(*** MAKROER BEGYNDER ***)

1 Makrodefinitioner

Dette afsnit indeholder dé makrodefinitioner, som vi vil gøre brug af i resten af rapporten. Definitionerne drejer sig for det meste om mængdeteoretiske begreber, f.eks. "ækvivalensklasse" og "partition". Til sidst i afsnittet formulerer vi hovedresultatet — at der til enhver ækvivalensrelation svarer en partition — som et formelt teorem.

1.1 Konnektiver

Ud fra de to basale konnektiver $[\dot{\neg}\,\mathsf{x}]$ og $[\mathsf{x} \Rightarrow \mathsf{y}]$ definerer vi konjunktion, disjunktion og dobbeltimplikation:

$$[\mathsf{x} \mathrel{\dot{\wedge}} \mathsf{y} \mathrel{\ddot{=}} \dot{\neg}\,(\mathsf{x} \Rightarrow \dot{\neg}\,\mathsf{y})]$$

$$[\mathsf{x} \mathrel{\dot{\vee}} \mathsf{y} \mathrel{\ddot{=}} \dot{\neg}\,\mathsf{x} \Rightarrow \mathsf{y}]$$

$$[\mathsf{x} \mathrel{\dot{\Leftrightarrow}} \mathsf{y} \mathrel{\ddot{=}} (\mathsf{x} \Rightarrow \mathsf{y}) \mathrel{\dot{\wedge}} (\mathsf{y} \Rightarrow \mathsf{x})]$$

1.2 Negerede formler

Det er ganske enkelt at definere negeret lighed $(\mathsf{x} \neq \mathsf{y})$ og negeret medlemskab $(\mathsf{x} \notin \mathsf{y})$:

$$[\mathsf{x} \neq \mathsf{y} \mathrel{\ddot{=}} \dot{\neg}\,\mathsf{x} {==} \mathsf{y}]$$

$$[\mathsf{x} \notin \mathsf{y} \mathrel{\ddot{=}} \dot{\neg}\,\mathsf{x} \in \mathsf{y}]$$ [1]

1.3 Delmængde

Mængden x er en delmængde af y hviss ethvert medlem af x også tilhører y:

$$[\mathsf{x} \subseteq \mathsf{y} \mathrel{\ddot{=}} (s \in \mathsf{x} \Rightarrow s \in \mathsf{y})]$$

1.4 Singleton-mængde

$[\{\mathsf{x}\}]$ er mængden, der indeholder x som sit eneste element. Vi definerer $[\{\mathsf{x}\}]$ ved at parre x med sig selv:

$$[\{\mathsf{x}\} \mathrel{\ddot{=}} \{\mathsf{x},\mathsf{x}\}]$$

1.5 Binær foreningsmængde og fællesmængde

Vi definerer foreningsmængden mellem to mængder x og y som følger:

$$[\mathsf{x} \cup \mathsf{y} \mathrel{\ddot{=}} \cup\{\{\mathsf{x}\},\{\mathsf{y}\}\}]$$

[1] Højresiderne i disse definitioner skal læses som hhv. $[\dot{\neg}\,(\mathsf{x}{==}\mathsf{y})]$ og $[\dot{\neg}\,(\mathsf{x} \in \mathsf{y})]$.

Fællesmængden mellem to mængder x og y er en delmængde af deres foreningsmængde:

$$[\mathsf{x} \cap \mathsf{y} \doteqdot \{\text{ph} \in \mathsf{x} \cup \mathsf{y} \mid \text{ph}_3 \in \mathsf{x} \dot{\wedge} \text{ph}_3 \in \mathsf{y}\}]$$

1.6 Relation

Det ordnede par $\langle \mathsf{x}, \mathsf{y} \rangle$ indeholder x som "førstekomponent" og y som "andenkomponent". Den følgende definition af $\langle \mathsf{x}, \mathsf{y} \rangle$ er den mest udbredte i litteraturen (se f.eks. afsnit 4.3 i [**?**] og afsnit 2.1 i [**?**]):

$$[\langle \mathsf{x}, \mathsf{y} \rangle \doteqdot \{\{\mathsf{x}\}, \{\mathsf{x}, \mathsf{y}\}\}]$$

Vi kan nu definere en "relation" som en mængde af ordnede par. Vi udtrykker denne definition ved at formalisere, hvad det vil sige, at x er relateret til y i kraft af relationen r:

$$[\mathsf{r}(\mathsf{x}, \mathsf{y}) \doteqdot \langle \mathsf{x}, \mathsf{y} \rangle \in \mathsf{r}]$$

Vi kommer faktisk ikke til at bruge disse to definitioner i rapporten; beviserne vil behandle $[\mathsf{r}(\mathsf{x}, \mathsf{y})]$ som en primitiv konstruktion. Men det er alligevel betryggende at have det formelle grundlag for relationsbegrebet på plads.

1.7 Ækvivalensrelation

At en relation er refleksiv på en mængde x vil sige, at alle elementer i x er relateret til sig selv:

$$[\mathrm{ReflRel}(\mathsf{r},\mathsf{x}) \ddot{=} \forall s\colon (s \in \mathsf{x} \Rightarrow \mathsf{r}(s,s))]$$

At en relation er symmetrisk på en mængde x vil sige, at alle elementer i x opfylder den følgende implikation:

$$[\mathrm{SymRel}(\mathsf{r},\mathsf{x}) \ddot{=} \forall s,t\colon (s \in \mathsf{x} \Rightarrow t \in \mathsf{x} \Rightarrow \mathsf{r}(s,t) \Rightarrow \mathsf{r}(t,s))]$$

At en relation er transitiv på en mængde x vil sige, at alle elementer i x opfylder den følgende implikation:

$$[\mathrm{TransRel}(\mathsf{r},\mathsf{x}) \ddot{=} \\ \forall s,t,u\colon (s \in \mathsf{x} \Rightarrow t \in \mathsf{x} \Rightarrow u \in \mathsf{x} \Rightarrow \mathsf{r}(s,t) \Rightarrow \mathsf{r}(t,u) \Rightarrow \mathsf{r}(s,u))]$$

Endelig er en ækvivalensrelation det samme som en relation, der er refleksiv, symmetrisk og transitiv:

$$[\mathrm{EqRel}(\mathsf{r},\mathsf{x}) \ddot{=} \mathrm{ReflRel}(\mathsf{r},\mathsf{x}) \dot{\wedge} \mathrm{SymRel}(\mathsf{r},\mathsf{x}) \dot{\wedge} \mathrm{TransRel}(\mathsf{r},\mathsf{x})]$$

1.8 Mængde-variable

Mange af rapportens beviser sker i forhold til en uspecificeret mængde. Vi vil referere til denne mængde med metavariablen $\mathcal{BS}$ og objektvariablen OBS:

$$[\mathcal{BS} \ddot{=} \underline{\mathsf{bs}}]$$
$$[\mathsf{OBS} \ddot{=} \overline{\mathsf{bs}}]$$[2]

Vi vil så vidt muligt bruge metavariablen, men i afsnit **??** og senere bliver det nødvendigt at gå over til objektvariablen.

1.9 Ækvivalensklasse

Lad r være en ækvivalensrelation defineret på bs, og lad x være et medlem af bs. Vi definerer ækvivalensklassen $[\mathsf{x} \in \mathsf{bs}]_\mathsf{r}$ som den delmængde af bs, hvis medlemmer står i forhold til x:

$$[[\mathsf{x} \in \mathsf{bs}]_\mathsf{r} \ddot{=} \{\mathrm{ph} \in \mathsf{bs} \mid \mathsf{r}(\mathrm{ph}_1,\mathsf{x})\}]$$

Ækvivalenssystemet bs/r er mængden af alle de ækvivalensklasser, som bs definerer på r. Vi definerer bs/r som en delmængde af potensmængden $\mathrm{P}(\mathsf{bs})$:

$$[\mathsf{bs}/\mathsf{r} \ddot{=} \{\mathrm{ph} \in \mathrm{P}(\mathsf{bs}) \mid \mathrm{Ex}_{20} \in \mathsf{bs} \dot{\wedge} [\mathrm{Ex}_{20} \in \mathsf{bs}]_\mathsf{r} == \mathrm{ph}_2\}]$$

[2] Navnene "$\mathcal{BS}$" og "OBS" står for hhv. for "big set" og "object big set". Konstruktionerne $[\underline{\mathsf{x}}]$ og $[\overline{\mathsf{x}}]$ omdanner x til hhv. en meta- og en objektvariabel. Variablen $[\mathsf{bs}]$ vil også blive brugt i nogle af de kommende definitioner, men ikke i selve beviserne.

1.10 Partition

En partition af en mængde bs er en mængde p, som opfylder tre krav:

1. Ingen af mængderne i p er tomme.
2. Alle mængderne i p er indbyrdes disjunkte.
3. Foreningsmængden af alle mængderne i p er lig med bs.

Den formelle version af denne definition ser således ud:

$$[\text{Partition}(\mathsf{p},\mathsf{bs}) \mathrel{\ddot{=}} (\forall s\colon (s \in \mathsf{p} \Rightarrow s \neq \emptyset)) \mathrel{\dot{\wedge}} \\ (\forall s,t\colon (s \in \mathsf{p} \Rightarrow t \in \mathsf{p} \Rightarrow s \neq t \Rightarrow s \cap t == \emptyset)) \mathrel{\dot{\wedge}} \\ \cup\, \mathsf{p} == \mathsf{bs}]$$

(*** MAKROER SLUTTER ***)

2 Deduktionsreglen

Dette bilag præsenterer dén version af deduktionsreglen fra [**?**], som jeg har gjort brug af. Underafsnit **??** forklarer, hvorfor jeg har ændret på den oprindelige regel, og underafsnit 2.1 indeholder selve den ændrede kode (som er skrevet i L).

2.1 Kode

Funktionen $[\text{Dedu}(\mathsf{p},\mathsf{c})]$ er en kopi af $[\text{Ded}(\mathsf{p},\mathsf{c})]$ fra [**?**]:

$$[\text{Dedu}(\mathsf{p},\mathsf{c}) \mathrel{\ddot{=}} \lambda\mathsf{x}.\text{Dedu}_0(\lceil\mathsf{p}\rceil,\lceil\mathsf{c}\rceil)]$$

Jeg har ændret funktionen $[\text{Ded}_0(\mathsf{p},\mathsf{c})]$, så den kalder $[\text{Dedu}_s(\text{Dedu}_7(\mathsf{p}),\mathsf{c},\mathsf{T})]$ i stedet for $[\text{Ded}_1(\text{Ded}_7(\mathsf{p}),\mathsf{c},\mathsf{T})]$:

$$[\text{Dedu}_0(\mathsf{p},\mathsf{c}) \mathrel{\dot{=}} \mathsf{c}!\text{Dedu}_8(\mathsf{p},\mathsf{T}) \mathrel{\ddot{\wedge}} \text{Dedu}_s(\text{Dedu}_7(\mathsf{p}),\mathsf{c},\mathsf{T})]$$

Funktionen $[\text{Dedu}_s(\mathsf{p},\mathsf{c},\mathsf{s})]$ giver straks kontrollen videre til $[\text{Ded}_1(\mathsf{p},\mathsf{c},\mathsf{s})]$ — medmindre p og c begynder med et antal identiske sidebetingelser. I så fald flyttes disse sidebetingelser fra p og c over til listen s, før kontrollen går videre til $[\text{Ded}_1(\mathsf{p},\mathsf{c},\mathsf{s})]$:

$$[\text{Dedu}_s(\mathsf{p},\mathsf{c},\mathsf{s}) \mathrel{\dot{=}} \textbf{if } \mathsf{p} \overset{\text{r}}{=} \lceil \mathsf{x} \Vdash \mathsf{y}\rceil \textbf{ then} \\ \mathsf{c} \overset{\text{r}}{=} \lceil \mathsf{x} \Vdash \mathsf{y}\rceil \wedge \mathsf{p}^1 \overset{\text{t}}{=} \mathsf{c}^1 \wedge \text{Dedu}_s(\mathsf{p}^2,\mathsf{c}^2,\mathsf{c}^1 :: \mathsf{s}) \textbf{ else} \\ \text{Dedu}_1(\mathsf{p},\mathsf{c},\mathsf{s})]$$

Fra og med $[\text{Ded}_1(\mathsf{p},\mathsf{c},\mathsf{s})]$ er koden kopieret fra appendikset til [**?**]:

$$[\text{Dedu}_1(\mathsf{p},\mathsf{c},\mathsf{s}) \mathrel{\dot{=}} \textbf{if } \mathsf{c} \overset{\text{r}}{=} \lceil \mathsf{x} \Vdash \mathsf{y}\rceil \textbf{ then } \text{Dedu}_1(\mathsf{p},\mathsf{c}^2,\mathsf{c}^1 :: \mathsf{s}) \textbf{ else } \text{Dedu}_2(\mathsf{p},\mathsf{c},\mathsf{s})]$$

$$[\text{Dedu}_2(\mathsf{p},\mathsf{c},\mathsf{s}) \mathrel{\dot{=}} \mathsf{s}! \\ \mathsf{p} \overset{\text{r}}{=} \lceil \mathsf{x} \vdash \mathsf{y}\rceil \wedge \mathsf{c} \overset{\text{r}}{=} \lceil \mathsf{x} \Rightarrow \mathsf{y}\rceil \begin{cases} \text{Dedu}_3(\mathsf{p}^1,\mathsf{c}^1,\mathsf{s},\mathsf{T}) \wedge \text{Dedu}_2(\mathsf{p}^2,\mathsf{c}^2,\mathsf{s}) \\ \text{Dedu}_4(\mathsf{p},\mathsf{c},\mathsf{s},\text{Dedu}_6(\mathsf{p},\mathsf{c},\mathsf{T},\mathsf{T})) \end{cases}]$$

$[\mathrm{Dedu}_3(\mathsf{p},\mathsf{c},\mathsf{s},\mathsf{b}) \doteq \textbf{if } \neg\mathsf{c} \overset{r}{=} \lceil\forall\mathsf{x}\colon\mathsf{y}\rceil \textbf{ then } \mathrm{Dedu}_4(\mathsf{p},\mathsf{c},\mathsf{s},\mathsf{b}) \textbf{ else}$
$\textbf{if } \mathsf{p} \overset{r}{=} \lceil\forall\mathsf{x}\colon\mathsf{y}\rceil \wedge \mathsf{p}^1 \overset{t}{=} \mathsf{c}^1 \textbf{ then } \mathrm{Dedu}_4(\mathsf{p},\mathsf{c},\mathsf{s},\mathsf{b}) \textbf{ else}$
$\mathrm{Dedu}_3(\mathsf{p},\mathsf{c}^2,\mathsf{s},(\mathsf{c}^1 :: \mathsf{c}^1) :: \mathsf{b})]$

$[\mathrm{Dedu}_4(\mathsf{p},\mathsf{c},\mathsf{s},\mathsf{b}) \doteq \mathsf{s!b!}$
$\textbf{if } \mathsf{p} \overset{r}{=} \lceil x\rceil \textbf{ then lookup}(\mathsf{p},\mathsf{b},\mathsf{T}) \overset{t}{=} \mathsf{c} \textbf{ else}$
$\textbf{if } \neg\mathsf{p} \overset{r}{=} \mathsf{c} \textbf{ then F else}$
$\textbf{if } \mathsf{p} \overset{r}{=} \lceil\forall\mathsf{x}\colon\mathsf{y}\rceil \textbf{ then } \mathsf{p}^1 \overset{t}{=} \mathsf{c}^1 \wedge \mathrm{Dedu}_4(\mathsf{p}^2,\mathsf{c}^2,\mathsf{s},(\mathsf{p}^1 :: \mathsf{p}^1) :: \mathsf{b}) \textbf{ else}$
$\textbf{if } \neg\mathsf{p} \overset{r}{=} \lceil\mathcal{X}\rceil \textbf{ then } \mathrm{Dedu}_4^*(\mathsf{p}^t,\mathsf{c}^t,\mathsf{s},\mathsf{b}) \textbf{ else}$
$\mathsf{p}^1 \overset{t}{=} \mathsf{c}^1 \wedge \mathrm{Dedu}_5(\mathsf{p},\mathsf{s},\mathsf{b})]$

$[\mathrm{Dedu}_4^*(\mathsf{p},\mathsf{c},\mathsf{s},\mathsf{b}) \doteq \mathsf{c!s!b!}\textbf{if } \mathsf{p} \textbf{ then T else } \mathrm{Dedu}_4(\mathsf{p}^h,\mathsf{c}^h,\mathsf{s},\mathsf{b}) \wedge \mathrm{Dedu}_4^*(\mathsf{p}^t,\mathsf{c}^t,\mathsf{s},\mathsf{b})]$

$[\mathrm{Dedu}_5(\mathsf{p},\mathsf{s},\mathsf{b}) \doteq \mathsf{p!s!}\textbf{if } \mathsf{b} \textbf{ then T else}$
$\langle\lceil\mathsf{x\#y}\rceil^h, \langle\lceil\lceil *\rceil\rceil^h, \mathsf{b}^{hh}\rangle, \langle\lceil\lceil\mathsf{x}\rceil\rceil^h, \mathsf{p}\rangle\rangle \in_t \mathsf{s} \wedge \mathrm{Dedu}_5(\mathsf{p},\mathsf{s},\mathsf{b}^t)]$

$[\mathrm{Dedu}_6(\mathsf{p},\mathsf{c},\mathsf{e},\mathsf{b}) \doteq \mathsf{p!c!b!e!}$
$\textbf{if } \mathsf{p} \overset{r}{=} \lceil x\rceil \textbf{ then } \mathsf{p} \in_t \mathsf{e} \begin{cases} \mathsf{b} \\ (\mathsf{p} :: \mathsf{c}) :: \mathsf{b} \end{cases} \textbf{ else}$
$\textbf{if } \neg\mathsf{p} \overset{r}{=} \mathsf{c} \textbf{ then T else}$
$\textbf{if } \mathsf{p} \overset{r}{=} \lceil\mathcal{A}\rceil \textbf{ then b else}$
$\textbf{if } \mathsf{p} \overset{r}{=} \lceil\forall\mathsf{x}\colon\mathsf{y}\rceil \textbf{ then } \mathrm{Dedu}_6(\mathsf{p}^2,\mathsf{c}^2,\mathsf{c}^1 :: \mathsf{e},\mathsf{b}) \textbf{ else}$
$\mathrm{Dedu}_6^*(\mathsf{p}^t,\mathsf{c}^t,\mathsf{e},\mathsf{b})]$

$[\mathrm{Dedu}_6^*(\mathsf{p},\mathsf{c},\mathsf{e},\mathsf{b}) \doteq \mathsf{p!c!b!e!}\textbf{if } \mathsf{p} \textbf{ then b else } \mathrm{Dedu}_6^*(\mathsf{p}^t,\mathsf{c}^t,\mathsf{e},\mathrm{Dedu}_6(\mathsf{p}^h,\mathsf{c}^h,\mathsf{e},\mathsf{b}))]$

$[\mathrm{Dedu}_7(\mathsf{p}) \doteq \mathsf{p} \overset{r}{=} \lceil\Pi\mathsf{x}\colon\mathsf{y}\rceil \begin{cases} \mathrm{Dedu}_7(\mathsf{p}^2) \\ \mathsf{p} \end{cases}]$

$[\mathrm{Dedu}_8(\mathsf{p},\mathsf{b}) \doteq$
$\textbf{if } \mathsf{p} \overset{r}{=} \lceil\Pi\mathsf{x}\colon\mathsf{y}\rceil \textbf{ then } \mathrm{Dedu}_8(\mathsf{p}^2,\mathsf{p}^1 :: \mathsf{b}) \textbf{ else}$
$\textbf{if } \mathsf{p} \overset{r}{=} \lceil\mathcal{A}\rceil \textbf{ then } \mathsf{p} \in_t \mathsf{b} \textbf{ else } \mathrm{Dedu}_8^*(\mathsf{p}^t,\mathsf{b})]$

$[\mathrm{Dedu}_8^*(\mathsf{p},\mathsf{b}) \doteq \mathsf{b!}\textbf{if } \mathsf{p} \textbf{ then T else } \mathrm{Dedu}_8(\mathsf{p}^h,\mathsf{b}) \ddot{\wedge} \mathrm{Dedu}_8^*(\mathsf{p}^t,\mathsf{b})]$

(*** EKSISTENS-VARIABLE ***)

$[\mathsf{x}^{\mathrm{Ex}} \doteq \mathsf{x} \overset{r}{=} \lceil\mathsf{x}_{\mathrm{Ex}}\rceil]$

Vi kan da definere de fire eksistens-variable, som denne rapport vil gøre brug af (jvf. bilag **??**):

$[\mathrm{Ex}_1 \ddot{=} \mathsf{a}_{\mathrm{Ex}}]$

$[\mathrm{Ex}_2 \ddot{=} \mathsf{b}_{\mathrm{Ex}}]$
$[\mathrm{Ex}_{10} \ddot{=} \mathsf{j}_{\mathrm{Ex}}]$
$[\mathrm{Ex}_{20} \ddot{=} \mathsf{t}_{\mathrm{Ex}}]$
$[\langle\mathsf{a}{\equiv}\mathsf{b}|\mathsf{x}{:==}\mathsf{t}\rangle_{\mathrm{Ex}} \ddot{=} \langle\lceil\mathsf{a}\rceil\equiv^0\lceil\mathsf{b}\rceil|\lceil\mathsf{x}\rceil{:==}\lceil\mathsf{t}\rceil\rangle_{\mathrm{Ex}}]$

$[\langle \mathsf{a}{\equiv}^0\mathsf{b}|\mathsf{x}{:}{=}{=}\mathsf{t}\rangle_{\mathrm{Ex}} \doteq \lambda \mathsf{c}.\mathsf{x}^{\mathrm{Ex}} \wedge \langle \mathsf{a}{\equiv}^1\mathsf{b}|\mathsf{x}{:}{=}{=}\mathsf{t}\rangle_{\mathrm{Ex}}]$

$[\langle \mathsf{a}{\equiv}^1\mathsf{b}|\mathsf{x}{:}{=}{=}\mathsf{t}\rangle_{\mathrm{Ex}} \doteq \mathsf{a!x!t!}$
if $\mathsf{b} \stackrel{\mathrm{r}}{=} \lceil \forall \mathsf{u}{:}\, \mathsf{v} \rceil$ **then** F **else**
if $\mathsf{b}^{\mathrm{Ex}} \wedge \mathsf{b} \stackrel{\mathrm{t}}{=} \mathsf{x}$ **then** $\mathsf{a} \stackrel{\mathrm{t}}{=} \mathsf{t}$ **else**
$\mathsf{a} \stackrel{\mathrm{r}}{=} \mathsf{b} \mathrel{\ddot{\wedge}} \langle \mathsf{a}^{\mathrm{t}}{\equiv}^*\mathsf{b}^{\mathrm{t}}|\mathsf{x}{:}{=}{=}\mathsf{t}\rangle_{\mathrm{Ex}}]$

$[\langle \mathsf{a}{\equiv}^*\mathsf{b}|\mathsf{x}{:}{=}{=}\mathsf{t}\rangle_{\mathrm{Ex}} \doteq \mathsf{b!x!t!}\mathrm{If}(\mathsf{a}, \mathsf{T}, \langle \mathsf{a}^{\mathrm{h}}{\equiv}^1\mathsf{b}^{\mathrm{h}}|\mathsf{x}{:}{=}{=}\mathsf{t}\rangle_{\mathrm{Ex}} \mathrel{\ddot{\wedge}} \langle \mathsf{a}^{\mathrm{t}}{\equiv}^*\mathsf{b}^{\mathrm{t}}|\mathsf{x}{:}{=}{=}\mathsf{t}\rangle_{\mathrm{Ex}})]$

(*** AKSIOMATISK SYSTEM ***)
[**Theory** ZFsub]

[ZFsub **rule** MP: $\Pi \mathcal{A}, \mathcal{B}{:}\ \mathcal{A} \Rightarrow \mathcal{B} \vdash \mathcal{A} \vdash \mathcal{B}$]

[ZFsub **rule** Gen: $\Pi \mathcal{X}, \mathcal{A}{:}\ \mathcal{A} \vdash \forall \mathcal{X}{:}\ \mathcal{A}$]

[ZFsub **rule** Repetition: $\Pi \mathcal{A}{:}\ \mathcal{A} \vdash \mathcal{A}$]

[ZFsub **rule** Neg: $\Pi \mathcal{A}, \mathcal{B}{:}\ \dot{\neg} \mathcal{B} \Rightarrow \mathcal{A} \vdash \dot{\neg} \mathcal{B} \Rightarrow \dot{\neg} \mathcal{A} \vdash \mathcal{B}$]

[ZFsub **rule** Ded: $\Pi \mathcal{A}, \mathcal{B}{:}\ \mathrm{Dedu}(\mathcal{A}, \mathcal{B}) \Vdash \mathcal{A} \vdash \mathcal{B}$]

[ZFsub **rule** ExistIntro: $\Pi \mathcal{X}, \mathcal{T}, \mathcal{A}, \mathcal{B}{:}\ \langle \mathcal{A}{\equiv}\mathcal{B}|\mathcal{X}{:}{=}{=}\mathcal{T}\rangle_{\mathrm{Ex}} \Vdash \mathcal{A} \vdash \mathcal{B}$]

[ZFsub **rule** Extensionality: $\Pi \mathcal{X}, \mathcal{Y}{:}\ \mathcal{X} {=}{=} \mathcal{Y} \mathrel{\dot{\Leftrightarrow}} \forall s{:}\, (s \in \mathcal{X} \mathrel{\dot{\Leftrightarrow}} s \in \mathcal{Y})$]

[ZFsub **rule** Ødef: $\Pi \mathcal{S}{:}\ \dot{\neg} \mathcal{S} \in \emptyset$]

[ZFsub **rule** PairDef: $\Pi \mathcal{S}, \mathcal{X}, \mathcal{Y}{:}\ \mathcal{S} \in \{\mathcal{X}, \mathcal{Y}\} \mathrel{\dot{\Leftrightarrow}} \mathcal{S} {=}{=} \mathcal{X} \mathrel{\dot{\vee}} \mathcal{S} {=}{=} \mathcal{Y}$]

[ZFsub **rule** UnionDef: $\Pi \mathcal{S}, \mathcal{X}{:}\ \mathcal{S} \in \cup \mathcal{X} \mathrel{\dot{\Leftrightarrow}} (\mathcal{S} \in \mathrm{Ex}_{10} \mathrel{\dot{\wedge}} \mathrm{Ex}_{10} \in \mathcal{X})$]

[ZFsub **rule** PowerDef: $\Pi \mathcal{S}, \mathcal{X}{:}\ \mathcal{S} \in \mathrm{P}(\mathcal{X}) \mathrel{\dot{\Leftrightarrow}} \forall s{:}\, (s \in \mathcal{S} \Rightarrow s \in \mathcal{X})$]

[ZFsub **rule** SeparationDef: $\Pi \mathcal{A}, \mathcal{B}, \mathcal{P}, \mathcal{X}, \mathcal{Z}{:}\ \mathcal{P}^{\mathrm{Ph}} \wedge \langle \mathcal{B}{\equiv}\mathcal{A}|\mathcal{P}{:}{=}{=}\mathcal{Z}\rangle_{\mathrm{Ph}} \Vdash$
$\mathcal{Z} \in \{\mathrm{ph} \in \mathcal{X} \mid \mathcal{A}\} \mathrel{\dot{\Leftrightarrow}} \mathcal{Z} \in \mathcal{X} \mathrel{\dot{\wedge}} \mathcal{B}$]

3 Udsagnslogisk bibliotek

I dette afsnit vil jeg bevise en samling af udsagnslogiske sandheder (eller "tautologier"), som vil blive brugt i de følgende afsnit. De fleste af disse tautologier har mange andre anvendelser end lige netop mængdelære. Beviserne er fordelt på syv underafsnit; figur 1 giver et overblik over, hvordan beviserne forholder sig til hinanden. Jeg vil kommentere de fleste af beviserne; dog er nogle af dem så tekniske, at jeg har ladet dem stå alene.

3.1 MP-lemmaer

Man får ofte brug for at anvende slutningsreglen MP flere gange i træk. Derfor vil jeg begynde med at vise fire lemmaer, der kan klare mellem 2 og 5 anvendelser af MP[3]:

[ZFsub **lemma** MP2: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{A}\Rightarrow\mathcal{B}\Rightarrow\mathcal{C}\vdash\mathcal{A}\vdash\mathcal{B}\vdash\mathcal{C}$]

[ZFsub **lemma** MP3: $\Pi\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}\colon \mathcal{A}\Rightarrow\mathcal{B}\Rightarrow\mathcal{C}\Rightarrow\mathcal{D}\vdash\mathcal{A}\vdash\mathcal{B}\vdash\mathcal{C}\vdash\mathcal{D}$]

[ZFsub **lemma** MP4: $\Pi\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D},\mathcal{E}\colon$
$\mathcal{A}\Rightarrow\mathcal{B}\Rightarrow\mathcal{C}\Rightarrow\mathcal{D}\Rightarrow\mathcal{E}\vdash\mathcal{A}\vdash\mathcal{B}\vdash\mathcal{C}\vdash\mathcal{D}\vdash\mathcal{E}$]

[ZFsub **lemma** MP5: $\Pi\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D},\mathcal{E},\mathcal{F}\colon$
$\mathcal{A}\Rightarrow\mathcal{B}\Rightarrow\mathcal{C}\Rightarrow\mathcal{D}\Rightarrow\mathcal{E}\Rightarrow\mathcal{F}\vdash\mathcal{A}\vdash\mathcal{B}\vdash\mathcal{C}\vdash\mathcal{D}\vdash\mathcal{E}\vdash\mathcal{F}$]

3.1.1 Det første bevis

Vi begynder med at bevise MP2:

[ZFsub **lemma** MP2: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{A}\Rightarrow\mathcal{B}\Rightarrow\mathcal{C}\vdash\mathcal{A}\vdash\mathcal{B}\vdash\mathcal{C}$]

ZFsub **proof of** MP2:

L01:	Arbitrary $\gg$	$\mathcal{A},\mathcal{B},\mathcal{C}$	;
L02:	Premise $\gg$	$\mathcal{A}\Rightarrow\mathcal{B}\Rightarrow\mathcal{C}$	;
L03:	Premise $\gg$	$\mathcal{A}$	;
L04:	Premise $\gg$	$\mathcal{B}$	;
L05:	MP $\rhd$ L02 $\rhd$ L03 $\gg$	$\mathcal{B}\Rightarrow\mathcal{C}$	;
L06:	MP $\rhd$ L05 $\rhd$ L04 $\gg$	$\mathcal{C}$	$\square$

Da dette er rapportens første bevis, vil jeg bringe nogle ekstra kommentarer[4]. Oven over beviset har jeg gentaget definitionen af det, der skal bevises; dette er kun for overblikkets skyld — det er ikke en formel nødvendighed. Selve beviset for MP2 består af seks linier, nummereret fra 1 til 6. En bevislinie kan have to former. Den første form er:

`Argumentation` $\gg$ `Konklusion`

hvor `Konklusion` er det som linien beviser, mens teksten i `Argumentation` udgør en begrundelse for, at `Konklusion` gælder. F.eks. siger linie 5, at meta-formlen [$\mathcal{B}\Rightarrow\mathcal{C}$] gælder, fordi den kan udledes fra slutningsreglen MP ved substitution. Argumentationen skal læses på den måde, at konklusionerne fra linie 2 og 3 bliver brugt som præmisser til MP. Den generelle betydning af konstruktionen [$\mathsf{x}\rhd\mathsf{y}$] er, at konklusionen fra linie y bliver brugt som præmis i forhold til x.

Den anden form, en bevislinie kan have, er:

`Nøgleord` $\gg$ `Konklusion`

[3] I afsnit **??** får vi faktisk brug for at anvende MP 6 gange i træk; men et eller andet sted skal man jo stoppe.

[4] Denne beskrivelse er en revideret udgave af afsnit 5.1 i [**?**].

hvor `Nøgleord` er et af de tre ord "Arbitrary", "Premise" eller "Side-condition". Betydningen af ordene "Premise" og "Side-condition" er åbenlys: De angiver, at liniens konklusion indgår som en præmis (hhv. sidebetingelse) i den sætning, der skal bevises. F.eks. siger bevisets linie 2, at MP2 bruger meta-formlen $[\mathcal{A} \Rightarrow \mathcal{B} \Rightarrow \mathcal{C}]$ som præmis. Når ordet "Arbitrary" bruges, består konklusionen af en liste af meta-variable (f.eks. $[\mathcal{A}, \mathcal{B}, \mathcal{C}]$ i linie 1). Ideen hermed er at udtrykke, at vi ikke antager noget om de pågældende meta-variable, og at vi derfor har ret til at binde dem med en meta-alkvantor i den sætning, der skal bevises. I det forhåndenværende bevis berettiger linien med "Arbitrary" altså, at MP2 er kvantificeret med $[\Pi \mathcal{A}, \mathcal{B}, \mathcal{C} \colon (\cdots)]$.

Alt dette har drejet sig om den formelle syntaks for et Logiweb bevis. Der er ikke så meget at sige om selve beviset for MP2; vi indkapsler simpelthen to på hinanden følgende anvendelser af MP.

3.1.2 Beviser for de andre MP-lemmaer

Beviserne for de øvrige MP-lemmaer er lige ud ad landevejen:

[ZFsub **lemma** MP3: $\Pi \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \colon \mathcal{A} \Rightarrow \mathcal{B} \Rightarrow \mathcal{C} \Rightarrow \mathcal{D} \vdash \mathcal{A} \vdash \mathcal{B} \vdash \mathcal{C} \vdash \mathcal{D}$]

ZFsub **proof of** MP3:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$	;
L02:	Premise $\gg$	$\mathcal{A} \Rightarrow \mathcal{B} \Rightarrow \mathcal{C} \Rightarrow \mathcal{D}$	;
L03:	Premise $\gg$	$\mathcal{A}$	;
L04:	Premise $\gg$	$\mathcal{B}$	;
L05:	Premise $\gg$	$\mathcal{C}$	;
L06:	MP2 $\rhd$ L02 $\rhd$ L03 $\rhd$ L04 $\gg$	$\mathcal{C} \Rightarrow \mathcal{D}$	;
L07:	MP $\rhd$ L06 $\rhd$ L05 $\gg$	$\mathcal{D}$	$\square$

[ZFsub **lemma** MP4: $\Pi \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E} \colon$
$\mathcal{A} \Rightarrow \mathcal{B} \Rightarrow \mathcal{C} \Rightarrow \mathcal{D} \Rightarrow \mathcal{E} \vdash \mathcal{A} \vdash \mathcal{B} \vdash \mathcal{C} \vdash \mathcal{D} \vdash \mathcal{E}$]

ZFsub **proof of** MP4:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$	;
L02:	Premise $\gg$	$\mathcal{A} \Rightarrow \mathcal{B} \Rightarrow \mathcal{C} \Rightarrow \mathcal{D} \Rightarrow \mathcal{E}$	;
L03:	Premise $\gg$	$\mathcal{A}$	;
L04:	Premise $\gg$	$\mathcal{B}$	;
L05:	Premise $\gg$	$\mathcal{C}$	;
L06:	Premise $\gg$	$\mathcal{D}$	;
L07:	MP2 $\rhd$ L02 $\rhd$ L03 $\rhd$ L04 $\gg$	$\mathcal{C} \Rightarrow \mathcal{D} \Rightarrow \mathcal{E}$	;
L08:	MP2 $\rhd$ L07 $\rhd$ L05 $\rhd$ L06 $\gg$	$\mathcal{E}$	$\square$

[ZFsub **lemma** MP5: $\Pi \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F} \colon$
$\mathcal{A} \Rightarrow \mathcal{B} \Rightarrow \mathcal{C} \Rightarrow \mathcal{D} \Rightarrow \mathcal{E} \Rightarrow \mathcal{F} \vdash \mathcal{A} \vdash \mathcal{B} \vdash \mathcal{C} \vdash \mathcal{D} \vdash \mathcal{E} \vdash \mathcal{F}$]

ZFsub **proof of** MP5:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}$	;
L02:	Premise $\gg$	$\mathcal{A} \Rightarrow \mathcal{B} \Rightarrow \mathcal{C} \Rightarrow \mathcal{D} \Rightarrow \mathcal{E} \Rightarrow \mathcal{F}$	;
L03:	Premise $\gg$	$\mathcal{A}$	;

L04:	Premise ≫	$\mathcal{B}$	;
L05:	Premise ≫	$\mathcal{C}$	;
L06:	Premise ≫	$\mathcal{D}$	;
L07:	Premise ≫	$\mathcal{E}$	;
L08:	MP3 ▷ L02 ▷ L03 ▷ L04 ▷ L05 ≫	$\mathcal{D} \Rightarrow \mathcal{E} \Rightarrow \mathcal{F}$	;
L09:	MP2 ▷ L08 ▷ L06 ▷ L07 ≫	$\mathcal{F}$	□

3.2 Implikation

Dette afsnit indeholder en række lemmaer vedr. implikation, grupperet i fire under-underafsnit.

3.2.1 Refleksivitet; blok-konstruktionen

Lemmaet AutoImply udsiger, at implikations-relationen er refleksiv:

[ZFsub **lemma** AutoImply: $\Pi\mathcal{A}\colon \mathcal{A} \Rightarrow \mathcal{A}$]

ZFsub **proof of** AutoImply:

L01:	Block ≫	Begin	;
L02:	Arbitrary ≫	$\mathcal{A}$	;
L03:	Premise ≫	$\mathcal{A}$	;
L04:	Repetition ▷ L03 ≫	$\mathcal{A}$	;
L05:	Block ≫	End	;
L06:	Arbitrary ≫	$\mathcal{A}$	;
L07:	Ded ▷ L05 ≫	$\mathcal{A} \Rightarrow \mathcal{A}$	□

Beviset for AutoImply indeholder to nye ting i forhold til de hidtidige beviser: En bevisblok, og en anvendelse af deduktions-reglen. En bevisblok er selvstændig enhed i et bevis; den afhænger ikke af den øvrige del af beviset. Den ovenstående bevisblok indeholder et bevis for lemmaet [$\Pi\mathcal{A}\colon \mathcal{A} \vdash \mathcal{A}$]. Pointen er nu, at blokkens sidste linie (linie 5) fungerer som en forkortelse for dette lemma. Vi kan da anvende deduktionsreglen på denne linie til at omdanne inferensen [$\Pi\mathcal{A}\colon \mathcal{A} \vdash \mathcal{A}$] til implikationen [$\mathcal{A} \Rightarrow \mathcal{A}$]. Det vigtigste formål med deduktionsreglen er netop, at vi let kan skifte fra inferens til implikation.

