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Up Help

am, $(\cdots)$, `Objekt-var`, `Ex-var`, `Ph-var`, `Værdi`, `Variabel`, Op$(*)$, Op$(*,*)$, $* \stackrel{..}{=} *$, ContainsEmpty$(*)$, Dedu$(*,*)$, Dedu$_0(*,*)$, Dedu$_s(*,*,*)$, Dedu$_1(*,*,*)$, Dedu$_2(*,*,*)$, Dedu$_3(*,*,*,*)$, Dedu$_4(*,*,*,*)$, Dedu$^*_4(*,*,*,*)$, Dedu$_5(*,*,*)$, Dedu$_6(*,*,*,*)$, Dedu$^*_6(*,*,*,*)$, Dedu$_7(*)$, Dedu$_8(*,*)$, Dedu$^*_8(*,*)$, Ex$_1$, Ex$_2$, Ex3, Ex$_{10}$, Ex$_{20}$, $*_{\mathrm{Ex}}$, $*^{\mathrm{Ex}}$, $\langle * \equiv * \mid * := *\rangle_{\mathrm{Ex}}$, $\langle * \equiv^0 * \mid * := *\rangle_{\mathrm{Ex}}$, $\langle * \equiv^1 * \mid * := *\rangle_{\mathrm{Ex}}$, $\langle * \equiv^* * \mid * := *\rangle_{\mathrm{Ex}}$, ph$_1$, ph$_2$, ph$_3$, $*_{\mathrm{Ph}}$, $*^{\mathrm{Ph}}$, $\langle * \equiv * \mid * := *\rangle_{\mathrm{Ph}}$, $\langle * \equiv^0 * \mid * := *\rangle_{\mathrm{Ph}}$, $\langle * \equiv^1 * \mid * := *\rangle_{\mathrm{Ph}}$, $\langle * \equiv^* * \mid * := *\rangle_{\mathrm{Ph}}$, bs, OBS, $\mathcal{BS}$, Ø, ZFsub, MP, Gen, Repetition, Neg, Ded, ExistIntro, Extensionality, Ødef, PairDef, UnionDef, PowerDef, SeparationDef, AddDoubleNeg, RemoveDoubleNeg, AndCommutativity, AutoImply, Contrapositive, FirstConjunct, SecondConjunct, FromContradiction, FromDisjuncts, IffCommutativity, IffFirst, IffSecond, ImplyTransitivity, JoinConjuncts, MP2, MP3, MP4, MP5, MT, NegativeMT, Technicality, Weakening, WeakenOr1, WeakenOr2, Formula2Pair, Pair2Formula, Formula2Union, Union2Formula, Formula2Sep, Sep2Formula, SubsetInPower, HelperPowerIsSub, PowerIsSub, (Switch)HelperPowerIsSub, (Switch)PowerIsSub, ToSetEquality, HelperToSetEquality(t), ToSetEquality(t), HelperFromSetEquality, FromSetEquality, HelperReflexivity, Reflexivity, HelperSymmetry, Symmetry, HelperTransitivity, Transitivity, ERisReflexive, ERisSymmetric, ERisTransitive, ØisSubset, HelperMemberNotØ, MemberNotØ, HelperUniqueØ, UniqueØ, ==Reflexivity, ==Symmetry, Helper==Transitivity, ==Transitivity, HelperTransferNotEq, TransferNotEq, HelperPairSubset, Helper(2)PairSubset, PairSubset, SamePair, SameSingleton, UnionSubset, SameUnion, SeparationSubset, SameSeparation, SameBinaryUnion, IntersectionSubset, SameIntersection, AutoMember, HelperEqSysNotØ, EqSysNotØ, HelperEqSubset, EqSubset, HelperEqNecessary, EqNecessary, HelperNoneEqNecessary, Helper(2)NoneEqNecessary, NoneEqNecessary, EqClassIsSubset, EqClassesAreDisjoint, AllDisjoint, AllDisjointImply, BSsubset, Union(BS/R)subset, UnionIdentity, EqSysIsPartition, (ϵ), (fx), (fy), (fz), (fv), var fv, (rx), (ry), (rz), (ru), ϵ, FX, FY, FZ, FU, FV, RX, RY, RZ, RU, 0, 1, (-1), 2, 1/2, 0f, 1f, 00, 01, leqReflexivity, leqAntisymmetryAxiom, leqTransitivityAxiom, leqTotality, leqAdditionAxiom, leqMultiplicationAxiom, plusAssociativity, plusCommutativity, Negative, plus0, timesAssociativity, timesCommutativity, ReciprocalAxiom, times1, Distribution, 0not1, equalityAxiom, eqLeqAxiom, eqAdditionAxiom, eqMultiplicationAxiom, SENC1, SENC2, IfThenElse(T), IfThenElse(F), From = f, To = f, From < f, To < f, PlusF, TimesF, MinusF, 0f, 1f, FromSF, ToSF, To == XX, From ==, To ==, From << XX, From << (1), From << (2), to << XX, From <<, To <<, FromInR, PlusR, TimesR, leqAntisymmetry, leqTransitivity, leqAddition, leqMultiplication, Reciprocal, Equality, eqLeq, eqAddition, eqMultiplication, ToNegatedImply, TND, ImplyNegation, FromNegations, From3Disjuncts, From2 ∗ 2Disjuncts, NegateDisjunct1, NegateDisjunct2, ExpandDisjuncts, eqReflexivity, eqSymmetry, eqTransitivity, eqTransitivity4, eqTransitivity5, eqTransitivity6, plus0Left, times1Left, lemma eqAdditionLeft, EqMultiplicationLeft, DistributionOut, Three2twoTerms, Three2threeTerms, Three2threeFactors, AddEquations, SubtractEquations, SubtractEquationsLeft, EqNegated, PositiveToRight(Eq), PositiveToLeft(Eq)(1term), NegativeToLeft(Eq), LessNeq, NeqSymmetry, NeqNegated, SubNeqRight, SubNeqLeft, NeqAddition, NeqMultiplication, UniqueNegative, DoubleMinus, LeqLessEq, LessLeq, FromLeqGeq, subLeqRight, subLeqLeft, Leq + 1, PositiveToRight(Leq), PositiveToRight(Leq)(1term), negativeToLeft(Leq), LeqAdditionLeft, leqSubtraction, leqSubtractionLeft, thirdGeq, LeqNegated, AddEquations(Leq), ThirdGeqSeries, LeqNeqLess, FromLess, ToLess, fromNotLess, toNotLess, NegativeLessPositive, leqLessTransitivity, LessLeqTransitivity, LessTransitivity, LessTotality, SubLessRight, SubLessLeft, LessAddition, LessAdditionLeft, LessMultiplication, LessMultiplicationLeft, LessDivision, AddEquations(Less), LessNegated, PositiveNegated, NonpositiveNegated, NegativeNegated, NonnegativeNegated, PositiveHalved, NonnegativeNumerical, NegativeNumerical, PositiveNumerical, lemma nonpositiveNumerical, |0| = 0, 0 <= |x|, SameNumerical, SignNumerical(+), SignNumerical, NumericalDifference, SplitNumericalSumHelper, splitNumericalSum(++), splitNumericalSum(−−), splitNumericalSum(+ − small), splitNumericalSum(+ − big), splitNumericalSum(+−), splitNumericalSum(−+), splitNumericalSum, insertMiddleTerm(Numerical), x + y = zBackwards, x ∗ y = zBackwards, x = x + (y − y), x = x + y − y, , insertMiddleTerm(Sum), insertMiddleTerm(Difference), x ∗ 0 + x = x, x ∗ 0 = 0, (−1) ∗ (−1) + (−1) ∗ 1 = 0, (−1) ∗ (−1) = 1, 0 < 1Helper, 0 < 1, 0 < 2, 0 < 1/2, TwoWholes, TwoHalves, −x − y = −(x + y), MinusNegated, Times(−1), Times(−1)Left, −0 = 0, SFsymmetry, SFtransitivity, = fToSameF, PlusF(Sym), TimesF(Sym), f2R(Plus), f2R(Times), PlusR(Sym), TimesR(Sym), LessLeq(R), eqLeq(R), SubLessRight(R), SubLessLeft(R), << TransitivityHelper(Q), << Transitivity, <<== Reflexivity, <<== AntisymmetryHelper(Q), <<== Antisymmetry, <<== Transitivity, Plus0f, Plus00, == Addition, == AdditionLeft, << Addition, <<== Addition, PlusAssociativity(F), PlusAssociativity(R), Negative(R), PlusCommutativity(F), PlusCommutativity(R), TimesAssociativity(F), TimesAssociativity(R), Times1f, Times01, TimesCommutativity(F), TimesCommutativity(R), Distribution(F), Distribution(R), R(∗), − − R(∗), rec∗, ∗/∗, ∗ ∩ ∗, ∗[∗], ∪∗, ∗ ∪ ∗, P(∗), {∗}, {∗, ∗}, ⟨∗, ∗⟩, −∗, $-_f$∗, ∗ ∈ ∗, ∗(∗, ∗), ReflRel(∗, ∗), SymRel(∗, ∗), TransRel(∗, ∗), EqRel(∗, ∗), [∗ ∈ ∗]$_*$, Partition(∗, ∗), ∗ ∗ ∗, ∗ $*_f$ ∗, ∗ ∗ ∗∗, ∗ + ∗,

$* - *$, $* +_{\mathrm{f}} *$, $* -_{\mathrm{f}} *$, $* + +*$, $\mathrm{R}(*) - -\mathrm{R}(*)$, $|*|$, $\mathrm{if}(*,*,*)$, $* = *$, $* \neq *$,
$* <= *$, $* < *$, $* =_{\mathrm{f}} *$, $* <_{\mathrm{f}} *$, $\mathrm{SF}(*,*)$, $* == *$, $* << *$, $* <<== *$, $*==*$,
$* \subseteq *$, $\dot{\neg} *$, $* \notin *$, $* \neq *$, $* \dot{\wedge} *$, $* \dot{\vee} *$, $* \dot{\Leftrightarrow} *$, $\{\mathrm{ph} \in * \mid *\}$,

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Preassociative

[am], [base], [bracket $*$ end bracket], [big bracket $*$ end bracket], [\$ $*$ \$], [**flush left** [$*$]], [x], [y], [z], [[$* \bowtie *$]], [[$* \stackrel{*}{\rightarrow} *$]], [pyk], [tex], [name], [prio], [$*$], [T], [if($*,*,*$)], [[$* \stackrel{*}{\Rightarrow} *$]], [val], [claim], [$\perp$], [f($*$)], [$(*)^{\mathrm{I}}$], [F], [$\underline{0}$], [$\underline{1}$], [$\underline{2}$], [$\underline{3}$], [$\underline{4}$], [$\underline{5}$], [$\underline{6}$], [$\underline{7}$], [$\underline{8}$], [$\underline{9}$], [0], [1], [2], [3], [4], [5], [6], [7], [8], [9], [a], [b], [c], [d], [e], [f], [g], [h], [i], [j], [k], [l], [m], [n], [o], [p], [q], [r], [s], [t], [u], [v], [w], [$(*)^{\mathrm{M}}$], [If($*,*,*$)], [array{$*$} $*$ end array], [l], [c], [r], [empty], [$\langle * \mid * := * \rangle$], [$\mathcal{M}(*)$], [$\tilde{\mathcal{U}}(*)$], [$\mathcal{U}(*)$], [$\mathcal{U}^{\mathrm{M}}(*)$], [**apply**($*,*$)], [$\mathbf{apply}_1(*,*)$], [identifier($*$)], [$\mathrm{identifier}_1(*,*)$], [array-plus($*,*$)], [array-remove($*,*,*$)], [array-put($*,*,*,*$)], [array-add($*,*,*,*,*$)], [bit($*,*$)], [$\mathrm{bit}_1(*,*)$], [rack], ["vector"], ["bibliography"], ["dictionary"], ["body"], ["codex"], ["expansion"], ["code"], ["cache"], ["diagnose"], ["pyk"], ["tex"], ["texname"], ["value"], ["message"], ["macro"], ["definition"], ["unpack"], ["claim"], ["priority"], ["lambda"], ["apply"], ["true"], ["if"], ["quote"], ["proclaim"], ["define"], ["introduce"], ["hide"], ["pre"], ["post"], [$\mathcal{E}(*,*,*)$], [$\mathcal{E}_2(*,*,*,*,*)$], [$\mathcal{E}_3(*,*,*,*)$], [$\mathcal{E}_4(*,*,*,*)$], [**lookup**($*,*,*$)], [**abstract**($*,*,*,*$)], [[$\lceil *$]], [$\mathcal{M}(*,*,*)$], [$\mathcal{M}_2(*,*,*,*)$], [$\mathcal{M}^*(*,*,*)$], [macro], [s_0], [**zip**($*,*$)], [$\mathbf{assoc}_1(*,*,*)$], [$(*)^{\mathbf{P}}$], [self], [[$* \stackrel{..}{=} *$]], [[$* \doteq *$]], [[$* \stackrel{\prime}{=} *$]], [[$* \stackrel{\text{pyk}}{=} *$]], [[$* \stackrel{\text{tex}}{=} *$]], [[$* \stackrel{\text{name}}{=} *$]], [**Priority table**[$*$]], [$\tilde{\mathcal{M}}_1$], [$\tilde{\mathcal{M}}_2(*)$], [$\tilde{\mathcal{M}}_3(*)$], [$\tilde{\mathcal{M}}_4(*,*,*,*)$], [$\tilde{\mathcal{M}}(*,*,*)$], [$\tilde{\mathcal{Q}}(*,*,*)$], [$\tilde{\mathcal{Q}}_2(*,*,*)$], [$\tilde{\mathcal{Q}}_3(*,*,*,*)$], [$\tilde{\mathcal{Q}}^*(*,*,*)$], [($*$)], [($*$)], [display($*$)], [statement($*$)], [[$*]^{\cdot}$], [[$*]^{-}$], [**aspect**($*,*$)], [**aspect**($*,*,*$)], [$\langle * \rangle$], [$\mathbf{tuple}_1(*)$], [$\mathbf{tuple}_2(*)$], [$\mathrm{let}_2(*,*)$], [$\mathrm{let}_1(*,*)$], [[$* \stackrel{\text{claim}}{=} *$]], [checker], [**check**($*,*$)], [$\mathbf{check}_2(*,*,*)$], [$\mathbf{check}_3(*,*,*)$], [$\mathbf{check}^*(*,*)$], [$\mathbf{check}^*_2(*,*,*)$], [[$*]^{\cdot}$], [[$*]^{-}$], [[$*]^{\circ}$], [msg], [[$* \stackrel{\text{msg}}{=} *$]], [<stmt>], [stmt], [[$* \stackrel{\text{stmt}}{=} *$]], [HeadNil′], [HeadPair′], [Transitivity′], [$\perp\!\!\!\perp$], [Contra′], [T'_{E}], [L_1], [$\underline{*}$], [$\mathcal{A}$], [$\mathcal{B}$], [$\mathcal{C}$], [$\mathcal{D}$], [$\mathcal{E}$], [$\mathcal{F}$], [$\mathcal{G}$], [$\mathcal{H}$], [$\mathcal{I}$], [$\mathcal{J}$], [$\mathcal{K}$], [$\mathcal{L}$], [$\mathcal{M}$], [$\mathcal{N}$], [$\mathcal{O}$], [$\mathcal{P}$], [$\mathcal{Q}$], [$\mathcal{R}$], [$\mathcal{S}$], [$\mathcal{T}$], [$\mathcal{U}$], [$\mathcal{V}$], [$\mathcal{W}$], [$\mathcal{X}$], [$\mathcal{Y}$], [$\mathcal{Z}$], [$\langle * \mid * := * \rangle$], [$\langle^* * \mid * := * \rangle$], [$\emptyset$], [Remainder], [$(*)^{\mathbf{V}}$], [intro($*,*,*,*$)], [intro($*,*,*$)], [error($*,*$)], [$\mathrm{error}_2(*,*)$], [proof($*,*,*$)], [$\mathrm{proof}_2(*,*)$], [$\mathcal{S}(*,*)$], [$\mathcal{S}^{\mathrm{I}}(*,*)$], [$\mathcal{S}^{\triangleright}(*,*)$], [$\mathcal{S}^{\triangleright}_1(*,*,*)$], [$\mathcal{S}^{\mathrm{E}}(*,*)$], [$\mathcal{S}^{\mathrm{E}}_1(*,*,*)$], [$\mathcal{S}^{+}(*,*)$], [$\mathcal{S}^{+}_1(*,*,*)$], [$\mathcal{S}^{-}(*,*)$], [$\mathcal{S}^{-}_1(*,*,*)$], [$\mathcal{S}^{*}(*,*)$], [$\mathcal{S}^{*}_1(*,*,*)$], [$\mathcal{S}^{*}_2(*,*,*,*)$], [$\mathcal{S}^{@}(*,*)$], [$\mathcal{S}^{@}_1(*,*,*)$], [$\mathcal{S}^{\vdash}(*,*)$], [$\mathcal{S}^{\vdash}_1(*,*,*,*)$], [$\mathcal{S}^{\Vdash}(*,*)$], [$\mathcal{S}^{\Vdash}_1(*,*,*,*)$], [$\mathcal{S}^{\mathrm{i.e.}}(*,*)$], [$\mathcal{S}^{\mathrm{i.e.}}_1(*,*,*,*)$], [$\mathcal{S}^{\mathrm{i.e.}}_2(*,*,*,*,*)$], [$\mathcal{S}^{\forall}(*,*)$], [$\mathcal{S}^{\forall}_1(*,*,*,*)$], [$\mathcal{S}^{;}(*,*)$], [$\mathcal{S}^{;}_1(*,*,*)$], [$\mathcal{S}^{;}_2(*,*,*,*)$], [$\mathcal{T}(*)$], [claims($*,*,*$)], [$\mathrm{claims}_2(*,*,*)$], [<proof>], [proof], [[**Lemma** $*: *$]], [[**Proof of** $*: *$]], [[$*$ **lemma** $*: *$]], [[$*$ **antilemma** $*: *$]], [[$*$ **rule** $*: *$]], [[$*$ **antirule** $*: *$]], [verifier], [$\mathcal{V}_1(*)$], [$\mathcal{V}_2(*,*)$], [$\mathcal{V}_3(*,*,*,*)$], [$\mathcal{V}_4(*,*)$], [$\mathcal{V}_5(*,*,*,*)$], [$\mathcal{V}_6(*,*,*,*)$], [$\mathcal{V}_7(*,*,*,*)$], [Cut($*,*$)], [$\mathrm{Head}_{\oplus}(*)$], [$\mathrm{Tail}_{\oplus}(*)$], [$\mathrm{rule}_1(*,*)$], [rule($*,*$)],

[Rule tactic], [Plus$(*,*)$], [[**Theory** $*$]], [theory$_2(*,*)$], [theory$_3(*,*)$], [theory$_4(*,*,*)$], [HeadNil$''$], [HeadPair$''$], [Transitivity$''$], [Contra$''$], [HeadNil], [HeadPair], [Transitivity], [Contra], [$\mathrm{T_E}$], [ragged right], [ragged right expansion], [parm$(*,*,*)$], [parm$^*(*,*,*)$], [inst$(*,*)$], [inst$^*(*,*)$], [occur$(*,*,*)$], [occur$^*(*,*,*)$], [unify$(*=*,*)$], [unify$^*(*=*,*)$], [unify$_2(*=*,*)$], [$\mathrm{L_a}$], [$\mathrm{L_b}$], [$\mathrm{L_c}$], [$\mathrm{L_d}$], [$\mathrm{L_e}$], [$\mathrm{L_f}$], [$\mathrm{L_g}$], [$\mathrm{L_h}$], [$\mathrm{L_i}$], [$\mathrm{L_j}$], [$\mathrm{L_k}$], [$\mathrm{L_l}$], [$\mathrm{L_m}$], [$\mathrm{L_n}$], [$\mathrm{L_o}$], [$\mathrm{L_p}$], [$\mathrm{L_q}$], [$\mathrm{L_r}$], [$\mathrm{L_s}$], [$\mathrm{L_t}$], [$\mathrm{L_u}$], [$\mathrm{L_v}$], [$\mathrm{L_w}$], [$\mathrm{L_x}$], [$\mathrm{L_y}$], [$\mathrm{L_z}$], [$\mathrm{L_A}$], [$\mathrm{L_B}$], [$\mathrm{L_C}$], [$\mathrm{L_D}$], [$\mathrm{L_E}$], [$\mathrm{L_F}$], [$\mathrm{L_G}$], [$\mathrm{L_H}$], [$\mathrm{L_I}$], [$\mathrm{L_J}$], [$\mathrm{L_K}$], [$\mathrm{L_L}$], [$\mathrm{L_M}$], [$\mathrm{L_N}$], [$\mathrm{L_O}$], [$\mathrm{L_P}$], [$\mathrm{L_Q}$], [$\mathrm{L_R}$], [$\mathrm{L_S}$], [$\mathrm{L_T}$], [$\mathrm{L_U}$], [$\mathrm{L_V}$], [$\mathrm{L_W}$], [$\mathrm{L_X}$], [$\mathrm{L_Y}$], [$\mathrm{L_Z}$], [$\mathrm{L_?}$], [Reflexivity], [Reflexivity$_1$], [Commutativity], [Commutativity$_1$], [<tactic>], [tactic], [[$* \stackrel{\text{tactic}}{=} *$]], [$\mathcal{P}(*,*,*)$], [$\mathcal{P}^*(*,*,*)$], [$\mathrm{p_0}$], [conclude$_1(*,*)$], [conclude$_2(*,*,*)$], [conclude$_3(*,*,*,*)$], [conclude$_4(*,*)$], [check], [[$* \stackrel{\circ}{=} *$]], [RootVisible$(*)$], [A], [R], [C], [T], [L], [$\{*\}$], [$\overline{*}$], [a], [b], [c], [d], [e], [f], [g], [h], [i], [j], [k], [l], [m], [n], [o], [p], [q], [r], [s], [t], [u], [v], [w], [x], [y], [z], [$\langle * \equiv * \mid * := * \rangle$], [$\langle * \equiv^0 * \mid * := * \rangle$], [$\langle * \equiv^1 * \mid * := * \rangle$], [$\langle * \equiv^* * \mid * := * \rangle$], [Ded$(*,*)$], [Ded$_0(*,*)$], [Ded$_1(*,*,*)$], [Ded$_2(*,*,*)$], [Ded$_3(*,*,*,*)$], [Ded$_4(*,*,*,*)$], [Ded$^*_4(*,*,*,*)$], [Ded$_5(*,*,*)$], [Ded$_6(*,*,*,*)$], [Ded$^*_6(*,*,*,*)$], [Ded$_7(*)$], [Ded$_8(*,*)$], [Ded$^*_8(*,*)$], [S], [Neg], [MP], [Gen], [Ded], [S1], [S2], [S3], [S4], [S5], [S6], [S7], [S8], [S9], [Repetition], [A1$'$], [A2$'$], [A4$'$], [A5$'$], [Prop 3.2a], [Prop 3.2b], [Prop 3.2c], [Prop 3.2d], [Prop 3.2e$_1$], [Prop 3.2e$_2$], [Prop 3.2e], [Prop 3.2f$_1$], [Prop 3.2f$_2$], [Prop 3.2f], [Prop 3.2g$_1$], [Prop 3.2g$_2$], [Prop 3.2g], [Prop 3.2h$_1$], [Prop 3.2h$_2$], [Prop 3.2h], [Block$_1(*,*,*)$], [Block$_2(*)$], [$(\cdots)$], [`Objekt-var`], [`Ex-var`], [`Ph-var`], [`Værdi`], [`Variabel`], [Op$(*)$], [Op$(*,*)$], [$* \ddot{=} *$], [ContainsEmpty$(*)$], [Dedu$(*,*)$], [Dedu$_0(*,*)$], [Dedu$_\mathrm{s}(*,*,*)$], [Dedu$_1(*,*,*)$], [Dedu$_2(*,*,*)$], [Dedu$_3(*,*,*,*)$], [Dedu$_4(*,*,*,*)$], [Dedu$^*_4(*,*,*,*)$], [Dedu$_5(*,*,*)$], [Dedu$_6(*,*,*,*)$], [Dedu$^*_6(*,*,*,*)$], [Dedu$_7(*)$], [Dedu$_8(*,*)$], [Dedu$^*_8(*,*)$], [Ex$_1$], [Ex$_2$], [Ex3], [Ex$_{10}$], [Ex$_{20}$], [$*_{\mathrm{Ex}}$], [$*^{\mathrm{Ex}}$], [$\langle * \equiv * \mid * :== * \rangle_{\mathrm{Ex}}$], [$\langle * \equiv^0 * \mid * :== * \rangle_{\mathrm{Ex}}$], [$\langle * \equiv^1 * \mid * :== * \rangle_{\mathrm{Ex}}$], [$\langle * \equiv^* * \mid * :== * \rangle_{\mathrm{Ex}}$], [ph$_1$], [ph$_2$], [ph$_3$], [$*_{\mathrm{Ph}}$], [$*^{\mathrm{Ph}}$], [$\langle * \equiv * \mid * :== * \rangle_{\mathrm{Ph}}$], [$\langle * \equiv^0 * \mid * :== * \rangle_{\mathrm{Ph}}$], [$\langle * \equiv^1 * \mid * :== * \rangle_{\mathrm{Ph}}$], [$\langle * \equiv^* * \mid * :== * \rangle_{\mathrm{Ph}}$], [bs], [OBS], [$\mathcal{BS}$], [Ø], [ZFsub], [MP], [Gen], [Repetition], [Neg], [Ded], [ExistIntro], [Extensionality], [Ødef], [PairDef], [UnionDef], [PowerDef], [SeparationDef], [AddDoubleNeg], [RemoveDoubleNeg], [AndCommutativity], [AutoImply], [Contrapositive], [FirstConjunct], [SecondConjunct], [FromContradiction], [FromDisjuncts], [IffCommutativity], [IffFirst], [IffSecond], [ImplyTransitivity], [JoinConjuncts], [MP2], [MP3], [MP4], [MP5], [MT], [NegativeMT], [Technicality], [Weakening], [WeakenOr1], [WeakenOr2], [Formula2Pair], [Pair2Formula], [Formula2Union], [Union2Formula], [Formula2Sep], [Sep2Formula], [SubsetInPower], [HelperPowerIsSub], [PowerIsSub], [(Switch)HelperPowerIsSub], [(Switch)PowerIsSub], [ToSetEquality], [HelperToSetEquality(t)], [ToSetEquality(t)], [HelperFromSetEquality], [FromSetEquality], [HelperReflexivity], [Reflexivity], [HelperSymmetry], [Symmetry], [HelperTransitivity], [Transitivity], [ERisReflexive], [ERisSymmetric], [ERisTransitive], [ØisSubset], [HelperMemberNotØ], [MemberNotØ], [HelperUniqueØ], [UniqueØ], [==Reflexivity], [==Symmetry], [Helper==Transitivity], [==Transitivity], [HelperTransferNotEq],

[TransferNotEq], [HelperPairSubset], [Helper(2)PairSubset], [PairSubset], [SamePair], [SameSingleton], [UnionSubset], [SameUnion], [SeparationSubset], [SameSeparation], [SameBinaryUnion], [IntersectionSubset], [SameIntersection], [AutoMember], [HelperEqSysNot$\emptyset$], [EqSysNot$\emptyset$], [HelperEqSubset], [EqSubset], [HelperEqNecessary], [EqNecessary], [HelperNoneEqNecessary], [Helper(2)NoneEqNecessary], [NoneEqNecessary], [EqClassIsSubset], [EqClassesAreDisjoint], [AllDisjoint], [AllDisjointImply], [BSsubset], [Union(BS/R)subset], [UnionIdentity], [EqSysIsPartition], [(ϵ)], [(fx)], [(fy)], [(fz)], [(fv)], [var fv], [(rx)], [(ry)], [(rz)], [(ru)], [ϵ], [FX], [FY], [FZ], [FU], [FV], [RX], [RY], [RZ], [RU], [0], [1], [(-1)], [2], [1/2], [0f], [1f], [00], [01], [leqReflexivity], [leqAntisymmetryAxiom], [leqTransitivityAxiom], [leqTotality], [leqAdditionAxiom], [leqMultiplicationAxiom], [plusAssociativity], [plusCommutativity], [Negative], [plus0], [timesAssociativity], [timesCommutativity], [ReciprocalAxiom], [times1], [Distribution], [0not1], [equalityAxiom], [eqLeqAxiom], [eqAdditionAxiom], [eqMultiplicationAxiom], [SENC1], [SENC2], [IfThenElse(T)], [IfThenElse(F)], [From = f], [To = f], [From < f], [To < f], [PlusF], [TimesF], [MinusF], [0f], [1f], [FromSF], [ToSF], [To == XX], [From ==], [To ==], [From << XX], [From << (1)], [From << (2)], [to << XX], [From <<], [To <<], [FromInR], [PlusR], [TimesR], [leqAntisymmetry], [leqTransitivity], [leqAddition], [leqMultiplication], [Reciprocal], [Equality], [eqLeq], [eqAddition], [eqMultiplication], [ToNegatedImply], [TND], [ImplyNegation], [FromNegations], [From3Disjuncts], [From2 ∗ 2Disjuncts], [NegateDisjunct1], [NegateDisjunct2], [ExpandDisjuncts], [eqReflexivity], [eqSymmetry], [eqTransitivity], [eqTransitivity4], [eqTransitivity5], [eqTransitivity6], [plus0Left], [times1Left], [lemma eqAdditionLeft], [EqMultiplicationLeft], [DistributionOut], [Three2twoTerms], [Three2threeTerms], [Three2threeFactors], [AddEquations], [SubtractEquations], [SubtractEquationsLeft], [EqNegated], [PositiveToRight(Eq)], [PositiveToLeft(Eq)(1term)], [NegativeToLeft(Eq)], [LessNeq], [NeqSymmetry], [NeqNegated], [SubNeqRight], [SubNeqLeft], [NeqAddition], [NeqMultiplication], [UniqueNegative], [DoubleMinus], [LeqLessEq], [LessLeq], [FromLeqGeq], [subLeqRight], [subLeqLeft], [Leq + 1], [PositiveToRight(Leq)], [PositiveToRight(Leq)(1term)], [negativeToLeft(Leq)], [LeqAdditionLeft], [leqSubtraction], [leqSubtractionLeft], [thirdGeq], [LeqNegated], [AddEquations(Leq)], [ThirdGeqSeries], [LeqNeqLess], [FromLess], [ToLess], [fromNotLess], [toNotLess], [NegativeLessPositive], [leqLessTransitivity], [LessLeqTransitivity], [LessTransitivity], [LessTotality], [SubLessRight], [SubLessLeft], [LessAddition], [LessAdditionLeft], [LessMultiplication], [LessMultiplicationLeft], [LessDivision], [AddEquations(Less)], [LessNegated], [PositiveNegated], [NonpositiveNegated], [NegativeNegated], [NonnegativeNegated], [PositiveHalved], [NonnegativeNumerical], [NegativeNumerical], [PositiveNumerical], [lemma nonpositiveNumerical], [$|0| = 0$], [$0 <= |x|$], [SameNumerical], [SignNumerical(+)], [SignNumerical], [NumericalDifference], [SplitNumericalSumHelper], [splitNumericalSum(++)], [splitNumericalSum(−−)], [splitNumericalSum(+ − small)],

[splitNumericalSum(+ − big)], [splitNumericalSum(+−)],
[splitNumericalSum(−+)], [splitNumericalSum],
[insertMiddleTerm(Numerical)], [x + y = zBackwards], [x ∗ y = zBackwards],
[x = x + (y − y)], [x = x + y − y], [], [insertMiddleTerm(Sum)],
[insertMiddleTerm(Difference)], [x ∗ 0 + x = x], [x ∗ 0 = 0],
[(−1) ∗ (−1) + (−1) ∗ 1 = 0], [(−1) ∗ (−1) = 1], [0 < 1Helper], [0 < 1], [0 < 2],
[0 < 1/2], [TwoWholes], [TwoHalves], [−x − y = −(x + y)], [MinusNegated],
[Times(−1)], [Times(−1)Left], [−0 = 0], [SFsymmetry], [SFtransitivity],
[= fToSameF], [PlusF(Sym)], [TimesF(Sym)], [f2R(Plus)], [f2R(Times)],
[PlusR(Sym)], [TimesR(Sym)], [LessLeq(R)], [eqLeq(R)], [SubLessRight(R)],
[SubLessLeft(R)], [<< TransitivityHelper(Q)], [<< Transitivity],
[<<== Reflexivity], [<<== AntisymmetryHelper(Q)],
[<<== Antisymmetry], [<<== Transitivity], [Plus0f], [Plus00], [== Addition],
[== AdditionLeft], [<< Addition], [<<== Addition], [PlusAssociativity(F)],
[PlusAssociativity(R)], [Negative(R)], [PlusCommutativity(F)],
[PlusCommutativity(R)], [TimesAssociativity(F)], [TimesAssociativity(R)],
[Times1f], [Times01], [TimesCommutativity(F)], [TimesCommutativity(R)],
[Distribution(F)], [Distribution(R)];
Preassociative
[∗_{∗}], [∗/indexintro(∗, ∗, ∗, ∗)], [∗/intro(∗, ∗, ∗)], [∗/bothintro(∗, ∗, ∗, ∗, ∗)],
[∗/nameintro(∗, ∗, ∗, ∗)], [∗′], [∗[∗]], [∗[∗→∗]], [∗[∗⇒∗]], [∗0], [∗1], [0b], [∗-color(∗)],
[∗-color*(∗)], [$*^{\mathrm{H}}$], [$*^{\mathrm{T}}$], [$*^{\mathrm{U}}$], [$*^{\mathrm{h}}$], [$*^{\mathrm{t}}$], [$*^{\mathrm{s}}$], [$*^{\mathrm{c}}$], [$*^{\mathrm{d}}$], [$*^{\mathrm{a}}$], [$*^{\mathrm{C}}$], [$*^{\mathrm{M}}$], [$*^{\mathrm{B}}$], [$*^{\mathrm{r}}$], [$*^{\mathrm{i}}$],
[$*^{\mathrm{d}}$], [$*^{\mathrm{R}}$], [$*^{0}$], [$*^{1}$], [$*^{2}$], [$*^{3}$], [$*^{4}$], [$*^{5}$], [$*^{6}$], [$*^{7}$], [$*^{8}$], [$*^{9}$], [$*^{\mathrm{E}}$], [$*^{\mathcal{V}}$], [$*^{\mathcal{C}}$], [$*^{\mathcal{C}^*}$],
[$*^{\mathrm{hide}}$];
Preassociative
[“ ∗ ”], [], [(∗)$^{\mathbf{t}}$], [string(∗) + ∗], [string(∗) ++ ∗], [
∗], [∗], [!∗], ["∗], [#∗], [$∗], [%∗], [&∗], [’∗], [(∗], [)∗], [∗∗], [+∗], [, ∗], [-∗], [.∗], [/∗],
[0∗], [1∗], [2∗], [3∗], [4∗], [5∗], [6∗], [7∗], [8∗], [9∗], [:∗], [; ∗], [<∗], [=∗], [>∗], [?∗],
[@∗], [A∗], [B∗], [C∗], [D∗], [E∗], [F∗], [G∗], [H∗], [I∗], [J∗], [K∗], [L∗], [M∗], [N∗],
[O∗], [P∗], [Q∗], [R∗], [S∗], [T∗], [U∗], [V∗], [W∗], [X∗], [Y∗], [Z∗], [[∗], [\∗], []∗], [ˆ∗],
[_∗], [‘∗], [a∗], [b∗], [c∗], [d∗], [e∗], [f∗], [g∗], [h∗], [i∗], [j∗], [k∗], [l∗], [m∗], [n∗], [o∗],
[p∗], [q∗], [r∗], [s∗], [t∗], [u∗], [v∗], [w∗], [x∗], [y∗], [z∗], [{∗], [|∗], [}∗], [˜∗],
[**Preassociative** ∗; ∗], [**Postassociative** ∗; ∗], [[∗], ∗], [priority ∗ end],
[newline ∗], [macro newline ∗], [MacroIndent(∗)];
Preassociative
[∗ ’ ∗], [∗ ‘ ∗];
Preassociative
[∗′], [R(∗)], [− − R(∗)], [rec∗];
Preassociative
[∗/∗], [∗ ∩ ∗], [∗[∗]];
Preassociative
[∪∗], [∗ ∪ ∗], [P(∗)];
Preassociative
[{∗}];
Preassociative
[{∗, ∗}], [⟨∗, ∗⟩], [−∗], [$-_{\mathrm{f}}$∗];

Preassociative
$[* \in *], [*(*, *)], [\mathrm{ReflRel}(*, *)], [\mathrm{SymRel}(*, *)], [\mathrm{TransRel}(*, *)], [\mathrm{EqRel}(*, *)],$
$[[* \in *]_*], [\mathrm{Partition}(*, *)];$
Preassociative
$[* \cdot *], [* \cdot_0 *], [* * *], [* *_{\mathrm{f}} *], [* * **];$
Preassociative
$[* + *], [* +_0 *], [* +_1 *], [* - *], [* -_0 *], [* -_1 *], [* + *], [* - *], [* +_{\mathrm{f}} *], [* -_{\mathrm{f}} *],$
$[* + +*], [\mathrm{R}(*) - -\mathrm{R}(*)];$
Preassociative
$[| * |], [\mathrm{if}(*, *, *)];$
Preassociative
$[* = *], [* \neq *], [* <= *], [* < *], [* =_{\mathrm{f}} *], [* <_{\mathrm{f}} *], [\mathrm{SF}(*, *)], [* == *], [* << *],$
$[* <<== *];$
Preassociative
$[* \cup \{*\}], [* \cup *], [*\backslash\{*\}];$
Postassociative
$[* \therefore *], [* \underline{\therefore} *], [* \underline{::} *], [* \underline{+2*} *], [* :: *], [* +2* *];$
Postassociative
$[*, *];$
Preassociative
$[* \stackrel{\mathrm{B}}{\approx} *], [* \stackrel{\mathrm{D}}{\approx} *], [* \stackrel{\mathrm{C}}{\approx} *], [* \stackrel{\mathrm{P}}{\approx} *], [* \approx *], [* = *], [* \stackrel{+}{\to} *], [* \stackrel{\mathrm{t}}{=} *], [* \stackrel{\mathrm{t}^*}{=} *], [* \stackrel{\mathrm{r}}{=} *],$
$[* \in_{\mathrm{t}} *], [* \subseteq_{\mathrm{T}} *], [* \stackrel{\mathrm{T}}{=} *], [* \stackrel{\mathrm{s}}{=} *], [* \text{ free in } *], [* \text{ free in}^* *], [* \text{ free for } * \text{ in } *],$
$[* \text{ free for}^* * \text{ in } *], [* \in_{\mathrm{c}} *], [* < *], [* <' *], [* \leq' *], [* = *], [* \neq *], [*^{\mathrm{var}}],$
$[*\#^0*], [*\#^1*], [*\#^**], [*{==}*], [* \subseteq *];$
Preassociative
$[\neg *], [\dot\neg *], [* \notin *], [* \neq *];$
Preassociative
$[* \wedge *], [* \ddot\wedge *], [* \tilde\wedge *], [* \wedge_{\mathrm{c}} *], [* \dot\wedge *];$
Preassociative
$[* \vee *], [* \parallel *], [* \ddot\vee *];$
Postassociative
$[* \dot\vee *];$
Preassociative
$[\exists *\colon *], [\forall *\colon *], [\forall_{\mathrm{obj}} *\colon *];$
Postassociative
$[* \ddot\Rightarrow *], [* \Rightarrow *], [* \Leftrightarrow *], [* \dot\Leftrightarrow *];$
Preassociative
$[\{\mathrm{ph} \in * \mid *\}];$
Postassociative
$[* : *], [* \text{ spy } *], [*!*];$
Preassociative
$[* \left\{ \begin{array}{c} * \\ * \end{array} \right.];$
Preassociative
$[\lambda * .*], [\Lambda * .*], [\Lambda *], [\textbf{if } * \textbf{ then } * \textbf{ else } *], [\textbf{let } * = * \textbf{ in } *], [\textbf{let } * \stackrel{..}{=} * \text{ in } *];$

Preassociative
$[*\#*]$;
Preassociative
$[*^{\mathrm{I}}], [*^{\triangleright}], [*^{\mathrm{V}}], [*^{+}], [*^{-}], [*^{*}]$;
Preassociative
$[* @ *], [* \rhd *], [* \unrhd\!\!\!\rhd *], [* \gg *], [* \trianglerighteq *]$;
Postassociative
$[* \vdash *], [* \Vdash *], [* \text{ i.e. } *]$;
Preassociative
$[\forall *\colon *], [\Pi *\colon *]$;
Postassociative
$[* \oplus *]$;
Postassociative
$[*; *]$;
Preassociative
$[* \text{ proves } *]$;
Preassociative
$[* \textbf{ proof of } * : *]$, $[\text{Line} * : * \gg *; *]$, $[\text{Last line} * \gg * \Box]$,
$[\text{Line} * : \text{Premise} \gg *; *]$, $[\text{Line} * : \text{Side-condition} \gg *; *]$, $[\text{Arbitrary} \gg *; *]$,
$[\text{Local} \gg * = *; *]$, $[\text{Begin} *; * : \text{End}; *]$, $[\text{Last block line} * \gg * ;]$,
$[\text{Arbitrary} \gg *; *]$;
Postassociative
$[* \mid *]$;
Postassociative
$[* , *], [*[*]*]$;
Preassociative
$[*\&*]$;
Preassociative
[**], [* linebreak[4] *], [**];]

$[\text{am} \overset{\text{tex}}{\to} \text{“am”}]$

$[\text{am} \overset{\text{pyk}}{\to} \text{“am”}]$

$(\cdots)$

$[(\cdots) \overset{\text{tex}}{\to}$ “(\cdots{})”]

$[(\cdots) \overset{\text{pyk}}{\to} \text{“cdots”}]$

`Objekt-var`

[`Objekt-var` $\overset{\text{tex}}{\to}$ “\texttt{Objekt-var}”]

[Objekt-var $\stackrel{\text{pyk}}{\to}$ "object-var"]

Ex-var

[Ex-var $\stackrel{\text{tex}}{\to}$ "\texttt{Ex-var}"]

[Ex-var $\stackrel{\text{pyk}}{\to}$ "ex-var"]

Ph-var

[Ph-var $\stackrel{\text{tex}}{\to}$ "\texttt{Ph-var}"]

[Ph-var $\stackrel{\text{pyk}}{\to}$ "ph-var"]

Værdi

[Værdi $\stackrel{\text{tex}}{\to}$ "\texttt{V\ae{}rdi}"]

[Værdi $\stackrel{\text{pyk}}{\to}$ "vaerdi"]

Variabel

[Variabel $\stackrel{\text{tex}}{\to}$ "\texttt{Variabel}"]

[Variabel $\stackrel{\text{pyk}}{\to}$ "variabel"]

Op($*$)

[Op(x) $\stackrel{\text{tex}}{\to}$ "Op(#1.
)"]

[Op($*$) $\stackrel{\text{pyk}}{\to}$ "op " end op"]

Op($*, *$)

[Op(x, y) $\stackrel{\text{tex}}{\to}$ "Op(#1.
, #2.
)"]

$[\mathrm{Op}(*,*) \overset{\mathrm{pyk}}{\to}$ "op2 " comma " end op2"$]$

$* \mathrel{\ddot{==}} *$

$[\mathsf{x} \mathrel{\ddot{==}} \mathsf{y} \overset{\mathrm{tex}}{\to}$ "#1.
\mathrel {\ddot {==}} #2."$]$

$[* \mathrel{\ddot{==}} * \overset{\mathrm{pyk}}{\to}$ "define-equal " comma " end equal"$]$

ContainsEmpty$(*)$

$[\mathrm{ContainsEmpty}(\mathsf{x}) \overset{\mathrm{tex}}{\to}$ "ContainsEmpty(#1.
)"$]$

$[\mathrm{ContainsEmpty}(*) \overset{\mathrm{pyk}}{\to}$ "contains-empty " end empty"$]$

Dedu$(*,*)$

$[\mathrm{Dedu}(\mathsf{p},\mathsf{c}) \overset{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathrm{Dedu}(\mathsf{p},\mathsf{c}) \ddot{=} \lambda \mathsf{x}.\mathrm{Dedu}_0(\lceil \mathsf{p}\rceil,\lceil \mathsf{c}\rceil)]\rceil)]$

$[\mathrm{Dedu}(\mathsf{x},\mathsf{y}) \overset{\mathrm{tex}}{\to}$ "
Dedu(#1.
, #2.
)"$]$

$[\mathrm{Dedu}(*,*) \overset{\mathrm{pyk}}{\to}$ "1deduction " conclude " end 1deduction"$]$

Dedu$_0(*,*)$

$[\mathrm{Dedu}_0(\mathsf{p},\mathsf{c}) \overset{\mathrm{val}}{\to} \mathsf{c}!\mathrm{If}(\mathrm{Dedu}_8(\mathsf{p},\mathsf{T}),\mathrm{Dedu}_s(\mathrm{Dedu}_7(\mathsf{p}),\mathsf{c},\mathsf{T}),\mathsf{F})]$

$[\mathrm{Dedu}_0(\mathsf{x},\mathsf{y}) \overset{\mathrm{tex}}{\to}$ "
Dedu_0(#1.
, #2.
)"$]$

$[\mathrm{Dedu}_0(*,*) \overset{\mathrm{pyk}}{\to}$ "1deduction zero " conclude " end 1deduction"$]$

$\mathrm{Dedu_s}(*, *, *)$

$[\mathrm{Dedu_s}(\mathsf{p}, \mathsf{c}, \mathsf{s}) \stackrel{\mathrm{val}}{\to} \mathrm{If}(\mathsf{p} \stackrel{\mathrm{r}}{=} \lceil \mathsf{x} \Vdash \mathsf{y} \rceil, \mathsf{c} \stackrel{\mathrm{r}}{=} \lceil \mathsf{x} \Vdash \mathsf{y} \rceil \wedge \mathsf{p}^1 \stackrel{\mathrm{t}}{=} \mathsf{c}^1 \wedge \mathrm{Dedu_s}(\mathsf{p}^2, \mathsf{c}^2, \mathsf{c}^1 :: \mathsf{s}),$
$\mathrm{Dedu_1}(\mathsf{p}, \mathsf{c}, \mathsf{s}))]$

$[\mathrm{Dedu_s}(\mathsf{x}, \mathsf{y}, \mathsf{z}) \stackrel{\mathrm{tex}}{\to}$ "Dedu_{s}(#1.
, #2.
, #3.
)"]

$[\mathrm{Dedu_s}(*, *, *) \stackrel{\mathrm{pyk}}{\to}$ "1deduction side " conclude " condition " end 1deduction"]

$\mathrm{Dedu_1}(*, *, *)$

$[\mathrm{Dedu_1}(\mathsf{p}, \mathsf{c}, \mathsf{s}) \stackrel{\mathrm{val}}{\to} \mathrm{If}(\mathsf{c} \stackrel{\mathrm{r}}{=} \lceil \mathsf{x} \Vdash \mathsf{y} \rceil, \mathrm{Dedu_1}(\mathsf{p}, \mathsf{c}^2, \mathsf{c}^1 :: \mathsf{s}), \mathrm{Dedu_2}(\mathsf{p}, \mathsf{c}, \mathsf{s}))]$

$[\mathrm{Dedu_1}(\mathsf{x}, \mathsf{y}, \mathsf{z}) \stackrel{\mathrm{tex}}{\to}$ "
Dedu_1(#1.
, #2.
, #3.
)"]

$[\mathrm{Dedu_1}(*, *, *) \stackrel{\mathrm{pyk}}{\to}$ "1deduction one " conclude " condition " end 1deduction"]

$\mathrm{Dedu_2}(*, *, *)$

$$[\mathrm{Dedu_2}(\mathsf{p}, \mathsf{c}, \mathsf{s}) \stackrel{\mathrm{val}}{\to} \mathsf{s}!\mathsf{p} \stackrel{\mathrm{r}}{=} \lceil \mathsf{x} \vdash \mathsf{y} \rceil \wedge \mathsf{c} \stackrel{\mathrm{r}}{=} \lceil \mathsf{x} \Rightarrow \mathsf{y} \rceil \begin{cases} \mathrm{Dedu_3}(\mathsf{p}^1, \mathsf{c}^1, \mathsf{s}, \mathsf{T}) \wedge \mathrm{Dedu_2}(\mathsf{p}^2, \mathsf{c}^2, \mathsf{s}) \\ \mathrm{Dedu_4}(\mathsf{p}, \mathsf{c}, \mathsf{s}, \mathrm{Dedu_6}(\mathsf{p}, \mathsf{c}, \mathsf{T}, \mathsf{T})) \end{cases}]$$

$[\mathrm{Dedu_2}(\mathsf{x}, \mathsf{y}, \mathsf{z}) \stackrel{\mathrm{tex}}{\to}$ "
Dedu_2(#1.
, #2.
, #3.
)"]

$[\mathrm{Dedu_2}(*, *, *) \stackrel{\mathrm{pyk}}{\to}$ "1deduction two " conclude " condition " end 1deduction"]

$\mathrm{Dedu_3}(*, *, *, *)$

$[\mathrm{Dedu_3}(\mathsf{p}, \mathsf{c}, \mathsf{s}, \mathsf{b}) \stackrel{\mathrm{val}}{\to} \mathrm{If}(\neg \mathsf{c} \stackrel{\mathrm{r}}{=} \lceil \forall_{\mathrm{obj}} \mathsf{x} \colon \mathsf{y} \rceil, \mathrm{Dedu_4}(\mathsf{p}, \mathsf{c}, \mathsf{s}, \mathsf{b}),$
$\mathrm{If}(\mathsf{p} \stackrel{\mathrm{r}}{=} \lceil \forall_{\mathrm{obj}} \mathsf{x} \colon \mathsf{y} \rceil \wedge \mathsf{p}^1 \stackrel{\mathrm{t}}{=} \mathsf{c}^1, \mathrm{Dedu_4}(\mathsf{p}, \mathsf{c}, \mathsf{s}, \mathsf{b}), \mathrm{Dedu_3}(\mathsf{p}, \mathsf{c}^2, \mathsf{s}, \mathsf{c}^1 :: \mathsf{c}^1 :: \mathsf{b}))))]$

$[\mathrm{Dedu}_3(\mathsf{x},\mathsf{y},\mathsf{z},\mathsf{u}) \stackrel{\mathrm{tex}}{\rightarrow}$ "
Dedu_3(#1.
, #2.
, #3.
, #4.
)"]

$[\mathrm{Dedu}_3(*,*,*,*) \stackrel{\mathrm{pyk}}{\rightarrow}$ "1deduction three " conclude " condition " bound " end 1deduction"]

$\mathrm{Dedu}_4(*,*,*,*)$

$[\mathrm{Dedu}_4(\mathsf{p},\mathsf{c},\mathsf{s},\mathsf{b}) \stackrel{\mathrm{val}}{\rightarrow} \mathsf{s}!\mathsf{b}!\mathrm{If}(\mathsf{p} \stackrel{\mathrm{r}}{=} \lceil \overline{\mathsf{x}} \rceil, \mathbf{lookup}(\mathsf{p},\mathsf{b},\mathsf{T}) \stackrel{\mathrm{t}}{=} \mathsf{c}, \mathrm{If}(\neg \mathsf{p} \stackrel{\mathrm{r}}{=} \mathsf{c}, \mathsf{F},$
$\mathrm{If}(\mathsf{p} \stackrel{\mathrm{r}}{=} \lceil \forall_{\mathrm{obj}} \mathsf{x}\colon \mathsf{y} \rceil, \mathsf{p}^1 \stackrel{\mathrm{t}}{=} \mathsf{c}^1 \wedge \mathrm{Dedu}_4(\mathsf{p}^2,\mathsf{c}^2,\mathsf{s},\mathsf{p}^1 :: \mathsf{p}^1 :: \mathsf{b}), \mathrm{If}(\neg \mathsf{p} \stackrel{\mathrm{r}}{=} \lceil \underline{\mathsf{x}} \rceil,$
$\mathrm{Dedu}_4^*(\mathsf{p}^{\mathrm{t}},\mathsf{c}^{\mathrm{t}},\mathsf{s},\mathsf{b}), \mathsf{p}^1 \stackrel{\mathrm{t}}{=} \mathsf{c}^1 \wedge \mathrm{Dedu}_5(\mathsf{p},\mathsf{s},\mathsf{b})))))]$

$[\mathrm{Dedu}_4(\mathsf{x},\mathsf{y},\mathsf{z},\mathsf{u}) \stackrel{\mathrm{tex}}{\rightarrow}$ "
Dedu_4(#1.
, #2.
, #3.
, #4.
)"]

$[\mathrm{Dedu}_4(*,*,*,*) \stackrel{\mathrm{pyk}}{\rightarrow}$ "1deduction four " conclude " condition " bound " end 1deduction"]

$\mathrm{Dedu}_4^*(*,*,*,*)$

$[\mathrm{Dedu}_4^*(\mathsf{p},\mathsf{c},\mathsf{s},\mathsf{b}) \stackrel{\mathrm{val}}{\rightarrow} \mathsf{c}!\mathsf{s}!\mathsf{b}!\mathrm{If}(\mathsf{p},\mathsf{T},\mathrm{Dedu}_4(\mathsf{p}^{\mathrm{h}},\mathsf{c}^{\mathrm{h}},\mathsf{s},\mathsf{b}) \wedge \mathrm{Dedu}_4^*(\mathsf{p}^{\mathrm{t}},\mathsf{c}^{\mathrm{t}},\mathsf{s},\mathsf{b}))]$

$[\mathrm{Dedu}_4^*(\mathsf{x},\mathsf{y},\mathsf{z},\mathsf{u}) \stackrel{\mathrm{tex}}{\rightarrow}$ "
Dedu_4^*(#1.
, #2.
, #3.
, #4.
)"]

$[\mathrm{Dedu}_4^*(*,*,*,*) \stackrel{\mathrm{pyk}}{\rightarrow}$ "1deduction four star " conclude " condition " bound " end 1deduction"]

$\mathrm{Dedu}_5(*, *, *)$

$[\mathrm{Dedu}_5(\mathsf{p}, \mathsf{s}, \mathsf{b}) \stackrel{\text{val}}{\to} \mathsf{p}!\mathsf{s}!\mathrm{If}(\mathsf{b}, \mathsf{T},$
$\lceil\lceil\mathsf{x}\rceil\#^0\lceil\mathsf{y}\rceil\rceil^{\mathrm{h}} :: \lceil\lceil*\rceil\rceil^{\mathrm{h}} :: \mathsf{b}^{\mathrm{hh}} :: \mathsf{T} :: \lceil\lceil\mathsf{x}\rceil\rceil^{\mathrm{h}} :: \mathsf{p} :: \mathsf{T} :: \mathsf{T} \in_{\mathrm{t}} \mathsf{s} \wedge \mathrm{Dedu}_5(\mathsf{p}, \mathsf{s}, \mathsf{b}^{\mathrm{t}}))]$

$[\mathrm{Dedu}_5(\mathsf{x}, \mathsf{y}, \mathsf{z}) \stackrel{\text{tex}}{\to}$ "
Dedu_5(#1.
, #2.
, #3.
)"]

$[\mathrm{Dedu}_5(*, *, *) \stackrel{\text{pyk}}{\to}$ "1deduction five " condition " bound " end 1deduction"]

$\mathrm{Dedu}_6(*, *, *, *)$

$[\mathrm{Dedu}_6(\mathsf{p}, \mathsf{c}, \mathsf{e}, \mathsf{b}) \stackrel{\text{val}}{\to} \mathsf{p}!\mathsf{c}!\mathsf{b}!\mathsf{e}!\mathrm{If}(\mathsf{p} \stackrel{\mathrm{r}}{=} \lceil\overline{\mathsf{x}}\rceil, \mathsf{p} \in_{\mathrm{t}} \mathsf{e} \begin{cases} \mathsf{b} \\ \mathsf{p} :: \mathsf{c} :: \mathsf{b} \end{cases}, \mathrm{If}(\neg\mathsf{p} \stackrel{\mathrm{r}}{=} \mathsf{c}, \mathsf{T},$
$\mathrm{If}(\mathsf{p} \stackrel{\mathrm{r}}{=} \lceil\underline{\mathsf{a}}\rceil, \mathsf{b}, \mathrm{If}(\mathsf{p} \stackrel{\mathrm{r}}{=} \lceil\forall_{\mathrm{obj}}\mathsf{x}\colon \mathsf{y}\rceil, \mathrm{Dedu}_6(\mathsf{p}^2, \mathsf{c}^2, \mathsf{c}^1 :: \mathsf{e}, \mathsf{b}), \mathrm{Dedu}_6^*(\mathsf{p}^{\mathrm{t}}, \mathsf{c}^{\mathrm{t}}, \mathsf{e}, \mathsf{b})))))]$

$[\mathrm{Dedu}_6(\mathsf{p}, \mathsf{c}, \mathsf{e}, \mathsf{b}) \stackrel{\text{tex}}{\to}$ "
Dedu_6(#1.
, #2.
, #3.
, #4.
)"]

$[\mathrm{Dedu}_6(*, *, *, *) \stackrel{\text{pyk}}{\to}$ "1deduction six " conclude " exception " bound " end 1deduction"]

$\mathrm{Dedu}_6^*(*, *, *, *)$

$[\mathrm{Dedu}_6^*(\mathsf{p}, \mathsf{c}, \mathsf{e}, \mathsf{b}) \stackrel{\text{val}}{\to} \mathsf{p}!\mathsf{c}!\mathsf{b}!\mathsf{e}!\mathrm{If}(\mathsf{p}, \mathsf{b}, \mathrm{Dedu}_6^*(\mathsf{p}^{\mathrm{t}}, \mathsf{c}^{\mathrm{t}}, \mathsf{e}, \mathrm{Dedu}_6(\mathsf{p}^{\mathrm{h}}, \mathsf{c}^{\mathrm{h}}, \mathsf{e}, \mathsf{b})))]$

$[\mathrm{Dedu}_6^*(\mathsf{p}, \mathsf{c}, \mathsf{e}, \mathsf{b}) \stackrel{\text{tex}}{\to}$ "
Dedu_6^*(#1.
, #2.
, #3.
, #4.
)"]

$[\mathrm{Dedu}_6^*(*, *, *, *) \stackrel{\text{pyk}}{\to}$ "1deduction six star " conclude " exception " bound " end 1deduction"]

$\mathrm{Dedu}_7(*)$

$[\mathrm{Dedu}_7(\mathsf{p}) \stackrel{\mathrm{val}}{\to} \mathsf{p} \stackrel{\mathrm{r}}{=} \lceil \forall \mathsf{x} \colon \mathsf{y} \rceil \begin{cases} \mathrm{Dedu}_7(\mathsf{p}^2) \\ \mathsf{p} \end{cases}]$

$[\mathrm{Dedu}_7(\mathsf{p}) \stackrel{\mathrm{tex}}{\to}$ "
Dedu_7(#1.
)"]

$[\mathrm{Dedu}_7(*) \stackrel{\mathrm{pyk}}{\to}$ "1deduction seven " end 1deduction"]

$\mathrm{Dedu}_8(*, *)$

$[\mathrm{Dedu}_8(\mathsf{p}, \mathsf{b}) \stackrel{\mathrm{val}}{\to} \mathrm{If}(\mathsf{p} \stackrel{\mathrm{r}}{=} \lceil \forall \mathsf{x} \colon \mathsf{y} \rceil, \mathrm{Dedu}_8(\mathsf{p}^2, \mathsf{p}^1 :: \mathsf{b}), \mathrm{If}(\mathsf{p} \stackrel{\mathrm{r}}{=} \lceil \underline{\mathsf{a}} \rceil, \mathsf{p} \in_{\mathrm{t}} \mathsf{b}, \mathrm{Dedu}_8^*(\mathsf{p}^{\mathrm{t}}, \mathsf{b})))]$

$[\mathrm{Dedu}_8(\mathsf{p}, \mathsf{b}) \stackrel{\mathrm{tex}}{\to}$ "
Dedu_8(#1.
, #2.
)"]

$[\mathrm{Dedu}_8(*, *) \stackrel{\mathrm{pyk}}{\to}$ "1deduction eight " bound " end 1deduction"]

$\mathrm{Dedu}_8^*(*, *)$

$[\mathrm{Dedu}_8^*(\mathsf{p}, \mathsf{b}) \stackrel{\mathrm{val}}{\to} \mathsf{b}!\mathrm{If}(\mathsf{p}, \mathsf{T}, \mathrm{If}(\mathrm{Dedu}_8(\mathsf{p}^{\mathrm{h}}, \mathsf{b}), \mathrm{Dedu}_8^*(\mathsf{p}^{\mathrm{t}}, \mathsf{b}), \mathsf{F}))]$

$[\mathrm{Dedu}_8^*(\mathsf{p}, \mathsf{b}) \stackrel{\mathrm{tex}}{\to}$ "
Dedu_8^*(#1.
, #2.
)"]

$[\mathrm{Dedu}_8^*(*, *) \stackrel{\mathrm{pyk}}{\to}$ "1deduction eight star " bound " end 1deduction"]

Ex_1

$[\mathrm{Ex}_1 \stackrel{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil [\mathrm{Ex}_1 \ddot{=} \mathsf{a}_{\mathrm{Ex}}] \rceil)]$

$[\mathrm{Ex}_1 \stackrel{\mathrm{tex}}{\to}$ "Ex_{1}"]

$[\mathrm{Ex}_1 \stackrel{\mathrm{pyk}}{\to}$ "ex1"]

Ex_2

$[\mathrm{Ex}_2 \overset{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathrm{Ex}_2 \ddot{=} \mathsf{b}_{\mathrm{Ex}}]\rceil)]$

$[\mathrm{Ex}_2 \overset{\mathrm{tex}}{\to}$ "Ex_{2}"]

$[\mathrm{Ex}_2 \overset{\mathrm{pyk}}{\to}$ "ex2"]

Ex3

$[\mathrm{Ex3} \overset{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathrm{Ex3} \ddot{=} \mathsf{c}_{\mathrm{Ex}}]\rceil)]$

$[\mathrm{Ex3} \overset{\mathrm{tex}}{\to}$ "Ex3"]

$[\mathrm{Ex3} \overset{\mathrm{pyk}}{\to}$ "ex3"]

Ex_{10}

$[\mathrm{Ex}_{10} \overset{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathrm{Ex}_{10} \ddot{=} \mathsf{j}_{\mathrm{Ex}}]\rceil)]$

$[\mathrm{Ex}_{10} \overset{\mathrm{tex}}{\to}$ "Ex_{10}"]

$[\mathrm{Ex}_{10} \overset{\mathrm{pyk}}{\to}$ "ex10"]

Ex_{20}

$[\mathrm{Ex}_{20} \overset{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathrm{Ex}_{20} \ddot{=} \mathsf{t}_{\mathrm{Ex}}]\rceil)]$

$[\mathrm{Ex}_{20} \overset{\mathrm{tex}}{\to}$ "Ex_{20}"]

$[\mathrm{Ex}_{20} \overset{\mathrm{pyk}}{\to}$ "ex20"]

$*_{\mathrm{Ex}}$

$[\mathsf{x}_{\mathrm{Ex}} \overset{\mathrm{tex}}{\to}$ "#1.
_{Ex}"]

$[*_{\mathrm{Ex}} \overset{\mathrm{pyk}}{\to}$ "existential var " end var"]

$*^{\mathrm{Ex}}$

$[\mathsf{x}^{\mathrm{Ex}} \overset{\mathrm{val}}{\to} \mathsf{x} \overset{\mathrm{r}}{=} \lceil \mathsf{x}_{\mathrm{Ex}} \rceil]$

$[\mathsf{x}^{\mathrm{Ex}} \stackrel{\mathrm{tex}}{\to}$ "#1.
^{Ex}"]

$[*^{\mathrm{Ex}} \stackrel{\mathrm{pyk}}{\to}$ "" is existential var"]

$\langle * \equiv * \mid * :== * \rangle_{\mathrm{Ex}}$

$[\langle \mathsf{a} \equiv \mathsf{b} | \mathsf{x} :== \mathsf{t} \rangle_{\mathrm{Ex}} \stackrel{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\langle \mathsf{a} \equiv \mathsf{b} | \mathsf{x} :== \mathsf{t} \rangle_{\mathrm{Ex}} \ddot{=} \langle \lceil \mathsf{a} \rceil \equiv^0 \lceil \mathsf{b} \rceil | \lceil \mathsf{x} \rceil :== \lceil \mathsf{t} \rceil \rangle_{\mathrm{Ex}}]\rceil)]$

$[\langle \mathsf{x} \equiv \mathsf{y} | \mathsf{z} :== \mathsf{u} \rangle_{\mathrm{Ex}} \stackrel{\mathrm{tex}}{\to}$ "\langle #1.
{\equiv} #2.
| #3.
{:==} #4.
\rangle_{Ex} "]

$[\langle * \equiv * \mid * :== * \rangle_{\mathrm{Ex}} \stackrel{\mathrm{pyk}}{\to}$ "exist-sub " is " where " is " end sub"]

$\langle * \equiv^0 * \mid * :== * \rangle_{\mathrm{Ex}}$

$[\langle \mathsf{a} \equiv^0 \mathsf{b} | \mathsf{x} :== \mathsf{t} \rangle_{\mathrm{Ex}} \stackrel{\mathrm{val}}{\to} \lambda \mathsf{c}.\mathsf{x}^{\mathrm{Ex}} \wedge \langle \mathsf{a} \equiv^1 \mathsf{b} | \mathsf{x} :== \mathsf{t} \rangle_{\mathrm{Ex}}]$

$[\langle \mathsf{x} \equiv^0 \mathsf{y} | \mathsf{z} :== \mathsf{u} \rangle_{\mathrm{Ex}} \stackrel{\mathrm{tex}}{\to}$ "\langle #1.
{\equiv}^0 #2.
| #3.
{:==} #4.
\rangle_{Ex} "]

$[\langle * \equiv^0 * \mid * :== * \rangle_{\mathrm{Ex}} \stackrel{\mathrm{pyk}}{\to}$ "exist-sub0 " is " where " is " end sub"]

$\langle * \equiv^1 * \mid * :== * \rangle_{\mathrm{Ex}}$

$[\langle \mathsf{a} \equiv^1 \mathsf{b} | \mathsf{x} :== \mathsf{t} \rangle_{\mathrm{Ex}} \stackrel{\mathrm{val}}{\to} \mathsf{a}!\mathsf{x}!\mathsf{t}!$
$\mathrm{If}(\mathsf{b} \stackrel{\mathrm{r}}{=} \lceil \forall_{\mathrm{obj}} \mathsf{u} \colon \mathsf{v} \rceil, \mathsf{F},$
$\mathrm{If}(\mathsf{b}^{\mathrm{Ex}} \wedge \mathsf{b} \stackrel{\mathrm{t}}{=} \mathsf{x}, \mathsf{a} \stackrel{\mathrm{t}}{=} \mathsf{t}, \mathrm{If}($
$\mathsf{a} \stackrel{\mathrm{r}}{=} \mathsf{b}, \langle \mathsf{a}^{\mathrm{t}} \equiv^* \mathsf{b}^{\mathrm{t}} | \mathsf{x} :== \mathsf{t} \rangle_{\mathrm{Ex}}, \mathsf{F})))]$

$[\langle \mathsf{x} \equiv^1 \mathsf{y} | \mathsf{z} :== \mathsf{u} \rangle_{\mathrm{Ex}} \stackrel{\mathrm{tex}}{\to}$ "\langle #1.
{\equiv}^1 #2.
| #3.
{:==} #4.
\rangle_{Ex} "]

$[\langle *\equiv^1 * \mid * :==* \rangle_{\mathrm{Ex}} \stackrel{\mathrm{pyk}}{\to}$ "exist-sub1 " is " where " is " end sub"]

$\langle *\equiv^* * \mid * :==* \rangle_{\mathrm{Ex}}$

$[\langle \mathsf{a}\equiv^*\mathsf{b}|\mathsf{x}:==\mathsf{t}\rangle_{\mathrm{Ex}} \stackrel{\mathrm{val}}{\to} \mathsf{b}!\mathsf{x}!\mathsf{t}!\mathrm{If}(\mathsf{a}, \mathsf{T}, \mathrm{If}(\langle \mathsf{a}^{\mathrm{h}}\equiv^1\mathsf{b}^{\mathrm{h}}|\mathsf{x}:==\mathsf{t}\rangle_{\mathrm{Ex}}, \langle \mathsf{a}^{\mathrm{t}}\equiv^*\mathsf{b}^{\mathrm{t}}|\mathsf{x}:==\mathsf{t}\rangle_{\mathrm{Ex}}, \mathsf{F}))]$

$[\langle \mathsf{x}\equiv^*\mathsf{y}|\mathsf{z}:==\mathsf{u}\rangle_{\mathrm{Ex}} \stackrel{\mathrm{tex}}{\to}$ "\langle #1.
{\equiv}ˆ∗ #2.
| #3.
{:==} #4.
\rangle_{Ex} "]

$[\langle *\equiv^* * \mid * :==* \rangle_{\mathrm{Ex}} \stackrel{\mathrm{pyk}}{\to}$ "exist-sub∗ " is " where " is " end sub"]

ph_1

$[\mathrm{ph}_1 \stackrel{\mathrm{tex}}{\to}$ "ph_{1}"]

$[\mathrm{ph}_1 \stackrel{\mathrm{pyk}}{\to}$ "placeholder-var1"]

ph_2

$[\mathrm{ph}_2 \stackrel{\mathrm{tex}}{\to}$ "ph_{2}"]

$[\mathrm{ph}_2 \stackrel{\mathrm{pyk}}{\to}$ "placeholder-var2"]

ph_3

$[\mathrm{ph}_3 \stackrel{\mathrm{tex}}{\to}$ "ph_{3}"]

$[\mathrm{ph}_3 \stackrel{\mathrm{pyk}}{\to}$ "placeholder-var3"]

$*_{\mathrm{Ph}}$

$[\mathsf{x}_{\mathrm{Ph}} \stackrel{\mathrm{tex}}{\to}$ "#1.
_{Ph} "]

$[*_{\mathrm{Ph}} \stackrel{\mathrm{pyk}}{\to}$ "placeholder-var " end var"]

$*^{\mathrm{Ph}}$

[$\mathsf{x}^{\mathrm{Ph}} \overset{\mathrm{tex}}{\to}$ "#1.
^{Ph}"]

[$*^{\mathrm{Ph}} \overset{\mathrm{pyk}}{\to}$ "" is placeholder-var"]

$\langle * \equiv * \mid * :== * \rangle_{\mathrm{Ph}}$

[$\langle \mathsf{x} \equiv \mathsf{y} | \mathsf{z} :== \mathsf{u} \rangle_{\mathrm{Ph}} \overset{\mathrm{tex}}{\to}$ "\langle #1.
{\equiv} #2.
| #3.
{:==} #4.
\rangle_{Ph} "]

[$\langle * \equiv * \mid * :== * \rangle_{\mathrm{Ph}} \overset{\mathrm{pyk}}{\to}$ "ph-sub " is " where " is " end sub"]

$\langle * \equiv^{0} * \mid * :== * \rangle_{\mathrm{Ph}}$

[$\langle \mathsf{x} \equiv^{0} \mathsf{y} | \mathsf{z} :== \mathsf{u} \rangle_{\mathrm{Ph}} \overset{\mathrm{tex}}{\to}$ "\langle #1.
{\equiv}^0 #2.
| #3.
{:==} #4.
\rangle_{Ph} "]

[$\langle * \equiv^{0} * \mid * :== * \rangle_{\mathrm{Ph}} \overset{\mathrm{pyk}}{\to}$ "ph-sub0 " is " where " is " end sub"]

$\langle * \equiv^{1} * \mid * :== * \rangle_{\mathrm{Ph}}$

[$\langle \mathsf{x} \equiv^{1} \mathsf{y} | \mathsf{z} :== \mathsf{u} \rangle_{\mathrm{Ph}} \overset{\mathrm{tex}}{\to}$ "\langle #1.
{\equiv}^1 #2.
| #3.
{:==} #4.
\rangle_{Ph} "]

[$\langle * \equiv^{1} * \mid * :== * \rangle_{\mathrm{Ph}} \overset{\mathrm{pyk}}{\to}$ "ph-sub1 " is " where " is " end sub"]

$\langle * \equiv^* * \mid * :== * \rangle_{\mathrm{Ph}}$

$[\langle \mathsf{x} \equiv^* \mathsf{y} | \mathsf{z} :== \mathsf{u} \rangle_{\mathrm{Ph}} \stackrel{\mathrm{tex}}{\to}$ "\langle #1.
{\equiv}^* #2.
| #3.
{:==} #4.
\rangle_{Ph} "]

$[\langle * \equiv^* * \mid * :== * \rangle_{\mathrm{Ph}} \stackrel{\mathrm{pyk}}{\to}$ "ph-sub* " is " where " is " end sub"]

bs

$[\mathsf{bs} \stackrel{\mathrm{tex}}{\to}$ "\mathsf {bs}"]

$[\mathsf{bs} \stackrel{\mathrm{pyk}}{\to}$ "var big set"]

OBS

$[\mathsf{OBS} \stackrel{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil [\mathsf{OBS} \ddot{=} \overline{\mathsf{bs}}] \rceil)]$

$[\mathsf{OBS} \stackrel{\mathrm{tex}}{\to}$ " \mathsf {OBS}"]

$[\mathsf{OBS} \stackrel{\mathrm{pyk}}{\to}$ "object big set"]

$\mathcal{BS}$

$[\mathcal{BS} \stackrel{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil [\mathcal{BS} \ddot{=} \underline{\mathsf{bs}}] \rceil)]$

$[\mathcal{BS} \stackrel{\mathrm{tex}}{\to}$ "{\cal BS}"]

$[\mathcal{BS} \stackrel{\mathrm{pyk}}{\to}$ "meta big set"]

Ø

$[\mathrm{\O} \stackrel{\mathrm{tex}}{\to}$ "\mathrm{\O}"]

$[\mathrm{\O} \stackrel{\mathrm{pyk}}{\to}$ "zermelo empty set"]

