX”]

[From << (1) $\stackrel{\text{tex}}{\to}$ “From<<(1)”]

[From << (2) $\stackrel{\text{tex}}{\to}$ “From<<(2)”]

[to << XX $\stackrel{\text{tex}}{\to}$ “to<<XX”]

[From <<$\stackrel{\text{tex}}{\to}$ “From<<”]

[To <<$\stackrel{\text{tex}}{\to}$ “To<<”]

[FromInR $\stackrel{\text{tex}}{\to}$ "FromInR"]

[PlusR $\stackrel{\text{tex}}{\to}$ "PlusR"]

[TimesR $\stackrel{\text{tex}}{\to}$ "TimesR"]

[leqAntisymmetry $\stackrel{\text{tex}}{\to}$ "leqAntisymmetry"]

[leqTransitivity $\stackrel{\text{tex}}{\to}$ "leqTransitivity"]

[leqAddition $\stackrel{\text{tex}}{\to}$ "leqAddition"]

[Reciprocal $\stackrel{\text{tex}}{\to}$ "Reciprocal"]

[Equality $\stackrel{\text{tex}}{\to}$ "Equality"]

[eqLeq $\stackrel{\text{tex}}{\to}$ "eqLeq"]

[eqAddition $\stackrel{\text{tex}}{\to}$ "eqAddition"]

[eqMultiplication $\stackrel{\text{tex}}{\to}$ "eqMultiplication"]

[ToNegatedImply $\stackrel{\text{tex}}{\to}$ "ToNegatedImply"]

[TND $\stackrel{\text{tex}}{\to}$ "TND"]

[ImplyNegation $\stackrel{\text{tex}}{\to}$ "ImplyNegation"]

[FromNegations $\stackrel{\text{tex}}{\to}$ "FromNegations"]

[From3Disjuncts $\stackrel{\text{tex}}{\to}$ "From3Disjuncts"]

[NegateDisjunct1 $\stackrel{\text{tex}}{\to}$ "NegateDisjunct1"]

[NegateDisjunct2 $\stackrel{\text{tex}}{\to}$ "NegateDisjunct2"]

[ExpandDisjuncts $\stackrel{\text{tex}}{\to}$ "ExpandDisjuncts"]

[From2 $*$ 2Disjuncts $\stackrel{\text{tex}}{\to}$ "From2$*$2Disjuncts"]

[eqReflexivity $\stackrel{\text{tex}}{\to}$ "eqReflexivity"]

[eqSymmetry $\stackrel{\text{tex}}{\to}$ "eqSymmetry"]

[eqTransitivity $\stackrel{\text{tex}}{\to}$ "eqTransitivity"]

[eqTransitivity4 $\stackrel{\text{tex}}{\to}$ "eqTransitivity4"]

[eqTransitivity5 $\stackrel{\text{tex}}{\to}$ "eqTransitivity5"] [eqTransitivity6 $\stackrel{\text{tex}}{\to}$ "eqTransitivity6"]

[plus0Left $\stackrel{\text{tex}}{\to}$ "plus0Left"]

[times1Left $\stackrel{\text{tex}}{\to}$ "times1Left"]

[EqMultiplicationLeft $\stackrel{\text{tex}}{\to}$ "EqMultiplicationLeft"]

[DistributionOut $\stackrel{\text{tex}}{\to}$ "DistributionOut"]

[Three2twoTerms $\stackrel{\text{tex}}{\to}$ "Three2twoTerms"]

[Three2threeTerms $\stackrel{\text{tex}}{\to}$ "Three2threeTerms"]

[Three2threeFactors $\stackrel{\text{tex}}{\to}$ "Three2threeFactors"]

[AddEquations $\stackrel{\text{tex}}{\to}$ "AddEquations"]

[SubtractEquations $\stackrel{\text{tex}}{\to}$ "SubtractEquations"]

[SubtractEquationsLeft $\stackrel{\text{tex}}{\to}$ "SubtractEquationsLeft"]

[EqNegated $\stackrel{\text{tex}}{\to}$ "EqNegated"]

[PositiveToRight(Eq) $\stackrel{\text{tex}}{\to}$ "PositiveToRight(Eq)"]

[PositiveToLeft(Eq)(1term) $\stackrel{\text{tex}}{\to}$ "PositiveToLeft(Eq)(1 term)"]

[NegativeToLeft(Eq) $\stackrel{\text{tex}}{\to}$ "NegativeToLeft(Eq)"]

[UniqueNegative $\stackrel{\text{tex}}{\to}$ "UniqueNegative"]

[DoubleMinus $\stackrel{\text{tex}}{\to}$ "DoubleMinus"]

[LessNeq $\stackrel{\text{tex}}{\to}$ "LessNeq"]