3.2.2 Transitivitet

Lemmaet ImplyTransitivity udsiger, at implikations-relationen er transitiv:

[ZFsub **lemma** ImplyTransitivity: $\Pi\mathcal{A},\mathcal{B},\mathcal{C}\colon \mathcal{A} \Rightarrow \mathcal{B} \vdash \mathcal{B} \Rightarrow \mathcal{C} \vdash \mathcal{A} \Rightarrow \mathcal{C}$]

Vi viser ImplyTransitivity ved hjælp af MP og deduktionsreglen:

ZFsub **proof of** ImplyTransitivity:

L01:	Block ≫	Begin	;
L02:	Arbitrary ≫	$\mathcal{A},\mathcal{B},\mathcal{C}$	;
L03:	Premise ≫	$\mathcal{A} \Rightarrow \mathcal{B}$	;

L04:	Premise ≫	$\quad\mathcal{B} \Rightarrow \mathcal{C}$	;
L05:	Premise ≫	$\quad\mathcal{A}$	;
L06:	MP ▷ L03 ▷ L05 ≫	$\quad\mathcal{B}$	;
L07:	MP ▷ L04 ▷ L06 ≫	$\quad\mathcal{C}$	;
L08:	Block ≫	End	;
L09:	Arbitrary ≫	$\mathcal{A}, \mathcal{B}, \mathcal{C}$	;
L10:	Premise ≫	$\mathcal{A} \Rightarrow \mathcal{B}$	;
L11:	Premise ≫	$\mathcal{B} \Rightarrow \mathcal{C}$	;
L12:	Ded ▷ L08 ≫	$(\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow (\mathcal{B} \Rightarrow \mathcal{C}) \Rightarrow \mathcal{A} \Rightarrow \mathcal{C}$	;
L13:	MP2 ▷ L12 ▷ L10 ▷ L11 ≫	$\mathcal{A} \Rightarrow \mathcal{C}$	□

3.2.3 Svækkelse

Vi får ofte brug for det følgende ræsonnement: Hvis formlen $\mathcal{A}$ gælder ubetinget, så gælder den også under antagelse af en vilkårlig anden formel $\mathcal{B}$. Lemmaet Weakening udtrykker dette ræsonnement som følger:

[ZFsub **lemma** Weakening: $\Pi\mathcal{A}, \mathcal{B}: \mathcal{B} \vdash \mathcal{A} \Rightarrow \mathcal{B}$]

Vi beviser Weakening ved hjælp af deduktionsreglen:

ZFsub **proof of** Weakening:

L01:	Block ≫	Begin	;
L02:	Arbitrary ≫	$\quad\mathcal{A}, \mathcal{B}$	;
L03:	Premise ≫	$\quad\mathcal{B}$	;
L04:	Premise ≫	$\quad\mathcal{A}$	;
L05:	Repetition ▷ L03 ≫	$\quad\mathcal{B}$	;
L06:	Block ≫	End	;
L07:	Arbitrary ≫	$\mathcal{A}, \mathcal{B}$	;
L08:	Ded ▷ L06 ≫	$\mathcal{B} \Rightarrow \mathcal{A} \Rightarrow \mathcal{B}$	;
L09:	Premise ≫	$\mathcal{B}$	;
L10:	MP ▷ L08 ▷ L09 ≫	$\mathcal{A} \Rightarrow \mathcal{B}$	□

3.2.4 Modsigelse

Det sidste lemma i dette afsnit vedrører strengt taget ikke implikation, men derimod inferens ($\mathsf{x} \vdash \mathsf{y}$). Lemmaet FromContradiction udsiger, at vi kan bevise hvad som helst, hvis vi har bevist to formler, der modsiger hinanden:

[ZFsub **lemma** FromContradiction: $\Pi\mathcal{A}, \mathcal{B}: \mathcal{A} \vdash \dot{\neg}\mathcal{A} \vdash \mathcal{B}$]

Beviset bruger Weakening og slutningsreglen Neg:

ZFsub **proof of** FromContradiction:

L01:	Arbitrary ≫	$\mathcal{A}, \mathcal{B}$	;
L02:	Premise ≫	$\mathcal{A}$	;
L03:	Premise ≫	$\dot{\neg}\mathcal{A}$	;
L04:	Weakening ▷ L02 ≫	$\dot{\neg}\mathcal{B} \Rightarrow \mathcal{A}$	;

L05: Weakening $\rhd$ L03 $\gg$ $\dot{\neg}\mathcal{B} \Rightarrow \dot{\neg}\mathcal{A}$;
L06: Neg $\rhd$ L04 $\rhd$ L05 $\gg$ $\mathcal{B}$ □

3.3 Håndtering af dobbeltnegationer

De to lemmaer RemoveDoubleNeg og AddDoubleNeg tillader os hhv. at fjerne og tilføje dobbeltnegationer. Jeg vil ikke kommentere beviserne:

[ZFsub **lemma** RemoveDoubleNeg: $\Pi\mathcal{A}$: $\dot{\neg}\dot{\neg}\mathcal{A} \vdash \mathcal{A}$]

ZFsub **proof of** RemoveDoubleNeg:
L01: Arbitrary $\gg$ $\mathcal{A}$;
L02: Premise $\gg$ $\dot{\neg}\dot{\neg}\mathcal{A}$;
L03: Weakening $\rhd$ L02 $\gg$ $\dot{\neg}\mathcal{A} \Rightarrow \dot{\neg}\dot{\neg}\mathcal{A}$;
L04: AutoImply $\gg$ $\dot{\neg}\mathcal{A} \Rightarrow \dot{\neg}\mathcal{A}$;
L05: Neg $\rhd$ L04 $\rhd$ L03 $\gg$ $\mathcal{A}$ □

[ZFsub **lemma** AddDoubleNeg: $\Pi\mathcal{A}$: $\mathcal{A} \vdash \dot{\neg}\dot{\neg}\mathcal{A}$]

ZFsub **proof of** AddDoubleNeg:
L01: Block $\gg$ Begin ;
L02: Arbitrary $\gg$ $\mathcal{A}$;
L03: Premise $\gg$ $\dot{\neg}\dot{\neg}\dot{\neg}\mathcal{A}$;
L04: RemoveDoubleNeg $\rhd$ L03 $\gg$ $\dot{\neg}\mathcal{A}$;
L05: Block $\gg$ End ;
L06: Arbitrary $\gg$ $\mathcal{A}$;
L07: Ded $\rhd$ L05 $\gg$ $\dot{\neg}\dot{\neg}\dot{\neg}\mathcal{A} \Rightarrow \dot{\neg}\mathcal{A}$;
L08: Premise $\gg$ $\mathcal{A}$;
L09: Weakening $\rhd$ L08 $\gg$ $\dot{\neg}\dot{\neg}\dot{\neg}\mathcal{A} \Rightarrow \mathcal{A}$;
L10: Neg $\rhd$ L09 $\rhd$ L07 $\gg$ $\dot{\neg}\dot{\neg}\mathcal{A}$ □

3.4 Modus tollens og beslægtede lemmaer

Hovedresultatet fra dette afsnit er slutningsreglen modus tollens, bevist som et lemma:

[ZFsub **lemma** MT: $\Pi\mathcal{A},\mathcal{B}$: $\mathcal{A} \Rightarrow \mathcal{B} \vdash \dot{\neg}\mathcal{B} \vdash \dot{\neg}\mathcal{A}$]

For at vise MT begynder vi med et teknisk lemma, der ikke har den store værdi i sig selv:

[ZFsub **lemma** Technicality: $\Pi\mathcal{A},\mathcal{B}$: $\mathcal{A} \Rightarrow \mathcal{B} \vdash \dot{\neg}\dot{\neg}\mathcal{A} \Rightarrow \mathcal{B}$]

ZFsub **proof of** Technicality:
L01: Block $\gg$ Begin ;
L02: Arbitrary $\gg$ $\mathcal{A},\mathcal{B}$;
L03: Premise $\gg$ $\mathcal{A} \Rightarrow \mathcal{B}$;
L04: Premise $\gg$ $\dot{\neg}\dot{\neg}\mathcal{A}$;

L05:	RemoveDoubleNeg $\rhd$ L04 $\gg$	$\quad\mathcal{A}$	;
L06:	MP $\rhd$ L03 $\rhd$ L05 $\gg$	$\quad\mathcal{B}$	;
L07:	Block $\gg$	End	;
L08:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}$	;
L09:	Ded $\rhd$ L07 $\gg$	$(\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow \dot{\neg}\,\dot{\neg}\,\mathcal{A} \Rightarrow \mathcal{B}$	;
L10:	Premise $\gg$	$\mathcal{A} \Rightarrow \mathcal{B}$	;
L11:	MP $\rhd$ L09 $\rhd$ L10 $\gg$	$\dot{\neg}\,\dot{\neg}\,\mathcal{A} \Rightarrow \mathcal{B}$	$\Box$

Uafhængigt af Technicality kan vi vise en version af MT, hvor $\mathcal{A}$ optræder i negeret form:

[ZFsub **lemma** NegativeMT: $\Pi \mathcal{A}, \mathcal{B}\colon \dot{\neg}\,\mathcal{A} \Rightarrow \mathcal{B} \vdash \dot{\neg}\,\mathcal{B} \vdash \mathcal{A}$]

ZFsub **proof of** NegativeMT:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}$	;
L02:	Premise $\gg$	$\dot{\neg}\,\mathcal{A} \Rightarrow \mathcal{B}$	;
L03:	Premise $\gg$	$\dot{\neg}\,\mathcal{B}$	;
L04:	Weakening $\rhd$ L03 $\gg$	$\dot{\neg}\,\mathcal{A} \Rightarrow \dot{\neg}\,\mathcal{B}$	;
L05:	Neg $\rhd$ L02 $\rhd$ L04 $\gg$	$\mathcal{A}$	$\Box$

Ud fra Technicality og NegativeMT kan vi nu vise MT:

[ZFsub **lemma** MT: $\Pi \mathcal{A}, \mathcal{B}\colon \mathcal{A} \Rightarrow \mathcal{B} \vdash \dot{\neg}\,\mathcal{B} \vdash \dot{\neg}\,\mathcal{A}$]

ZFsub **proof of** MT:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}$	;
L02:	Premise $\gg$	$\mathcal{A} \Rightarrow \mathcal{B}$	;
L03:	Premise $\gg$	$\dot{\neg}\,\mathcal{B}$	;
L04:	Technicality $\gg$	$\dot{\neg}\,\dot{\neg}\,\mathcal{A} \Rightarrow \mathcal{B}$	;
L05:	NegativeMT $\rhd$ L04 $\rhd$ L03 $\gg$	$\dot{\neg}\,\mathcal{A}$	$\Box$

Vi slutter dette underafsnit med en variant af MT, som erstatter en inferens med en implikation:

[ZFsub **lemma** Contrapositive: $\Pi \mathcal{A}, \mathcal{B}\colon \mathcal{A} \Rightarrow \mathcal{B} \vdash \dot{\neg}\,\mathcal{B} \Rightarrow \dot{\neg}\,\mathcal{A}$]

Når en inferens skal erstattes med en implikation, er det altid deduktionsreglen, der skal i spil:

ZFsub **proof of** Contrapositive:

L01:	Block $\gg$	Begin	;
L02:	Arbitrary $\gg$	$\quad\mathcal{A}, \mathcal{B}$	;
L03:	Premise $\gg$	$\quad\mathcal{A} \Rightarrow \mathcal{B}$	;
L04:	Premise $\gg$	$\quad\dot{\neg}\,\mathcal{B}$	;
L05:	MT $\rhd$ L03 $\rhd$ L04 $\gg$	$\quad\dot{\neg}\,\mathcal{A}$	;
L06:	Block $\gg$	End	;
L07:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}$	;
L08:	Premise $\gg$	$\mathcal{A} \Rightarrow \mathcal{B}$	;

L09:	Ded ▷ L06 ≫	$(\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow \dot\neg\,\mathcal{B} \Rightarrow \dot\neg\,\mathcal{A}$	;
L10:	MP ▷ L09 ▷ L08 ≫	$\dot\neg\,\mathcal{B} \Rightarrow \dot\neg\,\mathcal{A}$	□

3.5 Konjunktion

Hovedmålet med dette underafsnit er at konvertere mellem formlerne $\mathcal{A}$ og $\mathcal{B}$ og deres konjunktion $[\mathcal{A} \dot\wedge \mathcal{B}]$.

3.5.1 Forening af konjunkter

Vi begynder med at slå $\mathcal{A}$ og $\mathcal{B}$ sammen til $[\mathcal{A} \dot\wedge \mathcal{B}]$:

[ZFsub **lemma** JoinConjuncts: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A} \vdash \mathcal{B} \vdash \mathcal{A} \dot\wedge \mathcal{B}$]

Beviset for JoinConjuncts er af teknisk karakter. Vi viser den makroekspanderede form $[\dot\neg\,(\mathcal{A} \Rightarrow \dot\neg\,\mathcal{B})]$, som vi i bevisets sidste linie konverterer til $[\mathcal{A} \dot\wedge \mathcal{B}]$. Denne sidste linie er ikke nødvendig for bevischeckeren, men den gør beviset lidt nemmere at læse:

ZFsub **proof of** JoinConjuncts:

L01:	Block ≫	Begin	;
L02:	Arbitrary ≫	$\quad\mathcal{A},\mathcal{B}$	;
L03:	Premise ≫	$\quad\mathcal{A}$	;
L04:	Premise ≫	$\quad\mathcal{A} \Rightarrow \dot\neg\,\mathcal{B}$	;
L05:	MP ▷ L04 ▷ L03 ≫	$\quad\dot\neg\,\mathcal{B}$	;
L06:	Block ≫	End	;
L07:	Arbitrary ≫	$\mathcal{A},\mathcal{B}$	;
L08:	Ded ▷ L06 ≫	$\mathcal{A} \Rightarrow (\mathcal{A} \Rightarrow \dot\neg\,\mathcal{B}) \Rightarrow \dot\neg\,\mathcal{B}$	;
L09:	Premise ≫	$\mathcal{A}$	;
L10:	Premise ≫	$\mathcal{B}$	;
L11:	MP ▷ L08 ▷ L09 ≫	$(\mathcal{A} \Rightarrow \dot\neg\,\mathcal{B}) \Rightarrow \dot\neg\,\mathcal{B}$	;
L12:	AddDoubleNeg ▷ L10 ≫	$\dot\neg\,\dot\neg\,\mathcal{B}$	;
L13:	MT ▷ L11 ▷ L12 ≫	$\dot\neg\,(\mathcal{A} \Rightarrow \dot\neg\,\mathcal{B})$	;
L14:	Repetition ▷ L13 ≫	$\mathcal{A} \dot\wedge \mathcal{B}$	□

3.5.2 Udskilning af anden konjunkt

Tautologien SecondConjunct lader os udskille den anden konjunkt fra $[\mathcal{A} \dot\wedge \mathcal{B}]$. Jeg vil ikke kommentere beviset:

[ZFsub **lemma** SecondConjunct: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A} \dot\wedge \mathcal{B} \vdash \mathcal{B}$]

ZFsub **proof of** SecondConjunct:

L01:	Block ≫	Begin	;
L02:	Arbitrary ≫	$\quad\mathcal{A},\mathcal{B}$	;
L03:	Premise ≫	$\quad\dot\neg\,\mathcal{B}$	;
L04:	Weakening ▷ L03 ≫	$\quad\mathcal{A} \Rightarrow \dot\neg\,\mathcal{B}$	;
L05:	Block ≫	End	;

L06:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}$	;
L07:	Ded $\rhd$ L05 $\gg$	$\dot{\neg}\,\mathcal{B} \Rightarrow \mathcal{A} \Rightarrow \dot{\neg}\,\mathcal{B}$	;
L08:	Premise $\gg$	$\mathcal{A} \mathbin{\dot{\wedge}} \mathcal{B}$	;
L09:	Repetition $\rhd$ L08 $\gg$	$\dot{\neg}\,(\mathcal{A} \Rightarrow \dot{\neg}\,\mathcal{B})$	;
L10:	NegativeMT $\rhd$ L07 $\rhd$ L09 $\gg$	$\mathcal{B}$	$\Box$

3.5.3 Udskilning af første konjunkt

For at udskille $\mathcal{A}$ fra $[\mathcal{A} \mathbin{\dot{\wedge}} \mathcal{B}]$ viser vi først, at $[\mathcal{A} \mathbin{\dot{\wedge}} \mathcal{B}]$ er kommutativ. Jeg vil ikke kommentere beviset:

[ZFsub **lemma** AndCommutativity: $\Pi \mathcal{A}, \mathcal{B}\colon \mathcal{A} \mathbin{\dot{\wedge}} \mathcal{B} \vdash \mathcal{B} \mathbin{\dot{\wedge}} \mathcal{A}$]

ZFsub **proof of** AndCommutativity:

L01:	Block $\gg$	Begin	;
L02:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}$	;
L03:	Premise $\gg$	$\mathcal{B} \Rightarrow \dot{\neg}\,\mathcal{A}$	;
L04:	Premise $\gg$	$\mathcal{A}$	;
L05:	AddDoubleNeg $\rhd$ L04 $\gg$	$\dot{\neg}\,\dot{\neg}\,\mathcal{A}$	;
L06:	MT $\rhd$ L03 $\rhd$ L05 $\gg$	$\dot{\neg}\,\mathcal{B}$	;
L07:	Block $\gg$	End	;
L08:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}$	;
L09:	Ded $\rhd$ L07 $\gg$	$(\mathcal{B} \Rightarrow \dot{\neg}\,\mathcal{A}) \Rightarrow \mathcal{A} \Rightarrow \dot{\neg}\,\mathcal{B}$	;
L10:	Premise $\gg$	$\mathcal{A} \mathbin{\dot{\wedge}} \mathcal{B}$	;
L11:	Repetition $\gg$	$\dot{\neg}\,(\mathcal{A} \Rightarrow \dot{\neg}\,\mathcal{B})$	;
L12:	MT $\rhd$ L09 $\rhd$ L11 $\gg$	$\dot{\neg}\,(\mathcal{B} \Rightarrow \dot{\neg}\,\mathcal{A})$	;
L13:	Repetition $\rhd$ L12 $\gg$	$\mathcal{B} \mathbin{\dot{\wedge}} \mathcal{A}$	$\Box$

Nu er det let at udskille den første konjunkt fra $[\mathcal{A} \mathbin{\dot{\wedge}} \mathcal{B}]$: Først vender vi konjunktionen om til $[\mathcal{B} \mathbin{\dot{\wedge}} \mathcal{A}]$ ved hjælp af AndCommutativity, og så udskiller vi $\mathcal{A}$ ved hjælp af SecondConjunct:

[ZFsub **lemma** FirstConjunct: $\Pi \mathcal{A}, \mathcal{B}\colon \mathcal{A} \mathbin{\dot{\wedge}} \mathcal{B} \vdash \mathcal{A}$]

ZFsub **proof of** FirstConjunct:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}$	;
L02:	Premise $\gg$	$\mathcal{A} \mathbin{\dot{\wedge}} \mathcal{B}$	;
L03:	AndCommutativity $\rhd$ L02 $\gg$	$\mathcal{B} \mathbin{\dot{\wedge}} \mathcal{A}$	;
L04:	SecondConjunct $\rhd$ L03 $\gg$	$\mathcal{A}$	$\Box$

3.6 Dobbeltimplikation

I dette underafsnit viser vi tre enkle resultater vedr. dobbeltimplikation.

3.6.1 Brug sammen med modus ponens

De følgende to tautologier gør det let at bruge anvende slutningsreglen MP på dobbeltimplikationer. Beviserne er enkle og kræver ingen kommentarer:

[ZFsub **lemma** IffFirst: $\Pi \mathcal{A}, \mathcal{B}: \mathcal{A} \dot{\Leftrightarrow} \mathcal{B} \vdash \mathcal{B} \vdash \mathcal{A}$]

ZFsub **proof of** IffFirst:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}$	;
L02:	Premise $\gg$	$\mathcal{A} \dot{\Leftrightarrow} \mathcal{B}$	;
L03:	Premise $\gg$	$\mathcal{B}$	;
L04:	SecondConjunct $\rhd$ L02 $\gg$	$\mathcal{B} \Rightarrow \mathcal{A}$	;
L05:	MP $\rhd$ L04 $\rhd$ L03 $\gg$	$\mathcal{A}$	$\square$

[ZFsub **lemma** IffSecond: $\Pi \mathcal{A}, \mathcal{B}: \mathcal{A} \dot{\Leftrightarrow} \mathcal{B} \vdash \mathcal{A} \vdash \mathcal{B}$]

ZFsub **proof of** IffSecond:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}$	;
L02:	Premise $\gg$	$\mathcal{A} \dot{\Leftrightarrow} \mathcal{B}$	;
L03:	Premise $\gg$	$\mathcal{A}$	;
L04:	FirstConjunct $\rhd$ L02 $\gg$	$\mathcal{A} \Rightarrow \mathcal{B}$	;
L05:	MP $\rhd$ L04 $\rhd$ L03 $\gg$	$\mathcal{B}$	$\square$

3.6.2 Kommutativitet

Lemmaet IffCommutativity følger direkte af, at operatoren [$\mathsf{x} \dot{\wedge} \mathsf{y}$] er kommutativ:

[ZFsub **lemma** IffCommutativity: $\Pi \mathcal{A}, \mathcal{B}: \mathcal{A} \dot{\Leftrightarrow} \mathcal{B} \vdash \mathcal{B} \dot{\Leftrightarrow} \mathcal{A}$]

ZFsub **proof of** IffCommutativity:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}$	;
L02:	Premise $\gg$	$\mathcal{A} \dot{\Leftrightarrow} \mathcal{B}$	;
L03:	Repetition $\rhd$ L02 $\gg$	$(\mathcal{A} \Rightarrow \mathcal{B}) \dot{\wedge} (\mathcal{B} \Rightarrow \mathcal{A})$	;
L04:	AndCommutativity $\rhd$ L03 $\gg$	$(\mathcal{B} \Rightarrow \mathcal{A}) \dot{\wedge} (\mathcal{A} \Rightarrow \mathcal{B})$	;
L05:	Repetition $\rhd$ L04 $\gg$	$\mathcal{B} \dot{\Leftrightarrow} \mathcal{A}$	$\square$

3.7 Disjunktion

Dette underafsnit indeholder tre lemmaer vedr. disjunktion, som vi fordeler på to under-underafsnit.

3.7.1 Svækkelse

Givet en påstand $\mathcal{B}$ vil vi gerne udlede de svagere påstande [$\mathcal{A} \dot{\vee} \mathcal{B}$] og [$\mathcal{B} \dot{\vee} \mathcal{A}$]. Den første slutning varetages af lemmaet WeakenOr1:

[ZFsub **lemma** WeakenOr1: $\Pi \mathcal{A}, \mathcal{B}: \mathcal{B} \vdash \mathcal{A} \dot{\vee} \mathcal{B}$]

Beviset består af en simpel anvendelse af Weakening:

ZFsub **proof of** WeakenOr1:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}$	;
L02:	Premise $\gg$	$\mathcal{B}$	;

L03:	Weakening ⊳ L02 ≫	$\dot{\neg}\,\mathcal{A} \Rightarrow \mathcal{B}$	;
L04:	Repetition ⊳ L03 ≫	$\mathcal{A} \dot{\vee} \mathcal{B}$	□

Slutningen fra $\mathcal{A}$ til $[\mathcal{A} \dot{\vee} \mathcal{B}]$ varetages af lemmaet WeakenOr2:

[ZFsub **lemma** WeakenOr2: $\Pi \mathcal{A}, \mathcal{B} \colon \mathcal{A} \vdash \mathcal{A} \dot{\vee} \mathcal{B}$]

Kernen i beviset for WeakenOr2 er en anvendelse af FromContradiction:

ZFsub **proof of** WeakenOr2:

L01:	Block ≫	Begin	;
L02:	Arbitrary ≫	$\mathcal{A}, \mathcal{B}$	;
L03:	Premise ≫	$\mathcal{A}$	;
L04:	Premise ≫	$\dot{\neg}\,\mathcal{A}$	;
L05:	FromContradiction ⊳ L03 ⊳ L04 ≫	$\mathcal{B}$	;
L06:	Block ≫	End	;
L07:	Arbitrary ≫	$\mathcal{A}, \mathcal{B}$	;
L08:	Ded ⊳ L06 ≫	$\mathcal{A} \Rightarrow \dot{\neg}\,\mathcal{A} \Rightarrow \mathcal{B}$	;
L09:	Premise ≫	$\mathcal{A}$	;
L10:	MP ⊳ L08 ⊳ L09 ≫	$\dot{\neg}\,\mathcal{A} \Rightarrow \mathcal{B}$	;
L11:	Repetition ⊳ L10 ≫	$\mathcal{A} \dot{\vee} \mathcal{B}$	□

3.7.2 Slutning ud fra disjunktion

Lemmaet FromDisjuncts lader os drage slutninger ud fra en disjunktion:

[ZFsub **lemma** FromDisjuncts: $\Pi \mathcal{A}, \mathcal{B}, \mathcal{C} \colon \mathcal{A} \dot{\vee} \mathcal{B} \vdash \mathcal{A} \Rightarrow \mathcal{C} \vdash \mathcal{B} \Rightarrow \mathcal{C} \vdash \mathcal{C}$]

Om beviset vil jeg kun sige, at det er en ret elegant øvelse i bevisteknik:

ZFsub **proof of** FromDisjuncts:

L01:	Arbitrary ≫	$\mathcal{A}, \mathcal{B}, \mathcal{C}$	;
L02:	Premise ≫	$\mathcal{A} \dot{\vee} \mathcal{B}$	;
L03:	Premise ≫	$\mathcal{A} \Rightarrow \mathcal{C}$	;
L04:	Premise ≫	$\mathcal{B} \Rightarrow \mathcal{C}$	;
L05:	Repetition ⊳ L02 ≫	$\dot{\neg}\,\mathcal{A} \Rightarrow \mathcal{B}$	;
L06:	Contrapositive ⊳ L05 ≫	$\dot{\neg}\,\mathcal{B} \Rightarrow \dot{\neg}\,\dot{\neg}\,\mathcal{A}$	;
L07:	Technicality ⊳ L03 ≫	$\dot{\neg}\,\dot{\neg}\,\mathcal{A} \Rightarrow \mathcal{C}$	;
L08:	ImplyTransitivity⊳L06⊳L07 ≫	$\dot{\neg}\,\mathcal{B} \Rightarrow \mathcal{C}$	;
L09:	Contrapositive ⊳ L08 ≫	$\dot{\neg}\,\mathcal{C} \Rightarrow \dot{\neg}\,\dot{\neg}\,\mathcal{B}$	;
L10:	Contrapositive ⊳ L04 ≫	$\dot{\neg}\,\mathcal{C} \Rightarrow \dot{\neg}\,\mathcal{B}$	;
L11:	Neg ⊳ L10 ⊳ L09 ≫	$\mathcal{C}$	□

(*******************)

Priority table
Preassociative

[am], [base], [bracket $*$ end bracket], [big bracket $*$ end bracket], [\$ $*$ \$],
[**flush left** [$*$]], [x], [y], [z], [[$* \bowtie *$]], [[$* \stackrel{*}{\rightarrow} *$]], [pyk], [tex], [name], [prio], [$*$], [T],
[if($*,*,*$)], [[$* \stackrel{*}{\Rightarrow} *$]], [val], [claim], [$\perp$], [f($*$)], [$(*)^{\mathrm{I}}$], [F], [$\underline{0}$], [$\underline{1}$], [$\underline{2}$], [$\underline{3}$], [$\underline{4}$], [$\underline{5}$], [$\underline{6}$],
[$\underline{7}$], [$\underline{8}$], [$\underline{9}$], [0], [1], [2], [3], [4], [5], [6], [7], [8], [9], [a], [b], [c], [d], [e], [f], [g], [h], [i], [j],
[k], [l], [m], [n], [o], [p], [q], [r], [s], [t], [u], [v], [w], [$(*)^{\mathrm{M}}$], [If($*,*,*$)],
[array{$*$} $*$ end array], [l], [c], [r], [empty], [$\langle * \mid * := * \rangle$], [$\mathcal{M}(*)$], [$\tilde{\mathcal{U}}(*)$], [$\mathcal{U}(*)$],
[$\mathcal{U}^{\mathrm{M}}(*)$], [**apply**($*,*$)], [$\mathbf{apply}_1(*,*)$], [identifier($*$)], [identifier$_1$($*,*$)], [array-plus($*,*$)], [array-remove($*,*,*$)], [array-put($*,*,*,*$)], [array-add($*,*,*,*,*$)],
[bit($*,*$)], [bit$_1$($*,*$)], [rack], ["vector"], ["bibliography"], ["dictionary"],
["body"], ["codex"], ["expansion"], ["code"], ["cache"], ["diagnose"], ["pyk"],
["tex"], ["texname"], ["value"], ["message"], ["macro"], ["definition"],
["unpack"], ["claim"], ["priority"], ["lambda"], ["apply"], ["true"], ["if"],
["quote"], ["proclaim"], ["define"], ["introduce"], ["hide"], ["pre"], ["post"],
[$\mathcal{E}(*,*,*)$], [$\mathcal{E}_2(*,*,*,*,*)$], [$\mathcal{E}_3(*,*,*,*)$], [$\mathcal{E}_4(*,*,*,*)$], [**lookup**($*,*,*$)],
[**abstract**($*,*,*,*$)], [[$\lceil * \rceil$]], [$\mathcal{M}(*,*,*)$], [$\mathcal{M}_2(*,*,*,*)$], [$\mathcal{M}^*(*,*,*)$], [macro],
[s_0], [**zip**($*,*$)], [$\mathbf{assoc}_1(*,*,*)$], [$(*)^{\mathsf{P}}$], [self], [[$* \ddot{=} *$]], [[$* \doteq *$]], [[$* \acute{=} *$]],
[[$* \stackrel{\mathrm{pyk}}{=} *$]], [[$* \stackrel{\mathrm{tex}}{=} *$]], [[$* \stackrel{\mathrm{name}}{=} *$]], [**Priority table**[$*$]], [$\tilde{\mathcal{M}}_1$], [$\tilde{\mathcal{M}}_2(*)$], [$\tilde{\mathcal{M}}_3(*)$],
[$\tilde{\mathcal{M}}_4(*,*,*,*)$], [$\tilde{\mathcal{M}}(*,*,*)$], [$\tilde{\mathcal{Q}}(*,*,*)$], [$\tilde{\mathcal{Q}}_2(*,*,*)$], [$\tilde{\mathcal{Q}}_3(*,*,*,*)$], [$\tilde{\mathcal{Q}}^*(*,*,*)$],
[$(*)$], [$(*)$], [display($*$)], [statement($*$)], [$[*]^{\cdot}$], [$[*]^{-}$], [**aspect**($*,*$)],
[**aspect**($*,*,*$)], [$\langle * \rangle$], [$\mathbf{tuple}_1(*)$], [$\mathbf{tuple}_2(*)$], [let$_2$($*,*$)], [let$_1$($*,*$)],
[[$* \stackrel{\mathrm{claim}}{=} *$]], [checker], [**check**($*,*$)], [$\mathbf{check}_2(*,*,*)$], [$\mathbf{check}_3(*,*,*)$],
[$\mathbf{check}^*(*,*)$], [$\mathbf{check}^*_2(*,*,*)$], [$[*]^{\cdot}$], [$[*]^{-}$], [$[*]^{\circ}$], [msg], [[$* \stackrel{\mathrm{msg}}{=} *$]], [<stmt>],
[stmt], [[$* \stackrel{\mathrm{stmt}}{=} *$]], [HeadNil′], [HeadPair′], [Transitivity′], [$\perp\!\!\!\perp$], [Contra′], [T'_{E}],
[L_1], [$\underline{*}$], [$\mathcal{A}$], [$\mathcal{B}$], [$\mathcal{C}$], [$\mathcal{D}$], [$\mathcal{E}$], [$\mathcal{F}$], [$\mathcal{G}$], [$\mathcal{H}$], [$\mathcal{I}$], [$\mathcal{J}$], [$\mathcal{K}$], [$\mathcal{L}$], [$\mathcal{M}$], [$\mathcal{N}$], [$\mathcal{O}$], [$\mathcal{P}$], [$\mathcal{Q}$],
[$\mathcal{R}$], [$\mathcal{S}$], [$\mathcal{T}$], [$\mathcal{U}$], [$\mathcal{V}$], [$\mathcal{W}$], [$\mathcal{X}$], [$\mathcal{Y}$], [$\mathcal{Z}$], [$\langle * \mid * := * \rangle$], [$\langle^* * \mid * := * \rangle$], [$\emptyset$], [Remainder],
[$(*)^{\mathsf{V}}$], [intro($*,*,*,*$)], [intro($*,*,*$)], [error($*,*$)], [error$_2$($*,*$)], [proof($*,*,*$)],
[proof$_2$($*,*$)], [$\mathcal{S}(*,*)$], [$\mathcal{S}^{\mathrm{I}}(*,*)$], [$\mathcal{S}^{\rhd}(*,*)$], [$\mathcal{S}^{\rhd}_1(*,*,*)$], [$\mathcal{S}^{\mathrm{E}}(*,*)$], [$\mathcal{S}^{\mathrm{E}}_1(*,*,*)$],
[$\mathcal{S}^{+}(*,*)$], [$\mathcal{S}^{+}_1(*,*,*)$], [$\mathcal{S}^{-}(*,*)$], [$\mathcal{S}^{-}_1(*,*,*)$], [$\mathcal{S}^{*}(*,*)$], [$\mathcal{S}^{*}_1(*,*,*)$],
[$\mathcal{S}^{*}_2(*,*,*,*)$], [$\mathcal{S}^{@}(*,*)$], [$\mathcal{S}^{@}_1(*,*,*)$], [$\mathcal{S}^{\vdash}(*,*)$], [$\mathcal{S}^{\vdash}_1(*,*,*,*)$], [$\mathcal{S}^{\Vdash}(*,*)$],
[$\mathcal{S}^{\Vdash}_1(*,*,*,*)$], [$\mathcal{S}^{\mathrm{i.e.}}(*,*)$], [$\mathcal{S}^{\mathrm{i.e.}}_1(*,*,*,*)$], [$\mathcal{S}^{\mathrm{i.e.}}_2(*,*,*,*,*)$], [$\mathcal{S}^{\forall}(*,*)$],
[$\mathcal{S}^{\forall}_1(*,*,*,*)$], [$\mathcal{S}^{;}(*,*)$], [$\mathcal{S}^{;}_1(*,*,*)$], [$\mathcal{S}^{;}_2(*,*,*,*)$], [$\mathcal{T}(*)$], [claims($*,*,*$)],
[claims$_2$($*,*,*$)], [<proof>], [proof], [[**Lemma** $*: *$]], [[**Proof of** $*: *$]],
[[$*$ **lemma** $*: *$]], [[$*$ **antilemma** $*: *$]], [[$*$ **rule** $*: *$]], [[$*$ **antirule** $*: *$]],
[verifier], [$\mathcal{V}_1(*)$], [$\mathcal{V}_2(*,*)$], [$\mathcal{V}_3(*,*,*,*)$], [$\mathcal{V}_4(*,*)$], [$\mathcal{V}_5(*,*,*,*)$], [$\mathcal{V}_6(*,*,*,*)$],
[$\mathcal{V}_7(*,*,*,*)$], [Cut($*,*$)], [$\mathrm{Head}_\oplus(*)$], [$\mathrm{Tail}_\oplus(*)$], [rule$_1$($*,*$)], [rule($*,*$)],
[Rule tactic], [Plus($*,*$)], [[**Theory** $*$]], [theory$_2$($*,*$)], [theory$_3$($*,*$)],
[theory$_4$($*,*,*$)], [HeadNil″], [HeadPair″], [Transitivity″], [Contra″], [HeadNil],
[HeadPair], [Transitivity], [Contra], [T_{E}], [ragged right],
[ragged right expansion], [parm($*,*,*$)], [parm*($*,*,*$)], [inst($*,*$)],
[inst*($*,*$)], [occur($*,*,*$)], [occur*($*,*,*$)], [unify($* = *, *$)], [unify*($* = *, *$)],
[unify$_2$($* = *, *$)], [$\mathrm{L_a}$], [$\mathrm{L_b}$], [$\mathrm{L_c}$], [$\mathrm{L_d}$], [$\mathrm{L_e}$], [$\mathrm{L_f}$], [$\mathrm{L_g}$], [$\mathrm{L_h}$], [$\mathrm{L_i}$], [$\mathrm{L_j}$], [$\mathrm{L_k}$], [$\mathrm{L_l}$], [$\mathrm{L_m}$],
[$\mathrm{L_n}$], [$\mathrm{L_o}$], [$\mathrm{L_p}$], [$\mathrm{L_q}$], [$\mathrm{L_r}$], [$\mathrm{L_s}$], [$\mathrm{L_t}$], [$\mathrm{L_u}$], [$\mathrm{L_v}$], [$\mathrm{L_w}$], [$\mathrm{L_x}$], [$\mathrm{L_y}$], [$\mathrm{L_z}$], [$\mathrm{L_A}$], [$\mathrm{L_B}$], [$\mathrm{L_C}$],
[$\mathrm{L_D}$], [$\mathrm{L_E}$], [$\mathrm{L_F}$], [$\mathrm{L_G}$], [$\mathrm{L_H}$], [$\mathrm{L_I}$], [$\mathrm{L_J}$], [$\mathrm{L_K}$], [$\mathrm{L_L}$], [$\mathrm{L_M}$], [$\mathrm{L_N}$], [$\mathrm{L_O}$], [$\mathrm{L_P}$], [$\mathrm{L_Q}$], [$\mathrm{L_R}$],
[$\mathrm{L_S}$], [$\mathrm{L_T}$], [$\mathrm{L_U}$], [$\mathrm{L_V}$], [$\mathrm{L_W}$], [$\mathrm{L_X}$], [$\mathrm{L_Y}$], [$\mathrm{L_Z}$], [$\mathrm{L_?}$], [Reflexivity], [Reflexivity$_1$],

[Commutativity], [Commutativity$_1$], [<tactic>], [tactic], [[$* \stackrel{\text{tactic}}{=} *$]], [$\mathcal{P}(*,*,*)$], [$\mathcal{P}^*(*,*,*)$], [$\text{p}_0$], [conclude$_1(*,*)$], [conclude$_2(*,*,*)$], [conclude$_3(*,*,*,*)$], [conclude$_4(*,*)$], [check], [[$* \stackrel{\circ}{=} *$]], [RootVisible($*$)], [A], [R], [C], [T], [L], [$\{*\}$], [$\overline{*}$], [a], [b], [c], [d], [e], [f], [g], [h], [i], [j], [k], [l], [m], [n], [o], [p], [q], [r], [s], [t], [u], [v], [w], [x], [y], [z], [$\langle * \equiv * \mid * := * \rangle$], [$\langle * \equiv^0 * \mid * := * \rangle$], [$\langle * \equiv^1 * \mid * := * \rangle$], [$\langle * \equiv^* * \mid * := * \rangle$], [Ded$(*,*)$], [Ded$_0(*,*)$], [Ded$_1(*,*,*)$], [Ded$_2(*,*,*)$], [Ded$_3(*,*,*,*)$], [Ded$_4(*,*,*,*)$], [Ded$^*_4(*,*,*,*)$], [Ded$_5(*,*,*)$], [Ded$_6(*,*,*,*)$], [Ded$^*_6(*,*,*,*)$], [Ded$_7(*)$], [Ded$_8(*,*)$], [Ded$^*_8(*,*)$], [S], [Neg], [MP], [Gen], [Ded], [S1], [S2], [S3], [S4], [S5], [S6], [S7], [S8], [S9], [Repetition], [A1$'$], [A2$'$], [A4$'$], [A5$'$], [Prop 3.2a], [Prop 3.2b], [Prop 3.2c], [Prop 3.2d], [Prop 3.2e$_1$], [Prop 3.2e$_2$], [Prop 3.2e], [Prop 3.2f$_1$], [Prop 3.2f$_2$], [Prop 3.2f], [Prop 3.2g$_1$], [Prop 3.2g$_2$], [Prop 3.2g], [Prop 3.2h$_1$], [Prop 3.2h$_2$], [Prop 3.2h], [Block$_1(*,*,*)$], [Block$_2(*)$], [$(\cdots)$], [`Objekt-var`], [`Ex-var`], [`Ph-var`], [`Værdi`], [`Variabel`], [Op$(*)$], [Op$(*,*)$], [$* \stackrel{..}{==} *$], [ContainsEmpty$(*)$], [Dedu$(*,*)$], [Dedu$_0(*,*)$], [Dedu$_\text{s}(*,*,*)$], [Dedu$_1(*,*,*)$], [Dedu$_2(*,*,*)$], [Dedu$_3(*,*,*,*)$], [Dedu$_4(*,*,*,*)$], [Dedu$^*_4(*,*,*,*)$], [Dedu$_5(*,*,*)$], [Dedu$_6(*,*,*,*)$], [Dedu$^*_6(*,*,*,*)$], [Dedu$_7(*)$], [Dedu$_8(*,*)$], [Dedu$^*_8(*,*)$], [Ex$_1$], [Ex$_2$], [Ex3], [Ex$_{10}$], [Ex$_{20}$], [$*_{\text{Ex}}$], [$*^{\text{Ex}}$], [$\langle * \equiv * \mid * :== * \rangle_{\text{Ex}}$], [$\langle * \equiv^0 * \mid * :== * \rangle_{\text{Ex}}$], [$\langle * \equiv^1 * \mid * :== * \rangle_{\text{Ex}}$], [$\langle * \equiv^* * \mid * :== * \rangle_{\text{Ex}}$], [ph$_1$], [ph$_2$], [ph$_3$], [$*_{\text{Ph}}$], [$*^{\text{Ph}}$], [$\langle * \equiv * \mid * :== * \rangle_{\text{Ph}}$], [$\langle * \equiv^0 * \mid * :== * \rangle_{\text{Ph}}$], [$\langle * \equiv^1 * \mid * :== * \rangle_{\text{Ph}}$], [$\langle * \equiv^* * \mid * :== * \rangle_{\text{Ph}}$], [bs], [OBS], [$\mathcal{BS}$], [∅], [ZFsub], [MP], [Gen], [Repetition], [Neg], [Ded], [ExistIntro], [Extensionality], [∅def], [PairDef], [UnionDef], [PowerDef], [SeparationDef], [AddDoubleNeg], [RemoveDoubleNeg], [AndCommutativity], [AutoImply], [Contrapositive], [FirstConjunct], [SecondConjunct], [FromContradiction], [FromDisjuncts], [IffCommutativity], [IffFirst], [IffSecond], [ImplyTransitivity], [JoinConjuncts], [MP2], [MP3], [MP4], [MP5], [MT], [NegativeMT], [Technicality], [Weakening], [WeakenOr1], [WeakenOr2], [Formula2Pair], [Pair2Formula], [Formula2Union], [Union2Formula], [Formula2Sep], [Sep2Formula], [SubsetInPower], [HelperPowerIsSub], [PowerIsSub], [(Switch)HelperPowerIsSub], [(Switch)PowerIsSub], [ToSetEquality], [HelperToSetEquality(t)], [ToSetEquality(t)], [HelperFromSetEquality], [FromSetEquality], [HelperReflexivity], [Reflexivity], [HelperSymmetry], [Symmetry], [HelperTransitivity], [Transitivity], [ERisReflexive], [ERisSymmetric], [ERisTransitive], [∅isSubset], [HelperMemberNot∅], [MemberNot∅], [HelperUnique∅], [Unique∅], [==Reflexivity], [==Symmetry], [Helper==Transitivity], [==Transitivity], [HelperTransferNotEq], [TransferNotEq], [HelperPairSubset], [Helper(2)PairSubset], [PairSubset], [SamePair], [SameSingleton], [UnionSubset], [SameUnion], [SeparationSubset], [SameSeparation], [SameBinaryUnion], [IntersectionSubset], [SameIntersection], [AutoMember], [HelperEqSysNot∅], [EqSysNot∅], [HelperEqSubset], [EqSubset], [HelperEqNecessary], [EqNecessary], [HelperNoneEqNecessary], [Helper(2)NoneEqNecessary], [NoneEqNecessary], [EqClassIsSubset], [EqClassesAreDisjoint], [AllDisjoint], [AllDisjointImply], [BSsubset], [Union(BS/R)subset], [UnionIdentity], [EqSysIsPartition], [(ϵ)], [(fx)], [(fy)], [(fz)], [(fv)], [var fv], [(rx)], [(ry)], [(rz)], [(ru)], [ϵ], [FX], [FY], [FZ], [FU], [FV],