ZFsub

$[\mathrm{ZFsub} \stackrel{\mathrm{stmt}}{\rightarrow} \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \underline{x} * \underline{y} * \underline{z} = \underline{x} * \underline{y} * \underline{z} \oplus \forall\overline{(fx)}\colon \forall\overline{(fy)}\colon \mathrm{R}(\overline{(fx)}) << \mathrm{R}(\overline{(fy)}) \vdash$
$\overline{(fx)} <_{\mathrm{f}} \overline{(fy)} \oplus \forall\underline{m}\colon \forall\overline{(\epsilon)}\colon \forall\overline{(fx)}\colon \forall\overline{(fy)}\colon \overline{(fx)} <_{\mathrm{f}} \overline{(fy)} \vdash \dot{\neg}0 <= \overline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}0 = \overline{(\epsilon)} \vdash$
$c_{\mathrm{Ex}} <= \underline{m} \Rightarrow \overline{(fx)}[\underline{m}] <= \overline{(fy)}[\underline{m}] + -\overline{(\epsilon)} \oplus \forall\overline{(rx)}\colon \forall\overline{(ry)}\colon \forall\overline{(rz)}\colon \overline{(rx)} == \overline{(ry)} \vdash$
$\overline{(ry)} == \overline{(rz)} \vdash \overline{(rx)} == \overline{(rz)} \oplus \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \underline{x} <= \underline{y} \Rightarrow \underline{y} <= \underline{z} \Rightarrow \underline{x} <= \underline{z} \oplus$
$\forall\underline{x}\colon \forall\underline{t}\colon \forall\underline{a}\colon \forall\underline{b}\colon \langle\lceil\underline{a}\rceil{\equiv}^0\lceil\underline{b}\rceil|\lceil\underline{x}\rceil{:==}\lceil\underline{t}\rceil\rangle_{\mathrm{Ex}} \Vdash \underline{a} \vdash \underline{b} \oplus \forall\underline{m}\colon \forall\overline{(\epsilon)}\colon \forall\overline{(fx)}\colon \forall\overline{(fy)}\colon \dot{\neg}0 <=$
$\overline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}0 = \overline{(\epsilon)} \Rightarrow c_{\mathrm{Ex}} <= \underline{m} \Rightarrow \dot{\neg}\mathrm{if}(0 <=$
$\overline{(fx)}[\underline{m}] + -\overline{(fy)}[\underline{m}], \overline{(fx)}[\underline{m}] + -\overline{(fy)}[\underline{m}], -\overline{(fx)}[\underline{m}] + -\overline{(fy)}[\underline{m}]) <= \overline{(\epsilon)} \Rightarrow$
$\dot{\neg}\dot{\neg}\mathrm{if}(0 <= \overline{(fx)}[\underline{m}] + -\overline{(fy)}[\underline{m}], \overline{(fx)}[\underline{m}] + -\overline{(fy)}[\underline{m}], -\overline{(fx)}[\underline{m}] + -\overline{(fy)}[\underline{m}]) = \overline{(\epsilon)} \vdash$
$\mathrm{SF}(\overline{(fx)}, \overline{(fy)}) \oplus \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \underline{x} = \underline{y} \Rightarrow \underline{x} + \underline{z} = \underline{y} + \underline{z} \oplus \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \underline{x} + \underline{y} + \underline{z} =$
$\underline{x} + \underline{y} + \underline{z} \oplus \forall\underline{s}\colon \forall\underline{x}\colon \dot{\neg}\underline{s} \in \cup\underline{x} \Rightarrow \dot{\neg}\underline{s} \in j_{\mathrm{Ex}} \Rightarrow \dot{\neg}j_{\mathrm{Ex}} \in \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\underline{s} \in j_{\mathrm{Ex}} \Rightarrow \dot{\neg}j_{\mathrm{Ex}} \in \underline{x} \Rightarrow$
$\underline{s} \in \cup\underline{x} \oplus \forall\underline{m}\colon \forall\overline{(\epsilon)}\colon \forall\overline{(fx)}\colon \forall\overline{(fy)}\colon \forall\overline{(rx)}\colon \forall\overline{(ry)}\colon \overline{(rx)} << \overline{(ry)} \vdash \overline{(fx)} \in \overline{(rx)} \vdash \overline{(fy)} \in$
$\overline{(ry)} \vdash \dot{\neg}0 <= \overline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}0 = \overline{(\epsilon)} \vdash a_{\mathrm{Ex}} <= \underline{m} \Rightarrow \overline{(fx)}[\underline{m}] <= \overline{(fy)}[\underline{m}] + -\overline{(\epsilon)} \oplus$
$\forall\underline{a}\colon \forall\underline{x}\colon \forall\underline{y}\colon \underline{a} \vdash \mathrm{if}(\underline{a}, \underline{x}, \underline{y}) = \underline{x} \oplus \forall\underline{x}\colon \forall\underline{a}\colon \underline{a} \vdash \forall_{\mathrm{obj}}\underline{x}\colon \underline{a} \oplus \forall\underline{m}\colon \forall\overline{(fx)}\colon -_{\mathrm{f}}\overline{(fx)}[\underline{m}] =$
$-\overline{(fx)}[\underline{m}] \oplus \forall\overline{(fx)}\colon \forall\overline{(fy)}\colon \mathrm{R}(\overline{(fx)}) **\mathrm{R}(\overline{(fy)}) == \mathrm{R}(\overline{(fx)} *_{\mathrm{f}} \overline{(fy)}) \oplus \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \underline{x} * \underline{y} +$
$\underline{z} = \underline{x} * \underline{y} + \underline{x} * \underline{z} \oplus \forall\underline{x}\colon \underline{x} + -\underline{x} = 0 \oplus \forall\underline{a}\colon \forall\underline{b}\colon \forall\underline{p}\colon \forall\underline{x}\colon \forall\underline{z}\colon \underline{p}^{\mathrm{Ph}} \wedge \langle\underline{b}{\equiv}\underline{a}|\underline{p}{:==}\underline{z}\rangle_{\mathrm{Ph}} \Vdash$
$\dot{\neg}\underline{z} \in \{\mathrm{ph} \in \underline{x} \mid \underline{a}\} \Rightarrow \dot{\neg}\underline{z} \in \underline{x} \Rightarrow \dot{\neg}\underline{b} \Rightarrow \dot{\neg}\dot{\neg}\underline{z} \in \underline{x} \Rightarrow \dot{\neg}\underline{b} \Rightarrow \underline{z} \in \{\mathrm{ph} \in \underline{x} \mid \underline{a}\} \oplus$
$\forall\overline{(rx)}\colon \forall\overline{(ry)}\colon \overline{(rx)} << \overline{(ry)} \vdash t_{\mathrm{Ex}} \in \overline{(ry)} \oplus \forall\underline{m}\colon \forall\overline{(fx)}\colon \forall\overline{(fy)}\colon \overline{(fx)} =_{\mathrm{f}} \overline{(fy)} \vdash$
$\overline{(fx)}[\underline{m}] = \overline{(fy)}[\underline{m}] \oplus \forall\overline{(rx)}\colon \forall\overline{(ry)}\colon \overline{(rx)} == \overline{(ry)} \vdash \overline{(ry)} == \overline{(rx)} \oplus \forall\underline{x}\colon \underline{x} <= \underline{x} \oplus$
$\forall\underline{a}\colon \forall\underline{b}\colon \dot{\neg}\underline{b} \Rightarrow \underline{a} \vdash \dot{\neg}\underline{b} \Rightarrow \dot{\neg}\underline{a} \vdash \underline{b} \oplus \forall\underline{m}\colon 1\mathrm{f}[\underline{m}] = 1 \oplus \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \underline{x} = \underline{y} \Rightarrow \underline{x} = \underline{z} \Rightarrow$
$\underline{y} = \underline{z} \oplus \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \underline{x} <= \underline{y} \Rightarrow \underline{x} + \underline{z} <= \underline{y} + \underline{z} \oplus \forall\underline{s}\colon \dot{\neg}\underline{s} \in \emptyset \oplus$
$\forall\overline{(fx)}\colon \forall\overline{(fy)}\colon \mathrm{R}(\overline{(fx)}) == \mathrm{R}(\overline{(fy)}) \vdash \mathrm{SF}(\overline{(fx)}, \overline{(fy)}) \oplus \forall\overline{(fx)}\colon \forall\overline{(rx)}\colon \forall\overline{(ry)}\colon \overline{(rx)} ==$
$\overline{(ry)} \vdash \overline{(fx)} \in \overline{(rx)} \vdash \overline{(fx)} \in \overline{(ry)} \oplus \forall\underline{m}\colon \forall\overline{(fx)}\colon \forall\overline{(fy)}\colon \overline{(fx)} +_{\mathrm{f}} \overline{(fy)}[\underline{m}] =$
$\overline{(fx)}[\underline{m}] + \overline{(fy)}[\underline{m}] \oplus \forall\overline{(fx)}\colon \forall\overline{(fy)}\colon \overline{(fx)} \in \mathrm{R}(\overline{(fy)}) \vdash \mathrm{SF}(\overline{(fx)}, \overline{(fy)}) \oplus \forall\underline{x}\colon \dot{\neg}\underline{x} = 0 \Rightarrow$
$\underline{x} * \mathrm{rec}\underline{x} = 1 \oplus \forall\underline{x}\colon \underline{x} + 0 = \underline{x} \oplus \forall\underline{m}\colon \forall\underline{n}\colon \forall\overline{(\epsilon)}\colon \forall\overline{(fx)}\colon \forall\overline{(fy)}\colon \forall\overline{(rx)}\colon \forall\overline{(ry)}\colon \overline{(fx)} \in$
$\overline{(rx)} \Rightarrow \overline{(fy)} \in \overline{(ry)} \Rightarrow \dot{\neg}0 <= \overline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}0 = \overline{(\epsilon)} \Rightarrow a_{\mathrm{Ex}} <= \underline{m} \Rightarrow \overline{(fx)}[\underline{m}] <=$
$\overline{(fy)}[\underline{m}] + -\overline{(\epsilon)} \vdash \overline{(rx)} << \overline{(ry)} \oplus \forall\underline{m}\colon \forall\overline{(fx)}\colon \forall\overline{(fy)}\colon \overline{(fx)}[\underline{m}] = \overline{(fy)}[\underline{m}] \vdash \overline{(fx)} =_{\mathrm{f}}$
$\overline{(fy)} \oplus \forall\underline{x}\colon \forall\underline{y}\colon \underline{x} <= \underline{y} \Rightarrow \underline{y} <= \underline{x} \Rightarrow \underline{x} = \underline{y} \oplus \forall\underline{a}\colon \forall\underline{b}\colon \lambda x.\mathrm{Dedu}_0(\lceil\underline{a}\rceil, \lceil\underline{b}\rceil) \Vdash \underline{a} \vdash \underline{b} \oplus$
$\forall\underline{m}\colon \forall\overline{(\epsilon)}\colon \forall\overline{(fx)}\colon \forall\overline{(fy)}\colon \mathrm{SF}(\overline{(fx)}, \overline{(fy)}) \vdash \dot{\neg}0 <= \overline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}0 = \overline{(\epsilon)} \vdash c_{\mathrm{Ex}} <= \underline{m} \Rightarrow$
$\dot{\neg}\mathrm{if}(0 <= \overline{(fx)}[\underline{m}] + -\overline{(fy)}[\underline{m}], \overline{(fx)}[\underline{m}] + -\overline{(fy)}[\underline{m}], -\overline{(fx)}[\underline{m}] + -\overline{(fy)}[\underline{m}]) <= \overline{(\epsilon)} \Rightarrow$
$\dot{\neg}\dot{\neg}\mathrm{if}(0 <= \overline{(fx)}[\underline{m}] + -\overline{(fy)}[\underline{m}], \overline{(fx)}[\underline{m}] + -\overline{(fy)}[\underline{m}], -\overline{(fx)}[\underline{m}] + -\overline{(fy)}[\underline{m}]) = \overline{(\epsilon)} \oplus$
$\forall\underline{x}\colon \forall\underline{y}\colon \underline{x} = \underline{y} \Rightarrow \underline{x} <= \underline{y} \oplus \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon 0 <= \underline{z} \Rightarrow \underline{x} <= \underline{y} \Rightarrow \underline{x} * \underline{z} <= \underline{y} * \underline{z} \oplus$
$\forall\underline{s}\colon \forall\underline{x}\colon \forall\underline{y}\colon \dot{\neg}\underline{s} \in \{\underline{x}, \underline{y}\} \Rightarrow \dot{\neg}\underline{s}{==}\underline{x} \Rightarrow \underline{s}{==}\underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{s}{==}\underline{x} \Rightarrow \underline{s}{==}\underline{y} \Rightarrow \underline{s} \in$
$\{\underline{x}, \underline{y}\} \oplus \forall\overline{(fx)}\colon \forall\overline{(fy)}\colon \mathrm{SF}(\overline{(fx)}, \overline{(fy)}) \vdash \mathrm{R}(\overline{(fx)}) == \mathrm{R}(\overline{(fy)}) \oplus$
$\forall\overline{(fx)}\colon \forall\overline{(rx)}\colon \forall\overline{(ry)}\colon \overline{(rx)} == \overline{(ry)} \vdash \overline{(fx)} \in \overline{(ry)} \vdash \overline{(fx)} \in \overline{(rx)} \oplus \forall\underline{a}\colon \forall\underline{b}\colon \underline{a} \Rightarrow \underline{b} \vdash \underline{a} \vdash$
$\underline{b} \oplus \forall\underline{m}\colon \forall\overline{(fx)}\colon \forall\overline{(fy)}\colon \overline{(fx)} *_{\mathrm{f}} \overline{(fy)}[\underline{m}] = \overline{(fx)}[\underline{m}] * \overline{(fy)}[\underline{m}] \oplus$
$\forall\overline{(fx)}\colon \forall\overline{(fy)}\colon \mathrm{R}(\overline{(fx)} +_{\mathrm{f}} \overline{(fy)}) == \mathrm{R}(\overline{(fx)} +_{\mathrm{f}} \overline{(fy)}) \oplus \forall\underline{x}\colon \underline{x} * 1 = \underline{x} \oplus \forall\underline{x}\colon \forall\underline{y}\colon \underline{x} + \underline{y} =$
$\underline{y} + \underline{x} \oplus \forall\underline{s}\colon \forall\underline{x}\colon \dot{\neg}\underline{s} \in \mathrm{P}(\underline{x}) \Rightarrow \forall_{\mathrm{obj}}\bar{s}\colon \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \dot{\neg}\forall_{\mathrm{obj}}\bar{s}\colon \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \underline{s} \in$
$\mathrm{P}(\underline{x}) \oplus \forall\overline{(rx)}\colon \forall\overline{(ry)}\colon \overline{(rx)} << \overline{(ry)} \vdash j_{\mathrm{Ex}} \in \overline{(rx)} \oplus \forall\underline{a}\colon \forall\underline{x}\colon \forall\underline{y}\colon \dot{\neg}\underline{a} \vdash \mathrm{if}(\underline{a}, \underline{x}, \underline{y}) = \underline{y} \oplus$
$\forall\overline{(rx)}\colon \overline{(rx)} == \overline{(rx)} \oplus \forall\underline{a}\colon \underline{a} \vdash \underline{a} \oplus \forall\underline{m}\colon 0\mathrm{f}[\underline{m}] = 0 \oplus \dot{\neg}0 = 1 \oplus \forall\underline{x}\colon \forall\underline{y}\colon \dot{\neg}\underline{x} <= \underline{y} \Rightarrow$
$\underline{y} <= \underline{x} \oplus \forall\underline{x}\colon \forall\underline{y}\colon \dot{\neg}\underline{x}{==}\underline{y} \Rightarrow \forall_{\mathrm{obj}}\bar{s}\colon \dot{\neg}\bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y} \Rightarrow \dot{\neg}\bar{s} \in \underline{y} \Rightarrow \bar{s} \in \underline{x} \Rightarrow$
$\dot{\neg}\forall_{\mathrm{obj}}\bar{s}\colon \dot{\neg}\bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y} \Rightarrow \dot{\neg}\bar{s} \in \underline{y} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \underline{x}{==}\underline{y} \oplus$
$\forall\overline{(fx)}\colon \forall\overline{(fy)}\colon \forall\overline{(rx)}\colon \forall\overline{(ry)}\colon \overline{(fx)} \in \overline{(rx)} \Rightarrow \overline{(fy)} \in \overline{(ry)} \Rightarrow \mathrm{SF}(\overline{(fx)}, \overline{(fy)}) \vdash \overline{(rx)} ==$
$\overline{(ry)} \oplus \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \underline{x} = \underline{y} \Rightarrow \underline{x} * \underline{z} = \underline{y} * \underline{z} \oplus \forall\underline{m}\colon \forall\overline{(\epsilon)}\colon \forall\overline{(fx)}\colon \forall\overline{(fy)}\colon \dot{\neg}0 <= \overline{(\epsilon)} \Rightarrow$

$\dot{\neg}\,\dot{\neg}\, 0 = \underline{(\epsilon)} \Rightarrow \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \vdash \underline{(\mathrm{fx})} <_{\mathrm{f}} \underline{(\mathrm{fy})} \oplus \forall \underline{(\mathrm{fx})} \colon \forall \underline{(\mathrm{fy})} \colon \underline{(\mathrm{fx})} <_{\mathrm{f}} \underline{(\mathrm{fy})} \vdash \mathrm{R}(\underline{(\mathrm{fx})}) << \mathrm{R}(\underline{(\mathrm{fy})}) \oplus \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \underline{\mathsf{x}} * \underline{\mathsf{y}} = \underline{\mathsf{y}} * \underline{\mathsf{x}}]$

[ZFsub $\stackrel{\text{tex}}{\rightarrow}$ "ZFsub"]

[ZFsub $\stackrel{\text{pyk}}{\rightarrow}$ "system Q"]

MP

[MP $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[MP $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \underline{\mathsf{a}} \vdash \underline{\mathsf{b}}$]

[MP $\stackrel{\text{tex}}{\rightarrow}$ "MP"]

[MP $\stackrel{\text{pyk}}{\rightarrow}$ "1rule mp"]

Gen

[Gen $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[Gen $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{a}} \colon \underline{\mathsf{a}} \vdash \forall_{\mathrm{obj}} \underline{\mathsf{x}} \colon \underline{\mathsf{a}}$]

[Gen $\stackrel{\text{tex}}{\rightarrow}$ "Gen"]

[Gen $\stackrel{\text{pyk}}{\rightarrow}$ "1rule gen"]

Repetition

[Repetition $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[Repetition $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{\mathsf{a}} \colon \underline{\mathsf{a}} \vdash \underline{\mathsf{a}}$]

[Repetition $\stackrel{\text{tex}}{\rightarrow}$ "Repetition"]

[Repetition $\stackrel{\text{pyk}}{\rightarrow}$ "1rule repetition"]

Neg

[Neg $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[Neg $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{\mathsf{a}} \colon \forall \underline{\mathsf{b}} \colon \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}} \vdash \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{a}} \vdash \underline{\mathsf{b}}$]

[Neg $\stackrel{\text{tex}}{\rightarrow}$ "Neg"]

[Neg $\stackrel{\text{pyk}}{\rightarrow}$ "1rule ad absurdum"]

Ded

[Ded $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[Ded $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{a}\colon \forall \underline{b}\colon \lambda x.\mathrm{Dedu}_0(\lceil \underline{a} \rceil, \lceil \underline{b} \rceil) \Vdash \underline{a} \vdash \underline{b}$]

[Ded $\stackrel{\text{tex}}{\rightarrow}$ "Ded"]

[Ded $\stackrel{\text{pyk}}{\rightarrow}$ "1rule deduction"]

ExistIntro

[ExistIntro $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[ExistIntro $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{t}\colon \forall \underline{a}\colon \forall \underline{b}\colon \langle \lceil \underline{a} \rceil \equiv^0 \lceil \underline{b} \rceil | \lceil \underline{x} \rceil := = \lceil \underline{t} \rceil \rangle_{\mathrm{Ex}} \Vdash \underline{a} \vdash \underline{b}$]

[ExistIntro $\stackrel{\text{tex}}{\rightarrow}$ "ExistIntro"]

[ExistIntro $\stackrel{\text{pyk}}{\rightarrow}$ "1rule exist intro"]

Extensionality

[Extensionality $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[Extensionality $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \dot{\neg}\, \underline{x} == \underline{y} \Rightarrow \forall_{\mathrm{obj}} \bar{s}\colon \dot{\neg}\, \bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y} \Rightarrow \dot{\neg}\, \bar{s} \in \underline{y} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \dot{\neg} \forall_{\mathrm{obj}} \bar{s}\colon \dot{\neg}\, \bar{s} \in \underline{x} \Rightarrow \bar{s} \in \underline{y} \Rightarrow \dot{\neg}\, \bar{s} \in \underline{y} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \underline{x} == \underline{y}$]

[Extensionality $\stackrel{\text{tex}}{\rightarrow}$ "Extensionality"]

[Extensionality $\stackrel{\text{pyk}}{\rightarrow}$ "axiom extensionality"]

Ødef

[Ødef $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[Ødef $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{s}\colon \dot{\neg}\, \underline{s} \in \emptyset$]

[Ødef $\stackrel{\text{tex}}{\rightarrow}$ "\O{}def"]

[Ødef $\stackrel{\text{pyk}}{\rightarrow}$ "axiom empty set"]

PairDef

[PairDef $\stackrel{\text{proof}}{\to}$ Rule tactic]

[PairDef $\stackrel{\text{stmt}}{\to}$ ZFsub $\vdash \forall \underline{s}\colon \forall \underline{x}\colon \forall \underline{y}\colon \dot{\neg}\, \underline{s} \in \{\underline{x}, \underline{y}\} \Rightarrow \dot{\neg}\, \underline{s} == \underline{x} \Rightarrow \underline{s} == \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{s} == \underline{x} \Rightarrow \underline{s} == \underline{y} \Rightarrow \underline{s} \in \{\underline{x}, \underline{y}\}$]

[PairDef $\stackrel{\text{tex}}{\to}$ "PairDef"]

[PairDef $\stackrel{\text{pyk}}{\to}$ "axiom pair definition"]

UnionDef

[UnionDef $\stackrel{\text{proof}}{\to}$ Rule tactic]

[UnionDef $\stackrel{\text{stmt}}{\to}$ ZFsub $\vdash \forall \underline{s}\colon \forall \underline{x}\colon \dot{\neg}\, \underline{s} \in \cup \underline{x} \Rightarrow \dot{\neg}\, \underline{s} \in \mathsf{j}_{\mathrm{Ex}} \Rightarrow \dot{\neg}\, \mathsf{j}_{\mathrm{Ex}} \in \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{s} \in \mathsf{j}_{\mathrm{Ex}} \Rightarrow \dot{\neg}\, \mathsf{j}_{\mathrm{Ex}} \in \underline{x} \Rightarrow \underline{s} \in \cup \underline{x}$]

[UnionDef $\stackrel{\text{tex}}{\to}$ "UnionDef"]

[UnionDef $\stackrel{\text{pyk}}{\to}$ "axiom union definition"]

PowerDef

[PowerDef $\stackrel{\text{proof}}{\to}$ Rule tactic]

[PowerDef $\stackrel{\text{stmt}}{\to}$ ZFsub $\vdash \forall \underline{s}\colon \forall \underline{x}\colon \dot{\neg}\, \underline{s} \in \mathrm{P}(\underline{x}) \Rightarrow \forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \dot{\neg} \forall_{\mathrm{obj}} \bar{s}\colon \bar{s} \in \underline{s} \Rightarrow \bar{s} \in \underline{x} \Rightarrow \underline{s} \in \mathrm{P}(\underline{x})$]

[PowerDef $\stackrel{\text{tex}}{\to}$ "PowerDef"]

[PowerDef $\stackrel{\text{pyk}}{\to}$ "axiom power definition"]

SeparationDef

[SeparationDef $\stackrel{\text{proof}}{\to}$ Rule tactic]

[SeparationDef $\stackrel{\text{stmt}}{\to}$ ZFsub $\vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{p}\colon \forall \underline{x}\colon \forall \underline{z}\colon \underline{p}^{\mathrm{Ph}} \wedge \langle \underline{b} \equiv \underline{a} | \underline{p} \colon == \underline{z} \rangle_{\mathrm{Ph}} \Vdash \dot{\neg}\, \underline{z} \in \{\mathrm{ph} \in \underline{x} \mid \underline{a}\} \Rightarrow \dot{\neg}\, \underline{z} \in \underline{x} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{z} \in \underline{x} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{z} \in \{\mathrm{ph} \in \underline{x} \mid \underline{a}\}$]

[SeparationDef $\stackrel{\text{tex}}{\to}$ "SeparationDef"]

[SeparationDef $\stackrel{\text{pyk}}{\to}$ "axiom separation definition"]

AddDoubleNeg

$[\mathrm{AddDoubleNeg} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \dot{\neg}\,\dot{\neg}\,\dot{\neg}\,\underline{\mathsf{a}} \vdash \mathrm{RemoveDoubleNeg} \rhd \dot{\neg}\,\dot{\neg}\,\dot{\neg}\,\underline{\mathsf{a}} \gg \dot{\neg}\,\underline{\mathsf{a}}; \forall \underline{\mathsf{a}}\colon \mathrm{Ded} \rhd \forall \underline{\mathsf{a}}\colon \dot{\neg}\,\dot{\neg}\,\dot{\neg}\,\underline{\mathsf{a}} \vdash \dot{\neg}\,\underline{\mathsf{a}} \gg \dot{\neg}\,\dot{\neg}\,\dot{\neg}\,\underline{\mathsf{a}} \Rightarrow \dot{\neg}\,\underline{\mathsf{a}}; \underline{\mathsf{a}} \vdash \mathrm{Weakening} \rhd \underline{\mathsf{a}} \gg \dot{\neg}\,\dot{\neg}\,\dot{\neg}\,\underline{\mathsf{a}} \Rightarrow \underline{\mathsf{a}}; \mathrm{Neg} \rhd \dot{\neg}\,\dot{\neg}\,\dot{\neg}\,\underline{\mathsf{a}} \Rightarrow \underline{\mathsf{a}} \rhd \dot{\neg}\,\dot{\neg}\,\dot{\neg}\,\underline{\mathsf{a}} \Rightarrow \dot{\neg}\,\underline{\mathsf{a}} \gg \dot{\neg}\,\dot{\neg}\,\underline{\mathsf{a}}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{AddDoubleNeg} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \underline{\mathsf{a}} \vdash \dot{\neg}\,\dot{\neg}\,\underline{\mathsf{a}}]$

$[\mathrm{AddDoubleNeg} \stackrel{\mathrm{tex}}{\rightarrow}$ “AddDoubleNeg”$]$

$[\mathrm{AddDoubleNeg} \stackrel{\mathrm{pyk}}{\rightarrow}$ “prop lemma add double neg”$]$

RemoveDoubleNeg

$[\mathrm{RemoveDoubleNeg} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \dot{\neg}\,\dot{\neg}\,\underline{\mathsf{a}} \vdash \mathrm{Weakening} \rhd \dot{\neg}\,\dot{\neg}\,\underline{\mathsf{a}} \gg \dot{\neg}\,\underline{\mathsf{a}} \Rightarrow \dot{\neg}\,\dot{\neg}\,\underline{\mathsf{a}}; \mathrm{AutoImply} \gg \dot{\neg}\,\underline{\mathsf{a}} \Rightarrow \dot{\neg}\,\underline{\mathsf{a}}; \mathrm{Neg} \rhd \dot{\neg}\,\underline{\mathsf{a}} \Rightarrow \dot{\neg}\,\underline{\mathsf{a}} \rhd \dot{\neg}\,\underline{\mathsf{a}} \Rightarrow \dot{\neg}\,\dot{\neg}\,\underline{\mathsf{a}} \gg \underline{\mathsf{a}}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{RemoveDoubleNeg} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \dot{\neg}\,\dot{\neg}\,\underline{\mathsf{a}} \vdash \underline{\mathsf{a}}]$

$[\mathrm{RemoveDoubleNeg} \stackrel{\mathrm{tex}}{\rightarrow}$ “RemoveDoubleNeg”$]$

$[\mathrm{RemoveDoubleNeg} \stackrel{\mathrm{pyk}}{\rightarrow}$ “prop lemma remove double neg”$]$

AndCommutativity

$[\mathrm{AndCommutativity} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{b}} \Rightarrow \dot{\neg}\,\underline{\mathsf{a}} \vdash \underline{\mathsf{a}} \vdash \mathrm{AddDoubleNeg} \rhd \underline{\mathsf{a}} \gg \dot{\neg}\,\dot{\neg}\,\underline{\mathsf{a}}; \mathrm{MT} \rhd \underline{\mathsf{b}} \Rightarrow \dot{\neg}\,\underline{\mathsf{a}} \rhd \dot{\neg}\,\dot{\neg}\,\underline{\mathsf{a}} \gg \dot{\neg}\,\underline{\mathsf{b}}; \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \mathrm{Ded} \rhd \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{b}} \Rightarrow \dot{\neg}\,\underline{\mathsf{a}} \vdash \underline{\mathsf{a}} \vdash \dot{\neg}\,\underline{\mathsf{b}} \gg \underline{\mathsf{b}} \Rightarrow \dot{\neg}\,\underline{\mathsf{a}} \Rightarrow \underline{\mathsf{a}} \Rightarrow \dot{\neg}\,\underline{\mathsf{b}}; \dot{\neg}\,\underline{\mathsf{a}} \Rightarrow \dot{\neg}\,\underline{\mathsf{b}} \vdash \mathrm{Repetition} \gg \dot{\neg}\,\underline{\mathsf{a}} \Rightarrow \dot{\neg}\,\underline{\mathsf{b}}; \mathrm{MT} \rhd \underline{\mathsf{b}} \Rightarrow \dot{\neg}\,\underline{\mathsf{a}} \Rightarrow \underline{\mathsf{a}} \Rightarrow \dot{\neg}\,\underline{\mathsf{b}} \rhd \dot{\neg}\,\underline{\mathsf{a}} \Rightarrow \dot{\neg}\,\underline{\mathsf{b}} \gg \dot{\neg}\,\underline{\mathsf{b}} \Rightarrow \dot{\neg}\,\underline{\mathsf{a}}; \mathrm{Repetition} \rhd \dot{\neg}\,\underline{\mathsf{b}} \Rightarrow \dot{\neg}\,\underline{\mathsf{a}} \gg \dot{\neg}\,\underline{\mathsf{b}} \Rightarrow \dot{\neg}\,\underline{\mathsf{a}}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{AndCommutativity} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \dot{\neg}\,\underline{\mathsf{a}} \Rightarrow \dot{\neg}\,\underline{\mathsf{b}} \vdash \dot{\neg}\,\underline{\mathsf{b}} \Rightarrow \dot{\neg}\,\underline{\mathsf{a}}]$

$[\mathrm{AndCommutativity} \stackrel{\mathrm{tex}}{\rightarrow}$ “AndCommutativity”$]$

$[\mathrm{AndCommutativity} \stackrel{\mathrm{pyk}}{\rightarrow}$ “prop lemma and commutativity”$]$

AutoImply

$[\mathrm{AutoImply} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \underline{\mathsf{a}} \vdash \mathrm{Repetition} \rhd \underline{\mathsf{a}} \gg \underline{\mathsf{a}}; \forall \underline{\mathsf{a}}\colon \mathrm{Ded} \rhd \forall \underline{\mathsf{a}}\colon \underline{\mathsf{a}} \vdash \underline{\mathsf{a}} \gg \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{a}}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{AutoImply} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{a}}]$

$[\mathrm{AutoImply} \stackrel{\mathrm{tex}}{\rightarrow}$ “AutoImply”$]$

$[\mathrm{AutoImply} \overset{\mathrm{pyk}}{\to} \text{“prop lemma auto imply”}]$

Contrapositive

$[\mathrm{Contrapositive} \overset{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \dot{\neg}\, \underline{\mathsf{b}} \vdash \mathrm{MT} \rhd \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \rhd \dot{\neg}\, \underline{\mathsf{b}} \gg \dot{\neg}\, \underline{\mathsf{a}}; \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \mathrm{Ded} \rhd \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \dot{\neg}\, \underline{\mathsf{b}} \vdash \dot{\neg}\, \underline{\mathsf{a}} \gg \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{a}}; \mathrm{MP} \rhd \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{a}} \rhd \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \gg \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{a}} \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{Contrapositive} \overset{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{a}}]$

$[\mathrm{Contrapositive} \overset{\mathrm{tex}}{\to} \text{“Contrapositive”}]$

$[\mathrm{Contrapositive} \overset{\mathrm{pyk}}{\to} \text{“prop lemma contrapositive”}]$

FirstConjunct

$[\mathrm{FirstConjunct} \overset{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \vdash \mathrm{AndCommutativity} \rhd \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \gg \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{a}}; \mathrm{SecondConjunct} \rhd \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{a}} \gg \underline{\mathsf{a}} \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{FirstConjunct} \overset{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \vdash \underline{\mathsf{a}}]$

$[\mathrm{FirstConjunct} \overset{\mathrm{tex}}{\to} \text{“FirstConjunct”}]$

$[\mathrm{FirstConjunct} \overset{\mathrm{pyk}}{\to} \text{“prop lemma first conjunct”}]$

SecondConjunct

$[\mathrm{SecondConjunct} \overset{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \dot{\neg}\, \underline{\mathsf{b}} \vdash \mathrm{Weakening} \rhd \dot{\neg}\, \underline{\mathsf{b}} \gg \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}}; \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \mathrm{Ded} \rhd \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \dot{\neg}\, \underline{\mathsf{b}} \vdash \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \gg \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}}; \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \vdash \mathrm{Repetition} \rhd \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \gg \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}}; \mathrm{NegativeMT} \rhd \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \rhd \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \gg \underline{\mathsf{b}} \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{SecondConjunct} \overset{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \vdash \underline{\mathsf{b}}]$

$[\mathrm{SecondConjunct} \overset{\mathrm{tex}}{\to} \text{“SecondConjunct”}]$

$[\mathrm{SecondConjunct} \overset{\mathrm{pyk}}{\to} \text{“prop lemma second conjunct”}]$

FromContradiction

$[\mathrm{FromContradiction} \overset{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{a}} \vdash \dot{\neg}\, \underline{\mathsf{a}} \vdash \mathrm{Weakening} \rhd \underline{\mathsf{a}} \gg \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}}; \mathrm{Weakening} \rhd \dot{\neg}\, \underline{\mathsf{a}} \gg \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{a}}; \mathrm{Neg} \rhd \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}} \rhd \dot{\neg}\, \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \underline{\mathsf{a}} \gg \underline{\mathsf{b}} \rceil, \mathrm{p}_0, \mathsf{c})]$

[FromContradiction $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \vdash \dot{\neg}\, \underline{a} \vdash \underline{b}$]

[FromContradiction $\stackrel{\text{tex}}{\rightarrow}$ "FromContradiction"]

[FromContradiction $\stackrel{\text{pyk}}{\rightarrow}$ "prop lemma from contradiction"]

FromDisjuncts

[FromDisjuncts $\stackrel{\text{proof}}{\rightarrow}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \vdash \underline{a} \Rightarrow \underline{c} \vdash \underline{b} \Rightarrow \underline{c} \vdash$ Repetition $\rhd \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \gg \dot{\neg}\, \underline{a} \Rightarrow \underline{b}$; Contrapositive $\rhd \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \gg \dot{\neg}\, \underline{b} \Rightarrow \dot{\neg}\, \dot{\neg}\, \underline{a}$; Technicality $\rhd \underline{a} \Rightarrow \underline{c} \gg \dot{\neg}\, \dot{\neg}\, \underline{a} \Rightarrow \underline{c}$; ImplyTransitivity $\rhd \dot{\neg}\, \underline{b} \Rightarrow \dot{\neg}\, \dot{\neg}\, \underline{a} \rhd \dot{\neg}\, \dot{\neg}\, \underline{a} \Rightarrow \underline{c} \gg \dot{\neg}\, \underline{b} \Rightarrow \underline{c}$; Contrapositive $\rhd \dot{\neg}\, \underline{b} \Rightarrow \underline{c} \gg \dot{\neg}\, \underline{c} \Rightarrow \dot{\neg}\, \dot{\neg}\, \underline{b}$; Contrapositive $\rhd \underline{b} \Rightarrow \underline{c} \gg \dot{\neg}\, \underline{c} \Rightarrow \dot{\neg}\, \underline{b}$; Neg $\rhd \dot{\neg}\, \underline{c} \Rightarrow \dot{\neg}\, \underline{b} \rhd \dot{\neg}\, \underline{c} \Rightarrow \dot{\neg}\, \dot{\neg}\, \underline{b} \gg \underline{c}\rceil, \mathrm{p}_0, \mathsf{c})$]

[FromDisjuncts $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \vdash \underline{a} \Rightarrow \underline{c} \vdash \underline{b} \Rightarrow \underline{c} \vdash \underline{c}$]

[FromDisjuncts $\stackrel{\text{tex}}{\rightarrow}$ "FromDisjuncts"]

[FromDisjuncts $\stackrel{\text{pyk}}{\rightarrow}$ "prop lemma from disjuncts"]

IffCommutativity

[IffCommutativity $\stackrel{\text{proof}}{\rightarrow}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{a} \vdash$ Repetition $\rhd \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{a} \gg \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{a}$; AndCommutativity $\rhd \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{a} \gg \dot{\neg}\, \underline{b} \Rightarrow \underline{a} \Rightarrow \dot{\neg}\, \underline{a} \Rightarrow \underline{b}$; Repetition $\rhd \dot{\neg}\, \underline{b} \Rightarrow \underline{a} \Rightarrow \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \gg \dot{\neg}\, \underline{b} \Rightarrow \underline{a} \Rightarrow \dot{\neg}\, \underline{a} \Rightarrow \underline{b}\rceil, \mathrm{p}_0, \mathsf{c})$]

[IffCommutativity $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{a} \vdash \dot{\neg}\, \underline{b} \Rightarrow \underline{a} \Rightarrow \dot{\neg}\, \underline{a} \Rightarrow \underline{b}$]

[IffCommutativity $\stackrel{\text{tex}}{\rightarrow}$ "IffCommutativity"]

[IffCommutativity $\stackrel{\text{pyk}}{\rightarrow}$ "prop lemma iff commutativity"]

IffFirst

[IffFirst $\stackrel{\text{proof}}{\rightarrow}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{a} \vdash \underline{b} \vdash$ SecondConjunct $\rhd \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{a} \gg \underline{b} \Rightarrow \underline{a}$; MP $\rhd \underline{b} \Rightarrow \underline{a} \rhd \underline{b} \gg \underline{a}\rceil, \mathrm{p}_0, \mathsf{c})$]

[IffFirst $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{a} \vdash \underline{b} \vdash \underline{a}$]

[IffFirst $\stackrel{\text{tex}}{\rightarrow}$ "IffFirst"]

[IffFirst $\stackrel{\text{pyk}}{\rightarrow}$ "prop lemma iff first"]

IffSecond

[IffSecond $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{a} \vdash \underline{a} \vdash \text{FirstConjunct} \rhd \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{a} \gg \underline{a} \Rightarrow \underline{b}; \text{MP} \rhd \underline{a} \Rightarrow \underline{b} \rhd \underline{a} \gg \underline{b}\rceil, \text{p}_0, c)$]

[IffSecond $\stackrel{\text{stmt}}{\rightarrow}$ $\text{ZFsub} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{a} \vdash \underline{a} \vdash \underline{b}$]

[IffSecond $\stackrel{\text{tex}}{\rightarrow}$ “IffSecond”]

[IffSecond $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma iff second”]

ImplyTransitivity

[ImplyTransitivity $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \underline{a} \Rightarrow \underline{b} \vdash \underline{b} \Rightarrow \underline{c} \vdash \underline{a} \vdash \text{MP} \rhd \underline{a} \Rightarrow \underline{b} \rhd \underline{a} \gg \underline{b}; \text{MP} \rhd \underline{b} \Rightarrow \underline{c} \rhd \underline{b} \gg \underline{c}; \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \underline{a} \Rightarrow \underline{b} \vdash \underline{b} \Rightarrow \underline{c} \vdash \text{Ded} \rhd \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \underline{a} \Rightarrow \underline{b} \vdash \underline{b} \Rightarrow \underline{c} \vdash \underline{a} \vdash \underline{c} \gg \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{a} \Rightarrow \underline{c}; \text{MP2} \rhd \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{a} \Rightarrow \underline{c} \rhd \underline{a} \Rightarrow \underline{b} \rhd \underline{b} \Rightarrow \underline{c} \gg \underline{a} \Rightarrow \underline{c}\rceil, \text{p}_0, c)$]

[ImplyTransitivity $\stackrel{\text{stmt}}{\rightarrow}$ $\text{ZFsub} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \underline{a} \Rightarrow \underline{b} \vdash \underline{b} \Rightarrow \underline{c} \vdash \underline{a} \Rightarrow \underline{c}$]

[ImplyTransitivity $\stackrel{\text{tex}}{\rightarrow}$ “ImplyTransitivity”]

[ImplyTransitivity $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma imply transitivity”]

JoinConjuncts

[JoinConjuncts $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \underline{a} \vdash \underline{a} \Rightarrow \dot{\neg}\, \underline{b} \vdash \text{MP} \rhd \underline{a} \Rightarrow \dot{\neg}\, \underline{b} \rhd \underline{a} \gg \dot{\neg}\, \underline{b}; \forall \underline{a} \colon \forall \underline{b} \colon \text{Ded} \rhd \forall \underline{a} \colon \forall \underline{b} \colon \underline{a} \vdash \underline{a} \Rightarrow \dot{\neg}\, \underline{b} \vdash \dot{\neg}\, \underline{b} \gg \underline{a} \Rightarrow \underline{a} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \dot{\neg}\, \underline{b}; \underline{a} \vdash \underline{b} \vdash \text{MP} \rhd \underline{a} \Rightarrow \underline{a} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \dot{\neg}\, \underline{b} \rhd \underline{a} \gg \underline{a} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \dot{\neg}\, \underline{b}; \text{AddDoubleNeg} \rhd \underline{b} \gg \dot{\neg}\, \dot{\neg}\, \underline{b}; \text{MT} \rhd \underline{a} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \dot{\neg}\, \underline{b} \rhd \dot{\neg}\, \dot{\neg}\, \underline{b} \gg \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{b}; \text{Repetition} \rhd \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{b} \gg \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{b}\rceil, \text{p}_0, c)$]

[JoinConjuncts $\stackrel{\text{stmt}}{\rightarrow}$ $\text{ZFsub} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \underline{a} \vdash \underline{b} \vdash \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{b}$]

[JoinConjuncts $\stackrel{\text{tex}}{\rightarrow}$ “JoinConjuncts”]

[JoinConjuncts $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma join conjuncts”]

MP2

[MP2 $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \vdash \underline{a} \vdash \underline{b} \vdash \text{MP} \rhd \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \rhd \underline{a} \gg \underline{b} \Rightarrow \underline{c}; \text{MP} \rhd \underline{b} \Rightarrow \underline{c} \rhd \underline{b} \gg \underline{c}\rceil, \text{p}_0, c)$]

[MP2 $\stackrel{\text{stmt}}{\rightarrow}$ $\text{ZFsub} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \underline{a} \Rightarrow \underline{b} \Rightarrow \underline{c} \vdash \underline{a} \vdash \underline{b} \vdash \underline{c}$]

$[\mathrm{MP2} \stackrel{\mathrm{tex}}{\rightarrow} \text{“MP2”}]$

$[\mathrm{MP2} \stackrel{\mathrm{pyk}}{\rightarrow} \text{“prop lemma mp2”}]$

MP3

$[\mathrm{MP3} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \forall \underline{\mathsf{c}}\colon \forall \underline{\mathsf{d}}\colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \Rightarrow \underline{\mathsf{d}} \vdash \underline{\mathsf{a}} \vdash \underline{\mathsf{b}} \vdash \underline{\mathsf{c}} \vdash \mathrm{MP2} \rhd \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \Rightarrow \underline{\mathsf{d}} \rhd \underline{\mathsf{a}} \rhd \underline{\mathsf{b}} \gg \underline{\mathsf{c}} \Rightarrow \underline{\mathsf{d}}; \mathrm{MP} \rhd \underline{\mathsf{c}} \Rightarrow \underline{\mathsf{d}} \rhd \underline{\mathsf{c}} \gg \underline{\mathsf{d}} \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{MP3} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \forall \underline{\mathsf{c}}\colon \forall \underline{\mathsf{d}}\colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \Rightarrow \underline{\mathsf{d}} \vdash \underline{\mathsf{a}} \vdash \underline{\mathsf{b}} \vdash \underline{\mathsf{c}} \vdash \underline{\mathsf{d}}]$

$[\mathrm{MP3} \stackrel{\mathrm{tex}}{\rightarrow} \text{“MP3”}]$

$[\mathrm{MP3} \stackrel{\mathrm{pyk}}{\rightarrow} \text{“prop lemma mp3”}]$

MP4

$[\mathrm{MP4} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \forall \underline{\mathsf{c}}\colon \forall \underline{\mathsf{d}}\colon \forall \underline{\mathsf{e}}\colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \Rightarrow \underline{\mathsf{d}} \Rightarrow \underline{\mathsf{e}} \vdash \underline{\mathsf{a}} \vdash \underline{\mathsf{b}} \vdash \underline{\mathsf{c}} \vdash \underline{\mathsf{d}} \vdash \mathrm{MP2} \rhd \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \Rightarrow \underline{\mathsf{d}} \Rightarrow \underline{\mathsf{e}} \rhd \underline{\mathsf{a}} \rhd \underline{\mathsf{b}} \gg \underline{\mathsf{c}} \Rightarrow \underline{\mathsf{d}} \Rightarrow \underline{\mathsf{e}}; \mathrm{MP2} \rhd \underline{\mathsf{c}} \Rightarrow \underline{\mathsf{d}} \Rightarrow \underline{\mathsf{e}} \rhd \underline{\mathsf{c}} \rhd \underline{\mathsf{d}} \gg \underline{\mathsf{e}} \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{MP4} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \forall \underline{\mathsf{c}}\colon \forall \underline{\mathsf{d}}\colon \forall \underline{\mathsf{e}}\colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \Rightarrow \underline{\mathsf{d}} \Rightarrow \underline{\mathsf{e}} \vdash \underline{\mathsf{a}} \vdash \underline{\mathsf{b}} \vdash \underline{\mathsf{c}} \vdash \underline{\mathsf{d}} \vdash \underline{\mathsf{e}}]$

$[\mathrm{MP4} \stackrel{\mathrm{tex}}{\rightarrow} \text{“MP4”}]$

$[\mathrm{MP4} \stackrel{\mathrm{pyk}}{\rightarrow} \text{“prop lemma mp4”}]$

MP5

$[\mathrm{MP5} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \forall \underline{\mathsf{c}}\colon \forall \underline{\mathsf{d}}\colon \forall \underline{\mathsf{e}}\colon \forall \underline{\mathsf{f}}\colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \Rightarrow \underline{\mathsf{d}} \Rightarrow \underline{\mathsf{e}} \Rightarrow \underline{\mathsf{f}} \vdash \underline{\mathsf{a}} \vdash \underline{\mathsf{b}} \vdash \underline{\mathsf{c}} \vdash \underline{\mathsf{d}} \vdash \underline{\mathsf{e}} \vdash \mathrm{MP3} \rhd \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \Rightarrow \underline{\mathsf{d}} \Rightarrow \underline{\mathsf{e}} \Rightarrow \underline{\mathsf{f}} \rhd \underline{\mathsf{a}} \rhd \underline{\mathsf{b}} \rhd \underline{\mathsf{c}} \gg \underline{\mathsf{d}} \Rightarrow \underline{\mathsf{e}} \Rightarrow \underline{\mathsf{f}}; \mathrm{MP2} \rhd \underline{\mathsf{d}} \Rightarrow \underline{\mathsf{e}} \Rightarrow \underline{\mathsf{f}} \rhd \underline{\mathsf{d}} \rhd \underline{\mathsf{e}} \gg \underline{\mathsf{f}} \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{MP5} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \forall \underline{\mathsf{c}}\colon \forall \underline{\mathsf{d}}\colon \forall \underline{\mathsf{e}}\colon \forall \underline{\mathsf{f}}\colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{c}} \Rightarrow \underline{\mathsf{d}} \Rightarrow \underline{\mathsf{e}} \Rightarrow \underline{\mathsf{f}} \vdash \underline{\mathsf{a}} \vdash \underline{\mathsf{b}} \vdash \underline{\mathsf{c}} \vdash \underline{\mathsf{d}} \vdash \underline{\mathsf{e}} \vdash \underline{\mathsf{f}}]$

$[\mathrm{MP5} \stackrel{\mathrm{tex}}{\rightarrow} \text{“MP5”}]$

$[\mathrm{MP5} \stackrel{\mathrm{pyk}}{\rightarrow} \text{“prop lemma mp5”}]$

MT

$[\mathrm{MT} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \dot{\neg}\, \underline{\mathsf{b}} \vdash \mathrm{Technicality} \gg \dot{\neg}\, \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}; \mathrm{NegativeMT} \rhd \dot{\neg}\, \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \rhd \dot{\neg}\, \underline{\mathsf{b}} \gg \dot{\neg}\, \underline{\mathsf{a}} \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{MT} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \dot{\neg}\, \underline{\mathsf{b}} \vdash \dot{\neg}\, \underline{\mathsf{a}}]$

$[\mathrm{MT} \stackrel{\mathrm{tex}}{\rightarrow} \text{“MT”}]$

$[\mathrm{MT} \stackrel{\mathrm{pyk}}{\rightarrow} \text{“prop lemma mt”}]$

NegativeMT

$[\mathrm{NegativeMT} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \dot{\neg}\, \underline{\mathsf{b}} \vdash \mathrm{Weakening} \rhd \dot{\neg}\, \underline{\mathsf{b}} \gg \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}}; \mathrm{Neg} \rhd \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \rhd \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \dot{\neg}\, \underline{\mathsf{b}} \gg \underline{\mathsf{a}}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{NegativeMT} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \dot{\neg}\, \underline{\mathsf{b}} \vdash \underline{\mathsf{a}}]$

$[\mathrm{NegativeMT} \stackrel{\mathrm{tex}}{\rightarrow} \text{“NegativeMT”}]$

$[\mathrm{NegativeMT} \stackrel{\mathrm{pyk}}{\rightarrow} \text{“prop lemma negative mt”}]$

Technicality

$[\mathrm{Technicality} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \dot{\neg}\, \dot{\neg}\, \underline{\mathsf{a}} \vdash \mathrm{RemoveDoubleNeg} \rhd \dot{\neg}\, \dot{\neg}\, \underline{\mathsf{a}} \gg \underline{\mathsf{a}}; \mathrm{MP} \rhd \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \rhd \underline{\mathsf{a}} \gg \underline{\mathsf{b}}; \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \mathrm{Ded} \rhd \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \dot{\neg}\, \dot{\neg}\, \underline{\mathsf{a}} \vdash \underline{\mathsf{b}} \gg \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}; \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \mathrm{MP} \rhd \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \Rightarrow \dot{\neg}\, \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \rhd \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \gg \dot{\neg}\, \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{Technicality} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \vdash \dot{\neg}\, \dot{\neg}\, \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}]$

$[\mathrm{Technicality} \stackrel{\mathrm{tex}}{\rightarrow} \text{“Technicality”}]$

$[\mathrm{Technicality} \stackrel{\mathrm{pyk}}{\rightarrow} \text{“prop lemma technicality”}]$

Weakening

$[\mathrm{Weakening} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{b}} \vdash \underline{\mathsf{a}} \vdash \mathrm{Repetition} \rhd \underline{\mathsf{b}} \gg \underline{\mathsf{b}}; \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \mathrm{Ded} \rhd \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{b}} \vdash \underline{\mathsf{a}} \vdash \underline{\mathsf{b}} \gg \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}; \underline{\mathsf{b}} \vdash \mathrm{MP} \rhd \underline{\mathsf{b}} \Rightarrow \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}} \rhd \underline{\mathsf{b}} \gg \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{Weakening} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{b}}\colon \underline{\mathsf{b}} \vdash \underline{\mathsf{a}} \Rightarrow \underline{\mathsf{b}}]$

$[\mathrm{Weakening} \stackrel{\mathrm{tex}}{\rightarrow} \text{“Weakening”}]$

$[\mathrm{Weakening} \stackrel{\mathrm{pyk}}{\rightarrow} \text{“prop lemma weakening”}]$

WeakenOr1

[WeakenOr1 $\overset{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{b} \vdash \text{Weakening} \rhd \underline{b} \gg \dot{\neg}\, \underline{a} \Rightarrow \underline{b}; \text{Repetition} \rhd \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \gg \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \rceil, p_0, c)$]

[WeakenOr1 $\overset{\text{stmt}}{\rightarrow}$ $\text{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{b} \vdash \dot{\neg}\, \underline{a} \Rightarrow \underline{b}$]

[WeakenOr1 $\overset{\text{tex}}{\rightarrow}$ "WeakenOr1"]

[WeakenOr1 $\overset{\text{pyk}}{\rightarrow}$ "prop lemma weaken or first"]

WeakenOr2

[WeakenOr2 $\overset{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \vdash \dot{\neg}\, \underline{a} \vdash \text{FromContradiction} \rhd \underline{a} \rhd \dot{\neg}\, \underline{a} \gg \underline{b}; \forall \underline{a}\colon \forall \underline{b}\colon \text{Ded} \rhd \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \vdash \dot{\neg}\, \underline{a} \vdash \underline{b} \gg \underline{a} \Rightarrow \dot{\neg}\, \underline{a} \Rightarrow \underline{b}; \underline{a} \vdash \text{MP} \rhd \underline{a} \Rightarrow \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \rhd \underline{a} \gg \dot{\neg}\, \underline{a} \Rightarrow \underline{b}; \text{Repetition} \rhd \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \gg \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \rceil, p_0, c)$]

[WeakenOr2 $\overset{\text{stmt}}{\rightarrow}$ $\text{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \vdash \dot{\neg}\, \underline{a} \Rightarrow \underline{b}$]

[WeakenOr2 $\overset{\text{tex}}{\rightarrow}$ "WeakenOr2"]

[WeakenOr2 $\overset{\text{pyk}}{\rightarrow}$ "prop lemma weaken or second"]

Formula2Pair

[Formula2Pair $\overset{\text{tex}}{\rightarrow}$ "Formula2Pair"]

[Formula2Pair $\overset{\text{pyk}}{\rightarrow}$ "lemma formula2pair"]

Pair2Formula

[Pair2Formula $\overset{\text{tex}}{\rightarrow}$ "Pair2Formula"]

[Pair2Formula $\overset{\text{pyk}}{\rightarrow}$ "lemma pair2formula"]

Formula2Union

[Formula2Union $\overset{\text{tex}}{\rightarrow}$ "Formula2Union"]

[Formula2Union $\overset{\text{pyk}}{\rightarrow}$ "lemma formula2union"]

Union2Formula

[Union2Formula $\stackrel{\text{tex}}{\rightarrow}$ "Union2Formula"]

[Union2Formula $\stackrel{\text{pyk}}{\rightarrow}$ "lemma union2formula"]

Formula2Sep

[Formula2Sep $\stackrel{\text{tex}}{\rightarrow}$ "Formula2Sep"]

[Formula2Sep $\stackrel{\text{pyk}}{\rightarrow}$ "lemma formula2separation"]

Sep2Formula

[Sep2Formula $\stackrel{\text{tex}}{\rightarrow}$ "Sep2Formula"]

[Sep2Formula $\stackrel{\text{pyk}}{\rightarrow}$ "lemma separation2formula"]

SubsetInPower

[SubsetInPower $\stackrel{\text{tex}}{\rightarrow}$ "SubsetInPower"]

[SubsetInPower $\stackrel{\text{pyk}}{\rightarrow}$ "lemma subset in power set"]

HelperPowerIsSub

[HelperPowerIsSub $\stackrel{\text{tex}}{\rightarrow}$ "HelperPowerIsSub"]

[HelperPowerIsSub $\stackrel{\text{pyk}}{\rightarrow}$ "lemma power set is subset0"]

PowerIsSub

[PowerIsSub $\stackrel{\text{tex}}{\rightarrow}$ "PowerIsSub"]

[PowerIsSub $\stackrel{\text{pyk}}{\rightarrow}$ "lemma power set is subset"]

(Switch)HelperPowerIsSub

[(Switch)HelperPowerIsSub $\stackrel{\text{tex}}{\rightarrow}$ "(Switch)HelperPowerIsSub"]

[(Switch)HelperPowerIsSub $\stackrel{\text{pyk}}{\to}$ "lemma power set is subset0-switch"]

(Switch)PowerIsSub

[(Switch)PowerIsSub $\stackrel{\text{tex}}{\to}$ "(Switch)PowerIsSub"]

[(Switch)PowerIsSub $\stackrel{\text{pyk}}{\to}$ "lemma power set is subset-switch"]

ToSetEquality

[ToSetEquality $\stackrel{\text{tex}}{\to}$ "ToSetEquality"]

[ToSetEquality $\stackrel{\text{pyk}}{\to}$ "lemma set equality suff condition"]

HelperToSetEquality(t)

[HelperToSetEquality(t) $\stackrel{\text{tex}}{\to}$ "HelperToSetEquality(t)"]

[HelperToSetEquality(t) $\stackrel{\text{pyk}}{\to}$ "lemma set equality suff condition(t)0"]

ToSetEquality(t)

[ToSetEquality(t) $\stackrel{\text{tex}}{\to}$ "ToSetEquality(t)"]

[ToSetEquality(t) $\stackrel{\text{pyk}}{\to}$ "lemma set equality suff condition(t)"]

HelperFromSetEquality

[HelperFromSetEquality $\stackrel{\text{tex}}{\to}$ "HelperFromSetEquality"]

[HelperFromSetEquality $\stackrel{\text{pyk}}{\to}$ "lemma set equality skip quantifier"]

FromSetEquality

[FromSetEquality $\stackrel{\text{tex}}{\to}$ "FromSetEquality"]

[FromSetEquality $\stackrel{\text{pyk}}{\to}$ "lemma set equality nec condition"]

HelperReflexivity

[HelperReflexivity $\stackrel{\text{tex}}{\to}$ "HelperReflexivity"]

[HelperReflexivity $\stackrel{\text{pyk}}{\to}$ "lemma reflexivity0"]

Reflexivity

[Reflexivity $\stackrel{\text{tex}}{\to}$ "Reflexivity"]

[Reflexivity $\stackrel{\text{pyk}}{\to}$ "lemma reflexivity"]

HelperSymmetry

[HelperSymmetry $\stackrel{\text{tex}}{\to}$ "HelperSymmetry"]

[HelperSymmetry $\stackrel{\text{pyk}}{\to}$ "lemma symmetry0"]

Symmetry

[Symmetry $\stackrel{\text{tex}}{\to}$ "Symmetry"]

[Symmetry $\stackrel{\text{pyk}}{\to}$ "lemma symmetry"]

HelperTransitivity

[HelperTransitivity $\stackrel{\text{tex}}{\to}$ "HelperTransitivity"]

[HelperTransitivity $\stackrel{\text{pyk}}{\to}$ "lemma transitivity0"]

Transitivity

[Transitivity $\stackrel{\text{tex}}{\to}$ "Transitivity"]

[Transitivity $\stackrel{\text{pyk}}{\to}$ "lemma transitivity"]

ERisReflexive

[ERisReflexive $\stackrel{\text{tex}}{\to}$ "ERisReflexive"]

[ERisReflexive $\stackrel{\text{pyk}}{\rightarrow}$ "lemma er is reflexive"]

ERisSymmetric

[ERisSymmetric $\stackrel{\text{tex}}{\rightarrow}$ "ERisSymmetric"]

[ERisSymmetric $\stackrel{\text{pyk}}{\rightarrow}$ "lemma er is symmetric"]

ERisTransitive

[ERisTransitive $\stackrel{\text{tex}}{\rightarrow}$ "ERisTransitive"]

[ERisTransitive $\stackrel{\text{pyk}}{\rightarrow}$ "lemma er is transitive"]

ØisSubset

[ØisSubset $\stackrel{\text{tex}}{\rightarrow}$ "\O{}isSubset"]

[ØisSubset $\stackrel{\text{pyk}}{\rightarrow}$ "lemma empty set is subset"]

HelperMemberNotØ

[HelperMemberNotØ $\stackrel{\text{tex}}{\rightarrow}$ "HelperMemberNot\O{}"]

[HelperMemberNotØ $\stackrel{\text{pyk}}{\rightarrow}$ "lemma member not empty0"]

MemberNotØ

[MemberNotØ $\stackrel{\text{tex}}{\rightarrow}$ "MemberNot\O{}"]

[MemberNotØ $\stackrel{\text{pyk}}{\rightarrow}$ "lemma member not empty"]

HelperUniqueØ

[HelperUniqueØ $\stackrel{\text{tex}}{\rightarrow}$ "HelperUnique\O{}"]

[HelperUniqueØ $\stackrel{\text{pyk}}{\rightarrow}$ "lemma unique empty set0"]

Unique∅

[Unique∅ $\stackrel{\text{tex}}{\rightarrow}$ "Unique\O{}"]

[Unique∅ $\stackrel{\text{pyk}}{\rightarrow}$ "lemma unique empty set"]

==Reflexivity

[==Reflexivity $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[==Reflexivity $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{(\text{rx})}\colon \underline{(\text{rx})} == \underline{(\text{rx})}$]

[==Reflexivity $\stackrel{\text{tex}}{\rightarrow}$ "==\!{}Reflexivity"]

[==Reflexivity $\stackrel{\text{pyk}}{\rightarrow}$ "lemma ==Reflexivity"]

==Symmetry

[==Symmetry $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[==Symmetry $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{(\text{rx})}\colon \forall \underline{(\text{ry})}\colon \underline{(\text{rx})} == \underline{(\text{ry})} \vdash \underline{(\text{ry})} == \underline{(\text{rx})}$]

[==Symmetry $\stackrel{\text{tex}}{\rightarrow}$ "==\!{}Symmetry"]

[==Symmetry $\stackrel{\text{pyk}}{\rightarrow}$ "lemma ==Symmetry"]

Helper==Transitivity

[Helper==Transitivity $\stackrel{\text{tex}}{\rightarrow}$ "Helper\!{}==\!{}Transitivity"]

[Helper==Transitivity $\stackrel{\text{pyk}}{\rightarrow}$ "lemma ==Transitivity0"]

==Transitivity

[==Transitivity $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[==Transitivity $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{(\text{rx})}\colon \forall \underline{(\text{ry})}\colon \forall \underline{(\text{rz})}\colon \underline{(\text{rx})} == \underline{(\text{ry})} \vdash \underline{(\text{ry})} == \underline{(\text{rz})} \vdash \underline{(\text{rx})} == \underline{(\text{rz})}$]

[==Transitivity $\stackrel{\text{tex}}{\rightarrow}$ "\!{}==\!{}Transitivity"]

[==Transitivity $\stackrel{\text{pyk}}{\rightarrow}$ "lemma ==Transitivity"]

HelperTransferNotEq

[HelperTransferNotEq $\overset{\text{tex}}{\to}$ “HelperTransferNotEq”]

[HelperTransferNotEq $\overset{\text{pyk}}{\to}$ “lemma transfer ˜is0”]

TransferNotEq

[TransferNotEq $\overset{\text{tex}}{\to}$ “TransferNotEq”]

[TransferNotEq $\overset{\text{pyk}}{\to}$ “lemma transfer ˜is”]

HelperPairSubset

[HelperPairSubset $\overset{\text{tex}}{\to}$ “HelperPairSubset”]

[HelperPairSubset $\overset{\text{pyk}}{\to}$ “lemma pair subset0”]

Helper(2)PairSubset

[Helper(2)PairSubset $\overset{\text{tex}}{\to}$ “Helper(2)PairSubset”]

[Helper(2)PairSubset $\overset{\text{pyk}}{\to}$ “lemma pair subset1”]

PairSubset

[PairSubset $\overset{\text{tex}}{\to}$ “PairSubset”]

[PairSubset $\overset{\text{pyk}}{\to}$ “lemma pair subset”]

SamePair

[SamePair $\overset{\text{tex}}{\to}$ “SamePair”]

[SamePair $\overset{\text{pyk}}{\to}$ “lemma same pair”]

SameSingleton

[SameSingleton $\overset{\text{tex}}{\to}$ “SameSingleton”]

[SameSingleton $\overset{\text{pyk}}{\rightarrow}$ "lemma same singleton"]

UnionSubset

[UnionSubset $\overset{\text{tex}}{\rightarrow}$ "UnionSubset"]

[UnionSubset $\overset{\text{pyk}}{\rightarrow}$ "lemma union subset"]

SameUnion

[SameUnion $\overset{\text{tex}}{\rightarrow}$ "SameUnion"]

[SameUnion $\overset{\text{pyk}}{\rightarrow}$ "lemma same union"]

SeparationSubset

[SeparationSubset $\overset{\text{tex}}{\rightarrow}$ "SeparationSubset"]

[SeparationSubset $\overset{\text{pyk}}{\rightarrow}$ "lemma separation subset"]

SameSeparation

[SameSeparation $\overset{\text{tex}}{\rightarrow}$ "SameSeparation"]

[SameSeparation $\overset{\text{pyk}}{\rightarrow}$ "lemma same separation"]

SameBinaryUnion

[SameBinaryUnion $\overset{\text{tex}}{\rightarrow}$ "SameBinaryUnion"]

[SameBinaryUnion $\overset{\text{pyk}}{\rightarrow}$ "lemma same binary union"]

IntersectionSubset

[IntersectionSubset $\overset{\text{tex}}{\rightarrow}$ "IntersectionSubset"]

[IntersectionSubset $\overset{\text{pyk}}{\rightarrow}$ "lemma intersection subset"]

SameIntersection

[SameIntersection $\stackrel{\text{tex}}{\to}$ “SameIntersection”]

[SameIntersection $\stackrel{\text{pyk}}{\to}$ “lemma same intersection”]

AutoMember

[AutoMember $\stackrel{\text{tex}}{\to}$ “AutoMember”]

[AutoMember $\stackrel{\text{pyk}}{\to}$ “lemma auto member”]

HelperEqSysNot∅

[HelperEqSysNot∅ $\stackrel{\text{tex}}{\to}$ “HelperEqSysNot\O{}”]

[HelperEqSysNot∅ $\stackrel{\text{pyk}}{\to}$ “lemma eq-system not empty0”]

EqSysNot∅

[EqSysNot∅ $\stackrel{\text{tex}}{\to}$ “EqSysNot\O{}”]

[EqSysNot∅ $\stackrel{\text{pyk}}{\to}$ “lemma eq-system not empty”]

HelperEqSubset

[HelperEqSubset $\stackrel{\text{tex}}{\to}$ “HelperEqSubset”]

[HelperEqSubset $\stackrel{\text{pyk}}{\to}$ “lemma eq subset0”]

EqSubset

[EqSubset $\stackrel{\text{tex}}{\to}$ “EqSubset”]

[EqSubset $\stackrel{\text{pyk}}{\to}$ “lemma eq subset”]

HelperEqNecessary

[HelperEqNecessary $\stackrel{\text{tex}}{\to}$ “HelperEqNecessary”]

[HelperEqNecessary $\stackrel{\text{pyk}}{\rightarrow}$ "lemma equivalence nec condition0"]

EqNecessary

[EqNecessary $\stackrel{\text{tex}}{\rightarrow}$ "EqNecessary"]

[EqNecessary $\stackrel{\text{pyk}}{\rightarrow}$ "lemma equivalence nec condition"]

HelperNoneEqNecessary

[HelperNoneEqNecessary $\stackrel{\text{tex}}{\rightarrow}$ "HelperNoneEqNecessary"]

[HelperNoneEqNecessary $\stackrel{\text{pyk}}{\rightarrow}$ "lemma none-equivalence nec condition0"]

Helper(2)NoneEqNecessary

[Helper(2)NoneEqNecessary $\stackrel{\text{tex}}{\rightarrow}$ "Helper(2)NoneEqNecessary"]

[Helper(2)NoneEqNecessary $\stackrel{\text{pyk}}{\rightarrow}$ "lemma none-equivalence nec condition1"]

NoneEqNecessary

[NoneEqNecessary $\stackrel{\text{tex}}{\rightarrow}$ "NoneEqNecessary"]

[NoneEqNecessary $\stackrel{\text{pyk}}{\rightarrow}$ "lemma none-equivalence nec condition"]

EqClassIsSubset

[EqClassIsSubset $\stackrel{\text{tex}}{\rightarrow}$ "EqClassIsSubset"]

[EqClassIsSubset $\stackrel{\text{pyk}}{\rightarrow}$ "lemma equivalence class is subset"]

EqClassesAreDisjoint

[EqClassesAreDisjoint $\stackrel{\text{tex}}{\rightarrow}$ "EqClassesAreDisjoint"]

[EqClassesAreDisjoint $\stackrel{\text{pyk}}{\rightarrow}$ "lemma equivalence classes are disjoint"]

AllDisjoint

[AllDisjoint $\stackrel{\text{tex}}{\rightarrow}$ "AllDisjoint"]

[AllDisjoint $\stackrel{\text{pyk}}{\rightarrow}$ "lemma all disjoint"]

AllDisjointImply

[AllDisjointImply $\stackrel{\text{tex}}{\rightarrow}$ "AllDisjointImply"]

[AllDisjointImply $\stackrel{\text{pyk}}{\rightarrow}$ "lemma all disjoint-imply"]

BSsubset

[BSsubset $\stackrel{\text{tex}}{\rightarrow}$ "BSsubset"]

[BSsubset $\stackrel{\text{pyk}}{\rightarrow}$ "lemma bs subset union(bs/r)"]

Union(BS/R)subset

[Union(BS/R)subset $\stackrel{\text{tex}}{\rightarrow}$ "Union(BS/R)subset"]

[Union(BS/R)subset $\stackrel{\text{pyk}}{\rightarrow}$ "lemma union(bs/r) subset bs"]

UnionIdentity

[UnionIdentity $\stackrel{\text{tex}}{\rightarrow}$ "UnionIdentity"]

[UnionIdentity $\stackrel{\text{pyk}}{\rightarrow}$ "lemma union(bs/r) is bs"]

EqSysIsPartition

[EqSysIsPartition $\stackrel{\text{tex}}{\rightarrow}$ "EqSysIsPartition"]

[EqSysIsPartition $\stackrel{\text{pyk}}{\rightarrow}$ "theorem eq-system is partition"]

(ϵ)

[(ϵ) $\stackrel{\text{tex}}{\rightarrow}$ "(\epsilon)"]

$[(\epsilon) \stackrel{\text{pyk}}{\rightarrow}$ "var ep"$]$

(fx)

$[(\text{fx}) \stackrel{\text{tex}}{\rightarrow}$ "(fx)"$]$

$[(\text{fx}) \stackrel{\text{pyk}}{\rightarrow}$ "var fx"$]$

(fy)

$[(\text{fy}) \stackrel{\text{tex}}{\rightarrow}$ "(fy)"$]$

$[(\text{fy}) \stackrel{\text{pyk}}{\rightarrow}$ "var fy"$]$

(fz)

$[(\text{fz}) \stackrel{\text{tex}}{\rightarrow}$ "(fz)"$]$

$[(\text{fz}) \stackrel{\text{pyk}}{\rightarrow}$ "var fz"$]$

(fv)

$[(\text{fv}) \stackrel{\text{tex}}{\rightarrow}$ "(fv)"$]$

$[(\text{fv}) \stackrel{\text{pyk}}{\rightarrow}$ "var fu"$]$

var fv

$[\text{var fv} \stackrel{\text{pyk}}{\rightarrow}$ "var fv"$]$

(rx)

$[(\text{rx}) \stackrel{\text{tex}}{\rightarrow}$ "(rx)"$]$

$[(\text{rx}) \stackrel{\text{pyk}}{\rightarrow}$ "var rx"$]$

(ry)

$[(\mathrm{ry}) \stackrel{\mathrm{tex}}{\to} \text{“(ry)”}]$

$[(\mathrm{ry}) \stackrel{\mathrm{pyk}}{\to} \text{“var ry”}]$

(rz)

$[(\mathrm{rz}) \stackrel{\mathrm{tex}}{\to} \text{“(rz)”}]$

$[(\mathrm{rz}) \stackrel{\mathrm{pyk}}{\to} \text{“var rz”}]$

(ru)

$[(\mathrm{ru}) \stackrel{\mathrm{tex}}{\to} \text{“(ru)”}]$

$[(\mathrm{ru}) \stackrel{\mathrm{pyk}}{\to} \text{“var ru”}]$

ϵ

$[\epsilon \stackrel{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\epsilon \ddot{=} \underline{(\epsilon)}]\rceil)]$

$[\epsilon \stackrel{\mathrm{tex}}{\to} \text{“\textbackslash epsilon”}]$

$[\epsilon \stackrel{\mathrm{pyk}}{\to} \text{“meta ep”}]$

FX

$[\mathrm{FX} \stackrel{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathrm{FX} \ddot{=} \underline{(\mathrm{fx})}]\rceil)]$

$[\mathrm{FX} \stackrel{\mathrm{tex}}{\to} \text{“FX”}]$

$[\mathrm{FX} \stackrel{\mathrm{pyk}}{\to} \text{“meta fx”}]$

FY

$[\mathrm{FY} \stackrel{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathrm{FY} \ddot{=} \underline{(\mathrm{fy})}]\rceil)]$

$[\mathrm{FY} \stackrel{\mathrm{tex}}{\to} \text{“FY”}]$

$[\mathrm{FY} \stackrel{\mathrm{pyk}}{\to} \text{“meta fy”}]$

FZ

$[\mathrm{FZ} \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil [\mathrm{FZ} \ddot{=} \underline{(\mathrm{fz})}] \rceil)]$

$[\mathrm{FZ} \overset{\text{tex}}{\to} \text{“FZ”}]$

$[\mathrm{FZ} \overset{\text{pyk}}{\to} \text{“meta fz”}]$

FU

$[\mathrm{FU} \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil [\mathrm{FU} \ddot{=} \underline{(\mathrm{fv})}] \rceil)]$

$[\mathrm{FU} \overset{\text{tex}}{\to} \text{“FU”}]$

$[\mathrm{FU} \overset{\text{pyk}}{\to} \text{“meta fu”}]$

FV

$[\mathrm{FV} \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil [\mathrm{FV} \ddot{=} \underline{\mathrm{var\ fv}}] \rceil)]$

$[\mathrm{FV} \overset{\text{tex}}{\to} \text{“FV”}]$

$[\mathrm{FV} \overset{\text{pyk}}{\to} \text{“meta fv”}]$

RX

$[\mathrm{RX} \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil [\mathrm{RX} \ddot{=} \underline{(\mathrm{rx})}] \rceil)]$

$[\mathrm{RX} \overset{\text{tex}}{\to} \text{“RX”}]$

$[\mathrm{RX} \overset{\text{pyk}}{\to} \text{“meta rx”}]$

RY

$[\mathrm{RY} \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil [\mathrm{RY} \ddot{=} \underline{(\mathrm{ry})}] \rceil)]$

$[\mathrm{RY} \overset{\text{tex}}{\to} \text{“RY”}]$

$[\mathrm{RY} \overset{\text{pyk}}{\to} \text{“meta ry”}]$

RZ

$[\mathrm{RZ} \stackrel{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathrm{RZ} \ddot{=} \underline{(\mathrm{rz})}]\rceil)]$

$[\mathrm{RZ} \stackrel{\mathrm{tex}}{\to}$ "RZ"]

$[\mathrm{RZ} \stackrel{\mathrm{pyk}}{\to}$ "meta rz"]

RU

$[\mathrm{RU} \stackrel{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathrm{RU} \ddot{=} \underline{(\mathrm{ru})}]\rceil)]$

$[\mathrm{RU} \stackrel{\mathrm{tex}}{\to}$ "RU"]

$[\mathrm{RU} \stackrel{\mathrm{pyk}}{\to}$ "meta ru"]

0

$[0 \stackrel{\mathrm{tex}}{\to}$ "0"]

$[0 \stackrel{\mathrm{pyk}}{\to}$ "0"]

1

$[1 \stackrel{\mathrm{tex}}{\to}$ "1"]

$[1 \stackrel{\mathrm{pyk}}{\to}$ "1"]

(-1)

$[(-1) \stackrel{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[(-1) \ddot{=} -1]\rceil)]$

$[(-1) \stackrel{\mathrm{tex}}{\to}$ "(-1)"]

$[(-1) \stackrel{\mathrm{pyk}}{\to}$ "(-1)"]

2

$[2 \stackrel{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[2 \ddot{=} (1+1)]\rceil)]$

$[2 \stackrel{\mathrm{tex}}{\to}$ "2"]

$[2 \stackrel{\text{pyk}}{\rightarrow} \text{“2”}]$

1/2

$[1/2 \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[1/2 \ddot{=} \text{rec2}]\rceil)]$

$[1/2 \stackrel{\text{tex}}{\rightarrow} \text{“1/2”}]$

$[1/2 \stackrel{\text{pyk}}{\rightarrow} \text{“1/2”}]$

0f

$[0\text{f} \stackrel{\text{tex}}{\rightarrow} \text{“0f”}]$

$[0\text{f} \stackrel{\text{pyk}}{\rightarrow} \text{“0f”}]$

1f

$[1\text{f} \stackrel{\text{pyk}}{\rightarrow} \text{“1f”}]$

00

$[00 \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[00 \ddot{=} \text{R}(0\text{f})]\rceil)]$

$[00 \stackrel{\text{tex}}{\rightarrow} \text{“00”}]$

$[00 \stackrel{\text{pyk}}{\rightarrow} \text{“00”}]$

01

$[01 \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[01 \ddot{=} \text{R}(1\text{f})]\rceil)]$

$[01 \stackrel{\text{pyk}}{\rightarrow} \text{“01”}]$

leqReflexivity

$[\text{leqReflexivity} \stackrel{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[\text{leqReflexivity} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \underline{\mathsf{x}} <= \underline{\mathsf{x}}]$

[leqReflexivity $\stackrel{\text{tex}}{\rightarrow}$ "leqReflexivity"]

[leqReflexivity $\stackrel{\text{pyk}}{\rightarrow}$ "axiom leqReflexivity"]

leqAntisymmetryAxiom

[leqAntisymmetryAxiom $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[leqAntisymmetryAxiom $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} <= \underline{y} \Rightarrow \underline{y} <= \underline{x} \Rightarrow \underline{x} = \underline{y}$]

[leqAntisymmetryAxiom $\stackrel{\text{tex}}{\rightarrow}$ "leqAntisymmetryAxiom"]

[leqAntisymmetryAxiom $\stackrel{\text{pyk}}{\rightarrow}$ "axiom leqAntisymmetry"]

leqTransitivityAxiom

[leqTransitivityAxiom $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[leqTransitivityAxiom $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} <= \underline{y} \Rightarrow \underline{y} <= \underline{z} \Rightarrow \underline{x} <= \underline{z}$]

[leqTransitivityAxiom $\stackrel{\text{tex}}{\rightarrow}$ "leqTransitivityAxiom"]

[leqTransitivityAxiom $\stackrel{\text{pyk}}{\rightarrow}$ "axiom leqTransitivity"]

leqTotality

[leqTotality $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[leqTotality $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \underline{y} <= \underline{x}$]

[leqTotality $\stackrel{\text{tex}}{\rightarrow}$ "leqTotality"]

[leqTotality $\stackrel{\text{pyk}}{\rightarrow}$ "axiom leqTotality"]

leqAdditionAxiom

[leqAdditionAxiom $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[leqAdditionAxiom $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} <= \underline{y} \Rightarrow \underline{x} + \underline{z} <= \underline{y} + \underline{z}$]

[leqAdditionAxiom $\stackrel{\text{tex}}{\rightarrow}$ "leqAdditionAxiom"]

[leqAdditionAxiom $\stackrel{\text{pyk}}{\rightarrow}$ "axiom leqAddition"]

leqMultiplicationAxiom

[leqMultiplicationAxiom $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[leqMultiplicationAxiom $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x} \colon \forall \underline{y} \colon \forall \underline{z} \colon 0 <= \underline{z} \Rightarrow \underline{x} <= \underline{y} \Rightarrow \underline{x} * \underline{z} <= \underline{y} * \underline{z}$]

[leqMultiplicationAxiom $\stackrel{\text{tex}}{\rightarrow}$ "leqMultiplicationAxiom"]

[leqMultiplicationAxiom $\stackrel{\text{pyk}}{\rightarrow}$ "axiom leqMultiplication"]

plusAssociativity

[plusAssociativity $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[plusAssociativity $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x} \colon \forall \underline{y} \colon \forall \underline{z} \colon \underline{x} + \underline{y} + \underline{z} = \underline{x} + \underline{y} + \underline{z}$]

[plusAssociativity $\stackrel{\text{tex}}{\rightarrow}$ "plusAssociativity"]

[plusAssociativity $\stackrel{\text{pyk}}{\rightarrow}$ "axiom plusAssociativity"]

plusCommutativity

[plusCommutativity $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[plusCommutativity $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x} \colon \forall \underline{y} \colon \underline{x} + \underline{y} = \underline{y} + \underline{x}$]

[plusCommutativity $\stackrel{\text{tex}}{\rightarrow}$ "plusCommutativity"]

[plusCommutativity $\stackrel{\text{pyk}}{\rightarrow}$ "axiom plusCommutativity"]

Negative

[Negative $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[Negative $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x} \colon \underline{x} + -\underline{x} = 0$]

[Negative $\stackrel{\text{tex}}{\rightarrow}$ "Negative"]

[Negative $\stackrel{\text{pyk}}{\rightarrow}$ "axiom negative"]

plus0

[plus0 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[plus0 $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \underline{x} + 0 = \underline{x}$]

[plus0 $\stackrel{\text{tex}}{\rightarrow}$ "plus0"]

[plus0 $\stackrel{\text{pyk}}{\rightarrow}$ "axiom plus0"]

timesAssociativity

[timesAssociativity $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[timesAssociativity $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} * \underline{y} * \underline{z} = \underline{x} * \underline{y} * \underline{z}$]