[NeqSymmetry $\stackrel{\text{tex}}{\to}$ "NeqSymmetry"]

[NeqNegated $\stackrel{\text{tex}}{\to}$ "NeqNegated"]

[SubNeqRight $\stackrel{\text{tex}}{\to}$ "SubNeqRight"]

[SubNeqLeft $\stackrel{\text{tex}}{\to}$ "SubNeqLeft"]

[NeqAddition $\stackrel{\text{tex}}{\to}$ "NeqAddition"]

[NeqMultiplication $\stackrel{\text{tex}}{\to}$ "NeqMultiplication"]

[LeqLessEq $\stackrel{\text{tex}}{\to}$ "LeqLessEq"]

[LessLeq $\stackrel{\text{tex}}{\to}$ "LessLeq"]

[FromLeqGeq $\stackrel{\text{tex}}{\to}$ "FromLeqGeq"]

[subLeqRight $\stackrel{\text{tex}}{\to}$ "subLeqRight"]

[subLeqLeft $\stackrel{\text{tex}}{\to}$ "subLeqLeft"]

[Leq + 1 $\stackrel{\text{tex}}{\to}$ "Leq+1"]

[PositiveToRight(Leq) $\stackrel{\text{tex}}{\to}$ "PositiveToRight(Leq)"]

[PositiveToRight(Leq)(1term) $\stackrel{\text{tex}}{\to}$ "PositiveToRight(Leq)(1 term)"]

[LeqAdditionLeft $\stackrel{\text{tex}}{\to}$ "LeqAdditionLeft"]

[leqSubtraction $\stackrel{\text{tex}}{\to}$ "leqSubtraction"]

[leqSubtractionLeft $\stackrel{\text{tex}}{\to}$ "leqSubtractionLeft"]

[leqMultiplication $\stackrel{\text{tex}}{\to}$ "leqMultiplication"]

[thirdGeq $\stackrel{\text{tex}}{\to}$ "thirdGeq"]

[LeqNegated $\stackrel{\text{tex}}{\to}$ "LeqNegated"]

[AddEquations(Leq) $\stackrel{\text{tex}}{\to}$ "AddEquations(Leq)"]

[LeqNeqLess $\stackrel{\text{tex}}{\to}$ "LeqNeqLess"]

[FromLess $\stackrel{\text{tex}}{\to}$ "FromLess"]

[ToLess $\stackrel{\text{tex}}{\to}$ "ToLess"]

[fromNotLess $\stackrel{\text{tex}}{\to}$ "fromNotLess"]

[toNotLess $\stackrel{\text{tex}}{\to}$ "toNotLess"]

[LessAddition $\stackrel{\text{tex}}{\to}$ "LessAddition"]

[LessAdditionLeft $\stackrel{\text{tex}}{\to}$ "LessAdditionLeft"]

[LessMultiplication $\stackrel{\text{tex}}{\to}$ "LessMultiplication"]

[LessMultiplicationLeft $\stackrel{\text{tex}}{\to}$ "LessMultiplicationLeft"]

[LessDivision $\stackrel{\text{tex}}{\to}$ "LessDivision"]

[AddEquations(Less) $\stackrel{\text{tex}}{\to}$ "AddEquations(Less)"]

[NegativeLessPositive $\stackrel{\text{tex}}{\rightarrow}$ "NegativeLessPositive"]

[leqLessTransitivity $\stackrel{\text{tex}}{\rightarrow}$ "leqLessTransitivity"]

[LessLeqTransitivity $\stackrel{\text{tex}}{\rightarrow}$ "LessLeqTransitivity"]

[LessTransitivity $\stackrel{\text{tex}}{\rightarrow}$ "LessTransitivity"]

[LessTotality $\stackrel{\text{tex}}{\rightarrow}$ "LessTotality"]

[SubLessRight $\stackrel{\text{tex}}{\rightarrow}$ "SubLessRight"]

[SubLessLeft $\stackrel{\text{tex}}{\rightarrow}$ "SubLessLeft"]

[LessNegated $\stackrel{\text{tex}}{\rightarrow}$ "LessNegated"]

[PositiveNegated $\stackrel{\text{tex}}{\rightarrow}$ "PositiveNegated"]

[NonpositiveNegated $\stackrel{\text{tex}}{\rightarrow}$ "NonpositiveNegated"]

[NegativeNegated $\stackrel{\text{tex}}{\rightarrow}$ "NegativeNegated"]

[NonnegativeNegated $\stackrel{\text{tex}}{\rightarrow}$ "NonnegativeNegated"]

[PositiveHalved $\stackrel{\text{tex}}{\rightarrow}$ "PositiveHalved"]