[RX], [RY], [RZ], [RU], [0], [1], [(−1)], [2], [1/2], [0f], [1f], [00], [01], [leqReflexivity], [leqAntisymmetryAxiom], [leqTransitivityAxiom], [leqTotality], [leqAdditionAxiom], [leqMultiplicationAxiom], [plusAssociativity], [plusCommutativity], [Negative], [plus0], [timesAssociativity], [timesCommutativity], [ReciprocalAxiom], [times1], [Distribution], [0not1], [equalityAxiom], [eqLeqAxiom], [eqAdditionAxiom], [eqMultiplicationAxiom], [SENC1], [SENC2], [IfThenElse(T)], [IfThenElse(F)], [From = f], [To = f], [From < f], [To < f], [PlusF], [TimesF], [MinusF], [0f], [1f], [FromSF], [ToSF], [To == XX], [From ==], [To ==], [From << XX], [From << (1)], [From << (2)], [to << XX], [From <<], [To <<], [FromInR], [PlusR], [TimesR], [leqAntisymmetry], [leqTransitivity], [leqAddition], [leqMultiplication], [Reciprocal], [Equality], [eqLeq], [eqAddition], [eqMultiplication], [ToNegatedImply], [TND], [ImplyNegation], [FromNegations], [From3Disjuncts], [From2 ∗ 2Disjuncts], [NegateDisjunct1], [NegateDisjunct2], [ExpandDisjuncts], [eqReflexivity], [eqSymmetry], [eqTransitivity], [eqTransitivity4], [eqTransitivity5], [eqTransitivity6], [plus0Left], [times1Left], [lemma eqAdditionLeft], [EqMultiplicationLeft], [DistributionOut], [Three2twoTerms], [Three2threeTerms], [Three2threeFactors], [AddEquations], [SubtractEquations], [SubtractEquationsLeft], [EqNegated], [PositiveToRight(Eq)], [PositiveToLeft(Eq)(1term)], [NegativeToLeft(Eq)], [LessNeq], [NeqSymmetry], [NeqNegated], [SubNeqRight], [SubNeqLeft], [NeqAddition], [NeqMultiplication], [UniqueNegative], [DoubleMinus], [LeqLessEq], [LessLeq], [FromLeqGeq], [subLeqRight], [subLeqLeft], [Leq + 1], [PositiveToRight(Leq)], [PositiveToRight(Leq)(1term)], [negativeToLeft(Leq)], [LeqAdditionLeft], [leqSubtraction], [leqSubtractionLeft], [thirdGeq], [LeqNegated], [AddEquations(Leq)], [ThirdGeqSeries], [LeqNeqLess], [FromLess], [ToLess], [fromNotLess], [toNotLess], [NegativeLessPositive], [leqLessTransitivity], [LessLeqTransitivity], [LessTransitivity], [LessTotality], [SubLessRight], [SubLessLeft], [LessAddition], [LessAdditionLeft], [LessMultiplication], [LessMultiplicationLeft], [LessDivision], [AddEquations(Less)], [LessNegated], [PositiveNegated], [NonpositiveNegated], [NegativeNegated], [NonnegativeNegated], [PositiveHalved], [NonnegativeNumerical], [NegativeNumerical], [PositiveNumerical], [lemma nonpositiveNumerical], [|0| = 0], [0 <= |x|], [SameNumerical], [SignNumerical(+)], [SignNumerical], [NumericalDifference], [SplitNumericalSumHelper], [splitNumericalSum(++)], [splitNumericalSum(−−)], [splitNumericalSum(+ − small)], [splitNumericalSum(+ − big)], [splitNumericalSum(+−)], [splitNumericalSum(−+)], [splitNumericalSum], [insertMiddleTerm(Numerical)], [x + y = zBackwards], [x ∗ y = zBackwards], [x = x + (y − y)], [x = x + y − y], [], [insertMiddleTerm(Sum)], [insertMiddleTerm(Difference)], [x ∗ 0 + x = x], [x ∗ 0 = 0], [(−1) ∗ (−1) + (−1) ∗ 1 = 0], [(−1) ∗ (−1) = 1], [0 < 1Helper], [0 < 1], [0 < 2], [0 < 1/2], [TwoWholes], [TwoHalves], [−x − y = −(x + y)], [MinusNegated], [Times(−1)], [Times(−1)Left], [−0 = 0], [SFsymmetry], [SFtransitivity], [= fToSameF], [PlusF(Sym)], [TimesF(Sym)], [f2R(Plus)], [f2R(Times)], [PlusR(Sym)], [TimesR(Sym)], [LessLeq(R)], [eqLeq(R)], [SubLessRight(R)], [SubLessLeft(R)], [<< TransitivityHelper(Q)], [<< Transitivity], [<<== Reflexivity], [<<== AntisymmetryHelper(Q)], [<<== Antisymmetry], [<<== Transitivity], [Plus0f], [Plus00], [== Addition], [== AdditionLeft], [<< Addition], [<<== Addition], [PlusAssociativity(F)], [PlusAssociativity(R)], [Negative(R)], [PlusCommutativity(F)], [PlusCommutativity(R)], [TimesAssociativity(F)], [TimesAssociativity(R)], [Times1f], [Times01], [TimesCommutativity(F)], [TimesCommutativity(R)], [Distribution(F)], [Distribution(R)];

Preassociative

[$*_{\{*\}}$], [$*$/indexintro($*, *, *, *$)], [$*$/intro($*, *, *$)], [$*$/bothintro($*, *, *, *, *$)], [$*$/nameintro($*, *, *, *$)], [$*'$], [$*[\,*\,]$], [$*[*\to*]$], [$*[*\Rightarrow*]$], [$*0$], [$*1$], [0b], [$*$-color($*$)], [$*$-color*($*$)], [$*^{\mathrm{H}}$], [$*^{\mathrm{T}}$], [$*^{\mathrm{U}}$], [$*^{\mathrm{h}}$], [$*^{\mathrm{t}}$], [$*^{\mathrm{s}}$], [$*^{\mathrm{c}}$], [$*^{\mathrm{d}}$], [$*^{\mathrm{a}}$], [$*^{\mathrm{C}}$], [$*^{\mathrm{M}}$], [$*^{\mathrm{B}}$], [$*^{\mathrm{r}}$], [$*^{\mathrm{i}}$], [$*^{\mathrm{d}}$], [$*^{\mathrm{R}}$], [$*^{0}$], [$*^{1}$], [$*^{2}$], [$*^{3}$], [$*^{4}$], [$*^{5}$], [$*^{6}$], [$*^{7}$], [$*^{8}$], [$*^{9}$], [$*^{\mathrm{E}}$], [$*^{\mathcal{V}}$], [$*^{\mathcal{C}}$], [$*^{\mathcal{C}^*}$], [$*^{\mathrm{hide}}$];

Preassociative

[" * "], [], [($*$)$^{\mathbf{t}}$], [string($*$) + $*$], [string($*$) ++ $*$], [
*], [*], [!*], ["*], [#*], [$*], [%*], [&*], ['*], [(*], [)*], [**], [+*], [,*], [-*], [.*], [/*], [0*], [1*], [2*], [3*], [4*], [5*], [6*], [7*], [8*], [9*], [:*], [;*], [<*], [=*], [>*], [?*], [@*], [A*], [B*], [C*], [D*], [E*], [F*], [G*], [H*], [I*], [J*], [K*], [L*], [M*], [N*], [O*], [P*], [Q*], [R*], [S*], [T*], [U*], [V*], [W*], [X*], [Y*], [Z*], [[*], [*], []*], [^*], [_*], [`*], [a*], [b*], [c*], [d*], [e*], [f*], [g*], [h*], [i*], [j*], [k*], [l*], [m*], [n*], [o*], [p*], [q*], [r*], [s*], [t*], [u*], [v*], [w*], [x*], [y*], [z*], [{*], [|*], [}*], [~*], [**Preassociative** *; *], [**Postassociative** *; *], [[*], *], [priority * end], [newline *], [macro newline *], [MacroIndent(*)];

Preassociative

[* ' *], [* ` *];

Preassociative

[$*'$], [R($*$)], [$--$ R($*$)], [rec$*$];

Preassociative

[$*/*$], [$* \cap *$], [$*[*]$];

Preassociative

[$\cup *$], [$* \cup *$], [P($*$)];

Preassociative

[$\{*\}$];

Preassociative

[$\{*, *\}$], [$\langle *, * \rangle$], [$-*$], [$-_{\mathrm{f}}*$];

Preassociative

[$* \in *$], [$*(*, *)$], [ReflRel($*, *$)], [SymRel($*, *$)], [TransRel($*, *$)], [EqRel($*, *$)], [$[* \in *]_*$], [Partition($*, *$)];

Preassociative

[$* \cdot *$], [$* \cdot_0 *$], [$* * *$], [$* *_{\mathrm{f}} *$], [$* * **$];

Preassociative

[$* + *$], [$* +_0 *$], [$* +_1 *$], [$* - *$], [$* -_0 *$], [$* -_1 *$], [$* + *$], [$* - *$], [$* +_{\mathrm{f}} *$], [$* -_{\mathrm{f}} *$], [$* + +*$], [R($*$) $- -$R($*$)];

Preassociative

$[| * |], [\mathrm{if}(*,*,*)];$
Preassociative
$[* = *], [* \neq *], [* <= *], [* < *], [* =_{\mathrm{f}} *], [* <_{\mathrm{f}} *], [\mathrm{SF}(*,*)], [* == *], [* << *],$
$[* <<== *];$
Preassociative
$[* \cup \{*\}], [* \cup *], [* \backslash \{*\}];$
Postassociative
$[* \therefore *], [* \underline{\therefore} *], [* \underline{::} *], [* \underline{+2*} *], [* :: *], [* +2* *];$
Postassociative
$[*, *];$
Preassociative
$[* \overset{\mathrm{B}}{\approx} *], [* \overset{\mathrm{D}}{\approx} *], [* \overset{\mathrm{C}}{\approx} *], [* \overset{\mathrm{P}}{\approx} *], [* \approx *], [* = *], [* \overset{+}{\rightarrow} *], [* \overset{\mathrm{t}}{=} *], [* \overset{\mathrm{t}^*}{=} *], [* \overset{\mathrm{r}}{=} *],$
$[* \in_{\mathrm{t}} *], [* \subseteq_{\mathrm{T}} *], [* \overset{\mathrm{T}}{=} *], [* \overset{\mathrm{s}}{=} *], [* \text{ free in } *], [* \text{ free in}^* *], [* \text{ free for } * \text{ in } *],$
$[* \text{ free for}^* * \text{ in } *], [* \in_{\mathrm{c}} *], [* < *], [* <' *], [* \leq' *], [* = *], [* \neq *], [*^{\mathrm{var}}],$
$[*\#^0*], [*\#^1*], [*\#^**], [*{==}*], [* \subseteq *];$
Preassociative
$[\neg *], [\dot{\neg} *], [* \notin *], [* \neq *];$
Preassociative
$[* \wedge *], [* \ddot{\wedge} *], [* \tilde{\wedge} *], [* \wedge_{\mathrm{c}} *], [* \dot{\wedge} *];$
Preassociative
$[* \vee *], [* \parallel *], [* \ddot{\vee} *];$
Postassociative
$[* \dot{\vee} *];$
Preassociative
$[\exists *\colon *], [\forall *\colon *], [\forall_{\mathrm{obj}} *\colon *];$
Postassociative
$[* \ddot{\Rightarrow} *], [* \Rightarrow *], [* \Leftrightarrow *], [* \dot{\Leftrightarrow} *];$
Preassociative
$[\{\mathrm{ph} \in * \mid *\}];$
Postassociative
$[* : *], [* \text{ spy } *], [*!*];$
Preassociative
$[* \begin{cases} * \\ * \end{cases}];$
Preassociative
$[\lambda * .*], [\Lambda * .*], [\Lambda *], [\textbf{if } * \textbf{ then } * \textbf{ else } *], [\textbf{let } * = * \textbf{ in } *], [\textbf{let } * \ddot{=} * \text{ in } *];$
Preassociative
$[*\#*];$
Preassociative
$[*^{\mathrm{I}}], [*^{\triangleright}], [*^{\mathrm{V}}], [*^{+}], [*^{-}], [*^{*}];$
Preassociative
$[* @ *], [* \rhd *], [* \mathrel{|\!\!\triangleright} *], [* \gg *], [* \trianglerighteq *];$
Postassociative
$[* \vdash *], [* \Vdash *], [* \text{ i.e. } *];$
Preassociative

$[\forall *\colon *]$, $[\Pi *\colon *]$;
Postassociative
$[* \oplus *]$;
Postassociative
$[*; *]$;
Preassociative
[∗ proves ∗];
Preassociative
[∗ **proof of** ∗ : ∗], [Line ∗ : ∗ ≫ ∗; ∗], [Last line ∗ ≫ ∗ □],
[Line ∗ : Premise ≫ ∗; ∗], [Line ∗ : Side-condition ≫ ∗; ∗], [Arbitrary ≫ ∗; ∗],
[Local ≫ ∗ = ∗; ∗], [Begin ∗; ∗ : End; ∗], [Last block line ∗ ≫ ∗ ;],
[Arbitrary ≫ ∗; ∗];
Postassociative
$[* \mid *]$;
Postassociative
[∗ , ∗], [∗[∗]∗];
Preassociative
[∗&∗];
Preassociative
[∗\\∗], [∗ linebreak[4] ∗], [∗\\∗]; **End table**

A Pyk definitioner

$([(\cdots) \stackrel{\text{pyk}}{\to}$ "cdots"]

[`Objekt-var` $\stackrel{\text{pyk}}{\to}$ "object-var"]

[`Ex-var` $\stackrel{\text{pyk}}{\to}$ "ex-var"]

[`Ph-var` $\stackrel{\text{pyk}}{\to}$ "ph-var"]

[`Værdi` $\stackrel{\text{pyk}}{\to}$ "vaerdi"]

[`Variabel` $\stackrel{\text{pyk}}{\to}$ "variabel"]

[Op(∗) $\stackrel{\text{pyk}}{\to}$ "op " end op"]

[Op(∗, ∗) $\stackrel{\text{pyk}}{\to}$ "op2 " comma " end op2"]

[$* \stackrel{..}{==} *$ $\stackrel{\text{pyk}}{\to}$ "define-equal " comma " end equal"]

[ContainsEmpty(∗) $\stackrel{\text{pyk}}{\to}$ "contains-empty " end empty"]

[Dedu(∗, ∗) $\stackrel{\text{pyk}}{\to}$ "1deduction " conclude " end 1deduction"]

[$\mathrm{Dedu}_0(*, *)$ $\stackrel{\text{pyk}}{\to}$ "1deduction zero " conclude " end 1deduction"]

[$\mathrm{Dedu}_s(*, *, *)$ $\stackrel{\text{pyk}}{\to}$ "1deduction side " conclude " condition " end 1deduction"]

[$\mathrm{Dedu}_1(*, *, *)$ $\stackrel{\text{pyk}}{\to}$ "1deduction one " conclude " condition " end 1deduction"]

[$\mathrm{Dedu}_2(*, *, *)$ $\stackrel{\text{pyk}}{\to}$ "1deduction two " conclude " condition " end 1deduction"]

[$\mathrm{Dedu}_3(*, *, *, *)$ $\stackrel{\text{pyk}}{\to}$ "1deduction three " conclude " condition " bound " end 1deduction"]

[$\mathrm{Dedu}_4(*, *, *, *) \stackrel{\mathrm{pyk}}{\to}$ "1deduction four " conclude " condition " bound " end 1deduction"]

[$\mathrm{Dedu}_4^*(*, *, *, *) \stackrel{\mathrm{pyk}}{\to}$ "1deduction four star " conclude " condition " bound " end 1deduction"]

[$\mathrm{Dedu}_5(*, *, *) \stackrel{\mathrm{pyk}}{\to}$ "1deduction five " condition " bound " end 1deduction"]

[$\mathrm{Dedu}_6(*, *, *, *) \stackrel{\mathrm{pyk}}{\to}$ "1deduction six " conclude " exception " bound " end 1deduction"]

[$\mathrm{Dedu}_6^*(*, *, *, *) \stackrel{\mathrm{pyk}}{\to}$ "1deduction six star " conclude " exception " bound " end 1deduction"]

[$\mathrm{Dedu}_7(*) \stackrel{\mathrm{pyk}}{\to}$ "1deduction seven " end 1deduction"]

[$\mathrm{Dedu}_8(*, *) \stackrel{\mathrm{pyk}}{\to}$ "1deduction eight " bound " end 1deduction"]

[$\mathrm{Dedu}_8^*(*, *) \stackrel{\mathrm{pyk}}{\to}$ "1deduction eight star " bound " end 1deduction"]

[$\mathrm{Ex}_1 \stackrel{\mathrm{pyk}}{\to}$ "ex1"]

[$\mathrm{Ex}_2 \stackrel{\mathrm{pyk}}{\to}$ "ex2"]

[$\mathrm{Ex3} \stackrel{\mathrm{pyk}}{\to}$ "ex3"]

[$\mathrm{Ex}_{10} \stackrel{\mathrm{pyk}}{\to}$ "ex10"]

[$\mathrm{Ex}_{20} \stackrel{\mathrm{pyk}}{\to}$ "ex20"]

[$*_{\mathrm{Ex}} \stackrel{\mathrm{pyk}}{\to}$ "existential var " end var"]

[$*^{\mathrm{Ex}} \stackrel{\mathrm{pyk}}{\to}$ "" is existential var"]

[$\langle * \equiv * \,|\, * := = * \rangle_{\mathrm{Ex}} \stackrel{\mathrm{pyk}}{\to}$ "exist-sub " is " where " is " end sub"]

[$\langle * \equiv^0 * \,|\, * := = * \rangle_{\mathrm{Ex}} \stackrel{\mathrm{pyk}}{\to}$ "exist-sub0 " is " where " is " end sub"]

[$\langle * \equiv^1 * \,|\, * := = * \rangle_{\mathrm{Ex}} \stackrel{\mathrm{pyk}}{\to}$ "exist-sub1 " is " where " is " end sub"]

[$\langle * \equiv^* * \,|\, * := = * \rangle_{\mathrm{Ex}} \stackrel{\mathrm{pyk}}{\to}$ "exist-sub* " is " where " is " end sub"]

[$\mathrm{ph}_1 \stackrel{\mathrm{pyk}}{\to}$ "placeholder-var1"]

[$\mathrm{ph}_2 \stackrel{\mathrm{pyk}}{\to}$ "placeholder-var2"]

[$\mathrm{ph}_3 \stackrel{\mathrm{pyk}}{\to}$ "placeholder-var3"]

[$*_{\mathrm{Ph}} \stackrel{\mathrm{pyk}}{\to}$ "placeholder-var " end var"]

[$*^{\mathrm{Ph}} \stackrel{\mathrm{pyk}}{\to}$ "" is placeholder-var"]

[$\langle * \equiv * \,|\, * := = * \rangle_{\mathrm{Ph}} \stackrel{\mathrm{pyk}}{\to}$ "ph-sub " is " where " is " end sub"]

[$\langle * \equiv^0 * \,|\, * := = * \rangle_{\mathrm{Ph}} \stackrel{\mathrm{pyk}}{\to}$ "ph-sub0 " is " where " is " end sub"]

[$\langle * \equiv^1 * \,|\, * := = * \rangle_{\mathrm{Ph}} \stackrel{\mathrm{pyk}}{\to}$ "ph-sub1 " is " where " is " end sub"]

[$\langle * \equiv^* * \,|\, * := = * \rangle_{\mathrm{Ph}} \stackrel{\mathrm{pyk}}{\to}$ "ph-sub* " is " where " is " end sub"]

[$\mathsf{bs} \stackrel{\mathrm{pyk}}{\to}$ "var big set"]

[$\mathsf{OBS} \stackrel{\mathrm{pyk}}{\to}$ "object big set"]

[$\mathcal{BS} \stackrel{\mathrm{pyk}}{\to}$ "meta big set"]

[$\emptyset \stackrel{\mathrm{pyk}}{\to}$ "zermelo empty set"]

[ZFsub $\stackrel{\text{pyk}}{\rightarrow}$ “system Q”]

[MP $\stackrel{\text{pyk}}{\rightarrow}$ “1rule mp”]

[Gen $\stackrel{\text{pyk}}{\rightarrow}$ “1rule gen”]

[Repetition $\stackrel{\text{pyk}}{\rightarrow}$ “1rule repetition”]

[Neg $\stackrel{\text{pyk}}{\rightarrow}$ “1rule ad absurdum”]

[Ded $\stackrel{\text{pyk}}{\rightarrow}$ “1rule deduction”]

[ExistIntro $\stackrel{\text{pyk}}{\rightarrow}$ “1rule exist intro”]

[Extensionality $\stackrel{\text{pyk}}{\rightarrow}$ “axiom extensionality”]

[Ødef $\stackrel{\text{pyk}}{\rightarrow}$ “axiom empty set”]

[PairDef $\stackrel{\text{pyk}}{\rightarrow}$ “axiom pair definition”]

[UnionDef $\stackrel{\text{pyk}}{\rightarrow}$ “axiom union definition”]

[PowerDef $\stackrel{\text{pyk}}{\rightarrow}$ “axiom power definition”]

[SeparationDef $\stackrel{\text{pyk}}{\rightarrow}$ “axiom separation definition”]

[AddDoubleNeg $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma add double neg”]

[RemoveDoubleNeg $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma remove double neg”]

[AndCommutativity $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma and commutativity”]

[AutoImply $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma auto imply”]

[Contrapositive $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma contrapositive”]

[FirstConjunct $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma first conjunct”]

[SecondConjunct $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma second conjunct”]

[FromContradiction $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma from contradiction”]

[FromDisjuncts $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma from disjuncts”]

[IffCommutativity $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma iff commutativity”]

[IffFirst $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma iff first”]

[IffSecond $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma iff second”]

[ImplyTransitivity $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma imply transitivity”]

[JoinConjuncts $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma join conjuncts”]

[MP2 $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma mp2”]

[MP3 $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma mp3”]

[MP4 $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma mp4”]

[MP5 $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma mp5”]

[MT $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma mt”]

[NegativeMT $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma negative mt”]

[Technicality $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma technicality”]

[Weakening $\stackrel{\text{pyk}}{\rightarrow}$ “prop lemma weakening”]

[WeakenOr1 $\stackrel{\text{pyk}}{\to}$ "prop lemma weaken or first"]

[WeakenOr2 $\stackrel{\text{pyk}}{\to}$ "prop lemma weaken or second"]

[Formula2Pair $\stackrel{\text{pyk}}{\to}$ "lemma formula2pair"]

[Pair2Formula $\stackrel{\text{pyk}}{\to}$ "lemma pair2formula"]

[Formula2Union $\stackrel{\text{pyk}}{\to}$ "lemma formula2union"]

[Union2Formula $\stackrel{\text{pyk}}{\to}$ "lemma union2formula"]

[Formula2Sep $\stackrel{\text{pyk}}{\to}$ "lemma formula2separation"]

[Sep2Formula $\stackrel{\text{pyk}}{\to}$ "lemma separation2formula"]

[SubsetInPower $\stackrel{\text{pyk}}{\to}$ "lemma subset in power set"]

[HelperPowerIsSub $\stackrel{\text{pyk}}{\to}$ "lemma power set is subset0"]

[PowerIsSub $\stackrel{\text{pyk}}{\to}$ "lemma power set is subset"]

[(Switch)HelperPowerIsSub $\stackrel{\text{pyk}}{\to}$ "lemma power set is subset0-switch"]

[(Switch)PowerIsSub $\stackrel{\text{pyk}}{\to}$ "lemma power set is subset-switch"]

[ToSetEquality $\stackrel{\text{pyk}}{\to}$ "lemma set equality suff condition"]

[HelperToSetEquality(t) $\stackrel{\text{pyk}}{\to}$ "lemma set equality suff condition(t)0"]

[ToSetEquality(t) $\stackrel{\text{pyk}}{\to}$ "lemma set equality suff condition(t)"]

[HelperFromSetEquality $\stackrel{\text{pyk}}{\to}$ "lemma set equality skip quantifier"]

[FromSetEquality $\stackrel{\text{pyk}}{\to}$ "lemma set equality nec condition"]

[HelperReflexivity $\stackrel{\text{pyk}}{\to}$ "lemma reflexivity0"]

[Reflexivity $\stackrel{\text{pyk}}{\to}$ "lemma reflexivity"]

[HelperSymmetry $\stackrel{\text{pyk}}{\to}$ "lemma symmetry0"]

[Symmetry $\stackrel{\text{pyk}}{\to}$ "lemma symmetry"]

[HelperTransitivity $\stackrel{\text{pyk}}{\to}$ "lemma transitivity0"]

[Transitivity $\stackrel{\text{pyk}}{\to}$ "lemma transitivity"]

[ERisReflexive $\stackrel{\text{pyk}}{\to}$ "lemma er is reflexive"]

[ERisSymmetric $\stackrel{\text{pyk}}{\to}$ "lemma er is symmetric"]

[ERisTransitive $\stackrel{\text{pyk}}{\to}$ "lemma er is transitive"]

[ØisSubset $\stackrel{\text{pyk}}{\to}$ "lemma empty set is subset"]

[HelperMemberNotØ $\stackrel{\text{pyk}}{\to}$ "lemma member not empty0"]

[MemberNotØ $\stackrel{\text{pyk}}{\to}$ "lemma member not empty"]

[HelperUniqueØ $\stackrel{\text{pyk}}{\to}$ "lemma unique empty set0"]

[UniqueØ $\stackrel{\text{pyk}}{\to}$ "lemma unique empty set"]

[==Reflexivity $\stackrel{\text{pyk}}{\to}$ "lemma ==Reflexivity"]

[==Symmetry $\stackrel{\text{pyk}}{\to}$ "lemma ==Symmetry"]

[Helper==Transitivity $\stackrel{\text{pyk}}{\to}$ "lemma ==Transitivity0"]

[==Transitivity $\stackrel{\text{pyk}}{\to}$ "lemma ==Transitivity"]
[HelperTransferNotEq $\stackrel{\text{pyk}}{\to}$ "lemma transfer ~is0"]
[TransferNotEq $\stackrel{\text{pyk}}{\to}$ "lemma transfer ~is"]
[HelperPairSubset $\stackrel{\text{pyk}}{\to}$ "lemma pair subset0"]
[Helper(2)PairSubset $\stackrel{\text{pyk}}{\to}$ "lemma pair subset1"]
[PairSubset $\stackrel{\text{pyk}}{\to}$ "lemma pair subset"]
[SamePair $\stackrel{\text{pyk}}{\to}$ "lemma same pair"]
[SameSingleton $\stackrel{\text{pyk}}{\to}$ "lemma same singleton"]
[UnionSubset $\stackrel{\text{pyk}}{\to}$ "lemma union subset"]
[SameUnion $\stackrel{\text{pyk}}{\to}$ "lemma same union"]
[SeparationSubset $\stackrel{\text{pyk}}{\to}$ "lemma separation subset"]
[SameSeparation $\stackrel{\text{pyk}}{\to}$ "lemma same separation"]
[SameBinaryUnion $\stackrel{\text{pyk}}{\to}$ "lemma same binary union"]
[IntersectionSubset $\stackrel{\text{pyk}}{\to}$ "lemma intersection subset"]
[SameIntersection $\stackrel{\text{pyk}}{\to}$ "lemma same intersection"]
[AutoMember $\stackrel{\text{pyk}}{\to}$ "lemma auto member"]
[HelperEqSysNot∅ $\stackrel{\text{pyk}}{\to}$ "lemma eq-system not empty0"]
[EqSysNot∅ $\stackrel{\text{pyk}}{\to}$ "lemma eq-system not empty"]
[HelperEqSubset $\stackrel{\text{pyk}}{\to}$ "lemma eq subset0"]
[EqSubset $\stackrel{\text{pyk}}{\to}$ "lemma eq subset"]
[HelperEqNecessary $\stackrel{\text{pyk}}{\to}$ "lemma equivalence nec condition0"]
[EqNecessary $\stackrel{\text{pyk}}{\to}$ "lemma equivalence nec condition"]
[HelperNoneEqNecessary $\stackrel{\text{pyk}}{\to}$ "lemma none-equivalence nec condition0"]
[Helper(2)NoneEqNecessary $\stackrel{\text{pyk}}{\to}$ "lemma none-equivalence nec condition1"]
[NoneEqNecessary $\stackrel{\text{pyk}}{\to}$ "lemma none-equivalence nec condition"]
[EqClassIsSubset $\stackrel{\text{pyk}}{\to}$ "lemma equivalence class is subset"]
[EqClassesAreDisjoint $\stackrel{\text{pyk}}{\to}$ "lemma equivalence classes are disjoint"]
[AllDisjoint $\stackrel{\text{pyk}}{\to}$ "lemma all disjoint"]
[AllDisjointImply $\stackrel{\text{pyk}}{\to}$ "lemma all disjoint-imply"]
[BSsubset $\stackrel{\text{pyk}}{\to}$ "lemma bs subset union(bs/r)"]
[Union(BS/R)subset $\stackrel{\text{pyk}}{\to}$ "lemma union(bs/r) subset bs"]
[UnionIdentity $\stackrel{\text{pyk}}{\to}$ "lemma union(bs/r) is bs"]
[EqSysIsPartition $\stackrel{\text{pyk}}{\to}$ "theorem eq-system is partition"]
[(ϵ) $\stackrel{\text{pyk}}{\to}$ "var ep"]
[(fx) $\stackrel{\text{pyk}}{\to}$ "var fx"]

[(fy) $\stackrel{\text{pyk}}{\rightarrow}$ "var fy"]

[(fz) $\stackrel{\text{pyk}}{\rightarrow}$ "var fz"]

[(fv) $\stackrel{\text{pyk}}{\rightarrow}$ "var fu"]

[var fv $\stackrel{\text{pyk}}{\rightarrow}$ "var fv"]

[(rx) $\stackrel{\text{pyk}}{\rightarrow}$ "var rx"]

[(ry) $\stackrel{\text{pyk}}{\rightarrow}$ "var ry"]

[(rz) $\stackrel{\text{pyk}}{\rightarrow}$ "var rz"]

[(ru) $\stackrel{\text{pyk}}{\rightarrow}$ "var ru"]

[ϵ $\stackrel{\text{pyk}}{\rightarrow}$ "meta ep"]

[FX $\stackrel{\text{pyk}}{\rightarrow}$ "meta fx"]

[FY $\stackrel{\text{pyk}}{\rightarrow}$ "meta fy"]

[FZ $\stackrel{\text{pyk}}{\rightarrow}$ "meta fz"]

[FU $\stackrel{\text{pyk}}{\rightarrow}$ "meta fu"]

[FV $\stackrel{\text{pyk}}{\rightarrow}$ "meta fv"]

[RX $\stackrel{\text{pyk}}{\rightarrow}$ "meta rx"]

[RY $\stackrel{\text{pyk}}{\rightarrow}$ "meta ry"]

[RZ $\stackrel{\text{pyk}}{\rightarrow}$ "meta rz"]

[RU $\stackrel{\text{pyk}}{\rightarrow}$ "meta ru"]

[0 $\stackrel{\text{pyk}}{\rightarrow}$ "0"]

[1 $\stackrel{\text{pyk}}{\rightarrow}$ "1"]

[(-1) $\stackrel{\text{pyk}}{\rightarrow}$ "(-1)"]

[2 $\stackrel{\text{pyk}}{\rightarrow}$ "2"]

[1/2 $\stackrel{\text{pyk}}{\rightarrow}$ "1/2"]

[0f $\stackrel{\text{pyk}}{\rightarrow}$ "0f"]

[1f $\stackrel{\text{pyk}}{\rightarrow}$ "1f"]

[00 $\stackrel{\text{pyk}}{\rightarrow}$ "00"]

[01 $\stackrel{\text{pyk}}{\rightarrow}$ "01"]

[leqReflexivity $\stackrel{\text{pyk}}{\rightarrow}$ "axiom leqReflexivity"]

[leqAntisymmetryAxiom $\stackrel{\text{pyk}}{\rightarrow}$ "axiom leqAntisymmetry"]

[leqTransitivityAxiom $\stackrel{\text{pyk}}{\rightarrow}$ "axiom leqTransitivity"]

[leqTotality $\stackrel{\text{pyk}}{\rightarrow}$ "axiom leqTotality"]

[leqAdditionAxiom $\stackrel{\text{pyk}}{\rightarrow}$ "axiom leqAddition"]

[leqMultiplicationAxiom $\stackrel{\text{pyk}}{\rightarrow}$ "axiom leqMultiplication"]

[plusAssociativity $\stackrel{\text{pyk}}{\rightarrow}$ "axiom plusAssociativity"]

[plusCommutativity $\stackrel{\text{pyk}}{\rightarrow}$ "axiom plusCommutativity"]

[Negative $\stackrel{\text{pyk}}{\to}$ "axiom negative"]

[plus0 $\stackrel{\text{pyk}}{\to}$ "axiom plus0"]

[timesAssociativity $\stackrel{\text{pyk}}{\to}$ "axiom timesAssociativity"]

[timesCommutativity $\stackrel{\text{pyk}}{\to}$ "axiom timesCommutativity"]

[ReciprocalAxiom $\stackrel{\text{pyk}}{\to}$ "axiom reciprocal"]

[times1 $\stackrel{\text{pyk}}{\to}$ "axiom times1"]

[Distribution $\stackrel{\text{pyk}}{\to}$ "axiom distribution"]

[0not1 $\stackrel{\text{pyk}}{\to}$ "axiom 0not1"]

[equalityAxiom $\stackrel{\text{pyk}}{\to}$ "axiom equality"]

[eqLeqAxiom $\stackrel{\text{pyk}}{\to}$ "axiom eqLeq"]

[eqAdditionAxiom $\stackrel{\text{pyk}}{\to}$ "axiom eqAddition"]

[eqMultiplicationAxiom $\stackrel{\text{pyk}}{\to}$ "axiom eqMultiplication"]

[SENC1 $\stackrel{\text{pyk}}{\to}$ "lemma set equality nec condition(1)"]

[SENC2 $\stackrel{\text{pyk}}{\to}$ "lemma set equality nec condition(2)"]

[IfThenElse(T) $\stackrel{\text{pyk}}{\to}$ "1rule ifThenElse true"]

[IfThenElse(F) $\stackrel{\text{pyk}}{\to}$ "1rule ifThenElse false"]

[From = f $\stackrel{\text{pyk}}{\to}$ "1rule from=f"]

[To = f $\stackrel{\text{pyk}}{\to}$ "1rule to=f"]

[From < f $\stackrel{\text{pyk}}{\to}$ "1rule from<f"]

[To < f $\stackrel{\text{pyk}}{\to}$ "1rule to<f"]

[PlusF $\stackrel{\text{pyk}}{\to}$ "axiom plusF"]

[TimesF $\stackrel{\text{pyk}}{\to}$ "axiom timesF"]

[MinusF $\stackrel{\text{pyk}}{\to}$ "axiom minusF"]

[0f $\stackrel{\text{pyk}}{\to}$ "axiom 0f"]

[1f $\stackrel{\text{pyk}}{\to}$ "axiom 1f"]

[FromSF $\stackrel{\text{pyk}}{\to}$ "1rule fromSameF"]

[ToSF $\stackrel{\text{pyk}}{\to}$ "1rule toSameF"]

[To == XX $\stackrel{\text{pyk}}{\to}$ "1rule to==XX"]

[From ==$\stackrel{\text{pyk}}{\to}$ "1rule from=="]

[To ==$\stackrel{\text{pyk}}{\to}$ "1rule to=="]

[From << XX $\stackrel{\text{pyk}}{\to}$ "1rule from<<XX"]

[From << (1) $\stackrel{\text{pyk}}{\to}$ "1rule from<<XX(1)"]

[From << (2) $\stackrel{\text{pyk}}{\to}$ "1rule from<<XX(2)"]

[to << XX $\stackrel{\text{pyk}}{\to}$ "1rule to<<XX"]

[From <<$\stackrel{\text{pyk}}{\to}$ "1rule from<<"]

[To << $\stackrel{\text{pyk}}{\rightarrow}$ "1rule to<<"]

[FromInR $\stackrel{\text{pyk}}{\rightarrow}$ "1rule fromInR"]

[PlusR $\stackrel{\text{pyk}}{\rightarrow}$ "axiom plusR"]

[TimesR $\stackrel{\text{pyk}}{\rightarrow}$ "axiom timesR"]

[leqAntisymmetry $\stackrel{\text{pyk}}{\rightarrow}$ "lemma leqAntisymmetry"]

[leqTransitivity $\stackrel{\text{pyk}}{\rightarrow}$ "lemma leqTransitivity"]

[leqAddition $\stackrel{\text{pyk}}{\rightarrow}$ "lemma leqAddition"]

[leqMultiplication $\stackrel{\text{pyk}}{\rightarrow}$ "lemma leqMultiplication"]

[Reciprocal $\stackrel{\text{pyk}}{\rightarrow}$ "lemma reciprocal"]

[Equality $\stackrel{\text{pyk}}{\rightarrow}$ "lemma equality"]

[eqLeq $\stackrel{\text{pyk}}{\rightarrow}$ "lemma eqLeq"]

[eqAddition $\stackrel{\text{pyk}}{\rightarrow}$ "lemma eqAddition"]

[eqMultiplication $\stackrel{\text{pyk}}{\rightarrow}$ "lemma eqMultiplication"]

[ToNegatedImply $\stackrel{\text{pyk}}{\rightarrow}$ "prop lemma to negated imply"]

[TND $\stackrel{\text{pyk}}{\rightarrow}$ "prop lemma tertium non datur"]

[ImplyNegation $\stackrel{\text{pyk}}{\rightarrow}$ "prop lemma imply negation"]

[FromNegations $\stackrel{\text{pyk}}{\rightarrow}$ "prop lemma from negations"]

[From3Disjuncts $\stackrel{\text{pyk}}{\rightarrow}$ "prop lemma from three disjuncts"]

[From2 $*$ 2Disjuncts $\stackrel{\text{pyk}}{\rightarrow}$ "prop lemma from two times two disjuncts"]

[NegateDisjunct1 $\stackrel{\text{pyk}}{\rightarrow}$ "prop lemma negate first disjunct"]

[NegateDisjunct2 $\stackrel{\text{pyk}}{\rightarrow}$ "prop lemma negate second disjunct"]

[ExpandDisjuncts $\stackrel{\text{pyk}}{\rightarrow}$ "prop lemma expand disjuncts"]

[eqReflexivity $\stackrel{\text{pyk}}{\rightarrow}$ "lemma eqReflexivity"]

[eqSymmetry $\stackrel{\text{pyk}}{\rightarrow}$ "lemma eqSymmetry"]