[timesAssociativity $\stackrel{\text{tex}}{\rightarrow}$ "timesAssociativity"]

[timesAssociativity $\stackrel{\text{pyk}}{\rightarrow}$ "axiom timesAssociativity"]

timesCommutativity

[timesCommutativity $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[timesCommutativity $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} * \underline{y} = \underline{y} * \underline{x}$]

[timesCommutativity $\stackrel{\text{tex}}{\rightarrow}$ "timesCommutativity"]

[timesCommutativity $\stackrel{\text{pyk}}{\rightarrow}$ "axiom timesCommutativity"]

ReciprocalAxiom

[ReciprocalAxiom $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[ReciprocalAxiom $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \dot{\neg}\, \underline{x} = 0 \Rightarrow \underline{x} * \mathrm{rec}\underline{x} = 1$]

[ReciprocalAxiom $\stackrel{\text{tex}}{\rightarrow}$ "ReciprocalAxiom"]

[ReciprocalAxiom $\stackrel{\text{pyk}}{\rightarrow}$ "axiom reciprocal"]

times1

[times1 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic] [times1 $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{\mathsf{x}}\colon \underline{\mathsf{x}} * 1 = \underline{\mathsf{x}}$]

[times1 $\stackrel{\text{tex}}{\rightarrow}$ "times1"]

[times1 $\stackrel{\text{pyk}}{\rightarrow}$ "axiom times1"]

Distribution

[Distribution $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[Distribution $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \forall \underline{\mathsf{z}}\colon \underline{\mathsf{x}} * \underline{\mathsf{y}} + \underline{\mathsf{z}} = \underline{\mathsf{x}} * \underline{\mathsf{y}} + \underline{\mathsf{x}} * \underline{\mathsf{z}}$]

[Distribution $\stackrel{\text{tex}}{\rightarrow}$ "Distribution"]

[Distribution $\stackrel{\text{pyk}}{\rightarrow}$ "axiom distribution"]

0not1

[0not1 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[0not1 $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \dot{\neg}\, 0 = 1$]

[0not1 $\stackrel{\text{tex}}{\rightarrow}$ "0not1"]

[0not1 $\stackrel{\text{pyk}}{\rightarrow}$ "axiom 0not1"]

equalityAxiom

[equalityAxiom $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[equalityAxiom $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \forall \underline{\mathsf{z}}\colon \underline{\mathsf{x}} = \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{x}} = \underline{\mathsf{z}} \Rightarrow \underline{\mathsf{y}} = \underline{\mathsf{z}}$]

[equalityAxiom $\stackrel{\text{tex}}{\rightarrow}$ "equalityAxiom"]

[equalityAxiom $\stackrel{\text{pyk}}{\rightarrow}$ "axiom equality"]

eqLeqAxiom

[eqLeqAxiom $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[eqLeqAxiom $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \underline{\mathsf{x}} = \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{x}} <= \underline{\mathsf{y}}$]

[eqLeqAxiom $\stackrel{\text{tex}}{\rightarrow}$ "eqLeqAxiom"]

[eqLeqAxiom $\stackrel{\text{pyk}}{\rightarrow}$ "axiom eqLeq"]

eqAdditionAxiom

[eqAdditionAxiom $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[eqAdditionAxiom $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} \Rightarrow \underline{x} + \underline{z} = \underline{y} + \underline{z}$]

[eqAdditionAxiom $\stackrel{\text{tex}}{\rightarrow}$ "eqAdditionAxiom"]

[eqAdditionAxiom $\stackrel{\text{pyk}}{\rightarrow}$ "axiom eqAddition"]

eqMultiplicationAxiom

[eqMultiplicationAxiom $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[eqMultiplicationAxiom $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} \Rightarrow \underline{x} * \underline{z} = \underline{y} * \underline{z}$]

[eqMultiplicationAxiom $\stackrel{\text{tex}}{\rightarrow}$ "eqMultiplicationAxiom"]

[eqMultiplicationAxiom $\stackrel{\text{pyk}}{\rightarrow}$ "axiom eqMultiplication"]

SENC1

[SENC1 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[SENC1 $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{(fx)}\colon \forall \underline{(rx)}\colon \forall \underline{(ry)}\colon \underline{(rx)} == \underline{(ry)} \vdash \underline{(fx)} \in \underline{(rx)} \vdash \underline{(fx)} \in \underline{(ry)}$]

[SENC1 $\stackrel{\text{tex}}{\rightarrow}$ "SENC1"]

[SENC1 $\stackrel{\text{pyk}}{\rightarrow}$ "lemma set equality nec condition(1)"]

SENC2

[SENC2 $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[SENC2 $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{(fx)}\colon \forall \underline{(rx)}\colon \forall \underline{(ry)}\colon \underline{(rx)} == \underline{(ry)} \vdash \underline{(fx)} \in \underline{(ry)} \vdash \underline{(fx)} \in \underline{(rx)}$]

[SENC2 $\stackrel{\text{tex}}{\rightarrow}$ "SENC2"]

[SENC2 $\stackrel{\text{pyk}}{\rightarrow}$ "lemma set equality nec condition(2)"]

IfThenElse(T)

[IfThenElse(T) $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[IfThenElse(T) $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{a}\colon \forall \underline{x}\colon \forall \underline{y}\colon \underline{a} \vdash \text{if}(\underline{a}, \underline{x}, \underline{y}) = \underline{x}$]

[IfThenElse(T) $\stackrel{\text{tex}}{\rightarrow}$ "IfThenElse(T)"]

[IfThenElse(T) $\stackrel{\text{pyk}}{\rightarrow}$ "1rule ifThenElse true"]

IfThenElse(F)

[IfThenElse(F) $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[IfThenElse(F) $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{a}\colon \forall \underline{x}\colon \forall \underline{y}\colon \dot{\neg}\, \underline{a} \vdash \text{if}(\underline{a}, \underline{x}, \underline{y}) = \underline{y}$]

[IfThenElse(F) $\stackrel{\text{tex}}{\rightarrow}$ "IfThenElse(F)"]

[IfThenElse(F) $\stackrel{\text{pyk}}{\rightarrow}$ "1rule ifThenElse false"]

From = f

[From = f $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[From = f $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{m}\colon \forall \underline{(fx)}\colon \forall \underline{(fy)}\colon \underline{(fx)} =_f \underline{(fy)} \vdash \underline{(fx)}[\underline{m}] = \underline{(fy)}[\underline{m}]$]

[From = f $\stackrel{\text{tex}}{\rightarrow}$ "From=f"]

[From = f $\stackrel{\text{pyk}}{\rightarrow}$ "1rule from=f"]

To = f

[To = f $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[To = f $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{m}\colon \forall \underline{(fx)}\colon \forall \underline{(fy)}\colon \underline{(fx)}[\underline{m}] = \underline{(fy)}[\underline{m}] \vdash \underline{(fx)} =_f \underline{(fy)}$]

[To = f $\stackrel{\text{tex}}{\rightarrow}$ "To=f"]

[To = f $\stackrel{\text{pyk}}{\rightarrow}$ "1rule to=f"]

From < f

[From < f $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[From < f $\overset{\text{stmt}}{\to}$ $\text{ZFsub} \vdash \forall \underline{\mathsf{m}} \colon \forall \underline{(\epsilon)} \colon \forall \underline{(\text{fx})} \colon \forall \underline{(\text{fy})} \colon \underline{(\text{fx})} <_{\text{f}} \underline{(\text{fy})} \vdash \dot{\neg} 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{(\epsilon)} \vdash \mathsf{c}_{\text{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(\text{fx})}[\underline{\mathsf{m}}] <= \underline{(\text{fy})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)}$]

[From < f $\overset{\text{tex}}{\to}$ "From<f"]

[From < f $\overset{\text{pyk}}{\to}$ "1rule from<f"]

To < f

[To < f $\overset{\text{proof}}{\to}$ Rule tactic]

[To < f $\overset{\text{stmt}}{\to}$ $\text{ZFsub} \vdash \forall \underline{\mathsf{m}} \colon \forall \underline{(\epsilon)} \colon \forall \underline{(\text{fx})} \colon \forall \underline{(\text{fy})} \colon \dot{\neg} 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{(\epsilon)} \Rightarrow \mathsf{c}_{\text{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(\text{fx})}[\underline{\mathsf{m}}] <= \underline{(\text{fy})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \vdash \underline{(\text{fx})} <_{\text{f}} \underline{(\text{fy})}$]

[To < f $\overset{\text{tex}}{\to}$ "To<f"]

[To < f $\overset{\text{pyk}}{\to}$ "1rule to<f"]

PlusF

[PlusF $\overset{\text{proof}}{\to}$ Rule tactic]

[PlusF $\overset{\text{stmt}}{\to}$ $\text{ZFsub} \vdash \forall \underline{\mathsf{m}} \colon \forall \underline{(\text{fx})} \colon \forall \underline{(\text{fy})} \colon \underline{(\text{fx})} +_{\text{f}} \underline{(\text{fy})}[\underline{\mathsf{m}}] = \underline{(\text{fx})}[\underline{\mathsf{m}}] + \underline{(\text{fy})}[\underline{\mathsf{m}}]$]

[PlusF $\overset{\text{tex}}{\to}$ "PlusF"]

[PlusF $\overset{\text{pyk}}{\to}$ "axiom plusF"]

TimesF

[TimesF $\overset{\text{proof}}{\to}$ Rule tactic]

[TimesF $\overset{\text{stmt}}{\to}$ $\text{ZFsub} \vdash \forall \underline{\mathsf{m}} \colon \forall \underline{(\text{fx})} \colon \forall \underline{(\text{fy})} \colon \underline{(\text{fx})} *_{\text{f}} \underline{(\text{fy})}[\underline{\mathsf{m}}] = \underline{(\text{fx})}[\underline{\mathsf{m}}] * \underline{(\text{fy})}[\underline{\mathsf{m}}]$]

[TimesF $\overset{\text{tex}}{\to}$ "TimesF"]

[TimesF $\overset{\text{pyk}}{\to}$ "axiom timesF"]

MinusF

[MinusF $\overset{\text{proof}}{\to}$ Rule tactic]

[MinusF $\overset{\text{stmt}}{\to}$ $\text{ZFsub} \vdash \forall \underline{\mathsf{m}} \colon \forall \underline{(\text{fx})} \colon -_{\text{f}} \underline{(\text{fx})}[\underline{\mathsf{m}}] = -\underline{(\text{fx})}[\underline{\mathsf{m}}]$]

[MinusF $\overset{\text{tex}}{\to}$ "MinusF"]

[MinusF $\stackrel{\text{pyk}}{\rightarrow}$ “axiom minusF”]

0f

[0f $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[0f $\stackrel{\text{stmt}}{\rightarrow}$ $\text{ZFsub} \vdash \forall \underline{\mathsf{m}} \colon 0\text{f}[\underline{\mathsf{m}}] = 0$]

[0f $\stackrel{\text{tex}}{\rightarrow}$ “0f”]

[0f $\stackrel{\text{pyk}}{\rightarrow}$ “axiom 0f”]

1f

[1f $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[1f $\stackrel{\text{stmt}}{\rightarrow}$ $\text{ZFsub} \vdash \forall \underline{\mathsf{m}} \colon 1\text{f}[\underline{\mathsf{m}}] = 1$]

[1f $\stackrel{\text{tex}}{\rightarrow}$ “1f”]

[1f $\stackrel{\text{pyk}}{\rightarrow}$ “axiom 1f”]

FromSF

[FromSF $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[FromSF $\stackrel{\text{stmt}}{\rightarrow}$ $\text{ZFsub} \vdash \forall \underline{\mathsf{m}} \colon \forall \underline{(\epsilon)} \colon \forall \underline{(\text{fx})} \colon \forall \underline{(\text{fy})} \colon \text{SF}(\underline{(\text{fx})}, \underline{(\text{fy})}) \vdash \dot{\neg}\, 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\,\dot{\neg}\, 0 = \underline{(\epsilon)} \vdash \mathsf{c}_{\text{Ex}} <= \underline{\mathsf{m}} \Rightarrow \dot{\neg}\, \text{if}(0 <= \underline{(\text{fx})}[\underline{\mathsf{m}}] + -\underline{(\text{fy})}[\underline{\mathsf{m}}], \underline{(\text{fx})}[\underline{\mathsf{m}}] + -\underline{(\text{fy})}[\underline{\mathsf{m}}], -\underline{(\text{fx})}[\underline{\mathsf{m}}] + -\underline{(\text{fy})}[\underline{\mathsf{m}}]) <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\,\dot{\neg}\, \text{if}(0 <= \underline{(\text{fx})}[\underline{\mathsf{m}}] + -\underline{(\text{fy})}[\underline{\mathsf{m}}], \underline{(\text{fx})}[\underline{\mathsf{m}}] + -\underline{(\text{fy})}[\underline{\mathsf{m}}], -\underline{(\text{fx})}[\underline{\mathsf{m}}] + -\underline{(\text{fy})}[\underline{\mathsf{m}}]) = \underline{(\epsilon)}$]

[FromSF $\stackrel{\text{tex}}{\rightarrow}$ “FromSF”]

[FromSF $\stackrel{\text{pyk}}{\rightarrow}$ “1rule fromSameF”]

ToSF

[ToSF $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[ToSF $\stackrel{\text{stmt}}{\rightarrow}$ $\text{ZFsub} \vdash \forall \underline{\mathsf{m}} \colon \forall \underline{(\epsilon)} \colon \forall \underline{(\text{fx})} \colon \forall \underline{(\text{fy})} \colon \dot{\neg}\, 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\,\dot{\neg}\, 0 = \underline{(\epsilon)} \Rightarrow \mathsf{c}_{\text{Ex}} <= \underline{\mathsf{m}} \Rightarrow \dot{\neg}\, \text{if}(0 <= \underline{(\text{fx})}[\underline{\mathsf{m}}] + -\underline{(\text{fy})}[\underline{\mathsf{m}}], \underline{(\text{fx})}[\underline{\mathsf{m}}] + -\underline{(\text{fy})}[\underline{\mathsf{m}}], -\underline{(\text{fx})}[\underline{\mathsf{m}}] + -\underline{(\text{fy})}[\underline{\mathsf{m}}]) <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\,\dot{\neg}\, \text{if}(0 <= \underline{(\text{fx})}[\underline{\mathsf{m}}] + -\underline{(\text{fy})}[\underline{\mathsf{m}}], \underline{(\text{fx})}[\underline{\mathsf{m}}] + -\underline{(\text{fy})}[\underline{\mathsf{m}}], -\underline{(\text{fx})}[\underline{\mathsf{m}}] + -\underline{(\text{fy})}[\underline{\mathsf{m}}]) = \underline{(\epsilon)} \vdash \text{SF}(\underline{(\text{fx})}, \underline{(\text{fy})})$]

$[\text{ToSF} \overset{\text{tex}}{\rightarrow} \text{“ToSF”}]$

$[\text{ToSF} \overset{\text{pyk}}{\rightarrow} \text{“1rule toSameF”}]$

To == XX

$[\text{To} == \text{XX} \overset{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[\text{To} == \text{XX} \overset{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{(\text{fx})}: \forall \underline{(\text{fy})}: \forall \underline{(\text{rx})}: \forall \underline{(\text{ry})}: \underline{(\text{fx})} \in \underline{(\text{rx})} \Rightarrow \underline{(\text{fy})} \in \underline{(\text{ry})} \Rightarrow \text{SF}(\underline{(\text{fx})}, \underline{(\text{fy})}) \vdash \underline{(\text{rx})} == \underline{(\text{ry})}]$

$[\text{To} == \text{XX} \overset{\text{tex}}{\rightarrow} \text{“To==XX”}]$

$[\text{To} == \text{XX} \overset{\text{pyk}}{\rightarrow} \text{“1rule to==XX”}]$

From ==

$[\text{From} == \overset{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[\text{From} == \overset{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{(\text{fx})}: \forall \underline{(\text{fy})}: \text{R}(\underline{(\text{fx})}) == \text{R}(\underline{(\text{fy})}) \vdash \text{SF}(\underline{(\text{fx})}, \underline{(\text{fy})})]$

$[\text{From} == \overset{\text{tex}}{\rightarrow} \text{“From==”}]$

$[\text{From} == \overset{\text{pyk}}{\rightarrow} \text{“1rule from==”}]$

To ==

$[\text{To} == \overset{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[\text{To} == \overset{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{(\text{fx})}: \forall \underline{(\text{fy})}: \text{SF}(\underline{(\text{fx})}, \underline{(\text{fy})}) \vdash \text{R}(\underline{(\text{fx})}) == \text{R}(\underline{(\text{fy})})]$

$[\text{To} == \overset{\text{tex}}{\rightarrow} \text{“To==”}]$

$[\text{To} == \overset{\text{pyk}}{\rightarrow} \text{“1rule to==”}]$

From << XX

$[\text{From} << \text{XX} \overset{\text{proof}}{\rightarrow} \text{Rule tactic}]$

$[\text{From} << \text{XX} \overset{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{m}}: \forall \underline{(\epsilon)}: \forall \underline{(\text{fx})}: \forall \underline{(\text{fy})}: \forall \underline{(\text{rx})}: \forall \underline{(\text{ry})}: \underline{(\text{rx})} << \underline{(\text{ry})} \vdash \underline{(\text{fx})} \in \underline{(\text{rx})} \vdash \underline{(\text{fy})} \in \underline{(\text{ry})} \vdash \dot{\neg} 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg} \dot{\neg} 0 = \underline{(\epsilon)} \vdash \mathsf{a}_{\text{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(\text{fx})}[\underline{\mathsf{m}}] <= \underline{(\text{fy})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)}]$

$[\text{From} << \text{XX} \overset{\text{tex}}{\rightarrow} \text{“From<<XX”}]$

[From << XX $\stackrel{\text{pyk}}{\to}$ "1rule from<<XX"]

From << (1)

[From << (1) $\stackrel{\text{proof}}{\to}$ Rule tactic]

[From << (1) $\stackrel{\text{stmt}}{\to}$ $\text{ZFsub} \vdash \forall \underline{(\text{rx})}\colon \forall \underline{(\text{ry})}\colon \underline{(\text{rx})} << \underline{(\text{ry})} \vdash \mathsf{j}_{\text{Ex}} \in \underline{(\text{rx})}$]

[From << (1) $\stackrel{\text{tex}}{\to}$ "From<<(1)"]

[From << (1) $\stackrel{\text{pyk}}{\to}$ "1rule from<<XX(1)"]

From << (2)

[From << (2) $\stackrel{\text{proof}}{\to}$ Rule tactic]

[From << (2) $\stackrel{\text{stmt}}{\to}$ $\text{ZFsub} \vdash \forall \underline{(\text{rx})}\colon \forall \underline{(\text{ry})}\colon \underline{(\text{rx})} << \underline{(\text{ry})} \vdash \mathsf{t}_{\text{Ex}} \in \underline{(\text{ry})}$]

[From << (2) $\stackrel{\text{tex}}{\to}$ "From<<(2)"]

[From << (2) $\stackrel{\text{pyk}}{\to}$ "1rule from<<XX(2)"]

to << XX

[to << XX $\stackrel{\text{proof}}{\to}$ Rule tactic]

[to << XX $\stackrel{\text{stmt}}{\to}$ $\text{ZFsub} \vdash \forall \underline{\mathsf{m}}\colon \forall \underline{\mathsf{n}}\colon \forall \underline{(\epsilon)}\colon \forall \underline{(\text{fx})}\colon \forall \underline{(\text{fy})}\colon \forall \underline{(\text{rx})}\colon \forall \underline{(\text{ry})}\colon \underline{(\text{fx})} \in \underline{(\text{rx})} \Rightarrow \underline{(\text{fy})} \in \underline{(\text{ry})} \Rightarrow \dot{\neg} 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{(\epsilon)} \Rightarrow \mathsf{a}_{\text{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(\text{fx})}[\underline{\mathsf{m}}] <= \underline{(\text{fy})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \vdash \underline{(\text{rx})} << \underline{(\text{ry})}$]

[to << XX $\stackrel{\text{tex}}{\to}$ "to<<XX"]

[to << XX $\stackrel{\text{pyk}}{\to}$ "1rule to<<XX"]

From <<

[From << $\stackrel{\text{proof}}{\to}$ Rule tactic]

[From << $\stackrel{\text{stmt}}{\to}$ $\text{ZFsub} \vdash \forall \underline{(\text{fx})}\colon \forall \underline{(\text{fy})}\colon \text{R}(\underline{(\text{fx})}) << \text{R}(\underline{(\text{fy})}) \vdash \underline{(\text{fx})} <_{\text{f}} \underline{(\text{fy})}$]

[From << $\stackrel{\text{tex}}{\to}$ "From<<"]

[From << $\stackrel{\text{pyk}}{\to}$ "1rule from<<"]

To <<

[To << $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[To << $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{(\mathrm{fx})} \colon \forall \underline{(\mathrm{fy})} \colon \underline{(\mathrm{fx})} <_{\mathrm{f}} \underline{(\mathrm{fy})} \vdash \mathrm{R}(\underline{(\mathrm{fx})}) << \mathrm{R}(\underline{(\mathrm{fy})})$]

[To << $\stackrel{\text{tex}}{\rightarrow}$ “To<<”]

[To << $\stackrel{\text{pyk}}{\rightarrow}$ “1rule to<<”]

FromInR

[FromInR $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[FromInR $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{(\mathrm{fx})} \colon \forall \underline{(\mathrm{fy})} \colon \underline{(\mathrm{fx})} \in \mathrm{R}(\underline{(\mathrm{fy})}) \vdash \mathrm{SF}(\underline{(\mathrm{fx})}, \underline{(\mathrm{fy})})$]

[FromInR $\stackrel{\text{tex}}{\rightarrow}$ “FromInR”]

[FromInR $\stackrel{\text{pyk}}{\rightarrow}$ “1rule fromInR”]

PlusR

[PlusR $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[PlusR $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{(\mathrm{fx})} \colon \forall \underline{(\mathrm{fy})} \colon \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}(\underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})})$]

[PlusR $\stackrel{\text{tex}}{\rightarrow}$ “PlusR”]

[PlusR $\stackrel{\text{pyk}}{\rightarrow}$ “axiom plusR”]

TimesR

[TimesR $\stackrel{\text{proof}}{\rightarrow}$ Rule tactic]

[TimesR $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{(\mathrm{fx})} \colon \forall \underline{(\mathrm{fy})} \colon \mathrm{R}(\underline{(\mathrm{fx})}) * * \mathrm{R}(\underline{(\mathrm{fy})}) == \mathrm{R}(\underline{(\mathrm{fx})} *_{\mathrm{f}} \underline{(\mathrm{fy})})$]

[TimesR $\stackrel{\text{tex}}{\rightarrow}$ “TimesR”]

[TimesR $\stackrel{\text{pyk}}{\rightarrow}$ “axiom timesR”]

leqAntisymmetry

[leqAntisymmetry $\stackrel{\text{proof}}{\rightarrow}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \underline{\mathsf{x}} <= \underline{\mathsf{y}} \vdash \underline{\mathsf{y}} <= \underline{\mathsf{x}} \vdash$
leqAntisymmetryAxiom $\gg \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{y}} <= \underline{\mathsf{x}} \Rightarrow \underline{\mathsf{x}} = \underline{\mathsf{y}};$ MP2 $\rhd \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{y}} <=$

$\underline{x} \Rightarrow \underline{x} = \underline{y} \rhd \underline{x} <= \underline{y} \rhd \underline{y} <= \underline{x} \gg \underline{x} = \underline{y}\rceil, p_0, c)]$

$[\text{leqAntisymmetry} \overset{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \underline{x} <= \underline{y} \vdash \underline{y} <= \underline{x} \vdash \underline{x} = \underline{y}]$

[leqAntisymmetry $\overset{\text{tex}}{\rightarrow}$ "leqAntisymmetry"]

[leqAntisymmetry $\overset{\text{pyk}}{\rightarrow}$ "lemma leqAntisymmetry"]

leqTransitivity

$[\text{leqTransitivity} \overset{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \underline{x} <= \underline{y} \vdash \underline{y} <= \underline{z} \vdash \text{leqTransitivityAxiom} \gg \underline{x} <= \underline{y} \Rightarrow \underline{y} <= \underline{z} \Rightarrow \underline{x} <= \underline{z}; \text{MP2} \rhd \underline{x} <= \underline{y} \Rightarrow \underline{y} <= \underline{z} \Rightarrow \underline{x} <= \underline{z} \rhd \underline{x} <= \underline{y} \rhd \underline{y} <= \underline{z} \gg \underline{x} <= \underline{z}\rceil, p_0, c)]$

$[\text{leqTransitivity} \overset{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \underline{x} <= \underline{y} \vdash \underline{y} <= \underline{z} \vdash \underline{x} <= \underline{z}]$

[leqTransitivity $\overset{\text{tex}}{\rightarrow}$ "leqTransitivity"]

[leqTransitivity $\overset{\text{pyk}}{\rightarrow}$ "lemma leqTransitivity"]

leqAddition

$[\text{leqAddition} \overset{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \underline{x} <= \underline{y} \vdash \text{leqAdditionAxiom} \gg \underline{x} <= \underline{y} \Rightarrow \underline{x} + \underline{z} <= \underline{y} + \underline{z}; \text{MP} \rhd \underline{x} <= \underline{y} \Rightarrow \underline{x} + \underline{z} <= \underline{y} + \underline{z} \rhd \underline{x} <= \underline{y} \gg \underline{x} + \underline{z} <= \underline{y} + \underline{z}\rceil, p_0, c)]$

$[\text{leqAddition} \overset{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \underline{x} <= \underline{y} \vdash \underline{x} + \underline{z} <= \underline{y} + \underline{z}]$

[leqAddition $\overset{\text{tex}}{\rightarrow}$ "leqAddition"]

[leqAddition $\overset{\text{pyk}}{\rightarrow}$ "lemma leqAddition"]

leqMultiplication

$[\text{leqMultiplication} \overset{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon 0 <= \underline{z} \vdash \underline{x} <= \underline{y} \vdash \text{leqMultiplicationAxiom} \gg 0 <= \underline{z} \Rightarrow \underline{x} <= \underline{y} \Rightarrow \underline{x} * \underline{z} <= \underline{y} * \underline{z}; \text{MP2} \rhd 0 <= \underline{z} \Rightarrow \underline{x} <= \underline{y} \Rightarrow \underline{x} * \underline{z} <= \underline{y} * \underline{z} \rhd 0 <= \underline{z} \rhd \underline{x} <= \underline{y} \gg \underline{x} * \underline{z} <= \underline{y} * \underline{z}\rceil, p_0, c)]$

$[\text{leqMultiplication} \overset{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon 0 <= \underline{z} \vdash \underline{x} <= \underline{y} \vdash \underline{x} * \underline{z} <= \underline{y} * \underline{z}]$

[leqMultiplication $\overset{\text{tex}}{\rightarrow}$ "leqMultiplication"]

[leqMultiplication $\overset{\text{pyk}}{\rightarrow}$ "lemma leqMultiplication"]

Reciprocal

$[\text{Reciprocal} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \dot{\neg}\, \underline{\mathsf{x}} = 0 \vdash \text{ReciprocalAxiom} \gg \dot{\neg}\, \underline{\mathsf{x}} = 0 \Rightarrow \underline{\mathsf{x}} * \text{rec}\underline{\mathsf{x}} = 1; \text{MP} \rhd \dot{\neg}\, \underline{\mathsf{x}} = 0 \Rightarrow \underline{\mathsf{x}} * \text{rec}\underline{\mathsf{x}} = 1 \rhd \dot{\neg}\, \underline{\mathsf{x}} = 0 \gg \underline{\mathsf{x}} * \text{rec}\underline{\mathsf{x}} = 1\rceil, \text{p}_0, \mathsf{c})]$

$[\text{Reciprocal} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \dot{\neg}\, \underline{\mathsf{x}} = 0 \vdash \underline{\mathsf{x}} * \text{rec}\underline{\mathsf{x}} = 1]$

$[\text{Reciprocal} \stackrel{\text{tex}}{\rightarrow}$ "Reciprocal"$]$

$[\text{Reciprocal} \stackrel{\text{pyk}}{\rightarrow}$ "lemma reciprocal"$]$

Equality

$[\text{Equality} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \forall \underline{\mathsf{z}}\colon \underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \underline{\mathsf{x}} = \underline{\mathsf{z}} \vdash \text{equalityAxiom} \gg \underline{\mathsf{x}} = \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{x}} = \underline{\mathsf{z}} \Rightarrow \underline{\mathsf{y}} = \underline{\mathsf{z}}; \text{MP2} \rhd \underline{\mathsf{x}} = \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{x}} = \underline{\mathsf{z}} \Rightarrow \underline{\mathsf{y}} = \underline{\mathsf{z}} \rhd \underline{\mathsf{x}} = \underline{\mathsf{y}} \rhd \underline{\mathsf{x}} = \underline{\mathsf{z}} \gg \underline{\mathsf{y}} = \underline{\mathsf{z}}\rceil, \text{p}_0, \mathsf{c})]$

$[\text{Equality} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \forall \underline{\mathsf{z}}\colon \underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \underline{\mathsf{x}} = \underline{\mathsf{z}} \vdash \underline{\mathsf{y}} = \underline{\mathsf{z}}]$

$[\text{Equality} \stackrel{\text{tex}}{\rightarrow}$ "Equality"$]$

$[\text{Equality} \stackrel{\text{pyk}}{\rightarrow}$ "lemma equality"$]$

eqLeq

$[\text{eqLeq} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \text{eqLeqAxiom} \gg \underline{\mathsf{x}} = \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{x}} <= \underline{\mathsf{y}}; \text{MP} \rhd \underline{\mathsf{x}} = \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{x}} <= \underline{\mathsf{y}} \rhd \underline{\mathsf{x}} = \underline{\mathsf{y}} \gg \underline{\mathsf{x}} <= \underline{\mathsf{y}}\rceil, \text{p}_0, \mathsf{c})]$

$[\text{eqLeq} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \underline{\mathsf{x}} <= \underline{\mathsf{y}}]$

$[\text{eqLeq} \stackrel{\text{tex}}{\rightarrow}$ "eqLeq"$]$

$[\text{eqLeq} \stackrel{\text{pyk}}{\rightarrow}$ "lemma eqLeq"$]$

eqAddition

$[\text{eqAddition} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \forall \underline{\mathsf{z}}\colon \underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \text{eqAdditionAxiom} \gg \underline{\mathsf{x}} = \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{x}} + \underline{\mathsf{z}} = \underline{\mathsf{y}} + \underline{\mathsf{z}}; \text{MP} \rhd \underline{\mathsf{x}} = \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{x}} + \underline{\mathsf{z}} = \underline{\mathsf{y}} + \underline{\mathsf{z}} \rhd \underline{\mathsf{x}} = \underline{\mathsf{y}} \gg \underline{\mathsf{x}} + \underline{\mathsf{z}} = \underline{\mathsf{y}} + \underline{\mathsf{z}}\rceil, \text{p}_0, \mathsf{c})]$

$[\text{eqAddition} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \forall \underline{\mathsf{z}}\colon \underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \underline{\mathsf{x}} + \underline{\mathsf{z}} = \underline{\mathsf{y}} + \underline{\mathsf{z}}]$

$[\text{eqAddition} \stackrel{\text{tex}}{\rightarrow}$ "eqAddition"$]$

$[\text{eqAddition} \stackrel{\text{pyk}}{\rightarrow}$ "lemma eqAddition"$]$

eqMultiplication

[eqMultiplication $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} \vdash \text{eqMultiplicationAxiom} \gg \underline{x} = \underline{y} \Rightarrow \underline{x} * \underline{z} = \underline{y} * \underline{z}; \text{MP} \rhd \underline{x} = \underline{y} \Rightarrow \underline{x} * \underline{z} = \underline{y} * \underline{z} \rhd \underline{x} = \underline{y} \gg \underline{x} * \underline{z} = \underline{y} * \underline{z}\rceil, p_0, c)$]

[eqMultiplication $\stackrel{\text{stmt}}{\rightarrow}$ $\text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} \vdash \underline{x} * \underline{z} = \underline{y} * \underline{z}$]

[eqMultiplication $\stackrel{\text{tex}}{\rightarrow}$ "eqMultiplication"]

[eqMultiplication $\stackrel{\text{pyk}}{\rightarrow}$ "lemma eqMultiplication"]

ToNegatedImply

[ToNegatedImply $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \vdash \dot{\neg}\, \underline{b} \vdash \dot{\neg}\dot{\neg}\, \underline{a} \Rightarrow \underline{b} \vdash \text{RemoveDoubleNeg} \rhd \dot{\neg}\dot{\neg}\, \underline{a} \Rightarrow \underline{b} \gg \underline{a} \Rightarrow \underline{b}; \text{MP} \rhd \underline{a} \Rightarrow \underline{b} \rhd \underline{a} \gg \underline{b}; \text{FromContradiction} \rhd \underline{b} \rhd \dot{\neg}\, \underline{b} \gg \dot{\neg}\, \underline{a} \Rightarrow \underline{b}; \forall \underline{a}\colon \forall \underline{b}\colon \text{Ded} \rhd \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \vdash \dot{\neg}\, \underline{b} \vdash \dot{\neg}\dot{\neg}\, \underline{a} \Rightarrow \underline{b} \vdash \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \gg \underline{a} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\, \underline{a} \Rightarrow \underline{b}; \underline{a} \vdash \dot{\neg}\, \underline{b} \vdash \text{MP2} \rhd \underline{a} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \rhd \underline{a} \rhd \dot{\neg}\, \underline{b} \gg \dot{\neg}\dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\, \underline{a} \Rightarrow \underline{b}; \text{AutoImply} \gg \dot{\neg}\dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{a} \Rightarrow \underline{b}; \text{Neg} \rhd \dot{\neg}\dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \rhd \dot{\neg}\dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{a} \Rightarrow \underline{b} \gg \dot{\neg}\, \underline{a} \Rightarrow \underline{b}\rceil, p_0, c)$]

[ToNegatedImply $\stackrel{\text{stmt}}{\rightarrow}$ $\text{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \vdash \dot{\neg}\, \underline{b} \vdash \dot{\neg}\, \underline{a} \Rightarrow \underline{b}$]

[ToNegatedImply $\stackrel{\text{tex}}{\rightarrow}$ "ToNegatedImply"]

[ToNegatedImply $\stackrel{\text{pyk}}{\rightarrow}$ "prop lemma to negated imply"]

TND

[TND $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{a}\colon \text{AutoImply} \gg \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{a}; \text{Repetition} \rhd \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{a} \gg \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{a}\rceil, p_0, c)$]

[TND $\stackrel{\text{stmt}}{\rightarrow}$ $\text{ZFsub} \vdash \forall \underline{a}\colon \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{a}$]

[TND $\stackrel{\text{tex}}{\rightarrow}$ "TND"]

[TND $\stackrel{\text{pyk}}{\rightarrow}$ "prop lemma tertium non datur"]

ImplyNegation

[ImplyNegation $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{a}\colon \underline{a} \Rightarrow \dot{\neg}\, \underline{a} \vdash \text{AutoImply} \gg \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{a}; \text{TND} \gg \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{a}; \text{FromDisjuncts} \rhd \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{a} \rhd \underline{a} \Rightarrow \dot{\neg}\, \underline{a} \rhd \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{a} \gg \dot{\neg}\, \underline{a}\rceil, p_0, c)$]

$[\mathrm{ImplyNegation} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{a}\colon \underline{a} \Rightarrow \dot{\neg}\, \underline{a} \vdash \dot{\neg}\, \underline{a}]$

$[\mathrm{ImplyNegation} \stackrel{\mathrm{tex}}{\rightarrow} \text{“ImplyNegation”}]$

$[\mathrm{ImplyNegation} \stackrel{\mathrm{pyk}}{\rightarrow} \text{“prop lemma imply negation”}]$

FromNegations

$[\mathrm{FromNegations} \stackrel{\mathrm{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \Rightarrow \underline{b} \vdash \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \vdash \mathrm{TND} \gg \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{a}; \mathrm{FromDisjuncts} \rhd \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{a} \rhd \underline{a} \Rightarrow \underline{b} \rhd \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \gg \underline{b} \rceil, p_0, c)]$

$[\mathrm{FromNegations} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \underline{a} \Rightarrow \underline{b} \vdash \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \vdash \underline{b}]$

$[\mathrm{FromNegations} \stackrel{\mathrm{tex}}{\rightarrow} \text{“FromNegations”}]$

$[\mathrm{FromNegations} \stackrel{\mathrm{pyk}}{\rightarrow} \text{“prop lemma from negations”}]$

From3Disjuncts

$[\mathrm{From3Disjuncts} \stackrel{\mathrm{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{d}\colon \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{c} \vdash \underline{b} \Rightarrow \underline{d} \vdash \underline{c} \Rightarrow \underline{d} \vdash \dot{\neg}\, \underline{a} \vdash \mathrm{Repetition} \rhd \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{c} \gg \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{c}; \mathrm{MP} \rhd \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{c} \rhd \dot{\neg}\, \underline{a} \gg \dot{\neg}\, \underline{b} \Rightarrow \underline{c}; \mathrm{FromDisjuncts} \rhd \dot{\neg}\, \underline{b} \Rightarrow \underline{c} \rhd \underline{b} \Rightarrow \underline{d} \rhd \underline{c} \Rightarrow \underline{d} \gg \underline{d}; \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{d}\colon \mathrm{Ded} \rhd \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{d}\colon \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{c} \vdash \underline{b} \Rightarrow \underline{d} \vdash \underline{c} \Rightarrow \underline{d} \vdash \dot{\neg}\, \underline{a} \vdash \underline{d} \gg \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{b} \Rightarrow \underline{d} \Rightarrow \underline{c} \Rightarrow \underline{d} \Rightarrow \dot{\neg}\, \underline{a} \Rightarrow \underline{d}; \mathrm{AutoImply} \gg \underline{a} \Rightarrow \underline{d} \Rightarrow \underline{a} \Rightarrow \underline{d}; \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{c} \vdash \underline{a} \Rightarrow \underline{d} \vdash \underline{b} \Rightarrow \underline{d} \vdash \underline{c} \Rightarrow \underline{d} \vdash \mathrm{MP3} \rhd \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{b} \Rightarrow \underline{d} \Rightarrow \underline{c} \Rightarrow \underline{d} \Rightarrow \dot{\neg}\, \underline{a} \Rightarrow \underline{d} \rhd \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{c} \rhd \underline{b} \Rightarrow \underline{d} \rhd \underline{c} \Rightarrow \underline{d} \gg \dot{\neg}\, \underline{a} \Rightarrow \underline{d}; \mathrm{MP} \rhd \underline{a} \Rightarrow \underline{d} \Rightarrow \underline{a} \Rightarrow \underline{d} \rhd \underline{a} \Rightarrow \underline{d} \gg \underline{a} \Rightarrow \underline{d}; \mathrm{FromNegations} \rhd \underline{a} \Rightarrow \underline{d} \rhd \dot{\neg}\, \underline{a} \Rightarrow \underline{d} \gg \underline{d} \rceil, p_0, c)]$

$[\mathrm{From3Disjuncts} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{d}\colon \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \underline{c} \vdash \underline{a} \Rightarrow \underline{d} \vdash \underline{b} \Rightarrow \underline{d} \vdash \underline{c} \Rightarrow \underline{d} \vdash \underline{d}]$

$[\mathrm{From3Disjuncts} \stackrel{\mathrm{tex}}{\rightarrow} \text{“From3Disjuncts”}]$

$[\mathrm{From3Disjuncts} \stackrel{\mathrm{pyk}}{\rightarrow} \text{“prop lemma from three disjuncts”}]$

From2 ∗ 2Disjuncts

$[\mathrm{From2} * \mathrm{2Disjuncts} \stackrel{\mathrm{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{d}\colon \forall \underline{e}\colon \dot{\neg}\, \underline{c} \Rightarrow \underline{d} \vdash \underline{a} \Rightarrow \underline{c} \Rightarrow \underline{e} \vdash \underline{a} \Rightarrow \underline{d} \Rightarrow \underline{e} \vdash \underline{a} \vdash \mathrm{MP} \rhd \underline{a} \Rightarrow \underline{c} \Rightarrow \underline{e} \rhd \underline{a} \gg \underline{c} \Rightarrow \underline{e}; \mathrm{MP} \rhd \underline{a} \Rightarrow \underline{d} \Rightarrow \underline{e} \rhd \underline{a} \gg \underline{d} \Rightarrow \underline{e}; \mathrm{FromDisjuncts} \rhd \dot{\neg}\, \underline{c} \Rightarrow \underline{d} \rhd \underline{c} \Rightarrow \underline{e} \rhd \underline{d} \Rightarrow \underline{e} \gg \underline{e}; \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{d}\colon \forall \underline{e}\colon \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \vdash \dot{\neg}\, \underline{c} \Rightarrow \underline{d} \vdash \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{e} \vdash \underline{b} \Rightarrow \underline{d} \Rightarrow \underline{e} \vdash \dot{\neg}\, \underline{a} \vdash \mathrm{NegateDisjunct1} \rhd \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \rhd \dot{\neg}\, \underline{a} \gg \underline{b}; \mathrm{MP} \rhd \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{e} \rhd \underline{b} \gg \underline{c} \Rightarrow \underline{e}; \mathrm{MP} \rhd \underline{b} \Rightarrow \underline{d} \Rightarrow \underline{e} \rhd \underline{b} \gg \underline{d} \Rightarrow \underline{e}; \mathrm{FromDisjuncts} \rhd \dot{\neg}\, \underline{c} \Rightarrow \underline{d} \rhd \underline{c} \Rightarrow \underline{e} \rhd \underline{d} \Rightarrow \underline{e} \gg \underline{e}; \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{d}\colon \forall \underline{e}\colon \mathrm{Ded} \rhd \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{d}\colon \forall \underline{e}\colon \dot{\neg}\, \underline{c} \Rightarrow \underline{d} \vdash \underline{a} \Rightarrow \underline{c} \Rightarrow \underline{e} \vdash \underline{a} \Rightarrow \underline{d} \Rightarrow \underline{e} \vdash \underline{a} \vdash \underline{e} \gg \dot{\neg}\, \underline{c} \Rightarrow \underline{d} \Rightarrow \underline{a} \Rightarrow$

$\underline{c} \Rightarrow \underline{e} \Rightarrow \underline{a} \Rightarrow \underline{d} \Rightarrow \underline{e} \Rightarrow \underline{a} \Rightarrow \underline{e}; \mathrm{Ded} \rhd \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{d}\colon \forall \underline{e}\colon \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \vdash \dot{\neg}\, \underline{c} \Rightarrow \underline{d} \vdash \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{e} \vdash \underline{b} \Rightarrow \underline{d} \Rightarrow \underline{e} \vdash \dot{\neg}\, \underline{a} \vdash \underline{e} \gg \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\, \underline{c} \Rightarrow \underline{d} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{e} \Rightarrow \underline{b} \Rightarrow \underline{d} \Rightarrow \underline{e} \Rightarrow \dot{\neg}\, \underline{a} \Rightarrow \underline{e}; \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \vdash \dot{\neg}\, \underline{c} \Rightarrow \underline{d} \vdash \underline{a} \Rightarrow \underline{c} \Rightarrow \underline{e} \vdash \underline{a} \Rightarrow \underline{d} \Rightarrow \underline{e} \vdash \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{e} \vdash \underline{b} \Rightarrow \underline{d} \Rightarrow \underline{e} \vdash \mathrm{MP3} \rhd \dot{\neg}\, \underline{c} \Rightarrow \underline{d} \Rightarrow \underline{a} \Rightarrow \underline{c} \Rightarrow \underline{e} \Rightarrow \underline{a} \Rightarrow \underline{d} \Rightarrow \underline{e} \Rightarrow \underline{a} \Rightarrow \underline{e} \rhd \dot{\neg}\, \underline{c} \Rightarrow \underline{d} \rhd \underline{a} \Rightarrow \underline{c} \Rightarrow \underline{e} \rhd \underline{a} \Rightarrow \underline{d} \Rightarrow \underline{e} \gg \underline{a} \Rightarrow \underline{e}; \mathrm{MP4} \rhd \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\, \underline{c} \Rightarrow \underline{d} \Rightarrow \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{e} \Rightarrow \underline{b} \Rightarrow \underline{d} \Rightarrow \underline{e} \Rightarrow \dot{\neg}\, \underline{a} \Rightarrow \underline{e} \rhd \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \rhd \dot{\neg}\, \underline{c} \Rightarrow \underline{d} \rhd \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{e} \rhd \underline{b} \Rightarrow \underline{d} \Rightarrow \underline{e} \gg \dot{\neg}\, \underline{a} \Rightarrow \underline{e}; \mathrm{FromNegations} \rhd \underline{a} \Rightarrow \underline{e} \rhd \dot{\neg}\, \underline{a} \Rightarrow \underline{e} \gg \underline{e} \rceil, p_0, c)]$

$[\mathrm{From2} * \mathrm{2Disjuncts} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{d}\colon \forall \underline{e}\colon \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \vdash \dot{\neg}\, \underline{c} \Rightarrow \underline{d} \vdash \underline{a} \Rightarrow \underline{c} \Rightarrow \underline{e} \vdash \underline{a} \Rightarrow \underline{d} \Rightarrow \underline{e} \vdash \underline{b} \Rightarrow \underline{c} \Rightarrow \underline{e} \vdash \underline{b} \Rightarrow \underline{d} \Rightarrow \underline{e} \vdash \underline{e}]$

$[\mathrm{From2} * \mathrm{2Disjuncts} \overset{\mathrm{tex}}{\rightarrow}$ "From2∗2Disjuncts"$]$

$[\mathrm{From2} * \mathrm{2Disjuncts} \overset{\mathrm{pyk}}{\rightarrow}$ "prop lemma from two times two disjuncts"$]$

NegateDisjunct1

$[\mathrm{NegateDisjunct1} \overset{\mathrm{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \vdash \dot{\neg}\, \underline{a} \vdash \mathrm{Repetition} \rhd \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \gg \dot{\neg}\, \underline{a} \Rightarrow \underline{b}; \mathrm{MP} \rhd \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \rhd \dot{\neg}\, \underline{a} \gg \underline{b} \rceil, p_0, c)]$

$[\mathrm{NegateDisjunct1} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \vdash \dot{\neg}\, \underline{a} \vdash \underline{b}]$

$[\mathrm{NegateDisjunct1} \overset{\mathrm{tex}}{\rightarrow}$ "NegateDisjunct1"$]$

$[\mathrm{NegateDisjunct1} \overset{\mathrm{pyk}}{\rightarrow}$ "prop lemma negate first disjunct"$]$

NegateDisjunct2

$[\mathrm{NegateDisjunct2} \overset{\mathrm{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \vdash \dot{\neg}\, \underline{b} \vdash \mathrm{Repetition} \rhd \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \gg \dot{\neg}\, \underline{a} \Rightarrow \underline{b}; \mathrm{NegativeMT} \rhd \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \rhd \dot{\neg}\, \underline{b} \gg \underline{a} \rceil, p_0, c)]$

$[\mathrm{NegateDisjunct2} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \vdash \dot{\neg}\, \underline{b} \vdash \underline{a}]$

$[\mathrm{NegateDisjunct2} \overset{\mathrm{tex}}{\rightarrow}$ "NegateDisjunct2"$]$

$[\mathrm{NegateDisjunct2} \overset{\mathrm{pyk}}{\rightarrow}$ "prop lemma negate second disjunct"$]$

ExpandDisjuncts

$[\mathrm{ExpandDisjuncts} \overset{\mathrm{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{d}\colon \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \vdash \dot{\neg}\, \underline{c} \Rightarrow \underline{d} \vdash \dot{\neg}\, \underline{b} \vdash \dot{\neg}\, \underline{d} \vdash \mathrm{NegateDisjunct2} \rhd \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \rhd \dot{\neg}\, \underline{b} \gg \underline{a}; \mathrm{NegateDisjunct2} \rhd \dot{\neg}\, \underline{c} \Rightarrow \underline{d} \rhd \dot{\neg}\, \underline{d} \gg \underline{c}; \mathrm{JoinConjuncts} \rhd \underline{a} \rhd \underline{c} \gg \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{c}; \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{d}\colon \mathrm{Ded} \rhd \forall \underline{a}\colon \forall \underline{b}\colon \forall \underline{c}\colon \forall \underline{d}\colon \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \vdash \dot{\neg}\, \underline{c} \Rightarrow \underline{d} \vdash \dot{\neg}\, \underline{b} \vdash \dot{\neg}\, \underline{d} \vdash \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{c} \gg \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\, \underline{c} \Rightarrow \underline{d} \Rightarrow \dot{\neg}\, \underline{b} \Rightarrow \dot{\neg}\, \underline{d} \Rightarrow \dot{\neg}\, \underline{a} \Rightarrow \dot{\neg}\, \underline{c}; \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \vdash \dot{\neg}\, \underline{c} \Rightarrow \underline{d} \vdash \mathrm{MP2} \rhd \dot{\neg}\, \underline{a} \Rightarrow \underline{b} \Rightarrow \dot{\neg}\, \underline{c} \Rightarrow \underline{d} \Rightarrow$

$\dot\neg \underline{b} \Rightarrow \dot\neg \underline{d} \Rightarrow \dot\neg \underline{a} \Rightarrow \dot\neg \underline{c} \rhd \dot\neg \underline{a} \Rightarrow \underline{b} \rhd \dot\neg \underline{c} \Rightarrow \underline{d} \gg \dot\neg \underline{b} \Rightarrow \dot\neg \underline{d} \Rightarrow \dot\neg \underline{a} \Rightarrow \dot\neg \underline{c}; \mathrm{Repetition} \rhd \dot\neg \underline{b} \Rightarrow \dot\neg \underline{d} \Rightarrow \dot\neg \underline{a} \Rightarrow \dot\neg \underline{c} \gg \dot\neg \underline{b} \Rightarrow \dot\neg \underline{d} \Rightarrow \dot\neg \underline{a} \Rightarrow \dot\neg \underline{c}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{ExpandDisjuncts} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{a} \colon \forall \underline{b} \colon \forall \underline{c} \colon \forall \underline{d} \colon \dot\neg \underline{a} \Rightarrow \underline{b} \vdash \dot\neg \underline{c} \Rightarrow \underline{d} \vdash \dot\neg \underline{b} \Rightarrow \dot\neg \underline{d} \Rightarrow \dot\neg \underline{a} \Rightarrow \dot\neg \underline{c}]$

[ExpandDisjuncts $\overset{\mathrm{tex}}{\rightarrow}$ "ExpandDisjuncts"]

[ExpandDisjuncts $\overset{\mathrm{pyk}}{\rightarrow}$ "prop lemma expand disjuncts"]

eqReflexivity

$[\mathrm{eqReflexivity} \overset{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x} \colon \mathrm{leqReflexivity} \gg \underline{x} <= \underline{x}; \mathrm{leqAntisymmetry} \rhd \underline{x} <= \underline{x} \rhd \underline{x} <= \underline{x} \gg \underline{x} = \underline{x}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{eqReflexivity} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{x} \colon \underline{x} = \underline{x}]$

[eqReflexivity $\overset{\mathrm{tex}}{\rightarrow}$ "eqReflexivity"]

[eqReflexivity $\overset{\mathrm{pyk}}{\rightarrow}$ "lemma eqReflexivity"]

eqSymmetry

$[\mathrm{eqSymmetry} \overset{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x} \colon \forall \underline{y} \colon \underline{x} = \underline{y} \vdash \mathrm{eqReflexivity} \gg \underline{x} = \underline{x}; \mathrm{Equality} \rhd \underline{x} = \underline{y} \rhd \underline{x} = \underline{x} \gg \underline{y} = \underline{x}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{eqSymmetry} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{x} \colon \forall \underline{y} \colon \underline{x} = \underline{y} \vdash \underline{y} = \underline{x}]$

[eqSymmetry $\overset{\mathrm{tex}}{\rightarrow}$ "eqSymmetry"]

[eqSymmetry $\overset{\mathrm{pyk}}{\rightarrow}$ "lemma eqSymmetry"]

eqTransitivity

$[\mathrm{eqTransitivity} \overset{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x} \colon \forall \underline{y} \colon \forall \underline{z} \colon \underline{x} = \underline{y} \vdash \underline{y} = \underline{z} \vdash \mathrm{eqSymmetry} \rhd \underline{x} = \underline{y} \gg \underline{y} = \underline{x}; \mathrm{Equality} \rhd \underline{y} = \underline{x} \rhd \underline{y} = \underline{z} \gg \underline{x} = \underline{z}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{eqTransitivity} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{x} \colon \forall \underline{y} \colon \forall \underline{z} \colon \underline{x} = \underline{y} \vdash \underline{y} = \underline{z} \vdash \underline{x} = \underline{z}]$

[eqTransitivity $\overset{\mathrm{tex}}{\rightarrow}$ "eqTransitivity"]

[eqTransitivity $\overset{\mathrm{pyk}}{\rightarrow}$ "lemma eqTransitivity"]

eqTransitivity4

[eqTransitivity4 $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \forall \underline{u}\colon \underline{x} = \underline{y} \vdash \underline{y} = \underline{z} \vdash \underline{z} = \underline{u} \vdash \text{eqTransitivity} \rhd \underline{x} = \underline{y} \rhd \underline{y} = \underline{z} \gg \underline{x} = \underline{z}; \text{eqTransitivity} \rhd \underline{x} = \underline{z} \rhd \underline{z} = \underline{u} \gg \underline{x} = \underline{u}\rceil, p_0, c)$]

[eqTransitivity4 $\stackrel{\text{stmt}}{\rightarrow}$ $\text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \forall \underline{u}\colon \underline{x} = \underline{y} \vdash \underline{y} = \underline{z} \vdash \underline{z} = \underline{u} \vdash \underline{x} = \underline{u}$]

[eqTransitivity4 $\stackrel{\text{tex}}{\rightarrow}$ "eqTransitivity4"]

[eqTransitivity4 $\stackrel{\text{pyk}}{\rightarrow}$ "lemma eqTransitivity4"]

eqTransitivity5

[eqTransitivity5 $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \forall \underline{u}\colon \forall \underline{v}\colon \underline{x} = \underline{y} \vdash \underline{y} = \underline{z} \vdash \underline{z} = \underline{u} \vdash \underline{u} = \underline{v} \vdash \text{eqTransitivity4} \rhd \underline{x} = \underline{y} \rhd \underline{y} = \underline{z} \rhd \underline{z} = \underline{u} \gg \underline{x} = \underline{u}; \text{eqTransitivity} \rhd \underline{x} = \underline{u} \rhd \underline{u} = \underline{v} \gg \underline{x} = \underline{v}\rceil, p_0, c)$]

[eqTransitivity5 $\stackrel{\text{stmt}}{\rightarrow}$ $\text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \forall \underline{u}\colon \forall \underline{v}\colon \underline{x} = \underline{y} \vdash \underline{y} = \underline{z} \vdash \underline{z} = \underline{u} \vdash \underline{u} = \underline{v} \vdash \underline{x} = \underline{v}$]

[eqTransitivity5 $\stackrel{\text{tex}}{\rightarrow}$ "eqTransitivity5"]

[eqTransitivity5 $\stackrel{\text{pyk}}{\rightarrow}$ "lemma eqTransitivity5"]

eqTransitivity6

[eqTransitivity6 $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \forall \underline{u}\colon \forall \underline{v}\colon \forall \underline{w}\colon \underline{x} = \underline{y} \vdash \underline{y} = \underline{z} \vdash \underline{z} = \underline{u} \vdash \underline{u} = \underline{v} \vdash \underline{v} = \underline{w} \vdash \text{eqTransitivity5} \rhd \underline{x} = \underline{y} \rhd \underline{y} = \underline{z} \rhd \underline{z} = \underline{u} \rhd \underline{u} = \underline{v} \gg \underline{x} = \underline{v}; \text{eqTransitivity} \rhd \underline{x} = \underline{v} \rhd \underline{v} = \underline{w} \gg \underline{x} = \underline{w}\rceil, p_0, c)$]

[eqTransitivity6 $\stackrel{\text{stmt}}{\rightarrow}$ $\text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \forall \underline{u}\colon \forall \underline{v}\colon \forall \underline{w}\colon \underline{x} = \underline{y} \vdash \underline{y} = \underline{z} \vdash \underline{z} = \underline{u} \vdash \underline{u} = \underline{v} \vdash \underline{v} = \underline{w} \vdash \underline{x} = \underline{w}$]

[eqTransitivity6 $\stackrel{\text{tex}}{\rightarrow}$ "eqTransitivity6"]

[eqTransitivity6 $\stackrel{\text{pyk}}{\rightarrow}$ "lemma eqTransitivity6"]

plus0Left

[plus0Left $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \text{plus0} \gg \underline{x} + 0 = \underline{x}; \text{plusCommutativity} \gg 0 + \underline{x} = \underline{x} + 0; \text{eqTransitivity} \rhd 0 + \underline{x} = \underline{x} + 0 \rhd \underline{x} + 0 = \underline{x} \gg 0 + \underline{x} = \underline{x}\rceil, p_0, c)$]

[plus0Left $\stackrel{\text{stmt}}{\rightarrow}$ $\text{ZFsub} \vdash \forall \underline{x}\colon 0 + \underline{x} = \underline{x}$]

[plus0Left $\stackrel{\text{tex}}{\to}$ "plus0Left"]

[plus0Left $\stackrel{\text{pyk}}{\to}$ "lemma plus0Left"]

times1Left

[times1Left $\stackrel{\text{proof}}{\to}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \text{times1} \gg \underline{x} * 1 = \underline{x}; \text{timesCommutativity} \gg 1 * \underline{x} = \underline{x} * 1; \text{eqTransitivity} \rhd 1 * \underline{x} = \underline{x} * 1 \rhd \underline{x} * 1 = \underline{x} \gg 1 * \underline{x} = \underline{x}\rceil, p_0, c)$]

[times1Left $\stackrel{\text{stmt}}{\to}$ $\text{ZFsub} \vdash \forall \underline{x}\colon 1 * \underline{x} = \underline{x}$]

[times1Left $\stackrel{\text{tex}}{\to}$ "times1Left"]

[times1Left $\stackrel{\text{pyk}}{\to}$ "lemma times1Left"]

lemma eqAdditionLeft

[lemma eqAdditionLeft $\stackrel{\text{proof}}{\to}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} \vdash \text{eqAddition} \rhd \underline{x} = \underline{y} \gg \underline{x} + \underline{z} = \underline{y} + \underline{z}; \text{plusCommutativity} \gg \underline{z} + \underline{x} = \underline{x} + \underline{z}; \text{plusCommutativity} \gg \underline{y} + \underline{z} = \underline{z} + \underline{y}; \text{eqTransitivity4} \rhd \underline{z} + \underline{x} = \underline{x} + \underline{z} \rhd \underline{x} + \underline{z} = \underline{y} + \underline{z} \rhd \underline{y} + \underline{z} = \underline{z} + \underline{y} \gg \underline{z} + \underline{x} = \underline{z} + \underline{y}\rceil, p_0, c)$]

[lemma eqAdditionLeft $\stackrel{\text{stmt}}{\to}$ $\text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} \vdash \underline{z} + \underline{x} = \underline{z} + \underline{y}$]

[lemma eqAdditionLeft $\stackrel{\text{pyk}}{\to}$ "lemma eqAdditionLeft"]

EqMultiplicationLeft

[EqMultiplicationLeft $\stackrel{\text{proof}}{\to}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} \vdash \text{eqMultiplication} \rhd \underline{x} = \underline{y} \gg \underline{x} * \underline{z} = \underline{y} * \underline{z}; \text{timesCommutativity} \gg \underline{z} * \underline{x} = \underline{x} * \underline{z}; \text{timesCommutativity} \gg \underline{y} * \underline{z} = \underline{z} * \underline{y}; \text{eqTransitivity4} \rhd \underline{z} * \underline{x} = \underline{x} * \underline{z} \rhd \underline{x} * \underline{z} = \underline{y} * \underline{z} \rhd \underline{y} * \underline{z} = \underline{z} * \underline{y} \gg \underline{z} * \underline{x} = \underline{z} * \underline{y}\rceil, p_0, c)$]

[EqMultiplicationLeft $\stackrel{\text{stmt}}{\to}$ $\text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} \vdash \underline{z} * \underline{x} = \underline{z} * \underline{y}$]

[EqMultiplicationLeft $\stackrel{\text{tex}}{\to}$ "EqMultiplicationLeft"]

[EqMultiplicationLeft $\stackrel{\text{pyk}}{\to}$ "lemma eqMultiplicationLeft"]

DistributionOut

$[\text{DistributionOut} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \forall \underline{\mathsf{z}} \colon \text{Distribution} \gg \underline{\mathsf{x}} * \underline{\mathsf{y}} + \underline{\mathsf{z}} = \underline{\mathsf{x}} * \underline{\mathsf{y}} + \underline{\mathsf{x}} * \underline{\mathsf{z}}; \text{eqSymmetry} \rhd \underline{\mathsf{x}} * \underline{\mathsf{y}} + \underline{\mathsf{z}} = \underline{\mathsf{x}} * \underline{\mathsf{y}} + \underline{\mathsf{x}} * \underline{\mathsf{z}} \gg \underline{\mathsf{x}} * \underline{\mathsf{y}} + \underline{\mathsf{x}} * \underline{\mathsf{z}} = \underline{\mathsf{x}} * \underline{\mathsf{y}} + \underline{\mathsf{z}} \rceil, \text{p}_0, \mathsf{c})]$

$[\text{DistributionOut} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \forall \underline{\mathsf{z}} \colon \underline{\mathsf{x}} * \underline{\mathsf{y}} + \underline{\mathsf{x}} * \underline{\mathsf{z}} = \underline{\mathsf{x}} * \underline{\mathsf{y}} + \underline{\mathsf{z}}]$

$[\text{DistributionOut} \stackrel{\text{tex}}{\rightarrow}$ "DistributionOut"$]$

$[\text{DistributionOut} \stackrel{\text{pyk}}{\rightarrow}$ "lemma distributionOut"$]$

Three2twoTerms

$[\text{Three2twoTerms} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \forall \underline{\mathsf{z}} \colon \forall \underline{\mathsf{u}} \colon \underline{\mathsf{y}} + \underline{\mathsf{z}} = \underline{\mathsf{u}} \vdash \text{lemma eqAdditionLeft} \rhd \underline{\mathsf{y}} + \underline{\mathsf{z}} = \underline{\mathsf{u}} \gg \underline{\mathsf{x}} + \underline{\mathsf{y}} + \underline{\mathsf{z}} = \underline{\mathsf{x}} + \underline{\mathsf{u}}; \text{plusAssociativity} \gg \underline{\mathsf{x}} + \underline{\mathsf{y}} + \underline{\mathsf{z}} = \underline{\mathsf{x}} + \underline{\mathsf{y}} + \underline{\mathsf{z}}; \text{eqTransitivity} \rhd \underline{\mathsf{x}} + \underline{\mathsf{y}} + \underline{\mathsf{z}} = \underline{\mathsf{x}} + \underline{\mathsf{y}} + \underline{\mathsf{z}} \rhd \underline{\mathsf{x}} + \underline{\mathsf{y}} + \underline{\mathsf{z}} = \underline{\mathsf{x}} + \underline{\mathsf{u}} \gg \underline{\mathsf{x}} + \underline{\mathsf{y}} + \underline{\mathsf{z}} = \underline{\mathsf{x}} + \underline{\mathsf{u}} \rceil, \text{p}_0, \mathsf{c})]$

$[\text{Three2twoTerms} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \forall \underline{\mathsf{z}} \colon \forall \underline{\mathsf{u}} \colon \underline{\mathsf{y}} + \underline{\mathsf{z}} = \underline{\mathsf{u}} \vdash \underline{\mathsf{x}} + \underline{\mathsf{y}} + \underline{\mathsf{z}} = \underline{\mathsf{x}} + \underline{\mathsf{u}}]$

$[\text{Three2twoTerms} \stackrel{\text{tex}}{\rightarrow}$ "Three2twoTerms"$]$

$[\text{Three2twoTerms} \stackrel{\text{pyk}}{\rightarrow}$ "lemma three2twoTerms"$]$

Three2threeTerms

$[\text{Three2threeTerms} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \forall \underline{\mathsf{z}} \colon \text{plusCommutativity} \gg \underline{\mathsf{y}} + \underline{\mathsf{z}} = \underline{\mathsf{z}} + \underline{\mathsf{y}}; \text{Three2twoTerms} \rhd \underline{\mathsf{y}} + \underline{\mathsf{z}} = \underline{\mathsf{z}} + \underline{\mathsf{y}} \gg \underline{\mathsf{x}} + \underline{\mathsf{y}} + \underline{\mathsf{z}} = \underline{\mathsf{x}} + \underline{\mathsf{z}} + \underline{\mathsf{y}}; \text{plusAssociativity} \gg \underline{\mathsf{x}} + \underline{\mathsf{z}} + \underline{\mathsf{y}} = \underline{\mathsf{x}} + \underline{\mathsf{z}} + \underline{\mathsf{y}}; \text{eqSymmetry} \rhd \underline{\mathsf{x}} + \underline{\mathsf{z}} + \underline{\mathsf{y}} = \underline{\mathsf{x}} + \underline{\mathsf{z}} + \underline{\mathsf{y}} \gg \underline{\mathsf{x}} + \underline{\mathsf{z}} + \underline{\mathsf{y}} = \underline{\mathsf{x}} + \underline{\mathsf{z}} + \underline{\mathsf{y}}; \text{eqTransitivity} \rhd \underline{\mathsf{x}} + \underline{\mathsf{y}} + \underline{\mathsf{z}} = \underline{\mathsf{x}} + \underline{\mathsf{z}} + \underline{\mathsf{y}} \rhd \underline{\mathsf{x}} + \underline{\mathsf{z}} + \underline{\mathsf{y}} = \underline{\mathsf{x}} + \underline{\mathsf{z}} + \underline{\mathsf{y}} \gg \underline{\mathsf{x}} + \underline{\mathsf{y}} + \underline{\mathsf{z}} = \underline{\mathsf{x}} + \underline{\mathsf{z}} + \underline{\mathsf{y}} \rceil, \text{p}_0, \mathsf{c})]$

$[\text{Three2threeTerms} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \forall \underline{\mathsf{z}} \colon \underline{\mathsf{x}} + \underline{\mathsf{y}} + \underline{\mathsf{z}} = \underline{\mathsf{x}} + \underline{\mathsf{z}} + \underline{\mathsf{y}}]$

$[\text{Three2threeTerms} \stackrel{\text{tex}}{\rightarrow}$ "Three2threeTerms"$]$

$[\text{Three2threeTerms} \stackrel{\text{pyk}}{\rightarrow}$ "lemma three2threeTerms"$]$

Three2threeFactors

$[\text{Three2threeFactors} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \forall \underline{\mathsf{z}} \colon \forall \underline{\mathsf{u}} \colon \underline{\mathsf{y}} * \underline{\mathsf{z}} = \underline{\mathsf{u}} \vdash \text{EqMultiplicationLeft} \rhd \underline{\mathsf{y}} * \underline{\mathsf{z}} = \underline{\mathsf{u}} \gg \underline{\mathsf{x}} * \underline{\mathsf{y}} * \underline{\mathsf{z}} = \underline{\mathsf{x}} * \underline{\mathsf{u}}; \text{timesAssociativity} \gg \underline{\mathsf{x}} * \underline{\mathsf{y}} * \underline{\mathsf{z}} = \underline{\mathsf{x}} * \underline{\mathsf{y}} * \underline{\mathsf{z}}; \text{eqTransitivity} \rhd \underline{\mathsf{x}} * \underline{\mathsf{y}} * \underline{\mathsf{z}} = \underline{\mathsf{x}} * \underline{\mathsf{y}} * \underline{\mathsf{z}} \rhd \underline{\mathsf{x}} * \underline{\mathsf{y}} * \underline{\mathsf{z}} = \underline{\mathsf{x}} * \underline{\mathsf{u}} \gg \underline{\mathsf{x}} * \underline{\mathsf{y}} * \underline{\mathsf{z}} = \underline{\mathsf{x}} * \underline{\mathsf{u}} \rceil, \text{p}_0, \mathsf{c})]$

$[\text{Three2threeFactors} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \forall \underline{\mathsf{z}} \colon \forall \underline{\mathsf{u}} \colon \underline{\mathsf{y}} * \underline{\mathsf{z}} = \underline{\mathsf{u}} \vdash \underline{\mathsf{x}} * \underline{\mathsf{y}} * \underline{\mathsf{z}} = \underline{\mathsf{x}} * \underline{\mathsf{u}}]$

[Three2threeFactors $\stackrel{\text{tex}}{\to}$ "Three2threeFactors"]

[Three2threeFactors $\stackrel{\text{pyk}}{\to}$ "lemma three2twoFactors"]

AddEquations

[AddEquations $\stackrel{\text{proof}}{\to}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \forall \underline{u}\colon \underline{x} = \underline{y} \vdash \underline{z} = \underline{u} \vdash \mathrm{eqAddition} \rhd \underline{x} = \underline{y} \gg \underline{x} + \underline{z} = \underline{y} + \underline{z}; \mathrm{lemma\ eqAdditionLeft} \rhd \underline{z} = \underline{u} \gg \underline{y} + \underline{z} = \underline{y} + \underline{u}; \mathrm{eqTransitivity} \rhd \underline{x} + \underline{z} = \underline{y} + \underline{z} \rhd \underline{y} + \underline{z} = \underline{y} + \underline{u} \gg \underline{x} + \underline{z} = \underline{y} + \underline{u}\rceil, p_0, c)$]

[AddEquations $\stackrel{\text{stmt}}{\to}$ $\mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \forall \underline{u}\colon \underline{x} = \underline{y} \vdash \underline{z} = \underline{u} \vdash \underline{x} + \underline{z} = \underline{y} + \underline{u}$]

[AddEquations $\stackrel{\text{tex}}{\to}$ "AddEquations"]

[AddEquations $\stackrel{\text{pyk}}{\to}$ "lemma addEquations"]

SubtractEquations

[SubtractEquations $\stackrel{\text{proof}}{\to}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \forall \underline{u}\colon \underline{x} + \underline{z} = \underline{y} + \underline{u} \vdash \underline{z} = \underline{u} \vdash \mathrm{eqAddition} \rhd \underline{x} + \underline{z} = \underline{y} + \underline{u} \gg \underline{x} + \underline{z} + -\underline{z} = \underline{y} + \underline{u} + -\underline{z}; \mathrm{plus0Left} \gg 0 + \underline{z} = \underline{z}; \mathrm{eqTransitivity} \rhd 0 + \underline{z} = \underline{z} \rhd \underline{z} = \underline{u} \gg 0 + \underline{z} = \underline{u}; \mathrm{PositiveToRight(Eq)} \rhd 0 + \underline{z} = \underline{u} \gg 0 = \underline{u} + -\underline{z}; \mathrm{eqSymmetry} \rhd 0 = \underline{u} + -\underline{z} \gg \underline{u} + -\underline{z} = 0; \mathrm{lemma\ eqAdditionLeft} \rhd \underline{u} + -\underline{z} = 0 \gg \underline{y} + \underline{u} + -\underline{z} = \underline{y} + 0; \mathrm{plusAssociativity} \gg \underline{y} + \underline{u} + -\underline{z} = \underline{y} + \underline{u} + -\underline{z}; \mathrm{plus0} \gg \underline{y} + 0 = \underline{y}; \mathrm{eqTransitivity4} \rhd \underline{y} + \underline{u} + -\underline{z} = \underline{y} + \underline{u} + -\underline{z} \rhd \underline{y} + \underline{u} + -\underline{z} = \underline{y} + 0 \rhd \underline{y} + 0 = \underline{y} \gg \underline{y} + \underline{u} + -\underline{z} = \underline{y}; \mathrm{x} = \mathrm{x} + \mathrm{y} - \mathrm{y} \gg \underline{x} = \underline{x} + \underline{z} + -\underline{z}; \mathrm{eqTransitivity4} \rhd \underline{x} = \underline{x} + \underline{z} + -\underline{z} \rhd \underline{x} + \underline{z} + -\underline{z} = \underline{y} + \underline{u} + -\underline{z} \rhd \underline{y} + \underline{u} + -\underline{z} = \underline{y} \gg \underline{x} = \underline{y}\rceil, p_0, c)$]

[SubtractEquations $\stackrel{\text{stmt}}{\to}$ $\mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \forall \underline{u}\colon \underline{x} + \underline{z} = \underline{y} + \underline{u} \vdash \underline{z} = \underline{u} \vdash \underline{x} = \underline{y}$]

[SubtractEquations $\stackrel{\text{tex}}{\to}$ "SubtractEquations"]

[SubtractEquations $\stackrel{\text{pyk}}{\to}$ "lemma subtractEquations"]

SubtractEquationsLeft

[SubtractEquationsLeft $\stackrel{\text{proof}}{\to}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \forall \underline{u}\colon \underline{x} + \underline{z} = \underline{y} + \underline{u} \vdash \underline{x} = \underline{y} \vdash \mathrm{plusCommutativity} \gg \underline{z} + \underline{x} = \underline{x} + \underline{z}; \mathrm{plusCommutativity} \gg \underline{y} + \underline{u} = \underline{u} + \underline{y}; \mathrm{eqTransitivity4} \rhd \underline{z} + \underline{x} = \underline{x} + \underline{z} \rhd \underline{x} + \underline{z} = \underline{y} + \underline{u} \rhd \underline{y} + \underline{u} = \underline{u} + \underline{y} \gg \underline{z} + \underline{x} = \underline{u} + \underline{y}; \mathrm{SubtractEquations} \rhd \underline{z} + \underline{x} = \underline{u} + \underline{y} \rhd \underline{x} = \underline{y} \gg \underline{z} = \underline{u}\rceil, p_0, c)$]

[SubtractEquationsLeft $\stackrel{\text{stmt}}{\to}$ $\mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \forall \underline{u}\colon \underline{x} + \underline{z} = \underline{y} + \underline{u} \vdash \underline{x} = \underline{y} \vdash \underline{z} = \underline{u}$]

[SubtractEquationsLeft $\stackrel{\text{tex}}{\to}$ "SubtractEquationsLeft"]

$[\text{SubtractEquationsLeft} \stackrel{\text{pyk}}{\to} \text{“lemma subtractEquationsLeft”}]$

EqNegated

$[\text{EqNegated} \stackrel{\text{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \text{Negative} \gg \underline{\mathsf{x}} + -\underline{\mathsf{x}} = 0; \text{Negative} \gg \underline{\mathsf{y}} + -\underline{\mathsf{y}} = 0; \text{eqSymmetry} \rhd \underline{\mathsf{y}} + -\underline{\mathsf{y}} = 0 \gg 0 = \underline{\mathsf{y}} + -\underline{\mathsf{y}}; \text{eqTransitivity} \rhd \underline{\mathsf{x}} + -\underline{\mathsf{x}} = 0 \rhd 0 = \underline{\mathsf{y}} + -\underline{\mathsf{y}} \gg \underline{\mathsf{x}} + -\underline{\mathsf{x}} = \underline{\mathsf{y}} + -\underline{\mathsf{y}}; \text{SubtractEquationsLeft} \rhd \underline{\mathsf{x}} + -\underline{\mathsf{x}} = \underline{\mathsf{y}} + -\underline{\mathsf{y}} \rhd \underline{\mathsf{x}} = \underline{\mathsf{y}} \gg -\underline{\mathsf{x}} = -\underline{\mathsf{y}} \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\text{EqNegated} \stackrel{\text{stmt}}{\to} \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash -\underline{\mathsf{x}} = -\underline{\mathsf{y}}]$

$[\text{EqNegated} \stackrel{\text{tex}}{\to} \text{“EqNegated”}]$

$[\text{EqNegated} \stackrel{\text{pyk}}{\to} \text{“lemma eqNegated”}]$

PositiveToRight(Eq)

$[\text{PositiveToRight(Eq)} \stackrel{\text{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \forall \underline{\mathsf{z}}\colon \underline{\mathsf{x}} + \underline{\mathsf{y}} = \underline{\mathsf{z}} \vdash \text{eqAddition} \rhd \underline{\mathsf{x}} + \underline{\mathsf{y}} = \underline{\mathsf{z}} \gg \underline{\mathsf{x}} + \underline{\mathsf{y}} + -\underline{\mathsf{y}} = \underline{\mathsf{z}} + -\underline{\mathsf{y}}; \mathrm{x} = \mathrm{x} + \mathrm{y} - \mathrm{y} \gg \underline{\mathsf{x}} = \underline{\mathsf{x}} + \underline{\mathsf{y}} + -\underline{\mathsf{y}}; \text{eqTransitivity} \rhd \underline{\mathsf{x}} = \underline{\mathsf{x}} + \underline{\mathsf{y}} + -\underline{\mathsf{y}} \rhd \underline{\mathsf{x}} + \underline{\mathsf{y}} + -\underline{\mathsf{y}} = \underline{\mathsf{z}} + -\underline{\mathsf{y}} \gg \underline{\mathsf{x}} = \underline{\mathsf{z}} + -\underline{\mathsf{y}} \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\text{PositiveToRight(Eq)} \stackrel{\text{stmt}}{\to} \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \forall \underline{\mathsf{z}}\colon \underline{\mathsf{x}} + \underline{\mathsf{y}} = \underline{\mathsf{z}} \vdash \underline{\mathsf{x}} = \underline{\mathsf{z}} + -\underline{\mathsf{y}}]$

$[\text{PositiveToRight(Eq)} \stackrel{\text{tex}}{\to} \text{“PositiveToRight(Eq)”}]$

$[\text{PositiveToRight(Eq)} \stackrel{\text{pyk}}{\to} \text{“lemma positiveToRight(Eq)”}]$

PositiveToLeft(Eq)(1term)