[NonnegativeNumerical $\stackrel{\text{tex}}{\rightarrow}$ "NonnegativeNumerical"]

[NegativeNumerical $\stackrel{\text{tex}}{\rightarrow}$ "NegativeNumerical"]

[PositiveNumerical $\stackrel{\text{tex}}{\rightarrow}$ "PositiveNumerical"]

[$|0| = 0 \stackrel{\text{tex}}{\rightarrow}$ "|0|=0"]

[$0 <= |x| \stackrel{\text{tex}}{\rightarrow}$ "0<=|x|"]

[SameNumerical $\stackrel{\text{tex}}{\rightarrow}$ "SameNumerical"]

[SignNumerical(+) $\stackrel{\text{tex}}{\rightarrow}$ "SignNumerical(+)"]

[SignNumerical $\stackrel{\text{tex}}{\rightarrow}$ "SignNumerical"]

[NumericalDifference $\stackrel{\text{tex}}{\rightarrow}$ "NumericalDifference"]

[SplitNumericalSumHelper $\stackrel{\text{tex}}{\rightarrow}$ "SplitNumericalSumHelper"]

[splitNumericalSum(++) $\stackrel{\text{tex}}{\rightarrow}$ "splitNumericalSum(++)"]

[splitNumericalSum($--$) $\stackrel{\text{tex}}{\rightarrow}$ "splitNumericalSum(--)"]

[splitNumericalSum(+ − small) $\stackrel{\text{tex}}{\to}$ “splitNumericalSum(+-small)”]

[splitNumericalSum(+ − big) $\stackrel{\text{tex}}{\to}$ “splitNumericalSum(+-big)”]

[splitNumericalSum(+−) $\stackrel{\text{tex}}{\to}$ “splitNumericalSum(+-)”]

[splitNumericalSum(−+) $\stackrel{\text{tex}}{\to}$ “splitNumericalSum(-+)”]

[splitNumericalSum $\stackrel{\text{tex}}{\to}$ “splitNumericalSum”]

[insertMiddleTerm(Numerical) $\stackrel{\text{tex}}{\to}$ “insertMiddleTerm(Numerical)”]

[$\mathrm{x} + \mathrm{y} = \mathrm{z}$Backwards $\stackrel{\text{tex}}{\to}$ “x+y=zBackwards”]

[$\mathrm{x} * \mathrm{y} = \mathrm{z}$Backwards $\stackrel{\text{tex}}{\to}$ “x*y=zBackwards”]

[$\mathrm{x} = \mathrm{x} + (\mathrm{y} - \mathrm{y})$ $\stackrel{\text{tex}}{\to}$ “x=x+(y-y)”]

[$\mathrm{x} = \mathrm{x} + \mathrm{y} - \mathrm{y}$ $\stackrel{\text{tex}}{\to}$ “x=x+y-y”]

[$\stackrel{\text{tex}}{\to}$ “ ”]

[insertMiddleTerm(Sum) $\stackrel{\text{tex}}{\to}$ “insertMiddleTerm(Sum)”]

[insertMiddleTerm(Difference) $\stackrel{\text{tex}}{\to}$ “insertMiddleTerm(Difference)”]

[$\mathrm{x} * 0 + \mathrm{x} = \mathrm{x}$ $\stackrel{\text{tex}}{\to}$ “x*0+x=x”]

[$\mathrm{x} * 0 = 0$ $\stackrel{\text{tex}}{\to}$ “x*0=0”]

[$(-1) * (-1) + (-1) * 1 = 0$ $\stackrel{\text{tex}}{\to}$ “(-1)*(-1)+(-1)*1=0”]

[$(-1) * (-1) = 1$ $\stackrel{\text{tex}}{\to}$ “(-1)*(-1)=1”]

[$0 < 1$Helper $\stackrel{\text{tex}}{\to}$ “0<1Helper”]

[$0 < 1$ $\stackrel{\text{tex}}{\to}$ “0<1”]

[$0 < 2$ $\stackrel{\text{tex}}{\to}$ “0<2”]

[$0 < 1/2$ $\stackrel{\text{tex}}{\to}$ “0<1/2”]

[TwoWholes $\stackrel{\text{tex}}{\to}$ “TwoWholes”]

[TwoHalves $\stackrel{\text{tex}}{\to}$ “TwoHalves”]

[$-\mathrm{x} - \mathrm{y} = -(\mathrm{x} + \mathrm{y})$ $\stackrel{\text{tex}}{\to}$ “-x-y=-(x+y)”]

[MinusNegated $\stackrel{\text{tex}}{\to}$ “MinusNegated”]

[Times(−1) $\stackrel{\text{tex}}{\rightarrow}$ “Times(-1)”]