[eqTransitivity $\stackrel{\text{pyk}}{\rightarrow}$ "lemma eqTransitivity"]

[eqTransitivity4 $\stackrel{\text{pyk}}{\rightarrow}$ "lemma eqTransitivity4"]

[eqTransitivity5 $\stackrel{\text{pyk}}{\rightarrow}$ "lemma eqTransitivity5"]

[eqTransitivity6 $\stackrel{\text{pyk}}{\rightarrow}$ "lemma eqTransitivity6"]

[plus0Left $\stackrel{\text{pyk}}{\rightarrow}$ "lemma plus0Left"]

[times1Left $\stackrel{\text{pyk}}{\rightarrow}$ "lemma times1Left"]

[lemma eqAdditionLeft $\stackrel{\text{pyk}}{\rightarrow}$ "lemma eqAdditionLeft"]

[EqMultiplicationLeft $\stackrel{\text{pyk}}{\rightarrow}$ "lemma eqMultiplicationLeft"]

[DistributionOut $\stackrel{\text{pyk}}{\rightarrow}$ "lemma distributionOut"]

[Three2twoTerms $\stackrel{\text{pyk}}{\rightarrow}$ "lemma three2twoTerms"]

[Three2threeTerms $\stackrel{\text{pyk}}{\rightarrow}$ "lemma three2threeTerms"]

[Three2threeFactors $\stackrel{\text{pyk}}{\to}$ "lemma three2twoFactors"]

[AddEquations $\stackrel{\text{pyk}}{\to}$ "lemma addEquations"]

[SubtractEquations $\stackrel{\text{pyk}}{\to}$ "lemma subtractEquations"]

[SubtractEquationsLeft $\stackrel{\text{pyk}}{\to}$ "lemma subtractEquationsLeft"]

[EqNegated $\stackrel{\text{pyk}}{\to}$ "lemma eqNegated"]

[PositiveToRight(Eq) $\stackrel{\text{pyk}}{\to}$ "lemma positiveToRight(Eq)"]

[PositiveToLeft(Eq)(1term) $\stackrel{\text{pyk}}{\to}$ "lemma positiveToLeft(Eq)(1 term)"]

[NegativeToLeft(Eq) $\stackrel{\text{pyk}}{\to}$ "lemma negativeToLeft(Eq)"]

[LessNeq $\stackrel{\text{pyk}}{\to}$ "lemma lessNeq"]

[NeqSymmetry $\stackrel{\text{pyk}}{\to}$ "lemma neqSymmetry"]

[NeqNegated $\stackrel{\text{pyk}}{\to}$ "lemma neqNegated"]

[SubNeqRight $\stackrel{\text{pyk}}{\to}$ "lemma subNeqRight"]

[SubNeqLeft $\stackrel{\text{pyk}}{\to}$ "lemma subNeqLeft"]

[NeqAddition $\stackrel{\text{pyk}}{\to}$ "lemma neqAddition"]

[NeqMultiplication $\stackrel{\text{pyk}}{\to}$ "lemma neqMultiplication"]

[UniqueNegative $\stackrel{\text{pyk}}{\to}$ "lemma uniqueNegative"]

[DoubleMinus $\stackrel{\text{pyk}}{\to}$ "lemma doubleMinus"]

[LeqLessEq $\stackrel{\text{pyk}}{\to}$ "lemma leqLessEq"]

[LessLeq $\stackrel{\text{pyk}}{\to}$ "lemma lessLeq"]

[FromLeqGeq $\stackrel{\text{pyk}}{\to}$ "lemma from leqGeq"]

[subLeqRight $\stackrel{\text{pyk}}{\to}$ "lemma subLeqRight"]

[subLeqLeft $\stackrel{\text{pyk}}{\to}$ "lemma subLeqLeft"]

[Leq + 1 $\stackrel{\text{pyk}}{\to}$ "lemma leqPlus1"]

[PositiveToRight(Leq) $\stackrel{\text{pyk}}{\to}$ "lemma positiveToRight(Leq)"]

[PositiveToRight(Leq)(1term) $\stackrel{\text{pyk}}{\to}$ "lemma positiveToRight(Leq)(1 term)"]

[negativeToLeft(Leq) $\stackrel{\text{pyk}}{\to}$ "lemma negativeToLeft(Leq)"]

[LeqAdditionLeft $\stackrel{\text{pyk}}{\to}$ "lemma leqAdditionLeft"]

[leqSubtraction $\stackrel{\text{pyk}}{\to}$ "lemma leqSubtraction"]

[leqSubtractionLeft $\stackrel{\text{pyk}}{\to}$ "lemma leqSubtractionLeft"]

[thirdGeq $\stackrel{\text{pyk}}{\to}$ "lemma thirdGeq"]

[LeqNegated $\stackrel{\text{pyk}}{\to}$ "lemma leqNegated"]

[AddEquations(Leq) $\stackrel{\text{pyk}}{\to}$ "lemma addEquations(Leq)"]

[ThirdGeqSeries $\stackrel{\text{pyk}}{\to}$ "lemma thirdGeqSeries"]

[LeqNeqLess $\stackrel{\text{pyk}}{\to}$ "lemma leqNeqLess"]

[FromLess $\stackrel{\text{pyk}}{\to}$ "lemma fromLess"]

[ToLess $\stackrel{\text{pyk}}{\rightarrow}$ “lemma toLess”]

[fromNotLess $\stackrel{\text{pyk}}{\rightarrow}$ “lemma fromNotLess”]

[toNotLess $\stackrel{\text{pyk}}{\rightarrow}$ “lemma toNotLess”]

[NegativeLessPositive $\stackrel{\text{pyk}}{\rightarrow}$ “lemma negativeLessPositive”]

[leqLessTransitivity $\stackrel{\text{pyk}}{\rightarrow}$ “lemma leqLessTransitivity”]

[LessLeqTransitivity $\stackrel{\text{pyk}}{\rightarrow}$ “lemma lessLeqTransitivity”]

[LessTransitivity $\stackrel{\text{pyk}}{\rightarrow}$ “lemma lessTransitivity”]

[LessTotality $\stackrel{\text{pyk}}{\rightarrow}$ “lemma lessTotality”]

[SubLessRight $\stackrel{\text{pyk}}{\rightarrow}$ “lemma subLessRight”]

[SubLessLeft $\stackrel{\text{pyk}}{\rightarrow}$ “lemma subLessLeft”]

[LessAddition $\stackrel{\text{pyk}}{\rightarrow}$ “lemma lessAddition”]

[LessAdditionLeft $\stackrel{\text{pyk}}{\rightarrow}$ “lemma lessAdditionLeft”]

[LessMultiplication $\stackrel{\text{pyk}}{\rightarrow}$ “lemma lessMultiplication”]

[LessMultiplicationLeft $\stackrel{\text{pyk}}{\rightarrow}$ “lemma lessMultiplicationLeft”]

[LessDivision $\stackrel{\text{pyk}}{\rightarrow}$ “lemma lessDivision”]

[AddEquations(Less) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma addEquations(Less)”]

[LessNegated $\stackrel{\text{pyk}}{\rightarrow}$ “lemma lessNegated”]

[PositiveNegated $\stackrel{\text{pyk}}{\rightarrow}$ “lemma positiveNegated”]

[NonpositiveNegated $\stackrel{\text{pyk}}{\rightarrow}$ “lemma nonpositiveNegated”]

[NegativeNegated $\stackrel{\text{pyk}}{\rightarrow}$ “lemma negativeNegated”]

[NonnegativeNegated $\stackrel{\text{pyk}}{\rightarrow}$ “lemma nonnegativeNegated”]

[PositiveHalved $\stackrel{\text{pyk}}{\rightarrow}$ “lemma positiveHalved”]

[NonnegativeNumerical $\stackrel{\text{pyk}}{\rightarrow}$ “lemma nonnegativeNumerical”]

[NegativeNumerical $\stackrel{\text{pyk}}{\rightarrow}$ “lemma negativeNumerical”]

[PositiveNumerical $\stackrel{\text{pyk}}{\rightarrow}$ “lemma positiveNumerical”]

[lemma nonpositiveNumerical $\stackrel{\text{pyk}}{\rightarrow}$ “lemma nonpositiveNumerical”]

[$|0| = 0$ $\stackrel{\text{pyk}}{\rightarrow}$ “lemma |0|=0”]

[$0 <= |x|$ $\stackrel{\text{pyk}}{\rightarrow}$ “lemma 0<=|x|”]

[SameNumerical $\stackrel{\text{pyk}}{\rightarrow}$ “lemma sameNumerical”]

[SignNumerical(+) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma signNumerical(+)”]

[SignNumerical $\stackrel{\text{pyk}}{\rightarrow}$ “lemma signNumerical”]

[NumericalDifference $\stackrel{\text{pyk}}{\rightarrow}$ “lemma numericalDifference”]

[SplitNumericalSumHelper $\stackrel{\text{pyk}}{\rightarrow}$ “lemma splitNumericalSumHelper”]

[splitNumericalSum(++) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma splitNumericalSum(++)”]

[splitNumericalSum($--$) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma splitNumericalSum(--)”]

[splitNumericalSum(+ $-$ small) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma splitNumericalSum(+-, smallNegative)”]

[splitNumericalSum(+ $-$ big) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma splitNumericalSum(+-, bigNegative)”]

[splitNumericalSum(+$-$) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma splitNumericalSum(+-)”]

[splitNumericalSum($-$+) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma splitNumericalSum(-+)”]

[splitNumericalSum $\stackrel{\text{pyk}}{\rightarrow}$ “lemma splitNumericalSum”]

[insertMiddleTerm(Numerical) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma insertMiddleTerm(Numerical)”]

[$\mathrm{x} + \mathrm{y} =$ zBackwards $\stackrel{\text{pyk}}{\rightarrow}$ “lemma x+y=zBackwards”]

[$\mathrm{x} * \mathrm{y} =$ zBackwards $\stackrel{\text{pyk}}{\rightarrow}$ “lemma x*y=zBackwards”]

[$\mathrm{x} = \mathrm{x} + (\mathrm{y} - \mathrm{y})$ $\stackrel{\text{pyk}}{\rightarrow}$ “lemma x=x+(y-y)”]

[$\mathrm{x} = \mathrm{x} + \mathrm{y} - \mathrm{y}$ $\stackrel{\text{pyk}}{\rightarrow}$ “lemma x=x+y-y”]

[$\stackrel{\text{pyk}}{\rightarrow}$ “lemma x=x*y*(1/y)”]

[insertMiddleTerm(Sum) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma insertMiddleTerm(Sum)”]

[insertMiddleTerm(Difference) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma insertMiddleTerm(Difference)”]

[$\mathrm{x} * 0 + \mathrm{x} = \mathrm{x}$ $\stackrel{\text{pyk}}{\rightarrow}$ “lemma x*0+x=x”]

[$\mathrm{x} * 0 = 0$ $\stackrel{\text{pyk}}{\rightarrow}$ “lemma x*0=0”]

[$(-1) * (-1) + (-1) * 1 = 0$ $\stackrel{\text{pyk}}{\rightarrow}$ “lemma (-1)*(-1)+(-1)*1=0”]

[$(-1) * (-1) = 1$ $\stackrel{\text{pyk}}{\rightarrow}$ “lemma (-1)*(-1)=1”]

[$0 <$ 1Helper $\stackrel{\text{pyk}}{\rightarrow}$ “lemma 0<1Helper”]

[$0 < 1$ $\stackrel{\text{pyk}}{\rightarrow}$ “lemma 0<1”]

[$0 < 2$ $\stackrel{\text{pyk}}{\rightarrow}$ “lemma 0<2”]

[$0 < 1/2$ $\stackrel{\text{pyk}}{\rightarrow}$ “lemma 0<1/2”]

[TwoWholes $\stackrel{\text{pyk}}{\rightarrow}$ “lemma x+x=2*x”]

[TwoHalves $\stackrel{\text{pyk}}{\rightarrow}$ “lemma (1/2)x+(1/2)x=x”]

[$-\mathrm{x} - \mathrm{y} = -(\mathrm{x} + \mathrm{y})$ $\stackrel{\text{pyk}}{\rightarrow}$ “lemma -x-y=-(x+y)”]

[MinusNegated $\stackrel{\text{pyk}}{\rightarrow}$ “lemma minusNegated”]

[Times(-1) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma times(-1)”]

[Times(-1)Left $\stackrel{\text{pyk}}{\rightarrow}$ “lemma times(-1)Left”]

[$-0 = 0$ $\stackrel{\text{pyk}}{\rightarrow}$ “lemma -0=0”]

[SFsymmetry $\stackrel{\text{pyk}}{\rightarrow}$ “lemma sameFsymmetry”]

[SFtransitivity $\stackrel{\text{pyk}}{\rightarrow}$ “lemma sameFtransitivity”]

[= fToSameF $\stackrel{\text{pyk}}{\rightarrow}$ “lemma =f to sameF”]

[PlusF(Sym) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma plusF(Sym)”]

[TimesF(Sym) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma timesF(Sym)”]

[f2R(Plus) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma f2R(Plus)”] [f2R(Times) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma f2R(Times)”]

[PlusR(Sym) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma plusR(Sym)”]

[TimesR(Sym) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma timesR(Sym)”]

[LessLeq(R) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma lessLeq(R)”]

[eqLeq(R) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma eqLeq(R)”]

[SubLessRight(R) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma subLessRight(R)”]

[SubLessLeft(R) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma subLessLeft(R)”]

[<< TransitivityHelper(Q) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma <<TransitivityHelper(Q)”]

[<< Transitivity $\stackrel{\text{pyk}}{\rightarrow}$ “lemma <<Transitivity”]

[<<== Reflexivity $\stackrel{\text{pyk}}{\rightarrow}$ “lemma <<==Reflexivity”]

[<<== AntisymmetryHelper(Q) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma <<==AntisymmetryHelper(Q)”]

[<<== Antisymmetry $\stackrel{\text{pyk}}{\rightarrow}$ “lemma <<==Antisymmetry”]

[<<== Transitivity $\stackrel{\text{pyk}}{\rightarrow}$ “lemma <<==Transitivity”]

[Plus0f $\stackrel{\text{pyk}}{\rightarrow}$ “lemma plus0f”]

[Plus00 $\stackrel{\text{pyk}}{\rightarrow}$ “lemma plus00”]

[== Addition $\stackrel{\text{pyk}}{\rightarrow}$ “lemma ==Addition”]

[== AdditionLeft $\stackrel{\text{pyk}}{\rightarrow}$ “lemma ==AdditionLeft”]

[<< Addition $\stackrel{\text{pyk}}{\rightarrow}$ “lemma <<Addition”]

[<<== Addition $\stackrel{\text{pyk}}{\rightarrow}$ “lemma <<==Addition”]

[PlusAssociativity(F) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma plusAssociativity(F)”]

[PlusAssociativity(R) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma plusAssociativity(R)”]

[Negative(R) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma negative(R)”]

[PlusCommutativity(F) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma plusCommutativity(F)”]

[PlusCommutativity(R) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma plusCommutativity(R)”]

[TimesAssociativity(F) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma timesAssociativity(F)”]

[TimesAssociativity(R) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma timesAssociativity(R)”]

[Times1f $\stackrel{\text{pyk}}{\rightarrow}$ “lemma times1f”]

[Times01 $\stackrel{\text{pyk}}{\rightarrow}$ “lemma times01”]

[TimesCommutativity(F) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma timesCommutativity(F)”]

[TimesCommutativity(R) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma timesCommutativity(R)”]

[Distribution(F) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma distribution(F)”]

[Distribution(R) $\stackrel{\text{pyk}}{\rightarrow}$ “lemma distribution(R)”]

[R(∗) $\stackrel{\text{pyk}}{\rightarrow}$ “R(")”]

[− − R(∗) $\stackrel{\text{pyk}}{\rightarrow}$ “--R(")”]

[rec$\ast$ $\overset{\text{pyk}}{\to}$ “1/ "”]

[$\ast/\ast$ $\overset{\text{pyk}}{\to}$ “eq-system of " modulo "”]

[$\ast\cap\ast$ $\overset{\text{pyk}}{\to}$ “intersection " comma " end intersection”]

[$\ast[\ast]$ $\overset{\text{pyk}}{\to}$ “[" ; "]”]

[$\cup\ast$ $\overset{\text{pyk}}{\to}$ “union " end union”]

[$\ast\cup\ast$ $\overset{\text{pyk}}{\to}$ “binary-union " comma " end union”]

[$\mathrm{P}(\ast)$ $\overset{\text{pyk}}{\to}$ “power " end power”]

[$\{\ast\}$ $\overset{\text{pyk}}{\to}$ “zermelo singleton " end singleton”]

[$\{\ast,\ast\}$ $\overset{\text{pyk}}{\to}$ “zermelo pair " comma " end pair”]

[$\langle\ast,\ast\rangle$ $\overset{\text{pyk}}{\to}$ “zermelo ordered pair " comma " end pair”]

[$-\ast$ $\overset{\text{pyk}}{\to}$ “- "”]

[$-_{\mathrm{f}}\ast$ $\overset{\text{pyk}}{\to}$ “-f "”]

[$\ast\in\ast$ $\overset{\text{pyk}}{\to}$ “" in0 "”]

[$\ast(\ast,\ast)$ $\overset{\text{pyk}}{\to}$ “" is related to " under "”]

[$\mathrm{ReflRel}(\ast,\ast)$ $\overset{\text{pyk}}{\to}$ “" is reflexive relation in "”]

[$\mathrm{SymRel}(\ast,\ast)$ $\overset{\text{pyk}}{\to}$ “" is symmetric relation in "”]

[$\mathrm{TransRel}(\ast,\ast)$ $\overset{\text{pyk}}{\to}$ “" is transitive relation in "”]

[$\mathrm{EqRel}(\ast,\ast)$ $\overset{\text{pyk}}{\to}$ “" is equivalence relation in "”]

[$[\ast\in\ast]_{\ast}$ $\overset{\text{pyk}}{\to}$ “equivalence class of " in " modulo "”]

[$\mathrm{Partition}(\ast,\ast)$ $\overset{\text{pyk}}{\to}$ “" is partition of "”]

[$\ast\ast\ast$ $\overset{\text{pyk}}{\to}$ “" * "”]

[$\ast\ast_{\mathrm{f}}\ast$ $\overset{\text{pyk}}{\to}$ “" *f "”]

[$\ast\ast\ast\ast$ $\overset{\text{pyk}}{\to}$ “" ** "”]

[$\ast+\ast$ $\overset{\text{pyk}}{\to}$ “" + "”]

[$\ast-\ast$ $\overset{\text{pyk}}{\to}$ “" - "”]

[$\ast+_{\mathrm{f}}\ast$ $\overset{\text{pyk}}{\to}$ “" +f "”]

[$\ast-_{\mathrm{f}}\ast$ $\overset{\text{pyk}}{\to}$ “" -f "”]

[$\ast++\ast$ $\overset{\text{pyk}}{\to}$ “" ++ "”]

[$\mathrm{R}(\ast)--\mathrm{R}(\ast)$ $\overset{\text{pyk}}{\to}$ “R(") -- R(")”]

[$|\ast|$ $\overset{\text{pyk}}{\to}$ “| " |”]

[$\mathrm{if}(\ast,\ast,\ast)$ $\overset{\text{pyk}}{\to}$ “if(" , " , ")”]

[$\ast=\ast$ $\overset{\text{pyk}}{\to}$ “" = "”]

[$\ast\neq\ast$ $\overset{\text{pyk}}{\to}$ “" != "”]

[$\ast<=\ast$ $\overset{\text{pyk}}{\to}$ “" <= "”]

[$\ast<\ast$ $\overset{\text{pyk}}{\to}$ “" < "”]

$[* =_{\mathrm{f}} * \stackrel{\mathrm{pyk}}{\rightarrow}$ “" =f "”]

$[* <_{\mathrm{f}} * \stackrel{\mathrm{pyk}}{\rightarrow}$ “" <f "”]

$[\mathrm{SF}(*, *) \stackrel{\mathrm{pyk}}{\rightarrow}$ “" sameF "”]

$[* == * \stackrel{\mathrm{pyk}}{\rightarrow}$ “" == "”]

$[* << * \stackrel{\mathrm{pyk}}{\rightarrow}$ “" << "”]

$[* <<== * \stackrel{\mathrm{pyk}}{\rightarrow}$ “" <<== "”]

$[* {==} * \stackrel{\mathrm{pyk}}{\rightarrow}$ “" zermelo is "”]

$[* \subseteq * \stackrel{\mathrm{pyk}}{\rightarrow}$ “" is subset of "”]

$[\dot{\neg} * \stackrel{\mathrm{pyk}}{\rightarrow}$ “not0 "”]

$[* \notin * \stackrel{\mathrm{pyk}}{\rightarrow}$ “" zermelo ~in "”]

$[* \neq * \stackrel{\mathrm{pyk}}{\rightarrow}$ “" zermelo ~is "”]

$[* \dot{\wedge} * \stackrel{\mathrm{pyk}}{\rightarrow}$ “" and0 "”]

$[* \dot{\vee} * \stackrel{\mathrm{pyk}}{\rightarrow}$ “" or0 "”]

$[* \dot{\Leftrightarrow} * \stackrel{\mathrm{pyk}}{\rightarrow}$ “" iff "”]

$[\{ph \in * \mid *\} \stackrel{\mathrm{pyk}}{\rightarrow}$ “the set of ph in " such that " end set”]

$[\mathrm{am} \stackrel{\mathrm{pyk}}{\rightarrow}$ “am”]

$)^{\mathbf{P}}$

B TEX definitioner

[am $\stackrel{\text{tex}}{=}$ “am”]

[$(\cdots)$ $\stackrel{\text{tex}}{=}$ “(\cdots{})”]

[`Objekt-var` $\stackrel{\text{tex}}{=}$ “\texttt{Objekt-var}”]

[`Ex-var` $\stackrel{\text{tex}}{=}$ “\texttt{Ex-var}”]

[`Ph-var` $\stackrel{\text{tex}}{=}$ “\texttt{Ph-var}”]

[`Værdi` $\stackrel{\text{tex}}{=}$ “\texttt{V\ae{}rdi}”]

[`Variabel` $\stackrel{\text{tex}}{=}$ “\texttt{Variabel}”]

[$\mathrm{Op}(\mathsf{x})$ $\stackrel{\text{tex}}{=}$ “Op(#1.
)”]

[$\mathrm{Op}(\mathsf{x}, \mathsf{y})$ $\stackrel{\text{tex}}{=}$ “Op(#1.
, #2.
)”]

[$\mathsf{x} \mathrel{\ddot{==}} \mathsf{y}$ $\stackrel{\text{tex}}{=}$ “#1.
\mathrel {\ddot {==}} #2.”]

[$\mathrm{ContainsEmpty}(\mathsf{x})$ $\stackrel{\text{tex}}{=}$ “ContainsEmpty(#1.
)”]

[$\mathrm{Dedu}(\mathsf{x}, \mathsf{y})$ $\stackrel{\text{tex}}{=}$ “
Dedu(#1.
, #2.
)”]

[$\mathrm{Dedu}_0(\mathsf{x}, \mathsf{y})$ $\stackrel{\text{tex}}{=}$ “
Dedu_0(#1.
, #2.
)”]

[$\mathrm{Dedu}_{\mathrm{s}}(\mathsf{x}, \mathsf{y}, \mathsf{z})$ $\stackrel{\text{tex}}{=}$ “Dedu_{s}(#1.
, #2.
, #3.
)”]

[$\mathrm{Dedu}_1(\mathsf{x}, \mathsf{y}, \mathsf{z})$ $\stackrel{\text{tex}}{=}$ “
Dedu_1(#1.
, #2.
, #3.
)”]

[Dedu$_2$($\mathsf{x}, \mathsf{y}, \mathsf{z}$) $\stackrel{\text{tex}}{=}$ "
Dedu_2(#1.
, #2.
, #3.
)"]

[Dedu$_3$($\mathsf{x}, \mathsf{y}, \mathsf{z}, \mathsf{u}$) $\stackrel{\text{tex}}{=}$ "
Dedu_3(#1.
, #2.
, #3.
, #4.
)"]

[Dedu$_4$($\mathsf{x}, \mathsf{y}, \mathsf{z}, \mathsf{u}$) $\stackrel{\text{tex}}{=}$ "
Dedu_4(#1.
, #2.
, #3.
, #4.
)"]

[Dedu$_4^*$($\mathsf{x}, \mathsf{y}, \mathsf{z}, \mathsf{u}$) $\stackrel{\text{tex}}{=}$ "
Dedu_4^*(#1.
, #2.
, #3.
, #4.
)"]

[Dedu$_5$($\mathsf{x}, \mathsf{y}, \mathsf{z}$) $\stackrel{\text{tex}}{=}$ "
Dedu_5(#1.
, #2.
, #3.
)"]

[Dedu$_6$($\mathsf{p}, \mathsf{c}, \mathsf{e}, \mathsf{b}$) $\stackrel{\text{tex}}{=}$ "
Dedu_6(#1.
, #2.
, #3.
, #4.
)"]

[Dedu$_6^*$($\mathsf{p}, \mathsf{c}, \mathsf{e}, \mathsf{b}$) $\stackrel{\text{tex}}{=}$ "
Dedu_6^*(#1.
, #2.
, #3.
, #4.
)"]

[Dedu$_7$(p) $\stackrel{\text{tex}}{=}$ "
Dedu_7(#1.

)”]

[$\mathrm{Dedu}_8(\mathsf{p}, \mathsf{b}) \stackrel{\text{tex}}{=}$ “
Dedu_8(#1.
, #2.
)”]

[$\mathrm{Dedu}_8^*(\mathsf{p}, \mathsf{b}) \stackrel{\text{tex}}{=}$ “
Dedu_8ˆ∗(#1.
, #2.
)”]

[$\mathrm{Ex}_1 \stackrel{\text{tex}}{=}$ “Ex_{1}”]

[$\mathrm{Ex}_2 \stackrel{\text{tex}}{=}$ “Ex_{2}”]

[$\mathrm{Ex}_{10} \stackrel{\text{tex}}{=}$ “Ex_{10}”]

[$\mathrm{Ex}_{20} \stackrel{\text{tex}}{=}$ “Ex_{20}”]

[$\mathsf{x}_{\mathrm{Ex}} \stackrel{\text{tex}}{=}$ “#1.
_{Ex}”]

[$\mathsf{x}^{\mathrm{Ex}} \stackrel{\text{tex}}{=}$ “#1.
ˆ{Ex}”]

[$\langle \mathsf{x} {\equiv} \mathsf{y} | \mathsf{z} {:==} \mathsf{u} \rangle_{\mathrm{Ex}} \stackrel{\text{tex}}{=}$ “\langle #1.
{\equiv} #2.
| #3.
{:==} #4.
\rangle_{Ex} ”]

[$\langle \mathsf{x} {\equiv}^0 \mathsf{y} | \mathsf{z} {:==} \mathsf{u} \rangle_{\mathrm{Ex}} \stackrel{\text{tex}}{=}$ “\langle #1.
{\equiv}ˆ0 #2.
| #3.
{:==} #4.
\rangle_{Ex} ”]

[$\langle \mathsf{x} {\equiv}^1 \mathsf{y} | \mathsf{z} {:==} \mathsf{u} \rangle_{\mathrm{Ex}} \stackrel{\text{tex}}{=}$ “\langle #1.
{\equiv}ˆ1 #2.
| #3.
{:==} #4.
\rangle_{Ex} ”]

[$\langle \mathsf{x} {\equiv}^* \mathsf{y} | \mathsf{z} {:==} \mathsf{u} \rangle_{\mathrm{Ex}} \stackrel{\text{tex}}{=}$ “\langle #1.
{\equiv}ˆ∗ #2.
| #3.
{:==} #4.
\rangle_{Ex} ”]

[$\mathrm{ph}_1 \overset{\text{tex}}{=}$ "ph_{1}"]

[$\mathrm{ph}_2 \overset{\text{tex}}{=}$ "ph_{2}"]

[$\mathrm{ph}_3 \overset{\text{tex}}{=}$ "ph_{3}"]

[$\mathsf{x}_{\mathrm{Ph}} \overset{\text{tex}}{=}$ "#1.
_{Ph} "]

[$\mathsf{x}^{\mathrm{Ph}} \overset{\text{tex}}{=}$ "#1.
^{Ph}"]

[$\langle \mathsf{x} {\equiv} \mathsf{y} | \mathsf{z} {:==} \mathsf{u} \rangle_{\mathrm{Ph}} \overset{\text{tex}}{=}$ "\langle #1.
{\equiv} #2.
| #3.
{:==} #4.
\rangle_{Ph} "]

[$\langle \mathsf{x} {\equiv}^0 \mathsf{y} | \mathsf{z} {:==} \mathsf{u} \rangle_{\mathrm{Ph}} \overset{\text{tex}}{=}$ "\langle #1.
{\equiv}^0 #2.
| #3.
{:==} #4.
\rangle_{Ph} "]

[$\langle \mathsf{x} {\equiv}^1 \mathsf{y} | \mathsf{z} {:==} \mathsf{u} \rangle_{\mathrm{Ph}} \overset{\text{tex}}{=}$ "\langle #1.
{\equiv}^1 #2.
| #3.
{:==} #4.
\rangle_{Ph} "]

[$\langle \mathsf{x} {\equiv}^* \mathsf{y} | \mathsf{z} {:==} \mathsf{u} \rangle_{\mathrm{Ph}} \overset{\text{tex}}{=}$ "\langle #1.
{\equiv}^* #2.
| #3.
{:==} #4.
\rangle_{Ph} "]

[$\mathsf{bs} \overset{\text{tex}}{=}$ "\mathsf {bs}"]

[$\mathsf{OBS} \overset{\text{tex}}{=}$ " \mathsf {OBS}"]

[$\mathcal{BS} \overset{\text{tex}}{=}$ "{\cal BS}"]

[$\mathrm{\O} \overset{\text{tex}}{=}$ "\mathrm{\O}"]

[ZFsub $\overset{\text{tex}}{=}$ "ZFsub"]

[MP $\overset{\text{tex}}{=}$ "MP"]

[Gen $\overset{\text{tex}}{=}$ "Gen"]

[Repetition $\stackrel{\text{tex}}{=}$ “Repetition”]

[Neg $\stackrel{\text{tex}}{=}$ “Neg”]

[Ded $\stackrel{\text{tex}}{=}$ “Ded”]

[ExistIntro $\stackrel{\text{tex}}{=}$ “ExistIntro”]

[Extensionality $\stackrel{\text{tex}}{=}$ “Extensionality”]

[Ødef $\stackrel{\text{tex}}{=}$ “\O{}def”]

[PairDef $\stackrel{\text{tex}}{=}$ “PairDef”]

[UnionDef $\stackrel{\text{tex}}{=}$ “UnionDef”]

[PowerDef $\stackrel{\text{tex}}{=}$ “PowerDef”]

[SeparationDef $\stackrel{\text{tex}}{=}$ “SeparationDef”]

[AddDoubleNeg $\stackrel{\text{tex}}{=}$ “AddDoubleNeg”]

[RemoveDoubleNeg $\stackrel{\text{tex}}{=}$ “RemoveDoubleNeg”]

[AndCommutativity $\stackrel{\text{tex}}{=}$ “AndCommutativity”]

[AutoImply $\stackrel{\text{tex}}{=}$ “AutoImply”]

[Contrapositive $\stackrel{\text{tex}}{=}$ “Contrapositive”]

[FirstConjunct $\stackrel{\text{tex}}{=}$ “FirstConjunct”]

[SecondConjunct $\stackrel{\text{tex}}{=}$ “SecondConjunct”]

[FromContradiction $\stackrel{\text{tex}}{=}$ “FromContradiction”]

[FromDisjuncts $\stackrel{\text{tex}}{=}$ “FromDisjuncts”]

[IffCommutativity $\stackrel{\text{tex}}{=}$ “IffCommutativity”]

[IffFirst $\stackrel{\text{tex}}{=}$ “IffFirst”]

[IffSecond $\stackrel{\text{tex}}{=}$ “IffSecond”]

[ImplyTransitivity $\stackrel{\text{tex}}{=}$ “ImplyTransitivity”]

[JoinConjuncts $\stackrel{\text{tex}}{=}$ “JoinConjuncts”]

[MP2 $\stackrel{\text{tex}}{=}$ “MP2”]

$[\text{MP3} \stackrel{\text{tex}}{=} \text{“MP3”}]$

$[\text{MP4} \stackrel{\text{tex}}{=} \text{“MP4”}]$

$[\text{MP5} \stackrel{\text{tex}}{=} \text{“MP5”}]$

$[\text{MT} \stackrel{\text{tex}}{=} \text{“MT”}]$

$[\text{NegativeMT} \stackrel{\text{tex}}{=} \text{“NegativeMT”}]$

$[\text{Technicality} \stackrel{\text{tex}}{=} \text{“Technicality”}]$

$[\text{Weakening} \stackrel{\text{tex}}{=} \text{“Weakening”}]$

$[\text{WeakenOr1} \stackrel{\text{tex}}{=} \text{“WeakenOr1”}]$

$[\text{WeakenOr2} \stackrel{\text{tex}}{=} \text{“WeakenOr2”}]$

$[\text{Pair2Formula} \stackrel{\text{tex}}{=} \text{“Pair2Formula”}]$

$[\text{Formula2Pair} \stackrel{\text{tex}}{=} \text{“Formula2Pair”}]$

$[\text{Union2Formula} \stackrel{\text{tex}}{=} \text{“Union2Formula”}]$

$[\text{Formula2Union} \stackrel{\text{tex}}{=} \text{“Formula2Union”}]$

$[\text{Sep2Formula} \stackrel{\text{tex}}{=} \text{“Sep2Formula”}]$

$[\text{Formula2Sep} \stackrel{\text{tex}}{=} \text{“Formula2Sep”}]$

$[\text{SubsetInPower} \stackrel{\text{tex}}{=} \text{“SubsetInPower”}]$

$[\text{HelperPowerIsSub} \stackrel{\text{tex}}{=} \text{“HelperPowerIsSub”}]$

$[\text{PowerIsSub} \stackrel{\text{tex}}{=} \text{“PowerIsSub”}]$

$[\text{(Switch)HelperPowerIsSub} \stackrel{\text{tex}}{=} \text{“(Switch)HelperPowerIsSub”}]$

$[\text{(Switch)PowerIsSub} \stackrel{\text{tex}}{=} \text{“(Switch)PowerIsSub”}]$

$[\text{ToSetEquality} \stackrel{\text{tex}}{=} \text{“ToSetEquality”}]$

$[\text{HelperToSetEquality(t)} \stackrel{\text{tex}}{=} \text{“HelperToSetEquality(t)”}]$

$[\text{ToSetEquality(t)} \stackrel{\text{tex}}{=} \text{“ToSetEquality(t)”}]$

$[\text{HelperFromSetEquality} \stackrel{\text{tex}}{=} \text{“HelperFromSetEquality”}]$

$[\text{FromSetEquality} \stackrel{\text{tex}}{=} \text{“FromSetEquality”}]$

[HelperReflexivity $\stackrel{\text{tex}}{=}$ “HelperReflexivity”]

[Reflexivity $\stackrel{\text{tex}}{=}$ “Reflexivity”]

[HelperSymmetry $\stackrel{\text{tex}}{=}$ “HelperSymmetry”]

[Symmetry $\stackrel{\text{tex}}{=}$ “Symmetry”]

[HelperTransitivity $\stackrel{\text{tex}}{=}$ “HelperTransitivity”]

[Transitivity $\stackrel{\text{tex}}{=}$ “Transitivity”],

[ERisReflexive $\stackrel{\text{tex}}{=}$ “ERisReflexive”]

[ERisSymmetric $\stackrel{\text{tex}}{=}$ “ERisSymmetric”]

[ERisTransitive $\stackrel{\text{tex}}{=}$ “ERisTransitive”]

[ØisSubset $\stackrel{\text{tex}}{=}$ “\O{}isSubset”]

[HelperMemberNotØ $\stackrel{\text{tex}}{=}$ “HelperMemberNot\O{}”]

[MemberNotØ $\stackrel{\text{tex}}{=}$ “MemberNot\O{}”]

[HelperUniqueØ $\stackrel{\text{tex}}{=}$ “HelperUnique\O{}”]

[UniqueØ $\stackrel{\text{tex}}{=}$ “Unique\O{}”]

[==Reflexivity $\stackrel{\text{tex}}{=}$ “==\!{}Reflexivity”]

[==Symmetry $\stackrel{\text{tex}}{=}$ “==\!{}Symmetry”]

[Helper==Transitivity $\stackrel{\text{tex}}{=}$ “Helper\!{}==\!{}Transitivity”]

[==Transitivity $\stackrel{\text{tex}}{=}$ “\!{}==\!{}Transitivity”]

[HelperTransferNotEq $\stackrel{\text{tex}}{=}$ “HelperTransferNotEq”]

[TransferNotEq $\stackrel{\text{tex}}{=}$ “TransferNotEq”]

[HelperPairSubset $\stackrel{\text{tex}}{=}$ “HelperPairSubset”]

[Helper(2)PairSubset $\stackrel{\text{tex}}{=}$ “Helper(2)PairSubset”]

[PairSubset $\stackrel{\text{tex}}{=}$ “PairSubset”]

[SamePair $\stackrel{\text{tex}}{=}$ “SamePair”]

[SameSingleton $\stackrel{\text{tex}}{=}$ “SameSingleton”]

[UnionSubset $\stackrel{\text{tex}}{=}$ “UnionSubset”]

[SameUnion $\stackrel{\text{tex}}{=}$ “SameUnion”]

[SeparationSubset $\stackrel{\text{tex}}{=}$ “SeparationSubset”]

[SameSeparation $\stackrel{\text{tex}}{=}$ “SameSeparation”]

[SameBinaryUnion $\stackrel{\text{tex}}{=}$ “SameBinaryUnion”]

[IntersectionSubset $\stackrel{\text{tex}}{=}$ “IntersectionSubset”]

[SameIntersection $\stackrel{\text{tex}}{=}$ “SameIntersection”]

[AutoMember $\stackrel{\text{tex}}{=}$ “AutoMember”]

[HelperEqSysNot∅ $\stackrel{\text{tex}}{=}$ “HelperEqSysNot\O{}”]

[EqSysNot∅ $\stackrel{\text{tex}}{=}$ “EqSysNot\O{}”]

[HelperEqSubset $\stackrel{\text{tex}}{=}$ “HelperEqSubset”]

[EqSubset $\stackrel{\text{tex}}{=}$ “EqSubset”]

[EqNecessary $\stackrel{\text{tex}}{=}$ “EqNecessary”]

[HelperEqNecessary $\stackrel{\text{tex}}{=}$ “HelperEqNecessary”]

[HelperNoneEqNecessary $\stackrel{\text{tex}}{=}$ “HelperNoneEqNecessary”]

[Helper(2)NoneEqNecessary $\stackrel{\text{tex}}{=}$ “Helper(2)NoneEqNecessary”]

[NoneEqNecessary $\stackrel{\text{tex}}{=}$ “NoneEqNecessary”]

[EqClassIsSubset $\stackrel{\text{tex}}{=}$ “EqClassIsSubset”]

[EqClassesAreDisjoint $\stackrel{\text{tex}}{=}$ “EqClassesAreDisjoint”]

[AllDisjoint $\stackrel{\text{tex}}{=}$ “AllDisjoint”]

[AllDisjointImply $\stackrel{\text{tex}}{=}$ “AllDisjointImply”]

[BSsubset $\stackrel{\text{tex}}{=}$ “BSsubset”]

[Union(BS/R)subset $\stackrel{\text{tex}}{=}$ “Union(BS/R)subset”]

[UnionIdentity $\stackrel{\text{tex}}{=}$ “UnionIdentity”]

[EqSysIsPartition $\stackrel{\text{tex}}{=}$ “EqSysIsPartition”]

$[\mathsf{x}/\mathsf{y} \stackrel{\text{tex}}{=}$ “#1.
/ #2.”]

$[\mathsf{x} \cap \mathsf{y} \stackrel{\text{tex}}{=}$ “#1.
\cap #2.”]

$[\cup \mathsf{x} \stackrel{\text{tex}}{=}$ “\cup #1.”]

$[\mathsf{x} \cup \mathsf{y} \stackrel{\text{tex}}{=}$ “#1.
\mathrel{\cup} #2.”]