$[\text{PositiveToLeft(Eq)(1term)} \stackrel{\text{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \text{eqAddition} \rhd \underline{\mathsf{x}} = \underline{\mathsf{y}} \gg \underline{\mathsf{x}} + -\underline{\mathsf{y}} = \underline{\mathsf{y}} + -\underline{\mathsf{y}}; \text{Negative} \gg \underline{\mathsf{y}} + -\underline{\mathsf{y}} = 0; \text{eqTransitivity} \rhd \underline{\mathsf{x}} + -\underline{\mathsf{y}} = \underline{\mathsf{y}} + -\underline{\mathsf{y}} \rhd \underline{\mathsf{y}} + -\underline{\mathsf{y}} = 0 \gg \underline{\mathsf{x}} + -\underline{\mathsf{y}} = 0 \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\text{PositiveToLeft(Eq)(1term)} \stackrel{\text{stmt}}{\to} \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \underline{\mathsf{x}} + -\underline{\mathsf{y}} = 0]$

$[\text{PositiveToLeft(Eq)(1term)} \stackrel{\text{tex}}{\to} \text{“PositiveToLeft(Eq)(1 term)”}]$

$[\text{PositiveToLeft(Eq)(1term)} \stackrel{\text{pyk}}{\to} \text{“lemma positiveToLeft(Eq)(1 term)”}]$

NegativeToLeft(Eq)

[NegativeToLeft(Eq) $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} + -\underline{z} \vdash$ eqAddition $\rhd \underline{x} = \underline{y} + -\underline{z} \gg \underline{x} + \underline{z} = \underline{y} + -\underline{z} + \underline{z};$ Three2threeTerms $\gg$ $\underline{y} + -\underline{z} + \underline{z} = \underline{y} + \underline{z} + -\underline{z}; \mathrm{x} = \mathrm{x} + \mathrm{y} - \mathrm{y} \gg \underline{y} = \underline{y} + \underline{z} + -\underline{z};$ eqSymmetry $\rhd \underline{y} = \underline{y} + \underline{z} + -\underline{z} \gg \underline{y} + \underline{z} + -\underline{z} = \underline{y};$ eqTransitivity4 $\rhd \underline{x} + \underline{z} = \underline{y} + -\underline{z} + \underline{z} \rhd \underline{y} + -\underline{z} + \underline{z} = \underline{y} + \underline{z} + -\underline{z} \rhd \underline{y} + \underline{z} + -\underline{z} = \underline{y} \gg \underline{x} + \underline{z} = \underline{y}\rceil, \mathrm{p}_0, c)$]

[NegativeToLeft(Eq) $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} + -\underline{z} \vdash \underline{x} + \underline{z} = \underline{y}$]

[NegativeToLeft(Eq) $\stackrel{\text{tex}}{\rightarrow}$ "NegativeToLeft(Eq)"]

[NegativeToLeft(Eq) $\stackrel{\text{pyk}}{\rightarrow}$ "lemma negativeToLeft(Eq)"]

LessNeq

[LessNeq $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \vdash$ Repetition $\rhd \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \gg \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y};$ SecondConjunct $\rhd \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \gg \dot{\neg}\underline{x} = \underline{y}\rceil, \mathrm{p}_0, c)$]

[LessNeq $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \dot{\neg}\underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\underline{x} = \underline{y} \vdash \dot{\neg}\underline{x} = \underline{y}$]

[LessNeq $\stackrel{\text{tex}}{\rightarrow}$ "LessNeq"]

[LessNeq $\stackrel{\text{pyk}}{\rightarrow}$ "lemma lessNeq"]

NeqSymmetry

[NeqSymmetry $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \underline{y} = \underline{x} \vdash$ eqSymmetry $\rhd \underline{y} = \underline{x} \gg \underline{x} = \underline{y}; \forall \underline{x}\colon \forall \underline{y}\colon$ Ded $\rhd \forall \underline{x}\colon \forall \underline{y}\colon \underline{y} = \underline{x} \vdash \underline{x} = \underline{y} \gg \underline{y} = \underline{x} \Rightarrow \underline{x} = \underline{y}; \dot{\neg}\underline{x} = \underline{y} \vdash$ MT $\rhd \underline{y} = \underline{x} \Rightarrow \underline{x} = \underline{y} \rhd \dot{\neg}\underline{x} = \underline{y} \gg \dot{\neg}\underline{y} = \underline{x}\rceil, \mathrm{p}_0, c)$]

[NeqSymmetry $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \dot{\neg}\underline{x} = \underline{y} \vdash \dot{\neg}\underline{y} = \underline{x}$]

[NeqSymmetry $\stackrel{\text{tex}}{\rightarrow}$ "NeqSymmetry"]

[NeqSymmetry $\stackrel{\text{pyk}}{\rightarrow}$ "lemma neqSymmetry"]

NeqNegated

[NeqNegated $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \dot{\neg}\underline{x} = \underline{y} \vdash -\underline{x} = -\underline{y} \vdash$ EqNegated $\rhd -\underline{x} = -\underline{y} \gg - - \underline{x} = - - \underline{y};$ DoubleMinus $\gg - - \underline{x} = \underline{x};$ eqSymmetry $\rhd - - \underline{x} = \underline{x} \gg \underline{x} = - - \underline{x};$ DoubleMinus $\gg - - \underline{y} =$

$\underline{y}$; eqTransitivity4 $\rhd \underline{x} = --\underline{x} \rhd --\underline{x} = --\underline{y} \rhd --\underline{y} = \underline{y} \gg \underline{x} = \underline{y}$; FromContradiction $\rhd \underline{x} = \underline{y} \rhd \dot{\neg}\underline{x} = \underline{y} \gg \dot{\neg} -\underline{x} = -\underline{y}; \forall\underline{x}\colon \forall\underline{y}\colon$ Ded $\rhd \forall\underline{x}\colon \forall\underline{y}\colon \dot{\neg}\underline{x} = \underline{y} \vdash -\underline{x} = -\underline{y} \vdash \dot{\neg} -\underline{x} = -\underline{y} \gg \dot{\neg}\underline{x} = \underline{y} \Rightarrow -\underline{x} = -\underline{y} \Rightarrow \dot{\neg} -\underline{x} = -\underline{y}; \dot{\neg}\underline{x} = \underline{y} \vdash$ MP $\rhd \dot{\neg}\underline{x} = \underline{y} \Rightarrow -\underline{x} = -\underline{y} \Rightarrow \dot{\neg} -\underline{x} = -\underline{y} \rhd \dot{\neg}\underline{x} = \underline{y} \gg -\underline{x} = -\underline{y} \Rightarrow \dot{\neg} -\underline{x} = -\underline{y}$; ImplyNegation $\rhd -\underline{x} = -\underline{y} \Rightarrow \dot{\neg} -\underline{x} = -\underline{y} \gg \dot{\neg} -\underline{x} = -\underline{y}\rceil, \mathrm{p}_0, \mathsf{c})]$

[NeqNegated $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall\underline{x}\colon \forall\underline{y}\colon \dot{\neg}\underline{x} = \underline{y} \vdash \dot{\neg} -\underline{x} = -\underline{y}]$

[NeqNegated $\overset{\text{tex}}{\rightarrow}$ "NeqNegated"]

[NeqNegated $\overset{\text{pyk}}{\rightarrow}$ "lemma neqNegated"]

SubNeqRight

[SubNeqRight $\overset{\text{proof}}{\rightarrow}$ $\lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil$ZFsub $\vdash \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \underline{x} = \underline{y} \vdash \dot{\neg}\underline{z} = \underline{x} \vdash$ NeqSymmetry $\rhd \dot{\neg}\underline{z} = \underline{x} \gg \dot{\neg}\underline{x} = \underline{z}$; SubNeqLeft $\rhd \underline{x} = \underline{y} \rhd \dot{\neg}\underline{x} = \underline{z} \gg \dot{\neg}\underline{y} = \underline{z}$; NeqSymmetry $\rhd \dot{\neg}\underline{y} = \underline{z} \gg \dot{\neg}\underline{z} = \underline{y}\rceil, \mathrm{p}_0, \mathsf{c})]$

[SubNeqRight $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \underline{x} = \underline{y} \vdash \dot{\neg}\underline{z} = \underline{x} \vdash \dot{\neg}\underline{z} = \underline{y}]$

[SubNeqRight $\overset{\text{tex}}{\rightarrow}$ "SubNeqRight"]

[SubNeqRight $\overset{\text{pyk}}{\rightarrow}$ "lemma subNeqRight"]

SubNeqLeft

[SubNeqLeft $\overset{\text{proof}}{\rightarrow}$ $\lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil$ZFsub $\vdash \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \underline{x} = \underline{y} \vdash \dot{\neg}\underline{x} = \underline{z} \vdash$ equalityAxiom $\gg \underline{y} = \underline{x} \Rightarrow \underline{y} = \underline{z} \Rightarrow \underline{x} = \underline{z}$; eqSymmetry $\rhd \underline{x} = \underline{y} \gg \underline{y} = \underline{x}$; MP $\rhd \underline{y} = \underline{x} \Rightarrow \underline{y} = \underline{z} \Rightarrow \underline{x} = \underline{z} \rhd \underline{y} = \underline{x} \gg \underline{y} = \underline{z} \Rightarrow \underline{x} = \underline{z}$; Contrapositive $\rhd \underline{y} = \underline{z} \Rightarrow \underline{x} = \underline{z} \gg \dot{\neg}\underline{x} = \underline{z} \Rightarrow \dot{\neg}\underline{y} = \underline{z}$; MP $\rhd \dot{\neg}\underline{x} = \underline{z} \Rightarrow \dot{\neg}\underline{y} = \underline{z} \rhd \dot{\neg}\underline{x} = \underline{z} \gg \dot{\neg}\underline{y} = \underline{z}\rceil, \mathrm{p}_0, \mathsf{c})]$

[SubNeqLeft $\overset{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \underline{x} = \underline{y} \vdash \dot{\neg}\underline{x} = \underline{z} \vdash \dot{\neg}\underline{y} = \underline{z}]$

[SubNeqLeft $\overset{\text{tex}}{\rightarrow}$ "SubNeqLeft"]

[SubNeqLeft $\overset{\text{pyk}}{\rightarrow}$ "lemma subNeqLeft"]

NeqAddition

[NeqAddition $\overset{\text{proof}}{\rightarrow}$ $\lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil$ZFsub $\vdash \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \dot{\neg}\underline{x} = \underline{y} \vdash \underline{x} + \underline{z} = \underline{y} + \underline{z} \vdash$ eqReflexivity $\gg \underline{z} = \underline{z}$; SubtractEquations $\rhd \underline{x} + \underline{z} = \underline{y} + \underline{z} \rhd \underline{z} = \underline{z} \gg \underline{x} = \underline{y}$; FromContradiction $\rhd \underline{x} = \underline{y} \rhd \dot{\neg}\underline{x} = \underline{y} \gg \dot{\neg}\underline{x} + \underline{z} = \underline{y} + \underline{z}; \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon$ Ded $\rhd \forall\underline{x}\colon \forall\underline{y}\colon \forall\underline{z}\colon \dot{\neg}\underline{x} = \underline{y} \vdash \underline{x} + \underline{z} = \underline{y} + \underline{z} \vdash \dot{\neg}\underline{x} + \underline{z} = \underline{y} + \underline{z} \gg \dot{\neg}\underline{x} = \underline{y} \Rightarrow \underline{x} + \underline{z} = \underline{y} + \underline{z} \Rightarrow \dot{\neg}\underline{x} + \underline{z} = \underline{y} + \underline{z}; \dot{\neg}\underline{x} = \underline{y} \vdash$ MP $\rhd \dot{\neg}\underline{x} = \underline{y} \Rightarrow \underline{x} + \underline{z} =$

$\underline{y} + \underline{z} \Rightarrow \dot\neg \underline{x} + \underline{z} = \underline{y} + \underline{z} \rhd \dot\neg \underline{x} = \underline{y} \gg \underline{x} + \underline{z} = \underline{y} + \underline{z} \Rightarrow \dot\neg \underline{x} + \underline{z} = \underline{y} + \underline{z}; \mathrm{ImplyNegation} \rhd \underline{x} + \underline{z} = \underline{y} + \underline{z} \Rightarrow \dot\neg \underline{x} + \underline{z} = \underline{y} + \underline{z} \gg \dot\neg \underline{x} + \underline{z} = \underline{y} + \underline{z}\rceil, p_0, \mathsf{c})]$

$[\mathrm{NeqAddition} \stackrel{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{x} \colon \forall \underline{y} \colon \forall \underline{z} \colon \dot\neg \underline{x} = \underline{y} \vdash \dot\neg \underline{x} + \underline{z} = \underline{y} + \underline{z}]$

$[\mathrm{NeqAddition} \stackrel{\mathrm{tex}}{\to}$ "NeqAddition"$]$

$[\mathrm{NeqAddition} \stackrel{\mathrm{pyk}}{\to}$ "lemma neqAddition"$]$

NeqMultiplication

$[\mathrm{NeqMultiplication} \stackrel{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x} \colon \forall \underline{y} \colon \forall \underline{z} \colon \dot\neg \underline{z} = 0 \vdash \dot\neg \underline{x} = \underline{y} \vdash \underline{x} * \underline{z} = \underline{y} * \underline{z} \vdash \rhd \dot\neg \underline{z} = 0 \gg \underline{x} = \underline{x} * \underline{z} * \mathrm{rec}\underline{z}; \mathrm{eqMultiplication} \rhd \underline{x} * \underline{z} = \underline{y} * \underline{z} \gg \underline{x} * \underline{z} * \mathrm{rec}\underline{z} = \underline{y} * \underline{z} * \mathrm{rec}\underline{z}; \rhd \dot\neg \underline{z} = 0 \gg \underline{y} = \underline{y} * \underline{z} * \mathrm{rec}\underline{z}; \mathrm{eqSymmetry} \rhd \underline{y} = \underline{y} * \underline{z} * \mathrm{rec}\underline{z} \gg \underline{y} * \underline{z} * \mathrm{rec}\underline{z} = \underline{y}; \mathrm{eqTransitivity4} \rhd \underline{x} = \underline{x} * \underline{z} * \mathrm{rec}\underline{z} \rhd \underline{x} * \underline{z} * \mathrm{rec}\underline{z} = \underline{y} * \underline{z} * \mathrm{rec}\underline{z} \rhd \underline{y} * \underline{z} * \mathrm{rec}\underline{z} = \underline{y} \gg \underline{x} = \underline{y}; \mathrm{FromContradiction} \rhd \underline{x} = \underline{y} \rhd \dot\neg \underline{x} = \underline{y} \gg \dot\neg \underline{x} * \underline{z} = \underline{y} * \underline{z}; \forall \underline{x} \colon \forall \underline{y} \colon \forall \underline{z} \colon \mathrm{Ded} \rhd \forall \underline{x} \colon \forall \underline{y} \colon \forall \underline{z} \colon \dot\neg \underline{z} = 0 \vdash \dot\neg \underline{x} = \underline{y} \vdash \underline{x} * \underline{z} = \underline{y} * \underline{z} \vdash \dot\neg \underline{x} * \underline{z} = \underline{y} * \underline{z} \gg \dot\neg \underline{z} = 0 \Rightarrow \dot\neg \underline{x} = \underline{y} \Rightarrow \underline{x} * \underline{z} = \underline{y} * \underline{z} \Rightarrow \dot\neg \underline{x} * \underline{z} = \underline{y} * \underline{z}; \dot\neg \underline{z} = 0 \vdash \dot\neg \underline{x} = \underline{y} \vdash \mathrm{MP2} \rhd \dot\neg \underline{z} = 0 \Rightarrow \dot\neg \underline{x} = \underline{y} \Rightarrow \underline{x} * \underline{z} = \underline{y} * \underline{z} \Rightarrow \dot\neg \underline{x} * \underline{z} = \underline{y} * \underline{z} \rhd \dot\neg \underline{z} = 0 \rhd \dot\neg \underline{x} = \underline{y} \gg \underline{x} * \underline{z} = \underline{y} * \underline{z} \Rightarrow \dot\neg \underline{x} * \underline{z} = \underline{y} * \underline{z}; \mathrm{ImplyNegation} \rhd \underline{x} * \underline{z} = \underline{y} * \underline{z} \Rightarrow \dot\neg \underline{x} * \underline{z} = \underline{y} * \underline{z} \gg \dot\neg \underline{x} * \underline{z} = \underline{y} * \underline{z}\rceil, p_0, \mathsf{c})]$

$[\mathrm{NeqMultiplication} \stackrel{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{x} \colon \forall \underline{y} \colon \forall \underline{z} \colon \dot\neg \underline{z} = 0 \vdash \dot\neg \underline{x} = \underline{y} \vdash \dot\neg \underline{x} * \underline{z} = \underline{y} * \underline{z}]$

$[\mathrm{NeqMultiplication} \stackrel{\mathrm{tex}}{\to}$ "NeqMultiplication"$]$

$[\mathrm{NeqMultiplication} \stackrel{\mathrm{pyk}}{\to}$ "lemma neqMultiplication"$]$

UniqueNegative

$[\mathrm{UniqueNegative} \stackrel{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x} \colon \forall \underline{y} \colon \forall \underline{z} \colon \underline{x} + \underline{y} = 0 \vdash \underline{x} + \underline{z} = 0 \vdash \mathrm{plusCommutativity} \gg \underline{y} + \underline{x} = \underline{x} + \underline{y}; \mathrm{eqTransitivity} \rhd \underline{y} + \underline{x} = \underline{x} + \underline{y} \rhd \underline{x} + \underline{y} = 0 \gg \underline{y} + \underline{x} = 0; \mathrm{PositiveToRight(Eq)} \rhd \underline{y} + \underline{x} = 0 \gg \underline{y} = 0 + -\underline{x}; \mathrm{plusCommutativity} \gg \underline{z} + \underline{x} = \underline{x} + \underline{z}; \mathrm{eqTransitivity} \rhd \underline{z} + \underline{x} = \underline{x} + \underline{z} \rhd \underline{x} + \underline{z} = 0 \gg \underline{z} + \underline{x} = 0; \mathrm{PositiveToRight(Eq)} \rhd \underline{z} + \underline{x} = 0 \gg \underline{z} = 0 + -\underline{x}; \mathrm{eqSymmetry} \rhd \underline{z} = 0 + -\underline{x} \gg 0 + -\underline{x} = \underline{z}; \mathrm{eqTransitivity} \rhd \underline{y} = 0 + -\underline{x} \rhd 0 + -\underline{x} = \underline{z} \gg \underline{y} = \underline{z}\rceil, p_0, \mathsf{c})]$

$[\mathrm{UniqueNegative} \stackrel{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{x} \colon \forall \underline{y} \colon \forall \underline{z} \colon \underline{x} + \underline{y} = 0 \vdash \underline{x} + \underline{z} = 0 \vdash \underline{y} = \underline{z}]$

$[\mathrm{UniqueNegative} \stackrel{\mathrm{tex}}{\to}$ "UniqueNegative"$]$

$[\mathrm{UniqueNegative} \stackrel{\mathrm{pyk}}{\to}$ "lemma uniqueNegative"$]$

DoubleMinus

[DoubleMinus $\stackrel{\text{proof}}{\to} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall\underline{\mathsf{x}}\colon \mathrm{Negative} \gg -\underline{\mathsf{x}} + - - \underline{\mathsf{x}} = 0; \mathrm{x} + \mathrm{y} = \mathrm{zBackwards} \rhd -\underline{\mathsf{x}} + - - \underline{\mathsf{x}} = 0 \gg 0 = - - \underline{\mathsf{x}} + -\underline{\mathsf{x}}; \mathrm{NegativeToLeft(Eq)} \rhd 0 = - - \underline{\mathsf{x}} + -\underline{\mathsf{x}} \gg 0 + \underline{\mathsf{x}} = - - \underline{\mathsf{x}}; \mathrm{plus0Left} \gg 0 + \underline{\mathsf{x}} = \underline{\mathsf{x}}; \mathrm{Equality} \rhd 0 + \underline{\mathsf{x}} = - - \underline{\mathsf{x}} \rhd 0 + \underline{\mathsf{x}} = \underline{\mathsf{x}} \gg - - \underline{\mathsf{x}} = \underline{\mathsf{x}}\rceil, \mathrm{p}_0, \mathsf{c})$]

[DoubleMinus $\stackrel{\text{stmt}}{\to} \mathrm{ZFsub} \vdash \forall\underline{\mathsf{x}}\colon - - \underline{\mathsf{x}} = \underline{\mathsf{x}}$]

[DoubleMinus $\stackrel{\text{tex}}{\to}$ “DoubleMinus”]

[DoubleMinus $\stackrel{\text{pyk}}{\to}$ “lemma doubleMinus”]

LeqLessEq

[LeqLessEq $\stackrel{\text{proof}}{\to} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall\underline{\mathsf{x}}\colon \forall\underline{\mathsf{y}}\colon \underline{\mathsf{x}} <= \underline{\mathsf{y}} \vdash \dot{\neg}\dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \mathrm{fromNotLess} \rhd \dot{\neg}\dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}} \gg \underline{\mathsf{y}} <= \underline{\mathsf{x}}; \mathrm{leqAntisymmetry} \rhd \underline{\mathsf{x}} <= \underline{\mathsf{y}} \rhd \underline{\mathsf{y}} <= \underline{\mathsf{x}} \gg \underline{\mathsf{x}} = \underline{\mathsf{y}}; \forall\underline{\mathsf{x}}\colon \forall\underline{\mathsf{y}}\colon \mathrm{Ded} \rhd \forall\underline{\mathsf{x}}\colon \forall\underline{\mathsf{y}}\colon \underline{\mathsf{x}} <= \underline{\mathsf{y}} \vdash \dot{\neg}\dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \underline{\mathsf{x}} = \underline{\mathsf{y}} \gg \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{x}} = \underline{\mathsf{y}}; \underline{\mathsf{x}} <= \underline{\mathsf{y}} \vdash \mathrm{MP} \rhd \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{x}} = \underline{\mathsf{y}} \rhd \underline{\mathsf{x}} <= \underline{\mathsf{y}} \gg \dot{\neg}\dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{x}} = \underline{\mathsf{y}}; \mathrm{Repetition} \rhd \dot{\neg}\dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{x}} = \underline{\mathsf{y}} \gg \dot{\neg}\dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{x}} = \underline{\mathsf{y}}\rceil, \mathrm{p}_0, \mathsf{c})$]

[LeqLessEq $\stackrel{\text{stmt}}{\to} \mathrm{ZFsub} \vdash \forall\underline{\mathsf{x}}\colon \forall\underline{\mathsf{y}}\colon \underline{\mathsf{x}} <= \underline{\mathsf{y}} \vdash \dot{\neg}\dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{x}} = \underline{\mathsf{y}}$]

[LeqLessEq $\stackrel{\text{tex}}{\to}$ “LeqLessEq”]

[LeqLessEq $\stackrel{\text{pyk}}{\to}$ “lemma leqLessEq”]

LessLeq

[LessLeq $\stackrel{\text{proof}}{\to} \lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall\underline{\mathsf{x}}\colon \forall\underline{\mathsf{y}}\colon \dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \mathrm{Repetition} \rhd \dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}} \gg \dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}}; \mathrm{FirstConjunct} \rhd \dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}} \gg \underline{\mathsf{x}} <= \underline{\mathsf{y}}\rceil, \mathrm{p}_0, \mathsf{c})$]

[LessLeq $\stackrel{\text{stmt}}{\to} \mathrm{ZFsub} \vdash \forall\underline{\mathsf{x}}\colon \forall\underline{\mathsf{y}}\colon \dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \underline{\mathsf{x}} <= \underline{\mathsf{y}}$]

[LessLeq $\stackrel{\text{tex}}{\to}$ “LessLeq”]

[LessLeq $\stackrel{\text{pyk}}{\to}$ “lemma lessLeq”]

FromLeqGeq

$[\mathrm{FromLeqGeq} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{a}} \vdash \underline{\mathsf{y}} <= \underline{\mathsf{x}} \Rightarrow \underline{\mathsf{a}} \vdash \mathrm{leqTotality} \gg \dot{\neg}\, \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{y}} <= \underline{\mathsf{x}}; \mathrm{FromDisjuncts} \rhd \dot{\neg}\, \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{y}} <= \underline{\mathsf{x}} \rhd \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{a}} \rhd \underline{\mathsf{y}} <= \underline{\mathsf{x}} \Rightarrow \underline{\mathsf{a}} \gg \underline{\mathsf{a}} \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{FromLeqGeq} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{a}} \vdash \underline{\mathsf{y}} <= \underline{\mathsf{x}} \Rightarrow \underline{\mathsf{a}} \vdash \underline{\mathsf{a}}]$

$[\mathrm{FromLeqGeq} \stackrel{\mathrm{tex}}{\rightarrow}$ "FromLeqGeq"$]$

$[\mathrm{FromLeqGeq} \stackrel{\mathrm{pyk}}{\rightarrow}$ "lemma from leqGeq"$]$

subLeqRight

$[\mathrm{subLeqRight} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \forall \underline{\mathsf{z}}\colon \underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \underline{\mathsf{z}} <= \underline{\mathsf{x}} \vdash \mathrm{eqLeq} \rhd \underline{\mathsf{x}} = \underline{\mathsf{y}} \gg \underline{\mathsf{x}} <= \underline{\mathsf{y}}; \mathrm{leqTransitivity} \rhd \underline{\mathsf{z}} <= \underline{\mathsf{x}} \rhd \underline{\mathsf{x}} <= \underline{\mathsf{y}} \gg \underline{\mathsf{z}} <= \underline{\mathsf{y}} \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{subLeqRight} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \forall \underline{\mathsf{z}}\colon \underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \underline{\mathsf{z}} <= \underline{\mathsf{x}} \vdash \underline{\mathsf{z}} <= \underline{\mathsf{y}}]$

$[\mathrm{subLeqRight} \stackrel{\mathrm{tex}}{\rightarrow}$ "subLeqRight"$]$

$[\mathrm{subLeqRight} \stackrel{\mathrm{pyk}}{\rightarrow}$ "lemma subLeqRight"$]$

subLeqLeft

$[\mathrm{subLeqLeft} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \forall \underline{\mathsf{z}}\colon \underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \underline{\mathsf{x}} <= \underline{\mathsf{z}} \vdash \mathrm{eqSymmetry} \rhd \underline{\mathsf{x}} = \underline{\mathsf{y}} \gg \underline{\mathsf{y}} = \underline{\mathsf{x}}; \mathrm{eqLeq} \rhd \underline{\mathsf{y}} = \underline{\mathsf{x}} \gg \underline{\mathsf{y}} <= \underline{\mathsf{x}}; \mathrm{leqTransitivity} \rhd \underline{\mathsf{y}} <= \underline{\mathsf{x}} \rhd \underline{\mathsf{x}} <= \underline{\mathsf{z}} \gg \underline{\mathsf{y}} <= \underline{\mathsf{z}} \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{subLeqLeft} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \forall \underline{\mathsf{z}}\colon \underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \underline{\mathsf{x}} <= \underline{\mathsf{z}} \vdash \underline{\mathsf{y}} <= \underline{\mathsf{z}}]$

$[\mathrm{subLeqLeft} \stackrel{\mathrm{tex}}{\rightarrow}$ "subLeqLeft"$]$

$[\mathrm{subLeqLeft} \stackrel{\mathrm{pyk}}{\rightarrow}$ "lemma subLeqLeft"$]$

Leq + 1

$[\mathrm{Leq} + 1 \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \underline{\mathsf{x}} <= \underline{\mathsf{y}} \vdash 0 < 1 \gg \dot{\neg}\, 0 <= 1 \Rightarrow \dot{\neg}\, \dot{\neg}\, 0 = 1; \mathrm{LessAdditionLeft} \rhd \dot{\neg}\, 0 <= 1 \Rightarrow \dot{\neg}\, \dot{\neg}\, 0 = 1 \gg \dot{\neg}\, \underline{\mathsf{y}} + 0 <= \underline{\mathsf{y}} + 1 \Rightarrow \dot{\neg}\, \dot{\neg}\, \underline{\mathsf{y}} + 0 = \underline{\mathsf{y}} + 1; \mathrm{plus0} \gg \underline{\mathsf{y}} + 0 = \underline{\mathsf{y}}; \mathrm{SubLessLeft} \rhd \underline{\mathsf{y}} + 0 = \underline{\mathsf{y}} \rhd \dot{\neg}\, \underline{\mathsf{y}} + 0 <= \underline{\mathsf{y}} + 1 \Rightarrow \dot{\neg}\, \dot{\neg}\, \underline{\mathsf{y}} + 0 = \underline{\mathsf{y}} + 1 \gg \dot{\neg}\, \underline{\mathsf{y}} <= \underline{\mathsf{y}} + 1 \Rightarrow \dot{\neg}\, \dot{\neg}\, \underline{\mathsf{y}} = \underline{\mathsf{y}} + 1; \mathrm{leqLessTransitivity} \rhd \underline{\mathsf{x}} <= \underline{\mathsf{y}} \rhd \dot{\neg}\, \underline{\mathsf{y}} <= \underline{\mathsf{y}} + 1 \Rightarrow \dot{\neg}\, \dot{\neg}\, \underline{\mathsf{y}} = \underline{\mathsf{y}} + 1 \gg \dot{\neg}\, \underline{\mathsf{x}} <= \underline{\mathsf{y}} + 1 \Rightarrow \dot{\neg}\, \dot{\neg}\, \underline{\mathsf{x}} = \underline{\mathsf{y}} + 1 \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{Leq} + 1 \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \underline{\mathsf{x}} <= \underline{\mathsf{y}} \vdash \dot{\neg}\, \underline{\mathsf{x}} <= \underline{\mathsf{y}} + 1 \Rightarrow \dot{\neg}\, \dot{\neg}\, \underline{\mathsf{x}} = \underline{\mathsf{y}} + 1]$

[Leq + 1 $\stackrel{\text{tex}}{\to}$ "Leq+1"]

[Leq + 1 $\stackrel{\text{pyk}}{\to}$ "lemma leqPlus1"]

PositiveToRight(Leq)

[PositiveToRight(Leq) $\stackrel{\text{proof}}{\to}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \forall \underline{\mathsf{z}} \colon \underline{\mathsf{x}} + \underline{\mathsf{y}} <= \underline{\mathsf{z}} \vdash \mathrm{leqAddition} \rhd \underline{\mathsf{x}} + \underline{\mathsf{y}} <= \underline{\mathsf{z}} \gg \underline{\mathsf{x}} + \underline{\mathsf{y}} + -\underline{\mathsf{y}} <= \underline{\mathsf{z}} + -\underline{\mathsf{y}}; \mathrm{x} = \mathrm{x} + \mathrm{y} - \mathrm{y} \gg \underline{\mathsf{x}} = \underline{\mathsf{x}} + \underline{\mathsf{y}} + -\underline{\mathsf{y}}; \mathrm{eqSymmetry} \rhd \underline{\mathsf{x}} = \underline{\mathsf{x}} + \underline{\mathsf{y}} + -\underline{\mathsf{y}} \gg \underline{\mathsf{x}} + \underline{\mathsf{y}} + -\underline{\mathsf{y}} = \underline{\mathsf{x}}; \mathrm{subLeqLeft} \rhd \underline{\mathsf{x}} + \underline{\mathsf{y}} + -\underline{\mathsf{y}} = \underline{\mathsf{x}} \rhd \underline{\mathsf{x}} + \underline{\mathsf{y}} + -\underline{\mathsf{y}} <= \underline{\mathsf{z}} + -\underline{\mathsf{y}} \gg \underline{\mathsf{x}} <= \underline{\mathsf{z}} + -\underline{\mathsf{y}}\rceil, \mathrm{p}_0, \mathsf{c})$]

[PositiveToRight(Leq) $\stackrel{\text{stmt}}{\to}$ $\mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \forall \underline{\mathsf{z}} \colon \underline{\mathsf{x}} + \underline{\mathsf{y}} <= \underline{\mathsf{z}} \vdash \underline{\mathsf{x}} <= \underline{\mathsf{z}} + -\underline{\mathsf{y}}$]

[PositiveToRight(Leq) $\stackrel{\text{tex}}{\to}$ "PositiveToRight(Leq)"]

[PositiveToRight(Leq) $\stackrel{\text{pyk}}{\to}$ "lemma positiveToRight(Leq)"]

PositiveToRight(Leq)(1term)

[PositiveToRight(Leq)(1term) $\stackrel{\text{proof}}{\to}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{y}} \colon \forall \underline{\mathsf{z}} \colon \underline{\mathsf{y}} <= \underline{\mathsf{z}} \vdash \mathrm{plus0Left} \gg 0 + \underline{\mathsf{y}} = \underline{\mathsf{y}}; \mathrm{eqSymmetry} \rhd 0 + \underline{\mathsf{y}} = \underline{\mathsf{y}} \gg \underline{\mathsf{y}} = 0 + \underline{\mathsf{y}}; \mathrm{subLeqLeft} \rhd \underline{\mathsf{y}} = 0 + \underline{\mathsf{y}} \rhd \underline{\mathsf{y}} <= \underline{\mathsf{z}} \gg 0 + \underline{\mathsf{y}} <= \underline{\mathsf{z}}; \mathrm{PositiveToRight(Leq)} \rhd 0 + \underline{\mathsf{y}} <= \underline{\mathsf{z}} \gg 0 <= \underline{\mathsf{z}} + -\underline{\mathsf{y}}\rceil, \mathrm{p}_0, \mathsf{c})$]

[PositiveToRight(Leq)(1term) $\stackrel{\text{stmt}}{\to}$ $\mathrm{ZFsub} \vdash \forall \underline{\mathsf{y}} \colon \forall \underline{\mathsf{z}} \colon \underline{\mathsf{y}} <= \underline{\mathsf{z}} \vdash 0 <= \underline{\mathsf{z}} + -\underline{\mathsf{y}}$]

[PositiveToRight(Leq)(1term) $\stackrel{\text{tex}}{\to}$ "PositiveToRight(Leq)(1 term)"]

[PositiveToRight(Leq)(1term) $\stackrel{\text{pyk}}{\to}$ "lemma positiveToRight(Leq)(1 term)"]

negativeToLeft(Leq)

[negativeToLeft(Leq) $\stackrel{\text{proof}}{\to}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \forall \underline{\mathsf{z}} \colon \underline{\mathsf{x}} <= \underline{\mathsf{y}} + -\underline{\mathsf{z}} \vdash \mathrm{leqAddition} \rhd \underline{\mathsf{x}} <= \underline{\mathsf{y}} + -\underline{\mathsf{z}} \gg \underline{\mathsf{x}} + \underline{\mathsf{z}} <= \underline{\mathsf{y}} + -\underline{\mathsf{z}} + \underline{\mathsf{z}}; \mathrm{x} = \mathrm{x} + \mathrm{y} - \mathrm{y} \gg \underline{\mathsf{y}} = \underline{\mathsf{y}} + \underline{\mathsf{z}} + -\underline{\mathsf{z}}; \mathrm{Three2threeTerms} \gg \underline{\mathsf{y}} + \underline{\mathsf{z}} + -\underline{\mathsf{z}} = \underline{\mathsf{y}} + -\underline{\mathsf{z}} + \underline{\mathsf{z}}; \mathrm{eqTransitivity} \rhd \underline{\mathsf{y}} = \underline{\mathsf{y}} + \underline{\mathsf{z}} + -\underline{\mathsf{z}} \rhd \underline{\mathsf{y}} + \underline{\mathsf{z}} + -\underline{\mathsf{z}} = \underline{\mathsf{y}} + -\underline{\mathsf{z}} + \underline{\mathsf{z}} \gg \underline{\mathsf{y}} = \underline{\mathsf{y}} + -\underline{\mathsf{z}} + \underline{\mathsf{z}}; \mathrm{eqSymmetry} \rhd \underline{\mathsf{y}} = \underline{\mathsf{y}} + -\underline{\mathsf{z}} + \underline{\mathsf{z}} \gg \underline{\mathsf{y}} + -\underline{\mathsf{z}} + \underline{\mathsf{z}} = \underline{\mathsf{y}}; \mathrm{subLeqRight} \rhd \underline{\mathsf{y}} + -\underline{\mathsf{z}} + \underline{\mathsf{z}} = \underline{\mathsf{y}} \rhd \underline{\mathsf{x}} + \underline{\mathsf{z}} <= \underline{\mathsf{y}} + -\underline{\mathsf{z}} + \underline{\mathsf{z}} \gg \underline{\mathsf{x}} + \underline{\mathsf{z}} <= \underline{\mathsf{y}}\rceil, \mathrm{p}_0, \mathsf{c})$]

[negativeToLeft(Leq) $\stackrel{\text{stmt}}{\to}$ $\mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \forall \underline{\mathsf{z}} \colon \underline{\mathsf{x}} <= \underline{\mathsf{y}} + -\underline{\mathsf{z}} \vdash \underline{\mathsf{x}} + \underline{\mathsf{z}} <= \underline{\mathsf{y}}$]

[negativeToLeft(Leq) $\stackrel{\text{tex}}{\to}$ "negativeToLeft(Leq)"]

[negativeToLeft(Leq) $\stackrel{\text{pyk}}{\to}$ "lemma negativeToLeft(Leq)"]

LeqAdditionLeft

[LeqAdditionLeft $\stackrel{\text{proof}}{\rightarrow}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \forall \underline{\mathsf{z}}\colon \underline{\mathsf{x}} <= \underline{\mathsf{y}} \vdash \text{leqAddition} \rhd \underline{\mathsf{x}} <= \underline{\mathsf{y}} \gg \underline{\mathsf{x}} + \underline{\mathsf{z}} <= \underline{\mathsf{y}} + \underline{\mathsf{z}}; \text{plusCommutativity} \gg \underline{\mathsf{x}} + \underline{\mathsf{z}} = \underline{\mathsf{z}} + \underline{\mathsf{x}}; \text{plusCommutativity} \gg \underline{\mathsf{y}}+\underline{\mathsf{z}} = \underline{\mathsf{z}}+\underline{\mathsf{y}}; \text{subLeqLeft} \rhd \underline{\mathsf{x}}+\underline{\mathsf{z}} = \underline{\mathsf{z}}+\underline{\mathsf{x}} \rhd \underline{\mathsf{x}}+\underline{\mathsf{z}} <= \underline{\mathsf{y}}+\underline{\mathsf{z}} \gg \underline{\mathsf{z}}+\underline{\mathsf{x}} <= \underline{\mathsf{y}}+\underline{\mathsf{z}}; \text{subLeqRight} \rhd \underline{\mathsf{y}}+\underline{\mathsf{z}} = \underline{\mathsf{z}}+\underline{\mathsf{y}} \rhd \underline{\mathsf{z}}+\underline{\mathsf{x}} <= \underline{\mathsf{y}}+\underline{\mathsf{z}} \gg \underline{\mathsf{z}}+\underline{\mathsf{x}} <= \underline{\mathsf{z}}+\underline{\mathsf{y}} \rceil, \mathrm{p}_0, \mathsf{c})$]

[LeqAdditionLeft $\stackrel{\text{stmt}}{\rightarrow}$ $\text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \forall \underline{\mathsf{z}}\colon \underline{\mathsf{x}} <= \underline{\mathsf{y}} \vdash \underline{\mathsf{z}} + \underline{\mathsf{x}} <= \underline{\mathsf{z}} + \underline{\mathsf{y}}$]

[LeqAdditionLeft $\stackrel{\text{tex}}{\rightarrow}$ "LeqAdditionLeft"]

[LeqAdditionLeft $\stackrel{\text{pyk}}{\rightarrow}$ "lemma leqAdditionLeft"]

leqSubtraction

[leqSubtraction $\stackrel{\text{proof}}{\rightarrow}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \forall \underline{\mathsf{z}}\colon \underline{\mathsf{x}} + \underline{\mathsf{z}} <= \underline{\mathsf{y}} + \underline{\mathsf{z}} \vdash \text{leqAddition} \rhd \underline{\mathsf{x}} + \underline{\mathsf{z}} <= \underline{\mathsf{y}} + \underline{\mathsf{z}} \gg \underline{\mathsf{x}} + \underline{\mathsf{z}} + -\underline{\mathsf{z}} <= \underline{\mathsf{y}} + \underline{\mathsf{z}} + -\underline{\mathsf{z}}; \mathrm{x} = \mathrm{x} + \mathrm{y} - \mathrm{y} \gg \underline{\mathsf{x}} = \underline{\mathsf{x}} + \underline{\mathsf{z}} + -\underline{\mathsf{z}}; \text{eqSymmetry} \rhd \underline{\mathsf{x}} = \underline{\mathsf{x}} + \underline{\mathsf{z}} + -\underline{\mathsf{z}} \gg \underline{\mathsf{x}} + \underline{\mathsf{z}} + -\underline{\mathsf{z}} = \underline{\mathsf{x}}; \mathrm{x} = \mathrm{x} + \mathrm{y} - \mathrm{y} \gg \underline{\mathsf{y}} = \underline{\mathsf{y}} + \underline{\mathsf{z}} + -\underline{\mathsf{z}}; \text{eqSymmetry} \rhd \underline{\mathsf{y}} = \underline{\mathsf{y}} + \underline{\mathsf{z}} + -\underline{\mathsf{z}} \gg \underline{\mathsf{y}} + \underline{\mathsf{z}} + -\underline{\mathsf{z}} = \underline{\mathsf{y}}; \text{subLeqLeft} \rhd \underline{\mathsf{x}} + \underline{\mathsf{z}} + -\underline{\mathsf{z}} = \underline{\mathsf{x}} \rhd \underline{\mathsf{x}} + \underline{\mathsf{z}} + -\underline{\mathsf{z}} <= \underline{\mathsf{y}} + \underline{\mathsf{z}} + -\underline{\mathsf{z}} \gg \underline{\mathsf{x}} <= \underline{\mathsf{y}} + \underline{\mathsf{z}} + -\underline{\mathsf{z}}; \text{subLeqRight} \rhd \underline{\mathsf{y}} + \underline{\mathsf{z}} + -\underline{\mathsf{z}} = \underline{\mathsf{y}} \rhd \underline{\mathsf{x}} <= \underline{\mathsf{y}} + \underline{\mathsf{z}} + -\underline{\mathsf{z}} \gg \underline{\mathsf{x}} <= \underline{\mathsf{y}} \rceil, \mathrm{p}_0, \mathsf{c})$]

[leqSubtraction $\stackrel{\text{stmt}}{\rightarrow}$ $\text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \forall \underline{\mathsf{z}}\colon \underline{\mathsf{x}} + \underline{\mathsf{z}} <= \underline{\mathsf{y}} + \underline{\mathsf{z}} \vdash \underline{\mathsf{x}} <= \underline{\mathsf{y}}$]

[leqSubtraction $\stackrel{\text{tex}}{\rightarrow}$ "leqSubtraction"]

[leqSubtraction $\stackrel{\text{pyk}}{\rightarrow}$ "lemma leqSubtraction"]

leqSubtractionLeft

[leqSubtractionLeft $\stackrel{\text{proof}}{\rightarrow}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \forall \underline{\mathsf{z}}\colon \underline{\mathsf{z}} + \underline{\mathsf{x}} <= \underline{\mathsf{z}} + \underline{\mathsf{y}} \vdash \text{plusCommutativity} \gg \underline{\mathsf{z}} + \underline{\mathsf{x}} = \underline{\mathsf{x}} + \underline{\mathsf{z}}; \text{plusCommutativity} \gg \underline{\mathsf{z}} + \underline{\mathsf{y}} = \underline{\mathsf{y}} + \underline{\mathsf{z}}; \text{subLeqLeft} \rhd \underline{\mathsf{z}} + \underline{\mathsf{x}} = \underline{\mathsf{x}} + \underline{\mathsf{z}} \rhd \underline{\mathsf{z}} + \underline{\mathsf{x}} <= \underline{\mathsf{z}} + \underline{\mathsf{y}} \gg \underline{\mathsf{x}} + \underline{\mathsf{z}} <= \underline{\mathsf{z}} + \underline{\mathsf{y}}; \text{subLeqRight} \rhd \underline{\mathsf{z}} + \underline{\mathsf{y}} = \underline{\mathsf{y}} + \underline{\mathsf{z}} \rhd \underline{\mathsf{x}} + \underline{\mathsf{z}} <= \underline{\mathsf{z}} + \underline{\mathsf{y}} \gg \underline{\mathsf{x}} + \underline{\mathsf{z}} <= \underline{\mathsf{y}} + \underline{\mathsf{z}}; \text{leqSubtraction} \rhd \underline{\mathsf{x}} + \underline{\mathsf{z}} <= \underline{\mathsf{y}} + \underline{\mathsf{z}} \gg \underline{\mathsf{x}} <= \underline{\mathsf{y}} \rceil, \mathrm{p}_0, \mathsf{c})$]

[leqSubtractionLeft $\stackrel{\text{stmt}}{\rightarrow}$ $\text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \forall \underline{\mathsf{z}}\colon \underline{\mathsf{z}} + \underline{\mathsf{x}} <= \underline{\mathsf{z}} + \underline{\mathsf{y}} \vdash \underline{\mathsf{x}} <= \underline{\mathsf{y}}$]

[leqSubtractionLeft $\stackrel{\text{tex}}{\rightarrow}$ "leqSubtractionLeft"]

[leqSubtractionLeft $\stackrel{\text{pyk}}{\rightarrow}$ "lemma leqSubtractionLeft"]

thirdGeq

$[\mathrm{thirdGeq} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \underline{\mathsf{x}} <= \underline{\mathsf{y}} \vdash \mathrm{leqReflexivity} \gg \underline{\mathsf{y}} <= \underline{\mathsf{y}}; \mathrm{JoinConjuncts} \rhd \underline{\mathsf{x}} <= \underline{\mathsf{y}} \rhd \underline{\mathsf{y}} <= \underline{\mathsf{y}} \gg \dot{\neg}\, \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\, \underline{\mathsf{y}} <= \underline{\mathsf{y}}; \mathrm{ExistIntro} @ \mathsf{c}_{\mathrm{Ex}} @ \underline{\mathsf{y}} \rhd \dot{\neg}\, \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\, \underline{\mathsf{y}} <= \underline{\mathsf{y}} \gg \dot{\neg}\, \underline{\mathsf{x}} <= \mathsf{c}_{\mathrm{Ex}} \Rightarrow \dot{\neg}\, \underline{\mathsf{y}} <= \mathsf{c}_{\mathrm{Ex}}; \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \underline{\mathsf{y}} <= \underline{\mathsf{x}} \vdash \mathrm{leqReflexivity} \gg \underline{\mathsf{x}} <= \underline{\mathsf{x}}; \mathrm{JoinConjuncts} \rhd \underline{\mathsf{x}} <= \underline{\mathsf{x}} \rhd \underline{\mathsf{y}} <= \underline{\mathsf{x}} \gg \dot{\neg}\, \underline{\mathsf{x}} <= \underline{\mathsf{x}} \Rightarrow \dot{\neg}\, \underline{\mathsf{y}} <= \underline{\mathsf{x}}; \mathrm{ExistIntro} @ \mathsf{c}_{\mathrm{Ex}} @ \underline{\mathsf{x}} \rhd \dot{\neg}\, \underline{\mathsf{x}} <= \underline{\mathsf{x}} \Rightarrow \dot{\neg}\, \underline{\mathsf{y}} <= \underline{\mathsf{x}} \gg \dot{\neg}\, \underline{\mathsf{x}} <= \mathsf{c}_{\mathrm{Ex}} \Rightarrow \dot{\neg}\, \underline{\mathsf{y}} <= \mathsf{c}_{\mathrm{Ex}}; \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \mathrm{Ded} \rhd \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \underline{\mathsf{x}} <= \underline{\mathsf{y}} \vdash \dot{\neg}\, \underline{\mathsf{x}} <= \mathsf{c}_{\mathrm{Ex}} \Rightarrow \dot{\neg}\, \underline{\mathsf{y}} <= \mathsf{c}_{\mathrm{Ex}} \gg \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\, \underline{\mathsf{x}} <= \mathsf{c}_{\mathrm{Ex}} \Rightarrow \dot{\neg}\, \underline{\mathsf{y}} <= \mathsf{c}_{\mathrm{Ex}}; \mathrm{Ded} \rhd \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \underline{\mathsf{y}} <= \underline{\mathsf{x}} \vdash \dot{\neg}\, \underline{\mathsf{x}} <= \mathsf{c}_{\mathrm{Ex}} \Rightarrow \dot{\neg}\, \underline{\mathsf{y}} <= \mathsf{c}_{\mathrm{Ex}} \gg \underline{\mathsf{y}} <= \underline{\mathsf{x}} \Rightarrow \dot{\neg}\, \underline{\mathsf{x}} <= \mathsf{c}_{\mathrm{Ex}} \Rightarrow \dot{\neg}\, \underline{\mathsf{y}} <= \mathsf{c}_{\mathrm{Ex}}; \mathrm{leqTotality} \gg \dot{\neg}\, \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{y}} <= \underline{\mathsf{x}}; \mathrm{FromDisjuncts} \rhd \dot{\neg}\, \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{y}} <= \underline{\mathsf{x}} \rhd \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\, \underline{\mathsf{x}} <= \mathsf{c}_{\mathrm{Ex}} \Rightarrow \dot{\neg}\, \underline{\mathsf{y}} <= \mathsf{c}_{\mathrm{Ex}} \rhd \underline{\mathsf{y}} <= \underline{\mathsf{x}} \Rightarrow \dot{\neg}\, \underline{\mathsf{x}} <= \mathsf{c}_{\mathrm{Ex}} \Rightarrow \dot{\neg}\, \underline{\mathsf{y}} <= \mathsf{c}_{\mathrm{Ex}} \gg \dot{\neg}\, \underline{\mathsf{x}} <= \mathsf{c}_{\mathrm{Ex}} \Rightarrow \dot{\neg}\, \underline{\mathsf{y}} <= \mathsf{c}_{\mathrm{Ex}} \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{thirdGeq} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \dot{\neg}\, \underline{\mathsf{x}} <= \mathsf{c}_{\mathrm{Ex}} \Rightarrow \dot{\neg}\, \underline{\mathsf{y}} <= \mathsf{c}_{\mathrm{Ex}}]$

$[\mathrm{thirdGeq} \stackrel{\mathrm{tex}}{\rightarrow}$ “thirdGeq”$]$

$[\mathrm{thirdGeq} \stackrel{\mathrm{pyk}}{\rightarrow}$ “lemma thirdGeq”$]$

LeqNegated

$[\mathrm{LeqNegated} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \underline{\mathsf{x}} <= \underline{\mathsf{y}} \vdash \mathrm{leqAddition} \rhd \underline{\mathsf{x}} <= \underline{\mathsf{y}} \gg \underline{\mathsf{x}} + -\underline{\mathsf{x}} <= \underline{\mathsf{y}} + -\underline{\mathsf{x}}; \mathrm{Negative} \gg \underline{\mathsf{x}} + -\underline{\mathsf{x}} = 0; \mathrm{subLeqLeft} \rhd \underline{\mathsf{x}} + -\underline{\mathsf{x}} = 0 \rhd \underline{\mathsf{x}} + -\underline{\mathsf{x}} <= \underline{\mathsf{y}} + -\underline{\mathsf{x}} \gg 0 <= \underline{\mathsf{y}} + -\underline{\mathsf{x}}; \mathrm{plusCommutativity} \gg \underline{\mathsf{y}} + -\underline{\mathsf{x}} = -\underline{\mathsf{x}} + \underline{\mathsf{y}}; \mathrm{subLeqRight} \rhd \underline{\mathsf{y}} + -\underline{\mathsf{x}} = -\underline{\mathsf{x}} + \underline{\mathsf{y}} \rhd 0 <= \underline{\mathsf{y}} + -\underline{\mathsf{x}} \gg 0 <= -\underline{\mathsf{x}} + \underline{\mathsf{y}}; \mathrm{leqAddition} \rhd 0 <= -\underline{\mathsf{x}} + \underline{\mathsf{y}} \gg 0 + -\underline{\mathsf{y}} <= -\underline{\mathsf{x}} + \underline{\mathsf{y}} + -\underline{\mathsf{y}}; \mathrm{plus0Left} \gg 0 + -\underline{\mathsf{y}} = -\underline{\mathsf{y}}; \mathrm{x} = \mathrm{x} + \mathrm{y} - \mathrm{y} \gg -\underline{\mathsf{x}} = -\underline{\mathsf{x}} + \underline{\mathsf{y}} + -\underline{\mathsf{y}}; \mathrm{eqSymmetry} \rhd -\underline{\mathsf{x}} = -\underline{\mathsf{x}} + \underline{\mathsf{y}} + -\underline{\mathsf{y}} \gg -\underline{\mathsf{x}} + \underline{\mathsf{y}} + -\underline{\mathsf{y}} = -\underline{\mathsf{x}}; \mathrm{subLeqLeft} \rhd 0 + -\underline{\mathsf{y}} = -\underline{\mathsf{y}} \rhd 0 + -\underline{\mathsf{y}} <= -\underline{\mathsf{x}} + \underline{\mathsf{y}} + -\underline{\mathsf{y}} \gg -\underline{\mathsf{y}} <= -\underline{\mathsf{x}} + \underline{\mathsf{y}} + -\underline{\mathsf{y}}; \mathrm{subLeqRight} \rhd -\underline{\mathsf{x}} + \underline{\mathsf{y}} + -\underline{\mathsf{y}} = -\underline{\mathsf{x}} \rhd -\underline{\mathsf{y}} <= -\underline{\mathsf{x}} + \underline{\mathsf{y}} + -\underline{\mathsf{y}} \gg -\underline{\mathsf{y}} <= -\underline{\mathsf{x}} \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{LeqNegated} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \underline{\mathsf{x}} <= \underline{\mathsf{y}} \vdash -\underline{\mathsf{y}} <= -\underline{\mathsf{x}}]$

$[\mathrm{LeqNegated} \stackrel{\mathrm{tex}}{\rightarrow}$ “LeqNegated”$]$

$[\mathrm{LeqNegated} \stackrel{\mathrm{pyk}}{\rightarrow}$ “lemma leqNegated”$]$

AddEquations(Leq)

$[\mathrm{AddEquations(Leq)} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \forall \underline{\mathsf{z}}\colon \forall \underline{\mathsf{u}}\colon \underline{\mathsf{x}} <= \underline{\mathsf{y}} \vdash \underline{\mathsf{z}} <= \underline{\mathsf{u}} \vdash \mathrm{leqAddition} \rhd \underline{\mathsf{x}} <= \underline{\mathsf{y}} \gg \underline{\mathsf{x}} + \underline{\mathsf{z}} <= \underline{\mathsf{y}} + \underline{\mathsf{z}}; \mathrm{LeqAdditionLeft} \rhd \underline{\mathsf{z}} <= \underline{\mathsf{u}} \gg \underline{\mathsf{y}} + \underline{\mathsf{z}} <= \underline{\mathsf{y}} + \underline{\mathsf{u}}; \mathrm{leqTransitivity} \rhd \underline{\mathsf{x}} + \underline{\mathsf{z}} <= \underline{\mathsf{y}} + \underline{\mathsf{z}} \rhd \underline{\mathsf{y}} + \underline{\mathsf{z}} <= \underline{\mathsf{y}} + \underline{\mathsf{u}} \gg \underline{\mathsf{x}} + \underline{\mathsf{z}} <= \underline{\mathsf{y}} + \underline{\mathsf{u}} \rceil, \mathrm{p}_0, \mathsf{c})]$

[AddEquations(Leq) $\stackrel{\mathrm{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \forall \underline{\mathsf{z}}\colon \forall \underline{\mathsf{u}}\colon \underline{\mathsf{x}} <= \underline{\mathsf{y}} \vdash \underline{\mathsf{z}} <= \underline{\mathsf{u}} \vdash \underline{\mathsf{x}} + \underline{\mathsf{z}} <= \underline{\mathsf{y}} + \underline{\mathsf{u}}$]

[AddEquations(Leq) $\stackrel{\mathrm{tex}}{\rightarrow}$ "AddEquations(Leq)"]

[AddEquations(Leq) $\stackrel{\mathrm{pyk}}{\rightarrow}$ "lemma addEquations(Leq)"]

ThirdGeqSeries

[ThirdGeqSeries $\stackrel{\mathrm{proof}}{\rightarrow}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}}\colon \forall \underline{(\epsilon)}\colon \forall \underline{(fx)}\colon \forall \underline{(fy)}\colon \forall \underline{(fz)}\colon \forall \underline{(fv)}\colon \forall \underline{(rx)}\colon \forall \underline{(ry)}\colon \forall \underline{(rz)}\colon \forall \underline{(ru)}\colon \underline{(rx)} << \underline{(ry)} \vdash \underline{(rz)} << \underline{(ru)} \vdash \underline{(fx)} \in \underline{(rx)} \vdash \underline{(fy)} \in \underline{(ry)} \vdash \underline{(fz)} \in \underline{(rz)} \vdash \underline{(fv)} \in \underline{(ru)} \vdash \dot{\neg} 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{(\epsilon)} \vdash \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \vdash \mathrm{From} << \mathrm{XX} \rhd \underline{(rx)} << \underline{(ry)} \rhd \underline{(fx)} \in \underline{(rx)} \rhd \underline{(fy)} \in \underline{(ry)} \rhd \dot{\neg} 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{(\epsilon)} \gg \mathsf{a}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(fx)}[\underline{\mathsf{m}}] <= \underline{(fy)}[\underline{\mathsf{m}}] + -\underline{(\epsilon)}; \mathrm{From} << \mathrm{XX} \rhd \underline{(rz)} << \underline{(ru)} \rhd \underline{(fz)} \in \underline{(rz)} \rhd \underline{(fv)} \in \underline{(ru)} \rhd \dot{\neg} 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{(\epsilon)} \gg \mathsf{a}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(fz)}[\underline{\mathsf{m}}] <= \underline{(fv)}[\underline{\mathsf{m}}] + -\underline{(\epsilon)}; \mathrm{ExistIntro} @ \mathsf{b}_{\mathrm{Ex}} @ \mathsf{a}_{\mathrm{Ex}} \rhd \mathsf{a}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(fz)}[\underline{\mathsf{m}}] <= \underline{(fv)}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \gg \mathsf{b}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(fz)}[\underline{\mathsf{m}}] <= \underline{(fv)}[\underline{\mathsf{m}}] + -\underline{(\epsilon)}; \mathrm{thirdGeq} \gg \dot{\neg} \mathsf{a}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}} \Rightarrow \dot{\neg} \mathsf{b}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}}; \mathrm{FirstConjunct} \rhd \dot{\neg} \mathsf{a}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}} \Rightarrow \dot{\neg} \mathsf{b}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}} \gg \mathsf{a}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}}; \mathrm{leqTransitivity} \rhd \mathsf{a}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}} \rhd \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \gg \mathsf{a}_{\mathrm{Ex}} <= \underline{\mathsf{m}}; \mathrm{MP} \rhd \mathsf{a}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(fx)}[\underline{\mathsf{m}}] <= \underline{(fy)}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \rhd \mathsf{a}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \gg \underline{(fx)}[\underline{\mathsf{m}}] <= \underline{(fy)}[\underline{\mathsf{m}}] + -\underline{(\epsilon)}; \mathrm{SecondConjunct} \rhd \dot{\neg} \mathsf{a}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}} \Rightarrow \dot{\neg} \mathsf{b}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}} \gg \mathsf{b}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}}; \mathrm{leqTransitivity} \rhd \mathsf{b}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}} \rhd \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \gg \mathsf{b}_{\mathrm{Ex}} <= \underline{\mathsf{m}}; \mathrm{MP} \rhd \mathsf{b}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(fz)}[\underline{\mathsf{m}}] <= \underline{(fv)}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \rhd \mathsf{b}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \gg \underline{(fz)}[\underline{\mathsf{m}}] <= \underline{(fv)}[\underline{\mathsf{m}}] + -\underline{(\epsilon)}; \mathrm{JoinConjuncts} \rhd \underline{(fx)}[\underline{\mathsf{m}}] <= \underline{(fy)}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \rhd \underline{(fz)}[\underline{\mathsf{m}}] <= \underline{(fv)}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \gg \dot{\neg} \underline{(fx)}[\underline{\mathsf{m}}] <= \underline{(fy)}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \Rightarrow \dot{\neg} \underline{(fz)}[\underline{\mathsf{m}}] <= \underline{(fv)}[\underline{\mathsf{m}}] + -\underline{(\epsilon)}\rceil, \mathrm{p}_0, \mathsf{c})$]

[ThirdGeqSeries $\stackrel{\mathrm{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{\mathsf{m}}\colon \forall \underline{(\epsilon)}\colon \forall \underline{(fx)}\colon \forall \underline{(fy)}\colon \forall \underline{(fz)}\colon \forall \underline{(fv)}\colon \forall \underline{(rx)}\colon \forall \underline{(ry)}\colon \forall \underline{(rz)}\colon \forall \underline{(ru)}\colon \underline{(rx)} << \underline{(ry)} \vdash \underline{(rz)} << \underline{(ru)} \vdash \underline{(fx)} \in \underline{(rx)} \vdash \underline{(fy)} \in \underline{(ry)} \vdash \underline{(fz)} \in \underline{(rz)} \vdash \underline{(fv)} \in \underline{(ru)} \vdash \dot{\neg} 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{(\epsilon)} \vdash \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \vdash \dot{\neg} \underline{(fx)}[\underline{\mathsf{m}}] <= \underline{(fy)}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \Rightarrow \dot{\neg} \underline{(fz)}[\underline{\mathsf{m}}] <= \underline{(fv)}[\underline{\mathsf{m}}] + -\underline{(\epsilon)}$]

[ThirdGeqSeries $\stackrel{\mathrm{tex}}{\rightarrow}$ "ThirdGeqSeries"]

[ThirdGeqSeries $\stackrel{\mathrm{pyk}}{\rightarrow}$ "lemma thirdGeqSeries"]

LeqNeqLess

[LeqNeqLess $\stackrel{\mathrm{proof}}{\rightarrow}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \underline{\mathsf{x}} <= \underline{\mathsf{y}} \vdash \dot{\neg} \underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \mathrm{JoinConjuncts} \rhd \underline{\mathsf{x}} <= \underline{\mathsf{y}} \rhd \dot{\neg} \underline{\mathsf{x}} = \underline{\mathsf{y}} \gg \dot{\neg} \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg} \underline{\mathsf{x}} = \underline{\mathsf{y}}; \mathrm{Repetition} \rhd \dot{\neg} \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg} \underline{\mathsf{x}} = \underline{\mathsf{y}} \gg \dot{\neg} \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg} \underline{\mathsf{x}} = \underline{\mathsf{y}}\rceil, \mathrm{p}_0, \mathsf{c})$]

[LeqNeqLess $\stackrel{\mathrm{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \underline{\mathsf{x}} <= \underline{\mathsf{y}} \vdash \dot{\neg} \underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \dot{\neg} \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg} \underline{\mathsf{x}} = \underline{\mathsf{y}}$]

[LeqNeqLess $\stackrel{\text{tex}}{\to}$ "LeqNeqLess"]

[LeqNeqLess $\stackrel{\text{pyk}}{\to}$ "lemma leqNeqLess"]

FromLess

[FromLess $\stackrel{\text{proof}}{\to}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \underline{\mathsf{y}} <= \underline{\mathsf{x}} \vdash \mathrm{toNotLess} \rhd \underline{\mathsf{y}} <= \underline{\mathsf{x}} \gg \dot{\neg}\dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}}; \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \mathrm{Ded} \rhd \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \underline{\mathsf{y}} <= \underline{\mathsf{x}} \vdash \dot{\neg}\dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}} \gg \underline{\mathsf{y}} <= \underline{\mathsf{x}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}}; \dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \mathrm{AddDoubleNeg} \rhd \dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}} \gg \dot{\neg}\dot{\neg}\dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}}; \mathrm{MT} \rhd \underline{\mathsf{y}} <= \underline{\mathsf{x}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}} \rhd \dot{\neg}\dot{\neg}\dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}} \gg \dot{\neg}\underline{\mathsf{y}} <= \underline{\mathsf{x}} \rceil, \mathrm{p}_0, \mathsf{c})$]

[FromLess $\stackrel{\text{stmt}}{\to}$ $\mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \dot{\neg}\underline{\mathsf{y}} <= \underline{\mathsf{x}}$]

[FromLess $\stackrel{\text{tex}}{\to}$ "FromLess"]

[FromLess $\stackrel{\text{pyk}}{\to}$ "lemma fromLess"]

ToLess

[ToLess $\stackrel{\text{proof}}{\to}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \dot{\neg}\dot{\neg}\underline{\mathsf{y}} <= \underline{\mathsf{x}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{y}} = \underline{\mathsf{x}} \vdash \mathrm{fromNotLess} \rhd \dot{\neg}\dot{\neg}\underline{\mathsf{y}} <= \underline{\mathsf{x}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{y}} = \underline{\mathsf{x}} \gg \underline{\mathsf{x}} <= \underline{\mathsf{y}}; \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \mathrm{Ded} \rhd \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \dot{\neg}\dot{\neg}\underline{\mathsf{y}} <= \underline{\mathsf{x}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{y}} = \underline{\mathsf{x}} \vdash \underline{\mathsf{x}} <= \underline{\mathsf{y}} \gg \dot{\neg}\dot{\neg}\underline{\mathsf{y}} <= \underline{\mathsf{x}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{y}} = \underline{\mathsf{x}} \Rightarrow \underline{\mathsf{x}} <= \underline{\mathsf{y}}; \dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \vdash \mathrm{NegativeMT} \rhd \dot{\neg}\dot{\neg}\underline{\mathsf{y}} <= \underline{\mathsf{x}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{y}} = \underline{\mathsf{x}} \Rightarrow \underline{\mathsf{x}} <= \underline{\mathsf{y}} \rhd \dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \gg \dot{\neg}\underline{\mathsf{y}} <= \underline{\mathsf{x}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{y}} = \underline{\mathsf{x}} \rceil, \mathrm{p}_0, \mathsf{c})$]

[ToLess $\stackrel{\text{stmt}}{\to}$ $\mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \vdash \dot{\neg}\underline{\mathsf{y}} <= \underline{\mathsf{x}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{y}} = \underline{\mathsf{x}}$]

[ToLess $\stackrel{\text{tex}}{\to}$ "ToLess"]

[ToLess $\stackrel{\text{pyk}}{\to}$ "lemma toLess"]

fromNotLess

[fromNotLess $\stackrel{\text{proof}}{\to}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \dot{\neg}\dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \underline{\mathsf{x}} <= \underline{\mathsf{y}} \vdash \mathrm{Repetition} \rhd \dot{\neg}\dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}} \gg \dot{\neg}\dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}}; \mathrm{RemoveDoubleNeg} \rhd \dot{\neg}\dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}} \gg \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}}; \mathrm{MP} \rhd \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}} \rhd \underline{\mathsf{x}} <= \underline{\mathsf{y}} \gg \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}}; \mathrm{RemoveDoubleNeg} \rhd \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}} \gg \underline{\mathsf{x}} = \underline{\mathsf{y}}; \mathrm{eqSymmetry} \rhd \underline{\mathsf{x}} = \underline{\mathsf{y}} \gg \underline{\mathsf{y}} = \underline{\mathsf{x}}; \mathrm{eqLeq} \rhd \underline{\mathsf{y}} = \underline{\mathsf{x}} \gg \underline{\mathsf{y}} <= \underline{\mathsf{x}}; \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \mathrm{Ded} \rhd \forall \underline{\mathsf{x}} \colon \forall \underline{\mathsf{y}} \colon \dot{\neg}\dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \underline{\mathsf{x}} <= \underline{\mathsf{y}} \vdash \underline{\mathsf{y}} <= \underline{\mathsf{x}} \gg \dot{\neg}\dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{y}} <= \underline{\mathsf{x}}; \dot{\neg}\dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}} \vdash \mathrm{MP} \rhd \dot{\neg}\dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{y}} <= \underline{\mathsf{x}} \rhd \dot{\neg}\dot{\neg}\underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot{\neg}\dot{\neg}\underline{\mathsf{x}} = \underline{\mathsf{y}} \gg \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \underline{\mathsf{y}} <= \underline{\mathsf{x}}; \mathrm{AutoImply} \gg \underline{\mathsf{y}} <= \underline{\mathsf{x}} \Rightarrow \underline{\mathsf{y}} <= \underline{\mathsf{x}}; \mathrm{leqTotality} \gg$

$\dot\neg \underline{x} <= \underline{y} \Rightarrow \underline{y} <= \underline{x}$; FromDisjuncts $\rhd \dot\neg \underline{x} <= \underline{y} \Rightarrow \underline{y} <= \underline{x} \rhd \underline{x} <= \underline{y} \Rightarrow \underline{y} <= \underline{x} \rhd \underline{y} <= \underline{x} \Rightarrow \underline{y} <= \underline{x} \gg \underline{y} <= \underline{x}\rceil, p_0, c)]$

$[\text{fromNotLess} \overset{\text{stmt}}{\to} \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \dot\neg \dot\neg \underline{x} <= \underline{y} \Rightarrow \dot\neg \dot\neg \underline{x} = \underline{y} \vdash \underline{y} <= \underline{x}]$

$[\text{fromNotLess} \overset{\text{tex}}{\to}$ "fromNotLess"$]$

$[\text{fromNotLess} \overset{\text{pyk}}{\to}$ "lemma fromNotLess"$]$

toNotLess

$[\text{toNotLess} \overset{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} <= \underline{y} \vdash \underline{y} <= \underline{x} \vdash$ leqAntisymmetry $\rhd \underline{y} <= \underline{x} \rhd \underline{x} <= \underline{y} \gg \underline{y} = \underline{x}$; AddDoubleNeg $\rhd \underline{y} = \underline{x} \gg \dot\neg \dot\neg \underline{y} = \underline{x}; \forall \underline{x}\colon \forall \underline{y}\colon$ Ded $\rhd \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} <= \underline{y} \vdash \underline{y} <= \underline{x} \vdash \dot\neg \dot\neg \underline{y} = \underline{x} \gg \underline{x} <= \underline{y} \Rightarrow \underline{y} <= \underline{x} \Rightarrow \dot\neg \dot\neg \underline{y} = \underline{x}; \underline{x} <= \underline{y} \vdash$ MP $\rhd \underline{x} <= \underline{y} \Rightarrow \underline{y} <= \underline{x} \Rightarrow \dot\neg \dot\neg \underline{y} = \underline{x} \rhd \underline{x} <= \underline{y} \gg \underline{y} <= \underline{x} \Rightarrow \dot\neg \dot\neg \underline{y} = \underline{x}$; AddDoubleNeg $\rhd \underline{y} <= \underline{x} \Rightarrow \dot\neg \dot\neg \underline{y} = \underline{x} \gg \dot\neg \dot\neg \underline{y} <= \underline{x} \Rightarrow \dot\neg \dot\neg \underline{y} = \underline{x}$; Repetition $\rhd \dot\neg \dot\neg \underline{y} <= \underline{x} \Rightarrow \dot\neg \dot\neg \underline{y} = \underline{x} \gg \dot\neg \dot\neg \underline{y} <= \underline{x} \Rightarrow \dot\neg \dot\neg \underline{y} = \underline{x}$; Repetition $\rhd \dot\neg \dot\neg \underline{y} <= \underline{x} \Rightarrow \dot\neg \dot\neg \underline{y} = \underline{x} \gg \dot\neg \dot\neg \underline{y} <= \underline{x} \Rightarrow \dot\neg \dot\neg \underline{y} = \underline{x}\rceil, p_0, c)]$

$[\text{toNotLess} \overset{\text{stmt}}{\to} \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} <= \underline{y} \vdash \dot\neg \dot\neg \underline{y} <= \underline{x} \Rightarrow \dot\neg \dot\neg \underline{y} = \underline{x}]$

$[\text{toNotLess} \overset{\text{tex}}{\to}$ "toNotLess"$]$

$[\text{toNotLess} \overset{\text{pyk}}{\to}$ "lemma toNotLess"$]$

NegativeLessPositive

$[\text{NegativeLessPositive} \overset{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \dot\neg 0 <= \underline{x} \Rightarrow \dot\neg \dot\neg 0 = \underline{x} \vdash$ FirstConjunct $\rhd \dot\neg 0 <= \underline{x} \Rightarrow \dot\neg \dot\neg 0 = \underline{x} \gg 0 <= \underline{x}$; leqAddition $\rhd 0 <= \underline{x} \gg 0 + -\underline{x} <= \underline{x} + -\underline{x}$; plus0Left $\gg 0 + -\underline{x} = -\underline{x}$; Negative $\gg \underline{x} + -\underline{x} = 0$; subLeqLeft $\rhd 0 + -\underline{x} = -\underline{x} \rhd 0 + -\underline{x} <= \underline{x} + -\underline{x} \gg -\underline{x} <= \underline{x} + -\underline{x}$; subLeqRight $\rhd \underline{x} + -\underline{x} = 0 \rhd -\underline{x} <= \underline{x} + -\underline{x} \gg -\underline{x} <= 0$; leqLessTransitivity $\rhd -\underline{x} <= 0 \rhd \dot\neg 0 <= \underline{x} \Rightarrow \dot\neg \dot\neg 0 = \underline{x} \gg \dot\neg -\underline{x} <= \underline{x} \Rightarrow \dot\neg \dot\neg -\underline{x} = \underline{x}\rceil, p_0, c)]$

$[\text{NegativeLessPositive} \overset{\text{stmt}}{\to} \text{ZFsub} \vdash \forall \underline{x}\colon \dot\neg 0 <= \underline{x} \Rightarrow \dot\neg \dot\neg 0 = \underline{x} \vdash \dot\neg -\underline{x} <= \underline{x} \Rightarrow \dot\neg \dot\neg -\underline{x} = \underline{x}]$

$[\text{NegativeLessPositive} \overset{\text{tex}}{\to}$ "NegativeLessPositive"$]$

$[\text{NegativeLessPositive} \overset{\text{pyk}}{\to}$ "lemma negativeLessPositive"$]$

leqLessTransitivity

[leqLessTransitivity $\stackrel{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} <= \underline{y} \vdash \dot{\neg} \underline{y} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg} \underline{y} = \underline{z} \vdash \underline{x} = \underline{z} \vdash \text{FirstConjunct} \rhd \dot{\neg} \underline{y} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg} \underline{y} = \underline{z} \gg \underline{y} <= \underline{z}; \text{SecondConjunct} \rhd \dot{\neg} \underline{y} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg} \underline{y} = \underline{z} \gg \dot{\neg} \underline{y} = \underline{z}; \text{subLeqLeft} \rhd \underline{x} = \underline{z} \rhd \underline{x} <= \underline{y} \gg \underline{z} <= \underline{y}; \text{leqAntisymmetry} \rhd \underline{y} <= \underline{z} \rhd \underline{z} <= \underline{y} \gg \underline{y} = \underline{z}; \text{FromContradiction} \rhd \underline{y} = \underline{z} \rhd \dot{\neg} \underline{y} = \underline{z} \gg \dot{\neg} \underline{x} = \underline{z}; \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \text{Ded} \rhd \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} <= \underline{y} \vdash \dot{\neg} \underline{y} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg} \underline{y} = \underline{z} \vdash \underline{x} = \underline{z} \vdash \dot{\neg} \underline{x} = \underline{z} \gg \underline{x} <= \underline{y} \Rightarrow \dot{\neg} \underline{y} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg} \underline{y} = \underline{z} \Rightarrow \underline{x} = \underline{z} \Rightarrow \dot{\neg} \underline{x} = \underline{z}; \underline{x} <= \underline{y} \vdash \dot{\neg} \underline{y} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg} \underline{y} = \underline{z} \vdash \text{MP2} \rhd \underline{x} <= \underline{y} \Rightarrow \dot{\neg} \underline{y} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg} \underline{y} = \underline{z} \Rightarrow \underline{x} = \underline{z} \Rightarrow \dot{\neg} \underline{x} = \underline{z} \rhd \underline{x} <= \underline{y} \rhd \dot{\neg} \underline{y} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg} \underline{y} = \underline{z} \gg \underline{x} = \underline{z} \Rightarrow \dot{\neg} \underline{x} = \underline{z}; \text{ImplyNegation} \rhd \underline{x} = \underline{z} \Rightarrow \dot{\neg} \underline{x} = \underline{z} \gg \dot{\neg} \underline{x} = \underline{z}; \text{FirstConjunct} \rhd \dot{\neg} \underline{y} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg} \underline{y} = \underline{z} \gg \underline{y} <= \underline{z}; \text{leqTransitivity} \rhd \underline{x} <= \underline{y} \rhd \underline{y} <= \underline{z} \gg \underline{x} <= \underline{z}; \text{JoinConjuncts} \rhd \underline{x} <= \underline{z} \rhd \dot{\neg} \underline{x} = \underline{z} \gg \dot{\neg} \underline{x} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg} \underline{x} = \underline{z}\rceil, p_0, c)$]

[leqLessTransitivity $\stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} <= \underline{y} \vdash \dot{\neg} \underline{y} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg} \underline{y} = \underline{z} \vdash \dot{\neg} \underline{x} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg} \underline{x} = \underline{z}$]

[leqLessTransitivity $\stackrel{\text{tex}}{\rightarrow}$ "leqLessTransitivity"]

[leqLessTransitivity $\stackrel{\text{pyk}}{\rightarrow}$ "lemma leqLessTransitivity"]