[Times(−1)Left $\stackrel{\text{tex}}{\rightarrow}$ “Times(-1)Left”]

[−0 = 0 $\stackrel{\text{tex}}{\rightarrow}$ “-0=0”]

[negativeToLeft(Leq) $\stackrel{\text{tex}}{\rightarrow}$ “negativeToLeft(Leq)”]

[SFsymmetry $\stackrel{\text{tex}}{\rightarrow}$ “SFsymmetry”]

[SFtransitivity $\stackrel{\text{tex}}{\rightarrow}$ “SFtransitivity”]

[= fToSameF $\stackrel{\text{tex}}{\rightarrow}$ “=fToSameF ”]

[PlusF(Sym) $\stackrel{\text{tex}}{\rightarrow}$ “PlusF(Sym)”]

[TimesF(Sym) $\stackrel{\text{tex}}{\rightarrow}$ “TimesF(Sym)”]

[f2R(Plus) $\stackrel{\text{tex}}{\rightarrow}$ “f2R(Plus)”]

[f2R(Times) $\stackrel{\text{tex}}{\rightarrow}$ “f2R(Times)”]

[PlusR(Sym) $\stackrel{\text{tex}}{\rightarrow}$ “PlusR(Sym)”]

[TimesR(Sym) $\stackrel{\text{tex}}{\rightarrow}$ “TimesR(Sym)”]

[LessLeq(R) $\stackrel{\text{tex}}{\rightarrow}$ “LessLeq(R)”]

[eqLeq(R) $\stackrel{\text{tex}}{\rightarrow}$ “eqLeq(R)”]

[ThirdGeqSeries $\stackrel{\text{tex}}{\rightarrow}$ “ThirdGeqSeries”]

[SubLessRight(R) $\stackrel{\text{tex}}{\rightarrow}$ “SubLessRight(R)”]

[SubLessLeft(R) $\stackrel{\text{tex}}{\rightarrow}$ “SubLessLeft(R)”]

[<< TransitivityHelper(Q) $\stackrel{\text{tex}}{\rightarrow}$ “<<TransitivityHelper(Q)”]

[<< Transitivity $\stackrel{\text{tex}}{\rightarrow}$ “<<Transitivity”]

[<<== Reflexivity $\stackrel{\text{tex}}{\rightarrow}$ “<<==Reflexivity”]

[<<== AntisymmetryHelper(Q) $\stackrel{\text{tex}}{\rightarrow}$ “<<==AntisymmetryHelper(Q)”]

[<<== Antisymmetry $\stackrel{\text{tex}}{\rightarrow}$ “<<==Antisymmetry”]

[<<== Transitivity $\stackrel{\text{tex}}{\rightarrow}$ “<<==Transitivity”]

[Plus0f $\stackrel{\text{tex}}{\rightarrow}$ “Plus0f”]

[Plus00 $\stackrel{\text{tex}}{\to}$ “Plus00”]

[== Addition $\stackrel{\text{tex}}{\to}$ “==Addition”]

[== AdditionLeft $\stackrel{\text{tex}}{\to}$ “==AdditionLeft”]

[$<<$ Addition $\stackrel{\text{tex}}{\to}$ “$<<$Addition”]

[$<<$== Addition $\stackrel{\text{tex}}{\to}$ “$<<$==Addition”]

[PlusAssociativity(F) $\stackrel{\text{tex}}{\to}$ “PlusAssociativity(F)”]

[PlusAssociativity(R) $\stackrel{\text{tex}}{\to}$ “PlusAssociativity(R)”]

[Negative(R) $\stackrel{\text{tex}}{\to}$ “Negative(R)”]

[PlusCommutativity(F) $\stackrel{\text{tex}}{\to}$ “PlusCommutativity(F)”]

[PlusCommutativity(R) $\stackrel{\text{tex}}{\to}$ “PlusCommutativity(R)”]

[Times1f $\stackrel{\text{tex}}{\to}$ “Times1f”]

[Times01 $\stackrel{\text{tex}}{\to}$ “Times01”]

[TimesAssociativity(F) $\stackrel{\text{tex}}{\to}$ “TimesAssociativity(F)”]

[TimesAssociativity(R) $\stackrel{\text{tex}}{\to}$ “TimesAssociativity(R)”]

[TimesCommutativity(F) $\stackrel{\text{tex}}{\to}$ “TimesCommutativity(F)”]

[TimesCommutativity(R) $\stackrel{\text{tex}}{\to}$ “TimesCommutativity(R)”]

[Distribution(F) $\stackrel{\text{tex}}{\to}$ “Distribution(F)”]

[Distribution(R) $\stackrel{\text{tex}}{\to}$ “Distribution(R)”]