$[\mathrm{P}(\mathsf{x}) \stackrel{\text{tex}}{=}$ “P(#1.
)”]

$[\{\mathsf{x}\} \stackrel{\text{tex}}{=}$ “\{#1.
\}”]

$[\{\mathsf{x}, \mathsf{y}\} \stackrel{\text{tex}}{=}$ “\{#1.
, #2.
\}”]

$[\langle \mathsf{x}, \mathsf{y} \rangle \stackrel{\text{tex}}{=}$ “\langle #1.
, #2.
\rangle”],

$[\mathsf{x} \in \mathsf{y} \stackrel{\text{tex}}{=}$ “#1.
\mathrel{\in} #2.”]

$[\mathsf{z}(\mathsf{x}, \mathsf{y}) \stackrel{\text{tex}}{=}$ “#3.
(#1.
, #2.
)”]

$[\mathrm{ReflRel}(\mathsf{r}, \mathsf{x}) \stackrel{\text{tex}}{=}$ “ReflRel(#1.
, #2.
)”]

$[\mathrm{SymRel}(\mathsf{r}, \mathsf{x}) \stackrel{\text{tex}}{=}$ “SymRel(#1.
, #2.
)”]

$[\mathrm{TransRel}(\mathsf{r}, \mathsf{x}) \stackrel{\text{tex}}{=}$ “TransRel(#1.
, #2.
)”]

$[\mathrm{EqRel}(\mathsf{r}, \mathsf{x}) \stackrel{\text{tex}}{=}$ “EqRel(#1.
, #2.
)”]

[$[\mathsf{x} \in \mathsf{bs}]_{\mathsf{r}} \stackrel{\text{tex}}{=}$ “[#1.
\mathrel{\in} #2.
]_{#3.
}”]

[Partition$(\mathsf{x}, \mathsf{y}) \stackrel{\text{tex}}{=}$ “Partition(#1.
, #2.
)”]

[$\mathsf{x}{=}{=}\mathsf{y} \stackrel{\text{tex}}{=}$ “#1.
\!\mathrel{==}\! #2.”]

[$\mathsf{x} \subseteq \mathsf{y} \stackrel{\text{tex}}{=}$ “#1.
\mathrel{\subseteq} #2.”]

[$\dot{\neg}\, \mathsf{x} \stackrel{\text{tex}}{=}$ “\dot{\neg}\, #1.”]

[$\mathsf{x} \notin \mathsf{y} \stackrel{\text{tex}}{=}$ “#1.
\mathrel{\notin} #2.”]

[$\mathsf{x} \neq \mathsf{y} \stackrel{\text{tex}}{=}$ “#1.
\mathrel{\neq} #2.”]

[$\mathsf{x} \mathrel{\dot{\wedge}} \mathsf{y} \stackrel{\text{tex}}{=}$ “#1.
\mathrel{\dot{\wedge}} #2.”]

[$\mathsf{x} \mathrel{\dot{\vee}} \mathsf{y} \stackrel{\text{tex}}{=}$ “#1.
\mathrel{\dot{\vee}} #2.”]

[$\mathsf{x} \mathrel{\dot{\Leftrightarrow}} \mathsf{y} \stackrel{\text{tex}}{=}$ “#1.
\mathrel{\dot{\Leftrightarrow}} #2.”]

[$\{ph \in \mathsf{x} \mid \mathsf{a}\} \stackrel{\text{tex}}{=}$ “ \{ ph \mathrel{\in} #1.
\mid #2.
\}”]

————- RRRRRRRRRRRRRRR ————-

(*** aksiomer ***)

[ZFsub **rule** leqReflexivity: $\Pi \mathcal{X} \colon \mathcal{X} <= \mathcal{X}$]

[ZFsub **rule** leqAntisymmetryAxiom: $\Pi \mathcal{X}, \mathcal{Y} \colon \mathcal{X} <= \mathcal{Y} \Rightarrow \mathcal{Y} <= \mathcal{X} \Rightarrow \mathcal{X} = \mathcal{Y}$]

[ZFsub **rule** leqTransitivityAxiom: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z} \colon \mathcal{X} <= \mathcal{Y} \Rightarrow \mathcal{Y} <= \mathcal{Z} \Rightarrow \mathcal{X} <= \mathcal{Z}$]

[ZFsub **rule** leqTotality: $\Pi \mathcal{X}, \mathcal{Y} \colon \mathcal{X} <= \mathcal{Y} \mathrel{\dot{\vee}} \mathcal{Y} <= \mathcal{X}$]

[ZFsub **rule** leqAdditionAxiom: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z} \colon \mathcal{X} <= \mathcal{Y} \Rightarrow \mathcal{X} + \mathcal{Z} <= \mathcal{Y} + \mathcal{Z}$]

[ZFsub **rule** leqMultiplicationAxiom: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z} \colon 0 <= \mathcal{Z} \Rightarrow \mathcal{X} <= \mathcal{Y} \Rightarrow \mathcal{X} * \mathcal{Z} <= \mathcal{Y} * \mathcal{Z}$]

[ZFsub **rule** plusAssociativity: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z}\colon (\mathcal{X}+\mathcal{Y})+\mathcal{Z}=\mathcal{X}+(\mathcal{Y}+\mathcal{Z})$]
[ZFsub **rule** plusCommutativity: $\Pi\mathcal{X},\mathcal{Y}\colon \mathcal{X}+\mathcal{Y}=\mathcal{Y}+\mathcal{X}$]
[ZFsub **rule** Negative: $\Pi\mathcal{X}\colon \mathcal{X}+(-\mathcal{X})=0$]
[ZFsub **rule** plus0: $\Pi\mathcal{X}\colon \mathcal{X}+0=\mathcal{X}$]
[ZFsub **rule** timesAssociativity: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z}\colon (\mathcal{X}*\mathcal{Y})*\mathcal{Z}=\mathcal{X}*(\mathcal{Y}*\mathcal{Z})$]
[ZFsub **rule** timesCommutativity: $\Pi\mathcal{X},\mathcal{Y}\colon \mathcal{X}*\mathcal{Y}=\mathcal{Y}*\mathcal{X}$]
[ZFsub **rule** ReciprocalAxiom: $\Pi\mathcal{X}\colon \mathcal{X}\neq 0 \Rightarrow \mathcal{X}*\mathrm{rec}\mathcal{X}=1$]
[ZFsub **rule** times1: $\Pi\mathcal{X}\colon \mathcal{X}*1=\mathcal{X}$]
[ZFsub **rule** Distribution: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z}\colon \mathcal{X}*(\mathcal{Y}+\mathcal{Z})=(\mathcal{X}*\mathcal{Y})+(\mathcal{X}*\mathcal{Z})$]
[ZFsub **rule** 0not1: $0\neq 1$]
[ZFsub **rule** equalityAxiom: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z}\colon \mathcal{X}=\mathcal{Y}\Rightarrow \mathcal{X}=\mathcal{Z}\Rightarrow \mathcal{Y}=\mathcal{Z}$]
[ZFsub **rule** eqLeqAxiom: $\Pi\mathcal{X},\mathcal{Y}\colon \mathcal{X}=\mathcal{Y}\Rightarrow \mathcal{X}<=\mathcal{Y}$]
[ZFsub **rule** eqAdditionAxiom: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z}\colon \mathcal{X}=\mathcal{Y}\Rightarrow \mathcal{X}+\mathcal{Z}=\mathcal{Y}+\mathcal{Z}$]
[ZFsub **rule** eqMultiplicationAxiom: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z}\colon \mathcal{X}=\mathcal{Y}\Rightarrow \mathcal{X}*\mathcal{Z}=\mathcal{Y}*\mathcal{Z}$]
(*** XX snydeaksiomer ***)
[ZFsub **rule** ==Reflexivity: ΠRX: RX == RX]
[ZFsub **rule** ==Symmetry: ΠRX, RY: RX == RY $\vdash$ RY == RX]
[ZFsub **rule** == Transitivity: ΠRX, RY, RZ: RX == RY $\vdash$ RY == RZ $\vdash$ RX == RZ]

XX ikke 100procent identisk med originalen fra equivalence-relations [ZFsub **rul**
RX, RY: RX == RY $\vdash$ FX $\in$ RX $\vdash$ FX $\in$ RY]

XX boer bevises ud fra nummer 1 [ZFsub **rule** SENC2: ΠFX, RX, RY: RX == RY $\vdash$ FX $\in$ RY $\vdash$ FX $\in$ RX]

[ZFsub **rule** IfThenElse(T): $\Pi\mathcal{A},\mathcal{X},\mathcal{Y}\colon \mathcal{A}\vdash \mathrm{if}(\mathcal{A},\mathcal{X},\mathcal{Y})=\mathcal{X}$]
[ZFsub **rule** IfThenElse(F): $\Pi\mathcal{A},\mathcal{X},\mathcal{Y}\colon \dot{\neg}\,\mathcal{A}\vdash \mathrm{if}(\mathcal{A},\mathcal{X},\mathcal{Y})=\mathcal{Y}$]
[ZFsub **rule** FromSF: $\Pi\mathcal{M},\epsilon$, FX, FY: SF(FX, FY) $\vdash 0<\epsilon\vdash$ Ex3 $<=\mathcal{M}\Rightarrow$ |FX[$\mathcal{M}$] − FY[$\mathcal{M}$]| $<\epsilon$]
[ZFsub **rule** ToSF: $\Pi\mathcal{M},\epsilon$, FX, FY: $0<\epsilon\Rightarrow$ Ex3 $<=\mathcal{M}\Rightarrow$ |FX[$\mathcal{M}$] − FY[$\mathcal{M}$]| $<\epsilon\vdash$ SF(FX, FY)]
[ZFsub **rule** From = f: $\Pi\mathcal{M}$, FX, FY: FX $=_{\mathrm{f}}$ FY $\vdash$ FX[$\mathcal{M}$] = FY[$\mathcal{M}$]]

XX hm... det er nok med bare 1 meta m XX loesning: objektkvantor [ZFsub **rule**
f: $\Pi\mathcal{M}$, FX, FY: FX[$\mathcal{M}$] = FY[$\mathcal{M}$] $\vdash$ FX $=_{\mathrm{f}}$ FY]

[ZFsub **rule** From < f: $\Pi\mathcal{M},\epsilon$, FX, FY: FX $<_{\mathrm{f}}$ FY $\vdash 0<\epsilon\vdash$ Ex3 $<=\mathcal{M}\Rightarrow$ FX[$\mathcal{M}$] <= FY[$\mathcal{M}$] $-\,\epsilon$]
[ZFsub **rule** To < f: $\Pi\mathcal{M},\epsilon$, FX, FY: $0<\epsilon\Rightarrow$ Ex3 $<=\mathcal{M}\Rightarrow$ FX[$\mathcal{M}$] <= FY[$\mathcal{M}$] $-\,\epsilon\vdash$ FX $<_{\mathrm{f}}$ FY]
[ZFsub **rule** PlusF: $\Pi\mathcal{M}$, FX, FY: FX $+_{\mathrm{f}}$ FY[$\mathcal{M}$] = FX[$\mathcal{M}$] + FY[$\mathcal{M}$]]
[ZFsub **rule** MinusF: $\Pi\mathcal{M}$, FX: $-_{\mathrm{f}}$FX[$\mathcal{M}$] = −FX[$\mathcal{M}$]]
[ZFsub **rule** TimesF: $\Pi\mathcal{M}$, FX, FY: FX $*_{\mathrm{f}}$ FY[$\mathcal{M}$] = FX[$\mathcal{M}$] $*$ FY[$\mathcal{M}$]]
[ZFsub **rule** 0f: $\Pi\mathcal{M}$: 0f[$\mathcal{M}$] = 0]
[ZFsub **rule** 1f: $\Pi\mathcal{M}$: 1f[$\mathcal{M}$] = 1]
[ZFsub **rule** To == XX: ΠFX, FY, RX, RY: FX $\in$ RX $\Rightarrow$ FY $\in$ RY $\Rightarrow$ SF(FX, FY) $\vdash$ RX == RY]
[ZFsub **rule** From ==: ΠFX, FY: R(FX) == R(FY) $\vdash$ SF(FX, FY)]
[ZFsub **rule** To ==: ΠFX, FY: SF(FX, FY) $\vdash$ R(FX) == R(FY)]

[ZFsub **rule** From $<<$ XX: $\Pi \mathcal{M}, \epsilon, \mathrm{FX}, \mathrm{FY}, \mathrm{RX}, \mathrm{RY}\colon \mathrm{RX} << \mathrm{RY} \vdash \mathrm{FX} \in \mathrm{RX} \vdash \mathrm{FY} \in \mathrm{RY} \vdash 0 < \epsilon \vdash \mathrm{Ex}_1 <= \mathcal{M} \Rightarrow \mathrm{FX}[\mathcal{M}] <= \mathrm{FY}[\mathcal{M}] - \epsilon$]

[ZFsub **rule** From $<<$ (1): $\Pi \mathrm{RX}, \mathrm{RY}\colon \mathrm{RX} << \mathrm{RY} \vdash \mathrm{Ex}_{10} \in \mathrm{RX}$]

[ZFsub **rule** From $<<$ (2): $\Pi \mathrm{RX}, \mathrm{RY}\colon \mathrm{RX} << \mathrm{RY} \vdash \mathrm{Ex}_{20} \in \mathrm{RY}$]

[ZFsub **rule** to $<<$ XX: $\Pi \mathcal{M}, \mathcal{N}, \epsilon, \mathrm{FX}, \mathrm{FY}, \mathrm{RX}, \mathrm{RY}\colon \mathrm{FX} \in \mathrm{RX} \Rightarrow \mathrm{FY} \in \mathrm{RY} \Rightarrow 0 < \epsilon \Rightarrow \mathrm{Ex}_1 <= \mathcal{M} \Rightarrow \mathrm{FX}[\mathcal{M}] <= \mathrm{FY}[\mathcal{M}] - \epsilon \vdash \mathrm{RX} << \mathrm{RY}$]

[ZFsub **rule** From $<<$: $\Pi \mathrm{FX}, \mathrm{FY}\colon \mathrm{R(FX)} << \mathrm{R(FY)} \vdash \mathrm{FX} <_{\mathrm{f}} \mathrm{FY}$]

[ZFsub **rule** To $<<$: $\Pi \mathrm{FX}, \mathrm{FY}\colon \mathrm{FX} <_{\mathrm{f}} \mathrm{FY} \vdash \mathrm{R(FX)} << \mathrm{R(FY)}$]

[ZFsub **rule** FromInR: $\Pi \mathrm{FX}, \mathrm{FY}\colon \mathrm{FX} \in \mathrm{R(FY)} \vdash \mathrm{SF(FX, FY)}$]

[ZFsub **rule** PlusR: $\Pi \mathrm{FX}, \mathrm{FY}\colon \mathrm{R(FX)} + +\mathrm{R(FY)} == \mathrm{R(FX} +_{\mathrm{f}} \mathrm{FY)}$]

[ZFsub **rule** TimesR: $\Pi \mathrm{FX}, \mathrm{FY}\colon \mathrm{R(FX)} * *\mathrm{R(FY)} == \mathrm{R(FX} *_{\mathrm{f}} \mathrm{FY)}$]

(*** makroer ***)

[$\epsilon \doteq \underline{(\epsilon)}$] [$\mathrm{FX} \doteq \underline{(\mathrm{fx})}$] [$\mathrm{FY} \doteq \underline{(\mathrm{fy})}$] [$\mathrm{FZ} \doteq \underline{(\mathrm{fz})}$] [$\mathrm{FU} \doteq \underline{(\mathrm{fv})}$] [$\mathrm{FV} \doteq \underline{\mathrm{var\ fv}}$] [$\mathrm{RX} \doteq \underline{(\mathrm{rx})}$] [$\mathrm{RY} \doteq \underline{(\mathrm{ry})}$] [$\mathrm{RZ} \doteq \underline{(\mathrm{rz})}$] [$\mathrm{RU} \doteq \underline{(\mathrm{ru})}$]

[$\mathrm{Ex3} \doteq \mathsf{c}_{\mathrm{Ex}}$]

[$\mathsf{x} <<== \mathsf{y} \doteq \mathsf{x} << \mathsf{y} \dot{\vee} \mathsf{x} == \mathsf{y}$]

[$(-1) \doteq -1$]

[$2 \doteq (1+1)$]

[$1/2 \doteq \mathrm{rec2}$]

[$\mathsf{x} < \mathsf{y} \doteq \mathsf{x} <= \mathsf{y} \dot{\wedge} \mathsf{x} \neq \mathsf{y}$]

[$\mathsf{x} \neq \mathsf{y} \doteq \dot{\neg} \mathsf{x} = \mathsf{y}$]

[$\mathsf{x} - \mathsf{y} \doteq \mathsf{x} + (-\mathsf{y})$]

[$|\mathsf{x}| \doteq \mathrm{if}(0 <= \mathsf{x}, \mathsf{x}, -\mathsf{x})$]

[$00 \doteq \mathrm{R(0f)}$]

[$01 \doteq \mathrm{R(1f)}$]

[$\mathrm{R((fx))} + +\mathrm{R((fy))} \doteq \mathrm{R((fx)} +_{\mathrm{f}} \mathrm{(fy))}$]

[$- - \mathrm{R((fx))} \doteq \mathrm{R(}-_{\mathrm{f}}\mathrm{(fx))}$] XX noedvendig?

[$\mathrm{R((fx))} - -\mathrm{R((fy))} \doteq \mathrm{R((fx))} + +\mathrm{R(}-_{\mathrm{f}}\mathrm{(fy))}$] XX noedvendigt med [R( ) − R( )] konstruktionen?

(*** REGELLEMMAER ***)

[ZFsub **lemma** leqTransitivity: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}\colon \mathcal{X} <= \mathcal{Y} \vdash \mathcal{Y} <= \mathcal{Z} \vdash \mathcal{X} <= \mathcal{Z}$]

ZFsub **proof of** leqTransitivity:

L01:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}, \mathcal{Z}$	;
L02:	Premise $\gg$	$\mathcal{X} <= \mathcal{Y}$	;
L03:	Premise $\gg$	$\mathcal{Y} <= \mathcal{Z}$	;
L04:	leqTransitivityAxiom $\gg$	$\mathcal{X} <= \mathcal{Y} \Rightarrow \mathcal{Y} <= \mathcal{Z} \Rightarrow \mathcal{X} <= \mathcal{Z}$	;
L05:	MP2 $\rhd$ L04 $\rhd$ L02 $\rhd$ L03 $\gg$	$\mathcal{X} <= \mathcal{Z}$	$\square$

[ZFsub **lemma** leqAntisymmetry: $\Pi \mathcal{X}, \mathcal{Y}\colon \mathcal{X} <= \mathcal{Y} \vdash \mathcal{Y} <= \mathcal{X} \vdash \mathcal{X} = \mathcal{Y}$]

ZFsub **proof of** leqAntisymmetry:

L01:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}$	;
L02:	Premise $\gg$	$\mathcal{X} <= \mathcal{Y}$	;
L03:	Premise $\gg$	$\mathcal{Y} <= \mathcal{X}$	;
L04:	leqAntisymmetryAxiom $\gg$	$\mathcal{X} <= \mathcal{Y} \Rightarrow \mathcal{Y} <= \mathcal{X} \Rightarrow \mathcal{X} = \mathcal{Y}$	;
L05:	MP2 $\rhd$ L04 $\rhd$ L02 $\rhd$ L03 $\gg$	$\mathcal{X} = \mathcal{Y}$	$\square$

[ZFsub **lemma** leqAddition: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}\colon \mathcal{X} <= \mathcal{Y} \vdash \mathcal{X} + \mathcal{Z} <= \mathcal{Y} + \mathcal{Z}$]

ZFsub **proof of** leqAddition:

L01:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}, \mathcal{Z}$	;
L02:	Premise $\gg$	$\mathcal{X} <= \mathcal{Y}$	;
L03:	leqAdditionAxiom $\gg$	$\mathcal{X} <= \mathcal{Y} \Rightarrow \mathcal{X} + \mathcal{Z} <= \mathcal{Y} + \mathcal{Z}$	;
L04:	MP $\rhd$ L03 $\rhd$ L02 $\gg$	$\mathcal{X} + \mathcal{Z} <= \mathcal{Y} + \mathcal{Z}$	$\square$

[ZFsub **lemma** leqMultiplication: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}\colon 0 <= \mathcal{Z} \vdash \mathcal{X} <= \mathcal{Y} \vdash \mathcal{X} * \mathcal{Z} <= \mathcal{Y} * \mathcal{Z}$]

ZFsub **proof of** leqMultiplication:

L01:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}, \mathcal{Z}$	;
L02:	Premise $\gg$	$0 <= \mathcal{Z}$	;
L03:	Premise $\gg$	$\mathcal{X} <= \mathcal{Y}$	;
L04:	leqMultiplicationAxiom $\gg$	$0 <= \mathcal{Z} \Rightarrow \mathcal{X} <= \mathcal{Y} \Rightarrow \mathcal{X} * \mathcal{Z} <= \mathcal{Y} * \mathcal{Z}$	;
L05:	MP2 $\rhd$ L04 $\rhd$ L02 $\rhd$ L03 $\gg$	$\mathcal{X} * \mathcal{Z} <= \mathcal{Y} * \mathcal{Z}$	$\square$

[ZFsub **lemma** Reciprocal: $\Pi \mathcal{X}\colon \mathcal{X} \neq 0 \vdash \mathcal{X} * \mathrm{rec}\mathcal{X} = 1$]

ZFsub **proof of** Reciprocal:

L01:	Arbitrary $\gg$	$\mathcal{X}$	;
L02:	Premise $\gg$	$\mathcal{X} \neq 0$	;
L03:	ReciprocalAxiom $\gg$	$\mathcal{X} \neq 0 \Rightarrow \mathcal{X} * \mathrm{rec}\mathcal{X} = 1$	;
L04:	MP $\rhd$ L03 $\rhd$ L02 $\gg$	$\mathcal{X} * \mathrm{rec}\mathcal{X} = 1$	$\square$

[ZFsub **lemma** Equality: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}\colon \mathcal{X} = \mathcal{Y} \vdash \mathcal{X} = \mathcal{Z} \vdash \mathcal{Y} = \mathcal{Z}$]

ZFsub **proof of** Equality:

L01:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}, \mathcal{Z}$	;
L02:	Premise $\gg$	$\mathcal{X} = \mathcal{Y}$	;
L03:	Premise $\gg$	$\mathcal{X} = \mathcal{Z}$	;
L04:	equalityAxiom $\gg$	$\mathcal{X} = \mathcal{Y} \Rightarrow \mathcal{X} = \mathcal{Z} \Rightarrow \mathcal{Y} = \mathcal{Z}$	;
L05:	MP2 $\rhd$ L04 $\rhd$ L02 $\rhd$ L03 $\gg$	$\mathcal{Y} = \mathcal{Z}$	$\square$

[ZFsub **lemma** eqLeq: $\Pi \mathcal{X}, \mathcal{Y}\colon \mathcal{X} = \mathcal{Y} \vdash \mathcal{X} <= \mathcal{Y}$]

ZFsub **proof of** eqLeq:

L01:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}$	;
L02:	Premise $\gg$	$\mathcal{X} = \mathcal{Y}$	;
L03:	eqLeqAxiom $\gg$	$\mathcal{X} = \mathcal{Y} \Rightarrow \mathcal{X} <= \mathcal{Y}$	;
L04:	MP $\rhd$ L03 $\rhd$ L02 $\gg$	$\mathcal{X} <= \mathcal{Y}$	$\square$

[ZFsub **lemma** eqAddition: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}\colon \mathcal{X} = \mathcal{Y} \vdash \mathcal{X} + \mathcal{Z} = \mathcal{Y} + \mathcal{Z}$]

ZFsub **proof of** eqAddition:

L01:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}, \mathcal{Z}$	;
L02:	Premise $\gg$	$\mathcal{X} = \mathcal{Y}$	;
L03:	eqAdditionAxiom $\gg$	$\mathcal{X} = \mathcal{Y} \Rightarrow \mathcal{X} + \mathcal{Z} = \mathcal{Y} + \mathcal{Z}$	;
L04:	MP $\rhd$ L03 $\rhd$ L02 $\gg$	$\mathcal{X} + \mathcal{Z} = \mathcal{Y} + \mathcal{Z}$	$\square$

[ZFsub **lemma** eqMultiplication: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}\colon \mathcal{X} = \mathcal{Y} \vdash \mathcal{X} * \mathcal{Z} = \mathcal{Y} * \mathcal{Z}$]

ZFsub **proof of** eqMultiplication:

L01:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}, \mathcal{Z}$	;
L02:	Premise $\gg$	$\mathcal{X} = \mathcal{Y}$	;
L03:	eqMultiplicationAxiom $\gg$	$\mathcal{X} = \mathcal{Y} \Rightarrow \mathcal{X} * \mathcal{Z} = \mathcal{Y} * \mathcal{Z}$	;
L04:	MP $\rhd$ L03 $\rhd$ L02 $\gg$	$\mathcal{X} * \mathcal{Z} = \mathcal{Y} * \mathcal{Z}$	$\square$

(*** UDSAGNSLOGIK ***)

[ZFsub **lemma** ToNegatedImply: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A} \vdash \dot{\neg}\,\mathcal{B} \vdash \dot{\neg}\,(\mathcal{A} \Rightarrow \mathcal{B})$]

ZFsub **proof of** ToNegatedImply:

L01:	Block $\gg$	Begin	;
L02:	Arbitrary $\gg$	$\mathcal{A},\mathcal{B}$	;
L03:	Premise $\gg$	$\mathcal{A}$	;
L04:	Premise $\gg$	$\dot{\neg}\,\mathcal{B}$	;
L05:	Premise $\gg$	$\dot{\neg}\,\dot{\neg}\,(\mathcal{A} \Rightarrow \mathcal{B})$	;
L06:	RemoveDoubleNeg $\rhd$ L05 $\gg$	$\mathcal{A} \Rightarrow \mathcal{B}$	;
L07:	MP $\rhd$ L06 $\rhd$ L03 $\gg$	$\mathcal{B}$	;
L08:	FromContradiction $\rhd$ L07 $\rhd$ L04 $\gg$	$\dot{\neg}\,(\mathcal{A} \Rightarrow \mathcal{B})$	;
L09:	Block $\gg$	End	;
L10:	Arbitrary $\gg$	$\mathcal{A},\mathcal{B}$	;
L11:	Ded $\rhd$ L09 $\gg$	$\mathcal{A} \Rightarrow \dot{\neg}\,\mathcal{B} \Rightarrow \dot{\neg}\,\dot{\neg}\,(\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow \dot{\neg}\,(\mathcal{A} \Rightarrow \mathcal{B})$	;
L12:	Premise $\gg$	$\mathcal{A}$	;
L13:	Premise $\gg$	$\dot{\neg}\,\mathcal{B}$	;
L14:	MP2 $\rhd$ L11 $\rhd$ L12 $\rhd$ L13 $\gg$	$\dot{\neg}\,\dot{\neg}\,(\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow \dot{\neg}\,(\mathcal{A} \Rightarrow \mathcal{B})$	;
L15:	AutoImply $\gg$	$\dot{\neg}\,\dot{\neg}\,(\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow \dot{\neg}\,\dot{\neg}\,(\mathcal{A} \Rightarrow \mathcal{B})$	;
L16:	Neg $\rhd$ L14 $\rhd$ L15 $\gg$	$\dot{\neg}\,(\mathcal{A} \Rightarrow \mathcal{B})$	$\square$

[ZFsub **lemma** TND: $\Pi\mathcal{A}\colon \mathcal{A} \mathrel{\dot{\vee}} \dot{\neg}\,\mathcal{A}$]

ZFsub **proof of** TND:

L01:	Arbitrary $\gg$	$\mathcal{A}$	;
L02:	AutoImply $\gg$	$\dot{\neg}\,\mathcal{A} \Rightarrow \dot{\neg}\,\mathcal{A}$	;
L03:	Repetition $\rhd$ L02 $\gg$	$\mathcal{A} \mathrel{\dot{\vee}} \dot{\neg}\,\mathcal{A}$	$\square$

[ZFsub **lemma** FromNegations: $\Pi\mathcal{A},\mathcal{B}\colon \mathcal{A} \Rightarrow \mathcal{B} \vdash \dot{\neg}\,\mathcal{A} \Rightarrow \mathcal{B} \vdash \mathcal{B}$]

ZFsub **proof of** FromNegations:

L01:	Arbitrary $\gg$	$\mathcal{A},\mathcal{B}$	;
L02:	Premise $\gg$	$\mathcal{A} \Rightarrow \mathcal{B}$	;
L03:	Premise $\gg$	$\dot{\neg}\,\mathcal{A} \Rightarrow \mathcal{B}$	;
L04:	TND $\gg$	$\mathcal{A} \mathrel{\dot{\vee}} \dot{\neg}\,\mathcal{A}$	;
L05:	FromDisjuncts $\rhd$ L04 $\rhd$ L02 $\rhd$ L03 $\gg$	$\mathcal{B}$	$\square$

[ZFsub **lemma** ImplyNegation: $\Pi\mathcal{A}\colon \mathcal{A} \Rightarrow \dot{\neg}\,\mathcal{A} \vdash \dot{\neg}\,\mathcal{A}$]

ZFsub **proof of** ImplyNegation:

L01:	Arbitrary $\gg$	$\mathcal{A}$	;
L02:	Premise $\gg$	$\mathcal{A} \Rightarrow \dot{\neg}\,\mathcal{A}$	;
L03:	AutoImply $\gg$	$\dot{\neg}\,\mathcal{A} \Rightarrow \dot{\neg}\,\mathcal{A}$	;
L04:	TND $\gg$	$\mathcal{A} \mathrel{\dot{\vee}} \dot{\neg}\,\mathcal{A}$	;
L05:	FromDisjuncts $\rhd$ L04 $\rhd$ L02 $\rhd$ L03 $\gg$	$\dot{\neg}\,\mathcal{A}$	$\square$

[ZFsub **lemma** From3Disjuncts: $\Pi\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}\colon \mathcal{A} \mathrel{\dot{\vee}} \mathcal{B} \mathrel{\dot{\vee}} \mathcal{C} \vdash \mathcal{A} \Rightarrow \mathcal{D} \vdash \mathcal{B} \Rightarrow \mathcal{D} \vdash \mathcal{C} \Rightarrow \mathcal{D} \vdash \mathcal{D}$]

ZFsub **proof of** From3Disjuncts:

L01:	Block $\gg$	Begin	;
L02:	Arbitrary $\gg$	$\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$	;

L03:	Premise $\gg$	$\mathcal{A} \dot{\vee} \mathcal{B} \dot{\vee} \mathcal{C}$	;
L04:	Premise $\gg$	$\mathcal{B} \Rightarrow \mathcal{D}$	;
L05:	Premise $\gg$	$\mathcal{C} \Rightarrow \mathcal{D}$	;
L06:	Premise $\gg$	$\dot{\neg}\, \mathcal{A}$	;
L07:	Repetition $\rhd$ L03 $\gg$	$\dot{\neg}\, \mathcal{A} \Rightarrow (\mathcal{B} \dot{\vee} \mathcal{C})$	;
L08:	MP $\rhd$ L07 $\rhd$ L06 $\gg$	$\mathcal{B} \dot{\vee} \mathcal{C}$	;
L09:	FromDisjuncts $\rhd$ L08 $\rhd$ L04 $\rhd$ L05 $\gg$	$\mathcal{D}$	;
L10:	Block $\gg$	End	;
L11:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$	;
L12:	Ded $\rhd$ L10 $\gg$	$\mathcal{A} \dot{\vee} \mathcal{B} \dot{\vee} \mathcal{C} \Rightarrow (\mathcal{B} \Rightarrow \mathcal{D}) \Rightarrow (\mathcal{C} \Rightarrow \mathcal{D}) \Rightarrow \dot{\neg}\, \mathcal{A} \Rightarrow \mathcal{D}$	;
L13:	AutoImply $\gg$	$(\mathcal{A} \Rightarrow \mathcal{D}) \Rightarrow \mathcal{A} \Rightarrow \mathcal{D}$	;
L14:	Premise $\gg$	$\mathcal{A} \dot{\vee} \mathcal{B} \dot{\vee} \mathcal{C}$	;
L15:	Premise $\gg$	$\mathcal{A} \Rightarrow \mathcal{D}$	;
L16:	Premise $\gg$	$\mathcal{B} \Rightarrow \mathcal{D}$	;
L17:	Premise $\gg$	$\mathcal{C} \Rightarrow \mathcal{D}$	;
L18:	MP3 $\rhd$ L12 $\rhd$ L14 $\rhd$ L16 $\rhd$ L17 $\gg$	$\dot{\neg}\, \mathcal{A} \Rightarrow \mathcal{D}$	;
L19:	MP $\rhd$ L13 $\rhd$ L15 $\gg$	$\mathcal{A} \Rightarrow \mathcal{D}$	;
L20:	FromNegations $\rhd$ L19 $\rhd$ L18 $\gg$	$\mathcal{D}$	$\Box$

[ZFsub **lemma** NegateDisjunct1: $\Pi \mathcal{A}, \mathcal{B}\colon \mathcal{A} \dot{\vee} \mathcal{B} \vdash \dot{\neg}\, \mathcal{A} \vdash \mathcal{B}$]

ZFsub **proof of** NegateDisjunct1:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}$	;
L02:	Premise $\gg$	$\mathcal{A} \dot{\vee} \mathcal{B}$	;
L03:	Premise $\gg$	$\dot{\neg}\, \mathcal{A}$	;
L04:	Repetition $\rhd$ L02 $\gg$	$\dot{\neg}\, \mathcal{A} \Rightarrow \mathcal{B}$	;
L05:	MP $\rhd$ L04 $\rhd$ L03 $\gg$	$\mathcal{B}$	$\Box$

[ZFsub **lemma** NegateDisjunct2: $\Pi \mathcal{A}, \mathcal{B}\colon \mathcal{A} \dot{\vee} \mathcal{B} \vdash \dot{\neg}\, \mathcal{B} \vdash \mathcal{A}$]

ZFsub **proof of** NegateDisjunct2:

L01:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}$	;
L02:	Premise $\gg$	$\mathcal{A} \dot{\vee} \mathcal{B}$	;
L03:	Premise $\gg$	$\dot{\neg}\, \mathcal{B}$	;
L04:	Repetition $\rhd$ L02 $\gg$	$\dot{\neg}\, \mathcal{A} \Rightarrow \mathcal{B}$	;
L05:	NegativeMT $\rhd$ L04 $\rhd$ L03 $\gg$	$\mathcal{A}$	$\Box$

(***)

[ZFsub **lemma** ExpandDisjuncts: $\Pi \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}\colon \mathcal{A} \dot{\vee} \mathcal{B} \vdash \mathcal{C} \dot{\vee} \mathcal{D} \vdash \mathcal{B} \dot{\vee} \mathcal{D} \dot{\vee} (\mathcal{A} \dot{\wedge} \mathcal{C})$]

ZFsub **proof of** ExpandDisjuncts:

L01:	Block $\gg$	Begin	;
L02:	Arbitrary $\gg$	$\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$	;
L03:	Premise $\gg$	$\mathcal{A} \dot{\vee} \mathcal{B}$	;
L04:	Premise $\gg$	$\mathcal{C} \dot{\vee} \mathcal{D}$	;
L05:	Premise $\gg$	$\dot{\neg}\, \mathcal{B}$	;
L06:	Premise $\gg$	$\dot{\neg}\, \mathcal{D}$	;
L07:	NegateDisjunct2 $\rhd$ L03 $\rhd$ L05 $\gg$	$\mathcal{A}$	;
L08:	NegateDisjunct2 $\rhd$ L04 $\rhd$ L06 $\gg$	$\mathcal{C}$	;

L09:	JoinConjuncts ▷ L07 ▷ L08 ≫	$\mathcal{A} \dot{\wedge} \mathcal{C}$	;
L10:	Block ≫	End	;
L11:	Arbitrary ≫	$\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$	;
L12:	Ded ▷ L10 ≫	$\mathcal{A} \dot{\vee} \mathcal{B} \Rightarrow \mathcal{C} \dot{\vee} \mathcal{D} \Rightarrow \dot{\neg} \mathcal{B} \Rightarrow \dot{\neg} \mathcal{D} \Rightarrow \mathcal{A} \dot{\wedge} \mathcal{C}$	;
L13:	Premise ≫	$\mathcal{A} \dot{\vee} \mathcal{B}$	;
L14:	Premise ≫	$\mathcal{C} \dot{\vee} \mathcal{D}$	;
L15:	MP2 ▷ L12 ▷ L13 ▷ L14 ≫	$\dot{\neg} \mathcal{B} \Rightarrow \dot{\neg} \mathcal{D} \Rightarrow \mathcal{A} \dot{\wedge} \mathcal{C}$	;
L16:	Repetition ▷ L15 ≫	$\mathcal{B} \dot{\vee} \mathcal{D} \dot{\vee} (\mathcal{A} \dot{\wedge} \mathcal{C})$	□

[ZFsub **lemma** From2 ∗ 2Disjuncts: $\Pi \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E} \colon \mathcal{A} \dot{\vee} \mathcal{B} \vdash \mathcal{C} \dot{\vee} \mathcal{D} \vdash \mathcal{A} \Rightarrow \mathcal{C} \Rightarrow \mathcal{E} \vdash \mathcal{A} \Rightarrow \mathcal{D} \Rightarrow \mathcal{E} \vdash \mathcal{B} \Rightarrow \mathcal{C} \Rightarrow \mathcal{E} \vdash \mathcal{B} \Rightarrow \mathcal{D} \Rightarrow \mathcal{E} \vdash \mathcal{E}$]

ZFsub **proof of** From2 ∗ 2Disjuncts:

L01:	Block ≫	Begin	;
L02:	Arbitrary ≫	$\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$	;
L03:	Premise ≫	$\mathcal{C} \dot{\vee} \mathcal{D}$	;
L04:	Premise ≫	$\mathcal{A} \Rightarrow \mathcal{C} \Rightarrow \mathcal{E}$	;
L05:	Premise ≫	$\mathcal{A} \Rightarrow \mathcal{D} \Rightarrow \mathcal{E}$	;
L06:	Premise ≫	$\mathcal{A}$	;
L07:	MP ▷ L04 ▷ L06 ≫	$\mathcal{C} \Rightarrow \mathcal{E}$	;
L08:	MP ▷ L05 ▷ L06 ≫	$\mathcal{D} \Rightarrow \mathcal{E}$	;
L09:	FromDisjuncts ▷ L03 ▷ L07 ▷ L08 ≫	$\mathcal{E}$	;
L10:	Block ≫	End	;
L11:	Block ≫	Begin	;
L12:	Arbitrary ≫	$\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$	;
L13:	Premise ≫	$\mathcal{A} \dot{\vee} \mathcal{B}$	;
L14:	Premise ≫	$\mathcal{C} \dot{\vee} \mathcal{D}$	;
L15:	Premise ≫	$\mathcal{B} \Rightarrow \mathcal{C} \Rightarrow \mathcal{E}$	;
L16:	Premise ≫	$\mathcal{B} \Rightarrow \mathcal{D} \Rightarrow \mathcal{E}$	;
L17:	Premise ≫	$\dot{\neg} \mathcal{A}$	;
L18:	NegateDisjunct1 ▷ L13 ▷ L17 ≫	$\mathcal{B}$	;
L19:	MP ▷ L15 ▷ L18 ≫	$\mathcal{C} \Rightarrow \mathcal{E}$	;
L20:	MP ▷ L16 ▷ L18 ≫	$\mathcal{D} \Rightarrow \mathcal{E}$	;
L21:	FromDisjuncts ▷ L14 ▷ L19 ▷ L20 ≫	$\mathcal{E}$	;
L22:	Block ≫	End	;
L23:	Arbitrary ≫	$\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$	;
L24:	Ded ▷ L10 ≫	$\mathcal{C} \dot{\vee} \mathcal{D} \Rightarrow (\mathcal{A} \Rightarrow \mathcal{C} \Rightarrow \mathcal{E}) \Rightarrow (\mathcal{A} \Rightarrow \mathcal{D} \Rightarrow \mathcal{E}) \Rightarrow \mathcal{A} \Rightarrow \mathcal{E}$	;
L25:	Ded ▷ L22 ≫	$\mathcal{A} \dot{\vee} \mathcal{B} \Rightarrow \mathcal{C} \dot{\vee} \mathcal{D} \Rightarrow (\mathcal{B} \Rightarrow \mathcal{C} \Rightarrow \mathcal{E}) \Rightarrow (\mathcal{B} \Rightarrow \mathcal{D} \Rightarrow \mathcal{E}) \Rightarrow \dot{\neg} \mathcal{A} \Rightarrow \mathcal{E}$	;
L26:	Premise ≫	$\mathcal{A} \dot{\vee} \mathcal{B}$	;
L27:	Premise ≫	$\mathcal{C} \dot{\vee} \mathcal{D}$	;
L28:	Premise ≫	$\mathcal{A} \Rightarrow \mathcal{C} \Rightarrow \mathcal{E}$	;
L29:	Premise ≫	$\mathcal{A} \Rightarrow \mathcal{D} \Rightarrow \mathcal{E}$	;