LessLeqTransitivity

[LessLeqTransitivity $\stackrel{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \dot{\neg} \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg} \underline{x} = \underline{y} \vdash \underline{y} <= \underline{z} \vdash \underline{z} = \underline{x} \vdash \text{FirstConjunct} \rhd \dot{\neg} \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg} \underline{x} = \underline{y} \gg \underline{x} <= \underline{y}; \text{SecondConjunct} \rhd \dot{\neg} \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg} \underline{x} = \underline{y} \gg \dot{\neg} \underline{x} = \underline{y}; \text{subLeqRight} \rhd \underline{z} = \underline{x} \rhd \underline{y} <= \underline{z} \gg \underline{y} <= \underline{x}; \text{leqAntisymmetry} \rhd \underline{x} <= \underline{y} \rhd \underline{y} <= \underline{x} \gg \underline{x} = \underline{y}; \text{FromContradiction} \rhd \underline{x} = \underline{y} \rhd \dot{\neg} \underline{x} = \underline{y} \gg \dot{\neg} \underline{z} = \underline{x}; \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \text{Ded} \rhd \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \dot{\neg} \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg} \underline{x} = \underline{y} \vdash \underline{y} <= \underline{z} \vdash \underline{z} = \underline{x} \vdash \dot{\neg} \underline{z} = \underline{x} \gg \dot{\neg} \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg} \underline{x} = \underline{y} \Rightarrow \underline{y} <= \underline{z} \Rightarrow \underline{z} = \underline{x} \Rightarrow \dot{\neg} \underline{z} = \underline{x}; \dot{\neg} \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg} \underline{x} = \underline{y} \vdash \underline{y} <= \underline{z} \vdash \text{MP2} \rhd \dot{\neg} \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg} \underline{x} = \underline{y} \Rightarrow \underline{y} <= \underline{z} \Rightarrow \underline{z} = \underline{x} \Rightarrow \dot{\neg} \underline{z} = \underline{x} \rhd \dot{\neg} \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg} \underline{x} = \underline{y} \rhd \underline{y} <= \underline{z} \gg \underline{z} = \underline{x} \Rightarrow \dot{\neg} \underline{z} = \underline{x}; \text{ImplyNegation} \rhd \underline{z} = \underline{x} \Rightarrow \dot{\neg} \underline{z} = \underline{x} \gg \dot{\neg} \underline{z} = \underline{x}; \text{NeqSymmetry} \rhd \dot{\neg} \underline{z} = \underline{x} \gg \dot{\neg} \underline{x} = \underline{z}; \text{FirstConjunct} \rhd \dot{\neg} \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg} \underline{x} = \underline{y} \gg \underline{x} <= \underline{y}; \text{leqTransitivity} \rhd \underline{x} <= \underline{y} \rhd \underline{y} <= \underline{z} \gg \underline{x} <= \underline{z}; \text{JoinConjuncts} \rhd \underline{x} <= \underline{z} \rhd \dot{\neg} \underline{x} = \underline{z} \gg \dot{\neg} \underline{x} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg} \underline{x} = \underline{z}\rceil, p_0, c)$]

[LessLeqTransitivity $\stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \dot{\neg} \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg} \underline{x} = \underline{y} \vdash \underline{y} <= \underline{z} \vdash \dot{\neg} \underline{x} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg} \underline{x} = \underline{z}$]

[LessLeqTransitivity $\stackrel{\text{tex}}{\rightarrow}$ "LessLeqTransitivity"]

[LessLeqTransitivity $\stackrel{\text{pyk}}{\rightarrow}$ "lemma lessLeqTransitivity"]

LessTransitivity

$[\text{LessTransitivity} \stackrel{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{x} = \underline{y} \vdash \dot{\neg}\, \underline{y} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{y} = \underline{z} \vdash \text{FirstConjunct} \rhd \dot{\neg}\, \underline{y} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{y} = \underline{z} \gg \underline{y} <= \underline{z}; \text{LessLeqTransitivity} \rhd \dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{x} = \underline{y} \rhd \underline{y} <= \underline{z} \gg \dot{\neg}\, \underline{x} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{x} = \underline{z}\rceil, p_0, c)]$

$[\text{LessTransitivity} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{x} = \underline{y} \vdash \dot{\neg}\, \underline{y} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{y} = \underline{z} \vdash \dot{\neg}\, \underline{x} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{x} = \underline{z}]$

$[\text{LessTransitivity} \stackrel{\text{tex}}{\rightarrow}$ “LessTransitivity”$]$

$[\text{LessTransitivity} \stackrel{\text{pyk}}{\rightarrow}$ “lemma lessTransitivity”$]$

LessTotality

$[\text{LessTotality} \stackrel{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \dot{\neg}\dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{x} = \underline{y} \vdash \dot{\neg}\, \underline{x} = \underline{y} \vdash \text{fromNotLess} \rhd \dot{\neg}\dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{x} = \underline{y} \gg \underline{y} <= \underline{x}; \text{NeqSymmetry} \rhd \dot{\neg}\, \underline{x} = \underline{y} \gg \dot{\neg}\, \underline{y} = \underline{x}; \text{LeqNeqLess} \rhd \underline{y} <= \underline{x} \rhd \dot{\neg}\, \underline{y} = \underline{x} \gg \dot{\neg}\, \underline{y} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{y} = \underline{x}; \forall \underline{x}\colon \forall \underline{y}\colon \text{Ded} \rhd \forall \underline{x}\colon \forall \underline{y}\colon \dot{\neg}\dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{x} = \underline{y} \vdash \dot{\neg}\, \underline{x} = \underline{y} \vdash \dot{\neg}\, \underline{y} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{y} = \underline{x} \gg \dot{\neg}\dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{x} = \underline{y} \Rightarrow \dot{\neg}\, \underline{x} = \underline{y} \Rightarrow \dot{\neg}\, \underline{y} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{y} = \underline{x}; \text{Repetition} \rhd \dot{\neg}\dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{x} = \underline{y} \Rightarrow \dot{\neg}\, \underline{x} = \underline{y} \Rightarrow \dot{\neg}\, \underline{y} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{y} = \underline{x} \gg \dot{\neg}\dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{x} = \underline{y} \Rightarrow \dot{\neg}\, \underline{x} = \underline{y} \Rightarrow \dot{\neg}\, \underline{y} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{y} = \underline{x}\rceil, p_0, c)]$

$[\text{LessTotality} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \dot{\neg}\dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{x} = \underline{y} \Rightarrow \dot{\neg}\, \underline{x} = \underline{y} \Rightarrow \dot{\neg}\, \underline{y} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{y} = \underline{x}]$

$[\text{LessTotality} \stackrel{\text{tex}}{\rightarrow}$ “LessTotality”$]$

$[\text{LessTotality} \stackrel{\text{pyk}}{\rightarrow}$ “lemma lessTotality”$]$

SubLessRight

$[\text{SubLessRight} \stackrel{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} \vdash \dot{\neg}\, \underline{z} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{z} = \underline{x} \vdash \text{Repetition} \rhd \dot{\neg}\, \underline{z} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{z} = \underline{x} \gg \dot{\neg}\, \underline{z} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{z} = \underline{x}; \text{FirstConjunct} \rhd \dot{\neg}\, \underline{z} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{z} = \underline{x} \gg \underline{z} <= \underline{x}; \text{subLeqRight} \rhd \underline{x} = \underline{y} \rhd \underline{z} <= \underline{x} \gg \underline{z} <= \underline{y}; \text{SecondConjunct} \rhd \dot{\neg}\, \underline{z} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{z} = \underline{x} \gg \dot{\neg}\, \underline{z} = \underline{x}; \text{SubNeqRight} \rhd \underline{x} = \underline{y} \rhd \dot{\neg}\, \underline{z} = \underline{x} \gg \dot{\neg}\, \underline{z} = \underline{y}; \text{JoinConjuncts} \rhd \underline{z} <= \underline{y} \rhd \dot{\neg}\, \underline{z} = \underline{y} \gg \dot{\neg}\, \underline{z} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{z} = \underline{y}\rceil, p_0, c)]$

$[\text{SubLessRight} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} \vdash \dot{\neg}\, \underline{z} <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{z} = \underline{x} \vdash \dot{\neg}\, \underline{z} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{z} = \underline{y}]$

$[\text{SubLessRight} \stackrel{\text{tex}}{\rightarrow}$ “SubLessRight”$]$

[SubLessRight $\stackrel{\text{pyk}}{\to}$ "lemma subLessRight"]

SubLessLeft

[SubLessLeft $\stackrel{\text{proof}}{\to}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} \vdash \dot{\neg} \underline{x} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg} \underline{x} = \underline{z} \vdash \text{Repetition} \rhd \dot{\neg} \underline{x} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg} \underline{x} = \underline{z} \gg \dot{\neg} \underline{x} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg} \underline{x} = \underline{z}; \text{FirstConjunct} \rhd \dot{\neg} \underline{x} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg} \underline{x} = \underline{z} \gg \underline{x} <= \underline{z}; \text{subLeqLeft} \rhd \underline{x} = \underline{y} \rhd \underline{x} <= \underline{z} \gg \underline{y} <= \underline{z}; \text{SecondConjunct} \rhd \dot{\neg} \underline{x} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg} \underline{x} = \underline{z} \gg \dot{\neg} \underline{x} = \underline{z}; \text{SubNeqLeft} \rhd \underline{x} = \underline{y} \rhd \dot{\neg} \underline{x} = \underline{z} \gg \dot{\neg} \underline{y} = \underline{z}; \text{JoinConjuncts} \rhd \underline{y} <= \underline{z} \rhd \dot{\neg} \underline{y} = \underline{z} \gg \dot{\neg} \underline{y} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg} \underline{y} = \underline{z} \rceil, p_0, c)$]

[SubLessLeft $\stackrel{\text{stmt}}{\to}$ $\text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} = \underline{y} \vdash \dot{\neg} \underline{x} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg} \underline{x} = \underline{z} \vdash \dot{\neg} \underline{y} <= \underline{z} \Rightarrow \dot{\neg}\dot{\neg} \underline{y} = \underline{z}$]

[SubLessLeft $\stackrel{\text{tex}}{\to}$ "SubLessLeft"]

[SubLessLeft $\stackrel{\text{pyk}}{\to}$ "lemma subLessLeft"]

LessAddition

[LessAddition $\stackrel{\text{proof}}{\to}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \dot{\neg} \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg} \underline{x} = \underline{y} \vdash \text{LessLeq} \rhd \dot{\neg} \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg} \underline{x} = \underline{y} \gg \underline{x} <= \underline{y}; \text{leqAddition} \rhd \underline{x} <= \underline{y} \gg \underline{x} + \underline{z} <= \underline{y} + \underline{z}; \text{LessNeq} \rhd \dot{\neg} \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg} \underline{x} = \underline{y} \gg \dot{\neg} \underline{x} = \underline{y}; \text{NeqAddition} \rhd \dot{\neg} \underline{x} = \underline{y} \gg \dot{\neg} \underline{x} + \underline{z} = \underline{y} + \underline{z}; \text{JoinConjuncts} \rhd \underline{x} + \underline{z} <= \underline{y} + \underline{z} \rhd \dot{\neg} \underline{x} + \underline{z} = \underline{y} + \underline{z} \gg \dot{\neg} \underline{x} + \underline{z} <= \underline{y} + \underline{z} \Rightarrow \dot{\neg}\dot{\neg} \underline{x} + \underline{z} = \underline{y} + \underline{z} \rceil, p_0, c)$]

[LessAddition $\stackrel{\text{stmt}}{\to}$ $\text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \dot{\neg} \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg} \underline{x} = \underline{y} \vdash \dot{\neg} \underline{x} + \underline{z} <= \underline{y} + \underline{z} \Rightarrow \dot{\neg}\dot{\neg} \underline{x} + \underline{z} = \underline{y} + \underline{z}$]

[LessAddition $\stackrel{\text{tex}}{\to}$ "LessAddition"]

[LessAddition $\stackrel{\text{pyk}}{\to}$ "lemma lessAddition"]

LessAdditionLeft

[LessAdditionLeft $\stackrel{\text{proof}}{\to}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \dot{\neg} \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg} \underline{x} = \underline{y} \vdash \text{LessAddition} \rhd \dot{\neg} \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg} \underline{x} = \underline{y} \gg \dot{\neg} \underline{x} + \underline{z} <= \underline{y} + \underline{z} \Rightarrow \dot{\neg}\dot{\neg} \underline{x} + \underline{z} = \underline{y} + \underline{z}; \text{plusCommutativity} \gg \underline{x} + \underline{z} = \underline{z} + \underline{x}; \text{SubLessLeft} \rhd \underline{x} + \underline{z} = \underline{z} + \underline{x} \rhd \dot{\neg} \underline{x} + \underline{z} <= \underline{y} + \underline{z} \Rightarrow \dot{\neg}\dot{\neg} \underline{x} + \underline{z} = \underline{y} + \underline{z} \gg \dot{\neg} \underline{z} + \underline{x} <= \underline{y} + \underline{z} \Rightarrow \dot{\neg}\dot{\neg} \underline{z} + \underline{x} = \underline{y} + \underline{z}; \text{plusCommutativity} \gg \underline{y} + \underline{z} = \underline{z} + \underline{y}; \text{SubLessRight} \rhd \underline{y} + \underline{z} = \underline{z} + \underline{y} \rhd \dot{\neg} \underline{z} + \underline{x} <= \underline{y} + \underline{z} \Rightarrow \dot{\neg}\dot{\neg} \underline{z} + \underline{x} = \underline{y} + \underline{z} \gg \dot{\neg} \underline{z} + \underline{x} <= \underline{z} + \underline{y} \Rightarrow \dot{\neg}\dot{\neg} \underline{z} + \underline{x} = \underline{z} + \underline{y} \rceil, p_0, c)$]

[LessAdditionLeft $\stackrel{\text{stmt}}{\to}$ $\text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \dot{\neg} \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg} \underline{x} = \underline{y} \vdash \dot{\neg} \underline{z} + \underline{x} <= \underline{z} + \underline{y} \Rightarrow \dot{\neg}\dot{\neg} \underline{z} + \underline{x} = \underline{z} + \underline{y}$]

[LessAdditionLeft $\overset{\text{tex}}{\rightarrow}$ "LessAdditionLeft"]

[LessAdditionLeft $\overset{\text{pyk}}{\rightarrow}$ "lemma lessAdditionLeft"]

LessMultiplication

[LessMultiplication $\overset{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \dot\neg 0 <= \underline{z} \Rightarrow \dot\neg\dot\neg 0 = \underline{z} \vdash \dot\neg \underline{x} <= \underline{y} \Rightarrow \dot\neg\dot\neg \underline{x} = \underline{y} \vdash \mathrm{LessLeq} \rhd \dot\neg \underline{x} <= \underline{y} \Rightarrow \dot\neg\dot\neg \underline{x} = \underline{y} \gg \underline{x} <= \underline{y}; \mathrm{LessLeq} \rhd \dot\neg 0 <= \underline{z} \Rightarrow \dot\neg\dot\neg 0 = \underline{z} \gg 0 <= \underline{z}; \mathrm{leqMultiplication} \rhd 0 <= \underline{z} \rhd \underline{x} <= \underline{y} \gg \underline{x} * \underline{z} <= \underline{y} * \underline{z}; \mathrm{LessNeq} \rhd \dot\neg \underline{x} <= \underline{y} \Rightarrow \dot\neg\dot\neg \underline{x} = \underline{y} \gg \dot\neg \underline{x} = \underline{y}; \mathrm{LessNeq} \rhd \dot\neg 0 <= \underline{z} \Rightarrow \dot\neg\dot\neg 0 = \underline{z} \gg \dot\neg 0 = \underline{z}; \mathrm{NeqSymmetry} \rhd \dot\neg 0 = \underline{z} \gg \dot\neg \underline{z} = 0; \mathrm{NeqMultiplication} \rhd \dot\neg \underline{z} = 0 \rhd \dot\neg \underline{x} = \underline{y} \gg \dot\neg \underline{x} * \underline{z} = \underline{y} * \underline{z}; \mathrm{LeqNeqLess} \rhd \underline{x} * \underline{z} <= \underline{y} * \underline{z} \rhd \dot\neg \underline{x} * \underline{z} = \underline{y} * \underline{z} \gg \dot\neg \underline{x} * \underline{z} <= \underline{y} * \underline{z} \Rightarrow \dot\neg\dot\neg \underline{x} * \underline{z} = \underline{y} * \underline{z} \rceil, p_0, c)$]

[LessMultiplication $\overset{\text{stmt}}{\rightarrow}$ $\mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \dot\neg 0 <= \underline{z} \Rightarrow \dot\neg\dot\neg 0 = \underline{z} \vdash \dot\neg \underline{x} <= \underline{y} \Rightarrow \dot\neg\dot\neg \underline{x} = \underline{y} \vdash \dot\neg \underline{x} * \underline{z} <= \underline{y} * \underline{z} \Rightarrow \dot\neg\dot\neg \underline{x} * \underline{z} = \underline{y} * \underline{z}$]

[LessMultiplication $\overset{\text{tex}}{\rightarrow}$ "LessMultiplication"]

[LessMultiplication $\overset{\text{pyk}}{\rightarrow}$ "lemma lessMultiplication"]

LessMultiplicationLeft

[LessMultiplicationLeft $\overset{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \dot\neg 0 <= \underline{z} \Rightarrow \dot\neg\dot\neg 0 = \underline{z} \vdash \dot\neg \underline{x} <= \underline{y} \Rightarrow \dot\neg\dot\neg \underline{x} = \underline{y} \vdash \mathrm{LessMultiplication} \rhd \dot\neg 0 <= \underline{z} \Rightarrow \dot\neg\dot\neg 0 = \underline{z} \rhd \dot\neg \underline{x} <= \underline{y} \Rightarrow \dot\neg\dot\neg \underline{x} = \underline{y} \gg \dot\neg \underline{x} * \underline{z} <= \underline{y} * \underline{z} \Rightarrow \dot\neg\dot\neg \underline{x} * \underline{z} = \underline{y} * \underline{z}; \mathrm{timesCommutativity} \gg \underline{x} * \underline{z} = \underline{z} * \underline{x}; \mathrm{timesCommutativity} \gg \underline{y} * \underline{z} = \underline{z} * \underline{y}; \mathrm{SubLessLeft} \rhd \underline{x} * \underline{z} = \underline{z} * \underline{x} \rhd \dot\neg \underline{x} * \underline{z} <= \underline{y} * \underline{z} \Rightarrow \dot\neg\dot\neg \underline{x} * \underline{z} = \underline{y} * \underline{z} \gg \dot\neg \underline{z} * \underline{x} <= \underline{y} * \underline{z} \Rightarrow \dot\neg\dot\neg \underline{z} * \underline{x} = \underline{y} * \underline{z}; \mathrm{SubLessRight} \rhd \underline{y} * \underline{z} = \underline{z} * \underline{y} \rhd \dot\neg \underline{z} * \underline{x} <= \underline{y} * \underline{z} \Rightarrow \dot\neg\dot\neg \underline{z} * \underline{x} = \underline{y} * \underline{z} \gg \dot\neg \underline{z} * \underline{x} <= \underline{z} * \underline{y} \Rightarrow \dot\neg\dot\neg \underline{z} * \underline{x} = \underline{z} * \underline{y} \rceil, p_0, c)$]

[LessMultiplicationLeft $\overset{\text{stmt}}{\rightarrow}$ $\mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \dot\neg 0 <= \underline{z} \Rightarrow \dot\neg\dot\neg 0 = \underline{z} \vdash \dot\neg \underline{x} <= \underline{y} \Rightarrow \dot\neg\dot\neg \underline{x} = \underline{y} \vdash \dot\neg \underline{z} * \underline{x} <= \underline{z} * \underline{y} \Rightarrow \dot\neg\dot\neg \underline{z} * \underline{x} = \underline{z} * \underline{y}$]

[LessMultiplicationLeft $\overset{\text{tex}}{\rightarrow}$ "LessMultiplicationLeft"]

[LessMultiplicationLeft $\overset{\text{pyk}}{\rightarrow}$ "lemma lessMultiplicationLeft"]

LessDivision

[LessDivision $\overset{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon 0 <= \underline{z} \vdash \dot\neg \underline{x} * \underline{z} <= \underline{y} * \underline{z} \Rightarrow \dot\neg\dot\neg \underline{x} * \underline{z} = \underline{y} * \underline{z} \vdash \mathrm{FromLess} \rhd \dot\neg \underline{x} * \underline{z} <= \underline{y} * \underline{z} \Rightarrow \dot\neg\dot\neg \underline{x} * \underline{z} = \underline{y} * \underline{z} \gg \dot\neg \underline{y} * \underline{z} <= \underline{x} * \underline{z}; \mathrm{leqMultiplicationAxiom} \gg 0 <= \underline{z} \Rightarrow \underline{y} <= \underline{x} \Rightarrow \underline{y} * \underline{z} <= \underline{x} * \underline{z}; \mathrm{MP} \rhd 0 <= \underline{z} \Rightarrow \underline{y} <= \underline{x} \Rightarrow \underline{y} * \underline{z} <= \underline{x} * \underline{z} \rhd 0 <= \underline{z} \gg \underline{y} <= \underline{x} \Rightarrow \underline{y} * \underline{z} <=$

$\underline{x} * \underline{z}; \text{Contrapositive} \rhd \underline{y} <= \underline{x} \Rightarrow \underline{y} * \underline{z} <= \underline{x} * \underline{z} \gg \dot{\neg}\, \underline{y} * \underline{z} <= \underline{x} * \underline{z} \Rightarrow \dot{\neg}\, \underline{y} <= \underline{x}; \text{MP} \rhd \dot{\neg}\, \underline{y} * \underline{z} <= \underline{x} * \underline{z} \Rightarrow \dot{\neg}\, \underline{y} <= \underline{x} \rhd \dot{\neg}\, \underline{y} * \underline{z} <= \underline{x} * \underline{z} \gg \dot{\neg}\, \underline{y} <= \underline{x}; \text{ToLess} \rhd \dot{\neg}\, \underline{y} <= \underline{x} \gg \dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{x} = \underline{y}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\text{LessDivision} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon 0 <= \underline{z} \vdash \dot{\neg}\, \underline{x} * \underline{z} <= \underline{y} * \underline{z} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{x} * \underline{z} = \underline{y} * \underline{z} \vdash \dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{x} = \underline{y}]$

$[\text{LessDivision} \stackrel{\text{tex}}{\rightarrow}$ "LessDivision"]

$[\text{LessDivision} \stackrel{\text{pyk}}{\rightarrow}$ "lemma lessDivision"]

AddEquations(Less)

$[\text{AddEquations(Less)} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \forall \underline{u}\colon \dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{x} = \underline{y} \vdash \dot{\neg}\, \underline{z} <= \underline{u} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{z} = \underline{u} \vdash \text{LessAddition} \rhd \dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{x} = \underline{y} \gg \dot{\neg}\, \underline{x}+\underline{z} <= \underline{y}+\underline{z} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{x}+\underline{z} = \underline{y}+\underline{z}; \text{LessAdditionLeft} \rhd \dot{\neg}\, \underline{z} <= \underline{u} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{z} = \underline{u} \gg \dot{\neg}\, \underline{y}+\underline{z} <= \underline{y}+\underline{u} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{y}+\underline{z} = \underline{y}+\underline{u}; \text{LessTransitivity} \rhd \dot{\neg}\, \underline{x}+\underline{z} <= \underline{y}+\underline{z} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{x}+\underline{z} = \underline{y}+\underline{z} \rhd \dot{\neg}\, \underline{y}+\underline{z} <= \underline{y}+\underline{u} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{y}+\underline{z} = \underline{y}+\underline{u} \gg \dot{\neg}\, \underline{x}+\underline{z} <= \underline{y}+\underline{u} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{x}+\underline{z} = \underline{y}+\underline{u}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\text{AddEquations(Less)} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \forall \underline{u}\colon \dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{x} = \underline{y} \vdash \dot{\neg}\, \underline{z} <= \underline{u} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{z} = \underline{u} \vdash \dot{\neg}\, \underline{x} + \underline{z} <= \underline{y} + \underline{u} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{x} + \underline{z} = \underline{y} + \underline{u}]$

$[\text{AddEquations(Less)} \stackrel{\text{tex}}{\rightarrow}$ "AddEquations(Less)"]

$[\text{AddEquations(Less)} \stackrel{\text{pyk}}{\rightarrow}$ "lemma addEquations(Less)"]

LessNegated

$[\text{LessNegated} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{x} = \underline{y} \vdash \text{LessLeq} \rhd \dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{x} = \underline{y} \gg \underline{x} <= \underline{y}; \text{LeqNegated} \rhd \underline{x} <= \underline{y} \gg -\underline{y} <= -\underline{x}; \text{LessNeq} \rhd \dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{x} = \underline{y} \gg \dot{\neg}\, \underline{x} = \underline{y}; \text{NeqNegated} \rhd \dot{\neg}\, \underline{x} = \underline{y} \gg \dot{\neg} - \underline{x} = -\underline{y}; \text{NeqSymmetry} \rhd \dot{\neg} - \underline{x} = -\underline{y} \gg \dot{\neg} - \underline{y} = -\underline{x}; \text{LeqNeqLess} \rhd -\underline{y} <= -\underline{x} \rhd \dot{\neg} - \underline{y} = -\underline{x} \gg \dot{\neg} - \underline{y} <= -\underline{x} \Rightarrow \dot{\neg}\dot{\neg} - \underline{y} = -\underline{x}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\text{LessNegated} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \dot{\neg}\, \underline{x} <= \underline{y} \Rightarrow \dot{\neg}\dot{\neg}\, \underline{x} = \underline{y} \vdash \dot{\neg} - \underline{y} <= -\underline{x} \Rightarrow \dot{\neg}\dot{\neg} - \underline{y} = -\underline{x}]$

$[\text{LessNegated} \stackrel{\text{tex}}{\rightarrow}$ "LessNegated"]

$[\text{LessNegated} \stackrel{\text{pyk}}{\rightarrow}$ "lemma lessNegated"]

PositiveNegated

$[\mathrm{PositiveNegated} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \dot{\neg} 0 <= \underline{\mathsf{x}} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{\mathsf{x}} \vdash \mathrm{LessNegated} \rhd \dot{\neg} 0 <= \underline{\mathsf{x}} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{\mathsf{x}} \gg \dot{\neg} - \underline{\mathsf{x}} <= -0 \Rightarrow \dot{\neg}\dot{\neg} - \underline{\mathsf{x}} = -0; -0 = 0 \gg -0 = 0; \mathrm{SubLessRight} \rhd -0 = 0 \rhd \dot{\neg} - \underline{\mathsf{x}} <= -0 \Rightarrow \dot{\neg}\dot{\neg} - \underline{\mathsf{x}} = -0 \gg \dot{\neg} - \underline{\mathsf{x}} <= 0 \Rightarrow \dot{\neg}\dot{\neg} - \underline{\mathsf{x}} = 0 \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{PositiveNegated} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \dot{\neg} 0 <= \underline{\mathsf{x}} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{\mathsf{x}} \vdash \dot{\neg} - \underline{\mathsf{x}} <= 0 \Rightarrow \dot{\neg}\dot{\neg} - \underline{\mathsf{x}} = 0]$

$[\mathrm{PositiveNegated} \stackrel{\mathrm{tex}}{\rightarrow}$ "PositiveNegated"$]$

$[\mathrm{PositiveNegated} \stackrel{\mathrm{pyk}}{\rightarrow}$ "lemma positiveNegated"$]$

NonpositiveNegated

$[\mathrm{NonpositiveNegated} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \underline{\mathsf{x}} <= 0 \vdash \mathrm{LeqNegated} \rhd \underline{\mathsf{x}} <= 0 \gg -0 <= -\underline{\mathsf{x}}; -0 = 0 \gg -0 = 0; \mathrm{subLeqLeft} \rhd -0 = 0 \rhd -0 <= -\underline{\mathsf{x}} \gg 0 <= -\underline{\mathsf{x}} \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{NonpositiveNegated} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \underline{\mathsf{x}} <= 0 \vdash 0 <= -\underline{\mathsf{x}}]$

$[\mathrm{NonpositiveNegated} \stackrel{\mathrm{tex}}{\rightarrow}$ "NonpositiveNegated"$]$

$[\mathrm{NonpositiveNegated} \stackrel{\mathrm{pyk}}{\rightarrow}$ "lemma nonpositiveNegated"$]$

NegativeNegated

$[\mathrm{NegativeNegated} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \dot{\neg} \underline{\mathsf{x}} <= 0 \Rightarrow \dot{\neg}\dot{\neg} \underline{\mathsf{x}} = 0 \vdash \mathrm{LessNegated} \rhd \dot{\neg} \underline{\mathsf{x}} <= 0 \Rightarrow \dot{\neg}\dot{\neg} \underline{\mathsf{x}} = 0 \gg \dot{\neg} - 0 <= -\underline{\mathsf{x}} \Rightarrow \dot{\neg}\dot{\neg} - 0 = -\underline{\mathsf{x}}; -0 = 0 \gg -0 = 0; \mathrm{SubLessLeft} \rhd -0 = 0 \rhd \dot{\neg} - 0 <= -\underline{\mathsf{x}} \Rightarrow \dot{\neg}\dot{\neg} - 0 = -\underline{\mathsf{x}} \gg \dot{\neg} 0 <= -\underline{\mathsf{x}} \Rightarrow \dot{\neg}\dot{\neg} 0 = -\underline{\mathsf{x}} \rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{NegativeNegated} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \dot{\neg} \underline{\mathsf{x}} <= 0 \Rightarrow \dot{\neg}\dot{\neg} \underline{\mathsf{x}} = 0 \vdash \dot{\neg} 0 <= -\underline{\mathsf{x}} \Rightarrow \dot{\neg}\dot{\neg} 0 = -\underline{\mathsf{x}}]$

$[\mathrm{NegativeNegated} \stackrel{\mathrm{tex}}{\rightarrow}$ "NegativeNegated"$]$

$[\mathrm{NegativeNegated} \stackrel{\mathrm{pyk}}{\rightarrow}$ "lemma negativeNegated"$]$

NonnegativeNegated

$[\mathrm{NonnegativeNegated} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon 0 <= \underline{\mathsf{x}} \vdash \mathrm{LeqNegated} \rhd 0 <= \underline{\mathsf{x}} \gg -\underline{\mathsf{x}} <= -0; -0 = 0 \gg -0 = 0; \mathrm{subLeqRight} \rhd -0 = 0 \rhd -\underline{\mathsf{x}} <= -0 \gg$

$-\underline{x} <= 0\rceil, p_0, c)]$

[NonnegativeNegated $\stackrel{stmt}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}: 0 <= \underline{x} \vdash -\underline{x} <= 0$]

[NonnegativeNegated $\stackrel{tex}{\rightarrow}$ "NonnegativeNegated"]

[NonnegativeNegated $\stackrel{pyk}{\rightarrow}$ "lemma nonnegativeNegated"]

PositiveHalved

[PositiveHalved $\stackrel{proof}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}: \dot{\neg} 0 <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{x} \vdash 0 < 1/2 \gg \dot{\neg} 0 <= \text{rec1} + 1 \Rightarrow \dot{\neg}\dot{\neg} 0 = \text{rec1} + 1; \text{LessMultiplicationLeft} \rhd \dot{\neg} 0 <= \text{rec1} + 1 \Rightarrow \dot{\neg}\dot{\neg} 0 = \text{rec1} + 1 \rhd \dot{\neg} 0 <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{x} \gg \dot{\neg} \text{rec1} + 1 * 0 <= \text{rec1} + 1 * \underline{x} \Rightarrow \dot{\neg}\dot{\neg} \text{rec1} + 1 * 0 = \text{rec1} + 1 * \underline{x}; x * 0 = 0 \gg \text{rec1} + 1 * 0 = 0; \text{SubLessLeft} \rhd \text{rec1} + 1 * 0 = 0 \rhd \dot{\neg} \text{rec1} + 1 * 0 <= \text{rec1} + 1 * \underline{x} \Rightarrow \dot{\neg}\dot{\neg} \text{rec1} + 1 * 0 = \text{rec1} + 1 * \underline{x} \gg \dot{\neg} 0 <= \text{rec1} + 1 * \underline{x} \Rightarrow \dot{\neg}\dot{\neg} 0 = \text{rec1} + 1 * \underline{x}\rceil, p_0, c)$]

[PositiveHalved $\stackrel{stmt}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}: \dot{\neg} 0 <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{x} \vdash \dot{\neg} 0 <= \text{rec1} + 1 * \underline{x} \Rightarrow \dot{\neg}\dot{\neg} 0 = \text{rec1} + 1 * \underline{x}$]

[PositiveHalved $\stackrel{tex}{\rightarrow}$ "PositiveHalved"]

[PositiveHalved $\stackrel{pyk}{\rightarrow}$ "lemma positiveHalved"]

NonnegativeNumerical

[NonnegativeNumerical $\stackrel{proof}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}: 0 <= \underline{x} \vdash \text{IfThenElse(T)} \rhd 0 <= \underline{x} \gg \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \underline{x}; \text{Repetition} \rhd \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \underline{x} \gg \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \underline{x}\rceil, p_0, c)$]

[NonnegativeNumerical $\stackrel{stmt}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}: 0 <= \underline{x} \vdash \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \underline{x}$]

[NonnegativeNumerical $\stackrel{tex}{\rightarrow}$ "NonnegativeNumerical"]

[NonnegativeNumerical $\stackrel{pyk}{\rightarrow}$ "lemma nonnegativeNumerical"]

NegativeNumerical

[NegativeNumerical $\stackrel{proof}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}: \dot{\neg} \underline{x} <= 0 \Rightarrow \dot{\neg}\dot{\neg} \underline{x} = 0 \vdash \text{FromLess} \rhd \dot{\neg} \underline{x} <= 0 \Rightarrow \dot{\neg}\dot{\neg} \underline{x} = 0 \gg \dot{\neg} 0 <= \underline{x}; \text{IfThenElse(F)} \rhd \dot{\neg} 0 <= \underline{x} \gg \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = -\underline{x}; \text{Repetition} \rhd \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = -\underline{x} \gg \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = -\underline{x}\rceil, p_0, c)$]

[NegativeNumerical $\stackrel{stmt}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}: \dot{\neg} \underline{x} <= 0 \Rightarrow \dot{\neg}\dot{\neg} \underline{x} = 0 \vdash \text{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = -\underline{x}$]

$[\text{NegativeNumerical} \overset{\text{tex}}{\to} \text{“NegativeNumerical”}]$

$[\text{NegativeNumerical} \overset{\text{pyk}}{\to} \text{“lemma negativeNumerical”}]$

PositiveNumerical

$[\text{PositiveNumerical} \overset{\text{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \dot{\neg} 0 <= \underline{\mathsf{x}} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{\mathsf{x}} \vdash \text{LessLeq} \rhd \dot{\neg} 0 <= \underline{\mathsf{x}} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{\mathsf{x}} \gg 0 <= \underline{\mathsf{x}}; \text{NonnegativeNumerical} \rhd 0 <= \underline{\mathsf{x}} \gg \text{if}(0 <= \underline{\mathsf{x}}, \underline{\mathsf{x}}, -\underline{\mathsf{x}}) = \underline{\mathsf{x}} \rceil, \text{p}_0, \mathsf{c})]$

$[\text{PositiveNumerical} \overset{\text{stmt}}{\to} \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \dot{\neg} 0 <= \underline{\mathsf{x}} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{\mathsf{x}} \vdash \text{if}(0 <= \underline{\mathsf{x}}, \underline{\mathsf{x}}, -\underline{\mathsf{x}}) = \underline{\mathsf{x}}]$

$[\text{PositiveNumerical} \overset{\text{tex}}{\to} \text{“PositiveNumerical”}]$

$[\text{PositiveNumerical} \overset{\text{pyk}}{\to} \text{“lemma positiveNumerical”}]$

lemma nonpositiveNumerical

$[\text{lemma nonpositiveNumerical} \overset{\text{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \dot{\neg} \underline{\mathsf{x}} <= 0 \Rightarrow \dot{\neg}\dot{\neg} \underline{\mathsf{x}} = 0 \vdash \text{NegativeNumerical} \rhd \dot{\neg} \underline{\mathsf{x}} <= 0 \Rightarrow \dot{\neg}\dot{\neg} \underline{\mathsf{x}} = 0 \gg \text{if}(0 <= \underline{\mathsf{x}}, \underline{\mathsf{x}}, -\underline{\mathsf{x}}) = -\underline{\mathsf{x}}; \forall \underline{\mathsf{x}}\colon \underline{\mathsf{x}} = 0 \vdash \text{eqSymmetry} \rhd \underline{\mathsf{x}} = 0 \gg 0 = \underline{\mathsf{x}}; \text{eqLeq} \rhd 0 = \underline{\mathsf{x}} \gg 0 <= \underline{\mathsf{x}}; \text{NonnegativeNumerical} \rhd 0 <= \underline{\mathsf{x}} \gg \text{if}(0 <= \underline{\mathsf{x}}, \underline{\mathsf{x}}, -\underline{\mathsf{x}}) = \underline{\mathsf{x}}; -0 = 0 \gg -0 = 0; \text{eqSymmetry} \rhd -0 = 0 \gg 0 = -0; \text{EqNegated} \rhd 0 = \underline{\mathsf{x}} \gg -0 = -\underline{\mathsf{x}}; \text{eqTransitivity5} \rhd \text{if}(0 <= \underline{\mathsf{x}}, \underline{\mathsf{x}}, -\underline{\mathsf{x}}) = \underline{\mathsf{x}} \rhd \underline{\mathsf{x}} = 0 \rhd 0 = -0 \rhd -0 = -\underline{\mathsf{x}} \gg \text{if}(0 <= \underline{\mathsf{x}}, \underline{\mathsf{x}}, -\underline{\mathsf{x}}) = -\underline{\mathsf{x}}; \forall \underline{\mathsf{x}}\colon \text{Ded} \rhd \forall \underline{\mathsf{x}}\colon \dot{\neg} \underline{\mathsf{x}} <= 0 \Rightarrow \dot{\neg}\dot{\neg} \underline{\mathsf{x}} = 0 \vdash \text{if}(0 <= \underline{\mathsf{x}}, \underline{\mathsf{x}}, -\underline{\mathsf{x}}) = -\underline{\mathsf{x}} \gg \dot{\neg} \underline{\mathsf{x}} <= 0 \Rightarrow \dot{\neg}\dot{\neg} \underline{\mathsf{x}} = 0 \Rightarrow \text{if}(0 <= \underline{\mathsf{x}}, \underline{\mathsf{x}}, -\underline{\mathsf{x}}) = -\underline{\mathsf{x}}; \text{Ded} \rhd \forall \underline{\mathsf{x}}\colon \underline{\mathsf{x}} = 0 \vdash \text{if}(0 <= \underline{\mathsf{x}}, \underline{\mathsf{x}}, -\underline{\mathsf{x}}) = -\underline{\mathsf{x}} \gg \underline{\mathsf{x}} = 0 \Rightarrow \text{if}(0 <= \underline{\mathsf{x}}, \underline{\mathsf{x}}, -\underline{\mathsf{x}}) = -\underline{\mathsf{x}}; \underline{\mathsf{x}} <= 0 \vdash \text{LeqLessEq} \rhd \underline{\mathsf{x}} <= 0 \gg \dot{\neg}\dot{\neg} \underline{\mathsf{x}} <= 0 \Rightarrow \dot{\neg}\dot{\neg} \underline{\mathsf{x}} = 0 \Rightarrow \underline{\mathsf{x}} = 0; \text{FromDisjuncts} \rhd \dot{\neg}\dot{\neg} \underline{\mathsf{x}} <= 0 \Rightarrow \dot{\neg}\dot{\neg} \underline{\mathsf{x}} = 0 \Rightarrow \underline{\mathsf{x}} = 0 \rhd \dot{\neg} \underline{\mathsf{x}} <= 0 \Rightarrow \dot{\neg}\dot{\neg} \underline{\mathsf{x}} = 0 \Rightarrow \text{if}(0 <= \underline{\mathsf{x}}, \underline{\mathsf{x}}, -\underline{\mathsf{x}}) = -\underline{\mathsf{x}} \rhd \underline{\mathsf{x}} = 0 \Rightarrow \text{if}(0 <= \underline{\mathsf{x}}, \underline{\mathsf{x}}, -\underline{\mathsf{x}}) = -\underline{\mathsf{x}} \gg \text{if}(0 <= \underline{\mathsf{x}}, \underline{\mathsf{x}}, -\underline{\mathsf{x}}) = -\underline{\mathsf{x}} \rceil, \text{p}_0, \mathsf{c})]$

$[\text{lemma nonpositiveNumerical} \overset{\text{stmt}}{\to} \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \underline{\mathsf{x}} <= 0 \vdash \text{if}(0 <= \underline{\mathsf{x}}, \underline{\mathsf{x}}, -\underline{\mathsf{x}}) = -\underline{\mathsf{x}}]$

$[\text{lemma nonpositiveNumerical} \overset{\text{pyk}}{\to} \text{“lemma nonpositiveNumerical”}]$

$|0| = 0$

$[|0| = 0 \overset{\text{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \text{leqReflexivity} \gg 0 <= 0; \text{NonnegativeNumerical} \rhd 0 <= 0 \gg \text{if}(0 <= 0, 0, -0) = 0 \rceil, \text{p}_0, \mathsf{c})]$

$[|0| = 0 \overset{\text{stmt}}{\to} \text{ZFsub} \vdash \text{if}(0 <= 0, 0, -0) = 0]$

[$|0| = 0 \stackrel{\mathrm{tex}}{\rightarrow}$ "|0|=0"]

[$|0| = 0 \stackrel{\mathrm{pyk}}{\rightarrow}$ "lemma |0|=0"]

$0 <= |x|$

[$0 <= |x| \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x}\colon 0 <= \underline{x} \vdash \mathrm{NonnegativeNumerical} \rhd 0 <= \underline{x} \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \underline{x}; \mathrm{eqSymmetry} \rhd \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \underline{x} \gg \underline{x} = \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}); \mathrm{subLeqRight} \rhd \underline{x} = \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \rhd 0 <= \underline{x} \gg 0 <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}); \forall \underline{x}\colon \dot{\neg} 0 <= \underline{x} \vdash \mathrm{ToLess} \rhd \dot{\neg} 0 <= \underline{x} \gg \dot{\neg} \underline{x} <= 0 \Rightarrow \dot{\neg} \dot{\neg} \underline{x} = 0; \mathrm{NegativeNumerical} \rhd \dot{\neg} \underline{x} <= 0 \Rightarrow \dot{\neg} \dot{\neg} \underline{x} = 0 \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = -\underline{x}; \mathrm{eqSymmetry} \rhd \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = -\underline{x} \gg -\underline{x} = \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}); \mathrm{NegativeNegated} \rhd \dot{\neg} \underline{x} <= 0 \Rightarrow \dot{\neg} \dot{\neg} \underline{x} = 0 \gg \dot{\neg} 0 <= -\underline{x} \Rightarrow \dot{\neg} \dot{\neg} 0 = -\underline{x}; \mathrm{LessLeq} \rhd \dot{\neg} 0 <= -\underline{x} \Rightarrow \dot{\neg} \dot{\neg} 0 = -\underline{x} \gg 0 <= -\underline{x}; \mathrm{subLeqRight} \rhd -\underline{x} = \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \rhd 0 <= -\underline{x} \gg 0 <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}); \forall \underline{x}\colon \mathrm{Ded} \rhd \forall \underline{x}\colon 0 <= \underline{x} \vdash 0 <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \gg 0 <= \underline{x} \Rightarrow 0 <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}); \mathrm{Ded} \rhd \forall \underline{x}\colon \dot{\neg} 0 <= \underline{x} \vdash 0 <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \gg \dot{\neg} 0 <= \underline{x} \Rightarrow 0 <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}); \mathrm{FromNegations} \rhd 0 <= \underline{x} \Rightarrow 0 <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \rhd \dot{\neg} 0 <= \underline{x} \Rightarrow 0 <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \gg 0 <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \rceil, \mathrm{p}_0, \mathsf{c})$]

[$0 <= |x| \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{x}\colon 0 <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x})$]

[$0 <= |x| \stackrel{\mathrm{tex}}{\rightarrow}$ "0<=|x|"]

[$0 <= |x| \stackrel{\mathrm{pyk}}{\rightarrow}$ "lemma 0<=|x|"]

SameNumerical

[$\mathrm{SameNumerical} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon 0 <= \underline{x} \vdash \underline{x} = \underline{y} \vdash \mathrm{NonnegativeNumerical} \rhd 0 <= \underline{x} \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \underline{x}; \mathrm{subLeqRight} \rhd \underline{x} = \underline{y} \rhd 0 <= \underline{x} \gg 0 <= \underline{y}; \mathrm{NonnegativeNumerical} \rhd 0 <= \underline{y} \gg \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = \underline{y}; \mathrm{eqSymmetry} \rhd \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = \underline{y} \gg \underline{y} = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \mathrm{eqTransitivity4} \rhd \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \underline{x} \rhd \underline{x} = \underline{y} \rhd \underline{y} = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \forall \underline{x}\colon \forall \underline{y}\colon \dot{\neg} 0 <= \underline{x} \vdash \underline{x} = \underline{y} \vdash \mathrm{ToLess} \rhd \dot{\neg} 0 <= \underline{x} \gg \dot{\neg} \underline{x} <= 0 \Rightarrow \dot{\neg} \dot{\neg} \underline{x} = 0; \mathrm{NegativeNumerical} \rhd \dot{\neg} \underline{x} <= 0 \Rightarrow \dot{\neg} \dot{\neg} \underline{x} = 0 \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = -\underline{x}; \mathrm{SubLessLeft} \rhd \underline{x} = \underline{y} \rhd \dot{\neg} \underline{x} <= 0 \Rightarrow \dot{\neg} \dot{\neg} \underline{x} = 0 \gg \dot{\neg} \underline{y} <= 0 \Rightarrow \dot{\neg} \dot{\neg} \underline{y} = 0; \mathrm{NegativeNumerical} \rhd \dot{\neg} \underline{y} <= 0 \Rightarrow \dot{\neg} \dot{\neg} \underline{y} = 0 \gg \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = -\underline{y}; \mathrm{eqSymmetry} \rhd \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = -\underline{y} \gg -\underline{y} = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \mathrm{EqNegated} \rhd \underline{x} = \underline{y} \gg -\underline{x} = -\underline{y}; \mathrm{eqTransitivity4} \rhd \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = -\underline{x} \rhd -\underline{x} = -\underline{y} \rhd -\underline{y} = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} = \underline{y} \vdash \mathrm{Ded} \rhd \forall \underline{x}\colon \forall \underline{y}\colon 0 <= \underline{x} \vdash \underline{x} = \underline{y} \vdash \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg 0 <= \underline{x} \Rightarrow \underline{x} = \underline{y} \Rightarrow \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \mathrm{Ded} \rhd \forall \underline{x}\colon \forall \underline{y}\colon \dot{\neg} 0 <= \underline{x} \vdash \underline{x} = \underline{y} \vdash \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \dot{\neg} 0 <= \underline{x} \Rightarrow \underline{x} = \underline{y} \Rightarrow \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <=$

$\underline{y}, \underline{y}, -\underline{y}); \mathrm{FromNegations} \rhd 0 <= \underline{x} \Rightarrow \underline{x} = \underline{y} \Rightarrow \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \rhd \dot{\neg} 0 <= \underline{x} \Rightarrow \underline{x} = \underline{y} \Rightarrow \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \underline{x} = \underline{y} \Rightarrow \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \mathrm{MP} \rhd \underline{x} = \underline{y} \Rightarrow \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \rhd \underline{x} = \underline{y} \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})\rceil, p_0, \mathsf{c})]$

$[\mathrm{SameNumerical} \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{x} \colon \forall \underline{y} \colon \underline{x} = \underline{y} \vdash \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})]$

$[\mathrm{SameNumerical} \stackrel{\mathrm{tex}}{\rightarrow}$ "SameNumerical"$]$

$[\mathrm{SameNumerical} \stackrel{\mathrm{pyk}}{\rightarrow}$ "lemma sameNumerical"$]$

SignNumerical(+)

$[\mathrm{SignNumerical}(+) \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x} \colon \dot{\neg} 0 <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{x} \vdash \mathrm{PositiveNumerical} \rhd \dot{\neg} 0 <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{x} \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \underline{x}; \mathrm{PositiveNegated} \rhd \dot{\neg} 0 <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{x} \gg \dot{\neg} - \underline{x} <= 0 \Rightarrow \dot{\neg}\dot{\neg} - \underline{x} = 0; \mathrm{NegativeNumerical} \rhd \dot{\neg} - \underline{x} <= 0 \Rightarrow \dot{\neg}\dot{\neg} - \underline{x} = 0 \gg \mathrm{if}(0 <= -\underline{x}, -\underline{x}, - - \underline{x}) = - - \underline{x}; \mathrm{DoubleMinus} \gg - - \underline{x} = \underline{x}; \mathrm{eqTransitivity} \rhd \mathrm{if}(0 <= -\underline{x}, -\underline{x}, - - \underline{x}) = - - \underline{x} \rhd - - \underline{x} = \underline{x} \gg \mathrm{if}(0 <= -\underline{x}, -\underline{x}, - - \underline{x}) = \underline{x}; \mathrm{eqSymmetry} \rhd \mathrm{if}(0 <= -\underline{x}, -\underline{x}, - - \underline{x}) = \underline{x} \gg \underline{x} = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, - - \underline{x}); \mathrm{eqTransitivity} \rhd \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \underline{x} \rhd \underline{x} = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, - - \underline{x}) \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, - - \underline{x})\rceil, p_0, \mathsf{c})]$

$[\mathrm{SignNumerical}(+) \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{x} \colon \dot{\neg} 0 <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{x} \vdash \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, - - \underline{x})]$

$[\mathrm{SignNumerical}(+) \stackrel{\mathrm{tex}}{\rightarrow}$ "SignNumerical(+)"$]$

$[\mathrm{SignNumerical}(+) \stackrel{\mathrm{pyk}}{\rightarrow}$ "lemma signNumerical(+)"$]$

SignNumerical

$[\mathrm{SignNumerical} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x} \colon \dot{\neg} \underline{x} <= 0 \Rightarrow \dot{\neg}\dot{\neg} \underline{x} = 0 \vdash \mathrm{NegativeNegated} \rhd \dot{\neg} \underline{x} <= 0 \Rightarrow \dot{\neg}\dot{\neg} \underline{x} = 0 \gg \dot{\neg} 0 <= -\underline{x} \Rightarrow \dot{\neg}\dot{\neg} 0 = -\underline{x}; \mathrm{SignNumerical}(+) \rhd \dot{\neg} 0 <= -\underline{x} \Rightarrow \dot{\neg}\dot{\neg} 0 = -\underline{x} \gg \mathrm{if}(0 <= -\underline{x}, -\underline{x}, - - \underline{x}) = \mathrm{if}(0 <= - - \underline{x}, - - \underline{x}, - - - \underline{x}); \mathrm{DoubleMinus} \gg - - \underline{x} = \underline{x}; \mathrm{SameNumerical} \rhd - - \underline{x} = \underline{x} \gg \mathrm{if}(0 <= - - \underline{x}, - - \underline{x}, - - - \underline{x}) = \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}); \mathrm{eqTransitivity} \rhd \mathrm{if}(0 <= -\underline{x}, -\underline{x}, - - \underline{x}) = \mathrm{if}(0 <= - - \underline{x}, - - \underline{x}, - - - \underline{x}) \rhd \mathrm{if}(0 <= - - \underline{x}, - - \underline{x}, - - - \underline{x}) = \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \gg \mathrm{if}(0 <= -\underline{x}, -\underline{x}, - - \underline{x}) = \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}); \mathrm{eqSymmetry} \rhd \mathrm{if}(0 <= -\underline{x}, -\underline{x}, - - \underline{x}) = \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, - - \underline{x}); \forall \underline{x} \colon \underline{x} = 0 \vdash \mathrm{EqNegated} \rhd \underline{x} = 0 \gg -\underline{x} = -0; -0 = 0 \gg -0 = 0; \mathrm{eqSymmetry} \rhd \underline{x} = 0 \gg 0 = \underline{x}; \mathrm{eqTransitivity4} \rhd -\underline{x} = -0 \rhd -0 = 0 \rhd 0 = \underline{x} \gg -\underline{x} = \underline{x}; \mathrm{eqSymmetry} \rhd -\underline{x} = \underline{x} \gg \underline{x} = -\underline{x}; \mathrm{SameNumerical} \rhd \underline{x} = -\underline{x} \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, - - \underline{x}); \forall \underline{x} \colon \dot{\neg} 0 <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{x} \vdash \mathrm{SignNumerical}(+) \rhd \dot{\neg} 0 <= \underline{x} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{x} \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <=$

$-\underline{x}, -\underline{x}, --\underline{x}); \forall \underline{x}\colon \mathrm{Ded} \rhd \forall \underline{x}\colon \dot{\neg}\, \underline{x} <= 0 \Rightarrow \dot{\neg}\,\dot{\neg}\, \underline{x} = 0 \vdash \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) \gg \dot{\neg}\, \underline{x} <= 0 \Rightarrow \dot{\neg}\,\dot{\neg}\, \underline{x} = 0 \Rightarrow \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}); \mathrm{Ded} \rhd \forall \underline{x}\colon \underline{x} = 0 \vdash \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) \gg \underline{x} = 0 \Rightarrow \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}); \mathrm{Ded} \rhd \forall \underline{x}\colon \dot{\neg}\, 0 <= \underline{x} \Rightarrow \dot{\neg}\,\dot{\neg}\, 0 = \underline{x} \vdash \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) \gg \dot{\neg}\, 0 <= \underline{x} \Rightarrow \dot{\neg}\,\dot{\neg}\, 0 = \underline{x} \Rightarrow \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}); \mathrm{LessTotality} \gg \dot{\neg}\,\dot{\neg}\, \underline{x} <= 0 \Rightarrow \dot{\neg}\,\dot{\neg}\, \underline{x} = 0 \Rightarrow \dot{\neg}\, \underline{x} = 0 \Rightarrow \dot{\neg}\, 0 <= \underline{x} \Rightarrow \dot{\neg}\,\dot{\neg}\, 0 = \underline{x}; \mathrm{From3Disjuncts} \rhd \dot{\neg}\,\dot{\neg}\, \underline{x} <= 0 \Rightarrow \dot{\neg}\,\dot{\neg}\, \underline{x} = 0 \Rightarrow \dot{\neg}\, \underline{x} = 0 \Rightarrow \dot{\neg}\, 0 <= \underline{x} \Rightarrow \dot{\neg}\,\dot{\neg}\, 0 = \underline{x} \rhd \dot{\neg}\, \underline{x} <= 0 \Rightarrow \dot{\neg}\,\dot{\neg}\, \underline{x} = 0 \Rightarrow \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) \rhd \underline{x} = 0 \Rightarrow \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) \rhd \dot{\neg}\, 0 <= \underline{x} \Rightarrow \dot{\neg}\,\dot{\neg}\, 0 = \underline{x} \Rightarrow \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x})\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{SignNumerical} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{x}\colon \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x})]$

$[\mathrm{SignNumerical} \overset{\mathrm{tex}}{\rightarrow}$ "SignNumerical"$]$

$[\mathrm{SignNumerical} \overset{\mathrm{pyk}}{\rightarrow}$ "lemma signNumerical"$]$

NumericalDifference

$[\mathrm{NumericalDifference} \overset{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \mathrm{SignNumerical} \gg \mathrm{if}(0 <= \underline{x}+-\underline{y}, \underline{x}+-\underline{y}, -\underline{x}+-\underline{y}) = \mathrm{if}(0 <= -\underline{x}+-\underline{y}, -\underline{x}+-\underline{y}, --\underline{x}+-\underline{y}); \mathrm{MinusNegated} \gg -\underline{x}+-\underline{y} = \underline{y}+-\underline{x}; \mathrm{SameNumerical} \rhd -\underline{x}+-\underline{y} = \underline{y}+-\underline{x} \gg \mathrm{if}(0 <= -\underline{x}+-\underline{y}, -\underline{x}+-\underline{y}, --\underline{x}+-\underline{y}) = \mathrm{if}(0 <= \underline{y}+-\underline{x}, \underline{y}+-\underline{x}, -\underline{y}+-\underline{x}); \mathrm{eqTransitivity} \rhd \mathrm{if}(0 <= \underline{x}+-\underline{y}, \underline{x}+-\underline{y}, -\underline{x}+-\underline{y}) = \mathrm{if}(0 <= -\underline{x}+-\underline{y}, -\underline{x}+-\underline{y}, --\underline{x}+-\underline{y}) \rhd \mathrm{if}(0 <= -\underline{x}+-\underline{y}, -\underline{x}+-\underline{y}, --\underline{x}+-\underline{y}) = \mathrm{if}(0 <= \underline{y}+-\underline{x}, \underline{y}+-\underline{x}, -\underline{y}+-\underline{x}) \gg \mathrm{if}(0 <= \underline{x}+-\underline{y}, \underline{x}+-\underline{y}, -\underline{x}+-\underline{y}) = \mathrm{if}(0 <= \underline{y}+-\underline{x}, \underline{y}+-\underline{x}, -\underline{y}+-\underline{x})\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{NumericalDifference} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \mathrm{if}(0 <= \underline{x}+-\underline{y}, \underline{x}+-\underline{y}, -\underline{x}+-\underline{y}) = \mathrm{if}(0 <= \underline{y}+-\underline{x}, \underline{y}+-\underline{x}, -\underline{y}+-\underline{x})]$

$[\mathrm{NumericalDifference} \overset{\mathrm{tex}}{\rightarrow}$ "NumericalDifference"$]$

$[\mathrm{NumericalDifference} \overset{\mathrm{pyk}}{\rightarrow}$ "lemma numericalDifference"$]$

SplitNumericalSumHelper

$[\mathrm{SplitNumericalSumHelper} \overset{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \mathrm{if}(0 <= -\underline{x}+-\underline{y}, -\underline{x}+-\underline{y}, --\underline{x}+-\underline{y}) <= \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) + \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y}) \vdash \mathrm{SignNumerical} \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}); \mathrm{SignNumerical} \gg \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y}); \mathrm{AddEquations} \rhd \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) \rhd \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y}) \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) + \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y}); \mathrm{eqSymmetry} \rhd \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) + \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y}) \gg \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) + \mathrm{if}(0 <=$

$-\underline{y}, -\underline{y}, - - \underline{y}) = \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); -x - y = -(x+y) \gg -\underline{x} + -\underline{y} = -\underline{x} + \underline{y};$ SameNumerical $\rhd -\underline{x} + -\underline{y} = -\underline{x}+\underline{y} \gg \mathrm{if}(0 <= -\underline{x} + -\underline{y}, -\underline{x} + -\underline{y}, - - \underline{x} + -\underline{y}) = \mathrm{if}(0 <= -\underline{x}+\underline{y}, -\underline{x}+\underline{y}, - - \underline{x}+\underline{y});$ SignNumerical $\gg \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) = \mathrm{if}(0 <= -\underline{x}+\underline{y}, -\underline{x}+\underline{y}, - - \underline{x}+\underline{y});$ eqSymmetry $\rhd \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) = \mathrm{if}(0 <= -\underline{x}+\underline{y}, -\underline{x}+\underline{y}, - - \underline{x}+\underline{y}) \gg \mathrm{if}(0 <= -\underline{x}+\underline{y}, -\underline{x}+\underline{y}, - - \underline{x}+\underline{y}) = \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y});$ eqTransitivity $\rhd \mathrm{if}(0 <= -\underline{x}+-\underline{y}, -\underline{x}+-\underline{y}, - - \underline{x}+-\underline{y}) = \mathrm{if}(0 <= -\underline{x}+\underline{y}, -\underline{x}+\underline{y}, - - \underline{x}+\underline{y}) \rhd \mathrm{if}(0 <= -\underline{x}+\underline{y}, -\underline{x}+\underline{y}, - - \underline{x}+\underline{y}) = \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) \gg \mathrm{if}(0 <= -\underline{x}+-\underline{y}, -\underline{x}+-\underline{y}, - - \underline{x}+-\underline{y}) = \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y});$ subLeqRight $\rhd \mathrm{if}(0 <= -\underline{x}, -\underline{x}, - - \underline{x}) + \mathrm{if}(0 <= -\underline{y}, -\underline{y}, - - \underline{y}) = \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \rhd \mathrm{if}(0 <= -\underline{x}+-\underline{y}, -\underline{x}+-\underline{y}, - - \underline{x}+-\underline{y}) <= \mathrm{if}(0 <= -\underline{x}, -\underline{x}, - - \underline{x}) + \mathrm{if}(0 <= -\underline{y}, -\underline{y}, - - \underline{y}) \gg \mathrm{if}(0 <= -\underline{x}+-\underline{y}, -\underline{x}+-\underline{y}, - - \underline{x}+-\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y});$ subLeqLeft $\rhd \mathrm{if}(0 <= -\underline{x}+-\underline{y}, -\underline{x}+-\underline{y}, - - \underline{x}+-\underline{y}) = \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) \rhd \mathrm{if}(0 <= -\underline{x}+-\underline{y}, -\underline{x}+-\underline{y}, - - \underline{x}+-\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})\rceil, p_0, \mathsf{c})]$

[SplitNumericalSumHelper $\overset{\mathrm{stmt}}{\to}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon \mathrm{if}(0 <= -\underline{x}+-\underline{y}, -\underline{x}+-\underline{y}, - - \underline{x}+-\underline{y}) <= \mathrm{if}(0 <= -\underline{x}, -\underline{x}, - - \underline{x}) + \mathrm{if}(0 <= -\underline{y}, -\underline{y}, - - \underline{y}) \vdash \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})$]

[SplitNumericalSumHelper $\overset{\mathrm{tex}}{\to}$ "SplitNumericalSumHelper"]

[SplitNumericalSumHelper $\overset{\mathrm{pyk}}{\to}$ "lemma splitNumericalSumHelper"]

splitNumericalSum(++)

[splitNumericalSum(++) $\overset{\mathrm{proof}}{\to}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon 0 <= \underline{x} \vdash 0 <= \underline{y} \vdash$ AddEquations(Leq) $\rhd 0 <= \underline{x} \rhd 0 <= \underline{y} \gg 0 + 0 <= \underline{x} + \underline{y};$ plus0 $\gg 0 + 0 = 0;$ subLeqLeft $\rhd 0+0 = 0 \rhd 0+0 <= \underline{x}+\underline{y} \gg 0 <= \underline{x}+\underline{y};$ NonnegativeNumerical $\rhd 0 <= \underline{x}+\underline{y} \gg \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) = \underline{x}+\underline{y};$ NonnegativeNumerical $\rhd 0 <= \underline{x} \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \underline{x};$ NonnegativeNumerical $\rhd 0 <= \underline{y} \gg \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = \underline{y};$ AddEquations $\rhd \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \underline{x} \rhd \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = \underline{y} \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = \underline{x}+\underline{y};$ eqSymmetry $\rhd \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = \underline{x}+\underline{y} \gg \underline{x}+\underline{y} = \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y});$ eqTransitivity $\rhd \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) = \underline{x}+\underline{y} \rhd \underline{x}+\underline{y} = \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) = \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y});$ eqLeq $\rhd \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) = \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})\rceil, p_0, \mathsf{c})$]

[splitNumericalSum(++) $\overset{\mathrm{stmt}}{\to}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon 0 <= \underline{x} \vdash 0 <= \underline{y} \vdash \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})$]

$[\text{splitNumericalSum}(++) \overset{\text{tex}}{\rightarrow}$ "splitNumericalSum(++)"$]$

$[\text{splitNumericalSum}(++) \overset{\text{pyk}}{\rightarrow}$ "lemma splitNumericalSum(++)"$]$

splitNumericalSum(−−)

$[\text{splitNumericalSum}(--) \overset{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \underline{\mathsf{x}} <= 0 \vdash \underline{\mathsf{y}} <= 0 \vdash \text{NonpositiveNegated} \rhd \underline{\mathsf{x}} <= 0 \gg 0 <= -\underline{\mathsf{x}}; \text{NonpositiveNegated} \rhd \underline{\mathsf{y}} <= 0 \gg 0 <= -\underline{\mathsf{y}}; \text{splitNumericalSum}(++) \rhd 0 <= -\underline{\mathsf{x}} \rhd 0 <= -\underline{\mathsf{y}} \gg \text{if}(0 <= -\underline{\mathsf{x}} + -\underline{\mathsf{y}}, -\underline{\mathsf{x}} + -\underline{\mathsf{y}}, - - \underline{\mathsf{x}} + -\underline{\mathsf{y}}) <= \text{if}(0 <= -\underline{\mathsf{x}}, -\underline{\mathsf{x}}, - - \underline{\mathsf{x}}) + \text{if}(0 <= -\underline{\mathsf{y}}, -\underline{\mathsf{y}}, - - \underline{\mathsf{y}}); \text{SplitNumericalSumHelper} \rhd \text{if}(0 <= -\underline{\mathsf{x}} + -\underline{\mathsf{y}}, -\underline{\mathsf{x}} + -\underline{\mathsf{y}}, - - \underline{\mathsf{x}} + -\underline{\mathsf{y}}) <= \text{if}(0 <= -\underline{\mathsf{x}}, -\underline{\mathsf{x}}, - - \underline{\mathsf{x}}) + \text{if}(0 <= -\underline{\mathsf{y}}, -\underline{\mathsf{y}}, - - \underline{\mathsf{y}}) \gg \text{if}(0 <= \underline{\mathsf{x}} + \underline{\mathsf{y}}, \underline{\mathsf{x}} + \underline{\mathsf{y}}, -\underline{\mathsf{x}} + \underline{\mathsf{y}}) <= \text{if}(0 <= \underline{\mathsf{x}}, \underline{\mathsf{x}}, -\underline{\mathsf{x}}) + \text{if}(0 <= \underline{\mathsf{y}}, \underline{\mathsf{y}}, -\underline{\mathsf{y}})\rceil, \text{p}_0, \mathsf{c})]$

$[\text{splitNumericalSum}(--) \overset{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \underline{\mathsf{x}} <= 0 \vdash \underline{\mathsf{y}} <= 0 \vdash \text{if}(0 <= \underline{\mathsf{x}} + \underline{\mathsf{y}}, \underline{\mathsf{x}} + \underline{\mathsf{y}}, -\underline{\mathsf{x}} + \underline{\mathsf{y}}) <= \text{if}(0 <= \underline{\mathsf{x}}, \underline{\mathsf{x}}, -\underline{\mathsf{x}}) + \text{if}(0 <= \underline{\mathsf{y}}, \underline{\mathsf{y}}, -\underline{\mathsf{y}})]$

$[\text{splitNumericalSum}(--) \overset{\text{tex}}{\rightarrow}$ "splitNumericalSum(--)"$]$

$[\text{splitNumericalSum}(--) \overset{\text{pyk}}{\rightarrow}$ "lemma splitNumericalSum(--)"$]$

splitNumericalSum(+ − small)

$[\text{splitNumericalSum}(+-\text{small}) \overset{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon 0 <= \underline{\mathsf{x}} \vdash \underline{\mathsf{y}} <= 0 \vdash \text{if}(0 <= \underline{\mathsf{y}}, \underline{\mathsf{y}}, -\underline{\mathsf{y}}) <= \text{if}(0 <= \underline{\mathsf{x}}, \underline{\mathsf{x}}, -\underline{\mathsf{x}}) \vdash \text{LeqAdditionLeft} \rhd \underline{\mathsf{y}} <= 0 \gg \underline{\mathsf{x}} + \underline{\mathsf{y}} <= \underline{\mathsf{x}} + 0; \text{plus0} \gg \underline{\mathsf{x}} + 0 = \underline{\mathsf{x}}; \text{subLeqRight} \rhd \underline{\mathsf{x}} + 0 = \underline{\mathsf{x}} \rhd \underline{\mathsf{x}} + \underline{\mathsf{y}} <= \underline{\mathsf{x}} + 0 \gg \underline{\mathsf{x}} + \underline{\mathsf{y}} <= \underline{\mathsf{x}}; \text{PositiveToRight(Leq)(1term)} \rhd \text{if}(0 <= \underline{\mathsf{y}}, \underline{\mathsf{y}}, -\underline{\mathsf{y}}) <= \text{if}(0 <= \underline{\mathsf{x}}, \underline{\mathsf{x}}, -\underline{\mathsf{x}}) \gg 0 <= \text{if}(0 <= \underline{\mathsf{x}}, \underline{\mathsf{x}}, -\underline{\mathsf{x}}) + -\text{if}(0 <= \underline{\mathsf{y}}, \underline{\mathsf{y}}, -\underline{\mathsf{y}}); \text{lemma nonpositiveNumerical} \rhd \underline{\mathsf{y}} <= 0 \gg \text{if}(0 <= \underline{\mathsf{y}}, \underline{\mathsf{y}}, -\underline{\mathsf{y}}) = -\underline{\mathsf{y}}; \text{EqNegated} \rhd \text{if}(0 <= \underline{\mathsf{y}}, \underline{\mathsf{y}}, -\underline{\mathsf{y}}) = -\underline{\mathsf{y}} \gg -\text{if}(0 <= \underline{\mathsf{y}}, \underline{\mathsf{y}}, -\underline{\mathsf{y}}) = - - \underline{\mathsf{y}}; \text{DoubleMinus} \gg - - \underline{\mathsf{y}} = \underline{\mathsf{y}}; \text{eqTransitivity} \rhd -\text{if}(0 <= \underline{\mathsf{y}}, \underline{\mathsf{y}}, -\underline{\mathsf{y}}) = - - \underline{\mathsf{y}} \rhd - - \underline{\mathsf{y}} = \underline{\mathsf{y}} \gg -\text{if}(0 <= \underline{\mathsf{y}}, \underline{\mathsf{y}}, -\underline{\mathsf{y}}) = \underline{\mathsf{y}}; \text{NonnegativeNumerical} \rhd 0 <= \underline{\mathsf{x}} \gg \text{if}(0 <= \underline{\mathsf{x}}, \underline{\mathsf{x}}, -\underline{\mathsf{x}}) = \underline{\mathsf{x}}; \text{AddEquations} \rhd \text{if}(0 <= \underline{\mathsf{x}}, \underline{\mathsf{x}}, -\underline{\mathsf{x}}) = \underline{\mathsf{x}} \rhd -\text{if}(0 <= \underline{\mathsf{y}}, \underline{\mathsf{y}}, -\underline{\mathsf{y}}) = \underline{\mathsf{y}} \gg \text{if}(0 <= \underline{\mathsf{x}}, \underline{\mathsf{x}}, -\underline{\mathsf{x}}) + -\text{if}(0 <= \underline{\mathsf{y}}, \underline{\mathsf{y}}, -\underline{\mathsf{y}}) = \underline{\mathsf{x}} + \underline{\mathsf{y}}; \text{subLeqRight} \rhd \text{if}(0 <= \underline{\mathsf{x}}, \underline{\mathsf{x}}, -\underline{\mathsf{x}}) + -\text{if}(0 <= \underline{\mathsf{y}}, \underline{\mathsf{y}}, -\underline{\mathsf{y}}) = \underline{\mathsf{x}} + \underline{\mathsf{y}} \rhd 0 <= \text{if}(0 <= \underline{\mathsf{x}}, \underline{\mathsf{x}}, -\underline{\mathsf{x}}) + -\text{if}(0 <= \underline{\mathsf{y}}, \underline{\mathsf{y}}, -\underline{\mathsf{y}}) \gg 0 <= \underline{\mathsf{x}} + \underline{\mathsf{y}}; \text{NonnegativeNumerical} \rhd 0 <= \underline{\mathsf{x}} + \underline{\mathsf{y}} \gg \text{if}(0 <= \underline{\mathsf{x}} + \underline{\mathsf{y}}, \underline{\mathsf{x}} + \underline{\mathsf{y}}, -\underline{\mathsf{x}} + \underline{\mathsf{y}}) = \underline{\mathsf{x}} + \underline{\mathsf{y}}; \text{eqSymmetry} \rhd \text{if}(0 <= \underline{\mathsf{x}} + \underline{\mathsf{y}}, \underline{\mathsf{x}} + \underline{\mathsf{y}}, -\underline{\mathsf{x}} + \underline{\mathsf{y}}) = \underline{\mathsf{x}} + \underline{\mathsf{y}} \gg \underline{\mathsf{x}} + \underline{\mathsf{y}} = \text{if}(0 <= \underline{\mathsf{x}} + \underline{\mathsf{y}}, \underline{\mathsf{x}} + \underline{\mathsf{y}}, -\underline{\mathsf{x}} + \underline{\mathsf{y}}); \text{eqSymmetry} \rhd \text{if}(0 <= \underline{\mathsf{x}}, \underline{\mathsf{x}}, -\underline{\mathsf{x}}) = \underline{\mathsf{x}} \gg \underline{\mathsf{x}} = \text{if}(0 <= \underline{\mathsf{x}}, \underline{\mathsf{x}}, -\underline{\mathsf{x}}); \text{subLeqLeft} \rhd \underline{\mathsf{x}} + \underline{\mathsf{y}} = \text{if}(0 <= \underline{\mathsf{x}} + \underline{\mathsf{y}}, \underline{\mathsf{x}} + \underline{\mathsf{y}}, -\underline{\mathsf{x}} + \underline{\mathsf{y}}) \rhd \underline{\mathsf{x}} + \underline{\mathsf{y}} <= \underline{\mathsf{x}} \gg \text{if}(0 <= \underline{\mathsf{x}} + \underline{\mathsf{y}}, \underline{\mathsf{x}} + \underline{\mathsf{y}}, -\underline{\mathsf{x}} + \underline{\mathsf{y}}) <= \underline{\mathsf{x}}; \text{subLeqRight} \rhd \underline{\mathsf{x}} = \text{if}(0 <= \underline{\mathsf{x}}, \underline{\mathsf{x}}, -\underline{\mathsf{x}}) \rhd \text{if}(0 <= \underline{\mathsf{x}} + \underline{\mathsf{y}}, \underline{\mathsf{x}} + \underline{\mathsf{y}}, -\underline{\mathsf{x}} + \underline{\mathsf{y}}) <= \underline{\mathsf{x}} \gg \text{if}(0 <= \underline{\mathsf{x}} + \underline{\mathsf{y}}, \underline{\mathsf{x}} + \underline{\mathsf{y}}, -\underline{\mathsf{x}} + \underline{\mathsf{y}}) <= \text{if}(0 <= \underline{\mathsf{x}}, \underline{\mathsf{x}}, -\underline{\mathsf{x}})\rceil, \text{p}_0, \mathsf{c})]$

$[\text{splitNumericalSum}(+-\text{small}) \overset{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon 0 <= \underline{\mathsf{x}} \vdash \underline{\mathsf{y}} <= 0 \vdash \text{if}(0 <=$

$\underline{y}, \underline{y}, -\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \vdash \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x})]$

[splitNumericalSum(+ − small) $\overset{\text{tex}}{\to}$ "splitNumericalSum(+-small)"]

[splitNumericalSum(+ − small) $\overset{\text{pyk}}{\to}$ "lemma splitNumericalSum(+-, smallNegative)"]

splitNumericalSum(+ − big)

[splitNumericalSum(+ − big) $\overset{\text{proof}}{\to}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon 0 <= \underline{x} \vdash \underline{y} <= 0 \vdash \dot\neg\, \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) <= \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \Rightarrow \dot\neg\dot\neg\, \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \vdash \text{NonnegativeNegated} \rhd 0 <= \underline{x} \gg -\underline{x} <= 0; \text{NonpositiveNegated} \rhd \underline{y} <= 0 \gg 0 <= -\underline{y}; \text{SignNumerical} \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}); \text{SubLessLeft} \rhd \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) \rhd \dot\neg\, \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) <= \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \Rightarrow \dot\neg\dot\neg\, \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \dot\neg\, \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) <= \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \Rightarrow \dot\neg\dot\neg\, \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \text{SignNumerical} \gg \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y}); \text{SubLessRight} \rhd \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y}) \rhd \dot\neg\, \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) <= \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \Rightarrow \dot\neg\dot\neg\, \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \dot\neg\, \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) <= \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y}) \Rightarrow \dot\neg\dot\neg\, \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) = \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y}); \text{LessLeq} \rhd \dot\neg\, \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) <= \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y}) \Rightarrow \dot\neg\dot\neg\, \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) = \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y}) \gg \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) <= \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y}); \text{splitNumericalSum}(+-\text{small}) \rhd 0 <= -\underline{y} \rhd -\underline{x} <= 0 \rhd \mathrm{if}(0 <= -\underline{x}, -\underline{x}, --\underline{x}) <= \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y}) \gg \mathrm{if}(0 <= -\underline{y}+-\underline{x}, -\underline{y}+-\underline{x}, --\underline{y}+-\underline{x}) <= \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y}); \text{SignNumerical} \gg \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) = \mathrm{if}(0 <= -\underline{x}+\underline{y}, -\underline{x}+\underline{y}, --\underline{x}+\underline{y}); -x-y = -(x+y) \gg -\underline{x}+-\underline{y} = -\underline{x}+\underline{y}; \text{plusCommutativity} \gg -\underline{x}+-\underline{y} = -\underline{y}+-\underline{x}; \text{Equality} \rhd -\underline{x}+-\underline{y} = -\underline{x}+\underline{y} \rhd -\underline{x}+-\underline{y} = -\underline{y}+-\underline{x} \gg -\underline{x}+\underline{y} = -\underline{y}+-\underline{x}; \text{SameNumerical} \rhd -\underline{x}+\underline{y} = -\underline{y}+-\underline{x} \gg \mathrm{if}(0 <= -\underline{x}+\underline{y}, -\underline{x}+\underline{y}, --\underline{x}+\underline{y}) = \mathrm{if}(0 <= -\underline{y}+-\underline{x}, -\underline{y}+-\underline{x}, --\underline{y}+-\underline{x}); \text{eqTransitivity} \rhd \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) = \mathrm{if}(0 <= -\underline{x}+\underline{y}, -\underline{x}+\underline{y}, --\underline{x}+\underline{y}) \rhd \mathrm{if}(0 <= -\underline{x}+\underline{y}, -\underline{x}+\underline{y}, --\underline{x}+\underline{y}) = \mathrm{if}(0 <= -\underline{y}+-\underline{x}, -\underline{y}+-\underline{x}, --\underline{y}+-\underline{x}) \gg \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) = \mathrm{if}(0 <= -\underline{y}+-\underline{x}, -\underline{y}+-\underline{x}, --\underline{y}+-\underline{x}); \text{eqSymmetry} \rhd \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) = \mathrm{if}(0 <= -\underline{y}+-\underline{x}, -\underline{y}+-\underline{x}, --\underline{y}+-\underline{x}) \gg \mathrm{if}(0 <= -\underline{y}+-\underline{x}, -\underline{y}+-\underline{x}, --\underline{y}+-\underline{x}) = \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}); \text{eqSymmetry} \rhd \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y}) \gg \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y}) = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \text{subLeqLeft} \rhd \mathrm{if}(0 <= -\underline{y}+-\underline{x}, -\underline{y}+-\underline{x}, --\underline{y}+-\underline{x}) = \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) \rhd \mathrm{if}(0 <= -\underline{y}+-\underline{x}, -\underline{y}+-\underline{x}, --\underline{y}+-\underline{x}) <= \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y}) \gg \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y}); \text{subLeqRight} \rhd \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y}) = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \rhd \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= -\underline{y}, -\underline{y}, --\underline{y}) \gg \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})\rceil, p_0, c)]$

[splitNumericalSum(+ − big) $\overset{\text{stmt}}{\to}$ $\text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon 0 <= \underline{x} \vdash \underline{y} <= 0 \vdash \dot\neg\, \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) <= \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \Rightarrow \dot\neg\dot\neg\, \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \vdash$

$\mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})$]

[splitNumericalSum(+ − big) $\stackrel{\text{tex}}{\to}$ “splitNumericalSum(+-big)”]

[splitNumericalSum(+ − big) $\stackrel{\text{pyk}}{\to}$ “lemma splitNumericalSum(+-, bigNegative)”]

splitNumericalSum(+−)

[splitNumericalSum(+−) $\stackrel{\text{proof}}{\to}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \vdash 0 <= \underline{x} \vdash \underline{y} <= 0 \vdash$ splitNumericalSum(+ − small) $\rhd 0 <= \underline{x} \rhd \underline{y} <= 0 \rhd \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \gg \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}); 0 <= |x| \gg 0 <= \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y});$ LeqAdditionLeft $\rhd 0 <= \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + 0 <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y});$ plus0 $\gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + 0 = \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x});$ subLeqLeft $\rhd \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + 0 = \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \rhd \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + 0 <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y});$ leqTransitivity $\rhd \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \rhd \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \forall \underline{x}\colon \forall \underline{y}\colon \dot{\neg}\, \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \vdash 0 <= \underline{x} \vdash \underline{y} <= 0 \vdash$ ToLess $\rhd \dot{\neg}\, \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \gg \dot{\neg}\, \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) <= \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \Rightarrow \dot{\neg}\dot{\neg}\, \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y});$ splitNumericalSum(+ − big) $\rhd 0 <= \underline{x} \rhd \underline{y} <= 0 \rhd \dot{\neg}\, \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) <= \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \Rightarrow \dot{\neg}\dot{\neg}\, \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); 0 <= |x| \gg 0 <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x});$ leqAddition $\rhd 0 <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \gg 0 + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y});$ plus0Left $\gg 0 + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y});$ subLeqLeft $\rhd 0 + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) = \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \rhd 0 + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y});$ leqTransitivity $\rhd \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \rhd \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \forall \underline{x}\colon \forall \underline{y}\colon$ Ded $\rhd \forall \underline{x}\colon \forall \underline{y}\colon \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \vdash 0 <= \underline{x} \vdash \underline{y} <= 0 \vdash \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \Rightarrow 0 <= \underline{x} \Rightarrow \underline{y} <= 0 \Rightarrow \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y});$ Ded $\rhd \forall \underline{x}\colon \forall \underline{y}\colon \dot{\neg}\, \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \vdash 0 <= \underline{x} \vdash \underline{y} <= 0 \vdash \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \dot{\neg}\, \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \Rightarrow 0 <= \underline{x} \Rightarrow \underline{y} <= 0 \Rightarrow \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); 0 <= \underline{x} \vdash \underline{y} <= 0 \vdash$ FromNegations $\rhd \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \Rightarrow 0 <= \underline{x} \Rightarrow \underline{y} <= 0 \Rightarrow \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \rhd \dot{\neg}\, \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) \Rightarrow 0 <= \underline{x} \Rightarrow \underline{y} <= 0 \Rightarrow \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg 0 <=$

$\underline{x} \Rightarrow \underline{y} <= 0 \Rightarrow \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \mathrm{MP2} \rhd 0 <= \underline{x} \Rightarrow \underline{y} <= 0 \Rightarrow \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \rhd 0 <= \underline{x} \rhd \underline{y} <= 0 \gg \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{splitNumericalSum}(+-) \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{x} \colon \forall \underline{y} \colon 0 <= \underline{x} \vdash \underline{y} <= 0 \vdash \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})]$

$[\mathrm{splitNumericalSum}(+-) \stackrel{\mathrm{tex}}{\rightarrow}$ "splitNumericalSum(+-)"]

$[\mathrm{splitNumericalSum}(+-) \stackrel{\mathrm{pyk}}{\rightarrow}$ "lemma splitNumericalSum(+-)"]

splitNumericalSum(−+)

$[\mathrm{splitNumericalSum}(-+) \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x} \colon \forall \underline{y} \colon \underline{x} <= 0 \vdash 0 <= \underline{y} \vdash \mathrm{NonpositiveNegated} \rhd \underline{x} <= 0 \gg 0 <= -\underline{x}; \mathrm{NonnegativeNegated} \rhd 0 <= \underline{y} \gg -\underline{y} <= 0; \mathrm{splitNumericalSum}(+-) \rhd 0 <= -\underline{x} \rhd -\underline{y} <= 0 \gg \mathrm{if}(0 <= -\underline{x} + -\underline{y}, -\underline{x} + -\underline{y}, - - \underline{x} + -\underline{y}) <= \mathrm{if}(0 <= -\underline{x}, -\underline{x}, - - \underline{x}) + \mathrm{if}(0 <= -\underline{y}, -\underline{y}, - - \underline{y}); \mathrm{SplitNumericalSumHelper} \rhd \mathrm{if}(0 <= -\underline{x} + -\underline{y}, -\underline{x} + -\underline{y}, - - \underline{x} + -\underline{y}) <= \mathrm{if}(0 <= -\underline{x}, -\underline{x}, - - \underline{x}) + \mathrm{if}(0 <= -\underline{y}, -\underline{y}, - - \underline{y}) \gg \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{splitNumericalSum}(-+) \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{x} \colon \forall \underline{y} \colon \underline{x} <= 0 \vdash 0 <= \underline{y} \vdash \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})]$

$[\mathrm{splitNumericalSum}(-+) \stackrel{\mathrm{tex}}{\rightarrow}$ "splitNumericalSum(-+)"]

$[\mathrm{splitNumericalSum}(-+) \stackrel{\mathrm{pyk}}{\rightarrow}$ "lemma splitNumericalSum(-+)"]

splitNumericalSum

$[\mathrm{splitNumericalSum} \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x} \colon \forall \underline{y} \colon 0 <= \underline{x} \vdash 0 <= \underline{y} \vdash \mathrm{splitNumericalSum}(++) \rhd 0 <= \underline{x} \rhd 0 <= \underline{y} \gg \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \forall \underline{x} \colon \forall \underline{y} \colon 0 <= \underline{x} \vdash \underline{y} <= 0 \vdash \mathrm{splitNumericalSum}(+-) \rhd 0 <= \underline{x} \rhd \underline{y} <= 0 \gg \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \forall \underline{x} \colon \forall \underline{y} \colon \underline{x} <= 0 \vdash 0 <= \underline{y} \vdash \mathrm{splitNumericalSum}(-+) \rhd \underline{x} <= 0 \rhd 0 <= \underline{y} \gg \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \forall \underline{x} \colon \forall \underline{y} \colon \underline{x} <= 0 \vdash \underline{y} <= 0 \vdash \mathrm{splitNumericalSum}(--) \rhd \underline{x} <= 0 \rhd \underline{y} <= 0 \gg \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \forall \underline{x} \colon \forall \underline{y} \colon \mathrm{Ded} \rhd \forall \underline{x} \colon \forall \underline{y} \colon 0 <= \underline{x} \vdash 0 <= \underline{y} \vdash \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg 0 <= \underline{x} \Rightarrow 0 <= \underline{y} \Rightarrow \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \mathrm{Ded} \rhd \forall \underline{x} \colon \forall \underline{y} \colon 0 <= \underline{x} \vdash \underline{y} <= 0 \vdash \mathrm{if}(0 <= \underline{x} + \underline{y}, \underline{x} + \underline{y}, -\underline{x} + \underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg 0 <= \underline{x} \Rightarrow \underline{y} <= 0 \Rightarrow \mathrm{if}(0 <=$

$\underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x})+\mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \mathrm{Ded} \rhd \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} <= 0 \vdash 0 <= \underline{y} \vdash \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \underline{x} <= 0 \Rightarrow 0 <= \underline{y} \Rightarrow \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \mathrm{Ded} \rhd \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} <= 0 \vdash \underline{y} <= 0 \vdash \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \underline{x} <= 0 \Rightarrow \underline{y} <= 0 \Rightarrow \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \mathrm{FromLeqGeq} \rhd 0 <= \underline{x} \Rightarrow 0 <= \underline{y} \Rightarrow \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \rhd \underline{x} <= 0 \Rightarrow 0 <= \underline{y} \Rightarrow \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg 0 <= \underline{y} \Rightarrow \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x})+\mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \mathrm{FromLeqGeq} \rhd 0 <= \underline{x} \Rightarrow \underline{y} <= 0 \Rightarrow \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \rhd \underline{x} <= 0 \Rightarrow \underline{y} <= 0 \Rightarrow \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x})+\mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \underline{y} <= 0 \Rightarrow \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x})+\mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}); \mathrm{FromLeqGeq} \rhd 0 <= \underline{y} \Rightarrow \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \rhd \underline{y} <= 0 \Rightarrow \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y}) \gg \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{splitNumericalSum} \overset{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}, \underline{x}, -\underline{x}) + \mathrm{if}(0 <= \underline{y}, \underline{y}, -\underline{y})]$

[splitNumericalSum $\overset{\mathrm{tex}}{\to}$ “splitNumericalSum”]

[splitNumericalSum $\overset{\mathrm{pyk}}{\to}$ “lemma splitNumericalSum”]

insertMiddleTerm(Numerical)

$[\mathrm{insertMiddleTerm(Numerical)} \overset{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \mathrm{splitNumericalSum} \gg \mathrm{if}(0 <= \underline{x}+-\underline{z}+\underline{z}+\underline{y}, \underline{x}+-\underline{z}+\underline{z}+\underline{y}, -\underline{x}+-\underline{z}+\underline{z}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}+-\underline{z}, \underline{x}+-\underline{z}, -\underline{x}+-\underline{z}) + \mathrm{if}(0 <= \underline{z}+\underline{y}, \underline{z}+\underline{y}, -\underline{z}+\underline{y}); \mathrm{insertMiddleTerm(Sum)} \gg \underline{x}+\underline{y} = \underline{x}+-\underline{z}+\underline{z}+\underline{y}; \mathrm{SameNumerical} \rhd \underline{x}+\underline{y} = \underline{x}+-\underline{z}+\underline{z}+\underline{y} \gg \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) = \mathrm{if}(0 <= \underline{x}+-\underline{z}+\underline{z}+\underline{y}, \underline{x}+-\underline{z}+\underline{z}+\underline{y}, -\underline{x}+-\underline{z}+\underline{z}+\underline{y}); \mathrm{eqSymmetry} \rhd \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) = \mathrm{if}(0 <= \underline{x}+-\underline{z}+\underline{z}+\underline{y}, \underline{x}+-\underline{z}+\underline{z}+\underline{y}, -\underline{x}+-\underline{z}+\underline{z}+\underline{y}) \gg \mathrm{if}(0 <= \underline{x}+-\underline{z}+\underline{z}+\underline{y}, \underline{x}+-\underline{z}+\underline{z}+\underline{y}, -\underline{x}+-\underline{z}+\underline{z}+\underline{y}) = \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}); \mathrm{subLeqLeft} \rhd \mathrm{if}(0 <= \underline{x}+-\underline{z}+\underline{z}+\underline{y}, \underline{x}+-\underline{z}+\underline{z}+\underline{y}, -\underline{x}+-\underline{z}+\underline{z}+\underline{y}) = \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) \rhd \mathrm{if}(0 <= \underline{x}+-\underline{z}+\underline{z}+\underline{y}, \underline{x}+-\underline{z}+\underline{z}+\underline{y}, -\underline{x}+-\underline{z}+\underline{z}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}+-\underline{z}, \underline{x}+-\underline{z}, -\underline{x}+-\underline{z})+\mathrm{if}(0 <= \underline{z}+\underline{y}, \underline{z}+\underline{y}, -\underline{z}+\underline{y}) \gg \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}+-\underline{z}, \underline{x}+-\underline{z}, -\underline{x}+-\underline{z})+\mathrm{if}(0 <= \underline{z}+\underline{y}, \underline{z}+\underline{y}, -\underline{z}+\underline{y})\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{insertMiddleTerm(Numerical)} \overset{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \mathrm{if}(0 <= \underline{x}+\underline{y}, \underline{x}+\underline{y}, -\underline{x}+\underline{y}) <= \mathrm{if}(0 <= \underline{x}+-\underline{z}, \underline{x}+-\underline{z}, -\underline{x}+-\underline{z})+\mathrm{if}(0 <= \underline{z}+\underline{y}, \underline{z}+\underline{y}, -\underline{z}+\underline{y})]$

[insertMiddleTerm(Numerical) $\overset{\mathrm{tex}}{\to}$ “insertMiddleTerm(Numerical)”]

[insertMiddleTerm(Numerical) $\overset{\mathrm{pyk}}{\to}$ “lemma insertMiddleTerm(Numerical)”]

$x + y = zBackwards$

$[x + y = zBackwards \stackrel{proof}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil ZFsub \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} + \underline{y} = \underline{z} \vdash plusCommutativity \gg \underline{x} + \underline{y} = \underline{y} + \underline{x}; Equality \rhd \underline{x} + \underline{y} = \underline{z} \gg \underline{z} = \underline{y} + \underline{x}\rceil, p_0, c)]$

$[x + y = zBackwards \stackrel{stmt}{\rightarrow} ZFsub \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} + \underline{y} = \underline{z} \vdash \underline{z} = \underline{y} + \underline{x}]$

$[x + y = zBackwards \stackrel{tex}{\rightarrow}$ “x+y=zBackwards”$]$

$[x + y = zBackwards \stackrel{pyk}{\rightarrow}$ “lemma x+y=zBackwards”$]$

$x * y = zBackwards$

$[x * y = zBackwards \stackrel{proof}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil ZFsub \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} * \underline{y} = \underline{z} \vdash timesCommutativity \gg \underline{x} * \underline{y} = \underline{y} * \underline{x}; Equality \rhd \underline{x} * \underline{y} = \underline{z} \gg \underline{z} = \underline{y} * \underline{x}\rceil, p_0, c)]$

$[x * y = zBackwards \stackrel{stmt}{\rightarrow} ZFsub \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} * \underline{y} = \underline{z} \vdash \underline{z} = \underline{y} * \underline{x}]$

$[x * y = zBackwards \stackrel{tex}{\rightarrow}$ “x∗y=zBackwards”$]$

$[x * y = zBackwards \stackrel{pyk}{\rightarrow}$ “lemma x∗y=zBackwards”$]$

$x = x + (y - y)$

$[x = x + (y - y) \stackrel{proof}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil ZFsub \vdash \forall \underline{x}\colon \forall \underline{y}\colon plus0 \gg \underline{x} + 0 = \underline{x}; Negative \gg \underline{y} + -\underline{y} = 0; eqSymmetry \rhd \underline{y} + -\underline{y} = 0 \gg 0 = \underline{y} + -\underline{y}; lemma\ eqAdditionLeft \rhd 0 = \underline{y} + -\underline{y} \gg \underline{x} + 0 = \underline{x} + \underline{y} + -\underline{y}; Equality \rhd \underline{x} + 0 = \underline{x} \rhd \underline{x} + 0 = \underline{x} + \underline{y} + -\underline{y} \gg \underline{x} = \underline{x} + \underline{y} + -\underline{y}\rceil, p_0, c)]$

$[x = x + (y - y) \stackrel{stmt}{\rightarrow} ZFsub \vdash \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} = \underline{x} + \underline{y} + -\underline{y}]$

$[x = x + (y - y) \stackrel{tex}{\rightarrow}$ “x=x+(y-y)”$]$

$[x = x + (y - y) \stackrel{pyk}{\rightarrow}$ “lemma x=x+(y-y)”$]$

$x = x + y - y$

$[x = x + y - y \stackrel{proof}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil ZFsub \vdash \forall \underline{x}\colon \forall \underline{y}\colon x = x + (y - y) \gg \underline{x} = \underline{x}+\underline{y}+-\underline{y}; plusAssociativity \gg \underline{x}+\underline{y}+-\underline{y} = \underline{x}+\underline{y}+-\underline{y}; eqSymmetry \rhd \underline{x}+\underline{y}+-\underline{y} = \underline{x} + \underline{y} + -\underline{y} \gg \underline{x} + \underline{y} + -\underline{y} = \underline{x} + \underline{y} + -\underline{y}; eqTransitivity \rhd \underline{x} = \underline{x} + \underline{y} + -\underline{y} \rhd \underline{x} + \underline{y} + -\underline{y} = \underline{x} + \underline{y} + -\underline{y} \gg \underline{x} = \underline{x} + \underline{y} + -\underline{y}\rceil, p_0, c)]$

$[x = x + y - y \stackrel{stmt}{\rightarrow} ZFsub \vdash \forall \underline{x}\colon \forall \underline{y}\colon \underline{x} = \underline{x} + \underline{y} + -\underline{y}]$

$[x = x + y - y \stackrel{\text{tex}}{\to} \text{“x=x+y-y”}]$

$[x = x + y - y \stackrel{\text{pyk}}{\to} \text{“lemma x=x+y-y”}]$

$[\stackrel{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \dot{\neg}\, \underline{y} = 0 \vdash \text{times1} \gg \underline{x} * 1 = \underline{x}; \text{Reciprocal} \rhd \dot{\neg}\, \underline{y} = 0 \gg \underline{y} * \text{rec}\underline{y} = 1; \text{Three2threeFactors} \rhd \underline{y} * \text{rec}\underline{y} = 1 \gg \underline{x} * \underline{y} * \text{rec}\underline{y} = \underline{x} * 1; \text{eqTransitivity} \rhd \underline{x} * \underline{y} * \text{rec}\underline{y} = \underline{x} * 1 \rhd \underline{x} * 1 = \underline{x} \gg \underline{x} * \underline{y} * \text{rec}\underline{y} = \underline{x}; \text{eqSymmetry} \rhd \underline{x} * \underline{y} * \text{rec}\underline{y} = \underline{x} \gg \underline{x} = \underline{x} * \underline{y} * \text{rec}\underline{y} \rceil, p_0, c)]$

$[\stackrel{\text{stmt}}{\to} \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \dot{\neg}\, \underline{y} = 0 \vdash \underline{x} = \underline{x} * \underline{y} * \text{rec}\underline{y}]$

$[\stackrel{\text{tex}}{\to} \text{“ ”}]$

$[\stackrel{\text{pyk}}{\to} \text{“lemma x=x*y*(1/y)”}]$

insertMiddleTerm(Sum)

$[\text{insertMiddleTerm(Sum)} \stackrel{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon x = x + y - y \gg \underline{x} = \underline{x} + \underline{z} + -\underline{z}; \text{Three2threeTerms} \gg \underline{x} + \underline{z} + -\underline{z} = \underline{x} + -\underline{z} + \underline{z}; \text{eqTransitivity} \rhd \underline{x} = \underline{x} + \underline{z} + -\underline{z} \rhd \underline{x} + \underline{z} + -\underline{z} = \underline{x} + -\underline{z} + \underline{z} \gg \underline{x} = \underline{x} + -\underline{z} + \underline{z}; \text{eqAddition} \rhd \underline{x} = \underline{x} + -\underline{z} + \underline{z} \gg \underline{x} + \underline{y} = \underline{x} + -\underline{z} + \underline{z} + \underline{y}; \text{plusAssociativity} \gg \underline{x} + -\underline{z} + \underline{z} + \underline{y} = \underline{x} + -\underline{z} + \underline{z} + \underline{y}; \text{eqTransitivity} \rhd \underline{x} + \underline{y} = \underline{x} + -\underline{z} + \underline{z} + \underline{y} \rhd \underline{x} + -\underline{z} + \underline{z} + \underline{y} = \underline{x} + -\underline{z} + \underline{z} + \underline{y} \gg \underline{x} + \underline{y} = \underline{x} + -\underline{z} + \underline{z} + \underline{y} \rceil, p_0, c)]$

$[\text{insertMiddleTerm(Sum)} \stackrel{\text{stmt}}{\to} \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} + \underline{y} = \underline{x} + -\underline{z} + \underline{z} + \underline{y}]$

$[\text{insertMiddleTerm(Sum)} \stackrel{\text{tex}}{\to} \text{“insertMiddleTerm(Sum)”}]$

$[\text{insertMiddleTerm(Sum)} \stackrel{\text{pyk}}{\to} \text{“lemma insertMiddleTerm(Sum)”}]$

insertMiddleTerm(Difference)

$[\text{insertMiddleTerm(Difference)} \stackrel{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \text{insertMiddleTerm(Sum)} \gg \underline{x} + -\underline{y} = \underline{x} + --\underline{z} + -\underline{z} + -\underline{y}; \text{DoubleMinus} \gg --\underline{z} = \underline{z}; \text{lemma eqAdditionLeft} \rhd --\underline{z} = \underline{z} \gg \underline{x} + --\underline{z} = \underline{x} + \underline{z}; \text{plusCommutativity} \gg -\underline{z} + -\underline{y} = -\underline{y} + -\underline{z}; -x - y = -(x + y) \gg -\underline{y} + -\underline{z} = -\underline{y} + \underline{z}; \text{eqTransitivity} \rhd -\underline{z} + -\underline{y} = -\underline{y} + -\underline{z} \rhd -\underline{y} + -\underline{z} = -\underline{y} + \underline{z} \gg -\underline{z} + -\underline{y} = -\underline{y} + \underline{z}; \text{AddEquations} \rhd \underline{x} + --\underline{z} = \underline{x} + \underline{z} \rhd -\underline{z} + -\underline{y} = -\underline{y} + \underline{z} \gg \underline{x} + --\underline{z} + -\underline{z} + -\underline{y} = \underline{x} + \underline{z} + -\underline{y} + \underline{z}; \text{eqTransitivity} \rhd \underline{x} + -\underline{y} = \underline{x} + --\underline{z} + -\underline{z} + -\underline{y} \rhd \underline{x} + --\underline{z} + -\underline{z} + -\underline{y} = \underline{x} + \underline{z} + -\underline{y} + \underline{z} \gg \underline{x} + -\underline{y} = \underline{x} + \underline{z} + -\underline{y} + \underline{z} \rceil, p_0, c)]$

$[\text{insertMiddleTerm(Difference)} \stackrel{\text{stmt}}{\to} \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \forall \underline{z}\colon \underline{x} + -\underline{y} = \underline{x} + \underline{z} + -\underline{y} + \underline{z}]$

[insertMiddleTerm(Difference) $\stackrel{\text{tex}}{\rightarrow}$ “insertMiddleTerm(Difference)”]

[insertMiddleTerm(Difference) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma insertMiddleTerm(Difference)”]

$x * 0 + x = x$

[$x*0+x = x \stackrel{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \text{times1} \gg \underline{x}*1 = \underline{x}; \text{eqSymmetry} \rhd \underline{x}*1 = \underline{x} \gg \underline{x} = \underline{x}*1; \text{lemma eqAdditionLeft} \rhd \underline{x} = \underline{x}*1 \gg \underline{x}*0+\underline{x} = \underline{x}*0+\underline{x}*1; \text{Distribution} \gg \underline{x}*0+1 = \underline{x}*0+\underline{x}*1; \text{eqSymmetry} \rhd \underline{x}*0+1 = \underline{x}*0+\underline{x}*1 \gg \underline{x}*0+\underline{x}*1 = \underline{x}*0+1; \text{plus0Left} \gg 0+1 = 1; \text{EqMultiplicationLeft} \rhd 0+1 = 1 \gg \underline{x}*0+1 = \underline{x}*1; \text{eqTransitivity5} \rhd \underline{x}*0+\underline{x} = \underline{x}*0+\underline{x}*1 \rhd \underline{x}*0+\underline{x}*1 = \underline{x}*0+1 \rhd \underline{x}*0+1 = \underline{x}*1 \rhd \underline{x}*1 = \underline{x} \gg \underline{x}*0+\underline{x} = \underline{x}\rceil, p_0, c)$]

[$x*0+x = x \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{x}\colon \underline{x}*0+\underline{x} = \underline{x}$]

[$x*0+x = x \stackrel{\text{tex}}{\rightarrow}$ “x*0+x=x”]

[$x*0+x = x \stackrel{\text{pyk}}{\rightarrow}$ “lemma x*0+x=x”]

$x * 0 = 0$

[$x*0 = 0 \stackrel{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon x = x+(y-y) \gg \underline{x}*0 = \underline{x}*0+\underline{x}+-\underline{x}; \text{plusAssociativity} \gg \underline{x}*0+\underline{x}+-\underline{x} = \underline{x}*0+\underline{x}+-\underline{x}; \text{eqSymmetry} \rhd \underline{x}*0+\underline{x}+-\underline{x} = \underline{x}*0+\underline{x}+-\underline{x} \gg \underline{x}*0+\underline{x}+-\underline{x} = \underline{x}*0+\underline{x}+-\underline{x}; x*0+x = x \gg \underline{x}*0+\underline{x} = \underline{x}; \text{eqAddition} \rhd \underline{x}*0+\underline{x} = \underline{x} \gg \underline{x}*0+\underline{x}+-\underline{x} = \underline{x}+-\underline{x}; \text{Negative} \gg \underline{x}+-\underline{x} = 0; \text{eqTransitivity5} \rhd \underline{x}*0 = \underline{x}*0+\underline{x}+-\underline{x} \rhd \underline{x}*0+\underline{x}+-\underline{x} = \underline{x}*0+\underline{x}+-\underline{x} \rhd \underline{x}*0+\underline{x}+-\underline{x} = \underline{x}+-\underline{x} \rhd \underline{x}+-\underline{x} = 0 \gg \underline{x}*0 = 0\rceil, p_0, c)$]

[$x*0 = 0 \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{x}\colon \underline{x}*0 = 0$]

[$x*0 = 0 \stackrel{\text{tex}}{\rightarrow}$ “x*0=0”]

[$x*0 = 0 \stackrel{\text{pyk}}{\rightarrow}$ “lemma x*0=0”]

$(-1) * (-1) + (-1) * 1 = 0$

[$(-1)*(-1)+(-1)*1 = 0 \stackrel{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \text{DistributionOut} \gg -1*-1+-1*1 = -1*-1+1; \text{Negative} \gg 1+-1 = 0; \text{plusCommutativity} \gg -1+1 = 1+-1; \text{eqTransitivity} \rhd -1+1 = 1+-1 \rhd 1+-1 = 0 \gg -1+1 = 0; \text{EqMultiplicationLeft} \rhd -1+1 = 0 \gg -1*-1+1 = -1*0; x*0 = 0 \gg -1*0 = 0; \text{eqTransitivity4} \rhd -1*-1+-1*1 = -1*-1+1 \rhd -1*-1+1 = -1*0 \rhd -1*0 = 0 \gg -1*-1+-1*1 = 0\rceil, p_0, c)$]

[$(-1)*(-1)+(-1)*1 = 0 \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash -1*-1+-1*1 = 0$]

$[(-1)*(-1)+(-1)*1=0 \stackrel{\text{tex}}{\rightarrow}$ “(-1)*(-1)+(-1)*1=0”]

$[(-1)*(-1)+(-1)*1=0 \stackrel{\text{pyk}}{\rightarrow}$ “lemma (-1)*(-1)+(-1)*1=0”]

$(-1)*(-1)=1$

$[(-1)*(-1)=1 \stackrel{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash x = x + (y - y) \gg -1 * -1 = -1 * -1 + 1 + -1; \text{times1} \gg -1 * 1 = -1; \text{eqSymmetry} \rhd -1 * 1 = -1 \gg -1 = -1 * 1; \text{lemma eqAdditionLeft} \rhd -1 = -1 * 1 \gg 1 + -1 = 1 + -1 * 1; \text{lemma eqAdditionLeft} \rhd 1 + -1 = 1 + -1 * 1 \gg -1 * -1 + 1 + -1 = -1 * -1 + 1 + -1 * 1; \text{plusCommutativity} \gg 1 + -1 * 1 = -1*1+1; \text{lemma eqAdditionLeft} \rhd 1+-1*1 = -1*1+1 \gg -1*-1+1+-1*1 = -1 * -1 + -1 * 1 + 1; \text{plusAssociativity} \gg -1 * -1 + -1 * 1 + 1 = -1*-1+-1*1+1; \text{eqSymmetry} \rhd -1*-1+-1*1+1 = -1*-1+-1*1+1 \gg -1 * -1 + -1 * 1 + 1 = -1 * -1 + -1 * 1 + 1; (-1) * (-1) + (-1) * 1 = 0 \gg -1 * -1 + -1 * 1 = 0; \text{eqAddition} \rhd -1 * -1 + -1 * 1 = 0 \gg -1 * -1 + -1 * 1 + 1 = 0 + 1; \text{plus0Left} \gg 0 + 1 = 1; \text{eqTransitivity5} \rhd -1 * -1 = -1*-1+1+-1 \rhd -1*-1+1+-1 = -1*-1+1+-1*1 \rhd -1*-1+1+-1*1 = -1 * -1 + -1 * 1 + 1 \rhd -1 * -1 + -1 * 1 + 1 = -1 * -1 + -1 * 1 + 1 \gg -1 * -1 = -1 * -1 + -1 * 1 + 1; \text{eqTransitivity4} \rhd -1 * -1 = -1*-1+-1*1+1 \rhd -1*-1+-1*1+1 = 0+1 \rhd 0+1 = 1 \gg -1*-1 = 1 \rceil, p_0, c)]$

$[(-1)*(-1)=1 \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash -1 * -1 = 1]$

$[(-1)*(-1)=1 \stackrel{\text{tex}}{\rightarrow}$ “(-1)*(-1)=1”]

$[(-1)*(-1)=1 \stackrel{\text{pyk}}{\rightarrow}$ “lemma (-1)*(-1)=1”]

$0 < 1$Helper

$[0 < 1\text{Helper} \stackrel{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash 1 <= 0 \vdash \text{leqAddition} \rhd 1 <= 0 \gg 1+-1 <= 0+-1; \text{Negative} \gg 1+-1 = 0; \text{subLeqLeft} \rhd 1+-1 = 0 \rhd 1+-1 <= 0 + -1 \gg 0 <= 0 + -1; \text{plus0Left} \gg 0 + -1 = -1; \text{subLeqRight} \rhd 0 + -1 = -1 \rhd 0 <= 0 + -1 \gg 0 <= -1; \text{leqMultiplication} \rhd 0 <= -1 \rhd 0 <= -1 \gg 0 * -1 <= -1 * -1; x * 0 = 0 \gg -1 * 0 = 0; \text{timesCommutativity} \gg 0 * -1 = -1 * 0; \text{eqTransitivity} \rhd 0 * -1 = -1 * 0 \rhd -1 * 0 = 0 \gg 0 * -1 = 0; \text{subLeqLeft} \rhd 0 * -1 = 0 \rhd 0 * -1 <= -1 * -1 \gg 0 <= -1 * -1; (-1) * (-1) = 1 \gg -1 * -1 = 1; \text{subLeqRight} \rhd -1 * -1 = 1 \rhd 0 <= -1 * -1 \gg 0 <= 1; \text{Ded} \rhd 1 <= 0 \vdash 0 <= 1 \gg 1 <= 0 \Rightarrow 0 <= 1 \rceil, p_0, c)]$

$[0 < 1\text{Helper} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash 1 <= 0 \Rightarrow 0 <= 1]$

$[0 < 1\text{Helper} \stackrel{\text{tex}}{\rightarrow}$ “0<1Helper”]

$[0 < 1\text{Helper} \stackrel{\text{pyk}}{\rightarrow}$ “lemma 0<1Helper”]

$0 < 1$

$[0 < 1 \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \mathrm{leqTotality} \gg \dot{\neg} 0 <= 1 \Rightarrow 1 <= 0; \mathrm{AutoImply} \gg 0 <= 1 \Rightarrow 0 <= 1; 0 < 1\mathrm{Helper} \gg 1 <= 0 \Rightarrow 0 <= 1; \mathrm{FromDisjuncts} \rhd \dot{\neg} 0 <= 1 \Rightarrow 1 <= 0 \rhd 0 <= 1 \Rightarrow 0 <= 1 \rhd 1 <= 0 \Rightarrow 0 <= 1 \gg 0 <= 1; 0\mathrm{not}1 \gg \dot{\neg} 0 = 1; \mathrm{JoinConjuncts} \rhd 0 <= 1 \rhd \dot{\neg} 0 = 1 \gg \dot{\neg} 0 <= 1 \Rightarrow \dot{\neg} \dot{\neg} 0 = 1 \rceil, \mathrm{p}_0, \mathsf{c})]$

$[0 < 1 \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \dot{\neg} 0 <= 1 \Rightarrow \dot{\neg} \dot{\neg} 0 = 1]$

$[0 < 1 \stackrel{\mathrm{tex}}{\rightarrow}$ "0<1"$]$

$[0 < 1 \stackrel{\mathrm{pyk}}{\rightarrow}$ "lemma 0<1"$]$

$0 < 2$

$[0 < 2 \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash 0 < 1 \gg \dot{\neg} 0 <= 1 \Rightarrow \dot{\neg} \dot{\neg} 0 = 1; \mathrm{LessAddition} \rhd \dot{\neg} 0 <= 1 \Rightarrow \dot{\neg} \dot{\neg} 0 = 1 \gg \dot{\neg} 0 + 1 <= 1 + 1 \Rightarrow \dot{\neg} \dot{\neg} 0 + 1 = 1 + 1; \mathrm{plus0Left} \gg 0 + 1 = 1; \mathrm{SubLessLeft} \rhd 0 + 1 = 1 \rhd \dot{\neg} 0 + 1 <= 1 + 1 \Rightarrow \dot{\neg} \dot{\neg} 0 + 1 = 1 + 1 \gg \dot{\neg} 1 <= 1 + 1 \Rightarrow \dot{\neg} \dot{\neg} 1 = 1 + 1; \mathrm{LessTransitivity} \rhd \dot{\neg} 0 <= 1 \Rightarrow \dot{\neg} \dot{\neg} 0 = 1 \rhd \dot{\neg} 1 <= 1 + 1 \Rightarrow \dot{\neg} \dot{\neg} 1 = 1 + 1 \gg \dot{\neg} 0 <= 1 + 1 \Rightarrow \dot{\neg} \dot{\neg} 0 = 1 + 1 \rceil, \mathrm{p}_0, \mathsf{c})]$

$[0 < 2 \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \dot{\neg} 0 <= 1 + 1 \Rightarrow \dot{\neg} \dot{\neg} 0 = 1 + 1]$

$[0 < 2 \stackrel{\mathrm{tex}}{\rightarrow}$ "0<2"$]$

$[0 < 2 \stackrel{\mathrm{pyk}}{\rightarrow}$ "lemma 0<2"$]$

$0 < 1/2$

$[0 < 1/2 \stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash 0 < 2 \gg \dot{\neg} 0 <= 1 + 1 \Rightarrow \dot{\neg} \dot{\neg} 0 = 1 + 1; \mathrm{FirstConjunct} \rhd \dot{\neg} 0 <= 1 + 1 \Rightarrow \dot{\neg} \dot{\neg} 0 = 1 + 1 \gg 0 <= 1 + 1; \mathrm{SecondConjunct} \rhd \dot{\neg} 0 <= 1 + 1 \Rightarrow \dot{\neg} \dot{\neg} 0 = 1 + 1 \gg \dot{\neg} 0 = 1 + 1; \mathrm{NeqSymmetry} \rhd \dot{\neg} 0 = 1 + 1 \gg \dot{\neg} 1 + 1 = 0; 0 < 1 \gg \dot{\neg} 0 <= 1 \Rightarrow \dot{\neg} \dot{\neg} 0 = 1; \mathrm{x} * 0 = 0 \gg 1 + 1 * 0 = 0; \mathrm{x} * \mathrm{y} = \mathrm{zBackwards} \rhd 1 + 1 * 0 = 0 \gg 0 = 0 * 1 + 1; \mathrm{SubLessLeft} \rhd 0 = 0 * 1 + 1 \rhd \dot{\neg} 0 <= 1 \Rightarrow \dot{\neg} \dot{\neg} 0 = 1 \gg \dot{\neg} 0 * 1 + 1 <= 1 \Rightarrow \dot{\neg} \dot{\neg} 0 * 1 + 1 = 1; \mathrm{Reciprocal} \rhd \dot{\neg} 1 + 1 = 0 \gg 1 + 1 * \mathrm{rec}1 + 1 = 1; \mathrm{x} * \mathrm{y} = \mathrm{zBackwards} \rhd 1 + 1 * \mathrm{rec}1 + 1 = 1 \gg 1 = \mathrm{rec}1 + 1 * 1 + 1; \mathrm{SubLessRight} \rhd 1 = \mathrm{rec}1 + 1 * 1 + 1 \rhd \dot{\neg} 0 * 1 + 1 <= 1 \Rightarrow \dot{\neg} \dot{\neg} 0 * 1 + 1 = 1 \gg \dot{\neg} 0 * 1 + 1 <= \mathrm{rec}1 + 1 * 1 + 1 \Rightarrow \dot{\neg} \dot{\neg} 0 * 1 + 1 = \mathrm{rec}1 + 1 * 1 + 1; \mathrm{LessDivision} \rhd 0 <= 1 + 1 \rhd \dot{\neg} 0 * 1 + 1 <= \mathrm{rec}1 + 1 * 1 + 1 \Rightarrow \dot{\neg} \dot{\neg} 0 * 1 + 1 = \mathrm{rec}1 + 1 * 1 + 1 \gg \dot{\neg} 0 <= \mathrm{rec}1 + 1 \Rightarrow \dot{\neg} \dot{\neg} 0 = \mathrm{rec}1 + 1 \rceil, \mathrm{p}_0, \mathsf{c})]$

$[0 < 1/2 \stackrel{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \dot{\neg} 0 <= \mathrm{rec}1 + 1 \Rightarrow \dot{\neg} \dot{\neg} 0 = \mathrm{rec}1 + 1]$

$[0 < 1/2 \stackrel{\mathrm{tex}}{\rightarrow}$ "0<1/2"$]$

$[0 < 1/2 \stackrel{\text{pyk}}{\to}$ "lemma 0<1/2"$]$

TwoWholes

$[\text{TwoWholes} \stackrel{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \text{times1} \gg \underline{x} * 1 = \underline{x}; \text{eqSymmetry} \gg \underline{x} = \underline{x} * 1; \text{lemma eqAdditionLeft} \rhd \underline{x} = \underline{x} * 1 \gg \underline{x} + \underline{x} = \underline{x} + \underline{x} * 1; \text{eqAddition} \rhd \underline{x} = \underline{x} * 1 \gg \underline{x} + \underline{x} * 1 = \underline{x} * 1 + \underline{x} * 1; \text{eqTransitivity} \rhd \underline{x} + \underline{x} = \underline{x} + \underline{x} * 1 \rhd \underline{x} + \underline{x} * 1 = \underline{x} * 1 + \underline{x} * 1 \gg \underline{x} + \underline{x} = \underline{x} * 1 + \underline{x} * 1; \text{DistributionOut} \gg \underline{x} * 1 + \underline{x} * 1 = \underline{x} * 1 + 1; \text{Repetition} \rhd \underline{x} * 1 + \underline{x} * 1 = \underline{x} * 1 + 1 \gg \underline{x} * 1 + \underline{x} * 1 = \underline{x} * 1 + 1; \text{timesCommutativity} \gg \underline{x} * 1 + 1 = 1 + 1 * \underline{x}; \text{eqTransitivity4} \rhd \underline{x} + \underline{x} = \underline{x} * 1 + \underline{x} * 1 \rhd \underline{x} * 1 + \underline{x} * 1 = \underline{x} * 1 + 1 \rhd \underline{x} * 1 + 1 = 1 + 1 * \underline{x} \gg \underline{x} + \underline{x} = 1 + 1 * \underline{x} \rceil, p_0, c)]$

$[\text{TwoWholes} \stackrel{\text{stmt}}{\to} \text{ZFsub} \vdash \forall \underline{x}\colon \underline{x} + \underline{x} = 1 + 1 * \underline{x}]$

$[\text{TwoWholes} \stackrel{\text{tex}}{\to}$ "TwoWholes"$]$

$[\text{TwoWholes} \stackrel{\text{pyk}}{\to}$ "lemma x+x=2*x"$]$

TwoHalves

$[\text{TwoHalves} \stackrel{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon 0 < 2 \gg \dot{\neg} 0 <= 1 + 1 \Rightarrow \dot{\neg}\dot{\neg} 0 = 1 + 1; \text{LessNeq} \rhd \dot{\neg} 0 <= 1 + 1 \Rightarrow \dot{\neg}\dot{\neg} 0 = 1 + 1 \gg \dot{\neg} 0 = 1 + 1; \text{NeqSymmetry} \rhd \dot{\neg} 0 = 1 + 1 \gg \dot{\neg} 1 + 1 = 0; \text{TwoWholes} \gg \text{rec1} + 1 * \underline{x} + \text{rec1} + 1 * \underline{x} = 1 + 1 * \text{rec1} + 1 * \underline{x}; \text{timesAssociativity} \gg 1 + 1 * \text{rec1} + 1 * \underline{x} = 1 + 1 * \text{rec1} + 1 * \underline{x}; \text{eqSymmetry} \rhd 1 + 1 * \text{rec1} + 1 * \underline{x} = 1+1*\text{rec1}+1*\underline{x} \gg 1+1*\text{rec1}+1*\underline{x} = 1+1*\text{rec1}+1*\underline{x}; \text{Reciprocal} \rhd \dot{\neg} 1+1 = 0 \gg 1 + 1 * \text{rec1} + 1 = 1; \text{eqMultiplication} \rhd 1 + 1 * \text{rec1} + 1 = 1 \gg 1 + 1 * \text{rec1} + 1 * \underline{x} = 1 * \underline{x}; \text{times1Left} \gg 1 * \underline{x} = \underline{x}; \text{eqTransitivity5} \rhd \text{rec1} + 1 * \underline{x} + \text{rec1} + 1 * \underline{x} = 1 + 1 * \text{rec1} + 1 * \underline{x} \rhd 1 + 1 * \text{rec1} + 1 * \underline{x} = 1 + 1 * \text{rec1} + 1 * \underline{x} \rhd 1 + 1 * \text{rec1} + 1 * \underline{x} = 1 * \underline{x} \rhd 1 * \underline{x} = \underline{x} \gg \text{rec1} + 1 * \underline{x} + \text{rec1} + 1 * \underline{x} = \underline{x} \rceil, p_0, c)]$

$[\text{TwoHalves} \stackrel{\text{stmt}}{\to} \text{ZFsub} \vdash \forall \underline{x}\colon \text{rec1} + 1 * \underline{x} + \text{rec1} + 1 * \underline{x} = \underline{x}]$

$[\text{TwoHalves} \stackrel{\text{tex}}{\to}$ "TwoHalves"$]$

$[\text{TwoHalves} \stackrel{\text{pyk}}{\to}$ "lemma (1/2)x+(1/2)x=x"$]$

$-x - y = -(x + y)$

$[-x - y = -(x + y) \stackrel{\text{proof}}{\to} \lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{x}\colon \forall \underline{y}\colon \text{Times}(-1)\text{Left} \gg -1 * \underline{x} = -\underline{x}; \text{Times}(-1)\text{Left} \gg -1 * \underline{y} = -\underline{y}; \text{AddEquations} \rhd -1 * \underline{x} = -\underline{x} \rhd -1 * \underline{y} = -\underline{y} \gg -1 * \underline{x} + -1 * \underline{y} = -\underline{x} + -\underline{y}; \text{eqSymmetry} \rhd -1 * \underline{x} + -1 * \underline{y} = -\underline{x} + -\underline{y} \gg -\underline{x} + -\underline{y} = -1 * \underline{x} + -1 * \underline{y}; \text{DistributionOut} \gg -1 * \underline{x} + -1 * \underline{y} = -1 * \underline{x} + \underline{y}; \text{Times}(-1)\text{Left} \gg$

$-1 * \underline{x} + \underline{y} = -\underline{x} + \underline{y}$; eqTransitivity4 $\rhd -\underline{x} + -\underline{y} = -1 * \underline{x} + -1 * \underline{y} \rhd -1 * \underline{x} + -1 * \underline{y} = -1 * \underline{x} + \underline{y} \rhd -1 * \underline{x} + \underline{y} = -\underline{x} + \underline{y} \gg -\underline{x} + -\underline{y} = -\underline{x} + \underline{y}\rceil, p_0, c)]$

$[-x - y = -(x + y) \stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon -\underline{x} + -\underline{y} = -\underline{x} + \underline{y}]$

$[-x - y = -(x + y) \stackrel{\text{tex}}{\rightarrow}$ "-x-y=-(x+y)"]

$[-x - y = -(x + y) \stackrel{\text{pyk}}{\rightarrow}$ "lemma -x-y=-(x+y)"]

MinusNegated

[MinusNegated $\stackrel{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon$ DoubleMinus $\gg - - \underline{y} = \underline{y}$; eqAddition $\rhd - - \underline{y} = \underline{y} \gg - - \underline{y} + -\underline{x} = \underline{y} + -\underline{x}$; eqSymmetry $\rhd - - \underline{y} + -\underline{x} = \underline{y} + -\underline{x} \gg \underline{y} + -\underline{x} = - - \underline{y} + -\underline{x}$; $-x - y = -(x + y) \gg - - \underline{y} + -\underline{x} = - - \underline{y} + \underline{x}$; plusCommutativity $\gg -\underline{y} + \underline{x} = \underline{x} + -\underline{y}$; EqNegated $\rhd -\underline{y} + \underline{x} = \underline{x} + -\underline{y} \gg - - \underline{y} + \underline{x} = -\underline{x} + -\underline{y}$; eqTransitivity4 $\rhd \underline{y} + -\underline{x} = - - \underline{y} + -\underline{x} \rhd - - \underline{y} + -\underline{x} = - - \underline{y} + \underline{x} \rhd - - \underline{y} + \underline{x} = -\underline{x} + -\underline{y} \gg \underline{y} + -\underline{x} = -\underline{x} + -\underline{y}$; eqSymmetry $\rhd \underline{y} + -\underline{x} = -\underline{x} + -\underline{y} \gg -\underline{x} + -\underline{y} = \underline{y} + -\underline{x}\rceil, p_0, c)]$

[MinusNegated $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \forall \underline{y}\colon -\underline{x} + -\underline{y} = \underline{y} + -\underline{x}]$

[MinusNegated $\stackrel{\text{tex}}{\rightarrow}$ "MinusNegated"]

[MinusNegated $\stackrel{\text{pyk}}{\rightarrow}$ "lemma minusNegated"]

Times(−1)

[Times(-1) $\stackrel{\text{proof}}{\rightarrow} \lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{x}\colon$ Negative $\gg 1 + -1 = 0$; plusCommutativity $\gg -1 + 1 = 1 + -1$; eqTransitivity $\rhd -1 + 1 = 1 + -1 \rhd 1 + -1 = 0 \gg -1 + 1 = 0$; EqMultiplicationLeft $\rhd -1 + 1 = 0 \gg \underline{x} * -1 + 1 = \underline{x} * 0$; $x * 0 = 0 \gg \underline{x} * 0 = 0$; eqTransitivity $\rhd \underline{x} * -1 + 1 = \underline{x} * 0 \rhd \underline{x} * 0 = 0 \gg \underline{x} * -1 + 1 = 0$; Distribution $\gg \underline{x} * -1 + 1 = \underline{x} * -1 + \underline{x} * 1$; eqSymmetry $\rhd \underline{x} * -1 + 1 = \underline{x} * -1 + \underline{x} * 1 \gg \underline{x} * -1 + \underline{x} * 1 = \underline{x} * -1 + 1$; eqTransitivity $\rhd \underline{x} * -1 + \underline{x} * 1 = \underline{x} * -1 + 1 \rhd \underline{x} * -1 + 1 = 0 \gg \underline{x} * -1 + \underline{x} * 1 = 0$; PositiveToRight(Eq) $\rhd \underline{x} * -1 + \underline{x} * 1 = 0 \gg \underline{x} * -1 = 0 + -\underline{x} * 1$; plus0Left $\gg 0 + -\underline{x} * 1 = -\underline{x} * 1$; eqTransitivity $\rhd \underline{x} * -1 = 0 + -\underline{x} * 1 \rhd 0 + -\underline{x} * 1 = -\underline{x} * 1 \gg \underline{x} * -1 = -\underline{x} * 1$; times1 $\gg \underline{x} * 1 = \underline{x}$; EqNegated $\rhd \underline{x} * 1 = \underline{x} \gg -\underline{x} * 1 = -\underline{x}$; eqTransitivity $\rhd \underline{x} * -1 = -\underline{x} * 1 \rhd -\underline{x} * 1 = -\underline{x} \gg \underline{x} * -1 = -\underline{x}\rceil, p_0, c)]$

[Times(-1) $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{x}\colon \underline{x} * -1 = -\underline{x}]$

[Times(-1) $\stackrel{\text{tex}}{\rightarrow}$ "Times(-1)"]

[Times(-1) $\stackrel{\text{pyk}}{\rightarrow}$ "lemma times(-1)"]

Times(-1)Left

$[\text{Times}(-1)\text{Left} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon \text{Times}(-1) \gg \underline{\mathsf{x}} * -1 = -\underline{\mathsf{x}}; \text{timesCommutativity} \gg -1 * \underline{\mathsf{x}} = \underline{\mathsf{x}} * -1; \text{eqTransitivity} \rhd -1 * \underline{\mathsf{x}} = \underline{\mathsf{x}} * -1 \rhd \underline{\mathsf{x}} * -1 = -\underline{\mathsf{x}} \gg -1 * \underline{\mathsf{x}} = -\underline{\mathsf{x}}\rceil, \text{p}_0, \mathsf{c})]$

$[\text{Times}(-1)\text{Left} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{x}}\colon -1 * \underline{\mathsf{x}} = -\underline{\mathsf{x}}]$

$[\text{Times}(-1)\text{Left} \stackrel{\text{tex}}{\rightarrow}$ "Times(-1)Left"$]$

$[\text{Times}(-1)\text{Left} \stackrel{\text{pyk}}{\rightarrow}$ "lemma times(-1)Left"$]$

$-0 = 0$

$[-0 = 0 \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \text{Negative} \gg 0 + -0 = 0; \text{plus0} \gg 0 + 0 = 0; \text{UniqueNegative} \rhd 0 + -0 = 0 \rhd 0 + 0 = 0 \gg -0 = 0\rceil, \text{p}_0, \mathsf{c})]$

$[-0 = 0 \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash -0 = 0]$

$[-0 = 0 \stackrel{\text{tex}}{\rightarrow}$ "-0=0"$]$

$[-0 = 0 \stackrel{\text{pyk}}{\rightarrow}$ "lemma -0=0"$]$

SFsymmetry

$[\text{SFsymmetry} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \overline{(\epsilon)}\colon \forall \underline{\mathsf{m}}\colon \forall \overline{(\text{fx})}\colon \forall \overline{(\text{fy})}\colon \text{SF}(\overline{(\text{fx})}, \overline{(\text{fy})}) \vdash \dot{\neg}\, 0 <= \overline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}\, 0 = \overline{(\epsilon)} \vdash \mathsf{c}_{\text{Ex}} <= \underline{\mathsf{m}} \vdash \text{FromSF} \rhd \text{SF}(\overline{(\text{fx})}, \overline{(\text{fy})}) \rhd \dot{\neg}\, 0 <= \overline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}\, 0 = \overline{(\epsilon)} \gg \mathsf{c}_{\text{Ex}} <= \underline{\mathsf{m}} \Rightarrow \dot{\neg}\, \text{if}(0 <= \overline{(\text{fx})}[\underline{\mathsf{m}}] + -\overline{(\text{fy})}[\underline{\mathsf{m}}], \overline{(\text{fx})}[\underline{\mathsf{m}}] + -\overline{(\text{fy})}[\underline{\mathsf{m}}], -\overline{(\text{fx})}[\underline{\mathsf{m}}] + -\overline{(\text{fy})}[\underline{\mathsf{m}}]) <= \overline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}\, \text{if}(0 <= \overline{(\text{fx})}[\underline{\mathsf{m}}] + -\overline{(\text{fy})}[\underline{\mathsf{m}}], \overline{(\text{fx})}[\underline{\mathsf{m}}] + -\overline{(\text{fy})}[\underline{\mathsf{m}}], -\overline{(\text{fx})}[\underline{\mathsf{m}}] + -\overline{(\text{fy})}[\underline{\mathsf{m}}]) = \overline{(\epsilon)}; \text{MP} \rhd \mathsf{c}_{\text{Ex}} <= \underline{\mathsf{m}} \Rightarrow \dot{\neg}\, \text{if}(0 <= \overline{(\text{fx})}[\underline{\mathsf{m}}] + -\overline{(\text{fy})}[\underline{\mathsf{m}}], \overline{(\text{fx})}[\underline{\mathsf{m}}] + -\overline{(\text{fy})}[\underline{\mathsf{m}}], -\overline{(\text{fx})}[\underline{\mathsf{m}}] + -\overline{(\text{fy})}[\underline{\mathsf{m}}]) <= \overline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}\, \text{if}(0 <= \overline{(\text{fx})}[\underline{\mathsf{m}}] + -\overline{(\text{fy})}[\underline{\mathsf{m}}], \overline{(\text{fx})}[\underline{\mathsf{m}}] + -\overline{(\text{fy})}[\underline{\mathsf{m}}], -\overline{(\text{fx})}[\underline{\mathsf{m}}] + -\overline{(\text{fy})}[\underline{\mathsf{m}}]) = \overline{(\epsilon)} \rhd \mathsf{c}_{\text{Ex}} <= \underline{\mathsf{m}} \gg \dot{\neg}\, \text{if}(0 <= \overline{(\text{fx})}[\underline{\mathsf{m}}] + -\overline{(\text{fy})}[\underline{\mathsf{m}}], \overline{(\text{fx})}[\underline{\mathsf{m}}] + -\overline{(\text{fy})}[\underline{\mathsf{m}}], -\overline{(\text{fx})}[\underline{\mathsf{m}}] + -\overline{(\text{fy})}[\underline{\mathsf{m}}]) <= \overline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}\, \text{if}(0 <= \overline{(\text{fx})}[\underline{\mathsf{m}}] + -\overline{(\text{fy})}[\underline{\mathsf{m}}], \overline{(\text{fx})}[\underline{\mathsf{m}}] + -\overline{(\text{fy})}[\underline{\mathsf{m}}], -\overline{(\text{fx})}[\underline{\mathsf{m}}] + -\overline{(\text{fy})}[\underline{\mathsf{m}}]) = \overline{(\epsilon)}; \text{NumericalDifference} \gg \text{if}(0 <= \overline{(\text{fx})}[\underline{\mathsf{m}}] + -\overline{(\text{fy})}[\underline{\mathsf{m}}], \overline{(\text{fx})}[\underline{\mathsf{m}}] + -\overline{(\text{fy})}[\underline{\mathsf{m}}], -\overline{(\text{fx})}[\underline{\mathsf{m}}] + -\overline{(\text{fy})}[\underline{\mathsf{m}}]) = \text{if}(0 <= \overline{(\text{fy})}[\underline{\mathsf{m}}] + -\overline{(\text{fx})}[\underline{\mathsf{m}}], \overline{(\text{fy})}[\underline{\mathsf{m}}] + -\overline{(\text{fx})}[\underline{\mathsf{m}}], -\overline{(\text{fy})}[\underline{\mathsf{m}}] + -\overline{(\text{fx})}[\underline{\mathsf{m}}]); \text{SubLessLeft} \rhd \text{if}(0 <= \overline{(\text{fx})}[\underline{\mathsf{m}}] + -\overline{(\text{fy})}[\underline{\mathsf{m}}], \overline{(\text{fx})}[\underline{\mathsf{m}}] + -\overline{(\text{fy})}[\underline{\mathsf{m}}], -\overline{(\text{fx})}[\underline{\mathsf{m}}] + -\overline{(\text{fy})}[\underline{\mathsf{m}}]) = \text{if}(0 <= \overline{(\text{fy})}[\underline{\mathsf{m}}] + -\overline{(\text{fx})}[\underline{\mathsf{m}}], \overline{(\text{fy})}[\underline{\mathsf{m}}] + -\overline{(\text{fx})}[\underline{\mathsf{m}}], -\overline{(\text{fy})}[\underline{\mathsf{m}}] + -\overline{(\text{fx})}[\underline{\mathsf{m}}]) \rhd \dot{\neg}\, \text{if}(0 <= \overline{(\text{fx})}[\underline{\mathsf{m}}] + -\overline{(\text{fy})}[\underline{\mathsf{m}}], \overline{(\text{fx})}[\underline{\mathsf{m}}] + -\overline{(\text{fy})}[\underline{\mathsf{m}}], -\overline{(\text{fx})}[\underline{\mathsf{m}}] + -\overline{(\text{fy})}[\underline{\mathsf{m}}]) <= \overline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}\, \text{if}(0 <= \overline{(\text{fx})}[\underline{\mathsf{m}}] + -\overline{(\text{fy})}[\underline{\mathsf{m}}], \overline{(\text{fx})}[\underline{\mathsf{m}}] + -\overline{(\text{fy})}[\underline{\mathsf{m}}], -\overline{(\text{fx})}[\underline{\mathsf{m}}] + -\overline{(\text{fy})}[\underline{\mathsf{m}}]) = \overline{(\epsilon)} \gg \dot{\neg}\, \text{if}(0 <= \overline{(\text{fy})}[\underline{\mathsf{m}}] + -\overline{(\text{fx})}[\underline{\mathsf{m}}], \overline{(\text{fy})}[\underline{\mathsf{m}}] + -\overline{(\text{fx})}[\underline{\mathsf{m}}], -\overline{(\text{fy})}[\underline{\mathsf{m}}] + -\overline{(\text{fx})}[\underline{\mathsf{m}}]) <= \overline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}\, \text{if}(0 <= \overline{(\text{fy})}[\underline{\mathsf{m}}] + -\overline{(\text{fx})}[\underline{\mathsf{m}}], \overline{(\text{fy})}[\underline{\mathsf{m}}] + -\overline{(\text{fx})}[\underline{\mathsf{m}}], -\overline{(\text{fy})}[\underline{\mathsf{m}}] + -\overline{(\text{fx})}[\underline{\mathsf{m}}]) =$

$\underline{(\epsilon)}; \forall \underline{(\epsilon)} \colon \forall \underline{\mathsf{m}} \colon \forall \underline{(\mathrm{fx})} \colon \forall \underline{(\mathrm{fy})} \colon \mathrm{Ded} \rhd \forall \underline{(\epsilon)} \colon \forall \underline{\mathsf{m}} \colon \forall \underline{(\mathrm{fx})} \colon \forall \underline{(\mathrm{fy})} \colon \mathrm{SF}(\underline{(\mathrm{fx})}, \underline{(\mathrm{fy})}) \vdash \dot{\neg} 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{(\epsilon)} \vdash \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \vdash \dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}]) <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}]) =$

$\underline{(\epsilon)} \gg \mathrm{SF}(\underline{(\mathrm{fx})}, \underline{(\mathrm{fy})}) \Rightarrow \dot{\neg} 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{(\epsilon)} \Rightarrow \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}]) <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}]) =$

$\underline{(\epsilon)}; \mathrm{SF}(\underline{(\mathrm{fx})}, \underline{(\mathrm{fy})}) \vdash \mathrm{MP} \rhd \mathrm{SF}(\underline{(\mathrm{fx})}, \underline{(\mathrm{fy})}) \Rightarrow \dot{\neg} 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{(\epsilon)} \Rightarrow \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}]) <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}]) =$

$\underline{(\epsilon)} \rhd \mathrm{SF}(\underline{(\mathrm{fx})}, \underline{(\mathrm{fy})}) \gg \dot{\neg} 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{(\epsilon)} \Rightarrow \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}]) <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}]) =$

$\underline{(\epsilon)}; \mathrm{ToSF} \rhd \dot{\neg} 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{(\epsilon)} \Rightarrow \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}]) <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}]) =$

$\underline{(\epsilon)} \gg \mathrm{SF}(\underline{(\mathrm{fy})}, \underline{(\mathrm{fx})})\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{SFsymmetry} \stackrel{\mathrm{stmt}}{\to} \mathrm{ZFsub} \vdash \forall \underline{(\epsilon)} \colon \forall \underline{\mathsf{m}} \colon \forall \underline{(\mathrm{fx})} \colon \forall \underline{(\mathrm{fy})} \colon \mathrm{SF}(\underline{(\mathrm{fx})}, \underline{(\mathrm{fy})}) \vdash \mathrm{SF}(\underline{(\mathrm{fy})}, \underline{(\mathrm{fx})})]$

$[\mathrm{SFsymmetry} \stackrel{\mathrm{tex}}{\to}$ "SFsymmetry"]

$[\mathrm{SFsymmetry} \stackrel{\mathrm{pyk}}{\to}$ "lemma sameFsymmetry"]

SFtransitivity

$[\mathrm{SFtransitivity} \stackrel{\mathrm{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash$

$\forall \underline{\mathsf{m}} \colon \forall \underline{(\epsilon)} \colon \forall \underline{(\mathrm{fx})} \colon \forall \underline{(\mathrm{fy})} \colon \forall \underline{(\mathrm{fz})} \colon \mathrm{SF}(\underline{(\mathrm{fx})}, \underline{(\mathrm{fy})}) \vdash \mathrm{SF}(\underline{(\mathrm{fy})}, \underline{(\mathrm{fz})}) \vdash \dot{\neg} 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{(\epsilon)} \vdash \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \vdash \mathrm{PositiveHalved} \rhd \dot{\neg} 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{(\epsilon)} \gg \dot{\neg} 0 <= \mathrm{rec}1 + 1 * \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \mathrm{rec}1 + 1 * \underline{(\epsilon)}; \mathrm{FromSF} \rhd \mathrm{SF}(\underline{(\mathrm{fx})}, \underline{(\mathrm{fy})}) \rhd \dot{\neg} 0 <=$

$\mathrm{rec}1 + 1 * \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \mathrm{rec}1 + 1 * \underline{(\epsilon)} \gg \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) <= \mathrm{rec}1 + 1 * \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) =$

$\mathrm{rec}1 + 1 * \underline{(\epsilon)}; \mathrm{MP} \rhd \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) <= \mathrm{rec}1 + 1 * \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) =$

$\mathrm{rec}1 + 1 * \underline{(\epsilon)} \rhd \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \gg \dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) <= \mathrm{rec}1 + 1 * \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) =$

$\mathrm{rec}1 + 1 * \underline{(\epsilon)}; \mathrm{FromSF} \rhd \mathrm{SF}(\underline{(\mathrm{fy})}, \underline{(\mathrm{fz})}) \rhd \dot{\neg} 0 <= \mathrm{rec}1 + 1 * \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg} 0 = \mathrm{rec}1 + 1 * \underline{(\epsilon)} \gg \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}]) <= \mathrm{rec}1 + 1 * \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}]) =$

$\mathrm{rec}1 + 1 * \underline{(\epsilon)}; \mathrm{MP} \rhd \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fz})}[\underline{\mathsf{m}}]) <= \mathrm{rec}1 + 1 * \underline{(\epsilon)} \Rightarrow$

$\dot\neg\dot\neg$ if(0 <= (fy)[m] + −(fz)[m], (fy)[m] + −(fz)[m], −(fy)[m] + −(fz)[m]) =
rec1 + 1 ∗ (ϵ) ⊳ c_{Ex} <= m ≫ $\dot\neg$ if(0 <=
(fy)[m] + −(fz)[m], (fy)[m] + −(fz)[m], −(fy)[m] + −(fz)[m]) <= rec1 + 1 ∗ (ϵ) ⇒
$\dot\neg\dot\neg$ if(0 <= (fy)[m] + −(fz)[m], (fy)[m] + −(fz)[m], −(fy)[m] + −(fz)[m]) =
rec1 + 1 ∗ (ϵ); AddEquations(Less) ⊳ $\dot\neg$ if(0 <=
(fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) <= rec1 + 1 ∗ (ϵ) ⇒
$\dot\neg\dot\neg$ if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) =
rec1 + 1 ∗ (ϵ) ⊳ $\dot\neg$ if(0 <= (fy)[m] + −(fz)[m], (fy)[m] + −(fz)[m], −(fy)[m] +
−(fz)[m]) <= rec1 + 1 ∗ (ϵ) ⇒ $\dot\neg\dot\neg$ if(0 <=
(fy)[m] + −(fz)[m], (fy)[m] + −(fz)[m], −(fy)[m] + −(fz)[m]) = rec1 + 1 ∗ (ϵ) ≫
$\dot\neg$ if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) + if(0 <=
(fy)[m] + −(fz)[m], (fy)[m] + −(fz)[m], −(fy)[m] + −(fz)[m]) <=
rec1 + 1 ∗ (ϵ) + rec1 + 1 ∗ (ϵ) ⇒ $\dot\neg\dot\neg$ if(0 <=
(fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) + if(0 <=
(fy)[m] + −(fz)[m], (fy)[m] + −(fz)[m], −(fy)[m] + −(fz)[m]) =
rec1 + 1 ∗ (ϵ) + rec1 + 1 ∗ (ϵ); TwoHalves ≫ rec1 + 1 ∗ (ϵ) + rec1 + 1 ∗ (ϵ) =
(ϵ); SubLessRight ⊳ rec1 + 1 ∗ (ϵ) + rec1 + 1 ∗ (ϵ) = (ϵ) ⊳ $\dot\neg$ if(0 <=
(fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) + if(0 <=
(fy)[m] + −(fz)[m], (fy)[m] + −(fz)[m], −(fy)[m] + −(fz)[m]) <=
rec1 + 1 ∗ (ϵ) + rec1 + 1 ∗ (ϵ) ⇒ $\dot\neg\dot\neg$ if(0 <=
(fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) + if(0 <= (fy)[m] +
−(fz)[m], (fy)[m] + −(fz)[m], −(fy)[m] + −(fz)[m]) = rec1 + 1 ∗ (ϵ) + rec1 + 1 ∗ (ϵ) ≫
$\dot\neg$ if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) + if(0 <=
(fy)[m] + −(fz)[m], (fy)[m] + −(fz)[m], −(fy)[m] + −(fz)[m]) <= (ϵ) ⇒
$\dot\neg\dot\neg$ if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) + if(0 <=
(fy)[m] + −(fz)[m], (fy)[m] + −(fz)[m], −(fy)[m] + −(fz)[m]) =
(ϵ); insertMiddleTerm(Numerical) ≫ if(0 <=
(fx)[m] + −(fz)[m], (fx)[m] + −(fz)[m], −(fx)[m] + −(fz)[m]) <= if(0 <=
(fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) + if(0 <= (fy)[m] +
−(fz)[m], (fy)[m] + −(fz)[m], −(fy)[m] + −(fz)[m]); leqLessTransitivity ⊳ if(0 <=
(fx)[m] + −(fz)[m], (fx)[m] + −(fz)[m], −(fx)[m] + −(fz)[m]) <= if(0 <=
(fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) + if(0 <=
(fy)[m] + −(fz)[m], (fy)[m] + −(fz)[m], −(fy)[m] + −(fz)[m]) ⊳ $\dot\neg$ if(0 <=
(fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) + if(0 <=
(fy)[m] + −(fz)[m], (fy)[m] + −(fz)[m], −(fy)[m] + −(fz)[m]) <= (ϵ) ⇒
$\dot\neg\dot\neg$ if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) + if(0 <=
(fy)[m] + −(fz)[m], (fy)[m] + −(fz)[m], −(fy)[m] + −(fz)[m]) = (ϵ) ≫ $\dot\neg$ if(0 <=
(fx)[m] + −(fz)[m], (fx)[m] + −(fz)[m], −(fx)[m] + −(fz)[m]) <= (ϵ) ⇒
$\dot\neg\dot\neg$ if(0 <= (fx)[m] + −(fz)[m], (fx)[m] + −(fz)[m], −(fx)[m] + −(fz)[m]) =
(ϵ); ∀m: ∀(ϵ): ∀(fx): ∀(fy): ∀(fz): Ded ⊳ ∀m: ∀(ϵ): ∀(fx): ∀(fy): ∀(fz): SF((fx), (fy)) ⊢
SF((fy), (fz)) ⊢ $\dot\neg$ 0 <= (ϵ) ⇒ $\dot\neg\dot\neg$ 0 = (ϵ) ⊢ c_{Ex} <= m ⊢ $\dot\neg$ if(0 <=
(fx)[m] + −(fz)[m], (fx)[m] + −(fz)[m], −(fx)[m] + −(fz)[m]) <= (ϵ) ⇒
$\dot\neg\dot\neg$ if(0 <= (fx)[m] + −(fz)[m], (fx)[m] + −(fz)[m], −(fx)[m] + −(fz)[m]) = (ϵ) ≫
SF((fx), (fy)) ⇒ SF((fy), (fz)) ⇒ $\dot\neg$ 0 <= (ϵ) ⇒ $\dot\neg\dot\neg$ 0 = (ϵ) ⇒ c_{Ex} <= m ⇒
$\dot\neg$ if(0 <= (fx)[m] + −(fz)[m], (fx)[m] + −(fz)[m], −(fx)[m] + −(fz)[m]) <= (ϵ) ⇒
$\dot\neg\dot\neg$ if(0 <= (fx)[m] + −(fz)[m], (fx)[m] + −(fz)[m], −(fx)[m] + −(fz)[m]) =

(ϵ); SF((fx), (fy)) ⊢ SF((fy), (fz)) ⊢ MP2 ▷ SF((fx), (fy)) ⇒ SF((fy), (fz)) ⇒ ∸0 <= (ϵ) ⇒ ∸∸0 = (ϵ) ⇒ c_{Ex} <= m ⇒ ∸if(0 <= (fx)[m] + −(fz)[m], (fx)[m] + −(fz)[m], −(fx)[m] + −(fz)[m]) <= (ϵ) ⇒ ∸∸if(0 <= (fx)[m] + −(fz)[m], (fx)[m] + −(fz)[m], −(fx)[m] + −(fz)[m]) = (ϵ) ▷ SF((fx), (fy)) ▷ SF((fy), (fz)) ≫ ∸0 <= (ϵ) ⇒ ∸∸0 = (ϵ) ⇒ c_{Ex} <= m ⇒ ∸if(0 <= (fx)[m] + −(fz)[m], (fx)[m] + −(fz)[m], −(fx)[m] + −(fz)[m]) <= (ϵ) ⇒ ∸∸if(0 <= (fx)[m] + −(fz)[m], (fx)[m] + −(fz)[m], −(fx)[m] + −(fz)[m]) = (ϵ); ToSF ▷ ∸0 <= (ϵ) ⇒ ∸∸0 = (ϵ) ⇒ c_{Ex} <= m ⇒ ∸if(0 <= (fx)[m] + −(fz)[m], (fx)[m] + −(fz)[m], −(fx)[m] + −(fz)[m]) <= (ϵ) ⇒ ∸∸if(0 <= (fx)[m] + −(fz)[m], (fx)[m] + −(fz)[m], −(fx)[m] + −(fz)[m]) = (ϵ) ≫ SF((fx), (fz))⌉, p_0, c)]

[SFtransitivity $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub ⊢ ∀m: ∀(ϵ): ∀(fx): ∀(fy): ∀(fz): SF((fx), (fy)) ⊢ SF((fy), (fz)) ⊢ SF((fx), (fz))]

[SFtransitivity $\stackrel{\text{tex}}{\rightarrow}$ "SFtransitivity"]

[SFtransitivity $\stackrel{\text{pyk}}{\rightarrow}$ "lemma sameFtransitivity"]

= fToSameF

[= fToSameF $\stackrel{\text{proof}}{\rightarrow}$ λc.λx.𝒫(⌈ZFsub ⊢ ∀(ϵ): ∀m: ∀(fx): ∀(fy): (fx) $=_f$ (fy) ⊢ ∸0 <= (ϵ) ⇒ ∸∸0 = (ϵ) ⊢ From = f ▷ (fx) $=_f$ (fy) ≫ (fx)[m] = (fy)[m]; PositiveToLeft(Eq)(1term) ▷ (fx)[m] = (fy)[m] ≫ (fx)[m] + −(fy)[m] = 0; SameNumerical ▷ (fx)[m] + −(fy)[m] = 0 ≫ if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) = if(0 <= 0, 0, −0); |0| = 0 ≫ if(0 <= 0, 0, −0) = 0; eqTransitivity ▷ if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) = if(0 <= 0, 0, −0) ▷ if(0 <= 0, 0, −0) = 0 ≫ if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) = 0; eqSymmetry ▷ if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) = 0 ≫ 0 = if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]); SubLessLeft ▷ 0 = if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) ▷ ∸0 <= (ϵ) ⇒ ∸∸0 = (ϵ) ≫ ∸if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) <= (ϵ) ⇒ ∸∸if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) = (ϵ); Weakening ▷ ∸if(0 <= (fx)[m]+−(fy)[m], (fx)[m]+−(fy)[m], −(fx)[m]+−(fy)[m]) <= (ϵ) ⇒ ∸∸if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) = (ϵ) ≫ 0 <= m ⇒ ∸if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) <= (ϵ) ⇒ ∸∸if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) = (ϵ); ExistIntro @ c_{Ex} @ 0 ▷ 0 <= m ⇒ ∸if(0 <= (fx)[m]+−(fy)[m], (fx)[m]+−(fy)[m], −(fx)[m]+−(fy)[m]) <= (ϵ) ⇒ ∸∸if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) = (ϵ) ≫ c_{Ex} <= m ⇒ ∸if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) <=

$\underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) =$
$\underline{(\epsilon)}; \forall \underline{(\epsilon)}\colon \forall \underline{\mathsf{m}}\colon \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \mathrm{Ded} \rhd \forall \underline{(\epsilon)}\colon \forall \underline{\mathsf{m}}\colon \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \underline{(\mathrm{fx})} =_{\mathrm{f}} \underline{(\mathrm{fy})} \vdash \dot{\neg}0 <=$
$\underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}0 = \underline{(\epsilon)} \vdash \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \dot{\neg}\,\mathrm{if}(0 <=$
$\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) <= \underline{(\epsilon)} \Rightarrow$
$\dot{\neg}\dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) =$
$\underline{(\epsilon)} \gg \underline{(\mathrm{fx})} =_{\mathrm{f}} \underline{(\mathrm{fy})} \Rightarrow \dot{\neg}0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}0 = \underline{(\epsilon)} \Rightarrow \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \dot{\neg}\,\mathrm{if}(0 <=$
$\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) <= \underline{(\epsilon)} \Rightarrow$
$\dot{\neg}\dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) =$
$\underline{(\epsilon)}; \underline{(\mathrm{fx})} =_{\mathrm{f}} \underline{(\mathrm{fy})} \vdash \mathrm{MP} \rhd \underline{(\mathrm{fx})} =_{\mathrm{f}} \underline{(\mathrm{fy})} \Rightarrow \dot{\neg}0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}0 = \underline{(\epsilon)} \Rightarrow \mathsf{c}_{\mathrm{Ex}} <=$
$\underline{\mathsf{m}} \Rightarrow \dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) <=$
$\underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) =$
$\underline{(\epsilon)} \rhd \underline{(\mathrm{fx})} =_{\mathrm{f}} \underline{(\mathrm{fy})} \gg \dot{\neg}0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}0 = \underline{(\epsilon)} \Rightarrow \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \dot{\neg}\,\mathrm{if}(0 <=$
$\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) <= \underline{(\epsilon)} \Rightarrow$
$\dot{\neg}\dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) =$
$\underline{(\epsilon)}; \mathrm{ToSF} \rhd \dot{\neg}0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}0 = \underline{(\epsilon)} \Rightarrow \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \dot{\neg}\,\mathrm{if}(0 <=$
$\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) <= \underline{(\epsilon)} \Rightarrow$
$\dot{\neg}\dot{\neg}\,\mathrm{if}(0 <= \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}], -\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + -\underline{(\mathrm{fy})}[\underline{\mathsf{m}}]) =$
$\underline{(\epsilon)} \gg \mathrm{SF}(\underline{(\mathrm{fx})}, \underline{(\mathrm{fy})})\rceil, \mathrm{p}_0, \mathsf{c})]$

$[= \mathrm{fToSameF} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{(\epsilon)}\colon \forall \underline{\mathsf{m}}\colon \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \underline{(\mathrm{fx})} =_{\mathrm{f}} \underline{(\mathrm{fy})} \vdash \mathrm{SF}(\underline{(\mathrm{fx})}, \underline{(\mathrm{fy})})]$

$[= \mathrm{fToSameF} \overset{\mathrm{tex}}{\rightarrow}$ “=fToSameF ”]

$[= \mathrm{fToSameF} \overset{\mathrm{pyk}}{\rightarrow}$ “lemma =f to sameF”]

PlusF(Sym)

$[\mathrm{PlusF(Sym)} \overset{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}}\colon \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \mathrm{PlusF} \gg \underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] =$
$\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + \underline{(\mathrm{fy})}[\underline{\mathsf{m}}]; \mathrm{eqSymmetry} \rhd \underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] \gg$
$\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})}[\underline{\mathsf{m}}]\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{PlusF(Sym)} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}}\colon \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] + \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})}[\underline{\mathsf{m}}]]$

$[\mathrm{PlusF(Sym)} \overset{\mathrm{tex}}{\rightarrow}$ “PlusF(Sym)”]

$[\mathrm{PlusF(Sym)} \overset{\mathrm{pyk}}{\rightarrow}$ “lemma plusF(Sym)”]

TimesF(Sym)

$[\mathrm{TimesF(Sym)} \overset{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}}\colon \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \mathrm{TimesF} \gg$
$\underline{(\mathrm{fx})} *_{\mathrm{f}} \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fy})}[\underline{\mathsf{m}}]; \mathrm{eqSymmetry} \rhd \underline{(\mathrm{fx})} *_{\mathrm{f}} \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] =$
$\underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] \gg \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})} *_{\mathrm{f}} \underline{(\mathrm{fy})}[\underline{\mathsf{m}}]\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\mathrm{TimesF(Sym)} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}}\colon \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})} *_{\mathrm{f}} \underline{(\mathrm{fy})}[\underline{\mathsf{m}}]]$

[TimesF(Sym) $\stackrel{\text{tex}}{\rightarrow}$ "TimesF(Sym)"]