L30: Premise $\gg$ $\mathcal{B} \Rightarrow \mathcal{C} \Rightarrow \mathcal{E}$;
L31: Premise $\gg$ $\mathcal{B} \Rightarrow \mathcal{D} \Rightarrow \mathcal{E}$;
L32: MP3 $\rhd$ L24 $\rhd$ L27 $\rhd$ L28 $\rhd$ L29 $\gg$ $\mathcal{A} \Rightarrow \mathcal{E}$;
L33: MP4 $\rhd$ L25 $\rhd$ L26 $\rhd$ L27 $\rhd$
L30 $\rhd$ L31 $\gg$ $\dot{\neg} \mathcal{A} \Rightarrow \mathcal{E}$;
L34: FromNegations $\rhd$ L32 $\rhd$ L33 $\gg$ $\mathcal{E}$ □

(*** EQUALITY ***)

[ZFsub **lemma** eqReflexivity: $\Pi \mathcal{X}: \mathcal{X} = \mathcal{X}$]

ZFsub **proof of** eqReflexivity:

L01: Arbitrary $\gg$ $\mathcal{X}$;
L02: leqReflexivity $\gg$ $\mathcal{X} <= \mathcal{X}$;
L03: leqAntisymmetry $\rhd$ L02 $\rhd$ L02 $\gg$
$\mathcal{X} = \mathcal{X}$ □

[ZFsub **lemma** eqSymmetry: $\Pi \mathcal{X}, \mathcal{Y}: \mathcal{X} = \mathcal{Y} \vdash \mathcal{Y} = \mathcal{X}$]

ZFsub **proof of** eqSymmetry:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}$;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$;
L03: eqReflexivity $\gg$ $\mathcal{X} = \mathcal{X}$;
L04: Equality $\rhd$ L02 $\rhd$ L03 $\gg$ $\mathcal{Y} = \mathcal{X}$ □

[ZFsub **lemma** eqTransitivity: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}: \mathcal{X} = \mathcal{Y} \vdash \mathcal{Y} = \mathcal{Z} \vdash \mathcal{X} = \mathcal{Z}$]

ZFsub **proof of** eqTransitivity:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$;
L03: Premise $\gg$ $\mathcal{Y} = \mathcal{Z}$;
L04: eqSymmetry $\rhd$ L02 $\gg$ $\mathcal{Y} = \mathcal{X}$;
L05: Equality $\rhd$ L04 $\rhd$ L03 $\gg$ $\mathcal{X} = \mathcal{Z}$ □

[ZFsub **lemma** eqTransitivity4: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}: \mathcal{X} = \mathcal{Y} \vdash \mathcal{Y} = \mathcal{Z} \vdash \mathcal{Z} = \mathcal{U} \vdash \mathcal{X} = \mathcal{U}$]

ZFsub **proof of** eqTransitivity4:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}$;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$;
L03: Premise $\gg$ $\mathcal{Y} = \mathcal{Z}$;
L04: Premise $\gg$ $\mathcal{Z} = \mathcal{U}$;
L05: eqTransitivity $\rhd$ L02 $\rhd$ L03 $\gg$ $\mathcal{X} = \mathcal{Z}$;
L06: eqTransitivity $\rhd$ L05 $\rhd$ L04 $\gg$ $\mathcal{X} = \mathcal{U}$ □

[ZFsub **lemma** eqTransitivity5: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}, \mathcal{V}: \mathcal{X} = \mathcal{Y} \vdash \mathcal{Y} = \mathcal{Z} \vdash \mathcal{Z} = \mathcal{U} \vdash \mathcal{U} = \mathcal{V} \vdash \mathcal{X} = \mathcal{V}$]

ZFsub **proof of** eqTransitivity5:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}, \mathcal{V}$;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$;
L03: Premise $\gg$ $\mathcal{Y} = \mathcal{Z}$;
L04: Premise $\gg$ $\mathcal{Z} = \mathcal{U}$;
L05: Premise $\gg$ $\mathcal{U} = \mathcal{V}$;
L06: eqTransitivity4 $\rhd$ L02 $\rhd$ L03 $\rhd$
L04 $\gg$ $\mathcal{X} = \mathcal{U}$;
L07: eqTransitivity $\rhd$ L06 $\rhd$ L05 $\gg$ $\mathcal{X} = \mathcal{V}$ □

[ZFsub **lemma** eqTransitivity6: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{U},\mathcal{V},\mathcal{W}\colon \mathcal{X}=\mathcal{Y}\vdash\mathcal{Y}=\mathcal{Z}\vdash\mathcal{Z}=\mathcal{U}\vdash\mathcal{U}=\mathcal{V}\vdash\mathcal{V}=\mathcal{W}\vdash\mathcal{X}=\mathcal{W}$]

ZFsub **proof of** eqTransitivity6:

L01: Arbitrary ≫ $\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{U},\mathcal{V},\mathcal{W}$;
L02: Premise ≫ $\mathcal{X}=\mathcal{Y}$;
L03: Premise ≫ $\mathcal{Y}=\mathcal{Z}$;
L04: Premise ≫ $\mathcal{Z}=\mathcal{U}$;
L05: Premise ≫ $\mathcal{U}=\mathcal{V}$;
L06: Premise ≫ $\mathcal{V}=\mathcal{W}$;
L07: eqTransitivity5 ▷ L02 ▷ L03 ▷ L04 ▷ L05 ≫ $\mathcal{X}=\mathcal{V}$;
L08: eqTransitivity ▷ L07 ▷ L06 ≫ $\mathcal{X}=\mathcal{W}$ □

[ZFsub **lemma** plus0Left: $\Pi\mathcal{X}\colon 0+\mathcal{X}=\mathcal{X}$]

ZFsub **proof of** plus0Left:

L01: Arbitrary ≫ $\mathcal{X}$;
L02: plus0 ≫ $\mathcal{X}+0=\mathcal{X}$;
L03: plusCommutativity ≫ $0+\mathcal{X}=\mathcal{X}+0$;
L04: eqTransitivity ▷ L03 ▷ L02 ≫ $0+\mathcal{X}=\mathcal{X}$ □

[ZFsub **lemma** times1Left: $\Pi\mathcal{X}\colon 1*\mathcal{X}=\mathcal{X}$]

ZFsub **proof of** times1Left:

L01: Arbitrary ≫ $\mathcal{X}$;
L02: times1 ≫ $\mathcal{X}*1=\mathcal{X}$;
L03: timesCommutativity ≫ $1*\mathcal{X}=\mathcal{X}*1$;
L04: eqTransitivity ▷ L03 ▷ L02 ≫ $1*\mathcal{X}=\mathcal{X}$ □

[ZFsub **lemma** lemma eqAdditionLeft: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z}\colon \mathcal{X}=\mathcal{Y}\vdash\mathcal{Z}+\mathcal{X}=\mathcal{Z}+\mathcal{Y}$]

ZFsub **proof of** lemma eqAdditionLeft:

L01: Arbitrary ≫ $\mathcal{X},\mathcal{Y},\mathcal{Z}$;
L02: Premise ≫ $\mathcal{X}=\mathcal{Y}$;
L03: eqAddition ▷ L02 ≫ $\mathcal{X}+\mathcal{Z}=\mathcal{Y}+\mathcal{Z}$;
L04: plusCommutativity ≫ $\mathcal{Z}+\mathcal{X}=\mathcal{X}+\mathcal{Z}$;
L05: plusCommutativity ≫ $\mathcal{Y}+\mathcal{Z}=\mathcal{Z}+\mathcal{Y}$;
L06: eqTransitivity4 ▷ L04 ▷ L03 ▷ L05 ≫ $\mathcal{Z}+\mathcal{X}=\mathcal{Z}+\mathcal{Y}$ □

[ZFsub **lemma** EqMultiplicationLeft: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z}\colon \mathcal{X}=\mathcal{Y}\vdash\mathcal{Z}*\mathcal{X}=\mathcal{Z}*\mathcal{Y}$]

ZFsub **proof of** EqMultiplicationLeft:

L01: Arbitrary ≫ $\mathcal{X},\mathcal{Y},\mathcal{Z}$;
L02: Premise ≫ $\mathcal{X}=\mathcal{Y}$;
L03: eqMultiplication ▷ L02 ≫ $\mathcal{X}*\mathcal{Z}=\mathcal{Y}*\mathcal{Z}$;
L04: timesCommutativity ≫ $\mathcal{Z}*\mathcal{X}=\mathcal{X}*\mathcal{Z}$;
L05: timesCommutativity ≫ $\mathcal{Y}*\mathcal{Z}=\mathcal{Z}*\mathcal{Y}$;
L06: eqTransitivity4 ▷ L04 ▷ L03 ▷ L05 ≫ $\mathcal{Z}*\mathcal{X}=\mathcal{Z}*\mathcal{Y}$ □

[ZFsub **lemma** DistributionOut: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z}\colon \mathcal{X}*\mathcal{Y}+\mathcal{X}*\mathcal{Z}=\mathcal{X}*(\mathcal{Y}+\mathcal{Z})$]

ZFsub **proof of** DistributionOut:

L01: Arbitrary ≫ $\mathcal{X},\mathcal{Y},\mathcal{Z}$;
L02: Distribution ≫ $\mathcal{X}*(\mathcal{Y}+\mathcal{Z})=\mathcal{X}*\mathcal{Y}+\mathcal{X}*\mathcal{Z}$;

L03: eqSymmetry ▷ L02 ≫ $\mathcal{X} * \mathcal{Y} + \mathcal{X} * \mathcal{Z} = \mathcal{X} * (\mathcal{Y} + \mathcal{Z})$ □

[ZFsub **lemma** Three2twoTerms: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}: \mathcal{Y} + \mathcal{Z} = \mathcal{U} \vdash \mathcal{X} + \mathcal{Y} + \mathcal{Z} = \mathcal{X} + \mathcal{U}$]

ZFsub **proof of** Three2twoTerms:

L01: Arbitrary ≫ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}$;
L02: Premise ≫ $\mathcal{Y} + \mathcal{Z} = \mathcal{U}$;
L03: lemma eqAdditionLeft▷L02 ≫ $\mathcal{X} + (\mathcal{Y} + \mathcal{Z}) = \mathcal{X} + \mathcal{U}$;
L04: plusAssociativity ≫ $\mathcal{X} + \mathcal{Y} + \mathcal{Z} = \mathcal{X} + (\mathcal{Y} + \mathcal{Z})$;
L05: eqTransitivity ▷ L04 ▷ L03 ≫ $\mathcal{X} + \mathcal{Y} + \mathcal{Z} = \mathcal{X} + \mathcal{U}$ □

[ZFsub **lemma** Three2threeTerms: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}: \mathcal{X} + \mathcal{Y} + \mathcal{Z} = \mathcal{X} + \mathcal{Z} + \mathcal{Y}$]

ZFsub **proof of** Three2threeTerms:

L01: Arbitrary ≫ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$;
L02: plusCommutativity ≫ $\mathcal{Y} + \mathcal{Z} = \mathcal{Z} + \mathcal{Y}$;
L03: Three2twoTerms ▷ L02 ≫ $\mathcal{X} + \mathcal{Y} + \mathcal{Z} = \mathcal{X} + (\mathcal{Z} + \mathcal{Y})$;
L04: plusAssociativity ≫ $\mathcal{X} + \mathcal{Z} + \mathcal{Y} = \mathcal{X} + (\mathcal{Z} + \mathcal{Y})$;
L05: eqSymmetry ▷ L04 ≫ $\mathcal{X} + (\mathcal{Z} + \mathcal{Y}) = \mathcal{X} + \mathcal{Z} + \mathcal{Y}$;
L06: eqTransitivity ▷ L03 ▷ L05 ≫ $\mathcal{X} + \mathcal{Y} + \mathcal{Z} = \mathcal{X} + \mathcal{Z} + \mathcal{Y}$ □

[ZFsub **lemma** Three2threeFactors: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}: \mathcal{Y} * \mathcal{Z} = \mathcal{U} \vdash \mathcal{X} * \mathcal{Y} * \mathcal{Z} = \mathcal{X} * \mathcal{U}$]

ZFsub **proof of** Three2threeFactors:

L01: Arbitrary ≫ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}$;
L02: Premise ≫ $\mathcal{Y} * \mathcal{Z} = \mathcal{U}$;
L03: EqMultiplicationLeft ▷ L02 ≫ $\mathcal{X} * (\mathcal{Y} * \mathcal{Z}) = \mathcal{X} * \mathcal{U}$;
L04: timesAssociativity ≫ $\mathcal{X} * \mathcal{Y} * \mathcal{Z} = \mathcal{X} * (\mathcal{Y} * \mathcal{Z})$;
L05: eqTransitivity ▷ L04 ▷ L03 ≫ $\mathcal{X} * \mathcal{Y} * \mathcal{Z} = \mathcal{X} * \mathcal{U}$ □

[ZFsub **lemma** AddEquations: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}: \mathcal{X} = \mathcal{Y} \vdash \mathcal{Z} = \mathcal{U} \vdash \mathcal{X} + \mathcal{Z} = \mathcal{Y} + \mathcal{U}$]

ZFsub **proof of** AddEquations:

L01: Arbitrary ≫ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}$;
L02: Premise ≫ $\mathcal{X} = \mathcal{Y}$;
L03: Premise ≫ $\mathcal{Z} = \mathcal{U}$;
L04: eqAddition ▷ L02 ≫ $\mathcal{X} + \mathcal{Z} = \mathcal{Y} + \mathcal{Z}$;
L05: lemma eqAdditionLeft▷L03 ≫ $\mathcal{Y} + \mathcal{Z} = \mathcal{Y} + \mathcal{U}$;
L06: eqTransitivity ▷ L04 ▷ L05 ≫ $\mathcal{X} + \mathcal{Z} = \mathcal{Y} + \mathcal{U}$ □

[ZFsub **lemma** SubtractEquations: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}: \mathcal{X} + \mathcal{Z} = \mathcal{Y} + \mathcal{U} \vdash \mathcal{Z} = \mathcal{U} \vdash \mathcal{X} = \mathcal{Y}$]

ZFsub **proof of** SubtractEquations:

L01: Arbitrary ≫ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}$;
L02: Premise ≫ $\mathcal{X} + \mathcal{Z} = \mathcal{Y} + \mathcal{U}$;
L03: Premise ≫ $\mathcal{Z} = \mathcal{U}$;
L04: eqAddition ▷ L02 ≫ $\mathcal{X} + \mathcal{Z} - \mathcal{Z} = \mathcal{Y} + \mathcal{U} - \mathcal{Z}$;
L05: plus0Left ≫ $0 + \mathcal{Z} = \mathcal{Z}$;
L06: eqTransitivity ▷ L05 ▷ L03 ≫ $0 + \mathcal{Z} = \mathcal{U}$;
L07: PositiveToRight(Eq) ▷ L06 ≫ $0 = \mathcal{U} - \mathcal{Z}$;
L08: eqSymmetry ▷ L07 ≫ $\mathcal{U} - \mathcal{Z} = 0$;
L09: lemma eqAdditionLeft▷L08 ≫ $\mathcal{Y} + (\mathcal{U} - \mathcal{Z}) = \mathcal{Y} + 0$;

L10: plusAssociativity $\gg$ $\mathcal{Y} + \mathcal{U} - \mathcal{Z} = \mathcal{Y} + (\mathcal{U} - \mathcal{Z})$;
L11: plus0 $\gg$ $\mathcal{Y} + 0 = \mathcal{Y}$;
L12: eqTransitivity4 $\rhd$ L10 $\rhd$ L09 $\rhd$
L11 $\gg$ $\mathcal{Y} + \mathcal{U} - \mathcal{Z} = \mathcal{Y}$;
L13: x = x + y − y $\gg$ $\mathcal{X} = \mathcal{X} + \mathcal{Z} - \mathcal{Z}$;
L14: eqTransitivity4 $\rhd$ L13 $\rhd$ L04 $\rhd$
L12 $\gg$ $\mathcal{X} = \mathcal{Y}$ $\square$

[ZFsub **lemma** SubtractEquationsLeft: $\Pi\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}: \mathcal{X} + \mathcal{Z} = \mathcal{Y} + \mathcal{U} \vdash \mathcal{X} = \mathcal{Y} \vdash \mathcal{Z} = \mathcal{U}$]

ZFsub **proof of** SubtractEquationsLeft:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}$;
L02: Premise $\gg$ $\mathcal{X} + \mathcal{Z} = \mathcal{Y} + \mathcal{U}$;
L03: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$;
L04: plusCommutativity $\gg$ $\mathcal{Z} + \mathcal{X} = \mathcal{X} + \mathcal{Z}$;
L05: plusCommutativity $\gg$ $\mathcal{Y} + \mathcal{U} = \mathcal{U} + \mathcal{Y}$;
L06: eqTransitivity4 $\rhd$ L04 $\rhd$ L02 $\rhd$
L05 $\gg$ $\mathcal{Z} + \mathcal{X} = \mathcal{U} + \mathcal{Y}$;
L07: SubtractEquations $\rhd$ L06 $\rhd$
L03 $\gg$ $\mathcal{Z} = \mathcal{U}$ $\square$

[ZFsub **lemma** EqNegated: $\Pi\mathcal{X}, \mathcal{Y}: \mathcal{X} = \mathcal{Y} \vdash -\mathcal{X} = -\mathcal{Y}$]

ZFsub **proof of** EqNegated:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}$;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$;
L03: Negative $\gg$ $\mathcal{X} - \mathcal{X} = 0$;
L04: Negative $\gg$ $\mathcal{Y} - \mathcal{Y} = 0$;
L05: eqSymmetry $\rhd$ L04 $\gg$ $0 = \mathcal{Y} - \mathcal{Y}$;
L06: eqTransitivity $\rhd$ L03 $\rhd$ L05 $\gg$ $\mathcal{X} - \mathcal{X} = \mathcal{Y} - \mathcal{Y}$;
L07: SubtractEquationsLeft $\rhd$ L06 $\rhd$
L02 $\gg$ $-\mathcal{X} = -\mathcal{Y}$ $\square$

[ZFsub **lemma** PositiveToRight(Eq): $\Pi\mathcal{X}, \mathcal{Y}, \mathcal{Z}: \mathcal{X} + \mathcal{Y} = \mathcal{Z} \vdash \mathcal{X} = \mathcal{Z} - \mathcal{Y}$]

ZFsub **proof of** PositiveToRight(Eq):

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$;
L02: Premise $\gg$ $\mathcal{X} + \mathcal{Y} = \mathcal{Z}$;
L03: eqAddition $\rhd$ L02 $\gg$ $\mathcal{X} + \mathcal{Y} - \mathcal{Y} = \mathcal{Z} - \mathcal{Y}$;
L04: x = x + y − y $\gg$ $\mathcal{X} = \mathcal{X} + \mathcal{Y} - \mathcal{Y}$;
L05: eqTransitivity $\rhd$ L04 $\rhd$ L03 $\gg$ $\mathcal{X} = \mathcal{Z} - \mathcal{Y}$ $\square$

[ZFsub **lemma** PositiveToLeft(Eq)(1term): $\Pi\mathcal{X}, \mathcal{Y}: \mathcal{X} = \mathcal{Y} \vdash \mathcal{X} - \mathcal{Y} = 0$]

ZFsub **proof of** PositiveToLeft(Eq)(1term):

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}$;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$;
L03: eqAddition $\rhd$ L02 $\gg$ $\mathcal{X} - \mathcal{Y} = \mathcal{Y} - \mathcal{Y}$;
L04: Negative $\gg$ $\mathcal{Y} - \mathcal{Y} = 0$;
L05: eqTransitivity $\rhd$ L03 $\rhd$ L04 $\gg$ $\mathcal{X} - \mathcal{Y} = 0$ $\square$

[ZFsub **lemma** PositiveToRight(Leq)(1term): $\Pi\mathcal{Y}, \mathcal{Z}: \mathcal{Y} <= \mathcal{Z} \vdash 0 <= \mathcal{Z} - \mathcal{Y}$]

ZFsub **proof of** PositiveToRight(Leq)(1term):

L01:	Arbitrary $\gg$	$\mathcal{Y}, \mathcal{Z}$	;
L02:	Premise $\gg$	$\mathcal{Y} <= \mathcal{Z}$	;
L03:	plus0Left $\gg$	$0 + \mathcal{Y} = \mathcal{Y}$	;
L04:	eqSymmetry $\rhd$ L03 $\gg$	$\mathcal{Y} = 0 + \mathcal{Y}$	;
L05:	subLeqLeft $\rhd$ L04 $\rhd$ L02 $\gg$	$0 + \mathcal{Y} <= \mathcal{Z}$	;
L06:	PositiveToRight(Leq) $\rhd$ L05 $\gg$	$0 <= \mathcal{Z} - \mathcal{Y}$	$\Box$

[ZFsub **lemma** NegativeToLeft(Eq): $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}: \mathcal{X} = \mathcal{Y} - \mathcal{Z} \vdash \mathcal{X} + \mathcal{Z} = \mathcal{Y}$]

ZFsub **proof of** NegativeToLeft(Eq):

L01:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}, \mathcal{Z}$	;
L02:	Premise $\gg$	$\mathcal{X} = \mathcal{Y} - \mathcal{Z}$	;
L03:	eqAddition $\rhd$ L02 $\gg$	$\mathcal{X} + \mathcal{Z} = \mathcal{Y} - \mathcal{Z} + \mathcal{Z}$	;
L04:	Three2threeTerms $\gg$	$\mathcal{Y} - \mathcal{Z} + \mathcal{Z} = \mathcal{Y} + \mathcal{Z} - \mathcal{Z}$	;
L05:	x = x + y − y $\gg$	$\mathcal{Y} = \mathcal{Y} + \mathcal{Z} - \mathcal{Z}$	;
L06:	eqSymmetry $\rhd$ L05 $\gg$	$\mathcal{Y} + \mathcal{Z} - \mathcal{Z} = \mathcal{Y}$	;
L07:	eqTransitivity4 $\rhd$ L03 $\rhd$ L04 $\rhd$ L06 $\gg$	$\mathcal{X} + \mathcal{Z} = \mathcal{Y}$	$\Box$

(*** NO EQUALITY ***)

[ZFsub **lemma** LessNeq: $\Pi \mathcal{X}, \mathcal{Y}: \mathcal{X} < \mathcal{Y} \vdash \mathcal{X} \neq \mathcal{Y}$]

ZFsub **proof of** LessNeq:

L01:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}$	;
L02:	Premise $\gg$	$\mathcal{X} < \mathcal{Y}$	;
L03:	Repetition $\rhd$ L02 $\gg$	$\mathcal{X} <= \mathcal{Y} \dot{\wedge} \dot{\neg} (\mathcal{X} = \mathcal{Y})$	;
L04:	SecondConjunct $\rhd$ L03 $\gg$	$\mathcal{X} \neq \mathcal{Y}$	$\Box$

[ZFsub **lemma** NeqSymmetry: $\Pi \mathcal{X}, \mathcal{Y}: \mathcal{X} \neq \mathcal{Y} \vdash \mathcal{Y} \neq \mathcal{X}$]

ZFsub **proof of** NeqSymmetry:

L01:	Block $\gg$	Begin	;
L02:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}$	;
L03:	Premise $\gg$	$\mathcal{Y} = \mathcal{X}$	;
L04:	eqSymmetry $\rhd$ L03 $\gg$	$\mathcal{X} = \mathcal{Y}$	;
L05:	Block $\gg$	End	;
L06:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}$	;
L07:	Ded $\rhd$ L05 $\gg$	$\mathcal{Y} = \mathcal{X} \Rightarrow \mathcal{X} = \mathcal{Y}$	;
L08:	Premise $\gg$	$\mathcal{X} \neq \mathcal{Y}$	;
L09:	MT $\rhd$ L07 $\rhd$ L08 $\gg$	$\mathcal{Y} \neq \mathcal{X}$	$\Box$

[ZFsub **lemma** NeqNegated: $\Pi \mathcal{X}, \mathcal{Y}: \mathcal{X} \neq \mathcal{Y} \vdash -\mathcal{X} \neq -\mathcal{Y}$]

ZFsub **proof of** NeqNegated:

L01:	Block $\gg$	Begin	;
L02:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}$	;
L03:	Premise $\gg$	$\mathcal{X} \neq \mathcal{Y}$	;
L04:	Premise $\gg$	$-\mathcal{X} = -\mathcal{Y}$	;
L05:	EqNegated $\rhd$ L04 $\gg$	$- - \mathcal{X} = - - \mathcal{Y}$	;
L06:	DoubleMinus $\gg$	$- - \mathcal{X} = \mathcal{X}$	;
L07:	eqSymmetry $\rhd$ L06 $\gg$	$\mathcal{X} = - - \mathcal{X}$	;
L08:	DoubleMinus $\gg$	$- - \mathcal{Y} = \mathcal{Y}$	;

L09: eqTransitivity4 $\rhd$ L07 $\rhd$ L05 $\rhd$ L08 $\gg$ $\mathcal{X} = \mathcal{Y}$;
L10: FromContradiction $\rhd$ L09 $\rhd$ L03 $\gg$ $-\mathcal{X} \neq -\mathcal{Y}$;
L11: Block $\gg$ End ;
L12: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}$;
L13: Ded $\rhd$ L11 $\gg$ $\mathcal{X} \neq \mathcal{Y} \Rightarrow -\mathcal{X} = -\mathcal{Y} \Rightarrow \dot{\neg} - \mathcal{X} = -\mathcal{Y}$;
L14: Premise $\gg$ $\mathcal{X} \neq \mathcal{Y}$;
L15: MP $\rhd$ L13 $\rhd$ L14 $\gg$ $-\mathcal{X} = -\mathcal{Y} \Rightarrow \dot{\neg} - \mathcal{X} = -\mathcal{Y}$;
L16: ImplyNegation $\rhd$ L15 $\gg$ $\dot{\neg} - \mathcal{X} = -\mathcal{Y}$ □

[ZFsub **lemma** SubNeqRight: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}\colon \mathcal{X} = \mathcal{Y} \vdash \mathcal{Z} \neq \mathcal{X} \vdash \mathcal{Z} \neq \mathcal{Y}$]

ZFsub **proof of** SubNeqRight:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$;
L03: Premise $\gg$ $\mathcal{Z} \neq \mathcal{X}$;
L04: NeqSymmetry $\rhd$ L03 $\gg$ $\mathcal{X} \neq \mathcal{Z}$;
L05: SubNeqLeft $\rhd$ L02 $\rhd$ L04 $\gg$ $\mathcal{Y} \neq \mathcal{Z}$;
L06: NeqSymmetry $\rhd$ L05 $\gg$ $\mathcal{Z} \neq \mathcal{Y}$ □

[ZFsub **lemma** SubNeqLeft: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}\colon \mathcal{X} = \mathcal{Y} \vdash \mathcal{X} \neq \mathcal{Z} \vdash \mathcal{Y} \neq \mathcal{Z}$]

ZFsub **proof of** SubNeqLeft:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$;
L03: Premise $\gg$ $\mathcal{X} \neq \mathcal{Z}$;
L04: equalityAxiom $\gg$ $\mathcal{Y} = \mathcal{X} \Rightarrow \mathcal{Y} = \mathcal{Z} \Rightarrow \mathcal{X} = \mathcal{Z}$;
L05: eqSymmetry $\rhd$ L02 $\gg$ $\mathcal{Y} = \mathcal{X}$;
L06: MP $\rhd$ L04 $\rhd$ L05 $\gg$ $\mathcal{Y} = \mathcal{Z} \Rightarrow \mathcal{X} = \mathcal{Z}$;
L07: Contrapositive $\rhd$ L06 $\gg$ $\mathcal{X} \neq \mathcal{Z} \Rightarrow \mathcal{Y} \neq \mathcal{Z}$;
L08: MP $\rhd$ L07 $\rhd$ L03 $\gg$ $\mathcal{Y} \neq \mathcal{Z}$ □

[ZFsub **lemma** NeqAddition: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}\colon \mathcal{X} \neq \mathcal{Y} \vdash \mathcal{X} + \mathcal{Z} \neq \mathcal{Y} + \mathcal{Z}$]

ZFsub **proof of** NeqAddition:

L01: Block $\gg$ Begin ;
L02: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$;
L03: Premise $\gg$ $\mathcal{X} \neq \mathcal{Y}$;
L04: Premise $\gg$ $\mathcal{X} + \mathcal{Z} = \mathcal{Y} + \mathcal{Z}$;
L05: eqReflexivity $\gg$ $\mathcal{Z} = \mathcal{Z}$;
L06: SubtractEquations $\rhd$ L04 $\rhd$ L05 $\gg$ $\mathcal{X} = \mathcal{Y}$;
L07: FromContradiction $\rhd$ L06 $\rhd$ L03 $\gg$ $\mathcal{X} + \mathcal{Z} \neq \mathcal{Y} + \mathcal{Z}$;
L08: Block $\gg$ End ;
L09: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$;
L10: Ded $\rhd$ L08 $\gg$ $\mathcal{X} \neq \mathcal{Y} \Rightarrow \mathcal{X} + \mathcal{Z} = \mathcal{Y} + \mathcal{Z} \Rightarrow \mathcal{X} + \mathcal{Z} \neq \mathcal{Y} + \mathcal{Z}$;
L11: Premise $\gg$ $\mathcal{X} \neq \mathcal{Y}$;
L12: MP $\rhd$ L10 $\rhd$ L11 $\gg$ $\mathcal{X}+\mathcal{Z} = \mathcal{Y}+\mathcal{Z} \Rightarrow \mathcal{X}+\mathcal{Z} \neq \mathcal{Y}+\mathcal{Z}$;
L13: ImplyNegation $\rhd$ L12 $\gg$ $\mathcal{X} + \mathcal{Z} \neq \mathcal{Y} + \mathcal{Z}$ □

[ZFsub **lemma** NeqMultiplication: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z} \colon \mathcal{Z} \neq 0 \vdash \mathcal{X} \neq \mathcal{Y} \vdash \mathcal{X} * \mathcal{Z} \neq \mathcal{Y} * \mathcal{Z}$]

ZFsub **proof of** NeqMultiplication:

L01:	Block $\gg$	Begin	;
L02:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}, \mathcal{Z}$	;
L03:	Premise $\gg$	$\mathcal{Z} \neq 0$	;
L04:	Premise $\gg$	$\mathcal{X} \neq \mathcal{Y}$	;
L05:	Premise $\gg$	$\mathcal{X} * \mathcal{Z} = \mathcal{Y} * \mathcal{Z}$	;
L06:	$\rhd$L03 $\gg$	$\mathcal{X} = \mathcal{X} * \mathcal{Z} * \mathrm{rec}\mathcal{Z}$	;
L07:	eqMultiplication $\rhd$ L05 $\gg$	$\mathcal{X} * \mathcal{Z} * \mathrm{rec}\mathcal{Z} = \mathcal{Y} * \mathcal{Z} * \mathrm{rec}\mathcal{Z}$	;
L08:	$\rhd$L03 $\gg$	$\mathcal{Y} = \mathcal{Y} * \mathcal{Z} * \mathrm{rec}\mathcal{Z}$	;
L09:	eqSymmetry $\rhd$ L08 $\gg$	$\mathcal{Y} * \mathcal{Z} * \mathrm{rec}\mathcal{Z} = \mathcal{Y}$	;
L10:	eqTransitivity4 $\rhd$ L06 $\rhd$ L07 $\rhd$ L09 $\gg$	$\mathcal{X} = \mathcal{Y}$	;
L11:	FromContradiction $\rhd$ L10 $\rhd$ L04 $\gg$	$\mathcal{X} * \mathcal{Z} \neq \mathcal{Y} * \mathcal{Z}$	;
L12:	Block $\gg$	End	;
L13:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}, \mathcal{Z}$	;
L14:	Ded $\rhd$ L12 $\gg$	$\mathcal{Z} \neq 0 \Rightarrow \mathcal{X} \neq \mathcal{Y} \Rightarrow \mathcal{X} * \mathcal{Z} = \mathcal{Y} * \mathcal{Z} \Rightarrow \mathcal{X} * \mathcal{Z} \neq \mathcal{Y} * \mathcal{Z}$	;
L15:	Premise $\gg$	$\mathcal{Z} \neq 0$	;
L16:	Premise $\gg$	$\mathcal{X} \neq \mathcal{Y}$	;
L17:	MP2 $\rhd$ L14 $\rhd$ L15 $\rhd$ L16 $\gg$	$\mathcal{X} * \mathcal{Z} = \mathcal{Y} * \mathcal{Z} \Rightarrow \mathcal{X} * \mathcal{Z} \neq \mathcal{Y} * \mathcal{Z}$	;
L18:	ImplyNegation $\rhd$ L17 $\gg$	$\mathcal{X} * \mathcal{Z} \neq \mathcal{Y} * \mathcal{Z}$	$\square$

(*** NEGATIVE ***)

[ZFsub **lemma** UniqueNegative: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z} \colon \mathcal{X} + \mathcal{Y} = 0 \vdash \mathcal{X} + \mathcal{Z} = 0 \vdash \mathcal{Y} = \mathcal{Z}$]

ZFsub **proof of** UniqueNegative:

L01:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}, \mathcal{Z}$	;
L02:	Premise $\gg$	$\mathcal{X} + \mathcal{Y} = 0$	;
L03:	Premise $\gg$	$\mathcal{X} + \mathcal{Z} = 0$	;
L04:	plusCommutativity $\gg$	$\mathcal{Y} + \mathcal{X} = \mathcal{X} + \mathcal{Y}$	;
L05:	eqTransitivity $\rhd$ L04 $\rhd$ L02 $\gg$	$\mathcal{Y} + \mathcal{X} = 0$	;
L06:	PositiveToRight(Eq) $\rhd$ L05 $\gg$	$\mathcal{Y} = 0 - \mathcal{X}$	;
L07:	plusCommutativity $\gg$	$\mathcal{Z} + \mathcal{X} = \mathcal{X} + \mathcal{Z}$	;
L08:	eqTransitivity $\rhd$ L07 $\rhd$ L03 $\gg$	$\mathcal{Z} + \mathcal{X} = 0$	;
L09:	PositiveToRight(Eq) $\rhd$ L08 $\gg$	$\mathcal{Z} = 0 - \mathcal{X}$	;
L10:	eqSymmetry $\rhd$ L09 $\gg$	$0 - \mathcal{X} = \mathcal{Z}$	;
L11:	eqTransitivity $\rhd$ L06 $\rhd$ L10 $\gg$	$\mathcal{Y} = \mathcal{Z}$	$\square$

[ZFsub **lemma** DoubleMinus: $\Pi \mathcal{X} \colon - - \mathcal{X} = \mathcal{X}$]

ZFsub **proof of** DoubleMinus:

L01:	Arbitrary $\gg$	$\mathcal{X}$	;
L02:	Negative $\gg$	$-\mathcal{X} - -\mathcal{X} = 0$	;
L03:	x + y = zBackwards $\rhd$ L02 $\gg$	$0 = - - \mathcal{X} - \mathcal{X}$	;
L04:	NegativeToLeft(Eq) $\rhd$ L03 $\gg$	$0 + \mathcal{X} = - - \mathcal{X}$	;
L05:	plus0Left $\gg$	$0 + \mathcal{X} = \mathcal{X}$	;
L06:	Equality $\rhd$ L04 $\rhd$ L05 $\gg$	$- - \mathcal{X} = \mathcal{X}$	$\square$

(*** LEQ, nummer 1 af 2 ***)

[ZFsub **lemma** LeqLessEq: $\Pi \mathcal{X}, \mathcal{Y} \colon \mathcal{X} <= \mathcal{Y} \vdash \mathcal{X} < \mathcal{Y} \dot{\vee} \mathcal{X} = \mathcal{Y}$]

ZFsub **proof of** LeqLessEq:

L01: Block $\gg$ Begin ;
L02: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}$;
L03: Premise $\gg$ $\mathcal{X} <= \mathcal{Y}$;
L04: Premise $\gg$ $\dot{\neg} \mathcal{X} < \mathcal{Y}$;
L05: fromNotLess $\rhd$ L04 $\gg$ $\mathcal{Y} <= \mathcal{X}$;
L06: leqAntisymmetry $\rhd$ L03 $\rhd$ L05 $\gg$
$\mathcal{X} = \mathcal{Y}$;
L07: Block $\gg$ End ;
L08: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}$;
L09: Ded $\rhd$ L07 $\gg$ $\mathcal{X} <= \mathcal{Y} \Rightarrow \dot{\neg} \mathcal{X} < \mathcal{Y} \Rightarrow \mathcal{X} = \mathcal{Y}$;
L10: Premise $\gg$ $\mathcal{X} <= \mathcal{Y}$;
L11: MP $\rhd$ L09 $\rhd$ L10 $\gg$ $\dot{\neg} \mathcal{X} < \mathcal{Y} \Rightarrow \mathcal{X} = \mathcal{Y}$;
L12: Repetition $\rhd$ L11 $\gg$ $\mathcal{X} < \mathcal{Y} \dot{\vee} \mathcal{X} = \mathcal{Y}$ □

[ZFsub **lemma** LessLeq: $\Pi \mathcal{X}, \mathcal{Y} \colon \mathcal{X} < \mathcal{Y} \vdash \mathcal{X} <= \mathcal{Y}$]

ZFsub **proof of** LessLeq:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}$;
L02: Premise $\gg$ $\mathcal{X} < \mathcal{Y}$;
L03: Repetition $\rhd$ L02 $\gg$ $\mathcal{X} <= \mathcal{Y} \dot{\wedge} \dot{\neg} (\mathcal{X} = \mathcal{Y})$;
L04: FirstConjunct $\rhd$ L03 $\gg$ $\mathcal{X} <= \mathcal{Y}$ □

[ZFsub **lemma** FromLeqGeq: $\Pi \mathcal{A}, \mathcal{X}, \mathcal{Y} \colon \mathcal{X} <= \mathcal{Y} \Rightarrow \mathcal{A} \vdash \mathcal{Y} <= \mathcal{X} \Rightarrow \mathcal{A} \vdash \mathcal{A}$]

ZFsub **proof of** FromLeqGeq:

L01: Arbitrary $\gg$ $\mathcal{A}, \mathcal{X}, \mathcal{Y}$;
L02: Premise $\gg$ $\mathcal{X} <= \mathcal{Y} \Rightarrow \mathcal{A}$;
L03: Premise $\gg$ $\mathcal{Y} <= \mathcal{X} \Rightarrow \mathcal{A}$;
L04: leqTotality $\gg$ $\mathcal{X} <= \mathcal{Y} \dot{\vee} \mathcal{Y} <= \mathcal{X}$;
L05: FromDisjuncts $\rhd$ L04 $\rhd$ L02 $\rhd$
L03 $\gg$ $\mathcal{A}$ □

[ZFsub **lemma** subLeqRight: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z} \colon \mathcal{X} = \mathcal{Y} \vdash \mathcal{Z} <= \mathcal{X} \vdash \mathcal{Z} <= \mathcal{Y}$]

ZFsub **proof of** subLeqRight:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$;
L03: Premise $\gg$ $\mathcal{Z} <= \mathcal{X}$;
L04: eqLeq $\rhd$ L02 $\gg$ $\mathcal{X} <= \mathcal{Y}$;
L05: leqTransitivity $\rhd$ L03 $\rhd$ L04 $\gg$ $\mathcal{Z} <= \mathcal{Y}$ □

[ZFsub **lemma** subLeqLeft: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z} \colon \mathcal{X} = \mathcal{Y} \vdash \mathcal{X} <= \mathcal{Z} \vdash \mathcal{Y} <= \mathcal{Z}$]

ZFsub **proof of** subLeqLeft:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$;
L03: Premise $\gg$ $\mathcal{X} <= \mathcal{Z}$;
L04: eqSymmetry $\rhd$ L02 $\gg$ $\mathcal{Y} = \mathcal{X}$;
L05: eqLeq $\rhd$ L04 $\gg$ $\mathcal{Y} <= \mathcal{X}$;
L06: leqTransitivity $\rhd$ L05 $\rhd$ L03 $\gg$ $\mathcal{Y} <= \mathcal{Z}$ □

[ZFsub **lemma** Leq + 1: $\Pi\mathcal{X},\mathcal{Y}: \mathcal{X} <= \mathcal{Y} \vdash \mathcal{X} < \mathcal{Y} + 1$]

ZFsub **proof of** Leq + 1:

L01: Arbitrary $\gg$ $\mathcal{X},\mathcal{Y}$;

L02: Premise $\gg$ $\mathcal{X} <= \mathcal{Y}$;