[TimesF(Sym) $\stackrel{\text{pyk}}{\rightarrow}$ "lemma timesF(Sym)"]

f2R(Plus)

[f2R(Plus) $\stackrel{\text{proof}}{\rightarrow}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \text{(fx)}\colon \forall \text{(fy)}\colon \forall \text{(fz)}\colon \forall \text{(fv)}\colon \forall \text{var fv}\colon \text{SF}(\text{(fx)} +_{\mathrm{f}} \text{(fy)}, \text{(fz)}) \vdash \text{(fv)} \in \text{R}(\text{(fx)} +_{\mathrm{f}} \text{(fy)}) \vdash \text{var fv} \in \text{R}(\text{(fz)}) \vdash \text{FromInR} \rhd \text{(fv)} \in \text{R}(\text{(fx)} +_{\mathrm{f}} \text{(fy)}) \gg \text{SF}(\text{(fv)}, \text{(fx)} +_{\mathrm{f}} \text{(fy)}); \text{FromInR} \rhd \text{var fv} \in \text{R}(\text{(fz)}) \gg \text{SF}(\text{var fv}, \text{(fz)}); \text{SFsymmetry} \rhd \text{SF}(\text{var fv}, \text{(fz)}) \gg \text{SF}(\text{(fz)}, \text{var fv}); \text{SFtransitivity} \rhd \text{SF}(\text{(fv)}, \text{(fx)} +_{\mathrm{f}} \text{(fy)}) \rhd \text{SF}(\text{(fx)} +_{\mathrm{f}} \text{(fy)}, \text{(fz)}) \gg \text{SF}(\text{(fv)}, \text{(fz)}); \text{SFtransitivity} \rhd \text{SF}(\text{(fv)}, \text{(fz)}) \rhd \text{SF}(\text{(fz)}, \text{var fv}) \gg \text{SF}(\text{(fv)}, \text{var fv}); \forall \text{(fx)}\colon \forall \text{(fy)}\colon \forall \text{(fz)}\colon \forall \text{(fv)}\colon \forall \text{var fv}\colon \text{Ded} \rhd \forall \text{(fx)}\colon \forall \text{(fy)}\colon \forall \text{(fz)}\colon \forall \text{(fv)}\colon \forall \text{var fv}\colon \text{SF}(\text{(fx)} +_{\mathrm{f}} \text{(fy)}, \text{(fz)}) \vdash \text{(fv)} \in \text{R}(\text{(fx)} +_{\mathrm{f}} \text{(fy)}) \vdash \text{var fv} \in \text{R}(\text{(fz)}) \vdash \text{SF}(\text{(fv)}, \text{var fv}) \gg \text{SF}(\text{(fx)} +_{\mathrm{f}} \text{(fy)}, \text{(fz)}) \Rightarrow \text{(fv)} \in \text{R}(\text{(fx)} +_{\mathrm{f}} \text{(fy)}) \Rightarrow \text{var fv} \in \text{R}(\text{(fz)}) \Rightarrow \text{SF}(\text{(fv)}, \text{var fv}); \text{SF}(\text{(fx)} +_{\mathrm{f}} \text{(fy)}, \text{(fz)}) \vdash \text{MP} \rhd \text{SF}(\text{(fx)} +_{\mathrm{f}} \text{(fy)}, \text{(fz)}) \Rightarrow \text{(fv)} \in \text{R}(\text{(fx)} +_{\mathrm{f}} \text{(fy)}) \Rightarrow \text{var fv} \in \text{R}(\text{(fz)}) \Rightarrow \text{SF}(\text{(fv)}, \text{var fv}) \rhd \text{SF}(\text{(fx)} +_{\mathrm{f}} \text{(fy)}, \text{(fz)}) \gg \text{(fv)} \in \text{R}(\text{(fx)} +_{\mathrm{f}} \text{(fy)}) \Rightarrow \text{var fv} \in \text{R}(\text{(fz)}) \Rightarrow \text{SF}(\text{(fv)}, \text{var fv}); \text{To} == \text{XX} \rhd \text{(fv)} \in \text{R}(\text{(fx)} +_{\mathrm{f}} \text{(fy)}) \Rightarrow \text{var fv} \in \text{R}(\text{(fz)}) \Rightarrow \text{SF}(\text{(fv)}, \text{var fv}) \gg \text{R}(\text{(fx)} +_{\mathrm{f}} \text{(fy)}) == \text{R}(\text{(fz)}); \text{PlusR} \gg \text{R}(\text{(fx)} +_{\mathrm{f}} \text{(fy)}) == \text{R}(\text{(fx)} +_{\mathrm{f}} \text{(fy)}); == \text{Transitivity} \rhd \text{R}(\text{(fx)} +_{\mathrm{f}} \text{(fy)}) == \text{R}(\text{(fx)} +_{\mathrm{f}} \text{(fy)}) \rhd \text{R}(\text{(fx)} +_{\mathrm{f}} \text{(fy)}) == \text{R}(\text{(fz)}) \gg \text{R}(\text{(fx)} +_{\mathrm{f}} \text{(fy)}) == \text{R}(\text{(fz)})\rceil, \mathrm{p}_0, \mathsf{c})$]

[f2R(Plus) $\stackrel{\text{stmt}}{\rightarrow}$ $\text{ZFsub} \vdash \forall \text{(fx)}\colon \forall \text{(fy)}\colon \forall \text{(fz)}\colon \forall \text{(fv)}\colon \forall \text{var fv}\colon \text{SF}(\text{(fx)} +_{\mathrm{f}} \text{(fy)}, \text{(fz)}) \vdash \text{R}(\text{(fx)} +_{\mathrm{f}} \text{(fy)}) == \text{R}(\text{(fz)})$]

[f2R(Plus) $\stackrel{\text{tex}}{\rightarrow}$ "f2R(Plus)"]

[f2R(Plus) $\stackrel{\text{pyk}}{\rightarrow}$ "lemma f2R(Plus)"]

f2R(Times)

[f2R(Times) $\stackrel{\text{proof}}{\rightarrow}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \text{(fx)}\colon \forall \text{(fy)}\colon \forall \text{(fz)}\colon \text{SF}(\text{(fx)} *_{\mathrm{f}} \text{(fy)}, \text{(fz)}) \vdash \text{To} == \rhd \text{SF}(\text{(fx)} *_{\mathrm{f}} \text{(fy)}, \text{(fz)}) \gg \text{R}(\text{(fx)} *_{\mathrm{f}} \text{(fy)}) == \text{R}(\text{(fz)}); \text{TimesR} \gg \text{R}(\text{(fx)}) **\text{R}(\text{(fy)}) == \text{R}(\text{(fx)} *_{\mathrm{f}} \text{(fy)}); == \text{Transitivity} \rhd \text{R}(\text{(fx)}) **\text{R}(\text{(fy)}) == \text{R}(\text{(fx)} *_{\mathrm{f}} \text{(fy)}) \rhd \text{R}(\text{(fx)} *_{\mathrm{f}} \text{(fy)}) == \text{R}(\text{(fz)}) \gg \text{R}(\text{(fx)}) **\text{R}(\text{(fy)}) == \text{R}(\text{(fz)})\rceil, \mathrm{p}_0, \mathsf{c})$]

[f2R(Times) $\stackrel{\text{stmt}}{\rightarrow}$ $\text{ZFsub} \vdash \forall \text{(fx)}\colon \forall \text{(fy)}\colon \forall \text{(fz)}\colon \text{SF}(\text{(fx)} *_{\mathrm{f}} \text{(fy)}, \text{(fz)}) \vdash \text{R}(\text{(fx)}) **\text{R}(\text{(fy)}) == \text{R}(\text{(fz)})$]

[f2R(Times) $\stackrel{\text{tex}}{\rightarrow}$ "f2R(Times)"]

[f2R(Times) $\stackrel{\text{pyk}}{\rightarrow}$ "lemma f2R(Times)"]

PlusR(Sym)

[PlusR(Sym) $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{(\text{fx})} \colon \forall \underline{(\text{fy})} \colon \text{PlusR} \gg \text{R}((\text{fx}) +_{\text{f}} (\text{fy})) == \text{R}(\underline{(\text{fx})} +_{\text{f}} \underline{(\text{fy})}); == \text{Symmetry} \rhd \text{R}((\text{fx}) +_{\text{f}} (\text{fy})) == \text{R}(\underline{(\text{fx})} +_{\text{f}} \underline{(\text{fy})}) \gg \text{R}(\underline{(\text{fx})} +_{\text{f}} \underline{(\text{fy})}) == \text{R}((\text{fx}) +_{\text{f}} (\text{fy}))\rceil, \text{p}_0, c)$]

[PlusR(Sym) $\stackrel{\text{stmt}}{\rightarrow}$ $\text{ZFsub} \vdash \forall \underline{(\text{fx})} \colon \forall \underline{(\text{fy})} \colon \text{R}(\underline{(\text{fx})} +_{\text{f}} \underline{(\text{fy})}) == \text{R}((\text{fx}) +_{\text{f}} (\text{fy}))$]

[PlusR(Sym) $\stackrel{\text{tex}}{\rightarrow}$ "PlusR(Sym)"]

[PlusR(Sym) $\stackrel{\text{pyk}}{\rightarrow}$ "lemma plusR(Sym)"]

TimesR(Sym)

[TimesR(Sym) $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{(\text{fx})} \colon \forall \underline{(\text{fy})} \colon \text{TimesR} \gg \text{R}((\text{fx})) * * \text{R}((\text{fy})) == \text{R}(\underline{(\text{fx})} *_{\text{f}} \underline{(\text{fy})}); == \text{Symmetry} \rhd \text{R}(\underline{(\text{fx})}) * * \text{R}(\underline{(\text{fy})}) == \text{R}(\underline{(\text{fx})} *_{\text{f}} \underline{(\text{fy})}) \gg \text{R}(\underline{(\text{fx})} *_{\text{f}} \underline{(\text{fy})}) == \text{R}(\underline{(\text{fx})}) * * \text{R}(\underline{(\text{fy})})\rceil, \text{p}_0, c)$]

[TimesR(Sym) $\stackrel{\text{stmt}}{\rightarrow}$ $\text{ZFsub} \vdash \forall \underline{(\text{fx})} \colon \forall \underline{(\text{fy})} \colon \text{R}(\underline{(\text{fx})} *_{\text{f}} \underline{(\text{fy})}) == \text{R}(\underline{(\text{fx})}) * * \text{R}(\underline{(\text{fy})})$]

[TimesR(Sym) $\stackrel{\text{tex}}{\rightarrow}$ "TimesR(Sym)"]

[TimesR(Sym) $\stackrel{\text{pyk}}{\rightarrow}$ "lemma timesR(Sym)"]

LessLeq(R)

[LessLeq(R) $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{(\text{rx})} \colon \forall \underline{(\text{ry})} \colon \underline{(\text{rx})} << \underline{(\text{ry})} \vdash \text{WeakenOr2} \rhd \underline{(\text{rx})} << \underline{(\text{ry})} \gg \dot{\neg}\, \underline{(\text{rx})} << \underline{(\text{ry})} \Rightarrow \underline{(\text{rx})} == \underline{(\text{ry})}; \text{Repetition} \rhd \dot{\neg}\, \underline{(\text{rx})} << \underline{(\text{ry})} \Rightarrow \underline{(\text{rx})} == \underline{(\text{ry})} \gg \dot{\neg}\, \underline{(\text{rx})} << \underline{(\text{ry})} \Rightarrow \underline{(\text{rx})} == \underline{(\text{ry})}\rceil, \text{p}_0, c)$]

[LessLeq(R) $\stackrel{\text{stmt}}{\rightarrow}$ $\text{ZFsub} \vdash \forall \underline{(\text{rx})} \colon \forall \underline{(\text{ry})} \colon \underline{(\text{rx})} << \underline{(\text{ry})} \vdash \dot{\neg}\, \underline{(\text{rx})} << \underline{(\text{ry})} \Rightarrow \underline{(\text{rx})} == \underline{(\text{ry})}$]

[LessLeq(R) $\stackrel{\text{tex}}{\rightarrow}$ "LessLeq(R)"]

[LessLeq(R) $\stackrel{\text{pyk}}{\rightarrow}$ "lemma lessLeq(R)"]

eqLeq(R)

[eqLeq(R) $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{(\text{rx})} \colon \forall \underline{(\text{ry})} \colon \underline{(\text{rx})} == \underline{(\text{ry})} \vdash \text{WeakenOr1} \rhd \underline{(\text{rx})} == \underline{(\text{ry})} \gg \dot{\neg}\, \underline{(\text{rx})} << \underline{(\text{ry})} \Rightarrow \underline{(\text{rx})} == \underline{(\text{ry})}; \text{Repetition} \rhd \dot{\neg}\, \underline{(\text{rx})} << \underline{(\text{ry})} \Rightarrow \underline{(\text{rx})} == \underline{(\text{ry})} \gg \dot{\neg}\, \underline{(\text{rx})} << \underline{(\text{ry})} \Rightarrow \underline{(\text{rx})} == \underline{(\text{ry})}\rceil, \text{p}_0, c)$]

[eqLeq(R) $\overset{\text{stmt}}{\rightarrow}$ $\text{ZFsub} \vdash \forall \underline{(rx)}: \forall \underline{(ry)}: \underline{(rx)} == \underline{(ry)} \vdash \dot\neg \underline{(rx)} << \underline{(ry)} \Rightarrow \underline{(rx)} == \underline{(ry)}$]

[eqLeq(R) $\overset{\text{tex}}{\rightarrow}$ "eqLeq(R)"]

[eqLeq(R) $\overset{\text{pyk}}{\rightarrow}$ "lemma eqLeq(R)"]

SubLessRight(R)

[SubLessRight(R) $\overset{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{(\epsilon)}: \forall \underline{m}: \forall \underline{(fy)}: \forall \underline{(fz)}: \forall \underline{(rx)}: \forall \underline{(ry)}: \forall \underline{(rz)}: \underline{(rx)} == \underline{(ry)} \vdash \underline{(rz)} << \underline{(rx)} \vdash \underline{(fz)} \in \underline{(rz)} \vdash \underline{(fy)} \in \underline{(ry)} \vdash \dot\neg 0 <= \underline{(\epsilon)} \Rightarrow \dot\neg\dot\neg 0 = \underline{(\epsilon)} \vdash \text{SENC2} \rhd \underline{(rx)} == \underline{(ry)} \gg \underline{(fy)} \in \underline{(rx)}; \text{From} << \text{XX} \rhd \underline{(rz)} << \underline{(rx)} \rhd \underline{(fz)} \in \underline{(rz)} \rhd \underline{(fy)} \in \underline{(rx)} \rhd \dot\neg 0 <= \underline{(\epsilon)} \Rightarrow \dot\neg\dot\neg 0 = \underline{(\epsilon)} \gg a_{\text{Ex}} <= \underline{m} \Rightarrow \underline{(fz)}[\underline{m}] <= \underline{(fy)}[\underline{m}] + -\underline{(\epsilon)}; \forall \underline{(\epsilon)}: \forall \underline{m}: \forall \underline{(fy)}: \forall \underline{(fz)}: \forall \underline{(rx)}: \forall \underline{(ry)}: \forall \underline{(rz)}: \text{Ded} \rhd \forall \underline{(\epsilon)}: \forall \underline{m}: \forall \underline{(fy)}: \forall \underline{(fz)}: \forall \underline{(rx)}: \forall \underline{(ry)}: \forall \underline{(rz)}: \underline{(rx)} == \underline{(ry)} \vdash \underline{(rz)} << \underline{(rx)} \vdash \underline{(fz)} \in \underline{(rz)} \vdash \underline{(fy)} \in \underline{(ry)} \vdash \dot\neg 0 <= \underline{(\epsilon)} \Rightarrow \dot\neg\dot\neg 0 = \underline{(\epsilon)} \vdash a_{\text{Ex}} <= \underline{m} \Rightarrow \underline{(fz)}[\underline{m}] <= \underline{(fy)}[\underline{m}] + -\underline{(\epsilon)} \gg \underline{(rx)} == \underline{(ry)} \Rightarrow \underline{(rz)} << \underline{(rx)} \Rightarrow \underline{(fz)} \in \underline{(rz)} \Rightarrow \underline{(fy)} \in \underline{(ry)} \Rightarrow \dot\neg 0 <= \underline{(\epsilon)} \Rightarrow \dot\neg\dot\neg 0 = \underline{(\epsilon)} \Rightarrow a_{\text{Ex}} <= \underline{m} \Rightarrow \underline{(fz)}[\underline{m}] <= \underline{(fy)}[\underline{m}] + -\underline{(\epsilon)}; \underline{(rx)} == \underline{(ry)} \vdash \underline{(rz)} << \underline{(rx)} \vdash \text{MP2} \rhd \underline{(rx)} == \underline{(ry)} \Rightarrow \underline{(rz)} << \underline{(rx)} \Rightarrow \underline{(fz)} \in \underline{(rz)} \Rightarrow \underline{(fy)} \in \underline{(ry)} \Rightarrow \dot\neg 0 <= \underline{(\epsilon)} \Rightarrow \dot\neg\dot\neg 0 = \underline{(\epsilon)} \Rightarrow a_{\text{Ex}} <= \underline{m} \Rightarrow \underline{(fz)}[\underline{m}] <= \underline{(fy)}[\underline{m}] + -\underline{(\epsilon)} \rhd \underline{(rx)} == \underline{(ry)} \rhd \underline{(rz)} << \underline{(rx)} \gg \underline{(fz)} \in \underline{(rz)} \Rightarrow \underline{(fy)} \in \underline{(ry)} \Rightarrow \dot\neg 0 <= \underline{(\epsilon)} \Rightarrow \dot\neg\dot\neg 0 = \underline{(\epsilon)} \Rightarrow a_{\text{Ex}} <= \underline{m} \Rightarrow \underline{(fz)}[\underline{m}] <= \underline{(fy)}[\underline{m}] + -\underline{(\epsilon)}; \text{to} << \text{XX} \rhd \underline{(fz)} \in \underline{(rz)} \Rightarrow \underline{(fy)} \in \underline{(ry)} \Rightarrow \dot\neg 0 <= \underline{(\epsilon)} \Rightarrow \dot\neg\dot\neg 0 = \underline{(\epsilon)} \Rightarrow a_{\text{Ex}} <= \underline{m} \Rightarrow \underline{(fz)}[\underline{m}] <= \underline{(fy)}[\underline{m}] + -\underline{(\epsilon)} \gg \underline{(rz)} << \underline{(ry)}\rceil, p_0, c)$]

[SubLessRight(R) $\overset{\text{stmt}}{\rightarrow}$ $\text{ZFsub} \vdash \forall \underline{(\epsilon)}: \forall \underline{m}: \forall \underline{(fy)}: \forall \underline{(fz)}: \forall \underline{(rx)}: \forall \underline{(ry)}: \forall \underline{(rz)}: \underline{(rx)} == \underline{(ry)} \vdash \underline{(rz)} << \underline{(rx)} \vdash \underline{(rz)} << \underline{(ry)}$]

[SubLessRight(R) $\overset{\text{tex}}{\rightarrow}$ "SubLessRight(R)"]

[SubLessRight(R) $\overset{\text{pyk}}{\rightarrow}$ "lemma subLessRight(R)"]

SubLessLeft(R)

[SubLessLeft(R) $\overset{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{(\epsilon)}: \forall \underline{m}: \forall \underline{(fy)}: \forall \underline{(fz)}: \forall \underline{(rx)}: \forall \underline{(ry)}: \forall \underline{(rz)}: \underline{(rx)} == \underline{(ry)} \vdash \underline{(rx)} << \underline{(rz)} \vdash \underline{(fy)} \in \underline{(ry)} \vdash \underline{(fz)} \in \underline{(rz)} \vdash \dot\neg 0 <= \underline{(\epsilon)} \Rightarrow \dot\neg\dot\neg 0 = \underline{(\epsilon)} \vdash \text{SENC2} \rhd \underline{(rx)} == \underline{(ry)} \rhd \underline{(fy)} \in \underline{(ry)} \gg \underline{(fy)} \in \underline{(rx)}; \text{From} << \text{XX} \rhd \underline{(rx)} << \underline{(rz)} \rhd \underline{(fy)} \in \underline{(rx)} \rhd \underline{(fz)} \in \underline{(rz)} \rhd \dot\neg 0 <= \underline{(\epsilon)} \Rightarrow \dot\neg\dot\neg 0 = \underline{(\epsilon)} \gg a_{\text{Ex}} <= \underline{m} \Rightarrow \underline{(fy)}[\underline{m}] <= \underline{(fz)}[\underline{m}] + -\underline{(\epsilon)}; \forall \underline{(\epsilon)}: \forall \underline{m}: \forall \underline{(fy)}: \forall \underline{(fz)}: \forall \underline{(rx)}: \forall \underline{(ry)}: \forall \underline{(rz)}: \text{Ded} \rhd \forall \underline{(\epsilon)}: \forall \underline{m}: \forall \underline{(fy)}: \forall \underline{(fz)}: \forall \underline{(rx)}: \forall \underline{(ry)}: \forall \underline{(rz)}: \underline{(rx)} == \underline{(ry)} \vdash \underline{(rx)} << \underline{(rz)} \vdash \underline{(fy)} \in \underline{(ry)} \vdash \underline{(fz)} \in \underline{(rz)} \vdash \dot\neg 0 <= \underline{(\epsilon)} \Rightarrow \dot\neg\dot\neg 0 = \underline{(\epsilon)} \vdash a_{\text{Ex}} <= \underline{m} \Rightarrow \underline{(fy)}[\underline{m}] <= \underline{(fz)}[\underline{m}] + -\underline{(\epsilon)} \gg \underline{(rx)} == \underline{(ry)} \Rightarrow \underline{(rx)} << \underline{(rz)} \Rightarrow \underline{(fy)} \in \underline{(ry)} \Rightarrow \underline{(fz)} \in \underline{(rz)} \Rightarrow$

$\dot\neg 0 <= \underline{(\epsilon)} \Rightarrow \dot\neg\dot\neg 0 = \underline{(\epsilon)} \Rightarrow \mathsf{a}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)}; \underline{(\mathrm{rx})} == \underline{(\mathrm{ry})} \vdash \underline{(\mathrm{rx})} << \underline{(\mathrm{rz})} \vdash \mathrm{MP2} \rhd \underline{(\mathrm{rx})} == \underline{(\mathrm{ry})} \Rightarrow \underline{(\mathrm{rx})} << \underline{(\mathrm{rz})} \Rightarrow \underline{(\mathrm{fy})} \in \underline{(\mathrm{ry})} \Rightarrow \underline{(\mathrm{fz})} \in \underline{(\mathrm{rz})} \Rightarrow \dot\neg 0 <= \underline{(\epsilon)} \Rightarrow \dot\neg\dot\neg 0 = \underline{(\epsilon)} \Rightarrow \mathsf{a}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \rhd \underline{(\mathrm{rx})} == \underline{(\mathrm{ry})} \rhd \underline{(\mathrm{rx})} << \underline{(\mathrm{rz})} \gg \underline{(\mathrm{fy})} \in \underline{(\mathrm{ry})} \Rightarrow \underline{(\mathrm{fz})} \in \underline{(\mathrm{rz})} \Rightarrow \dot\neg 0 <= \underline{(\epsilon)} \Rightarrow \dot\neg\dot\neg 0 = \underline{(\epsilon)} \Rightarrow \mathsf{a}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)}; \mathrm{to} << \mathrm{XX} \rhd \underline{(\mathrm{fy})} \in \underline{(\mathrm{ry})} \Rightarrow \underline{(\mathrm{fz})} \in \underline{(\mathrm{rz})} \Rightarrow \dot\neg 0 <= \underline{(\epsilon)} \Rightarrow \dot\neg\dot\neg 0 = \underline{(\epsilon)} \Rightarrow \mathsf{a}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(\mathrm{fy})}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \gg \underline{(\mathrm{ry})} << \underline{(\mathrm{rz})}\rceil, \mathrm{p}_0, \mathsf{c})]$

[SubLessLeft(R) $\stackrel{\mathrm{stmt}}{\rightarrow}$ ZFsub $\vdash \forall\underline{(\epsilon)}\colon \forall\underline{\mathsf{m}}\colon \forall\underline{(\mathrm{fy})}\colon \forall\underline{(\mathrm{fz})}\colon \forall\underline{(\mathrm{rx})}\colon \forall\underline{(\mathrm{ry})}\colon \forall\underline{(\mathrm{rz})}\colon \underline{(\mathrm{rx})} == \underline{(\mathrm{ry})} \vdash \underline{(\mathrm{rx})} << \underline{(\mathrm{rz})} \vdash \underline{(\mathrm{ry})} << \underline{(\mathrm{rz})}$]

[SubLessLeft(R) $\stackrel{\mathrm{tex}}{\rightarrow}$ "SubLessLeft(R)"]

[SubLessLeft(R) $\stackrel{\mathrm{pyk}}{\rightarrow}$ "lemma subLessLeft(R)"]

<< TransitivityHelper(Q)

[<< TransitivityHelper(Q) $\stackrel{\mathrm{proof}}{\rightarrow}$ $\lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall\underline{\mathsf{x}}\colon \forall\underline{\mathsf{y}}\colon \forall\underline{\mathsf{z}}\colon \forall\underline{\mathsf{u}}\colon \underline{\mathsf{x}} <= \underline{\mathsf{y}} + -\underline{\mathsf{u}} \vdash \underline{\mathsf{y}} <= \underline{\mathsf{z}} + -\underline{\mathsf{u}} \vdash \dot\neg 0 <= \underline{\mathsf{u}} \Rightarrow \dot\neg\dot\neg 0 = \underline{\mathsf{u}} \vdash \mathrm{PositiveNegated} \rhd \dot\neg 0 <= \underline{\mathsf{u}} \Rightarrow \dot\neg\dot\neg 0 = \underline{\mathsf{u}} \gg \dot\neg - \underline{\mathsf{u}} <= 0 \Rightarrow \dot\neg\dot\neg - \underline{\mathsf{u}} = 0; \mathrm{LessAdditionLeft} \rhd \dot\neg - \underline{\mathsf{u}} <= 0 \Rightarrow \dot\neg\dot\neg - \underline{\mathsf{u}} = 0 \gg \dot\neg \underline{\mathsf{y}} + -\underline{\mathsf{u}} <= \underline{\mathsf{y}} + 0 \Rightarrow \dot\neg\dot\neg \underline{\mathsf{y}} + -\underline{\mathsf{u}} = \underline{\mathsf{y}} + 0; \mathrm{plus0} \gg \underline{\mathsf{y}} + 0 = \underline{\mathsf{y}}; \mathrm{SubLessRight} \rhd \underline{\mathsf{y}} + 0 = \underline{\mathsf{y}} \rhd \dot\neg \underline{\mathsf{y}} + -\underline{\mathsf{u}} <= \underline{\mathsf{y}} + 0 \Rightarrow \dot\neg\dot\neg \underline{\mathsf{y}} + -\underline{\mathsf{u}} = \underline{\mathsf{y}} + 0 \gg \dot\neg \underline{\mathsf{y}} + -\underline{\mathsf{u}} <= \underline{\mathsf{y}} \Rightarrow \dot\neg\dot\neg \underline{\mathsf{y}} + -\underline{\mathsf{u}} = \underline{\mathsf{y}}; \mathrm{leqLessTransitivity} \rhd \underline{\mathsf{x}} <= \underline{\mathsf{y}} + -\underline{\mathsf{u}} \rhd \dot\neg \underline{\mathsf{y}} + -\underline{\mathsf{u}} <= \underline{\mathsf{y}} \Rightarrow \dot\neg\dot\neg \underline{\mathsf{y}} + -\underline{\mathsf{u}} = \underline{\mathsf{y}} \gg \dot\neg \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot\neg\dot\neg \underline{\mathsf{x}} = \underline{\mathsf{y}}; \mathrm{LessLeqTransitivity} \rhd \dot\neg \underline{\mathsf{x}} <= \underline{\mathsf{y}} \Rightarrow \dot\neg\dot\neg \underline{\mathsf{x}} = \underline{\mathsf{y}} \rhd \underline{\mathsf{y}} <= \underline{\mathsf{z}} + -\underline{\mathsf{u}} \gg \dot\neg \underline{\mathsf{x}} <= \underline{\mathsf{z}} + -\underline{\mathsf{u}} \Rightarrow \dot\neg\dot\neg \underline{\mathsf{x}} = \underline{\mathsf{z}} + -\underline{\mathsf{u}}; \mathrm{LessLeq} \rhd \dot\neg \underline{\mathsf{x}} <= \underline{\mathsf{z}} + -\underline{\mathsf{u}} \Rightarrow \dot\neg\dot\neg \underline{\mathsf{x}} = \underline{\mathsf{z}} + -\underline{\mathsf{u}} \gg \underline{\mathsf{x}} <= \underline{\mathsf{z}} + -\underline{\mathsf{u}}\rceil, \mathrm{p}_0, \mathsf{c})$]

[<< TransitivityHelper(Q) $\stackrel{\mathrm{stmt}}{\rightarrow}$ ZFsub $\vdash \forall\underline{\mathsf{x}}\colon \forall\underline{\mathsf{y}}\colon \forall\underline{\mathsf{z}}\colon \forall\underline{\mathsf{u}}\colon \underline{\mathsf{x}} <= \underline{\mathsf{y}} + -\underline{\mathsf{u}} \vdash \underline{\mathsf{y}} <= \underline{\mathsf{z}} + -\underline{\mathsf{u}} \vdash \dot\neg 0 <= \underline{\mathsf{u}} \Rightarrow \dot\neg\dot\neg 0 = \underline{\mathsf{u}} \vdash \underline{\mathsf{x}} <= \underline{\mathsf{z}} + -\underline{\mathsf{u}}$]

[<< TransitivityHelper(Q) $\stackrel{\mathrm{tex}}{\rightarrow}$ "<<TransitivityHelper(Q)"]

[<< TransitivityHelper(Q) $\stackrel{\mathrm{pyk}}{\rightarrow}$ "lemma <<TransitivityHelper(Q)"]

<< Transitivity

[<< Transitivity $\stackrel{\mathrm{proof}}{\rightarrow}$ $\lambda\mathsf{c}.\lambda\mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall\underline{\mathsf{m}}\colon \forall\underline{(\epsilon)}\colon \forall\underline{(\mathrm{fx})}\colon \forall\underline{(\mathrm{fz})}\colon \forall\underline{(\mathrm{rx})}\colon \forall\underline{(\mathrm{ry})}\colon \forall\underline{(\mathrm{rz})}\colon \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})} \vdash \underline{(\mathrm{ry})} << \underline{(\mathrm{rz})} \vdash \underline{(\mathrm{fx})} \in \underline{(\mathrm{rx})} \vdash \underline{(\mathrm{fz})} \in \underline{(\mathrm{rz})} \vdash \dot\neg 0 <= \underline{(\epsilon)} \Rightarrow \dot\neg\dot\neg 0 = \underline{(\epsilon)} \vdash \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \vdash \mathrm{From} << (2) \rhd \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})} \gg \mathsf{t}_{\mathrm{Ex}} \in \underline{(\mathrm{ry})}; \mathrm{ThirdGeqSeries} \rhd \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})} \rhd \underline{(\mathrm{ry})} << \underline{(\mathrm{rz})} \rhd \underline{(\mathrm{fx})} \in \underline{(\mathrm{rx})} \rhd \mathsf{t}_{\mathrm{Ex}} \in \underline{(\mathrm{ry})} \rhd \mathsf{t}_{\mathrm{Ex}} \in \underline{(\mathrm{ry})} \rhd \underline{(\mathrm{fz})} \in \underline{(\mathrm{rz})} \rhd \dot\neg 0 <= \underline{(\epsilon)} \Rightarrow \dot\neg\dot\neg 0 = \underline{(\epsilon)} \rhd \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \gg \dot\neg \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] <= \mathsf{t}_{\mathrm{Ex}}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \Rightarrow \dot\neg \mathsf{t}_{\mathrm{Ex}}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)}; \mathrm{FirstConjunct} \rhd \dot\neg \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] <= \mathsf{t}_{\mathrm{Ex}}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \Rightarrow \dot\neg \mathsf{t}_{\mathrm{Ex}}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \gg \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] <= \mathsf{t}_{\mathrm{Ex}}[\underline{\mathsf{m}}] + -\underline{(\epsilon)}; \mathrm{SecondConjunct} \rhd \dot\neg \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] <=$

$\mathsf{t}_{\mathrm{Ex}}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \Rightarrow \dot\neg\, \mathsf{t}_{\mathrm{Ex}}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \gg \mathsf{t}_{\mathrm{Ex}}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)}; <<$ TransitivityHelper(Q) $\rhd \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] <= \mathsf{t}_{\mathrm{Ex}}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \rhd \mathsf{t}_{\mathrm{Ex}}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \rhd \dot\neg 0 <= \underline{(\epsilon)} \Rightarrow \dot\neg\dot\neg 0 = \underline{(\epsilon)} \gg \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)}; \forall \underline{\mathsf{m}}\colon \forall \underline{(\epsilon)}\colon \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fz})}\colon \forall \underline{(\mathrm{rx})}\colon \forall \underline{(\mathrm{ry})}\colon \forall \underline{(\mathrm{rz})}\colon$ Ded $\rhd \forall \underline{\mathsf{m}}\colon \forall \underline{(\epsilon)}\colon \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fz})}\colon \forall \underline{(\mathrm{rx})}\colon \forall \underline{(\mathrm{ry})}\colon \forall \underline{(\mathrm{rz})}\colon \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})} \vdash \underline{(\mathrm{ry})} << \underline{(\mathrm{rz})} \vdash \underline{(\mathrm{fx})} \in \underline{(\mathrm{rx})} \vdash \underline{(\mathrm{fz})} \in \underline{(\mathrm{rz})} \vdash \dot\neg 0 <= \underline{(\epsilon)} \Rightarrow \dot\neg\dot\neg 0 = \underline{(\epsilon)} \vdash \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \vdash \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \gg \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})} \Rightarrow \underline{(\mathrm{ry})} << \underline{(\mathrm{rz})} \Rightarrow \underline{(\mathrm{fx})} \in \underline{(\mathrm{rx})} \Rightarrow \underline{(\mathrm{fz})} \in \underline{(\mathrm{rz})} \Rightarrow \dot\neg 0 <= \underline{(\epsilon)} \Rightarrow \dot\neg\dot\neg 0 = \underline{(\epsilon)} \Rightarrow \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)}; \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})} \vdash \underline{(\mathrm{ry})} << \underline{(\mathrm{rz})} \vdash$ MP2 $\rhd \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})} \Rightarrow \underline{(\mathrm{ry})} << \underline{(\mathrm{rz})} \Rightarrow \underline{(\mathrm{fx})} \in \underline{(\mathrm{rx})} \Rightarrow \underline{(\mathrm{fz})} \in \underline{(\mathrm{rz})} \Rightarrow \dot\neg 0 <= \underline{(\epsilon)} \Rightarrow \dot\neg\dot\neg 0 = \underline{(\epsilon)} \Rightarrow \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \rhd \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})} \rhd \underline{(\mathrm{ry})} << \underline{(\mathrm{rz})} \gg \underline{(\mathrm{fx})} \in \underline{(\mathrm{rx})} \Rightarrow \underline{(\mathrm{fz})} \in \underline{(\mathrm{rz})} \Rightarrow \dot\neg 0 <= \underline{(\epsilon)} \Rightarrow \dot\neg\dot\neg 0 = \underline{(\epsilon)} \Rightarrow \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)};$ ExistIntro @ a_{Ex} @ $\mathsf{c}_{\mathrm{Ex}} \rhd \underline{(\mathrm{fx})} \in \underline{(\mathrm{rx})} \Rightarrow \underline{(\mathrm{fz})} \in \underline{(\mathrm{rz})} \Rightarrow \dot\neg 0 <= \underline{(\epsilon)} \Rightarrow \dot\neg\dot\neg 0 = \underline{(\epsilon)} \Rightarrow \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \gg \underline{(\mathrm{fx})} \in \underline{(\mathrm{rx})} \Rightarrow \underline{(\mathrm{fz})} \in \underline{(\mathrm{rz})} \Rightarrow \dot\neg 0 <= \underline{(\epsilon)} \Rightarrow \dot\neg\dot\neg 0 = \underline{(\epsilon)} \Rightarrow \mathsf{a}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)};$ to $<<$ XX $\rhd \underline{(\mathrm{fx})} \in \underline{(\mathrm{rx})} \Rightarrow \underline{(\mathrm{fz})} \in \underline{(\mathrm{rz})} \Rightarrow \dot\neg 0 <= \underline{(\epsilon)} \Rightarrow \dot\neg\dot\neg 0 = \underline{(\epsilon)} \Rightarrow \mathsf{a}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] <= \underline{(\mathrm{fz})}[\underline{\mathsf{m}}] + -\underline{(\epsilon)} \gg \underline{(\mathrm{rx})} << \underline{(\mathrm{rz})}\rceil, \mathrm{p}_0, \mathsf{c})]$

[$<<$ Transitivity $\overset{\mathrm{stmt}}{\to}$ ZFsub $\vdash \forall \underline{\mathsf{m}}\colon \forall \underline{(\epsilon)}\colon \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fz})}\colon \forall \underline{(\mathrm{rx})}\colon \forall \underline{(\mathrm{ry})}\colon \forall \underline{(\mathrm{rz})}\colon \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})} \vdash \underline{(\mathrm{ry})} << \underline{(\mathrm{rz})} \vdash \underline{(\mathrm{rx})} << \underline{(\mathrm{rz})}$]

[$<<$ Transitivity $\overset{\mathrm{tex}}{\to}$ "<<Transitivity"]

[$<<$ Transitivity $\overset{\mathrm{pyk}}{\to}$ "lemma <<Transitivity"]

<<== Reflexivity

[$<<==$ Reflexivity $\overset{\mathrm{proof}}{\to}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{(\mathrm{rx})}\colon ==$Reflexivity $\gg \underline{(\mathrm{rx})} == \underline{(\mathrm{rx})};$ eqLeq(R) $\rhd \underline{(\mathrm{rx})} == \underline{(\mathrm{rx})} \gg \dot\neg\, \underline{(\mathrm{rx})} << \underline{(\mathrm{rx})} \Rightarrow \underline{(\mathrm{rx})} == \underline{(\mathrm{rx})}\rceil, \mathrm{p}_0, \mathsf{c})$]

[$<<==$ Reflexivity $\overset{\mathrm{stmt}}{\to}$ ZFsub $\vdash \forall \underline{(\mathrm{rx})}\colon \dot\neg\, \underline{(\mathrm{rx})} << \underline{(\mathrm{rx})} \Rightarrow \underline{(\mathrm{rx})} == \underline{(\mathrm{rx})}$]

[$<<==$ Reflexivity $\overset{\mathrm{tex}}{\to}$ "<<==Reflexivity"]

[$<<==$ Reflexivity $\overset{\mathrm{pyk}}{\to}$ "lemma <<==Reflexivity"]

<<== AntisymmetryHelper(Q)

[$<<==$ AntisymmetryHelper(Q) $\overset{\mathrm{proof}}{\to}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{\mathsf{a}}\colon \forall \underline{\mathsf{x}}\colon \forall \underline{\mathsf{y}}\colon \forall \underline{\mathsf{z}}\colon \dot\neg 0 <= \underline{\mathsf{z}} \Rightarrow \dot\neg\dot\neg 0 = \underline{\mathsf{z}} \vdash \underline{\mathsf{x}} <= \underline{\mathsf{y}} + -\underline{\mathsf{z}} \vdash \underline{\mathsf{y}} <= \underline{\mathsf{x}} + -\underline{\mathsf{z}} \vdash$ leqAddition $\rhd \underline{\mathsf{x}} <= \underline{\mathsf{y}} + -\underline{\mathsf{z}} \gg \underline{\mathsf{x}} + \underline{\mathsf{z}} <= \underline{\mathsf{y}} + -\underline{\mathsf{z}} + \underline{\mathsf{z}};$ plusAssociativity $\gg \underline{\mathsf{y}} + -\underline{\mathsf{z}} + \underline{\mathsf{z}} = \underline{\mathsf{y}} + -\underline{\mathsf{z}} + \underline{\mathsf{z}};$ plusCommutativity $\gg -\underline{\mathsf{z}} + \underline{\mathsf{z}} = \underline{\mathsf{z}} + -\underline{\mathsf{z}};$ lemma eqAdditionLeft $\rhd -\underline{\mathsf{z}} + \underline{\mathsf{z}} = \underline{\mathsf{z}} + -\underline{\mathsf{z}} \gg \underline{\mathsf{y}} + -\underline{\mathsf{z}} + \underline{\mathsf{z}} = \underline{\mathsf{y}} + \underline{\mathsf{z}} + -\underline{\mathsf{z}};$ x = x + (y − y) $\gg \underline{\mathsf{y}} = \underline{\mathsf{y}} + \underline{\mathsf{z}} + -\underline{\mathsf{z}};$ eqSymmetry $\rhd \underline{\mathsf{y}} = \underline{\mathsf{y}} + \underline{\mathsf{z}} + -\underline{\mathsf{z}} \gg \underline{\mathsf{y}} + \underline{\mathsf{z}} + -\underline{\mathsf{z}} =$

$\underline{\mathsf{y}}$; eqTransitivity4 $\rhd \underline{\mathsf{y}}+-\underline{\mathsf{z}}+\underline{\mathsf{z}} = \underline{\mathsf{y}}+-\underline{\mathsf{z}}+\underline{\mathsf{z}} \rhd \underline{\mathsf{y}}+-\underline{\mathsf{z}}+\underline{\mathsf{z}} = \underline{\mathsf{y}}+\underline{\mathsf{z}}+-\underline{\mathsf{z}} \rhd \underline{\mathsf{y}}+\underline{\mathsf{z}}+-\underline{\mathsf{z}} = \underline{\mathsf{y}} \gg \underline{\mathsf{y}}+-\underline{\mathsf{z}}+\underline{\mathsf{z}} = \underline{\mathsf{y}}$; subLeqRight $\rhd \underline{\mathsf{y}}+-\underline{\mathsf{z}}+\underline{\mathsf{z}} = \underline{\mathsf{y}} \rhd \underline{\mathsf{x}}+\underline{\mathsf{z}} <= \underline{\mathsf{y}}+-\underline{\mathsf{z}}+\underline{\mathsf{z}} \gg \underline{\mathsf{x}}+\underline{\mathsf{z}} <= \underline{\mathsf{y}}$; leqTransitivity $\rhd \underline{\mathsf{x}}+\underline{\mathsf{z}} <= \underline{\mathsf{y}} \rhd \underline{\mathsf{y}} <= \underline{\mathsf{x}}+-\underline{\mathsf{z}} \gg \underline{\mathsf{x}}+\underline{\mathsf{z}} <= \underline{\mathsf{x}}+-\underline{\mathsf{z}}$; leqSubtractionLeft $\rhd \underline{\mathsf{x}}+\underline{\mathsf{z}} <= \underline{\mathsf{x}}+-\underline{\mathsf{z}} \gg \underline{\mathsf{z}} <= -\underline{\mathsf{z}}$; toNotLess $\rhd \underline{\mathsf{z}} <= -\underline{\mathsf{z}} \gg \dot{\neg}\dot{\neg} - \underline{\mathsf{z}} <= \underline{\mathsf{z}} \Rightarrow \dot{\neg}\dot{\neg} - \underline{\mathsf{z}} = \underline{\mathsf{z}}$; NegativeLessPositive $\rhd \dot{\neg} 0 <= \underline{\mathsf{z}} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{\mathsf{z}} \gg \dot{\neg} - \underline{\mathsf{z}} <= \underline{\mathsf{z}} \Rightarrow \dot{\neg}\dot{\neg} - \underline{\mathsf{z}} = \underline{\mathsf{z}}$; FromContradiction $\rhd \dot{\neg} - \underline{\mathsf{z}} <= \underline{\mathsf{z}} \Rightarrow \dot{\neg}\dot{\neg} - \underline{\mathsf{z}} = \underline{\mathsf{z}} \rhd \dot{\neg}\dot{\neg} - \underline{\mathsf{z}} <= \underline{\mathsf{z}} \Rightarrow \dot{\neg}\dot{\neg} - \underline{\mathsf{z}} = \underline{\mathsf{z}} \gg \underline{\mathsf{a}}\rceil, p_0, \mathsf{c})]$

[<<== AntisymmetryHelper(Q) $\stackrel{\mathrm{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{\mathsf{a}}: \forall \underline{\mathsf{x}}: \forall \underline{\mathsf{y}}: \forall \underline{\mathsf{z}}: \dot{\neg} 0 <= \underline{\mathsf{z}} \Rightarrow \dot{\neg}\dot{\neg} 0 = \underline{\mathsf{z}} \vdash \underline{\mathsf{x}} <= \underline{\mathsf{y}} + -\underline{\mathsf{z}} \vdash \underline{\mathsf{y}} <= \underline{\mathsf{x}} + -\underline{\mathsf{z}} \vdash \underline{\mathsf{a}}]$

[<<== AntisymmetryHelper(Q) $\stackrel{\mathrm{tex}}{\rightarrow}$ "<<==AntisymmetryHelper(Q)"]

[<<== AntisymmetryHelper(Q) $\stackrel{\mathrm{pyk}}{\rightarrow}$ "lemma <<==AntisymmetryHelper(Q)"]

<<== Antisymmetry

[<<== Antisymmetry $\stackrel{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{(\mathrm{rx})}: \forall \underline{(\mathrm{ry})}: \dot{\neg} \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})} \Rightarrow \underline{(\mathrm{rx})} == \underline{(\mathrm{ry})} \vdash \dot{\neg} \underline{(\mathrm{ry})} << \underline{(\mathrm{rx})} \Rightarrow \underline{(\mathrm{ry})} == \underline{(\mathrm{rx})} \vdash$ Repetition $\rhd \dot{\neg} \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})} \Rightarrow \underline{(\mathrm{rx})} == \underline{(\mathrm{ry})} \gg \dot{\neg} \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})} \Rightarrow \underline{(\mathrm{rx})} == \underline{(\mathrm{ry})}$; Repetition $\rhd \dot{\neg} \underline{(\mathrm{ry})} << \underline{(\mathrm{rx})} \Rightarrow \underline{(\mathrm{ry})} == \underline{(\mathrm{rx})} \gg \dot{\neg} \underline{(\mathrm{ry})} << \underline{(\mathrm{rx})} \Rightarrow \underline{(\mathrm{ry})} == \underline{(\mathrm{rx})}$; ExpandDisjuncts $\rhd \dot{\neg} \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})} \Rightarrow \underline{(\mathrm{rx})} == \underline{(\mathrm{ry})} \rhd \dot{\neg} \underline{(\mathrm{ry})} << \underline{(\mathrm{rx})} \Rightarrow \underline{(\mathrm{ry})} == \underline{(\mathrm{rx})} \gg \dot{\neg} \underline{(\mathrm{rx})} == \underline{(\mathrm{ry})} \Rightarrow \dot{\neg} \underline{(\mathrm{ry})} == \underline{(\mathrm{rx})} \Rightarrow \dot{\neg} \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})} \Rightarrow \dot{\neg} \underline{(\mathrm{ry})} << \underline{(\mathrm{rx})}$; AutoImply $\gg \underline{(\mathrm{rx})} == \underline{(\mathrm{ry})} \Rightarrow \underline{(\mathrm{rx})} == \underline{(\mathrm{ry})}$; $\forall \underline{(\mathrm{rx})}: \forall \underline{(\mathrm{ry})}: \underline{(\mathrm{ry})} == \underline{(\mathrm{rx})} \vdash$==Symmetry $\rhd \underline{(\mathrm{ry})} == \underline{(\mathrm{rx})} \gg \underline{(\mathrm{rx})} == \underline{(\mathrm{ry})}$; Ded $\rhd \forall \underline{(\mathrm{rx})}: \forall \underline{(\mathrm{ry})}: \underline{(\mathrm{ry})} == \underline{(\mathrm{rx})} \vdash \underline{(\mathrm{rx})} == \underline{(\mathrm{ry})} \gg \underline{(\mathrm{ry})} == \underline{(\mathrm{rx})} \Rightarrow \underline{(\mathrm{rx})} == \underline{(\mathrm{ry})}$; $\forall \underline{(\mathrm{rx})}: \forall \underline{(\mathrm{ry})}: \dot{\neg} \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})} \Rightarrow \dot{\neg} \underline{(\mathrm{ry})} << \underline{(\mathrm{rx})} \vdash$ FirstConjunct $\rhd \dot{\neg} \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})} \Rightarrow \dot{\neg} \underline{(\mathrm{ry})} << \underline{(\mathrm{rx})} \gg \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})}$; From << (1) $\rhd \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})} \gg \mathsf{j}_{\mathrm{Ex}} \in \underline{(\mathrm{rx})}$; From << (2) $\rhd \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})} \gg \mathsf{t}_{\mathrm{Ex}} \in \underline{(\mathrm{ry})}$; $0 < 1 \gg \dot{\neg} 0 <= 1 \Rightarrow \dot{\neg}\dot{\neg} 0 = 1$; From << XX $\rhd \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})} \rhd \mathsf{j}_{\mathrm{Ex}} \in \underline{(\mathrm{rx})} \rhd \mathsf{t}_{\mathrm{Ex}} \in \underline{(\mathrm{ry})} \rhd \dot{\neg} 0 <= 1 \Rightarrow \dot{\neg}\dot{\neg} 0 = 1 \gg \mathsf{a}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}} \Rightarrow \mathsf{j}_{\mathrm{Ex}}[\mathsf{c}_{\mathrm{Ex}}] <= \mathsf{t}_{\mathrm{Ex}}[\mathsf{c}_{\mathrm{Ex}}] + -1$; SecondConjunct $\rhd \dot{\neg} \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})} \Rightarrow \dot{\neg} \underline{(\mathrm{ry})} << \underline{(\mathrm{rx})} \gg \underline{(\mathrm{ry})} << \underline{(\mathrm{rx})}$; From << XX $\rhd \underline{(\mathrm{ry})} << \underline{(\mathrm{rx})} \rhd \mathsf{t}_{\mathrm{Ex}} \in \underline{(\mathrm{ry})} \rhd \mathsf{j}_{\mathrm{Ex}} \in \underline{(\mathrm{rx})} \rhd \dot{\neg} 0 <= 1 \Rightarrow \dot{\neg}\dot{\neg} 0 = 1 \gg \mathsf{a}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}} \Rightarrow \mathsf{t}_{\mathrm{Ex}}[\mathsf{c}_{\mathrm{Ex}}] <= \mathsf{j}_{\mathrm{Ex}}[\mathsf{c}_{\mathrm{Ex}}] + -1$; ExistIntro @ b_{Ex} @ $\mathsf{a}_{\mathrm{Ex}} \rhd \mathsf{a}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}} \Rightarrow \mathsf{t}_{\mathrm{Ex}}[\mathsf{c}_{\mathrm{Ex}}] <= \mathsf{j}_{\mathrm{Ex}}[\mathsf{c}_{\mathrm{Ex}}] + -1 \gg \mathsf{b}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}} \Rightarrow \mathsf{t}_{\mathrm{Ex}}[\mathsf{c}_{\mathrm{Ex}}] <= \mathsf{j}_{\mathrm{Ex}}[\mathsf{c}_{\mathrm{Ex}}] + -1$; thirdGeq $\gg \dot{\neg} \mathsf{a}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}} \Rightarrow \dot{\neg} \mathsf{b}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}}$; FirstConjunct $\rhd \dot{\neg} \mathsf{a}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}} \Rightarrow \dot{\neg} \mathsf{b}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}} \gg \mathsf{a}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}}$; MP $\rhd \mathsf{a}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}} \Rightarrow \mathsf{j}_{\mathrm{Ex}}[\mathsf{c}_{\mathrm{Ex}}] <= \mathsf{t}_{\mathrm{Ex}}[\mathsf{c}_{\mathrm{Ex}}] + -1 \rhd \mathsf{a}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}} \gg \mathsf{j}_{\mathrm{Ex}}[\mathsf{c}_{\mathrm{Ex}}] <= \mathsf{t}_{\mathrm{Ex}}[\mathsf{c}_{\mathrm{Ex}}] + -1$; SecondConjunct $\rhd \dot{\neg} \mathsf{a}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}} \Rightarrow \dot{\neg} \mathsf{b}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}} \gg \mathsf{b}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}}$; MP $\rhd \mathsf{b}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}} \Rightarrow \mathsf{t}_{\mathrm{Ex}}[\mathsf{c}_{\mathrm{Ex}}] <= \mathsf{j}_{\mathrm{Ex}}[\mathsf{c}_{\mathrm{Ex}}] + -1 \rhd \mathsf{b}_{\mathrm{Ex}} <= \mathsf{c}_{\mathrm{Ex}} \gg \mathsf{t}_{\mathrm{Ex}}[\mathsf{c}_{\mathrm{Ex}}] <= \mathsf{j}_{\mathrm{Ex}}[\mathsf{c}_{\mathrm{Ex}}] + -1$; <<== AntisymmetryHelper(Q) $\rhd \dot{\neg} 0 <= 1 \Rightarrow \dot{\neg}\dot{\neg} 0 = 1 \rhd \mathsf{j}_{\mathrm{Ex}}[\mathsf{c}_{\mathrm{Ex}}] <= \mathsf{t}_{\mathrm{Ex}}[\mathsf{c}_{\mathrm{Ex}}] + -1 \rhd \mathsf{t}_{\mathrm{Ex}}[\mathsf{c}_{\mathrm{Ex}}] <= \mathsf{j}_{\mathrm{Ex}}[\mathsf{c}_{\mathrm{Ex}}] + -1 \gg \underline{(\mathrm{rx})} == \underline{(\mathrm{ry})}$; Ded $\rhd \forall \underline{(\mathrm{rx})}: \forall \underline{(\mathrm{ry})}: \dot{\neg} \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})} \Rightarrow \dot{\neg} \underline{(\mathrm{ry})} << \underline{(\mathrm{rx})} \vdash \underline{(\mathrm{rx})} == \underline{(\mathrm{ry})} \gg \dot{\neg} \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})} \Rightarrow \dot{\neg} \underline{(\mathrm{ry})} << \underline{(\mathrm{rx})} \Rightarrow \underline{(\mathrm{rx})} == \underline{(\mathrm{ry})}$; From3Disjuncts $\rhd \dot{\neg} \underline{(\mathrm{rx})} == \underline{(\mathrm{ry})} \Rightarrow \dot{\neg} \underline{(\mathrm{ry})} == \underline{(\mathrm{rx})} \Rightarrow \dot{\neg} \underline{(\mathrm{rx})} << \underline{(\mathrm{ry})} \Rightarrow$

∸¬(ry) << (rx) ▷ (rx) == (ry) ⇒ (rx) == (ry) ▷ (ry) == (rx) ⇒ (rx) == (ry) ▷ ∸¬(rx) << (ry) ⇒ ∸¬(ry) << (rx) ⇒ (rx) == (ry) ≫ (rx) == (ry)⌉, p_0, c)]

[<<== Antisymmetry $\overset{\text{stmt}}{\to}$ ZFsub ⊢ ∀(rx): ∀(ry): ∸¬(rx) << (ry) ⇒ (rx) == (ry) ⊢ ∸¬(ry) << (rx) ⇒ (ry) == (rx) ⊢ (rx) == (ry)]

[<<== Antisymmetry $\overset{\text{tex}}{\to}$ “<<==Antisymmetry”]

[<<== Antisymmetry $\overset{\text{pyk}}{\to}$ “lemma <<==Antisymmetry”]

<<== Transitivity

[<<== Transitivity $\overset{\text{proof}}{\to}$ λc.λx.𝒫(⌈ZFsub ⊢ ∀(rx): ∀(ry): ∀(rz): (rx) << (ry) ⊢ (ry) << (rz) ⊢<< Transitivity ▷ (rx) << (ry) ▷ (ry) << (rz) ≫ (rx) << (rz); LessLeq(R) ▷ (rx) << (rz) ≫ ∸¬(rx) << (rz) ⇒ (rx) == (rz); ∀(rx): ∀(ry): ∀(rz): (rx) << (ry) ⊢ (ry) == (rz) ⊢ SubLessRight(R) ▷ (ry) == (rz) ▷ (rx) << (ry) ≫ (rx) << (rz); LessLeq(R) ▷ (rx) << (rz) ≫ ∸¬(rx) << (rz) ⇒ (rx) == (rz); ∀(rx): ∀(ry): ∀(rz): (rx) == (ry) ⊢ (ry) << (rz) ⊢== Symmetry ▷ (rx) == (ry) ≫ (ry) == (rx); SubLessLeft(R) ▷ (ry) == (rx) ▷ (ry) << (rz) ≫ (rx) << (rz); LessLeq(R) ▷ (rx) << (rz) ≫ ∸¬(rx) << (rz) ⇒ (rx) == (rz); ∀(rx): ∀(ry): ∀(rz): (rx) == (ry) ⊢ (ry) == (rz) ⊢ == Transitivity ▷ (rx) == (ry) ▷ (ry) == (rz) ≫ (rx) == (rz); eqLeq(R) ▷ (rx) == (rz) ≫ ∸¬(rx) << (rz) ⇒ (rx) == (rz); ∀(rx): ∀(ry): ∀(rz): ∸¬(rx) << (ry) ⇒ (rx) == (ry) ⊢ ∸¬(ry) << (rz) ⇒ (ry) == (rz) ⊢ Repetition ▷ ∸¬(rx) << (ry) ⇒ (rx) == (ry) ≫ ∸¬(rx) << (ry) ⇒ (rx) == (ry); Repetition ▷ ∸¬(ry) << (rz) ⇒ (ry) == (rz) ≫ ∸¬(ry) << (rz) ⇒ (ry) == (rz); Ded ▷ ∀(rx): ∀(ry): ∀(rz): (rx) << (ry) ⊢ (ry) << (rz) ⊢ ∸¬(rx) << (rz) ⇒ (rx) == (rz) ≫ (rx) << (ry) ⇒ (ry) << (rz) ⇒ ∸¬(rx) << (rz) ⇒ (rx) == (rz); Ded ▷ ∀(rx): ∀(ry): ∀(rz): (rx) << (ry) ⊢ (ry) == (rz) ⊢ ∸¬(rx) << (rz) ⇒ (rx) == (rz) ≫ (rx) << (ry) ⇒ (ry) == (rz) ⇒ ∸¬(rx) << (rz) ⇒ (rx) == (rz); Ded ▷ ∀(rx): ∀(ry): ∀(rz): (rx) == (ry) ⊢ (ry) << (rz) ⊢ ∸¬(rx) << (rz) ⇒ (rx) == (rz) ≫ (rx) == (ry) ⇒ (ry) << (rz) ⇒ ∸¬(rx) << (rz) ⇒ (rx) == (rz); Ded ▷ ∀(rx): ∀(ry): ∀(rz): (rx) == (ry) ⊢ (ry) == (rz) ⊢ ∸¬(rx) << (rz) ⇒ (rx) == (rz) ≫ (rx) == (ry) ⇒ (ry) == (rz) ⇒ ∸¬(rx) << (rz) ⇒ (rx) == (rz); From2 ∗ 2Disjuncts ▷ ∸¬(rx) << (ry) ⇒ (rx) == (ry) ▷ ∸¬(ry) << (rz) ⇒ (ry) == (rz) ▷ (rx) << (ry) ⇒ (ry) << (rz) ⇒ ∸¬(rx) << (rz) ⇒ (rx) == (rz) ▷ (rx) << (ry) ⇒ (ry) == (rz) ⇒ ∸¬(rx) << (rz) ⇒ (rx) == (rz) ▷ (rx) == (ry) ⇒ (ry) << (rz) ⇒ ∸¬(rx) << (rz) ⇒ (rx) == (rz) ▷ (rx) == (ry) ⇒ (ry) == (rz) ⇒ ∸¬(rx) << (rz) ⇒ (rx) == (rz) ≫ ∸¬(rx) << (rz) ⇒ (rx) == (rz)⌉, p_0, c)]

[<<== Transitivity $\overset{\text{stmt}}{\to}$ ZFsub ⊢ ∀(rx): ∀(ry): ∀(rz): ∸¬(rx) << (ry) ⇒ (rx) == (ry) ⊢ ∸¬(ry) << (rz) ⇒ (ry) == (rz) ⊢ ∸¬(rx) << (rz) ⇒ (rx) == (rz)]

[<<== Transitivity $\overset{\text{tex}}{\to}$ “<<==Transitivity”]

[<<== Transitivity $\overset{\text{pyk}}{\to}$ "lemma <<==Transitivity"]

Plus0f

[Plus0f $\overset{\text{proof}}{\to}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{m}\colon \forall \underline{(fx)}\colon \text{PlusF} \gg \underline{(fx)} +_f 0f[\underline{m}] = \underline{(fx)}[\underline{m}] + 0f[\underline{m}]; 0f \gg 0f[\underline{m}] = 0; \text{lemma eqAdditionLeft} \rhd 0f[\underline{m}] = 0 \gg \underline{(fx)}[\underline{m}] + 0f[\underline{m}] = \underline{(fx)}[\underline{m}] + 0; \text{plus0} \gg \underline{(fx)}[\underline{m}] + 0 = \underline{(fx)}[\underline{m}]; \text{eqTransitivity4} \rhd \underline{(fx)} +_f 0f[\underline{m}] = \underline{(fx)}[\underline{m}] + 0f[\underline{m}] \rhd \underline{(fx)}[\underline{m}] + 0f[\underline{m}] = \underline{(fx)}[\underline{m}] + 0 \rhd \underline{(fx)}[\underline{m}] + 0 = \underline{(fx)}[\underline{m}] \gg \underline{(fx)} +_f 0f[\underline{m}] = \underline{(fx)}[\underline{m}]; \text{To} = f \rhd \underline{(fx)} +_f 0f[\underline{m}] = \underline{(fx)}[\underline{m}] \gg \underline{(fx)} +_f 0f =_f \underline{(fx)}\rceil, p_0, c)$]

[Plus0f $\overset{\text{stmt}}{\to}$ $\text{ZFsub} \vdash \forall \underline{m}\colon \forall \underline{(fx)}\colon \underline{(fx)} +_f 0f =_f \underline{(fx)}$]

[Plus0f $\overset{\text{tex}}{\to}$ "Plus0f"]

[Plus0f $\overset{\text{pyk}}{\to}$ "lemma plus0f"]

Plus00

[Plus00 $\overset{\text{proof}}{\to}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{(fx)}\colon \text{Plus0f} \gg \underline{(fx)} +_f 0f =_f \underline{(fx)}; = \text{fToSameF} \rhd \underline{(fx)} +_f 0f =_f \underline{(fx)} \gg \text{SF}(\underline{(fx)} +_f 0f, \underline{(fx)}); \text{f2R(Plus)} \rhd \text{SF}(\underline{(fx)} +_f 0f, \underline{(fx)}) \gg \text{R}(\underline{(fx)} +_f (fy)) == \text{R}(\underline{(fx)}); \text{Repetition} \rhd \text{R}((fx) +_f (fy)) == \text{R}(\underline{(fx)}) \gg \text{R}((fx) +_f (fy)) == \text{R}(\underline{(fx)})\rceil, p_0, c)$]

[Plus00 $\overset{\text{stmt}}{\to}$ $\text{ZFsub} \vdash \forall \underline{(fx)}\colon \text{R}((fx) +_f (fy)) == \text{R}(\underline{(fx)})$]

[Plus00 $\overset{\text{tex}}{\to}$ "Plus00"]

[Plus00 $\overset{\text{pyk}}{\to}$ "lemma plus00"]

== Addition

[== Addition $\overset{\text{proof}}{\to}$ $\lambda c.\lambda x.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{m}\colon \forall \underline{(\epsilon)}\colon \forall \underline{(fx)}\colon \forall \underline{(fy)}\colon \forall \underline{(fz)}\colon \text{R}(\underline{(fx)}) == \text{R}(\underline{(fy)}) \vdash \dot{\neg}\, 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}\, 0 = \underline{(\epsilon)} \vdash c_{\text{Ex}} <= \underline{m} \vdash \text{From} == \rhd \text{R}(\underline{(fx)}) == \text{R}(\underline{(fy)}) \gg \text{SF}(\underline{(fx)}, \underline{(fy)}); \text{FromSF} \rhd \text{SF}(\underline{(fx)}, \underline{(fy)}) \rhd \dot{\neg}\, 0 <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}\, 0 = \underline{(\epsilon)} \gg c_{\text{Ex}} <= \underline{m} \Rightarrow \dot{\neg}\, \text{if}(0 <= \underline{(fx)}[\underline{m}] + -\underline{(fy)}[\underline{m}], \underline{(fx)}[\underline{m}] + -\underline{(fy)}[\underline{m}], -\underline{(fx)}[\underline{m}] + -\underline{(fy)}[\underline{m}]) <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}\, \text{if}(0 <= \underline{(fx)}[\underline{m}] + -\underline{(fy)}[\underline{m}], \underline{(fx)}[\underline{m}] + -\underline{(fy)}[\underline{m}], -\underline{(fx)}[\underline{m}] + -\underline{(fy)}[\underline{m}]) = \underline{(\epsilon)}; \text{MP} \rhd c_{\text{Ex}} <= \underline{m} \Rightarrow \dot{\neg}\, \text{if}(0 <= \underline{(fx)}[\underline{m}] + -\underline{(fy)}[\underline{m}], \underline{(fx)}[\underline{m}] + -\underline{(fy)}[\underline{m}], -\underline{(fx)}[\underline{m}] + -\underline{(fy)}[\underline{m}]) <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}\, \text{if}(0 <= \underline{(fx)}[\underline{m}] + -\underline{(fy)}[\underline{m}], \underline{(fx)}[\underline{m}] + -\underline{(fy)}[\underline{m}], -\underline{(fx)}[\underline{m}] + -\underline{(fy)}[\underline{m}]) = \underline{(\epsilon)} \rhd c_{\text{Ex}} <= \underline{m} \gg \dot{\neg}\, \text{if}(0 <= \underline{(fx)}[\underline{m}] + -\underline{(fy)}[\underline{m}], \underline{(fx)}[\underline{m}] + -\underline{(fy)}[\underline{m}], -\underline{(fx)}[\underline{m}] + -\underline{(fy)}[\underline{m}]) <= \underline{(\epsilon)} \Rightarrow \dot{\neg}\dot{\neg}\, \text{if}(0 <= \underline{(fx)}[\underline{m}] + -\underline{(fy)}[\underline{m}], \underline{(fx)}[\underline{m}] + -\underline{(fy)}[\underline{m}], -\underline{(fx)}[\underline{m}] + -\underline{(fy)}[\underline{m}]) = \underline{(\epsilon)}; \text{insertMiddleTerm(Difference)} \gg \underline{(fx)}[\underline{m}] + -\underline{(fy)}[\underline{m}] =$

(fx)[m] + (fz)[m] + −(fy)[m] + (fz)[m]; PlusF ≫ (fx) $+_{f}$ (fz)[m] =
(fx)[m] + (fz)[m]; PlusF ≫ (fy) $+_{f}$ (fz)[m] =
(fy)[m] + (fz)[m]; EqNegated ▷ (fy) $+_{f}$ (fz)[m] = (fy)[m] + (fz)[m] ≫
−(fy) $+_{f}$ (fz)[m] = −(fy)[m] + (fz)[m]; AddEquations ▷ (fx) $+_{f}$ (fz)[m] =
(fx)[m] + (fz)[m] ▷ −(fy) $+_{f}$ (fz)[m] = −(fy)[m] + (fz)[m] ≫
(fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$ (fz)[m] = (fx)[m] + (fz)[m] + −(fy)[m] +
(fz)[m]; eqSymmetry ▷ (fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$ (fz)[m] =
(fx)[m] + (fz)[m] + −(fy)[m] + (fz)[m] ≫ (fx)[m] + (fz)[m] + −(fy)[m] + (fz)[m] =
(fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$ (fz)[m]; eqTransitivity ▷ (fx)[m] + −(fy)[m] =
(fx)[m] + (fz)[m] + −(fy)[m] + (fz)[m] ▷ (fx)[m] + (fz)[m] + −(fy)[m] + (fz)[m] =
(fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$ (fz)[m] ≫ (fx)[m] + −(fy)[m] = (fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$
(fz)[m]; SameNumerical ▷ (fx)[m] + −(fy)[m] = (fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$ (fz)[m] ≫
if(0 <= (fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) = if(0 <=
(fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$ (fz)[m], (fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$ (fz)[m], −(fx) $+_{f}$
(fz)[m] + −(fy) $+_{f}$ (fz)[m]); SubLessLeft ▷ if(0 <= (fx)[m] + −(fy)[m], (fx)[m] +
−(fy)[m], −(fx)[m] + −(fy)[m]) = if(0 <= (fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$ (fz)[m], (fx) $+_{f}$
(fz)[m] + −(fy) $+_{f}$ (fz)[m], −(fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$ (fz)[m]) ▷ ¬̇if(0 <=
(fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) <= (ϵ) ⇒ ¬̇¬̇if(0 <=
(fx)[m] + −(fy)[m], (fx)[m] + −(fy)[m], −(fx)[m] + −(fy)[m]) = (ϵ) ≫ ¬̇if(0 <=
(fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$ (fz)[m], (fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$ (fz)[m], −(fx) $+_{f}$
(fz)[m] + −(fy) $+_{f}$ (fz)[m]) <= (ϵ) ⇒ ¬̇¬̇if(0 <= (fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$
(fz)[m], (fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$ (fz)[m], −(fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$ (fz)[m]) =
(ϵ); ∀m: ∀(ϵ): ∀(fx): ∀(fy): ∀(fz): Ded ▷ ∀m: ∀(ϵ): ∀(fx): ∀(fy): ∀(fz): R((fx)) ==
R((fy)) ⊢ ¬̇0 <= (ϵ) ⇒ ¬̇¬̇0 = (ϵ) ⊢ c$_{\mathrm{Ex}}$ <= m ⊢ ¬̇if(0 <=
(fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$ (fz)[m], (fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$ (fz)[m], −(fx) $+_{f}$ (fz)[m] +
−(fy) $+_{f}$ (fz)[m]) <= (ϵ) ⇒ ¬̇¬̇if(0 <= (fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$ (fz)[m], (fx) $+_{f}$
(fz)[m] + −(fy) $+_{f}$ (fz)[m], −(fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$ (fz)[m]) = (ϵ) ≫
R((fx)) == R((fy)) ⇒ ¬̇0 <= (ϵ) ⇒ ¬̇¬̇0 = (ϵ) ⇒ c$_{\mathrm{Ex}}$ <= m ⇒ ¬̇if(0 <=
(fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$ (fz)[m], (fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$ (fz)[m], −(fx) $+_{f}$
(fz)[m] + −(fy) $+_{f}$ (fz)[m]) <= (ϵ) ⇒ ¬̇¬̇if(0 <= (fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$
(fz)[m], (fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$ (fz)[m], −(fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$ (fz)[m]) =
(ϵ); R((fx)) == R((fy)) ⊢ MP ▷ R((fx)) == R((fy)) ⇒ ¬̇0 <= (ϵ) ⇒ ¬̇¬̇0 =
(ϵ) ⇒ c$_{\mathrm{Ex}}$ <= m ⇒ ¬̇if(0 <= (fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$ (fz)[m], (fx) $+_{f}$ (fz)[m] +
−(fy) $+_{f}$ (fz)[m], −(fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$ (fz)[m]) <= (ϵ) ⇒ ¬̇¬̇if(0 <=
(fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$ (fz)[m], (fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$ (fz)[m], −(fx) $+_{f}$
(fz)[m] + −(fy) $+_{f}$ (fz)[m]) = (ϵ) ▷ R((fx)) == R((fy)) ≫ ¬̇0 <= (ϵ) ⇒ ¬̇¬̇0 =
(ϵ) ⇒ c$_{\mathrm{Ex}}$ <= m ⇒ ¬̇if(0 <= (fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$ (fz)[m], (fx) $+_{f}$ (fz)[m] +
−(fy) $+_{f}$ (fz)[m], −(fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$ (fz)[m]) <= (ϵ) ⇒ ¬̇¬̇if(0 <=
(fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$ (fz)[m], (fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$ (fz)[m], −(fx) $+_{f}$
(fz)[m] + −(fy) $+_{f}$ (fz)[m]) = (ϵ); ToSF ▷ ¬̇0 <= (ϵ) ⇒ ¬̇¬̇0 = (ϵ) ⇒ c$_{\mathrm{Ex}}$ <=
m ⇒ ¬̇if(0 <= (fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$ (fz)[m], (fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$
(fz)[m], −(fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$ (fz)[m]) <= (ϵ) ⇒ ¬̇¬̇if(0 <=
(fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$ (fz)[m], (fx) $+_{f}$ (fz)[m] + −(fy) $+_{f}$ (fz)[m], −(fx) $+_{f}$
(fz)[m] + −(fy) $+_{f}$ (fz)[m]) = (ϵ) ≫ SF((fx) $+_{f}$ (fz), (fy) $+_{f}$ (fz)); To ==
▷SF((fx) $+_{f}$ (fz), (fy) $+_{f}$ (fz)) ≫ R((fx) $+_{f}$ (fz)) == R((fy) $+_{f}$ (fz)); PlusR ≫
R((fx) $+_{f}$ (fy)) == R((fx) $+_{f}$ (fz)); PlusR ≫ R((fx) $+_{f}$ (fy)) ==

$\mathrm{R}((\mathrm{fy}) +_{\mathrm{f}} (\mathrm{fz}));$ == Symmetry $\rhd \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}((\mathrm{fy}) +_{\mathrm{f}} (\mathrm{fz})) \gg \mathrm{R}((\mathrm{fy}) +_{\mathrm{f}} (\mathrm{fz})) == \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}));$ == Transitivity $\rhd \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fz})) \rhd \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fz})) == \mathrm{R}((\mathrm{fy}) +_{\mathrm{f}} (\mathrm{fz})) \gg \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}((\mathrm{fy}) +_{\mathrm{f}} (\mathrm{fz}));$ == Transitivity $\rhd \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}((\mathrm{fy}) +_{\mathrm{f}} (\mathrm{fz})) \rhd \mathrm{R}((\mathrm{fy}) +_{\mathrm{f}} (\mathrm{fz})) == \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) \gg \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}))\rceil, \mathrm{p}_0, \mathsf{c})]$

$[== \text{Addition} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \mathsf{m}\colon \forall (\epsilon)\colon \forall (\mathrm{fx})\colon \forall (\mathrm{fy})\colon \forall (\mathrm{fz})\colon \mathrm{R}((\mathrm{fx})) == \mathrm{R}((\mathrm{fy})) \vdash \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}))]$

[== Addition $\stackrel{\text{tex}}{\rightarrow}$ "==Addition"]

[== Addition $\stackrel{\text{pyk}}{\rightarrow}$ "lemma ==Addition"]

== AdditionLeft

$[== \text{AdditionLeft} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall (\mathrm{fx})\colon \forall (\mathrm{fy})\colon \forall (\mathrm{fz})\colon \mathrm{R}((\mathrm{fx})) == \mathrm{R}((\mathrm{fy})) \vdash$ == Addition $\rhd \mathrm{R}((\mathrm{fx})) == \mathrm{R}((\mathrm{fy})) \gg \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}));$ PlusCommutativity(R) $\gg \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}));$ PlusCommutativity(R) $\gg \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}));$ == Transitivity $\rhd \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) \rhd \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) \gg \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}));$ == Transitivity $\rhd \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) \rhd \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) \gg \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}))\rceil, \mathrm{p}_0, \mathsf{c})]$

$[== \text{AdditionLeft} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall (\mathrm{fx})\colon \forall (\mathrm{fy})\colon \forall (\mathrm{fz})\colon \mathrm{R}((\mathrm{fx})) == \mathrm{R}((\mathrm{fy})) \vdash \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}))]$

[== AdditionLeft $\stackrel{\text{tex}}{\rightarrow}$ "==AdditionLeft"]

[== AdditionLeft $\stackrel{\text{pyk}}{\rightarrow}$ "lemma ==AdditionLeft"]

<< Addition

$[<< \text{Addition} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \mathsf{m}\colon \forall (\epsilon)\colon \forall (\mathrm{fx})\colon \forall (\mathrm{fy})\colon \forall (\mathrm{fz})\colon \mathrm{R}((\mathrm{fx})) << \mathrm{R}((\mathrm{fy})) \vdash \dot{\neg} 0 <= (\epsilon) \Rightarrow \dot{\neg}\dot{\neg} 0 = (\epsilon) \vdash \mathsf{c}_{\text{Ex}} <= \mathsf{m} \vdash$ From $<< \rhd \mathrm{R}((\mathrm{fx})) << \mathrm{R}((\mathrm{fy})) \gg (\mathrm{fx}) <_{\mathrm{f}} (\mathrm{fy});$ From $<$ f $\rhd (\mathrm{fx}) <_{\mathrm{f}} (\mathrm{fy}) \rhd \dot{\neg} 0 <= (\epsilon) \Rightarrow \dot{\neg}\dot{\neg} 0 = (\epsilon) \gg \mathsf{c}_{\text{Ex}} <= \mathsf{m} \Rightarrow (\mathrm{fx})[\mathsf{m}] <= (\mathrm{fy})[\mathsf{m}] + -(\epsilon);$ MP $\rhd \mathsf{c}_{\text{Ex}} <= \mathsf{m} \Rightarrow (\mathrm{fx})[\mathsf{m}] <= (\mathrm{fy})[\mathsf{m}] + -(\epsilon) \rhd \mathsf{c}_{\text{Ex}} <= \mathsf{m} \gg (\mathrm{fx})[\mathsf{m}] <= (\mathrm{fy})[\mathsf{m}] + -(\epsilon);$ leqAddition $\rhd (\mathrm{fx})[\mathsf{m}] <= (\mathrm{fy})[\mathsf{m}] + -(\epsilon) \gg (\mathrm{fx})[\mathsf{m}] + (\mathrm{fz})[\mathsf{m}] <= (\mathrm{fy})[\mathsf{m}] + -(\epsilon) + (\mathrm{fz})[\mathsf{m}];$ PlusF(Sym) $\gg (\mathrm{fx})[\mathsf{m}] + (\mathrm{fz})[\mathsf{m}] = (\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fz})[\mathsf{m}];$ subLeqLeft $\rhd (\mathrm{fx})[\mathsf{m}] + (\mathrm{fz})[\mathsf{m}] = (\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fz})[\mathsf{m}] \rhd (\mathrm{fx})[\mathsf{m}] + (\mathrm{fz})[\mathsf{m}] <= (\mathrm{fy})[\mathsf{m}] + -(\epsilon) + (\mathrm{fz})[\mathsf{m}] \gg (\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fz})[\mathsf{m}] <= (\mathrm{fy})[\mathsf{m}] + -(\epsilon) + (\mathrm{fz})[\mathsf{m}];$ Three2threeTerms $\gg (\mathrm{fy})[\mathsf{m}] + -(\epsilon) + (\mathrm{fz})[\mathsf{m}] = (\mathrm{fy})[\mathsf{m}] + (\mathrm{fz})[\mathsf{m}] + -(\epsilon);$ PlusF(Sym) $\gg (\mathrm{fy})[\mathsf{m}] + (\mathrm{fz})[\mathsf{m}] = (\mathrm{fy}) +_{\mathrm{f}} (\mathrm{fz})[\mathsf{m}];$ eqAddition $\rhd (\mathrm{fy})[\mathsf{m}] + (\mathrm{fz})[\mathsf{m}] = (\mathrm{fy}) +_{\mathrm{f}} (\mathrm{fz})[\mathsf{m}] \gg (\mathrm{fy})[\mathsf{m}] + (\mathrm{fz})[\mathsf{m}] + -(\epsilon) = (\mathrm{fy}) +_{\mathrm{f}} (\mathrm{fz})[\mathsf{m}] + -(\epsilon);$ eqTransitivity $\rhd (\mathrm{fy})[\mathsf{m}] + -(\epsilon) + (\mathrm{fz})[\mathsf{m}] =$