L03: $0 < 1 \gg$ $0 < 1$;

L04: LessAdditionLeft $\rhd$ L03 $\gg$ $\mathcal{Y} + 0 < \mathcal{Y} + 1$;

L05: plus0 $\gg$ $\mathcal{Y} + 0 = \mathcal{Y}$;

L06: SubLessLeft $\rhd$ L05 $\rhd$ L04 $\gg$ $\mathcal{Y} < \mathcal{Y} + 1$;

L07: leqLessTransitivity $\rhd$ L02 $\rhd$ L06 $\gg$ $\mathcal{X} < \mathcal{Y} + 1$ $\Box$

[ZFsub **lemma** PositiveToRight(Leq): $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z}: \mathcal{X} + \mathcal{Y} <= \mathcal{Z} \vdash \mathcal{X} <= \mathcal{Z} - \mathcal{Y}$]

ZFsub **proof of** PositiveToRight(Leq):

L01: Arbitrary $\gg$ $\mathcal{X},\mathcal{Y},\mathcal{Z}$;

L02: Premise $\gg$ $\mathcal{X} + \mathcal{Y} <= \mathcal{Z}$;

L03: leqAddition $\rhd$ L02 $\gg$ $\mathcal{X} + \mathcal{Y} - \mathcal{Y} <= \mathcal{Z} - \mathcal{Y}$;

L04: $\mathrm{x} = \mathrm{x} + \mathrm{y} - \mathrm{y} \gg$ $\mathcal{X} = \mathcal{X} + \mathcal{Y} - \mathcal{Y}$;

L05: eqSymmetry $\rhd$ L04 $\gg$ $\mathcal{X} + \mathcal{Y} - \mathcal{Y} = \mathcal{X}$;

L06: subLeqLeft $\rhd$ L05 $\rhd$ L03 $\gg$ $\mathcal{X} <= \mathcal{Z} - \mathcal{Y}$ $\Box$

[ZFsub **lemma** LeqAdditionLeft: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z}: \mathcal{X} <= \mathcal{Y} \vdash \mathcal{Z} + \mathcal{X} <= \mathcal{Z} + \mathcal{Y}$]

ZFsub **proof of** LeqAdditionLeft:

L01: Arbitrary $\gg$ $\mathcal{X},\mathcal{Y},\mathcal{Z}$;

L02: Premise $\gg$ $\mathcal{X} <= \mathcal{Y}$;

L03: leqAddition $\rhd$ L02 $\gg$ $\mathcal{X} + \mathcal{Z} <= \mathcal{Y} + \mathcal{Z}$;

L04: plusCommutativity $\gg$ $\mathcal{X} + \mathcal{Z} = \mathcal{Z} + \mathcal{X}$;

L05: plusCommutativity $\gg$ $\mathcal{Y} + \mathcal{Z} = \mathcal{Z} + \mathcal{Y}$;

L06: subLeqLeft $\rhd$ L04 $\rhd$ L03 $\gg$ $\mathcal{Z} + \mathcal{X} <= \mathcal{Y} + \mathcal{Z}$;

L07: subLeqRight $\rhd$ L05 $\rhd$ L06 $\gg$ $\mathcal{Z} + \mathcal{X} <= \mathcal{Z} + \mathcal{Y}$ $\Box$

[ZFsub **lemma** leqSubtraction: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z}: \mathcal{X} + \mathcal{Z} <= \mathcal{Y} + \mathcal{Z} \vdash \mathcal{X} <= \mathcal{Y}$]

ZFsub **proof of** leqSubtraction:

L01: Arbitrary $\gg$ $\mathcal{X},\mathcal{Y},\mathcal{Z}$;

L02: Premise $\gg$ $\mathcal{X} + \mathcal{Z} <= \mathcal{Y} + \mathcal{Z}$;

L03: leqAddition $\rhd$ L02 $\gg$ $\mathcal{X} + \mathcal{Z} - \mathcal{Z} <= \mathcal{Y} + \mathcal{Z} - \mathcal{Z}$;

L04: $\mathrm{x} = \mathrm{x} + \mathrm{y} - \mathrm{y} \gg$ $\mathcal{X} = \mathcal{X} + \mathcal{Z} - \mathcal{Z}$;

L05: eqSymmetry $\rhd$ L04 $\gg$ $\mathcal{X} + \mathcal{Z} - \mathcal{Z} = \mathcal{X}$;

L06: $\mathrm{x} = \mathrm{x} + \mathrm{y} - \mathrm{y} \gg$ $\mathcal{Y} = \mathcal{Y} + \mathcal{Z} - \mathcal{Z}$;

L07: eqSymmetry $\rhd$ L06 $\gg$ $\mathcal{Y} + \mathcal{Z} - \mathcal{Z} = \mathcal{Y}$;

L08: subLeqLeft $\rhd$ L05 $\rhd$ L03 $\gg$ $\mathcal{X} <= \mathcal{Y} + \mathcal{Z} - \mathcal{Z}$;

L09: subLeqRight $\rhd$ L07 $\rhd$ L08 $\gg$ $\mathcal{X} <= \mathcal{Y}$ $\Box$

[ZFsub **lemma** leqSubtractionLeft: $\Pi\mathcal{X},\mathcal{Y},\mathcal{Z}: \mathcal{Z} + \mathcal{X} <= \mathcal{Z} + \mathcal{Y} \vdash \mathcal{X} <= \mathcal{Y}$]

ZFsub **proof of** leqSubtractionLeft:

L01: Arbitrary $\gg$ $\mathcal{X},\mathcal{Y},\mathcal{Z}$;

L02: Premise $\gg$ $\mathcal{Z} + \mathcal{X} <= \mathcal{Z} + \mathcal{Y}$;

L03: plusCommutativity $\gg$ $\mathcal{Z} + \mathcal{X} = \mathcal{X} + \mathcal{Z}$;

L04: plusCommutativity $\gg$ $\mathcal{Z} + \mathcal{Y} = \mathcal{Y} + \mathcal{Z}$;

L05: subLeqLeft $\rhd$ L03 $\rhd$ L02 $\gg$ $\mathcal{X} + \mathcal{Z} <= \mathcal{Z} + \mathcal{Y}$;

L06: subLeqRight $\rhd$ L04 $\rhd$ L05 $\gg$ $\mathcal{X} + \mathcal{Z} <= \mathcal{Y} + \mathcal{Z}$;
L07: leqSubtraction $\rhd$ L06 $\gg$ $\mathcal{X} <= \mathcal{Y}$ $\square$

[ZFsub **lemma** thirdGeq: $\Pi \mathcal{X}, \mathcal{Y} \colon \mathcal{X} <= \mathrm{Ex3} \mathrel{\dot\wedge} \mathcal{Y} <= \mathrm{Ex3}$]

ZFsub **proof of** thirdGeq:

L01: Block $\gg$ Begin ;
L02: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}$;
L03: Premise $\gg$ $\mathcal{X} <= \mathcal{Y}$;
L04: leqReflexivity $\gg$ $\mathcal{Y} <= \mathcal{Y}$;
L05: JoinConjuncts $\rhd$ L03 $\rhd$ L04 $\gg$ $\mathcal{X} <= \mathcal{Y} \mathrel{\dot\wedge} \mathcal{Y} <= \mathcal{Y}$;
L06: ExistIntro @ Ex3 @ $\mathcal{Y}$ $\rhd$ L05 $\gg$ $\mathcal{X} <= \mathrm{Ex3} \mathrel{\dot\wedge} \mathcal{Y} <= \mathrm{Ex3}$;
L07: Block $\gg$ End ;
L08: Block $\gg$ Begin ;
L09: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}$;
L10: Premise $\gg$ $\mathcal{Y} <= \mathcal{X}$;
L11: leqReflexivity $\gg$ $\mathcal{X} <= \mathcal{X}$;
L12: JoinConjuncts $\rhd$ L11 $\rhd$ L10 $\gg$ $\mathcal{X} <= \mathcal{X} \mathrel{\dot\wedge} \mathcal{Y} <= \mathcal{X}$;
L13: ExistIntro @ Ex3 @ $\mathcal{X}$ $\rhd$ L12 $\gg$ $\mathcal{X} <= \mathrm{Ex3} \mathrel{\dot\wedge} \mathcal{Y} <= \mathrm{Ex3}$;
L14: Block $\gg$ End ;
L15: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}$;
L16: Ded $\rhd$ L07 $\gg$ $\mathcal{X} <= \mathcal{Y} \Rightarrow \mathcal{X} <= \mathrm{Ex3} \mathrel{\dot\wedge} \mathcal{Y} <= \mathrm{Ex3}$;
L17: Ded $\rhd$ L14 $\gg$ $\mathcal{Y} <= \mathcal{X} \Rightarrow \mathcal{X} <= \mathrm{Ex3} \mathrel{\dot\wedge} \mathcal{Y} <= \mathrm{Ex3}$;
L18: leqTotality $\gg$ $\mathcal{X} <= \mathcal{Y} \mathrel{\dot\vee} \mathcal{Y} <= \mathcal{X}$;
L19: FromDisjuncts $\rhd$ L18 $\rhd$ L16 $\rhd$ L17 $\gg$ $\mathcal{X} <= \mathrm{Ex3} \mathrel{\dot\wedge} \mathcal{Y} <= \mathrm{Ex3}$ $\square$

[ZFsub **lemma** LeqNegated: $\Pi \mathcal{X}, \mathcal{Y} \colon \mathcal{X} <= \mathcal{Y} \vdash -\mathcal{Y} <= -\mathcal{X}$]

ZFsub **proof of** LeqNegated:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}$;
L02: Premise $\gg$ $\mathcal{X} <= \mathcal{Y}$;
L03: leqAddition $\rhd$ L02 $\gg$ $\mathcal{X} - \mathcal{X} <= \mathcal{Y} - \mathcal{X}$;
L04: Negative $\gg$ $\mathcal{X} - \mathcal{X} = 0$;
L05: subLeqLeft $\rhd$ L04 $\rhd$ L03 $\gg$ $0 <= \mathcal{Y} - \mathcal{X}$;
L06: plusCommutativity $\gg$ $\mathcal{Y} - \mathcal{X} = -\mathcal{X} + \mathcal{Y}$;
L07: subLeqRight $\rhd$ L06 $\rhd$ L05 $\gg$ $0 <= -\mathcal{X} + \mathcal{Y}$;
L08: leqAddition $\rhd$ L07 $\gg$ $0 - \mathcal{Y} <= -\mathcal{X} + \mathcal{Y} - \mathcal{Y}$;
L09: plus0Left $\gg$ $0 - \mathcal{Y} = -\mathcal{Y}$;
L10: $\mathrm{x} = \mathrm{x} + \mathrm{y} - \mathrm{y}$ $\gg$ $-\mathcal{X} = -\mathcal{X} + \mathcal{Y} - \mathcal{Y}$;
L11: eqSymmetry $\rhd$ L10 $\gg$ $-\mathcal{X} + \mathcal{Y} - \mathcal{Y} = -\mathcal{X}$;
L12: subLeqLeft $\rhd$ L09 $\rhd$ L08 $\gg$ $-\mathcal{Y} <= -\mathcal{X} + \mathcal{Y} - \mathcal{Y}$;
L13: subLeqRight $\rhd$ L11 $\rhd$ L12 $\gg$ $-\mathcal{Y} <= -\mathcal{X}$ $\square$

[ZFsub **lemma** AddEquations(Leq): $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U} \colon \mathcal{X} <= \mathcal{Y} \vdash \mathcal{Z} <= \mathcal{U} \vdash \mathcal{X} + \mathcal{Z} <= \mathcal{Y} + \mathcal{U}$]

ZFsub **proof of** AddEquations(Leq):

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}$;
L02: Premise $\gg$ $\mathcal{X} <= \mathcal{Y}$;

L03:	Premise $\gg$	$\mathcal{Z} <= \mathcal{U}$	;
L04:	leqAddition $\rhd$ L02 $\gg$	$\mathcal{X} + \mathcal{Z} <= \mathcal{Y} + \mathcal{Z}$	;
L05:	LeqAdditionLeft $\rhd$ L03 $\gg$	$\mathcal{Y} + \mathcal{Z} <= \mathcal{Y} + \mathcal{U}$	;
L06:	leqTransitivity $\rhd$ L04 $\rhd$ L05 $\gg$	$\mathcal{X} + \mathcal{Z} <= \mathcal{Y} + \mathcal{U}$	$\square$

(*** LESS ***)

[ZFsub **lemma** LeqNeqLess: $\Pi \mathcal{X}, \mathcal{Y} \colon \mathcal{X} <= \mathcal{Y} \vdash \mathcal{X} \neq \mathcal{Y} \vdash \mathcal{X} < \mathcal{Y}$]

ZFsub **proof of** LeqNeqLess:

L01:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}$	;
L02:	Premise $\gg$	$\mathcal{X} <= \mathcal{Y}$	;
L03:	Premise $\gg$	$\mathcal{X} \neq \mathcal{Y}$	;
L04:	JoinConjuncts $\rhd$ L02 $\rhd$ L03 $\gg$	$\mathcal{X} <= \mathcal{Y} \dot{\wedge} \mathcal{X} \neq \mathcal{Y}$	;
L05:	Repetition $\rhd$ L04 $\gg$	$\mathcal{X} < \mathcal{Y}$	$\square$

[ZFsub **lemma** FromLess: $\Pi \mathcal{X}, \mathcal{Y} \colon \mathcal{X} < \mathcal{Y} \vdash \dot{\neg} \mathcal{Y} <= \mathcal{X}$]

ZFsub **proof of** FromLess:

L01:	Block $\gg$	Begin	;
L02:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}$	;
L03:	Premise $\gg$	$\mathcal{Y} <= \mathcal{X}$	;
L04:	toNotLess $\rhd$ L03 $\gg$	$\dot{\neg} \mathcal{X} < \mathcal{Y}$	;
L05:	Block $\gg$	End	;
L06:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}$	;
L07:	Ded $\rhd$ L05 $\gg$	$\mathcal{Y} <= \mathcal{X} \Rightarrow \dot{\neg} \mathcal{X} < \mathcal{Y}$	;
L08:	Premise $\gg$	$\mathcal{X} < \mathcal{Y}$	;
L09:	AddDoubleNeg $\rhd$ L08 $\gg$	$\dot{\neg} \dot{\neg} \mathcal{X} < \mathcal{Y}$	;
L10:	MT $\rhd$ L07 $\rhd$ L09 $\gg$	$\dot{\neg} \mathcal{Y} <= \mathcal{X}$	$\square$

[ZFsub **lemma** ToLess: $\Pi \mathcal{X}, \mathcal{Y} \colon \dot{\neg} \mathcal{X} <= \mathcal{Y} \vdash \mathcal{Y} < \mathcal{X}$]

ZFsub **proof of** ToLess:

L01:	Block $\gg$	Begin	;
L02:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}$	;
L03:	Premise $\gg$	$\dot{\neg} \mathcal{Y} < \mathcal{X}$	;
L04:	fromNotLess $\rhd$ L03 $\gg$	$\mathcal{X} <= \mathcal{Y}$	;
L05:	Block $\gg$	End	;
L06:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}$	;
L07:	Ded $\rhd$ L05 $\gg$	$\dot{\neg} \mathcal{Y} < \mathcal{X} \Rightarrow \mathcal{X} <= \mathcal{Y}$	;
L08:	Premise $\gg$	$\dot{\neg} \mathcal{X} <= \mathcal{Y}$	;
L09:	NegativeMT $\rhd$ L07 $\rhd$ L08 $\gg$	$\mathcal{Y} < \mathcal{X}$	$\square$

[ZFsub **lemma** fromNotLess: $\Pi \mathcal{X}, \mathcal{Y} \colon \dot{\neg} (\mathcal{X} < \mathcal{Y}) \vdash \mathcal{Y} <= \mathcal{X}$]

ZFsub **proof of** fromNotLess:

L01:	Block $\gg$	Begin	;
L02:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}$	;
L03:	Premise $\gg$	$\dot{\neg} (\mathcal{X} < \mathcal{Y})$	;
L04:	Premise $\gg$	$\mathcal{X} <= \mathcal{Y}$	;
L05:	Repetition $\rhd$ L03 $\gg$	$\dot{\neg} \dot{\neg} (\mathcal{X} <= \mathcal{Y} \Rightarrow \dot{\neg} \mathcal{X} \neq \mathcal{Y})$	;
L06:	RemoveDoubleNeg $\rhd$ L05 $\gg$	$\mathcal{X} <= \mathcal{Y} \Rightarrow \dot{\neg} \mathcal{X} \neq \mathcal{Y}$	;
L07:	MP $\rhd$ L06 $\rhd$ L04 $\gg$	$\dot{\neg} \mathcal{X} \neq \mathcal{Y}$	;
L08:	RemoveDoubleNeg $\rhd$ L07 $\gg$	$\mathcal{X} = \mathcal{Y}$	;
L09:	eqSymmetry $\rhd$ L08 $\gg$	$\mathcal{Y} = \mathcal{X}$	;

L10:	eqLeq ⊳ L09 ≫	$\mathcal{Y} <= \mathcal{X}$	;
L11:	Block ≫	End	;
L12:	Arbitrary ≫	$\mathcal{X}, \mathcal{Y}$	;
L13:	Ded ⊳ L11 ≫	$\dot{\neg} \mathcal{X} < \mathcal{Y} \Rightarrow \mathcal{X} <= \mathcal{Y} \Rightarrow \mathcal{Y} <= \mathcal{X}$	;
L14:	Premise ≫	$\dot{\neg} \mathcal{X} < \mathcal{Y}$	;
L15:	MP ⊳ L13 ⊳ L14 ≫	$\mathcal{X} <= \mathcal{Y} \Rightarrow \mathcal{Y} <= \mathcal{X}$	;
L16:	AutoImply ≫	$\mathcal{Y} <= \mathcal{X} \Rightarrow \mathcal{Y} <= \mathcal{X}$	;
L17:	leqTotality ≫	$\mathcal{X} <= \mathcal{Y} \dot{\vee} \mathcal{Y} <= \mathcal{X}$	;
L18:	FromDisjuncts ⊳ L17 ⊳ L15 ⊳ L16 ≫	$\mathcal{Y} <= \mathcal{X}$	□

[ZFsub **lemma** toNotLess: $\Pi \mathcal{X}, \mathcal{Y}\colon \mathcal{X} <= \mathcal{Y} \vdash \dot{\neg} \mathcal{Y} < \mathcal{X}$]

ZFsub **proof of** toNotLess:

L01:	Block ≫	Begin	;
L02:	Arbitrary ≫	$\mathcal{X}, \mathcal{Y}$	;
L03:	Premise ≫	$\mathcal{X} <= \mathcal{Y}$	;
L04:	Premise ≫	$\mathcal{Y} <= \mathcal{X}$	;
L05:	leqAntisymmetry ⊳ L04 ⊳ L03 ≫	$\mathcal{Y} = \mathcal{X}$	;
L06:	AddDoubleNeg ⊳ L05 ≫	$\dot{\neg} \dot{\neg} \mathcal{Y} = \mathcal{X}$	;
L07:	Block ≫	End	;
L08:	Arbitrary ≫	$\mathcal{X}, \mathcal{Y}$	;
L09:	Ded ⊳ L07 ≫	$\mathcal{X} <= \mathcal{Y} \Rightarrow \mathcal{Y} <= \mathcal{X} \Rightarrow \dot{\neg} \dot{\neg} \mathcal{Y} = \mathcal{X}$	;
L10:	Premise ≫	$\mathcal{X} <= \mathcal{Y}$	;
L11:	MP ⊳ L09 ⊳ L10 ≫	$\mathcal{Y} <= \mathcal{X} \Rightarrow \dot{\neg} \dot{\neg} \mathcal{Y} = \mathcal{X}$	;
L12:	AddDoubleNeg ⊳ L11 ≫	$\dot{\neg} \dot{\neg} (\mathcal{Y} <= \mathcal{X} \Rightarrow \dot{\neg} \dot{\neg} \mathcal{Y} = \mathcal{X})$	;
L13:	Repetition ⊳ L12 ≫	$\dot{\neg} (\mathcal{Y} <= \mathcal{X} \dot{\wedge} \dot{\neg} \mathcal{Y} = \mathcal{X})$	;
L14:	Repetition ⊳ L13 ≫	$\dot{\neg} \mathcal{Y} < \mathcal{X}$	□

[ZFsub **lemma** LessAddition: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}\colon \mathcal{X} < \mathcal{Y} \vdash \mathcal{X} + \mathcal{Z} < \mathcal{Y} + \mathcal{Z}$]

ZFsub **proof of** LessAddition:

L01:	Arbitrary ≫	$\mathcal{X}, \mathcal{Y}, \mathcal{Z}$	;
L02:	Premise ≫	$\mathcal{X} < \mathcal{Y}$	;
L03:	LessLeq ⊳ L02 ≫	$\mathcal{X} <= \mathcal{Y}$	;
L04:	leqAddition ⊳ L03 ≫	$\mathcal{X} + \mathcal{Z} <= \mathcal{Y} + \mathcal{Z}$	;
L05:	LessNeq ⊳ L02 ≫	$\mathcal{X} \neq \mathcal{Y}$	;
L06:	NeqAddition ⊳ L05 ≫	$\mathcal{X} + \mathcal{Z} \neq \mathcal{Y} + \mathcal{Z}$	;
L07:	JoinConjuncts ⊳ L04 ⊳ L06 ≫	$\mathcal{X} + \mathcal{Z} < \mathcal{Y} + \mathcal{Z}$	□

[ZFsub **lemma** LessAdditionLeft: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}\colon \mathcal{X} < \mathcal{Y} \vdash \mathcal{Z} + \mathcal{X} < \mathcal{Z} + \mathcal{Y}$]

ZFsub **proof of** LessAdditionLeft:

L01:	Arbitrary ≫	$\mathcal{X}, \mathcal{Y}, \mathcal{Z}$	;
L02:	Premise ≫	$\mathcal{X} < \mathcal{Y}$	;
L03:	LessAddition ⊳ L02 ≫	$\mathcal{X} + \mathcal{Z} < \mathcal{Y} + \mathcal{Z}$	;
L04:	plusCommutativity ≫	$\mathcal{X} + \mathcal{Z} = \mathcal{Z} + \mathcal{X}$	;
L05:	SubLessLeft ⊳ L04 ⊳ L03 ≫	$\mathcal{Z} + \mathcal{X} < \mathcal{Y} + \mathcal{Z}$	;
L06:	plusCommutativity ≫	$\mathcal{Y} + \mathcal{Z} = \mathcal{Z} + \mathcal{Y}$	;

L07: SubLessRight $\rhd$ L06 $\rhd$ L05 $\gg$ $\mathcal{Z} + \mathcal{X} < \mathcal{Z} + \mathcal{Y}$ $\square$

[ZFsub **lemma** LessMultiplication: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}$: $0 < \mathcal{Z} \vdash \mathcal{X} < \mathcal{Y} \vdash \mathcal{X} * \mathcal{Z} < \mathcal{Y} * \mathcal{Z}$]

ZFsub **proof of** LessMultiplication:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$;
L02: Premise $\gg$ $0 < \mathcal{Z}$;
L03: Premise $\gg$ $\mathcal{X} < \mathcal{Y}$;
L04: LessLeq $\rhd$ L03 $\gg$ $\mathcal{X} <= \mathcal{Y}$;
L05: LessLeq $\rhd$ L02 $\gg$ $0 <= \mathcal{Z}$;
L06: leqMultiplication $\rhd$ L05 $\rhd$ L04 $\gg$ $\mathcal{X} * \mathcal{Z} <= \mathcal{Y} * \mathcal{Z}$;
L07: LessNeq $\rhd$ L03 $\gg$ $\mathcal{X} \neq \mathcal{Y}$;
L08: LessNeq $\rhd$ L02 $\gg$ $0 \neq \mathcal{Z}$;
L09: NeqSymmetry $\rhd$ L08 $\gg$ $\mathcal{Z} \neq 0$;
L10: NeqMultiplication $\rhd$ L09 $\rhd$ L07 $\gg$ $\mathcal{X} * \mathcal{Z} \neq \mathcal{Y} * \mathcal{Z}$;
L11: LeqNeqLess $\rhd$ L06 $\rhd$ L10 $\gg$ $\mathcal{X} * \mathcal{Z} < \mathcal{Y} * \mathcal{Z}$ $\square$

[ZFsub **lemma** LessMultiplicationLeft: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}$: $0 < \mathcal{Z} \vdash \mathcal{X} < \mathcal{Y} \vdash \mathcal{Z} * \mathcal{X} < \mathcal{Z} * \mathcal{Y}$]

ZFsub **proof of** LessMultiplicationLeft:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$;
L02: Premise $\gg$ $0 < \mathcal{Z}$;
L03: Premise $\gg$ $\mathcal{X} < \mathcal{Y}$;
L04: LessMultiplication $\rhd$ L02 $\rhd$ L03 $\gg$ $\mathcal{X} * \mathcal{Z} < \mathcal{Y} * \mathcal{Z}$;
L05: timesCommutativity $\gg$ $\mathcal{X} * \mathcal{Z} = \mathcal{Z} * \mathcal{X}$;
L06: timesCommutativity $\gg$ $\mathcal{Y} * \mathcal{Z} = \mathcal{Z} * \mathcal{Y}$;
L07: SubLessLeft $\rhd$ L05 $\rhd$ L04 $\gg$ $\mathcal{Z} * \mathcal{X} < \mathcal{Y} * \mathcal{Z}$;
L08: SubLessRight $\rhd$ L06 $\rhd$ L07 $\gg$ $\mathcal{Z} * \mathcal{X} < \mathcal{Z} * \mathcal{Y}$ $\square$

[ZFsub **lemma** LessDivision: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}$: $0 <= \mathcal{Z} \vdash \mathcal{X} * \mathcal{Z} < \mathcal{Y} * \mathcal{Z} \vdash \mathcal{X} < \mathcal{Y}$]

ZFsub **proof of** LessDivision:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$;
L02: Premise $\gg$ $0 <= \mathcal{Z}$;
L03: Premise $\gg$ $\mathcal{X} * \mathcal{Z} < \mathcal{Y} * \mathcal{Z}$;
L04: FromLess $\rhd$ L03 $\gg$ $\dot{\neg} \mathcal{Y} * \mathcal{Z} <= \mathcal{X} * \mathcal{Z}$;
L05: leqMultiplicationAxiom $\gg$ $0 <= \mathcal{Z} \Rightarrow \mathcal{Y} <= \mathcal{X} \Rightarrow \mathcal{Y} * \mathcal{Z} <= \mathcal{X} * \mathcal{Z}$;
L06: MP $\rhd$ L05 $\rhd$ L02 $\gg$ $\mathcal{Y} <= \mathcal{X} \Rightarrow \mathcal{Y} * \mathcal{Z} <= \mathcal{X} * \mathcal{Z}$;
L07: Contrapositive $\rhd$ L06 $\gg$ $\dot{\neg} \mathcal{Y} * \mathcal{Z} <= \mathcal{X} * \mathcal{Z} \Rightarrow \dot{\neg} \mathcal{Y} <= \mathcal{X}$;
L08: MP $\rhd$ L07 $\rhd$ L04 $\gg$ $\dot{\neg} \mathcal{Y} <= \mathcal{X}$;
L09: ToLess $\rhd$ L08 $\gg$ $\mathcal{X} < \mathcal{Y}$ $\square$

[ZFsub **lemma** AddEquations(Less): $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}$: $\mathcal{X} < \mathcal{Y} \vdash \mathcal{Z} < \mathcal{U} \vdash \mathcal{X} + \mathcal{Z} < \mathcal{Y} + \mathcal{U}$]

ZFsub **proof of** AddEquations(Less):

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}$;
L02: Premise $\gg$ $\mathcal{X} < \mathcal{Y}$;

L03: Premise ≫ $\mathcal{Z} < \mathcal{U}$;
L04: LessAddition ▷ L02 ≫ $\mathcal{X} + \mathcal{Z} < \mathcal{Y} + \mathcal{Z}$;
L05: LessAdditionLeft ▷ L03 ≫ $\mathcal{Y} + \mathcal{Z} < \mathcal{Y} + \mathcal{U}$;
L06: LessTransitivity ▷ L04 ▷ L05 ≫ $\mathcal{X} + \mathcal{Z} < \mathcal{Y} + \mathcal{U}$ □

[ZFsub **lemma** leqLessTransitivity: $\Pi\mathcal{X}, \mathcal{Y}, \mathcal{Z}{:}\, \mathcal{X} <= \mathcal{Y} \vdash \mathcal{Y} < \mathcal{Z} \vdash \mathcal{X} < \mathcal{Z}$]

ZFsub **proof of** leqLessTransitivity:

L01: Block ≫ Begin ;
L02: Arbitrary ≫ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$;
L03: Premise ≫ $\mathcal{X} <= \mathcal{Y}$;
L04: Premise ≫ $\mathcal{Y} < \mathcal{Z}$;
L05: Premise ≫ $\mathcal{X} = \mathcal{Z}$;
L06: FirstConjunct ▷ L04 ≫ $\mathcal{Y} <= \mathcal{Z}$;
L07: SecondConjunct ▷ L04 ≫ $\mathcal{Y} \neq \mathcal{Z}$;
L08: subLeqLeft ▷ L05 ▷ L03 ≫ $\mathcal{Z} <= \mathcal{Y}$;
L09: leqAntisymmetry ▷ L06 ▷ L08 ≫ $\mathcal{Y} = \mathcal{Z}$;
L10: FromContradiction ▷ L09 ▷ L07 ≫ $\mathcal{X} \neq \mathcal{Z}$;
L11: Block ≫ End ;
L12: Arbitrary ≫ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$;
L13: Ded ▷ L11 ≫ $\mathcal{X} <= \mathcal{Y} \Rightarrow \mathcal{Y} < \mathcal{Z} \Rightarrow \mathcal{X} = \mathcal{Z} \Rightarrow \mathcal{X} \neq \mathcal{Z}$;
L14: Premise ≫ $\mathcal{X} <= \mathcal{Y}$;
L15: Premise ≫ $\mathcal{Y} < \mathcal{Z}$;
L16: MP2 ▷ L13 ▷ L14 ▷ L15 ≫ $\mathcal{X} = \mathcal{Z} \Rightarrow \mathcal{X} \neq \mathcal{Z}$;
L17: ImplyNegation ▷ L16 ≫ $\mathcal{X} \neq \mathcal{Z}$;
L18: FirstConjunct ▷ L15 ≫ $\mathcal{Y} <= \mathcal{Z}$;
L19: leqTransitivity ▷ L14 ▷ L18 ≫ $\mathcal{X} <= \mathcal{Z}$;
L20: JoinConjuncts ▷ L19 ▷ L17 ≫ $\mathcal{X} < \mathcal{Z}$ □

[ZFsub **lemma** LessLeqTransitivity: $\Pi\mathcal{X}, \mathcal{Y}, \mathcal{Z}{:}\, \mathcal{X} < \mathcal{Y} \vdash \mathcal{Y} <= \mathcal{Z} \vdash \mathcal{X} < \mathcal{Z}$]

ZFsub **proof of** LessLeqTransitivity:

L01: Block ≫ Begin ;
L02: Arbitrary ≫ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$;
L03: Premise ≫ $\mathcal{X} < \mathcal{Y}$;
L04: Premise ≫ $\mathcal{Y} <= \mathcal{Z}$;
L05: Premise ≫ $\mathcal{Z} = \mathcal{X}$;
L06: FirstConjunct ▷ L03 ≫ $\mathcal{X} <= \mathcal{Y}$;
L07: SecondConjunct ▷ L03 ≫ $\mathcal{X} \neq \mathcal{Y}$;
L08: subLeqRight ▷ L05 ▷ L04 ≫ $\mathcal{Y} <= \mathcal{X}$;
L09: leqAntisymmetry ▷ L06 ▷ L08 ≫ $\mathcal{X} = \mathcal{Y}$;
L10: FromContradiction ▷ L09 ▷ L07 ≫ $\mathcal{Z} \neq \mathcal{X}$;
L11: Block ≫ End ;
L12: Arbitrary ≫ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$;
L13: Ded ▷ L11 ≫ $\mathcal{X} < \mathcal{Y} \Rightarrow \mathcal{Y} <= \mathcal{Z} \Rightarrow \mathcal{Z} = \mathcal{X} \Rightarrow \mathcal{Z} \neq \mathcal{X}$;

L14: Premise $\gg$ $\mathcal{X} < \mathcal{Y}$;
L15: Premise $\gg$ $\mathcal{Y} <= \mathcal{Z}$;
L16: MP2 $\rhd$ L13 $\rhd$ L14 $\rhd$ L15 $\gg$ $\mathcal{Z} = \mathcal{X} \Rightarrow \mathcal{Z} \neq \mathcal{X}$;
L17: ImplyNegation $\rhd$ L16 $\gg$ $\mathcal{Z} \neq \mathcal{X}$;
L18: NeqSymmetry $\rhd$ L17 $\gg$ $\mathcal{X} \neq \mathcal{Z}$;
L19: FirstConjunct $\rhd$ L14 $\gg$ $\mathcal{X} <= \mathcal{Y}$;
L20: leqTransitivity $\rhd$ L19 $\rhd$ L15 $\gg$ $\mathcal{X} <= \mathcal{Z}$;
L21: JoinConjuncts $\rhd$ L20 $\rhd$ L18 $\gg$ $\mathcal{X} < \mathcal{Z}$ $\square$

[ZFsub **lemma** LessTransitivity: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}\colon \mathcal{X} < \mathcal{Y} \vdash \mathcal{Y} < \mathcal{Z} \vdash \mathcal{X} < \mathcal{Z}$]

ZFsub **proof of** LessTransitivity:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$;
L02: Premise $\gg$ $\mathcal{X} < \mathcal{Y}$;
L03: Premise $\gg$ $\mathcal{Y} < \mathcal{Z}$;
L04: FirstConjunct $\rhd$ L03 $\gg$ $\mathcal{Y} <= \mathcal{Z}$;
L05: LessLeqTransitivity $\rhd$ L02 $\rhd$ L04 $\gg$ $\mathcal{X} < \mathcal{Z}$ $\square$

[ZFsub **lemma** LessTotality: $\Pi \mathcal{X}, \mathcal{Y}\colon \mathcal{X} < \mathcal{Y} \dot\vee \mathcal{X} = \mathcal{Y} \dot\vee \mathcal{Y} < \mathcal{X}$]

ZFsub **proof of** LessTotality:

L01: Block $\gg$ Begin ;
L02: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}$;
L03: Premise $\gg$ $\dot\neg \mathcal{X} < \mathcal{Y}$;
L04: Premise $\gg$ $\mathcal{X} \neq \mathcal{Y}$;
L05: fromNotLess $\rhd$ L03 $\gg$ $\mathcal{Y} <= \mathcal{X}$;
L06: NeqSymmetry $\rhd$ L04 $\gg$ $\mathcal{Y} \neq \mathcal{X}$;
L07: LeqNeqLess $\rhd$ L05 $\rhd$ L06 $\gg$ $\mathcal{Y} < \mathcal{X}$;
L08: Block $\gg$ End ;
L09: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}$;
L10: Ded $\rhd$ L08 $\gg$ $\dot\neg \mathcal{X} < \mathcal{Y} \Rightarrow \mathcal{X} \neq \mathcal{Y} \Rightarrow \mathcal{Y} < \mathcal{X}$;
L11: Repetition $\rhd$ L10 $\gg$ $\mathcal{X} < \mathcal{Y} \dot\vee \mathcal{X} = \mathcal{Y} \dot\vee \mathcal{Y} < \mathcal{X}$ $\square$

[ZFsub **lemma** SubLessRight: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}\colon \mathcal{X} = \mathcal{Y} \vdash \mathcal{Z} < \mathcal{X} \vdash \mathcal{Z} < \mathcal{Y}$]

ZFsub **proof of** SubLessRight:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$;
L03: Premise $\gg$ $\mathcal{Z} < \mathcal{X}$;
L04: Repetition $\rhd$ L03 $\gg$ $\mathcal{Z} <= \mathcal{X} \dot\wedge \mathcal{Z} \neq \mathcal{X}$;
L05: FirstConjunct $\rhd$ L04 $\gg$ $\mathcal{Z} <= \mathcal{X}$;
L06: subLeqRight $\rhd$ L02 $\rhd$ L05 $\gg$ $\mathcal{Z} <= \mathcal{Y}$;
L07: SecondConjunct $\rhd$ L04 $\gg$ $\mathcal{Z} \neq \mathcal{X}$;
L08: SubNeqRight $\rhd$ L02 $\rhd$ L07 $\gg$ $\mathcal{Z} \neq \mathcal{Y}$;
L09: JoinConjuncts $\rhd$ L06 $\rhd$ L08 $\gg$ $\mathcal{Z} < \mathcal{Y}$ $\square$

[ZFsub **lemma** SubLessLeft: $\Pi \mathcal{X}, \mathcal{Y}, \mathcal{Z}\colon \mathcal{X} = \mathcal{Y} \vdash \mathcal{X} < \mathcal{Z} \vdash \mathcal{Y} < \mathcal{Z}$]

ZFsub **proof of** SubLessLeft:

L01: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$;
L02: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$;
L03: Premise $\gg$ $\mathcal{X} < \mathcal{Z}$;
L04: Repetition $\rhd$ L03 $\gg$ $\mathcal{X} <= \mathcal{Z} \dot\wedge \mathcal{X} \neq \mathcal{Z}$;

L05:	FirstConjunct $\rhd$ L04 $\gg$	$\mathcal{X} <= \mathcal{Z}$	;
L06:	subLeqLeft $\rhd$ L02 $\rhd$ L05 $\gg$	$\mathcal{Y} <= \mathcal{Z}$	;
L07:	SecondConjunct $\rhd$ L04 $\gg$	$\mathcal{X} \neq \mathcal{Z}$	;
L08:	SubNeqLeft $\rhd$ L02 $\rhd$ L07 $\gg$	$\mathcal{Y} \neq \mathcal{Z}$	;
L09:	JoinConjuncts $\rhd$ L06 $\rhd$ L08 $\gg$	$\mathcal{Y} < \mathcal{Z}$	$\Box$

[ZFsub **lemma** NegativeLessPositive: $\Pi\mathcal{X}\colon 0 < \mathcal{X} \vdash -\mathcal{X} < \mathcal{X}$]

ZFsub **proof of** NegativeLessPositive:

L01:	Arbitrary $\gg$	$\mathcal{X}$	;
L02:	Premise $\gg$	$0 < \mathcal{X}$	;
L03:	FirstConjunct $\rhd$ L02 $\gg$	$0 <= \mathcal{X}$	;
L04:	leqAddition $\rhd$ L03 $\gg$	$0 - \mathcal{X} <= \mathcal{X} - \mathcal{X}$	;
L05:	plus0Left $\gg$	$0 - \mathcal{X} = -\mathcal{X}$	;
L06:	Negative $\gg$	$\mathcal{X} - \mathcal{X} = 0$	;
L07:	subLeqLeft $\rhd$ L05 $\rhd$ L04 $\gg$	$-\mathcal{X} <= \mathcal{X} - \mathcal{X}$	;
L08:	subLeqRight $\rhd$ L06 $\rhd$ L07 $\gg$	$-\mathcal{X} <= 0$	;
L09:	leqLessTransitivity $\rhd$ L08 $\rhd$ L02 $\gg$	$-\mathcal{X} < \mathcal{X}$	$\Box$

[ZFsub **lemma** LessNegated: $\Pi\mathcal{X},\mathcal{Y}\colon \mathcal{X} < \mathcal{Y} \vdash -\mathcal{Y} < -\mathcal{X}$]

ZFsub **proof of** LessNegated:

L01:	Arbitrary $\gg$	$\mathcal{X}, \mathcal{Y}$	;
L02:	Premise $\gg$	$\mathcal{X} < \mathcal{Y}$	;
L03:	LessLeq $\rhd$ L02 $\gg$	$\mathcal{X} <= \mathcal{Y}$	;
L04:	LeqNegated $\rhd$ L03 $\gg$	$-\mathcal{Y} <= -\mathcal{X}$	;
L05:	LessNeq $\rhd$ L02 $\gg$	$\mathcal{X} \neq \mathcal{Y}$	;
L06:	NeqNegated $\rhd$ L05 $\gg$	$\dot{\neg} - \mathcal{X} = -\mathcal{Y}$	;
L07:	NeqSymmetry $\rhd$ L06 $\gg$	$\dot{\neg} - \mathcal{Y} = -\mathcal{X}$	;
L08:	LeqNeqLess $\rhd$ L04 $\rhd$ L07 $\gg$	$-\mathcal{Y} < -\mathcal{X}$	$\Box$

[ZFsub **lemma** PositiveNegated: $\Pi\mathcal{X}\colon 0 < \mathcal{X} \vdash -\mathcal{X} < 0$]

ZFsub **proof of** PositiveNegated:

L01:	Arbitrary $\gg$	$\mathcal{X}$	;
L02:	Premise $\gg$	$0 < \mathcal{X}$	;
L03:	LessNegated $\rhd$ L02 $\gg$	$-\mathcal{X} < -0$	;
L04:	$-0 = 0 \gg$	$-0 = 0$	;
L05:	SubLessRight $\rhd$ L04 $\rhd$ L03 $\gg$	$-\mathcal{X} < 0$	$\Box$

[ZFsub **lemma** NonpositiveNegated: $\Pi\mathcal{X}\colon \mathcal{X} <= 0 \vdash 0 <= -\mathcal{X}$]

ZFsub **proof of** NonpositiveNegated:

L01:	Arbitrary $\gg$	$\mathcal{X}$	;
L02:	Premise $\gg$	$\mathcal{X} <= 0$	;
L03:	LeqNegated $\rhd$ L02 $\gg$	$-0 <= -\mathcal{X}$	;
L04:	$-0 = 0 \gg$	$-0 = 0$	;
L05:	subLeqLeft $\rhd$ L04 $\rhd$ L03 $\gg$	$0 <= -\mathcal{X}$	$\Box$

[ZFsub **lemma** NegativeNegated: $\Pi\mathcal{X}\colon \mathcal{X} < 0 \vdash 0 < -\mathcal{X}$]

ZFsub **proof of** NegativeNegated:

L01:	Arbitrary $\gg$	$\mathcal{X}$	;
L02:	Premise $\gg$	$\mathcal{X} < 0$	;
L03:	LessNegated $\rhd$ L02 $\gg$	$-0 < -\mathcal{X}$	;

L04: $-0 = 0 \gg$ $-0 = 0$;
L05: SubLessLeft $\rhd$ L04 $\rhd$ L03 $\gg$ $0 < -\mathcal{X}$ $\square$

[ZFsub **lemma** NonnegativeNegated: $\Pi\mathcal{X}\colon 0 <= \mathcal{X} \vdash -\mathcal{X} <= 0$]

ZFsub **proof of** NonnegativeNegated:

L01: Arbitrary $\gg$ $\mathcal{X}$;
L02: Premise $\gg$ $0 <= \mathcal{X}$;
L03: LeqNegated $\rhd$ L02 $\gg$ $-\mathcal{X} <= -0$;
L04: $-0 = 0 \gg$ $-0 = 0$;
L05: subLeqRight $\rhd$ L04 $\rhd$ L03 $\gg$ $-\mathcal{X} <= 0$ $\square$

[ZFsub **lemma** PositiveHalved: $\Pi\mathcal{X}\colon 0 < \mathcal{X} \vdash 0 < 1/2 * \mathcal{X}$]

ZFsub **proof of** PositiveHalved:

L01: Arbitrary $\gg$ $\mathcal{X}$;
L02: Premise $\gg$ $0 < \mathcal{X}$;
L03: $0 < 1/2 \gg$ $0 < 1/2$;
L04: LessMultiplicationLeft $\rhd$ L03 $\rhd$ L02 $\gg$ $1/2 * 0 < 1/2 * \mathcal{X}$;
L05: $\mathrm{x} * 0 = 0 \gg$ $1/2 * 0 = 0$;
L06: SubLessLeft $\rhd$ L05 $\rhd$ L04 $\gg$ $0 < 1/2 * \mathcal{X}$ $\square$

(*** NUMERISK ***)

[ZFsub **lemma** NonnegativeNumerical: $\Pi\mathcal{X}\colon 0 <= \mathcal{X} \vdash |\mathcal{X}| = \mathcal{X}$]

ZFsub **proof of** NonnegativeNumerical:

L01: Arbitrary $\gg$ $\mathcal{X}$;
L02: Premise $\gg$ $0 <= \mathcal{X}$;
L03: IfThenElse(T) $\rhd$ L02 $\gg$ $\mathrm{if}(0 <= \mathcal{X}, \mathcal{X}, -\mathcal{X}) = \mathcal{X}$;
L04: Repetition $\rhd$ L03 $\gg$ $|\mathcal{X}| = \mathcal{X}$ $\square$

[ZFsub **lemma** PositiveNumerical: $\Pi\mathcal{X}\colon 0 < \mathcal{X} \vdash |\mathcal{X}| = \mathcal{X}$]

ZFsub **proof of** PositiveNumerical:

L01: Arbitrary $\gg$ $\mathcal{X}$;
L02: Premise $\gg$ $0 < \mathcal{X}$;
L03: LessLeq $\rhd$ L02 $\gg$ $0 <= \mathcal{X}$;
L04: NonnegativeNumerical $\rhd$ L03 $\gg$ $|\mathcal{X}| = \mathcal{X}$ $\square$

[ZFsub **lemma** NegativeNumerical: $\Pi\mathcal{X}\colon \mathcal{X} < 0 \vdash |\mathcal{X}| = -\mathcal{X}$]

ZFsub **proof of** NegativeNumerical:

L01: Arbitrary $\gg$ $\mathcal{X}$;
L02: Premise $\gg$ $\mathcal{X} < 0$;
L03: FromLess $\rhd$ L02 $\gg$ $\dot\neg\, 0 <= \mathcal{X}$;
L04: IfThenElse(F) $\rhd$ L03 $\gg$ $\mathrm{if}(0 <= \mathcal{X}, \mathcal{X}, -\mathcal{X}) = -\mathcal{X}$;
L05: Repetition $\rhd$ L04 $\gg$ $|\mathcal{X}| = -\mathcal{X}$ $\square$

[ZFsub **lemma** lemma nonpositiveNumerical: $\Pi\mathcal{X}\colon \mathcal{X} <= 0 \vdash |\mathcal{X}| = -\mathcal{X}$]

ZFsub **proof of** lemma nonpositiveNumerical:

L01: Block $\gg$ Begin ;
L02: Arbitrary $\gg$ $\mathcal{X}$;
L03: Premise $\gg$ $\mathcal{X} < 0$;
L04: NegativeNumerical $\rhd$ L03 $\gg$ $|\mathcal{X}| = -\mathcal{X}$;
L05: Block $\gg$ End ;

L06:	Block ≫	Begin	;
L07:	Arbitrary ≫	$\mathcal{X}$	;
L08:	Premise ≫	$\mathcal{X} = 0$	;
L09:	eqSymmetry ⊳ L08 ≫	$0 = \mathcal{X}$	;
L10:	eqLeq ⊳ L09 ≫	$0 <= \mathcal{X}$	;
L11:	NonnegativeNumerical ⊳ L10 ≫	$\|\mathcal{X}\| = \mathcal{X}$	;
L12:	$-0 = 0$ ≫	$-0 = 0$	;
L13:	eqSymmetry ⊳ L12 ≫	$0 = -0$	;
L14:	EqNegated ⊳ L09 ≫	$-0 = -\mathcal{X}$	;
L15:	eqTransitivity5 ⊳ L11 ⊳ L08 ⊳ L13 ⊳ L14 ≫	$\|\mathcal{X}\| = -\mathcal{X}$	;
L16:	Block ≫	End	;
L17:	Arbitrary ≫	$\mathcal{X}$	;
L18:	Ded ⊳ L05 ≫	$\mathcal{X} < 0 \Rightarrow \|\mathcal{X}\| = -\mathcal{X}$	;
L19:	Ded ⊳ L16 ≫	$\mathcal{X} = 0 \Rightarrow \|\mathcal{X}\| = -\mathcal{X}$	;
L20:	Premise ≫	$\mathcal{X} <= 0$	;
L21:	LeqLessEq ⊳ L20 ≫	$\mathcal{X} < 0 \dot\vee \mathcal{X} = 0$	;
L22:	FromDisjuncts ⊳ L21 ⊳ L18 ⊳ L19 ≫	$\|\mathcal{X}\| = -\mathcal{X}$	□

[ZFsub **lemma** $|0| = 0$: $|0| = 0$]

ZFsub **proof of** $|0| = 0$:

L01:	leqReflexivity ≫	$0 <= 0$	;
L02:	NonnegativeNumerical ⊳ L01 ≫	$\|0\| = 0$	□

[ZFsub **lemma** $0 <= |\mathrm{x}|$: $\Pi\mathcal{X}$: $0 <= |\mathcal{X}|$]

ZFsub **proof of** $0 <= |\mathrm{x}|$:

L01:	Block ≫	Begin	;
L02:	Arbitrary ≫	$\mathcal{X}$	;
L03:	Premise ≫	$0 <= \mathcal{X}$	;
L04:	NonnegativeNumerical ⊳ L03 ≫	$\|\mathcal{X}\| = \mathcal{X}$	;
L05:	eqSymmetry ⊳ L04 ≫	$\mathcal{X} = \|\mathcal{X}\|$	;
L06:	subLeqRight ⊳ L05 ⊳ L03 ≫	$0 <= \|\mathcal{X}\|$	;
L07:	Block ≫	End	;
L08:	Block ≫	Begin	;
L09:	Arbitrary ≫	$\mathcal{X}$	;
L10:	Premise ≫	$\dot\neg 0 <= \mathcal{X}$	;
L11:	ToLess ⊳ L10 ≫	$\mathcal{X} < 0$	;
L12:	NegativeNumerical ⊳ L11 ≫	$\|\mathcal{X}\| = -\mathcal{X}$	;
L13:	eqSymmetry ⊳ L12 ≫	$-\mathcal{X} = \|\mathcal{X}\|$	;
L14:	NegativeNegated ⊳ L11 ≫	$0 < -\mathcal{X}$	;
L15:	LessLeq ⊳ L14 ≫	$0 <= -\mathcal{X}$	;
L16:	subLeqRight ⊳ L13 ⊳ L15 ≫	$0 <= \|\mathcal{X}\|$	;
L17:	Block ≫	End	;
L18:	Arbitrary ≫	$\mathcal{X}$	;

L19: Ded $\rhd$ L07 $\gg$ $0 <= \mathcal{X} \Rightarrow 0 <= |\mathcal{X}|$;
L20: Ded $\rhd$ L17 $\gg$ $\dot{\neg}\, 0 <= \mathcal{X} \Rightarrow 0 <= |\mathcal{X}|$;
L21: FromNegations $\rhd$ L19 $\rhd$ L20 $\gg$ $0 <= |\mathcal{X}|$ $\square$

[ZFsub **lemma** SameNumerical: $\Pi\mathcal{X}, \mathcal{Y}\colon \mathcal{X} = \mathcal{Y} \vdash |\mathcal{X}| = |\mathcal{Y}|$]

ZFsub **proof of** SameNumerical:

L01: Block $\gg$ Begin ;
L02: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}$;
L03: Premise $\gg$ $0 <= \mathcal{X}$;
L04: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$;
L05: NonnegativeNumerical $\rhd$ L03 $\gg$ $|\mathcal{X}| = \mathcal{X}$;
L06: subLeqRight $\rhd$ L04 $\rhd$ L03 $\gg$ $0 <= \mathcal{Y}$;
L07: NonnegativeNumerical $\rhd$ L06 $\gg$ $|\mathcal{Y}| = \mathcal{Y}$;
L08: eqSymmetry $\rhd$ L07 $\gg$ $\mathcal{Y} = |\mathcal{Y}|$;
L09: eqTransitivity4 $\rhd$ L05 $\rhd$ L04 $\rhd$ L08 $\gg$ $|\mathcal{X}| = |\mathcal{Y}|$;
L10: Block $\gg$ End ;
L11: Block $\gg$ Begin ;
L12: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}$;
L13: Premise $\gg$ $\dot{\neg}\, 0 <= \mathcal{X}$;
L14: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$;
L15: ToLess $\rhd$ L13 $\gg$ $\mathcal{X} < 0$;
L16: NegativeNumerical $\rhd$ L15 $\gg$ $|\mathcal{X}| = -\mathcal{X}$;
L17: SubLessLeft $\rhd$ L14 $\rhd$ L15 $\gg$ $\mathcal{Y} < 0$;
L18: NegativeNumerical $\rhd$ L17 $\gg$ $|\mathcal{Y}| = -\mathcal{Y}$;
L19: eqSymmetry $\rhd$ L18 $\gg$ $-\mathcal{Y} = |\mathcal{Y}|$;
L20: EqNegated $\rhd$ L14 $\gg$ $-\mathcal{X} = -\mathcal{Y}$;
L21: eqTransitivity4 $\rhd$ L16 $\rhd$ L20 $\rhd$ L19 $\gg$ $|\mathcal{X}| = |\mathcal{Y}|$;
L22: Block $\gg$ End ;
L23: Arbitrary $\gg$ $\mathcal{X}, \mathcal{Y}$;
L24: Premise $\gg$ $\mathcal{X} = \mathcal{Y}$;
L25: Ded $\rhd$ L10 $\gg$ $0 <= \mathcal{X} \Rightarrow \mathcal{X} = \mathcal{Y} \Rightarrow |\mathcal{X}| = |\mathcal{Y}|$;
L26: Ded $\rhd$ L22 $\gg$ $\dot{\neg}\, 0 <= \mathcal{X} \Rightarrow \mathcal{X} = \mathcal{Y} \Rightarrow |\mathcal{X}| = |\mathcal{Y}|$;
L27: FromNegations $\rhd$ L25 $\rhd$ L26 $\gg$ $\mathcal{X} = \mathcal{Y} \Rightarrow |\mathcal{X}| = |\mathcal{Y}|$;
L28: MP $\rhd$ L27 $\rhd$ L24 $\gg$ $|\mathcal{X}| = |\mathcal{Y}|$ $\square$

[ZFsub **lemma** SignNumerical(+): $\Pi\mathcal{X}\colon 0 < \mathcal{X} \vdash |\mathcal{X}| = |-\mathcal{X}|$]

ZFsub **proof of** SignNumerical(+):

L01: Arbitrary $\gg$ $\mathcal{X}$;
L02: Premise $\gg$ $0 < \mathcal{X}$;
L03: PositiveNumerical $\rhd$ L02 $\gg$ $|\mathcal{X}| = \mathcal{X}$;
L04: PositiveNegated $\rhd$ L02 $\gg$ $-\mathcal{X} < 0$;
L05: NegativeNumerical $\rhd$ L04 $\gg$ $|-\mathcal{X}| = --\mathcal{X}$;
L06: DoubleMinus $\gg$ $--\mathcal{X} = \mathcal{X}$;

L07:	eqTransitivity ▷ L05 ▷ L06 ≫	$\lvert -\mathcal{X}\rvert = \mathcal{X}$	;
L08:	eqSymmetry ▷ L07 ≫	$\mathcal{X} = \lvert -\mathcal{X}\rvert$	;
L09:	eqTransitivity ▷ L03 ▷ L08 ≫	$\lvert\mathcal{X}\rvert = \lvert -\mathcal{X}\rvert$	□

[ZFsub **lemma** SignNumerical: $\Pi\mathcal{X}\colon \lvert\mathcal{X}\rvert = \lvert -\mathcal{X}\rvert$]

ZFsub **proof of** SignNumerical:

L01:	Block ≫	Begin	;
L02:	Arbitrary ≫	$\mathcal{X}$	;
L03:	Premise ≫	$\mathcal{X} < 0$	;
L04:	NegativeNegated ▷ L03 ≫	$0 < -\mathcal{X}$	;
L05:	SignNumerical(+) ▷ L04 ≫	$\lvert -\mathcal{X}\rvert = \lvert --\mathcal{X}\rvert$	;
L06:	DoubleMinus ≫	$--\mathcal{X} = \mathcal{X}$	;
L07:	SameNumerical ▷ L06 ≫	$\lvert --\mathcal{X}\rvert = \lvert\mathcal{X}\rvert$	;
L08:	eqTransitivity ▷ L05 ▷ L07 ≫	$\lvert -\mathcal{X}\rvert = \lvert\mathcal{X}\rvert$	;
L09:	eqSymmetry ▷ L08 ≫	$\lvert\mathcal{X}\rvert = \lvert -\mathcal{X}\rvert$	;
L10:	Block ≫	End	;
L11:	Block ≫	Begin	;
L12:	Arbitrary ≫	$\mathcal{X}$	;
L03:	Premise ≫	$\mathcal{X} = 0$	;
L04:	EqNegated ▷ L03 ≫	$-\mathcal{X} = -0$	;
L05:	$-0 = 0$ ≫	$-0 = 0$	;
L06:	eqSymmetry ▷ L03 ≫	$0 = \mathcal{X}$	;
L07:	eqTransitivity4 ▷ L04 ▷ L05 ▷ L06 ≫	$-\mathcal{X} = \mathcal{X}$	;
L08:	eqSymmetry ▷ L07 ≫	$\mathcal{X} = -\mathcal{X}$	;
L13:	SameNumerical ▷ L08 ≫	$\lvert\mathcal{X}\rvert = \lvert -\mathcal{X}\rvert$	;
L14:	Block ≫	End	;
L15:	Block ≫	Begin	;
L16:	Arbitrary ≫	$\mathcal{X}$	;
L03:	Premise ≫	$0 < \mathcal{X}$	;
L17:	SignNumerical(+) ▷ L03 ≫	$\lvert\mathcal{X}\rvert = \lvert -\mathcal{X}\rvert$	;
L18:	Block ≫	End	;
L19:	Arbitrary ≫	$\mathcal{X}$	;
L20:	Ded ▷ L10 ≫	$\mathcal{X} < 0 \Rightarrow \lvert\mathcal{X}\rvert = \lvert -\mathcal{X}\rvert$	;
L21:	Ded ▷ L14 ≫	$\mathcal{X} = 0 \Rightarrow \lvert\mathcal{X}\rvert = \lvert -\mathcal{X}\rvert$	;
L22:	Ded ▷ L18 ≫	$0 < \mathcal{X} \Rightarrow \lvert\mathcal{X}\rvert = \lvert -\mathcal{X}\rvert$	;
L23:	LessTotality ≫	$\mathcal{X} < 0 \mathbin{\dot\vee} \mathcal{X} = 0 \mathbin{\dot\vee} 0 < \mathcal{X}$	;
L24:	From3Disjuncts ▷ L23 ▷ L20 ▷ L21 ▷ L22 ≫	$\lvert\mathcal{X}\rvert = \lvert -\mathcal{X}\rvert$	□

[ZFsub **lemma** NumericalDifference: $\Pi\mathcal{X},\mathcal{Y}\colon \lvert\mathcal{X}-\mathcal{Y}\rvert = \lvert\mathcal{Y}-\mathcal{X}\rvert$]

ZFsub **proof of** NumericalDifference:

L01:	Arbitrary ≫	$\mathcal{X},\mathcal{Y}$	;
L02:	SignNumerical ≫	$\lvert\mathcal{X}-\mathcal{Y}\rvert = \lvert -(\mathcal{X}-\mathcal{Y})\rvert$	;
L03:	MinusNegated ≫	$-(\mathcal{X}-\mathcal{Y}) = \mathcal{Y}-\mathcal{X}$	;
L04:	SameNumerical ▷ L03 ≫	$\lvert -(\mathcal{X}-\mathcal{Y})\rvert = \lvert\mathcal{Y}-\mathcal{X}\rvert$	;
L05:	eqTransitivity ▷ L02 ▷ L04 ≫	$\lvert\mathcal{X}-\mathcal{Y}\rvert = \lvert\mathcal{Y}-\mathcal{X}\rvert$	□

[ZFsub **lemma** SplitNumericalSumHelper: $\Pi\mathcal{X},\mathcal{Y}\colon |-\mathcal{X}-\mathcal{Y}| <= |-\mathcal{X}| + |-\mathcal{Y}| \vdash |\mathcal{X}+\mathcal{Y}| <= |\mathcal{X}| + |\mathcal{Y}|$]

ZFsub **proof of** SplitNumericalSumHelper:

L01: Arbitrary $\gg$ $\mathcal{X},\mathcal{Y}$;

L02: Premise $\gg$ $|-\mathcal{X}-\mathcal{Y}| <= |-\mathcal{X}| + |-\mathcal{Y}|$;

L03: SignNumerical $\gg$ $|\mathcal{X}| = |-\mathcal{X}|$;

L04: SignNumerical $\gg$ $|\mathcal{Y}| = |-\mathcal{Y}|$;

L05: AddEquations $\rhd$ L03 $\rhd$ L04 $\gg$ $|\mathcal{X}| + |\mathcal{Y}| = |-\mathcal{X}| + |-\mathcal{Y}|$;

L06: eqSymmetry $\rhd$ L05 $\gg$ $|-\mathcal{X}| + |-\mathcal{Y}| = |\mathcal{X}| + |\mathcal{Y}|$;

L07: $-\mathrm{x}-\mathrm{y} = -(\mathrm{x}+\mathrm{y})$ $\gg$ $-\mathcal{X}-\mathcal{Y} = -(\mathcal{X}+\mathcal{Y})$;

L08: SameNumerical $\rhd$ L07 $\gg$ $|-\mathcal{X}-\mathcal{Y}| = |-(\mathcal{X}+\mathcal{Y})|$;

L09: SignNumerical $\gg$ $|\mathcal{X}+\mathcal{Y}| = |-(\mathcal{X}+\mathcal{Y})|$;

L10: eqSymmetry $\rhd$ L09 $\gg$ $|-(\mathcal{X}+\mathcal{Y})| = |\mathcal{X}+\mathcal{Y}|$;

L11: eqTransitivity $\rhd$ L08 $\rhd$ L10 $\gg$ $|-\mathcal{X}-\mathcal{Y}| = |\mathcal{X}+\mathcal{Y}|$;

L12: subLeqRight $\rhd$ L06 $\rhd$ L02 $\gg$ $|-\mathcal{X}-\mathcal{Y}| <= |\mathcal{X}| + |\mathcal{Y}|$;

L13: subLeqLeft $\rhd$ L11 $\rhd$ L12 $\gg$ $|\mathcal{X}+\mathcal{Y}| <= |\mathcal{X}| + |\mathcal{Y}|$ $\Box$

[ZFsub **lemma** splitNumericalSum(++): $\Pi\mathcal{X},\mathcal{Y}\colon 0 <= \mathcal{X} \vdash 0 <= \mathcal{Y} \vdash |\mathcal{X}+\mathcal{Y}| <= |\mathcal{X}| + |\mathcal{Y}|$]

ZFsub **proof of** splitNumericalSum(++):

L01: Arbitrary $\gg$ $\mathcal{X},\mathcal{Y}$;

L02: Premise $\gg$ $0 <= \mathcal{X}$;

L03: Premise $\gg$ $0 <= \mathcal{Y}$;

L04: AddEquations(Leq) $\rhd$ L02 $\rhd$ L03 $\gg$ $0+0 <= \mathcal{X}+\mathcal{Y}$;

L05: plus0 $\gg$ $0+0 = 0$;

L06: subLeqLeft $\rhd$ L05 $\rhd$ L04 $\gg$ $0 <= \mathcal{X}+\mathcal{Y}$;

L07: NonnegativeNumerical $\rhd$ L06 $\gg$ $|\mathcal{X}+\mathcal{Y}| = \mathcal{X}+\mathcal{Y}$;

L08: NonnegativeNumerical $\rhd$ L02 $\gg$ $|\mathcal{X}| = \mathcal{X}$;

L09: NonnegativeNumerical $\rhd$ L03 $\gg$ $|\mathcal{Y}| = \mathcal{Y}$;

L10: AddEquations $\rhd$ L08 $\rhd$ L09 $\gg$ $|\mathcal{X}| + |\mathcal{Y}| = \mathcal{X}+\mathcal{Y}$;

L11: eqSymmetry $\rhd$ L10 $\gg$ $\mathcal{X}+\mathcal{Y} = |\mathcal{X}| + |\mathcal{Y}|$;

L12: eqTransitivity $\rhd$ L07 $\rhd$ L11 $\gg$ $|\mathcal{X}+\mathcal{Y}| = |\mathcal{X}| + |\mathcal{Y}|$;

L13: eqLeq $\rhd$ L12 $\gg$ $|\mathcal{X}+\mathcal{Y}| <= |\mathcal{X}| + |\mathcal{Y}|$ $\Box$

[ZFsub **lemma** splitNumericalSum(−−): $\Pi\mathcal{X},\mathcal{Y}\colon \mathcal{X} <= 0 \vdash \mathcal{Y} <= 0 \vdash |\mathcal{X}+\mathcal{Y}| <= |\mathcal{X}| + |\mathcal{Y}|$]

ZFsub **proof of** splitNumericalSum(−−):

L01: Arbitrary $\gg$ $\mathcal{X},\mathcal{Y}$;

L02: Premise $\gg$ $\mathcal{X} <= 0$;

L03: Premise $\gg$ $\mathcal{Y} <= 0$;

L04: NonpositiveNegated $\rhd$ L02 $\gg$ $0 <= -\mathcal{X}$;

L05: NonpositiveNegated $\rhd$ L03 $\gg$ $0 <= -\mathcal{Y}$;

L06: splitNumericalSum(++) $\rhd$ L04 $\rhd$ L05 $\gg$ $|-\mathcal{X}-\mathcal{Y}| <= |-\mathcal{X}| + |-\mathcal{Y}|$;

L07:	SplitNumericalSumHelper ▷ L06 ≫	$\|\mathcal{X}+\mathcal{Y}\| <= \|\mathcal{X}\|+\|\mathcal{Y}\|$	□

[ZFsub **lemma** splitNumericalSum(+ − small): $\Pi\mathcal{X},\mathcal{Y}\colon 0 <= \mathcal{X} \vdash \mathcal{Y} <= 0 \vdash |\mathcal{Y}| <= |\mathcal{X}| \vdash |\mathcal{X}+\mathcal{Y}| <= |\mathcal{X}|$]

ZFsub **proof of** splitNumericalSum(+ − small):

L01:	Arbitrary ≫	$\mathcal{X},\mathcal{Y}$	;
L02:	Premise ≫	$0 <= \mathcal{X}$	;
L03:	Premise ≫	$\mathcal{Y} <= 0$	;
L04:	Premise ≫	$\|\mathcal{Y}\| <= \|\mathcal{X}\|$	;
L05:	LeqAdditionLeft ▷ L03 ≫	$\mathcal{X}+\mathcal{Y} <= \mathcal{X}+0$	;
L06:	plus0 ≫	$\mathcal{X}+0 = \mathcal{X}$	;
L07:	subLeqRight ▷ L06 ▷ L05 ≫	$\mathcal{X}+\mathcal{Y} <= \mathcal{X}$	;
L08:	PositiveToRight(Leq)(1term) ▷ L04 ≫	$0 <= \|\mathcal{X}\| - \|\mathcal{Y}\|$	;
L09:	lemma nonpositiveNumerical ▷ L03 ≫	$\|\mathcal{Y}\| = -\mathcal{Y}$	;
L10:	EqNegated ▷ L09 ≫	$-\|\mathcal{Y}\| = - - \mathcal{Y}$	;
L11:	DoubleMinus ≫	$- - \mathcal{Y} = \mathcal{Y}$	;
L12:	eqTransitivity ▷ L10 ▷ L11 ≫	$-\|\mathcal{Y}\| = \mathcal{Y}$	;
L13:	NonnegativeNumerical ▷ L02 ≫	$\|\mathcal{X}\| = \mathcal{X}$	;
L14:	AddEquations ▷ L13 ▷ L12 ≫	$\|\mathcal{X}\| - \|\mathcal{Y}\| = \mathcal{X}+\mathcal{Y}$	;
L15:	subLeqRight ▷ L14 ▷ L08 ≫	$0 <= \mathcal{X}+\mathcal{Y}$	;
L16:	NonnegativeNumerical ▷ L15 ≫	$\|\mathcal{X}+\mathcal{Y}\| = \mathcal{X}+\mathcal{Y}$	;
L17:	eqSymmetry ▷ L16 ≫	$\mathcal{X}+\mathcal{Y} = \|\mathcal{X}+\mathcal{Y}\|$	;
L18:	eqSymmetry ▷ L13 ≫	$\mathcal{X} = \|\mathcal{X}\|$	;
L19:	subLeqLeft ▷ L17 ▷ L07 ≫	$\|\mathcal{X}+\mathcal{Y}\| <= \mathcal{X}$	;
L20:	subLeqRight ▷ L18 ▷ L19 ≫	$\|\mathcal{X}+\mathcal{Y}\| <= \|\mathcal{X}\|$	□

[ZFsub **lemma** splitNumericalSum(+ − big): $\Pi\mathcal{X},\mathcal{Y}\colon 0 <= \mathcal{X} \vdash \mathcal{Y} <= 0 \vdash |\mathcal{X}| < |\mathcal{Y}| \vdash |\mathcal{X}+\mathcal{Y}| <= |\mathcal{Y}|$]

ZFsub **proof of** splitNumericalSum(+ − big):

L01:	Arbitrary ≫	$\mathcal{X},\mathcal{Y}$	;
L02:	Premise ≫	$0 <= \mathcal{X}$	;
L03:	Premise ≫	$\mathcal{Y} <= 0$	;
L04:	Premise ≫	$\|\mathcal{X}\| < \|\mathcal{Y}\|$	;
L05:	NonnegativeNegated ▷ L02 ≫	$-\mathcal{X} <= 0$	;
L06:	NonpositiveNegated ▷ L03 ≫	$0 <= -\mathcal{Y}$	;
L07:	SignNumerical ≫	$\|\mathcal{X}\| = \|-\mathcal{X}\|$	;
L08:	SubLessLeft ▷ L07 ▷ L04 ≫	$\|-\mathcal{X}\| < \|\mathcal{Y}\|$	;
L09:	SignNumerical ≫	$\|\mathcal{Y}\| = \|-\mathcal{Y}\|$	;
L10:	SubLessRight ▷ L09 ▷ L08 ≫	$\|-\mathcal{X}\| < \|-\mathcal{Y}\|$	;
L11:	LessLeq ▷ L10 ≫	$\|-\mathcal{X}\| <= \|-\mathcal{Y}\|$	;
L12:	splitNumericalSum(+ − small) ▷ L06 ▷ L05 ▷ L11 ≫	$\|-\mathcal{Y}-\mathcal{X}\| <= \|-\mathcal{Y}\|$	;
L13:	SignNumerical ≫	$\|\mathcal{X}+\mathcal{Y}\| = \|-(\mathcal{X}+\mathcal{Y})\|$	;

L14:	$-\mathrm{x}-\mathrm{y}=-(\mathrm{x}+\mathrm{y}) \gg$	$-\mathcal{X}-\mathcal{Y}=-(\mathcal{X}+\mathcal{Y})$	;
L15:	plusCommutativity $\gg$	$-\mathcal{X}-\mathcal{Y}=-\mathcal{Y}-\mathcal{X}$	;
L16:	Equality $\triangleright$ L14 $\triangleright$ L15 $\gg$	$-(\mathcal{X}+\mathcal{Y})=-\mathcal{Y}-\mathcal{X}$	;
L17:	SameNumerical $\triangleright$ L16 $\gg$	$\|-(\mathcal{X}+\mathcal{Y})\|=\|-\mathcal{Y}-\mathcal{X}\|$	;
L18:	eqTransitivity $\triangleright$ L13 $\triangleright$ L17 $\gg$	$\|\mathcal{X}+\mathcal{Y}\|=\|-\mathcal{Y}-\mathcal{X}\|$	;
L19:	eqSymmetry $\triangleright$ L18 $\gg$	$\|-\mathcal{Y}-\mathcal{X}\|=\|\mathcal{X}+\mathcal{Y}\|$	;
L20:	eqSymmetry $\triangleright$ L09 $\gg$	$\|-\mathcal{Y}\|=\|\mathcal{Y}\|$	;
L21:	subLeqLeft $\triangleright$ L19 $\triangleright$ L12 $\gg$	$\|\mathcal{X}+\mathcal{Y}\| <= \|-\mathcal{Y}\|$	;
L22:	subLeqRight $\triangleright$ L20 $\triangleright$ L21 $\gg$	$\|\mathcal{X}+\mathcal{Y}\| <= \|\mathcal{Y}\|$	$\square$

[ZFsub **lemma** splitNumericalSum(+−): $\Pi\mathcal{X},\mathcal{Y}\colon 0 <= \mathcal{X} \vdash \mathcal{Y} <= 0 \vdash |\mathcal{X}+\mathcal{Y}| <= |\mathcal{X}|+|\mathcal{Y}|$]

ZFsub **proof of** splitNumericalSum(+−):

L01:	Block $\gg$	Begin	;
L02:	Arbitrary $\gg$	$\mathcal{X},\mathcal{Y}$	;
L03:	Premise $\gg$	$\|\mathcal{Y}\| <= \|\mathcal{X}\|$	;
L04:	Premise $\gg$	$0 <= \mathcal{X}$	;
L05:	Premise $\gg$	$\mathcal{Y} <= 0$	;
L06:	splitNumericalSum(+ − small) $\triangleright$ L04 $\triangleright$ L05 $\triangleright$ L03 $\gg$	$\|\mathcal{X}+\mathcal{Y}\| <= \|\mathcal{X}\|$	;
L07:	$0 <= \|\mathrm{x}\| \gg$	$0 <= \|\mathcal{Y}\|$	;
L08:	LeqAdditionLeft $\triangleright$ L07 $\gg$	$\|\mathcal{X}\|+0 <= \|\mathcal{X}\|+\|\mathcal{Y}\|$	;
L09:	plus0 $\gg$	$\|\mathcal{X}\|+0=\|\mathcal{X}\|$	;
L10:	subLeqLeft $\triangleright$ L09 $\triangleright$ L08 $\gg$	$\|\mathcal{X}\| <= \|\mathcal{X}\|+\|\mathcal{Y}\|$	;
L11:	leqTransitivity $\triangleright$ L06 $\triangleright$ L10 $\gg$	$\|\mathcal{X}+\mathcal{Y}\| <= \|\mathcal{X}\|+\|\mathcal{Y}\|$	;
L12:	Block $\gg$	End	;
L13:	Block $\gg$	Begin	;
L14:	Arbitrary $\gg$	$\mathcal{X},\mathcal{Y}$	;
L15:	Premise $\gg$	$\dot{\neg}\,\|\mathcal{Y}\| <= \|\mathcal{X}\|$	;
L16:	Premise $\gg$	$0 <= \mathcal{X}$	;
L17:	Premise $\gg$	$\mathcal{Y} <= 0$	;
L18:	ToLess $\triangleright$ L15 $\gg$	$\|\mathcal{X}\| < \|\mathcal{Y}\|$	;
L19:	splitNumericalSum(+ − big) $\triangleright$ L16 $\triangleright$ L17 $\triangleright$ L18 $\gg$	$\|\mathcal{X}+\mathcal{Y}\| <= \|\mathcal{Y}\|$	;
L20:	$0 <= \|\mathrm{x}\| \gg$	$0 <= \|\mathcal{X}\|$	;
L21:	leqAddition $\triangleright$ L20 $\gg$	$0+\|\mathcal{Y}\| <= \|\mathcal{X}\|+\|\mathcal{Y}\|$	;
L22:	plus0Left $\gg$	$0+\|\mathcal{Y}\|=\|\mathcal{Y}\|$	;
L23:	subLeqLeft $\triangleright$ L22 $\triangleright$ L21 $\gg$	$\|\mathcal{Y}\| <= \|\mathcal{X}\|+\|\mathcal{Y}\|$	;
L24:	leqTransitivity $\triangleright$ L19 $\triangleright$ L23 $\gg$	$\|\mathcal{X}+\mathcal{Y}\| <= \|\mathcal{X}\|+\|\mathcal{Y}\|$	;
L25:	Block $\gg$	End	;
L26:	Arbitrary $\gg$	$\mathcal{X},\mathcal{Y}$	;
L27:	Ded $\triangleright$ L12 $\gg$	$\|\mathcal{Y}\| <= \|\mathcal{X}\| \Rightarrow 0 <= \mathcal{X} \Rightarrow \mathcal{Y} <= 0 \Rightarrow \|\mathcal{X}+\mathcal{Y}\| <= \|\mathcal{X}\|+\|\mathcal{Y}\|$	;
L28:	Ded $\triangleright$ L25 $\gg$	$\dot{\neg}\,\|\mathcal{Y}\| <= \|\mathcal{X}\| \Rightarrow 0 <= \mathcal{X} \Rightarrow \mathcal{Y} <= 0 \Rightarrow \|\mathcal{X}+\mathcal{Y}\| <= \|\mathcal{X}\|+\|\mathcal{Y}\|$	;
L29:	Premise $\gg$	$0 <= \mathcal{X}$	;
L30:	Premise $\gg$	$\mathcal{Y} <= 0$	;

L31: FromNegations $\rhd$ L27 $\rhd$ L28 $\gg$ $0 <= \mathcal{X} \Rightarrow \mathcal{Y} <= 0 \Rightarrow |\mathcal{X}+\mathcal{Y}| <= |\mathcal{X}|+|\mathcal{Y}|$;

L32: MP2 $\rhd$ L31 $\rhd$ L29 $\rhd$ L30 $\gg$ $|\mathcal{X}+\mathcal{Y}| <= |\mathcal{X}|+|\mathcal{Y}|$ □

[ZFsub **lemma** splitNumericalSum(−+): $\Pi\mathcal{X},\mathcal{Y}\colon \mathcal{X} <= 0 \vdash 0 <= \mathcal{Y} \vdash |\mathcal{X}+\mathcal{Y}| <= |\mathcal{X}|+|\mathcal{Y}|$]

ZFsub **proof of** splitNumericalSum(−+):

L01: Arbitrary $\gg$ $\mathcal{X},\mathcal{Y}$;

L02: Premise $\gg$ $\mathcal{X} <= 0$;

L03: Premise $\gg$ $0 <= \mathcal{Y}$;

L04: NonpositiveNegated $\rhd$ L02 $\gg$ $0 <= -\mathcal{X}$;

L05: NonnegativeNegated $\rhd$ L03 $\gg$ $-\mathcal{Y} <= 0$;

L06: splitNumericalSum(+−) $\rhd$ L04 $\rhd$ L05 $\gg$ $|-\mathcal{X}-\mathcal{Y}| <= |-\mathcal{X}|+|-\mathcal{Y}|$;

L07: SplitNumericalSumHelper $\rhd$ L06 $\gg$ $|\mathcal{X}+\mathcal{Y}| <= |\mathcal{X}|+|\mathcal{Y}|$ □

[ZFsub **lemma** splitNumericalSum: $\Pi\mathcal{X},\mathcal{Y}\colon |\mathcal{X}+\mathcal{Y}| <= |\mathcal{X}|+|\mathcal{Y}|$]

ZFsub **proof of** splitNumericalSum:

L01: Block $\gg$ Begin ;

L02: Arbitrary $\gg$ $\mathcal{X},\mathcal{Y}$;

L03: Premise $\gg$ $0 <= \mathcal{X}$;

L04: Premise $\gg$ $0 <= \mathcal{Y}$;

L05: splitNumericalSum(++) $\rhd$ L03 $\rhd$ L04 $\gg$ $|\mathcal{X}+\mathcal{Y}| <= |\mathcal{X}|+|\mathcal{Y}|$;

L06: Block $\gg$ End ;

L07: Block $\gg$ Begin ;

L08: Arbitrary $\gg$ $\mathcal{X},\mathcal{Y}$;

L09: Premise $\gg$ $0 <= \mathcal{X}$;

L10: Premise $\gg$ $\mathcal{Y} <= 0$;

L11: splitNumericalSum(+−) $\rhd$ L09 $\rhd$ L10 $\gg$ $|\mathcal{X}+\mathcal{Y}| <= |\mathcal{X}|+|\mathcal{Y}|$;

L12: Block $\gg$ End ;

L13: Block $\gg$ Begin ;

L14: Arbitrary $\gg$ $\mathcal{X},\mathcal{Y}$;

L15: Premise $\gg$ $\mathcal{X} <= 0$;

L16: Premise $\gg$ $0 <= \mathcal{Y}$;

L17: splitNumericalSum(−+) $\rhd$ L15 $\rhd$ L16 $\gg$ $|\mathcal{X}+\mathcal{Y}| <= |\mathcal{X}|+|\mathcal{Y}|$;

L18: Block $\gg$ End ;

L19: Block $\gg$ Begin ;

L20: Arbitrary $\gg$ $\mathcal{X},\mathcal{Y}$;

L21: Premise $\gg$ $\mathcal{X} <= 0$;

L22: Premise $\gg$ $\mathcal{Y} <= 0$;

L23: splitNumericalSum(−−) $\rhd$ L21 $\rhd$ L22 $\gg$ $|\mathcal{X}+\mathcal{Y}| <= |\mathcal{X}|+|\mathcal{Y}|$;

L24: Block $\gg$ End ;

L25: Arbitrary $\gg$ $\mathcal{X},\mathcal{Y}$;