$(fy)[\underline{\mathsf{m}}] + (fz)[\underline{\mathsf{m}}] + -(\epsilon) \rhd (fy)[\underline{\mathsf{m}}] + (fz)[\underline{\mathsf{m}}] + -(\epsilon) = (fy) +_{\mathrm{f}} (fz)[\underline{\mathsf{m}}] + -(\epsilon) \gg (fy)[\underline{\mathsf{m}}] + -(\epsilon) + (fz)[\underline{\mathsf{m}}] = (fy) +_{\mathrm{f}} (fz)[\underline{\mathsf{m}}] + -(\epsilon); \mathrm{subLeqRight} \rhd (fy)[\underline{\mathsf{m}}] + -(\epsilon) + (fz)[\underline{\mathsf{m}}] = (fy) +_{\mathrm{f}} (fz)[\underline{\mathsf{m}}] + -(\epsilon) \rhd (fx) +_{\mathrm{f}} (fz)[\underline{\mathsf{m}}] <= (fy)[\underline{\mathsf{m}}] + -(\epsilon) + (fz)[\underline{\mathsf{m}}] \gg (fx) +_{\mathrm{f}} (fz)[\underline{\mathsf{m}}] <= (fy) +_{\mathrm{f}} (fz)[\underline{\mathsf{m}}] + -(\epsilon); \forall \underline{\mathsf{m}} \colon \forall (\epsilon) \colon \forall (fx) \colon \forall (fy) \colon \forall (fz) \colon \mathrm{Ded} \rhd \forall \underline{\mathsf{m}} \colon \forall (\epsilon) \colon \forall (fx) \colon \forall (fy) \colon \forall (fz) \colon R((fx)) << R((fy)) \vdash \dot{\neg} 0 <= (\epsilon) \Rightarrow \dot{\neg}\dot{\neg} 0 = (\epsilon) \vdash \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \vdash (fx) +_{\mathrm{f}} (fz)[\underline{\mathsf{m}}] <= (fy) +_{\mathrm{f}} (fz)[\underline{\mathsf{m}}] + -(\epsilon) \gg R((fx)) << R((fy)) \Rightarrow \dot{\neg} 0 <= (\epsilon) \Rightarrow \dot{\neg}\dot{\neg} 0 = (\epsilon) \Rightarrow \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow (fx) +_{\mathrm{f}} (fz)[\underline{\mathsf{m}}] <= (fy) +_{\mathrm{f}} (fz)[\underline{\mathsf{m}}] + -(\epsilon); R((fx)) << R((fy)) \vdash \mathrm{MP} \rhd R((fx)) << R((fy)) \Rightarrow \dot{\neg} 0 <= (\epsilon) \Rightarrow \dot{\neg}\dot{\neg} 0 = (\epsilon) \Rightarrow \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow (fx) +_{\mathrm{f}} (fz)[\underline{\mathsf{m}}] <= (fy) +_{\mathrm{f}} (fz)[\underline{\mathsf{m}}] + -(\epsilon) \rhd R((fx)) << R((fy)) \gg \dot{\neg} 0 <= (\epsilon) \Rightarrow \dot{\neg}\dot{\neg} 0 = (\epsilon) \Rightarrow \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow (fx) +_{\mathrm{f}} (fz)[\underline{\mathsf{m}}] <= (fy) +_{\mathrm{f}} (fz)[\underline{\mathsf{m}}] + -(\epsilon); \mathrm{To} < f \rhd \dot{\neg} 0 <= (\epsilon) \Rightarrow \dot{\neg}\dot{\neg} 0 = (\epsilon) \Rightarrow \mathsf{c}_{\mathrm{Ex}} <= \underline{\mathsf{m}} \Rightarrow (fx) +_{\mathrm{f}} (fz)[\underline{\mathsf{m}}] <= (fy) +_{\mathrm{f}} (fz)[\underline{\mathsf{m}}] + -(\epsilon) \gg (fx) +_{\mathrm{f}} (fz) <_{\mathrm{f}} (fy) +_{\mathrm{f}} (fz); \mathrm{To} << \rhd (fx) +_{\mathrm{f}} (fz) <_{\mathrm{f}} (fy) +_{\mathrm{f}} (fz) \gg R((fx) +_{\mathrm{f}} (fz)) << R((fy) +_{\mathrm{f}} (fz)); \mathrm{PlusR}(\mathrm{Sym}) \gg R((fx) +_{\mathrm{f}} (fz)) == R((fx) +_{\mathrm{f}} (fy)); \mathrm{PlusR}(\mathrm{Sym}) \gg R((fy) +_{\mathrm{f}} (fz)) == R((fx) +_{\mathrm{f}} (fy)); \mathrm{SubLessLeft}(R) \rhd R((fx) +_{\mathrm{f}} (fz)) == R((fx) +_{\mathrm{f}} (fy)) \rhd R((fx) +_{\mathrm{f}} (fz)) << R((fy) +_{\mathrm{f}} (fz)) \gg R((fx) +_{\mathrm{f}} (fy)) << R((fy) +_{\mathrm{f}} (fz)); \mathrm{SubLessRight}(R) \rhd R((fy) +_{\mathrm{f}} (fz)) == R((fx) +_{\mathrm{f}} (fy)) \rhd R((fx) +_{\mathrm{f}} (fy)) << R((fy) +_{\mathrm{f}} (fz)) \gg R((fx) +_{\mathrm{f}} (fy)) << R((fx) +_{\mathrm{f}} (fy)) \rceil, p_0, \mathsf{c})]$

$[<<\ \mathrm{Addition} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}} \colon \forall (\epsilon) \colon \forall (fx) \colon \forall (fy) \colon \forall (fz) \colon R((fx)) << R((fy)) \vdash R((fx) +_{\mathrm{f}} (fy)) << R((fx) +_{\mathrm{f}} (fy))]$

$[<<\ \mathrm{Addition} \overset{\mathrm{tex}}{\rightarrow}$ "<<Addition"$]$

$[<<\ \mathrm{Addition} \overset{\mathrm{pyk}}{\rightarrow}$ "lemma <<Addition"$]$

<<== Addition

$[<<==\ \mathrm{Addition} \overset{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall (fx) \colon \forall (fy) \colon \forall (fz) \colon R((fx)) << R((fy)) \vdash <<\ \mathrm{Addition} \rhd R((fx)) << R((fy)) \gg R((fx) +_{\mathrm{f}} (fy)) << R((fx) +_{\mathrm{f}} (fy)); \mathrm{LessLeq}(R) \rhd R((fx) +_{\mathrm{f}} (fy)) << R((fx) +_{\mathrm{f}} (fy)) \gg \dot{\neg} R((fx) +_{\mathrm{f}} (fy)) << R((fx) +_{\mathrm{f}} (fy)) \Rightarrow R((fx) +_{\mathrm{f}} (fy)) == R((fx) +_{\mathrm{f}} (fy)); \forall (fx) \colon \forall (fy) \colon \forall (fz) \colon R((fx)) == R((fy)) \vdash ==\ \mathrm{Addition} \rhd R((fx)) == R((fy)) \gg R((fx) +_{\mathrm{f}} (fy)) == R((fx) +_{\mathrm{f}} (fy)); \mathrm{eqLeq}(R) \rhd R((fx) +_{\mathrm{f}} (fy)) == R((fx) +_{\mathrm{f}} (fy)) \gg \dot{\neg} R((fx) +_{\mathrm{f}} (fy)) << R((fx) +_{\mathrm{f}} (fy)) \Rightarrow R((fx) +_{\mathrm{f}} (fy)) == R((fx) +_{\mathrm{f}} (fy)); \forall (fx) \colon \forall (fy) \colon \forall (fz) \colon \mathrm{Ded} \rhd \forall (fx) \colon \forall (fy) \colon \forall (fz) \colon R((fx)) << R((fy)) \vdash \dot{\neg} R((fx) +_{\mathrm{f}} (fy)) << R((fx) +_{\mathrm{f}} (fy)) \Rightarrow R((fx) +_{\mathrm{f}} (fy)) == R((fx) +_{\mathrm{f}} (fy)) \gg R((fx)) << R((fy)) \Rightarrow \dot{\neg} R((fx) +_{\mathrm{f}} (fy)) << R((fx) +_{\mathrm{f}} (fy)) \Rightarrow R((fx) +_{\mathrm{f}} (fy)) == R((fx) +_{\mathrm{f}} (fy)); \mathrm{Ded} \rhd \forall (fx) \colon \forall (fy) \colon \forall (fz) \colon R((fx)) == R((fy)) \vdash \dot{\neg} R((fx) +_{\mathrm{f}} (fy)) << R((fx) +_{\mathrm{f}} (fy)) \Rightarrow R((fx) +_{\mathrm{f}} (fy)) == R((fx) +_{\mathrm{f}} (fy)) \gg R((fx)) == R((fy)) \Rightarrow \dot{\neg} R((fx) +_{\mathrm{f}} (fy)) << R((fx) +_{\mathrm{f}} (fy)) \Rightarrow R((fx) +_{\mathrm{f}} (fy)) == R((fx) +_{\mathrm{f}} (fy)); \dot{\neg} R((fx)) << R((fy)) \Rightarrow R((fx)) == R((fy)) \vdash \mathrm{Repetition} \rhd \dot{\neg} R((fx)) << R((fy)) \Rightarrow R((fx)) == R((fy)) \gg \dot{\neg} R((fx)) << R((fy)) \Rightarrow R((fx)) == R((fy)); \mathrm{FromDisjuncts} \rhd \dot{\neg} R((fx)) << R((fy)) \Rightarrow R((fx)) == R((fy)) \rhd R((fx)) << R((fy)) \Rightarrow \dot{\neg} R((fx) +_{\mathrm{f}} (fy)) <<$

$\mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) \Rightarrow \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) \rhd \mathrm{R}((\mathrm{fx})) == \mathrm{R}((\mathrm{fy})) \Rightarrow \dot{\neg}\, \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) << \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) \Rightarrow \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) \gg \dot{\neg}\, \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) << \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) \Rightarrow \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}))\rceil, \mathrm{p}_0, \mathsf{c})]$

$[<<== \text{Addition} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \forall \underline{(\mathrm{fz})}\colon \dot{\neg}\, \mathrm{R}((\mathrm{fx})) << \mathrm{R}(\underline{(\mathrm{fy})}) \Rightarrow \mathrm{R}(\underline{(\mathrm{fx})}) == \mathrm{R}(\underline{(\mathrm{fy})}) \vdash \dot{\neg}\, \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) << \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) \Rightarrow \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}))]$

$[<<== \text{Addition} \stackrel{\text{tex}}{\rightarrow}$ "<<==Addition"$]$

$[<<== \text{Addition} \stackrel{\text{pyk}}{\rightarrow}$ "lemma <<==Addition"$]$

PlusAssociativity(F)

$[\text{PlusAssociativity(F)} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{m}}\colon \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \forall \underline{(\mathrm{fz})}\colon \text{PlusF} \gg (\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}) +_{\mathrm{f}} (\mathrm{fz})[\underline{\mathsf{m}}] = (\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})[\underline{\mathsf{m}}] + (\mathrm{fz})[\underline{\mathsf{m}}]; \text{PlusF} \gg (\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})[\underline{\mathsf{m}}] = (\mathrm{fx})[\underline{\mathsf{m}}] + (\mathrm{fy})[\underline{\mathsf{m}}]; \text{eqAddition} \rhd (\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})[\underline{\mathsf{m}}] = (\mathrm{fx})[\underline{\mathsf{m}}] + (\mathrm{fy})[\underline{\mathsf{m}}] \gg (\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})[\underline{\mathsf{m}}] + (\mathrm{fz})[\underline{\mathsf{m}}] = (\mathrm{fx})[\underline{\mathsf{m}}] + (\mathrm{fy})[\underline{\mathsf{m}}] + (\mathrm{fz})[\underline{\mathsf{m}}]; \text{plusAssociativity} \gg (\mathrm{fx})[\underline{\mathsf{m}}] + (\mathrm{fy})[\underline{\mathsf{m}}] + (\mathrm{fz})[\underline{\mathsf{m}}] = (\mathrm{fx})[\underline{\mathsf{m}}] + (\mathrm{fy})[\underline{\mathsf{m}}] + (\mathrm{fz})[\underline{\mathsf{m}}]; \text{PlusF(Sym)} \gg (\mathrm{fy})[\underline{\mathsf{m}}] + (\mathrm{fz})[\underline{\mathsf{m}}] = (\mathrm{fy}) +_{\mathrm{f}} (\mathrm{fz})[\underline{\mathsf{m}}]; \text{lemma eqAdditionLeft} \rhd (\mathrm{fy})[\underline{\mathsf{m}}] + (\mathrm{fz})[\underline{\mathsf{m}}] = (\mathrm{fy}) +_{\mathrm{f}} (\mathrm{fz})[\underline{\mathsf{m}}] \gg (\mathrm{fx})[\underline{\mathsf{m}}] + (\mathrm{fy})[\underline{\mathsf{m}}] + (\mathrm{fz})[\underline{\mathsf{m}}] = (\mathrm{fx})[\underline{\mathsf{m}}] + (\mathrm{fy}) +_{\mathrm{f}} (\mathrm{fz})[\underline{\mathsf{m}}]; \text{PlusF(Sym)} \gg (\mathrm{fx})[\underline{\mathsf{m}}] + (\mathrm{fy}) +_{\mathrm{f}} (\mathrm{fz})[\underline{\mathsf{m}}] = (\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}) +_{\mathrm{f}} (\mathrm{fz})[\underline{\mathsf{m}}]; \text{eqTransitivity6} \rhd (\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}) +_{\mathrm{f}} (\mathrm{fz})[\underline{\mathsf{m}}] = (\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})[\underline{\mathsf{m}}] + (\mathrm{fz})[\underline{\mathsf{m}}] \rhd (\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})[\underline{\mathsf{m}}] + (\mathrm{fz})[\underline{\mathsf{m}}] = (\mathrm{fx})[\underline{\mathsf{m}}] + (\mathrm{fy})[\underline{\mathsf{m}}] + (\mathrm{fz})[\underline{\mathsf{m}}] \rhd (\mathrm{fx})[\underline{\mathsf{m}}] + (\mathrm{fy})[\underline{\mathsf{m}}] + (\mathrm{fz})[\underline{\mathsf{m}}] = (\mathrm{fx})[\underline{\mathsf{m}}] + (\mathrm{fy})[\underline{\mathsf{m}}] + (\mathrm{fz})[\underline{\mathsf{m}}] \rhd (\mathrm{fx})[\underline{\mathsf{m}}] + (\mathrm{fy})[\underline{\mathsf{m}}] + (\mathrm{fz})[\underline{\mathsf{m}}] = (\mathrm{fx})[\underline{\mathsf{m}}] + (\mathrm{fy}) +_{\mathrm{f}} (\mathrm{fz})[\underline{\mathsf{m}}] \rhd (\mathrm{fx})[\underline{\mathsf{m}}] + (\mathrm{fy}) +_{\mathrm{f}} (\mathrm{fz})[\underline{\mathsf{m}}] = (\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}) +_{\mathrm{f}} (\mathrm{fz})[\underline{\mathsf{m}}] \gg (\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}) +_{\mathrm{f}} (\mathrm{fz})[\underline{\mathsf{m}}] = (\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}) +_{\mathrm{f}} (\mathrm{fz})[\underline{\mathsf{m}}]; \text{To} = \mathrm{f} \rhd (\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}) +_{\mathrm{f}} (\mathrm{fz})[\underline{\mathsf{m}}] = (\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}) +_{\mathrm{f}} (\mathrm{fz})[\underline{\mathsf{m}}] \gg \underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})} +_{\mathrm{f}} \underline{(\mathrm{fz})} =_{\mathrm{f}} \underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})} +_{\mathrm{f}} \underline{(\mathrm{fz})}\rceil, \mathrm{p}_0, \mathsf{c})]$

$[\text{PlusAssociativity(F)} \stackrel{\text{stmt}}{\rightarrow} \text{ZFsub} \vdash \forall \underline{\mathsf{m}}\colon \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \forall \underline{(\mathrm{fz})}\colon \underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})} +_{\mathrm{f}} \underline{(\mathrm{fz})} =_{\mathrm{f}} \underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})} +_{\mathrm{f}} \underline{(\mathrm{fz})}]$

$[\text{PlusAssociativity(F)} \stackrel{\text{tex}}{\rightarrow}$ "PlusAssociativity(F)"$]$

$[\text{PlusAssociativity(F)} \stackrel{\text{pyk}}{\rightarrow}$ "lemma plusAssociativity(F)"$]$

PlusAssociativity(R)

$[\text{PlusAssociativity(R)} \stackrel{\text{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \forall \underline{(\mathrm{fz})}\colon \text{PlusAssociativity(F)} \gg (\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}) +_{\mathrm{f}} (\mathrm{fz}) =_{\mathrm{f}} (\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}) +_{\mathrm{f}} (\mathrm{fz}); = \text{fToSameF} \rhd (\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}) +_{\mathrm{f}} (\mathrm{fz}) =_{\mathrm{f}} (\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}) +_{\mathrm{f}} (\mathrm{fz}) \gg \text{SF}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}) +_{\mathrm{f}} (\mathrm{fz}), (\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}) +_{\mathrm{f}} (\mathrm{fz})); \text{f2R(Plus)} \rhd \text{SF}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}) +_{\mathrm{f}} (\mathrm{fz}), (\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}) +_{\mathrm{f}} (\mathrm{fz})) \gg \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})) == \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} \underline{(\mathrm{fy})} +_{\mathrm{f}} \underline{(\mathrm{fz})}); \text{PlusR(Sym)} \gg \mathrm{R}(\underline{(\mathrm{fx})} +_{\mathrm{f}} \underline{(\mathrm{fy})} +_{\mathrm{f}} \underline{(\mathrm{fz})}) == \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy})); ==$

$\text{Transitivity} \rhd \mathrm{R}(\underline{(fx)} +_f \underline{(fy)} +_f \underline{(fz)}) == \mathrm{R}((fx) +_f (fy)) \rhd \mathrm{R}((fx) +_f (fy)) == \mathrm{R}(\underline{(fx)} +_f \underline{(fy)} +_f \underline{(fz)}) \gg \overline{\mathrm{R}(\underline{(fx)} +_f \underline{(fy)} +_f \underline{(fz)}) == \mathrm{R}(\underline{(fx)} +_f \underline{(fy)} +_f \underline{(fz)})}\rceil, p_0, \mathsf{c})]$

$[\text{PlusAssociativity(R)} \overset{\text{stmt}}{\to} \text{ZFsub} \vdash \forall \underline{(fx)}\colon \forall \underline{(fy)}\colon \forall \underline{(fz)}\colon \mathrm{R}(\underline{(fx)} +_f \underline{(fy)} +_f \underline{(fz)}) == \mathrm{R}(\underline{(fx)} +_f \underline{(fy)} +_f \underline{(fz)})]$

$[\text{PlusAssociativity(R)} \overset{\text{tex}}{\to}$ "PlusAssociativity(R)"]

$[\text{PlusAssociativity(R)} \overset{\text{pyk}}{\to}$ "lemma plusAssociativity(R)"]

Negative(R)

$[\text{Negative(R)} \overset{\text{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{m}}\colon \forall \underline{(fx)}\colon \text{PlusF} \gg \underline{(fx)} +_f -_f \underline{(fx)}[\underline{\mathsf{m}}] = \underline{(fx)}[\underline{\mathsf{m}}] + -_f \underline{(fx)}[\underline{\mathsf{m}}]; \text{MinusF} \gg -_f \underline{(fx)}[\underline{\mathsf{m}}] = -\underline{(fx)}[\underline{\mathsf{m}}]; \text{lemma eqAdditionLeft} \rhd \overline{-_f \underline{(fx)}[\underline{\mathsf{m}}] = -\underline{(fx)}[\underline{\mathsf{m}}]} \gg \underline{(fx)}[\underline{\mathsf{m}}] + -_f \underline{(fx)}[\underline{\mathsf{m}}] = \underline{(fx)}[\underline{\mathsf{m}}] + -\underline{(fx)}[\underline{\mathsf{m}}]; \text{Negative} \gg \underline{(fx)}[\underline{\mathsf{m}}] + -\underline{(fx)}[\underline{\mathsf{m}}] = 0; 0f \gg 0f[\underline{\mathsf{m}}] = 0; \text{eqSymmetry} \rhd \overline{0f[\underline{\mathsf{m}}] = 0} \gg 0 = 0f[\underline{\mathsf{m}}]; \text{eqTransitivity5} \rhd \underline{(fx)} +_f -_f \underline{(fx)}[\underline{\mathsf{m}}] = \underline{(fx)}[\underline{\mathsf{m}}] + -_f \underline{(fx)}[\underline{\mathsf{m}}] \rhd \underline{(fx)}[\underline{\mathsf{m}}] + -_f \underline{(fx)}[\underline{\mathsf{m}}] = \underline{(fx)}[\underline{\mathsf{m}}] + -\underline{(fx)}[\underline{\mathsf{m}}] \rhd \underline{(fx)}[\underline{\mathsf{m}}] + -\underline{(fx)}[\underline{\mathsf{m}}] = 0 \rhd 0 = 0f[\underline{\mathsf{m}}] \gg \underline{(fx)} +_f -_f \underline{(fx)}[\underline{\mathsf{m}}] = 0f[\underline{\mathsf{m}}]; \text{To} = f \rhd \overline{\underline{(fx)} +_f -_f \underline{(fx)}[\underline{\mathsf{m}}] = 0f[\underline{\mathsf{m}}]} \gg \underline{(fx)} +_f -_f \underline{(fx)} =_f 0f; = \text{fToSameF} \rhd \overline{\underline{(fx)} +_f -_f \underline{(fx)} =_f 0f} \gg \text{SF}(\underline{(fx)} +_f -_f \underline{(fx)}, 0f); \text{f2R(Plus)} \rhd \overline{\text{SF}(\underline{(fx)} +_f -_f \underline{(fx)}, 0f)} \gg \mathrm{R}((fx) +_f (fy)) == \mathrm{R}(0f); \text{Repetition} \rhd \overline{\mathrm{R}((fx) +_f (fy)) == \mathrm{R}(0f)} \gg \mathrm{R}((fx) +_f (fy)) == \mathrm{R}(0f)\rceil, p_0, \mathsf{c})]$

$[\text{Negative(R)} \overset{\text{stmt}}{\to} \text{ZFsub} \vdash \forall \underline{\mathsf{m}}\colon \forall \underline{(fx)}\colon \mathrm{R}((fx) +_f (fy)) == \mathrm{R}(0f)]$

$[\text{Negative(R)} \overset{\text{tex}}{\to}$ "Negative(R)"]

$[\text{Negative(R)} \overset{\text{pyk}}{\to}$ "lemma negative(R)"]

PlusCommutativity(F)

$[\text{PlusCommutativity(F)} \overset{\text{proof}}{\to} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \text{ZFsub} \vdash \forall \underline{\mathsf{m}}\colon \forall \underline{(fx)}\colon \forall \underline{(fy)}\colon \text{PlusF} \gg \underline{(fx)} +_f \underline{(fy)}[\underline{\mathsf{m}}] = \underline{(fx)}[\underline{\mathsf{m}}] + \underline{(fy)}[\underline{\mathsf{m}}]; \text{plusCommutativity} \gg \underline{(fx)}[\underline{\mathsf{m}}] + \underline{(fy)}[\underline{\mathsf{m}}] = \underline{(fy)}[\underline{\mathsf{m}}] + \underline{(fx)}[\underline{\mathsf{m}}]; \text{PlusF(Sym)} \gg \underline{(fy)}[\underline{\mathsf{m}}] + \underline{(fx)}[\underline{\mathsf{m}}] = \underline{(fy)} +_f \underline{(fx)}[\underline{\mathsf{m}}]; \text{eqTransitivity4} \rhd \underline{(fx)} +_f \underline{(fy)}[\underline{\mathsf{m}}] = \underline{(fx)}[\underline{\mathsf{m}}] + \underline{(fy)}[\underline{\mathsf{m}}] \rhd \underline{(fx)}[\underline{\mathsf{m}}] + \underline{(fy)}[\underline{\mathsf{m}}] = \underline{(fy)}[\underline{\mathsf{m}}] + \underline{(fx)}[\underline{\mathsf{m}}] \rhd \underline{(fy)}[\underline{\mathsf{m}}] + \underline{(fx)}[\underline{\mathsf{m}}] = \underline{(fy)} +_f \underline{(fx)}[\underline{\mathsf{m}}] \gg \underline{(fx)} +_f \underline{(fy)}[\underline{\mathsf{m}}] = \underline{(fy)} +_f \underline{(fx)}[\underline{\mathsf{m}}]; \text{To} = f \rhd \overline{\underline{(fx)} +_f \underline{(fy)}[\underline{\mathsf{m}}] = \underline{(fy)} +_f \underline{(fx)}[\underline{\mathsf{m}}]} \gg \underline{(fx)} +_f \underline{(fy)} =_f \underline{(fy)} +_f \underline{(fx)}\rceil, p_0, \mathsf{c})]$

$[\text{PlusCommutativity(F)} \overset{\text{stmt}}{\to} \text{ZFsub} \vdash \forall \underline{\mathsf{m}}\colon \forall \underline{(fx)}\colon \forall \underline{(fy)}\colon \underline{(fx)} +_f \underline{(fy)} =_f \underline{(fy)} +_f \underline{(fx)}]$

$[\text{PlusCommutativity(F)} \overset{\text{tex}}{\to}$ "PlusCommutativity(F)"]

$[\text{PlusCommutativity(F)} \overset{\text{pyk}}{\to}$ "lemma plusCommutativity(F)"]

PlusCommutativity(R)

[PlusCommutativity(R) $\stackrel{\text{proof}}{\rightarrow}$ λc.λx.$\mathcal{P}$($\lceil$ZFsub $\vdash$ $\forall$(fx): $\forall$(fy): PlusCommutativity(F) $\gg$ (fx) $+_{\mathrm{f}}$ (fy) $=_{\mathrm{f}}$ (fy) $+_{\mathrm{f}}$ (fx); = fToSameF $\rhd$ (fx) $+_{\mathrm{f}}$ (fy) $=_{\mathrm{f}}$ (fy) $+_{\mathrm{f}}$ (fx) $\gg$ SF((fx) $+_{\mathrm{f}}$ (fy), (fy) $+_{\mathrm{f}}$ (fx)); f2R(Plus) $\rhd$ SF((fx) $+_{\mathrm{f}}$ (fy), (fy) $+_{\mathrm{f}}$ (fx)) $\gg$ R((fx) $+_{\mathrm{f}}$ (fy)) == R((fy) $+_{\mathrm{f}}$ (fx)); PlusR(Sym) $\gg$ R((fy) $+_{\mathrm{f}}$ (fx)) == R((fx) $+_{\mathrm{f}}$ (fy)); == Transitivity $\rhd$ R((fx) $+_{\mathrm{f}}$ (fy)) == R((fy) $+_{\mathrm{f}}$ (fx)) $\rhd$ R((fy) $+_{\mathrm{f}}$ (fx)) == R((fx) $+_{\mathrm{f}}$ (fy)) $\gg$ R((fx) $+_{\mathrm{f}}$ (fy)) == R((fx) $+_{\mathrm{f}}$ (fy))$\rceil$, p_0, c)]

[PlusCommutativity(R) $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash$ $\forall$(fx): $\forall$(fy): R((fx) $+_{\mathrm{f}}$ (fy)) == R((fx) $+_{\mathrm{f}}$ (fy))]

[PlusCommutativity(R) $\stackrel{\text{tex}}{\rightarrow}$ "PlusCommutativity(R)"]

[PlusCommutativity(R) $\stackrel{\text{pyk}}{\rightarrow}$ "lemma plusCommutativity(R)"]

TimesAssociativity(F)

[TimesAssociativity(F) $\stackrel{\text{proof}}{\rightarrow}$ λc.λx.$\mathcal{P}$($\lceil$ZFsub $\vdash$ $\forall$m: $\forall$(fx): $\forall$(fy): $\forall$(fz): TimesF $\gg$ (fx) $*_{\mathrm{f}}$ (fy) $*_{\mathrm{f}}$ (fz)[m] = (fx) $*_{\mathrm{f}}$ (fy)[m] $*$ (fz)[m]; TimesF $\gg$ (fx) $*_{\mathrm{f}}$ (fy)[m] = (fx)[m] $*$ (fy)[m]; eqMultiplication $\rhd$ (fx) $*_{\mathrm{f}}$ (fy)[m] = (fx)[m] $*$ (fy)[m] $\gg$ (fx) $*_{\mathrm{f}}$ (fy)[m] $*$ (fz)[m] = (fx)[m] $*$ (fy)[m] $*$ (fz)[m]; timesAssociativity $\gg$ (fx)[m] $*$ (fy)[m] $*$ (fz)[m] = (fx)[m] $*$ (fy)[m] $*$ (fz)[m]; TimesF(Sym) $\gg$ (fy)[m] $*$ (fz)[m] = (fy) $*_{\mathrm{f}}$ (fz)[m]; EqMultiplicationLeft $\rhd$ (fy)[m] $*$ (fz)[m] = (fy) $*_{\mathrm{f}}$ (fz)[m] $\gg$ (fx)[m] $*$ (fy)[m] $*$ (fz)[m] = (fx)[m] $*$ (fy) $*_{\mathrm{f}}$ (fz)[m]; TimesF(Sym) $\gg$ (fx)[m] $*$ (fy) $*_{\mathrm{f}}$ (fz)[m] = (fx) $*_{\mathrm{f}}$ (fy) $*_{\mathrm{f}}$ (fz)[m]; eqTransitivity6 $\rhd$ (fx) $*_{\mathrm{f}}$ (fy) $*_{\mathrm{f}}$ (fz)[m] = (fx) $*_{\mathrm{f}}$ (fy)[m] $*$ (fz)[m] $\rhd$ (fx) $*_{\mathrm{f}}$ (fy)[m] $*$ (fz)[m] = (fx)[m] $*$ (fy)[m] $*$ (fz)[m] $\rhd$ (fx)[m] $*$ (fy)[m] $*$ (fz)[m] = (fx)[m] $*$ (fy)[m] $*$ (fz)[m] $\rhd$ (fx)[m] $*$ (fy)[m] $*$ (fz)[m] = (fx)[m] $*$ (fy) $*_{\mathrm{f}}$ (fz)[m] $\rhd$ (fx)[m] $*$ (fy) $*_{\mathrm{f}}$ (fz)[m] = (fx) $*_{\mathrm{f}}$ (fy) $*_{\mathrm{f}}$ (fz)[m] $\gg$ (fx) $*_{\mathrm{f}}$ (fy) $*_{\mathrm{f}}$ (fz)[m] = (fx) $*_{\mathrm{f}}$ (fy) $*_{\mathrm{f}}$ (fz)[m]; To = f $\rhd$ (fx) $*_{\mathrm{f}}$ (fy) $*_{\mathrm{f}}$ (fz)[m] = (fx) $*_{\mathrm{f}}$ (fy) $*_{\mathrm{f}}$ (fz)[m] $\gg$ (fx) $*_{\mathrm{f}}$ (fy) $*_{\mathrm{f}}$ (fz) $=_{\mathrm{f}}$ (fx) $*_{\mathrm{f}}$ (fy) $*_{\mathrm{f}}$ (fz)$\rceil$, p_0, c)]

[TimesAssociativity(F) $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash$ $\forall$m: $\forall$(fx): $\forall$(fy): $\forall$(fz): (fx) $*_{\mathrm{f}}$ (fy) $*_{\mathrm{f}}$ (fz) $=_{\mathrm{f}}$ (fx) $*_{\mathrm{f}}$ (fy) $*_{\mathrm{f}}$ (fz)]

[TimesAssociativity(F) $\stackrel{\text{tex}}{\rightarrow}$ "TimesAssociativity(F)"]

[TimesAssociativity(F) $\stackrel{\text{pyk}}{\rightarrow}$ "lemma timesAssociativity(F)"]

TimesAssociativity(R)

[TimesAssociativity(R) $\stackrel{\text{proof}}{\rightarrow}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \forall \underline{(\mathrm{fz})}\colon \mathrm{TimesAssociativity(F)} \gg \underline{(\mathrm{fx})} *_{\mathrm{f}} \underline{(\mathrm{fy})} *_{\mathrm{f}} \underline{(\mathrm{fz})} =_{\mathrm{f}} \underline{(\mathrm{fx})} *_{\mathrm{f}} \underline{(\mathrm{fy})} *_{\mathrm{f}} \underline{(\mathrm{fz})}; = \mathrm{fToSameF} \rhd \underline{(\mathrm{fx})} *_{\mathrm{f}} \underline{(\mathrm{fy})} *_{\mathrm{f}} \underline{(\mathrm{fz})} =_{\mathrm{f}} \underline{(\mathrm{fx})} *_{\mathrm{f}} \underline{(\mathrm{fy})} *_{\mathrm{f}} \underline{(\mathrm{fz})} \gg \mathrm{SF}(\underline{(\mathrm{fx})} *_{\mathrm{f}} \underline{(\mathrm{fy})} *_{\mathrm{f}} \underline{(\mathrm{fz})}, \underline{(\mathrm{fx})} *_{\mathrm{f}} \underline{(\mathrm{fy})} *_{\mathrm{f}} \underline{(\mathrm{fz})}); \mathrm{f2R(Times)} \rhd \mathrm{SF}(\underline{(\mathrm{fx})} *_{\mathrm{f}} \underline{(\mathrm{fy})} *_{\mathrm{f}} \underline{(\mathrm{fz})}, \underline{(\mathrm{fx})} *_{\mathrm{f}} \underline{(\mathrm{fy})} *_{\mathrm{f}} \underline{(\mathrm{fz})}) \gg \mathrm{R}(\underline{(\mathrm{fx})} *_{\mathrm{f}} \underline{(\mathrm{fy})}) * * \mathrm{R}(\underline{(\mathrm{fz})}) == \mathrm{R}(\underline{(\mathrm{fx})} *_{\mathrm{f}} \underline{(\mathrm{fy})} *_{\mathrm{f}} \underline{(\mathrm{fz})}); \mathrm{TimesR(Sym)} \gg \mathrm{R}(\underline{(\mathrm{fx})} *_{\mathrm{f}} \underline{(\mathrm{fy})} *_{\mathrm{f}} \underline{(\mathrm{fz})}) == \mathrm{R}(\underline{(\mathrm{fx})} *_{\mathrm{f}} \underline{(\mathrm{fy})}) * * \mathrm{R}(\underline{(\mathrm{fz})}); == \mathrm{Transitivity} \rhd \mathrm{R}(\underline{(\mathrm{fx})} *_{\mathrm{f}} \underline{(\mathrm{fy})} *_{\mathrm{f}} \underline{(\mathrm{fz})}) == \mathrm{R}(\underline{(\mathrm{fx})} *_{\mathrm{f}} \underline{(\mathrm{fy})}) * * \mathrm{R}(\underline{(\mathrm{fz})}) \rhd \mathrm{R}(\underline{(\mathrm{fx})} *_{\mathrm{f}} \underline{(\mathrm{fy})}) * * \mathrm{R}(\underline{(\mathrm{fz})}) == \mathrm{R}(\underline{(\mathrm{fx})} *_{\mathrm{f}} \underline{(\mathrm{fy})} *_{\mathrm{f}} \underline{(\mathrm{fz})}) \gg \mathrm{R}(\underline{(\mathrm{fx})} *_{\mathrm{f}} \underline{(\mathrm{fy})} *_{\mathrm{f}} \underline{(\mathrm{fz})}) == \mathrm{R}(\underline{(\mathrm{fx})} *_{\mathrm{f}} \underline{(\mathrm{fy})} *_{\mathrm{f}} \underline{(\mathrm{fz})})\rceil, \mathrm{p}_0, \mathsf{c})$]

[TimesAssociativity(R) $\stackrel{\text{stmt}}{\rightarrow}$ $\mathrm{ZFsub} \vdash \forall \underline{(\mathrm{fx})}\colon \forall \underline{(\mathrm{fy})}\colon \forall \underline{(\mathrm{fz})}\colon \mathrm{R}(\underline{(\mathrm{fx})} *_{\mathrm{f}} \underline{(\mathrm{fy})} *_{\mathrm{f}} \underline{(\mathrm{fz})}) == \mathrm{R}(\underline{(\mathrm{fx})} *_{\mathrm{f}} \underline{(\mathrm{fy})} *_{\mathrm{f}} \underline{(\mathrm{fz})})$]

[TimesAssociativity(R) $\stackrel{\text{tex}}{\rightarrow}$ “TimesAssociativity(R)”]

[TimesAssociativity(R) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma timesAssociativity(R)”]

Times1f

[Times1f $\stackrel{\text{proof}}{\rightarrow}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}}\colon \forall \underline{(\mathrm{fx})}\colon \mathrm{TimesF} \gg \underline{(\mathrm{fx})} *_{\mathrm{f}} 1\mathrm{f}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * 1\mathrm{f}[\underline{\mathsf{m}}]; 1\mathrm{f} \gg 1\mathrm{f}[\underline{\mathsf{m}}] = 1; \mathrm{EqMultiplicationLeft} \rhd 1\mathrm{f}[\underline{\mathsf{m}}] = 1 \gg \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * 1\mathrm{f}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * 1; \mathrm{times1} \gg \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * 1 = \underline{(\mathrm{fx})}[\underline{\mathsf{m}}]; \mathrm{eqTransitivity4} \rhd \underline{(\mathrm{fx})} *_{\mathrm{f}} 1\mathrm{f}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * 1\mathrm{f}[\underline{\mathsf{m}}] \rhd \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * 1\mathrm{f}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * 1 \rhd \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] * 1 = \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] \gg \underline{(\mathrm{fx})} *_{\mathrm{f}} 1\mathrm{f}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})}[\underline{\mathsf{m}}]; \mathrm{To} = \mathrm{f} \rhd \underline{(\mathrm{fx})} *_{\mathrm{f}} 1\mathrm{f}[\underline{\mathsf{m}}] = \underline{(\mathrm{fx})}[\underline{\mathsf{m}}] \gg \underline{(\mathrm{fx})} *_{\mathrm{f}} 1\mathrm{f} =_{\mathrm{f}} \underline{(\mathrm{fx})}\rceil, \mathrm{p}_0, \mathsf{c})$]

[Times1f $\stackrel{\text{stmt}}{\rightarrow}$ $\mathrm{ZFsub} \vdash \forall \underline{\mathsf{m}}\colon \forall \underline{(\mathrm{fx})}\colon \underline{(\mathrm{fx})} *_{\mathrm{f}} 1\mathrm{f} =_{\mathrm{f}} \underline{(\mathrm{fx})}$]

[Times1f $\stackrel{\text{tex}}{\rightarrow}$ “Times1f”]

[Times1f $\stackrel{\text{pyk}}{\rightarrow}$ “lemma times1f”]

Times01

[Times01 $\stackrel{\text{proof}}{\rightarrow}$ $\lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{(\mathrm{fx})}\colon \mathrm{Times1f} \gg \underline{(\mathrm{fx})} *_{\mathrm{f}} 1\mathrm{f} =_{\mathrm{f}} \underline{(\mathrm{fx})}; = \mathrm{fToSameF} \rhd \underline{(\mathrm{fx})} *_{\mathrm{f}} 1\mathrm{f} =_{\mathrm{f}} \underline{(\mathrm{fx})} \gg \mathrm{SF}(\underline{(\mathrm{fx})} *_{\mathrm{f}} 1\mathrm{f}, \underline{(\mathrm{fx})}); \mathrm{f2R(Times)} \rhd \mathrm{SF}(\underline{(\mathrm{fx})} *_{\mathrm{f}} 1\mathrm{f}, \underline{(\mathrm{fx})}) \gg \mathrm{R}(\underline{(\mathrm{fx})}) * * \mathrm{R}(1\mathrm{f}) == \mathrm{R}(\underline{(\mathrm{fx})}); \mathrm{Repetition} \rhd \mathrm{R}(\underline{(\mathrm{fx})}) * * \mathrm{R}(1\mathrm{f}) == \mathrm{R}(\underline{(\mathrm{fx})}) \gg \mathrm{R}(\underline{(\mathrm{fx})}) * * \mathrm{R}(1\mathrm{f}) == \mathrm{R}(\underline{(\mathrm{fx})})\rceil, \mathrm{p}_0, \mathsf{c})$]

[Times01 $\stackrel{\text{stmt}}{\rightarrow}$ $\mathrm{ZFsub} \vdash \forall \underline{(\mathrm{fx})}\colon \mathrm{R}(\underline{(\mathrm{fx})}) * * \mathrm{R}(1\mathrm{f}) == \mathrm{R}(\underline{(\mathrm{fx})})$]

[Times01 $\stackrel{\text{tex}}{\rightarrow}$ “Times01”]

[Times01 $\stackrel{\text{pyk}}{\rightarrow}$ “lemma times01”]

TimesCommutativity(F)

[TimesCommutativity(F) $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{m}\colon \forall \underline{(fx)}\colon \forall \underline{(fy)}\colon$ TimesF $\gg$ $\underline{(fx)} *_f \underline{(fy)}[\underline{m}] = \underline{(fx)}[\underline{m}] * \underline{(fy)}[\underline{m}]$; timesCommutativity $\gg$ $\underline{(fx)}[\underline{m}] * \underline{(fy)}[\underline{m}] = \underline{(fy)}[\underline{m}] * \underline{(fx)}[\underline{m}]$; TimesF(Sym) $\gg$ $\underline{(fy)}[\underline{m}] * \underline{(fx)}[\underline{m}] = \underline{(fy)} *_f \underline{(fx)}[\underline{m}]$; eqTransitivity4 $\rhd$ $\underline{(fx)} *_f \underline{(fy)}[\underline{m}] = \underline{(fx)}[\underline{m}] * \underline{(fy)}[\underline{m}] \rhd \underline{(fx)}[\underline{m}] * \underline{(fy)}[\underline{m}] = \underline{(fy)}[\underline{m}] * \underline{(fx)}[\underline{m}] \rhd \underline{(fy)}[\underline{m}] * \underline{(fx)}[\underline{m}] = \underline{(fy)} *_f \underline{(fx)}[\underline{m}] \gg \underline{(fx)} *_f \underline{(fy)}[\underline{m}] = \underline{(fy)} *_f \underline{(fx)}[\underline{m}]$; To = f $\rhd$ $\underline{(fx)} *_f \underline{(fy)}[\underline{m}] = \underline{(fy)} *_f \underline{(fx)}[\underline{m}] \gg \underline{(fx)} *_f \underline{(fy)} =_f \underline{(fy)} *_f \underline{(fx)}\rceil, p_0, c)$]

[TimesCommutativity(F) $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{m}\colon \forall \underline{(fx)}\colon \forall \underline{(fy)}\colon \underline{(fx)} *_f \underline{(fy)} =_f \underline{(fy)} *_f \underline{(fx)}$]

[TimesCommutativity(F) $\stackrel{\text{tex}}{\rightarrow}$ "TimesCommutativity(F)"]

[TimesCommutativity(F) $\stackrel{\text{pyk}}{\rightarrow}$ "lemma timesCommutativity(F)"]

TimesCommutativity(R)

[TimesCommutativity(R) $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash$ $\forall \underline{(fx)}\colon \forall \underline{(fy)}\colon$ TimesCommutativity(F) $\gg$ $\underline{(fx)} *_f \underline{(fy)} =_f \underline{(fy)} *_f \underline{(fx)}$; = fToSameF $\rhd$ $\underline{(fx)} *_f \underline{(fy)} =_f \underline{(fy)} *_f \underline{(fx)} \gg$ SF$(\underline{(fx)} *_f \underline{(fy)}, \underline{(fy)} *_f \underline{(fx)})$; f2R(Times) $\rhd$ SF$(\underline{(fx)} *_f \underline{(fy)}, \underline{(fy)} *_f \underline{(fx)}) \gg$ R$(\underline{(fx)}) * *$R$(\underline{(fy)})$ == R$(\underline{(fy)} *_f \underline{(fx)})$; TimesR(Sym) $\gg$ R$(\underline{(fy)} *_f \underline{(fx)})$ == R$(\underline{(fy)}) * *$R$(\underline{(fx)})$; == Transitivity $\rhd$ R$(\underline{(fx)}) * *$R$(\underline{(fy)})$ == R$(\underline{(fy)} *_f \underline{(fx)}) \rhd$ R$(\underline{(fy)} *_f \underline{(fx)})$ == R$(\underline{(fy)}) * *$R$(\underline{(fx)}) \gg$ R$(\underline{(fx)}) * *$R$(\underline{(fy)})$ == R$(\underline{(fy)}) * *$R$(\underline{(fx)})\rceil, p_0, c)$]

[TimesCommutativity(R) $\stackrel{\text{stmt}}{\rightarrow}$ ZFsub $\vdash \forall \underline{(fx)}\colon \forall \underline{(fy)}\colon$ R$(\underline{(fx)}) * *$R$(\underline{(fy)})$ == R$(\underline{(fy)}) * *$R$(\underline{(fx)})$]

[TimesCommutativity(R) $\stackrel{\text{tex}}{\rightarrow}$ "TimesCommutativity(R)"]

[TimesCommutativity(R) $\stackrel{\text{pyk}}{\rightarrow}$ "lemma timesCommutativity(R)"]

Distribution(F)

[Distribution(F) $\stackrel{\text{proof}}{\rightarrow}$ $\lambda c.\lambda x.\mathcal{P}(\lceil$ZFsub $\vdash \forall \underline{(fx)}\colon \forall \underline{(fy)}\colon \forall \underline{(fz)}\colon$ TimesF $\gg$ $\underline{(fx)} *_f \underline{(fy)} +_f \underline{(fz)}[\underline{m}] = \underline{(fx)}[\underline{m}] * \underline{(fy)} +_f \underline{(fz)}[\underline{m}]$; PlusF $\gg$ $\underline{(fy)} +_f \underline{(fz)}[\underline{m}] = \underline{(fy)}[\underline{m}] + \underline{(fz)}[\underline{m}]$; EqMultiplicationLeft $\rhd$ $\underline{(fy)} +_f \underline{(fz)}[\underline{m}] = \underline{(fy)}[\underline{m}] + \underline{(fz)}[\underline{m}] \gg \underline{(fx)}[\underline{m}] * \underline{(fy)} +_f \underline{(fz)}[\underline{m}] = \underline{(fx)}[\underline{m}] * \underline{(fy)}[\underline{m}] + \underline{(fz)}[\underline{m}]$; Distribution $\gg$ $\underline{(fx)}[\underline{m}] * \underline{(fy)}[\underline{m}] + \underline{(fz)}[\underline{m}] = \underline{(fx)}[\underline{m}] * \underline{(fy)}[\underline{m}] + \underline{(fx)}[\underline{m}] * \underline{(fz)}[\underline{m}]$; TimesF(Sym) $\gg$ $\underline{(fx)}[\underline{m}] * \underline{(fy)}[\underline{m}] = \underline{(fx)} *_f \underline{(fy)}[\underline{m}]$; TimesF(Sym) $\gg$ $\underline{(fx)}[\underline{m}] * \underline{(fz)}[\underline{m}] = \underline{(fx)} *_f \underline{(fz)}[\underline{m}]$; AddEquations $\rhd$ $\underline{(fx)}[\underline{m}] * \underline{(fy)}[\underline{m}] = \underline{(fx)} *_f \underline{(fy)}[\underline{m}] \rhd \underline{(fx)}[\underline{m}] * \underline{(fz)}[\underline{m}] = \underline{(fx)} *_f \underline{(fz)}[\underline{m}] \gg \underline{(fx)}[\underline{m}] * \underline{(fy)}[\underline{m}] + \underline{(fx)}[\underline{m}] * \underline{(fz)}[\underline{m}] =$

$\overline{(fx)} *_f \overline{(fy)}[\underline{m}] + \overline{(fx)} *_f \overline{(fz)}[\underline{m}]; \mathrm{PlusF(Sym)} \gg \overline{(fx)} *_f \overline{(fy)}[\underline{m}] + \overline{(fx)} *_f \overline{(fz)}[\underline{m}] = \overline{(fx)} *_f \overline{(fy)} +_f \overline{(fx)} *_f \overline{(fz)}[\underline{m}]; \mathrm{eqTransitivity6} \rhd \overline{(fx)} *_f \overline{(fy)} +_f \overline{(fz)}[\underline{m}] = \overline{(fx)}[\underline{m}] * \overline{(fy)} +_f \overline{(fz)}[\underline{m}] \rhd \overline{(fx)}[\underline{m}] * \overline{(fy)} +_f \overline{(fz)}[\underline{m}] = \overline{(fx)}[\underline{m}] * \overline{(fy)}[\underline{m}] + \overline{(fz)}[\underline{m}] \rhd \overline{(fx)}[\underline{m}] * \overline{(fy)}[\underline{m}] + \overline{(fz)}[\underline{m}] = \overline{(fx)}[\underline{m}] * \overline{(fy)}[\underline{m}] + \overline{(fx)}[\underline{m}] * \overline{(fz)}[\underline{m}] \rhd \overline{(fx)}[\underline{m}] * \overline{(fy)}[\underline{m}] + \overline{(fx)}[\underline{m}] * \overline{(fz)}[\underline{m}] = \overline{(fx)} *_f \overline{(fy)}[\underline{m}] + \overline{(fx)} *_f \overline{(fz)}[\underline{m}] \rhd \overline{(fx)} *_f \overline{(fy)}[\underline{m}] + \overline{(fx)} *_f \overline{(fz)}[\underline{m}] = \overline{(fx)} *_f \overline{(fy)} +_f \overline{(fx)} *_f \overline{(fz)}[\underline{m}] \gg \overline{(fx)} *_f \overline{(fy)} +_f \overline{(fz)}[\underline{m}] = \overline{(fx)} *_f \overline{(fy)} +_f \overline{(fx)} *_f \overline{(fz)}[\underline{m}]; \mathrm{To} = \mathrm{f} \rhd \overline{(fx)} *_f \overline{(fy)} +_f \overline{(fz)}[\underline{m}] = \overline{(fx)} *_f \overline{(fy)} +_f \overline{(fx)} *_f \overline{(fz)}[\underline{m}] \gg \underline{(fx)} *_f \underline{(fy)} +_f \underline{(fz)} =_f \underline{(fx)} *_f \underline{(fy)} +_f \underline{(fx)} *_f \underline{(fz)} \rceil, p_0, \mathsf{c})]$

$[\mathrm{Distribution(F)} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{(fx)} \colon \forall \underline{(fy)} \colon \forall \underline{(fz)} \colon \underline{(fx)} *_f \underline{(fy)} +_f \underline{(fz)} =_f \underline{(fx)} *_f \underline{(fy)} +_f \underline{(fx)} *_f \underline{(fz)}]$

[Distribution(F) $\overset{\mathrm{tex}}{\rightarrow}$ "Distribution(F)"]

[Distribution(F) $\overset{\mathrm{pyk}}{\rightarrow}$ "lemma distribution(F)"]

Distribution(R)

$[\mathrm{Distribution(R)} \overset{\mathrm{proof}}{\rightarrow} \lambda \mathsf{c}.\lambda \mathsf{x}.\mathcal{P}(\lceil \mathrm{ZFsub} \vdash \forall \underline{(fx)} \colon \forall \underline{(fy)} \colon \forall \underline{(fz)} \colon \mathrm{Distribution(F)} \gg \underline{(fx)} *_f \underline{(fy)} +_f \underline{(fz)} =_f \underline{(fx)} *_f \underline{(fy)} +_f \underline{(fx)} *_f \underline{(fz)}; = \mathrm{fToSameF} \rhd \overline{(fx)} *_f \overline{(fy)} +_f \overline{(fz)} =_f \overline{(fx)} *_f \overline{(fy)} +_f \overline{(fx)} *_f \overline{(fz)} \gg \mathrm{SF}(\overline{(fx)} *_f \overline{(fy)} +_f \overline{(fz)}, \overline{(fx)} *_f \overline{(fy)} +_f \overline{(fx)} *_f \overline{(fz)}); \mathrm{f2R(Times)} \rhd \mathrm{SF}(\overline{(fx)} *_f \overline{(fy)} +_f \overline{(fz)}, \overline{(fx)} *_f \overline{(fy)} +_f \overline{(fx)} *_f \overline{(fz)}) \gg \mathrm{R}(\overline{(fx)}) * * \mathrm{R}(\overline{(fy)} +_f \overline{(fz)}) == \mathrm{R}(\overline{(fx)} *_f \overline{(fy)} +_f \overline{(fx)} *_f \overline{(fz)}); \mathrm{PlusR(Sym)} \gg \mathrm{R}(\overline{(fx)} *_f \overline{(fy)} +_f \overline{(fx)} *_f \overline{(fz)}) == \mathrm{R}(\overline{(fx)} +_f \overline{(fy)}); == \mathrm{Transitivity} \rhd \mathrm{R}(\overline{(fx)}) * * \mathrm{R}(\overline{(fy)} +_f \overline{(fz)}) == \mathrm{R}(\overline{(fx)} *_f \overline{(fy)} +_f \overline{(fx)} *_f \overline{(fz)}) \rhd \mathrm{R}(\overline{(fx)} *_f \overline{(fy)} +_f \overline{(fx)} *_f \overline{(fz)}) == \mathrm{R}(\overline{(fx)} +_f \overline{(fy)}) \gg \mathrm{R}(\underline{(fx)}) * * \mathrm{R}(\underline{(fy)} +_f \underline{(fz)}) == \mathrm{R}((fx) +_f (fy)) \rceil, p_0, \mathsf{c})]$

$[\mathrm{Distribution(R)} \overset{\mathrm{stmt}}{\rightarrow} \mathrm{ZFsub} \vdash \forall \underline{(fx)} \colon \forall \underline{(fy)} \colon \forall \underline{(fz)} \colon \mathrm{R}(\underline{(fx)}) * * \mathrm{R}(\underline{(fy)} +_f \underline{(fz)}) == \mathrm{R}((fx) +_f (fy))]$

[Distribution(R) $\overset{\mathrm{tex}}{\rightarrow}$ "Distribution(R)"]

[Distribution(R) $\overset{\mathrm{pyk}}{\rightarrow}$ "lemma distribution(R)"]

R(∗)

[R(x) $\overset{\mathrm{tex}}{\rightarrow}$ "R(#1.
)"]

[R(∗) $\overset{\mathrm{pyk}}{\rightarrow}$ "R(")"]

$--\mathrm{R}(*)$

$[--\mathrm{R}((\mathrm{fx})) \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil[--\mathrm{R}((\mathrm{fx})) \ddot{=} \mathrm{R}(-_{\mathrm{f}}(\mathrm{fx}))]\rceil)]$

$[--\mathrm{R}(\mathsf{x}) \stackrel{\text{tex}}{\rightarrow}$ "--R(#1.
)"]

$[--\mathrm{R}(*) \stackrel{\text{pyk}}{\rightarrow}$ "--R(")"]

rec$*$

[rec$\mathsf{x} \stackrel{\text{tex}}{\rightarrow}$ "rec#1."]

[rec$* \stackrel{\text{pyk}}{\rightarrow}$ "1/ ""]

$*/*$

$[\mathsf{bs}/\mathsf{r} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil[\mathsf{bs}/\mathsf{r} \ddot{=} \{\mathrm{ph} \in \mathrm{P}(\mathsf{bs}) \mid \mathrm{Ex}_{20} \in \mathsf{bs} \dot{\wedge} [\mathrm{Ex}_{20} \in \mathsf{bs}]_{\mathsf{r}} == \mathrm{ph}_2\}]\rceil)]$

$[\mathsf{x}/\mathsf{y} \stackrel{\text{tex}}{\rightarrow}$ "#1.
/ #2."]

$[*/* \stackrel{\text{pyk}}{\rightarrow}$ "eq-system of " modulo ""]

$* \cap *$

$[\mathsf{x} \cap \mathsf{y} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil[\mathsf{x} \cap \mathsf{y} \ddot{=} \{\mathrm{ph} \in \mathsf{x} \cup \mathsf{y} \mid \mathrm{ph}_3 \in \mathsf{x} \dot{\wedge} \mathrm{ph}_3 \in \mathsf{y}\}]\rceil)]$

$[\mathsf{x} \cap \mathsf{y} \stackrel{\text{tex}}{\rightarrow}$ "#1.
\cap #2."]

$[* \cap * \stackrel{\text{pyk}}{\rightarrow}$ "intersection " comma " end intersection"]

$*[*]$

$[\mathsf{x}[\mathsf{y}] \stackrel{\text{tex}}{\rightarrow}$ "#1.
[#2.
]"]

$[*[*] \stackrel{\text{pyk}}{\rightarrow}$ "[" ; "]"]

$\cup *$

$[\cup \mathsf{x} \stackrel{\text{tex}}{\to}$ "\cup #1."]

$[\cup * \stackrel{\text{pyk}}{\to}$ "union " end union"]

$* \cup *$

$[\mathsf{x} \cup \mathsf{y} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil [\mathsf{x} \cup \mathsf{y} \stackrel{..}{=} \cup\{\{\mathsf{x}\}, \{\mathsf{y}\}\}] \rceil)]$

$[\mathsf{x} \cup \mathsf{y} \stackrel{\text{tex}}{\to}$ "#1.
\mathrel{\cup} #2."]

$[* \cup * \stackrel{\text{pyk}}{\to}$ "binary-union " comma " end union"]

P(*)

[P(x) $\stackrel{\text{tex}}{\to}$ "P(#1.
)"]

[P(*) $\stackrel{\text{pyk}}{\to}$ "power " end power"]

{*}

$[\{\mathsf{x}\} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil [\{\mathsf{x}\} \stackrel{..}{=} \{\mathsf{x}, \mathsf{x}\}] \rceil)]$

$[\{\mathsf{x}\} \stackrel{\text{tex}}{\to}$ "\{#1.
\}"]

$[\{*\} \stackrel{\text{pyk}}{\to}$ "zermelo singleton " end singleton"]

{*, *}

$[\{\mathsf{x}, \mathsf{y}\} \stackrel{\text{tex}}{\to}$ "\{#1.
, #2.
\}"]

$[\{*, *\} \stackrel{\text{pyk}}{\to}$ "zermelo pair " comma " end pair"]

$\langle *, * \rangle$

$[\langle \mathsf{x}, \mathsf{y} \rangle \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil [\langle \mathsf{x}, \mathsf{y} \rangle \doteq \{\{\mathsf{x}\}, \{\mathsf{x}, \mathsf{y}\}\}] \rceil)]$

[$\langle \mathsf{x}, \mathsf{y} \rangle \overset{\text{tex}}{\to}$ "\langle #1.
, #2.
\rangle"]

[$\langle *, * \rangle \overset{\text{pyk}}{\to}$ "zermelo ordered pair " comma " end pair"]

$-*$

[$-\mathsf{x} \overset{\text{tex}}{\to}$ "-#1."]

[$-* \overset{\text{pyk}}{\to}$ "- ""]

$-_{\mathrm{f}}*$

[$-_{\mathrm{f}}\mathsf{x} \overset{\text{tex}}{\to}$ "-_{f}#1."]

[$-_{\mathrm{f}}* \overset{\text{pyk}}{\to}$ "-f ""]

$* \in *$

[$\mathsf{x} \in \mathsf{y} \overset{\text{tex}}{\to}$ "#1.
\mathrel{\in} #2."]

[$* \in * \overset{\text{pyk}}{\to}$ "" in0 ""]

$*(*, *)$

$[\mathsf{r}(\mathsf{x}, \mathsf{y}) \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil [\mathsf{r}(\mathsf{x}, \mathsf{y}) \doteq \langle \mathsf{x}, \mathsf{y} \rangle \in \mathsf{r}] \rceil)]$

[$\mathsf{z}(\mathsf{x}, \mathsf{y}) \overset{\text{tex}}{\to}$ "#3.
(#1.
, #2.
)"]

[$*(*, *) \overset{\text{pyk}}{\to}$ "" is related to " under ""]

$\mathrm{ReflRel}(*,*)$

$[\mathrm{ReflRel}(\mathsf{r},\mathsf{x}) \stackrel{\text{macro}}{\to} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathrm{ReflRel}(\mathsf{r},\mathsf{x}) \stackrel{..}{=} \forall s\colon (s \in \mathsf{x} \Rightarrow \mathsf{r}(s,s))]\rceil)]$

[ReflRel(r, x) $\stackrel{\text{tex}}{\to}$ "ReflRel(#1.
, #2.
)"]

[ReflRel(∗, ∗) $\stackrel{\text{pyk}}{\to}$ "" is reflexive relation in """]

$\mathrm{SymRel}(*,*)$

$[\mathrm{SymRel}(\mathsf{r},\mathsf{x}) \stackrel{\text{macro}}{\to} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathrm{SymRel}(\mathsf{r},\mathsf{x}) \stackrel{..}{=} \forall s,$
$t\colon (s \in \mathsf{x} \Rightarrow t \in \mathsf{x} \Rightarrow \mathsf{r}(s,t) \Rightarrow \mathsf{r}(t,s))]\rceil)]$

[SymRel(r, x) $\stackrel{\text{tex}}{\to}$ "SymRel(#1.
, #2.
)"]

[SymRel(∗, ∗) $\stackrel{\text{pyk}}{\to}$ "" is symmetric relation in """]

$\mathrm{TransRel}(*,*)$

$[\mathrm{TransRel}(\mathsf{r},\mathsf{x}) \stackrel{\text{macro}}{\to} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathrm{TransRel}(\mathsf{r},\mathsf{x}) \stackrel{..}{=}$
$\forall s,t,u\colon (s \in \mathsf{x} \Rightarrow t \in \mathsf{x} \Rightarrow u \in \mathsf{x} \Rightarrow \mathsf{r}(s,t) \Rightarrow \mathsf{r}(t,u) \Rightarrow \mathsf{r}(s,u))]\rceil)]$

[TransRel(r, x) $\stackrel{\text{tex}}{\to}$ "TransRel(#1.
, #2.
)"]

[TransRel(∗, ∗) $\stackrel{\text{pyk}}{\to}$ "" is transitive relation in """]

$\mathrm{EqRel}(*,*)$

$[\mathrm{EqRel}(\mathsf{r},\mathsf{x}) \stackrel{\text{macro}}{\to} \lambda\mathsf{t}.\lambda\mathsf{s}.\lambda\mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathrm{EqRel}(\mathsf{r},\mathsf{x}) \stackrel{..}{=} \mathrm{ReflRel}(\mathsf{r},\mathsf{x}) \dot{\wedge}$
$\mathrm{SymRel}(\mathsf{r},\mathsf{x}) \dot{\wedge} \mathrm{TransRel}(\mathsf{r},\mathsf{x})]\rceil)]$

[EqRel(r, x) $\stackrel{\text{tex}}{\to}$ "EqRel(#1.
, #2.
)"]

[EqRel(∗, ∗) $\stackrel{\text{pyk}}{\to}$ "" is equivalence relation in """]

$[* \in *]_*$

$[[\mathsf{x} \in \mathsf{bs}]_\mathsf{r} \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil [[\mathsf{x} \in \mathsf{bs}]_\mathsf{r} \ddot{=} \{\mathrm{ph} \in \mathsf{bs} \mid \mathsf{r}(\mathrm{ph}_1,\mathsf{x})\}]\rceil)]$

$[[\mathsf{x} \in \mathsf{bs}]_\mathsf{r} \overset{\text{tex}}{\to}$ "[#1.
\mathrel{\in} #2.
]_{#3.
}"]

$[[* \in *]_* \overset{\text{pyk}}{\to}$ "equivalence class of " in " modulo ""]

Partition(∗, ∗)

$[\mathrm{Partition}(\mathsf{p},\mathsf{bs}) \overset{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil [\mathrm{Partition}(\mathsf{p},\mathsf{bs}) \ddot{=} (\forall s\colon (s \in \mathsf{p} \Rightarrow s \neq$
$\emptyset)) \dot{\wedge}$
$(\forall s,t\colon (s \in \mathsf{p} \Rightarrow t \in \mathsf{p} \Rightarrow s \neq t \Rightarrow s \cap t == \emptyset)) \dot{\wedge}$
$\cup\, \mathsf{p} == \mathsf{bs}]\rceil)]$

$[\mathrm{Partition}(\mathsf{x},\mathsf{y}) \overset{\text{tex}}{\to}$ "Partition(#1.
, #2.
)"]

$[\mathrm{Partition}(*,*) \overset{\text{pyk}}{\to}$ "" is partition of ""]

$* * *$

$[\mathsf{x} * \mathsf{y} \overset{\text{tex}}{\to}$ "#1.
*#2."]

$[* * * \overset{\text{pyk}}{\to}$ "" * ""]

$* *_\mathrm{f} *$

$[(\mathrm{fx}) *_\mathrm{f} (\mathrm{fy}) \overset{\text{tex}}{\to}$ "#1.
*_{f}#2."]

$[* *_\mathrm{f} * \overset{\text{pyk}}{\to}$ "" *f ""]

$* * **$

$[\mathsf{x} * * \mathsf{y} \overset{\text{tex}}{\to}$ "#1.
**#2."]

$[* * ** \overset{\text{pyk}}{\rightarrow}$ “" ** "”]

$* + *$

$[\mathsf{x} + \mathsf{y} \overset{\text{tex}}{\rightarrow}$ “#1.
+#2.”]

$[* + * \overset{\text{pyk}}{\rightarrow}$ “" + "”]

$* - *$

$[\mathsf{x} - \mathsf{y} \overset{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil[\mathsf{x} - \mathsf{y} \ddot{=} \mathsf{x} + (-\mathsf{y})]\rceil)]$

$[\mathsf{x} - \mathsf{y} \overset{\text{tex}}{\rightarrow}$ “#1.
-#2.”]

$[* - * \overset{\text{pyk}}{\rightarrow}$ “" - "”]

$* +_{\mathrm{f}} *$

$[(\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}) \overset{\text{tex}}{\rightarrow}$ “#1.
+_{f}#2.”]

$[* +_{\mathrm{f}} * \overset{\text{pyk}}{\rightarrow}$ “" +f "”]

$* -_{\mathrm{f}} *$

$[(\mathrm{fx}) -_{\mathrm{f}} (\mathrm{fy}) \overset{\text{tex}}{\rightarrow}$ “#1.
-_{f}#2.”]

$[* -_{\mathrm{f}} * \overset{\text{pyk}}{\rightarrow}$ “" -f "”]

$* + +*$

$[\mathrm{R}((\mathrm{fx})) + +\mathrm{R}((\mathrm{fy})) \overset{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil[\mathrm{R}((\mathrm{fx})) + +\mathrm{R}((\mathrm{fy})) \ddot{=} \mathrm{R}((\mathrm{fx}) +_{\mathrm{f}} (\mathrm{fy}))]\rceil)]$

$[\mathsf{x} + +\mathsf{y} \overset{\text{tex}}{\rightarrow}$ “#1.
++#2.”]

$[* + +* \overset{\text{pyk}}{\rightarrow}$ “" ++ "”]

$\mathrm{R}(*) - -\mathrm{R}(*)$

$[\mathrm{R}((\mathrm{fx})) - -\mathrm{R}((\mathrm{fy})) \stackrel{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil[\mathrm{R}((\mathrm{fx})) - -\mathrm{R}((\mathrm{fy})) \ddot{=} \mathrm{R}((\mathrm{fx})) + +\mathrm{R}(-_{\mathrm{f}}(\mathrm{fy}))]\rceil)]$

$[\mathrm{R}((\mathrm{fx})) - -\mathrm{R}((\mathrm{fy})) \stackrel{\mathrm{tex}}{\to}$ "R(#1.
) -- R(#2.
)"]

$[\mathrm{R}(*) - -\mathrm{R}(*) \stackrel{\mathrm{pyk}}{\to}$ "R(") -- R(")"]

$| * |$

$[|\mathsf{x}| \stackrel{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil[|\mathsf{x}| \ddot{=} \mathrm{if}(0 <= \mathsf{x}, \mathsf{x}, -\mathsf{x})]\rceil)]$

$[|\mathsf{x}| \stackrel{\mathrm{tex}}{\to}$ "|#1.|"]

$[| * | \stackrel{\mathrm{pyk}}{\to}$ "| " |"]

$\mathrm{if}(*, *, *)$

$[\mathrm{if}(\mathsf{x}, \mathsf{y}, \mathsf{z}) \stackrel{\mathrm{tex}}{\to}$ "if(#1.
, #2.
, #3.
)"]

$[\mathrm{if}(*, *, *) \stackrel{\mathrm{pyk}}{\to}$ "if(" , " , ")"]

$* = *$

$[\mathsf{x} = \mathsf{y} \stackrel{\mathrm{tex}}{\to}$ "#1.
= #2."]

$[* = * \stackrel{\mathrm{pyk}}{\to}$ "" = ""]

$* \neq *$

$[\mathsf{x} \neq \mathsf{y} \stackrel{\mathrm{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil[\mathsf{x} \neq \mathsf{y} \ddot{=} \dot{\neg} \mathsf{x} = \mathsf{y}]\rceil)]$

$[\mathsf{x} \neq \mathsf{y} \stackrel{\mathrm{tex}}{\to}$ "#1.
\neq #2."]

$[* \neq * \stackrel{\mathrm{pyk}}{\to}$ "" != ""]

$* <= *$

$[\mathsf{x} <= \mathsf{y} \stackrel{\text{tex}}{\to}$ "#1.
<= #2."]

$[* <= * \stackrel{\text{pyk}}{\to}$ "" <= ""]

$* < *$

$[\mathsf{x} < \mathsf{y} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil[\mathsf{x} < \mathsf{y} \ddot{=} \mathsf{x} <= \mathsf{y} \dot{\wedge} \mathsf{x} \neq \mathsf{y}]\rceil)]$

$[\mathsf{x} < \mathsf{y} \stackrel{\text{tex}}{\to}$ "#1.
< #2."]

$[* < * \stackrel{\text{pyk}}{\to}$ "" < ""]

$* =_{\mathrm{f}} *$

$[\mathsf{x} =_{\mathrm{f}} \mathsf{y} \stackrel{\text{tex}}{\to}$ "#1.
=_{f}#2."]

$[* =_{\mathrm{f}} * \stackrel{\text{pyk}}{\to}$ "" =f ""]

$* <_{\mathrm{f}} *$

$[\mathsf{x} <_{\mathrm{f}} \mathsf{y} \stackrel{\text{tex}}{\to}$ "#1.
<_{f}#2."]

$[* <_{\mathrm{f}} * \stackrel{\text{pyk}}{\to}$ "" <f ""]

$\mathrm{SF}(*, *)$

$[\mathrm{SF}(\mathsf{x}, \mathsf{y}) \stackrel{\text{tex}}{\to}$ "SF(#1.
, #2.
)"]

$[\mathrm{SF}(*, *) \stackrel{\text{pyk}}{\to}$ "" sameF ""]

$* == *$

$[\mathsf{x} == \mathsf{y} \stackrel{\text{tex}}{\to}$ "#1.
== #2."]

$[* == * \stackrel{\text{pyk}}{\to}$ "" == ""]

$* << *$

$[\mathsf{x} << \mathsf{y} \stackrel{\text{tex}}{\to}$ "#1.
<< #2."]

$[* << * \stackrel{\text{pyk}}{\to}$ "" << ""]

$* <<== *$

$[\mathsf{x} <<== \mathsf{y} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil [\mathsf{x} <<== \mathsf{y} \ddot{=} \mathsf{x} << \mathsf{y} \dot{\vee} \mathsf{x} == \mathsf{y}] \rceil)]$

$[\mathsf{x} <<== \mathsf{y} \stackrel{\text{tex}}{\to}$ "#1.
<<== #2."]

$[* <<== * \stackrel{\text{pyk}}{\to}$ "" <<== ""]

$*{==}*$

$[\mathsf{x}{==}\mathsf{y} \stackrel{\text{tex}}{\to}$ "#1.
\!\mathrel{==}\! #2."]

$[*{==}* \stackrel{\text{pyk}}{\to}$ "" zermelo is ""]

$* \subseteq *$

$[\mathsf{x} \subseteq \mathsf{y} \stackrel{\text{macro}}{\to} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t}, \mathsf{s}, \mathsf{c}, \lceil [\mathsf{x} \subseteq \mathsf{y} \ddot{=} (s \in \mathsf{x} \Rightarrow s \in \mathsf{y})] \rceil)]$

$[\mathsf{x} \subseteq \mathsf{y} \stackrel{\text{tex}}{\to}$ "#1.
\mathrel{\subseteq} #2."]

$[* \subseteq * \stackrel{\text{pyk}}{\to}$ "" is subset of ""]

$\dot{\neg} *$

$[\dot{\neg} \mathsf{x} \stackrel{\text{tex}}{\to}$ "\dot{\neg}\, #1."]

$[\dot{\neg} * \stackrel{\text{pyk}}{\rightarrow}$ "not0 ""]

$* \notin *$

$[\mathsf{x} \notin \mathsf{y} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \notin \mathsf{y} \ddot{=} \dot{\neg}\mathsf{x} \in \mathsf{y}]\rceil)]$

$[\mathsf{x} \notin \mathsf{y} \stackrel{\text{tex}}{\rightarrow}$ "#1.
\mathrel{\notin} #2."]

$[* \notin * \stackrel{\text{pyk}}{\rightarrow}$ "" zermelo ~in ""]

$* \neq *$

$[\mathsf{x} \neq \mathsf{y} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \neq \mathsf{y} \ddot{=} \dot{\neg}\mathsf{x}{=}{=}\mathsf{y}]\rceil)]$

$[\mathsf{x} \neq \mathsf{y} \stackrel{\text{tex}}{\rightarrow}$ "#1.
\mathrel{\neq} #2."]

$[* \neq * \stackrel{\text{pyk}}{\rightarrow}$ "" zermelo ~is ""]

$* \dot{\wedge} *$

$[\mathsf{x} \dot{\wedge} \mathsf{y} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \dot{\wedge} \mathsf{y} \ddot{=} \dot{\neg}(\mathsf{x} \Rightarrow \dot{\neg}\mathsf{y})]\rceil)]$

$[\mathsf{x} \dot{\wedge} \mathsf{y} \stackrel{\text{tex}}{\rightarrow}$ "#1.
\mathrel{\dot{\wedge}} #2."]

$[* \dot{\wedge} * \stackrel{\text{pyk}}{\rightarrow}$ "" and0 ""]

$* \dot{\vee} *$

$[\mathsf{x} \dot{\vee} \mathsf{y} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \dot{\vee} \mathsf{y} \ddot{=} \dot{\neg}\mathsf{x} \Rightarrow \mathsf{y}]\rceil)]$

$[\mathsf{x} \dot{\vee} \mathsf{y} \stackrel{\text{tex}}{\rightarrow}$ "#1.
\mathrel{\dot{\vee}} #2."]

$[* \dot{\vee} * \stackrel{\text{pyk}}{\rightarrow}$ "" or0 ""]

$* \dot{\Leftrightarrow} *$

$[\mathsf{x} \dot{\Leftrightarrow} \mathsf{y} \stackrel{\text{macro}}{\rightarrow} \lambda \mathsf{t}.\lambda \mathsf{s}.\lambda \mathsf{c}.\tilde{\mathcal{M}}_4(\mathsf{t},\mathsf{s},\mathsf{c},\lceil[\mathsf{x} \dot{\Leftrightarrow} \mathsf{y} \ddot{=} (\mathsf{x} \Rightarrow \mathsf{y}) \dot{\wedge} (\mathsf{y} \Rightarrow \mathsf{x})]\rceil)]$

[$\mathsf{x} \dot{\Leftrightarrow} \mathsf{y} \stackrel{\text{tex}}{\to}$ "#1.
\mathrel{\dot{\Leftrightarrow}} #2."]

[$* \dot{\Leftrightarrow} * \stackrel{\text{pyk}}{\to}$ "" iff ""]

$\{ph \in * \mid *\}$

[$\{ph \in \mathsf{x} \mid \mathsf{a}\} \stackrel{\text{tex}}{\to}$ " \{ ph \mathrel{\in} #1.
\mid #2.
\}"]

[$\{ph \in * \mid *\} \stackrel{\text{pyk}}{\to}$ "the set of ph in " such that " end set"]

The pyk compiler, version 0.grue.20060417+ by Klaus Grue
GRD-2006-09-15.UTC:09:33:20.992497 = MJD-53993.TAI:09:33:53.992497 =
LGT-4665029633992497e-6